A combinatorial approach to Specht module cohomology

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Introduction

Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_s) \vdash d$ so $p\lambda = (p\lambda_1, p\lambda_2, \dots, p\lambda_s) \vdash pd$. We recently proved the following "generic cohomology" type theorem:

Theorem

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$$\mathrm{H}^1(\Sigma_{pd}, S^{p\lambda}) \cong \mathrm{H}^1(\Sigma_{p^2d}, S^{p^2\lambda}).$$

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The proof is not constructive. We transfer the problem (requiring $p > 2$) to an algebraic group setting (where $p\lambda$ corresponds to a Frobenius twist) and use work of Doty and Andersen.

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Degree 0 cohomology: In his 1978 Lecture notes, James computed

$$\mathrm{H}^0(\Sigma_d, S^\lambda) = \mathrm{Hom}_{\Sigma_d}(k, S^\lambda)$$

in a combinatorial way using his famous Kernel Intersection Theorem.

Degree 1 cohomology

Recall that $\mathrm{H}^1(\Sigma_d, S^\lambda) \cong \mathrm{Ext}^1_{\Sigma_d}(k, S^\lambda)$ measures equivalence classes of short exact sequences:

$$0 \to S^\lambda \to U \to k \to 0.$$

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For any such nonsplit SES we prove that (if $p > 2$) U embeds in the permutation module M^λ, which lets us apply combinatorial techniques to compute cohomology. The technique explicitly fails in characteristic two.

Young tableaux and tabloids

Definition

Let λ be a partition of d. A λ- tableau t is an assignment of the numbers $\{1, 2, \ldots, d\}$ to the boxes in the Young diagram for λ.

For example if $\lambda = (3, 2, 1)$ then:

$$t = \begin{matrix} 1 & 4 & 2 \\ 6 & 3 & \\ 5 & & \end{matrix}$$

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Definition

A tabloid $\{t\}$ is an equivalence class of tableau under row permutations..

For $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_s) \vdash d$ there is a Young subgroup

$$\Sigma_\lambda \cong \Sigma_{\lambda_1} \times \cdots \times \Sigma_{\lambda_s}.$$

Over any field one can define the permutation module

$$M^\lambda := \mathrm{Ind}_{\Sigma_\lambda}^{\Sigma_d} k.$$

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The λ-tabloids give a basis for the module M^λ. For example:

$$M^{(2,2)} = \langle \{ \begin{smallmatrix} 1 & 2 \\ 3 & 4 \end{smallmatrix} \}, \{ \begin{smallmatrix} 1 & 3 \\ 2 & 4 \end{smallmatrix} \}, \{ \begin{smallmatrix} 1 & 4 \\ 2 & 3 \end{smallmatrix} \}, \{ \begin{smallmatrix} 2 & 3 \\ 1 & 4 \end{smallmatrix} \}, \{ \begin{smallmatrix} 2 & 4 \\ 1 & 3 \end{smallmatrix} \}, \{ \begin{smallmatrix} 3 & 4 \\ 1 & 2 \end{smallmatrix} \} \rangle$$

Specht modules

To each $\lambda \vdash d$ there is a Specht module $S^\lambda \subseteq M^\lambda$ such that:

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- M^λ / S^λ has a filtration by Specht modules S^μ with $\mu > \lambda$ and known multiplicities.
- $\mathrm{H}^i(\Sigma_d, (S^\lambda)^*) = 0$ for $1 \leq i \leq p - 3$.

Kernel Intersection Theorem

As part of his construction of a Specht filtration of M^{λ}, Gordon James proved the following alternate characterization of $S^{\lambda} \subseteq M^{\lambda}$:

Theorem (Kernel Intersection Theorem)

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The theorem actually says one can check only certain maps ψ to M^μ's where μ and λ differ in only two rows. The maps ψ are easy to describe combinatorially. So given a vector $u \in M^\lambda$ one has an explicit combinatorial test for whether $u \in S^\lambda$.

James' computation of $H^0(\Sigma_d, S^\lambda)$

Let $f_\lambda \in M^\lambda$ denote the sum of all λ-tabloids. Then $\mathrm{Hom}_{\Sigma_d}(k, M^\lambda) \cong k$ with image spanned by f_λ.

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Since $S^\lambda \subseteq M^\lambda$, to determine if $\mathrm{Hom}_{\Sigma_d}(k, S^\lambda) \neq 0$ one must test if $f_\lambda \in S^\lambda$. James used the K.I.T. to test this.

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For an integer t let $l_p(t)$ be the least nonnegative integer satisfying $t < p^{l_p(t)}$. James proved:

Theorem

$\mathrm{H}^0(\Sigma_d, S^\lambda)$ *is zero unless* $\lambda_i \equiv -1 \mod p^{l_p(\lambda_{i+1})}$ *for all* i, *in which case it is one-dimensional.*

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For example $\mathrm{H}^1(\Sigma_{64}, S^{(26,17,8,8,2,2,1)}) \neq 0$ if $p = 3$.

