Vanishing Ranges for the Cohomology of Finite Groups of Lie Type

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The Problem

- k - algebraically closed field of characteristic $p > 0$
- G - reductive algebraic group over k (split over $\mathbb{F}_p$)
- $q = p^r$ for an integer $r \geq 1$
- $G(\mathbb{F}_q)$ - associated finite Chevalley group

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Question

What is the least positive i such that

$$\mathrm{H}^i(G(\mathbb{F}_q), k) \neq 0?$$

Quillen, 1972

- $\mathrm{H}^i(GL_n(\mathbb{F}_q), k) = 0$ for $0 < i < r(p-1)$.
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- For SL_2: $C = \frac{p-1}{2}$; no other values given.
- Not sharp in general (cf. Carlson, 1983)

More Vanishing Ranges

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- $B(\mathbb{F}_q) \subset GL_n(\mathbb{F}_q)$
- $\mathrm{H}^i(B(\mathbb{F}_q), k) = 0$ for $0 < i < r(2p - 3)$

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- $\mathrm{H}^{r(2p-3)}(B(\mathbb{F}_q), k) \neq 0$

Notation

- G - simple and simply connected
- Φ - root system
- T - torus
- B - Borel subgroup (corresponding to negative roots)
- U - its unipotent radical
- W - Weyl group
- $X(T)$ - weights
- $X(T)_+$ - dominant weights
- For $\lambda \in X(T)_+$, $H^0(\lambda) := \mathrm{ind}_B^G(\lambda)$ - the costandard module
- For $\lambda \in X(T)_+$, $V(\lambda) := H^0(\lambda^*)^*$ - the Weyl or standard module
- w_0 - longest word W
- $\lambda^* := -w_0(\lambda)$

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Step 1: $\mathrm{H}^i(G(\mathbb{F}_q), k) \simeq \mathrm{H}^i(G, \mathrm{ind}_{G(\mathbb{F}_q)}^G(k))$

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Step 5: Use inductive arguments to get vanishing for higher r.

The Filtration

Proposition

As a G-module, $\operatorname{ind}_{G(\mathbb{F}_q)}^{G}(k)$ *has a filtration with factors of the form* $H^0(\lambda) \otimes H^0(\lambda^*)^{(r)}$ *with multiplicity one for each* $\lambda \in X(T)_+$.

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"Proof:"
Consider $k[G]$ as a $G \times G$-module structure via the left and right regular actions.
As a G-module via $G \to G \times G$,
Known Fact: $k[G]$ has a filtration by $H^0(\lambda) \otimes H^0(\lambda^*)$.

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New action: $G \to G \times G \overset{Id \times Fr^r}{\to} G \times G$

Key Step: Show that $\mathrm{ind}_{G(\mathbb{F}_q)}^{G}(k) \simeq k[G]$ with this new action.

Vanishing Criteria

Theorem

Let m be the least positive integer such that there exists $\nu \in X(T)_+$ with $\mathrm{H}^m(G, H^0(\nu) \otimes H^0(\nu^*)^{(r)}) \neq 0$. Let $\lambda \in X(T)_+$ be such that $\mathrm{H}^m(G, H^0(\lambda) \otimes H^0(\lambda^*)^{(r)}) \neq 0$. Suppose $\mathrm{H}^{m+1}(G, H^0(\nu) \otimes H^0(\nu^*)^{(r)}) = 0$ for all $\nu < \lambda$ that are linked to λ. Then

(i) $\mathrm{H}^i(G(\mathbb{F}_q), k) = 0$ *for* $0 < i < m$;

(ii) $\mathrm{H}^m(G(\mathbb{F}_q), k) \neq 0$;

(iii) *if, in addition,* $\mathrm{H}^m(G, H^0(\nu) \otimes H^0(\nu^*)^{(r)}) = 0$ *for all* $\nu \in X(T)_+$ *with* $\nu \neq \lambda$, *then* $\mathrm{H}^m(G(\mathbb{F}_q), k) \cong \mathrm{H}^m(G, H^0(\lambda) \otimes H^0(\lambda^*)^{(r)})$.

