Geometric Categorification

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Finite Dimensional $\mathfrak{sl}_2$ modules

An action of $U(\mathfrak{sl}_2)$ on a finite dimensional vector space V consists of

- a weight space decomposition $V = \bigoplus_{\lambda \in \mathbb{Z}} V(\lambda)$,
- linear maps

$$e(\lambda) : V(\lambda - 1) \to V(\lambda + 1)$$

$$f(\lambda) : V(\lambda + 1) \to V(\lambda - 1)$$

for each λ. These satisfy

$$e(\lambda - 1)f(\lambda - 1) = \lambda \mathrm{Id}_{V(\lambda)} + f(\lambda + 1)e(\lambda + 1).$$

The element $s = \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right] \in SL_2$ acts on V, giving isomorphisms

$$s : V(-\lambda) \to V(\lambda).$$

$\mathfrak{sl}_2$ Categorification

Frameworks for $\mathfrak{sl}_2$ categorification have been proposed by Chuang-Rouquier, Lauda, Khovanov-Lauda, and Rouquier. They define (a collection of closely related) categories $U_2(\mathfrak{sl}_2)$ which categorify the enveloping algebra $U(\mathfrak{sl}_2)$.

Our Goal: Construct geometric representations of $U_2(\mathfrak{sl}_2)$.

Ordinary representations of $\mathfrak{sl}_2$ have been constructed geometrically by Lusztig, Ginzburg, Nakajima, and others. So it is natural to look for geometric examples of categorified representations.

Geometric $\mathfrak{sl}_2$ Categorification

- Weight spaces get replaced by varieties:

$$V(\lambda) \mapsto Y(\lambda).$$

- Linear transformations get replaced by Fourier-Mukai Kernels

$$e \mapsto \mathcal{E}, \quad f \mapsto \mathcal{F},$$

$$\mathcal{E}(\lambda), \mathcal{F}(\lambda) \in \mathcal{D}^b\left(Y(\lambda-1) \times Y(\lambda+1)\right).$$

  These kernels are required to satisfy $\mathfrak{sl}_2$ relations, but only at the level of cohomology of complexes.

- We require the existence of deformations

$$\tilde{Y}(\lambda) \longrightarrow \mathbb{A}^1$$

  with some special properties.

From Geometry to Categorification

One main virtue of geometric $\mathfrak{sl}_2$ categorifications is that they give rise to categorified representations.

Theorem (Cautis-Kamnitzer-L)

A geometric $\mathfrak{sl}_2$ categorification induces a representation of $\mathrm{U}_2(\mathfrak{sl}_2)$ *on*

$$\bigoplus_{\lambda} \mathcal{D}^b\left(Y(\lambda)\right).$$

The functors E *and* F *are induced by the kernels* $\mathcal{E}$ *and* $\mathcal{F}$*, while the natural transformations (*X *and* T*,* y *and* ψ*, or dots and crosses) are constructed using the deformations* $\tilde{Y}(\lambda)$*.*

A Basic Example

Fix $N \in \mathbb{N}$. For $0 \leq k \leq N$, set

$$Y(2k-N) = T^{\star}Gr(k,N) \cong$$

$$\{(x,V) : x \in M_N(\mathbb{C}), 0 \subset V \subset \mathbb{C}^N, \dim(V) = k \text{ and } \mathbb{C}^N \xrightarrow{x} V \xrightarrow{x} 0\}$$

There are tautological bundles

- V,
- $\mathbb{C}^N/V$

on $Y(\lambda)$.
There are also natural deformations $\tilde{Y}(\lambda)$, given by varying the action of x on V and $\mathbb{C}^N/V$.

Hecke Correpondences

For $r \geq 0$, define $W^r(\lambda) \subset Y(\lambda - r) \times Y(\lambda + r)$ by

$$W^r(\lambda) := \{(x, V, V') : 0 \subset V \subset V' \subset \mathbb{C}^N; \mathbb{C}^N \xrightarrow{x} V \text{ and } V' \xrightarrow{x} 0\}.$$

Projections:

$$\pi_1 : (x, V, V') \mapsto (x, V), \ \ \pi_2 : (x, V, V') \mapsto (x, V').$$

Tautological bundles on $W^r(\lambda)$:

- $V := \pi_1^*(V)$
- $V' := \pi_2^*(V)$

Inclusions:

$$0 \subset V \subset V' \subset \mathbb{C}^N \cong \mathcal{O}_{W^r(\lambda)}^{\oplus N}.$$

