Irreducible representations of Khovanov-Lauda-Rouquier algebras of finite type

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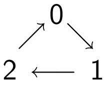

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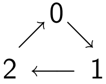

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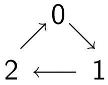

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- $\mathbf{W} := \bigsqcup_{d \geq 0} I^d$ (*words* in the alphabet I);
- for $\alpha \in Q_+$, define *words of weight* α:

$$\mathbf{W}^\alpha := \{\mathbf{i} = (i_1, \ldots, i_d) \in \mathbf{W} \mid \alpha_{i_1} + \cdots + \alpha_{i_d} = \alpha\}.$$

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Each R_α (= a block of R_d) is a unital F-algebra generated by

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and some relations. For example, the relations say that $e(\mathbf{i})$'s are mutually orthogonal idempotents which sum to 1, that e's and y's commute, there is an important relation

$$e(\mathbf{i})\psi_r = \psi_r e((r, r+1) \cdot \mathbf{i}),$$

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for certain explicit polynomial $Q_{i_r,i_{r+1}}$ depending on $c_{i_r,i_{r+1}}$ and orientation (i.e. on how i_r and i_{r+1} are connected in the Dynkin diagram).

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A ($\mathbb{Z}$-)*grading* on R_α is defined by prescribing explicit degrees to the generators $e(\mathbf{i})$, $y_r e(\mathbf{i})$, and $\psi_r e(\mathbf{i})$.

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Motivation 2: Brundan-K.'08 constructed explicit isomorphisms between the usual *cyclotomic Hecke algebras* H_d^Λ and the corresponding *cyclotomic quotients* R_d^Λ of R_d for C "of type A":

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This sheds some new light on the classical representation theory of Hecke algebras and symmetric groups, for example allowing us to grade the corresponding irreducible modules and Specht modules, study *graded decomposition numbers*, and so on.

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Theorem (Brundan-K.'08)

Let the Cartan matrix C be of type

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So representation theory of $R_d(A_\infty)$ is equivalent to representation theory of affine Hecke algebras in characteristic zero (or with generic parameter), while representation theory of $R_d(A_{e-1}^{(1)})$ is equivalent to modular representation theory in "characteristic e".

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(It was noticed by Khovanov and Lauda that irreducible R_d-modules are always finite dimensional.)

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Notation: if $\mathbf{i}$ is the highest word of an irreducible R_α-module L, we denote L by $L(\mathbf{i})$.

Word theory (continued)

Definition

A word $\mathbf{i} \in \mathbf{W}^\alpha$ is called *dominant* if and only if it occurs as a highest word of some (irreducible) R_α-module. The set of all dominant words in $\mathbf{W}^\alpha$ is denoted by $\mathbf{W}^\alpha_+$.

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The goal now is to describe the set of dominant words and to construct the simple modules as heads of certain standard modules.

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Classical fact: every word $\mathbf{i}$ has a unique factorization

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Thus we are reduced to describing only dominant Lyndon words, which we call *minuscule* words.

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(iii) *Let* $\beta \in \Phi^+$ *and*

$$C(\beta) = \{(\beta_1, \beta_2) \in \Phi^+ \times \Phi^+ \mid \beta_1 + \beta_2 = \beta, \ \mathbf{i}_{\beta_1} < \mathbf{i}_{\beta_2}\}.$$

Then $\mathbf{i}_\beta = \max\{\mathbf{i}_{\beta_1}\mathbf{i}_{\beta_2} \mid (\beta_1, \beta_2) \in C(\beta)\}$.

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E.g. if you choose one of the two natural orderings in type A_n:

$$1 < 2 < \cdots < n \qquad \text{or} \qquad 1 > 2 > \cdots > n$$

then the minuscule modules are just one-dimensional and they correspond to Zelevinsky segments.

Cuspidal modules (continued)

Originally, we were able to construct the miniscule modules in all types other than E_8 and F_4 (for some specific natural choice of the ordering on I). In type E_8 we could construct them for all but 12 positive positive roots.

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A special case of their result which improves our work in type E_8 is

Theorem (Hill-Melvin-Mondragon'09)

In simply laced types (ADE), there is an order on I, for which all minuscule modules are homogeneous, *and hence can be explicitly constructed using K.-Ram'08.*

Induction

Let $\alpha, \beta \in Q_+$. There is an obvious (non-unital) algebra embedding of $R_\alpha \otimes R_\beta$ into the $R_{\alpha+\beta}$ mapping $e(\mathbf{i}) \otimes e(\mathbf{j})$ to $e(\mathbf{ij})$.

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$$e_{\alpha,\beta} = \sum_{\mathbf{i} \in \mathbf{W}^\alpha, \mathbf{j} \in \mathbf{W}^\beta} e(\mathbf{ij}).$$

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Consider the functor

$$\mathrm{Ind}_{\alpha,\beta}^{\alpha+\beta} := R_{\alpha+\beta} e_{\alpha,\beta} \otimes_{R_\alpha \otimes R_\beta} ? : R_\alpha \otimes R_\beta\text{-Mod} \to R_{\alpha+\beta}\text{-Mod}\,.$$

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For $\alpha, \beta \in Q_+$, $M \in \mathrm{Rep}(R_\alpha)$ and $N \in \mathrm{Rep}(R_\beta)$, we denote

$$M \circ N := \mathrm{Ind}_{\alpha,\beta}^{\alpha+\beta}(M \boxtimes N).$$

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(iii) *(Generalized Kato) If $\mathbf{i} = \mathbf{j}^n$ for a good Lyndon word $\mathbf{j}$, then $L(\mathbf{i}) = \Delta(\mathbf{i})$.*

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Remark. A different classification of irreducible modules over KLR algebras was obtained by Lauda and Vazirani.