The New Goal

Study $\mathrm{H}^i(G, H^0(\lambda) \otimes H^0(\lambda^*)^{(r)}) \simeq \mathrm{Ext}^i_G(V(\lambda)^{(r)}, H^0(\lambda))$.

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Lemma

If $\mathrm{H}^i(G_r, H^0(\lambda))^{(-r)}$ *admits a good filtration, then*

$$\begin{aligned}\mathrm{H}^i(G, H^0(\lambda) \otimes H^0(\lambda^*)^{(r)}) &\simeq \mathrm{Ext}^i_G(V(\lambda)^{(r)}, H^0(\lambda)) \\ &\simeq \mathrm{Hom}_G(V(\lambda), \mathrm{H}^i(G_r, H^0(\lambda))^{(-r)})\end{aligned}$$

Hypothesis holds for $r = 1$ and $p > h$ (the Coxeter number).
For $r > 1$, ???

Root Combinatorics

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Theorem (Andersen-Jantzen 1986, Kumar-Lauritzen-Thomsen 1999)

For $p > h$,

$$\mathrm{H}^i(G_1, H^0(\lambda))^{(-1)} = \begin{cases} \mathrm{ind}_B^G(S^{\frac{i-\ell(w)}{2}}(\mathfrak{u}^*) \otimes \mu) & \text{if } \lambda = w \cdot 0 + p\mu \\ 0 & \text{otherwise,} \end{cases}$$

where $\mathfrak{u} = \mathrm{Lie}(U)$.

Note: since $p > h$ and λ is dominant, μ must also be dominant.

Root Combinatorics - Continued

Say $\lambda = p\mu + w \cdot 0$,

$$\begin{aligned}\mathrm{Hom}_G(V(\lambda), \mathrm{H}^i(G_1, H^0(\lambda))^{(-1)}) &\\ &\simeq \mathrm{Hom}_G(V(\lambda), \mathrm{ind}_B^G\, S^{\frac{i-\ell(w)}{2}}(\mathfrak{u}^*) \otimes \mu) \\ &\simeq \mathrm{Hom}_B(V(\lambda), S^{\frac{i-\ell(w)}{2}}(\mathfrak{u}^*) \otimes \mu)\end{aligned}$$

Therefore, $\lambda - \mu = (p-1)\mu + w \cdot 0$ is a sum of $\frac{i-\ell(w)}{2}$ positive roots.

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Therefore, $\lambda - \mu = (p-1)\mu + w \cdot 0$ is a sum of $\frac{i-\ell(w)}{2}$ positive roots.

Lemma

Assume that $p > h$. Assume $\mathrm{H}^i(G, H^0(\lambda) \otimes H^0(\lambda^*)^{(1)}) \neq 0$. Then $i \geq (p-1)\langle \mu, \tilde{\alpha}^\vee \rangle - 1$, where $\tilde{\alpha}$ is the longest root.

Note: $\langle \mu, \tilde{\alpha}^\vee \rangle \geq 1$

Generic Vanishing Theorem

Theorem

Assume that $p > h$. Then

(a) $\mathrm{H}^i(G, H^0(\lambda) \otimes H^0(\lambda^*)^{(r)}) = 0$ *for* $0 < i < r(p-2)$ *and* $\lambda \in X(T)_+$;

(b) $\mathrm{H}^i(G(\mathbb{F}_q), k) = 0$ *for* $0 < i < r(p-2)$.

Note: The $r > 1$ case requires working with $\mathrm{Ext}^i_G(V(\lambda)^{(r)}, H^0(\nu))$ for possibly distinct λ, ν.

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While better than previous bounds in most cases: $\approx r\left(\frac{p-1}{2}\right)$, these are still not sharp in general.

Finding Sharp Bounds

For $\lambda = p\mu + w \cdot 0$, if $\langle \mu, \tilde{\alpha}^\vee \rangle \geq 2$, then $i \geq 2(p-1) - 1 = 2p - 3$.

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In general, the sharp bound will lie between $p - 2$ and $2p - 3$.

If it is strictly less than $2p - 3$, μ is necessarily a fundamental dominant weight. And in fact certain specific such weights.

The task is then to study $\mathrm{H}^i(G, H^0(\lambda) \otimes H^0(\lambda^*)^{(1)})$ for those weights and see whether it in fact is non-zero.