Kernels $\mathcal{E}(\lambda), \mathcal{F}(\lambda)$ for the Basic Example

Define the kernel $\mathcal{E}^{(r)}(\lambda) \in D(Y(\lambda - r) \times Y(\lambda + r))$ by

$$\mathcal{E}^{(r)}(\lambda) := \mathcal{O}_{W^r(\lambda)} \otimes \det(V'/V)^{-\lambda}.$$

Similarly, $\mathcal{F}^{(r)}(\lambda) \in D(Y(\lambda + r) \times Y(\lambda - r))$ is defined by

$$\mathcal{F}^{(r)}(\lambda) := \mathcal{O}_{W^r(\lambda)} \otimes \det(\mathbb{C}^N/V')^{-r} \det(V)^r.$$

Functors E and F from kernels $\mathcal{E}$ and $\mathcal{F}$.

A kernel $\mathcal{A} \in D(Y(\lambda) \times Y(\lambda'))$ induces a functor

$$\Phi_{\mathcal{A}} : D(Y(\lambda)) \longrightarrow D(Y(\lambda'))$$

given by

$$y \mapsto \pi_{2*}(\pi_1^*(y) \otimes \mathcal{A}).$$

Thus the kernels $\mathcal{E}^{(r)}(\lambda), \mathcal{F}^{(r)}(\lambda)$ give rise to functors

$$\mathrm{E}^{(r)}(\lambda) := \Phi_{\mathcal{E}^{(r)}(\lambda)} : D(Y(\lambda - r)) \longrightarrow D(Y(\lambda + r))$$

$$\mathrm{F}^{(r)}(\lambda) := \Phi_{\mathcal{F}^{(r)}(\lambda)} : D(Y(\lambda + r)) \longrightarrow D(Y(\lambda - r)).$$

Relations satisfied by the Es and Fs

- The E's and F's are biadjoint up to shifts.
- E's (and hence F's) compose as

$$\mathsf{E}(\lambda + r) \circ \mathsf{E}^{(r)}(\lambda - 1)) \cong \mathsf{E}^{(r+1)}(\lambda) \otimes IH^*(\mathbb{P}^r)$$

  via an explicit isomorphism built from the data.
- If $\lambda \leq 0$ then

$$\mathsf{F}(\lambda + 1) \circ \mathsf{E}(\lambda + 1) \cong \mathsf{E}(\lambda - 1) \circ \mathsf{F}(\lambda - 1) \oplus \mathrm{Id} \otimes IH^\star(\mathbb{P}^{-\lambda-1})$$

  via an explicit isomorphism built from the data. (Similarly for $\lambda \geq 0$.)

Application: Equivalences

Given any geometric $\mathfrak{sl}_2$ categorification, we build a complex of functors

$$\Theta_* : \mathcal{D}(-\lambda) \to \mathcal{D}(\lambda).$$

Fix $\lambda \geq 0$. For $r = 0, \ldots, N - \lambda$, set

$$\Theta_r = \mathsf{E}^{(\lambda+r)}(-r)\mathsf{F}^{(r)}(-\lambda - r)[-r].$$

The differential $\Theta_r \to \Theta_{r-1}$ is constructed using units and counits of adjunctions between $\mathsf{E}^{(k)}$ and $\mathsf{F}^{(k)}$.
The complex Θ_* categorifies the action of the reflection element $s \in SL_2$, and was considered first by Chuang-Rouquier.

Equivalences

Theorem (Cautis-Kamnitzer-L)

The complex Θ_* is an equivalence between the opposite $\mathfrak{sl}_2$ weight space categories $\mathcal{D}(-\lambda)$ and $\mathcal{D}(\lambda)$.

Applied to the basic example:

Corollary

The complex Θ_* gives an equivalence

$$\Theta_* : D(T^\star(Gr(k,N)) \longrightarrow D(T^\star(Gr(N-k,N)).$$

This answers questions posed by Kawamata and Namikawa.

From $\mathfrak{sl}_2$ to $\mathfrak{g}$

There are analogous definitions of geometric categorification when $\mathfrak{g}$ is a Kac-Moody Lie algebra.

- Examples: Nakajima Quiver Varieties.
- Conjecturally, geometric categorifications induce representations of $U_2(\mathfrak{g})$.
- Braid group actions: For each root $\mathfrak{sl}_2$ inside $\mathfrak{g}$, we have an equivalence Θ_{i*} given by the Chuang-Rouquier complex.

Theorem (Cautis-Kamnitzer)

The equivalences $\{\Theta_{i*}\}$ coming from a geometric $\mathfrak{g}$ categorifcation define an action of the braid group.