Assume now $p > 2$. The key observation, which follows easily from the fact that $\mathrm{H}^1(\Sigma_d, k) = 0$ in odd characteristic, is:

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Suppose U is a nonsplit extension of S^λ by k. Then U embeds in M^λ.

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Determining if $\mathrm{H}^1(\Sigma_d, S^\lambda) \neq 0$ is equivalent to finding such a u.

An example

For a two-part partition $\lambda = (\lambda_1, \lambda_2)$ we will denote a λ-tabloid uniquely by its second row. For example if $\lambda = (3, 3)$ we denote:

$$\{t\} = \begin{matrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{matrix} \text{ by } \overline{246}.$$

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The relevant maps for the K.I.T. are:

$$\begin{aligned} \psi_{2,0}(\overline{246}) &= \overline{\emptyset} \in M^{(6)} \\ \psi_{2,1}(\overline{246}) &= \overline{2} + \overline{4} + \overline{6} \in M^{(5,1)} \\ \psi_{2,2}(\overline{246}) &= \overline{24} + \overline{26} + \overline{46} \in M^{(4,2)} \end{aligned}$$

A sample nonzero $H^1(\Sigma_d, S^\lambda)$.

Let $p = 3$ and $\lambda = (3,3)$. Define $u \in M^{(3,3)}$ by:

$$\begin{aligned} u &= \overline{134} + \overline{135} + \overline{136} + \overline{145} + \overline{146} + \overline{156} \\ &+ \overline{234} + \overline{235} + \overline{236} + \overline{245} + \overline{246} + \overline{256} \\ &- \overline{123} - \overline{124} - \overline{125} - \overline{126} \end{aligned}$$

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Finally check that

$$\psi_{2,1}(af_{(3,3)} - u) = (-a-1)f_{(6)}, \;\; \psi_{2,0}(af_{(3,3)} - u) = (a-1)f_{(5,1)}.$$

The calculations on the previous slide show:

Proposition

In characteristic three, $\mathrm{H}^1(\Sigma_6, S^{(3,3)}) \neq 0$ *and the subspace of* $M^{(3,3)}$ *spanned by* S^λ *and* u *is a nonsplit extension of* $S^{(3,3)}$ *by* k*.*

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- Warning: The choice of u is not unique, since any $u + v, v \in S^\lambda$ also works. So to prove general results one must be "strategic" in the choice of u.

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- Our main goal is to realize the isomorphism $\mathrm{H}^1(\Sigma_{pd}, S^{p\lambda}) \cong \mathrm{H}^1(\Sigma_{p^2d}, S^{p^2\lambda})$. Start with a $u \in M^{p\lambda}$ which "works" for $S^{p\lambda}$ and give a procedure to construct one for $S^{p^2\lambda}$.

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- Use the technique to do other cohomology calculations.
- It would be much harder to use this method to show $\mathrm{H}^1(\Sigma_d, S^\lambda)$ is at most one-dimensional.

A more general example

Let $\lambda = (p^a, p^a)$. Let

$$v_i = \sum \{\{t\} \in M^{(p^a,p^a)} \mid \text{Exactly i of } \{1, 2, \ldots, p^a - 1\} \text{ lie in row two.}\}$$

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Proposition

For u as above,

$$\psi_{2,0}(u) = \overline{\emptyset}, \ \ \psi_{2,i}(u) = 0 \ \forall i > 0.$$

Thus $\mathrm{H}^1(\Sigma_{2p^a}, S^{(p^a,p^a)}) \neq 0$ *and* $\langle S^\lambda, u \rangle$ *gives a nonsplit extension.*

Another example

Let $\lambda = (p^b - 1, p^a) \vdash$ for $a < b$. Let:

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Remark: The proofs of the previous two examples both involve binomial coefficients and combinatorial identities, the first is much more elaborate than the second. The coefficients in the image of the various $\psi_{2,i}$'s are sums of products of binomial coefficients.

Future Directions

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Conjecture

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_s) \vdash d$ and suppose $a \equiv -1 \bmod p^{l_p(\lambda_1)}$. Then:

$$\mathrm{H}^1(\Sigma_d, S^\lambda) \cong \mathrm{H}^1(\Sigma_{d+a}, S^{(a,\lambda_1,\ldots,\lambda_s)}).$$

The corresponding result for H^0 is clear from James' condition.

Problem

For $\lambda \neq (d)$ one can show

$$\mathrm{H}^0(\Sigma_d, S^\lambda) \neq 0 \Rightarrow \mathrm{H}^1(\Sigma_d, S^\lambda) \neq 0.$$

Prove this by constructing a u that works for each λ satisfying James' condition.

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Problem

Compute $H^1(\Sigma_d, S^\lambda)$. I have a conjecture that it is at most one-dimensional, and a conjecture for which λ it is nonzero.