In part, use Kostant Partition Functions.

From the above description of G_1-cohomology, we can more precisely say:

$$\dim \mathrm{H}^i(G, H^0(\lambda) \otimes H^0(\lambda^*)^{(1)}) = \sum_{u \in W} (-1)^{\ell(u)} P_{\frac{i-\ell(w)}{2}}(u \cdot \lambda - \mu)$$

$P_k(\nu)$ is the dimension of the ν weight space in $S^k(\mathfrak{u}^*)$.

Alternatively, $P_k(\nu)$ is the number of distinct ways that ν can be expressed as a sum of exactly k positive roots.

Theorem

Assume that $p > n + 1$.

(a) *(Generic case) If $p > n + 2$ and $n > 3$, then*
   (i) $\mathrm{H}^i(G(\mathbb{F}_p), k) = 0$ *for* $0 < i < 2p - 3$*;*
   (ii) $\mathrm{H}^{2p-3}(G(\mathbb{F}_p), k) = k$*.*

(b) *If $p = n + 2$, then*
   (i) $\mathrm{H}^i(G(\mathbb{F}_p), k) = 0$ *for* $0 < i < p - 2$*;*
   (ii) $\mathrm{H}^{p-2}(G(\mathbb{F}_p), k) = k \oplus k$*.*

Φ of Type A_n, $n \geq 2$ - Continued

Theorem

Assume that $p > n+1$.

(c) *If $n = 2$ and 3 divides $p-1$, then*
  - (i) $\mathrm{H}^i(G(\mathbb{F}_p), k) = 0$ *for* $0 < i < 2p-6$;
  - (ii) $\mathrm{H}^{2p-6}(G(\mathbb{F}_p), k) = k \oplus k$.

(d) *If $n = 2$ and 3 does not divide $p-1$, then*
  - (i) $\mathrm{H}^i(G(\mathbb{F}_p), k) = 0$ *for* $0 < i < 2p-3$;
  - (ii) $\mathrm{H}^{2p-3}(G(\mathbb{F}_p), k) = k$.

(e) *If $n = 3$ and $p > 5$, then*
  - (i) $\mathrm{H}^i(G(\mathbb{F}_p), k) = 0$ *for* $0 < i < 2p-6$;
  - (ii) $\mathrm{H}^{2p-6}(G(\mathbb{F}_p), k) = k$.

Φ of Type A_n - $r > 1$

Theorem

Assume that $p > 2(n+1)$. *Suppose further that* $p > 11$ *whenever* $n = 4$. *Then*

(a) $\mathrm{H}^i(G(\mathbb{F}_p), k) = 0$ *for* $0 < i < r(2p-3)$;

(b) $\mathrm{H}^{r(2p-3)}(G(\mathbb{F}_p), k) \neq 0$.

Φ of Type B_n, $n \geq 3$ - Under Construction

Guess

Assume that $p > 2n$.

(a) *(Generic case) If $n \geq 7$, $n = 6$ with $p \neq 13$, or $n = 5$ with $p \neq 11, 13$ then*
  (i) $\mathrm{H}^i(G(\mathbb{F}_p), k) = 0$ *for* $0 < i < 2p - 3$;
  (ii) $\mathrm{H}^{2p-3}(G(\mathbb{F}_p), k) \neq 0$.

(b) *If $n = 3, 4$, then*
  (i) $\mathrm{H}^i(G(\mathbb{F}_p), k) = 0$ *for* $0 < i < 2p - 8$;
  (ii) $\mathrm{H}^{2p-8}(G(\mathbb{F}_p), k) \neq 0$.

(c) *If $n = 5$ and $p = 11$, then*
  (i) $\mathrm{H}^i(G(\mathbb{F}_p), k) = 0$ *for* $0 < i < 2p - 7$;
  (ii) $\mathrm{H}^{2p-7}(G(\mathbb{F}_p), k) \neq 0$.

(c) *If $n = 5, 6$ and $p = 13$, then*
  (i) $\mathrm{H}^i(G(\mathbb{F}_p), k) = 0$ *for* $0 < i < 2p - 5$;
  (ii) $\mathrm{H}^{2p-5}(G(\mathbb{F}_p), k) \neq 0$.

Note: For the small n cases, the vanishing ranges can are known.

Theorem

Assume that $p > 2n$. Then

(a) $\mathrm{H}^i(G(\mathbb{F}_q), k) = 0$ *for* $0 < i < r(p-2)$;

(b) $\mathrm{H}^{r(p-2)}(G(\mathbb{F}_q), k) \neq 0$.

Note: The only dominant weight with $\mathrm{H}^{p-2}(G, H^0(\lambda) \otimes H^0(\lambda^*)^{(1)}) \neq 0$ is $\lambda = (p-2n)\omega_1$, where ω_1 is the first fundamental dominant weight.

Theorem

Assume that $p > 2n - 2$. Then

(a) $\mathrm{H}^i(G(\mathbb{F}_p), k) = 0$ *for* $0 < i < 2p - 2n$.

(b) $\mathrm{H}^{2p-2n}(G(\mathbb{F}_p), k) = \begin{cases} k \text{ if } n \geq 5 \\ k \oplus k \oplus k \text{ if } n = 4. \end{cases}$

Note: The "special" weight is $\lambda = (p - 2n + 2)\omega_1$

We should be able to extend this to $r > 1$.

Theorem

Assume that $p > 12$.

(a) *If $p \neq 13, 19$, then*
  (i) $\mathrm{H}^i(G(\mathbb{F}_p), k) = 0$ *for* $0 < i < 2p - 3$.
  (ii) $\mathrm{H}^{2p-3}(G(\mathbb{F}_p), k) \neq 0$.

(b) *If $p = 13$, then*
  (i) $\mathrm{H}^i(G(\mathbb{F}_p), k) = 0$ *for* $0 < i < 16$.
  (ii) $\mathrm{H}^{16}(G(\mathbb{F}_p), k) \neq 0$.

(c) *If $p = 19$, then*
  (i) $\mathrm{H}^i(G(\mathbb{F}_p), k) = 0$ *for* $0 < i < 33??$.
  (ii) $\mathrm{H}^{??}(G(\mathbb{F}_p), k) \neq 0$.

We should be able to extend this to $r > 1$.

Φ of type E_7

Theorem

Assume that $p > 18$.

(a) *If $p \neq 19, 23$, then*

   (i) $\mathrm{H}^i(G(\mathbb{F}_p), k) = 0$ *for* $0 < i < 2p - 3$.

   (ii) $\mathrm{H}^{2p-3}(G(\mathbb{F}_p), k) \neq 0$.

(b) *If $p = 19$, then*

   (i) $\mathrm{H}^i(G(\mathbb{F}_p), k) = 0$ *for* $0 < i < 27$.

   (ii) $\mathrm{H}^{27}(G(\mathbb{F}_p), k) \neq 0$.

(c) *If $p = 23$, then*

   (i) $\mathrm{H}^i(G(\mathbb{F}_p), k) = 0$ *for* $0 < i < 39??$.

   (ii) $\mathrm{H}^{??}(G(\mathbb{F}_p), k) \neq 0$.

We should be able to extend this to $r > 1$.

Φ of type E_8

Theorem

Assume that $p > 30$.

(a) $\mathrm{H}^i(G(\mathbb{F}_q), k) = 0$ *for* $0 < i < r(2p-3)$.

(b) $\mathrm{H}^{r(2p-3)}(G(\mathbb{F}_q), k) \neq 0$.

Note: The root lattice equals the weight lattice here.

More generally, this should be the answer whenever Φ is simply laced and the group G is of adjoint type (as opposed to simply connected).

Assume that $p > 12$.

Know: $\mathrm{H}^i(G(\mathbb{F}_p), k) = 0$ for $0 < i < 2p - 9$.

Guess: $\mathrm{H}^{2p-9}(G(\mathbb{F}_p), k) = 0$ and $\mathrm{H}^{2p-8}(G(\mathbb{F}_p), k) \neq 0$

Φ of type G_2

Theorem

Assume that $p > 6$. *Then*

(i) $\mathrm{H}^i(G(\mathbb{F}_p), k) = 0$ *for* $0 < i < 2p - 8$.

(ii) $\mathrm{H}^{2p-8}(G(\mathbb{F}_p), k) \neq 0$.