You have 50 minutes. Make sure that you have 4 pages of questions. You should aim to spend about 12 minutes per page. Make sure that you show your work and justify your answers. If it looks like you just came up with an answer out of nowhere, you won’t receive any credit. You don’t need to simplify, and in fact you should think carefully about whether you are making things easier or harder when you simplify.

If you need more space you may work on the back of the pages. Be sure to indicate where your work is if this isn’t obvious.

Remember that you have signed an honor code and that academic misconduct will probably result in a grade of 0 on this exam.

Good luck!

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1. Indicate whether each of the following statements is true or false. No explanation is needed on this problem only.

(a) If $A$ and $B$ are equivalent matrices, then any linear relation satisfied by the rows of $B$ is also satisfied by the rows of $A$.
   False

(b) If $V$ and $W$ are subspaces of $\mathbb{R}^n$, and $V$ is a subset of $W$, then $\dim V \leq \dim W$.
   True

(c) If $A$ and $B$ are equivalent matrices and $B$ is in echelon form, then the nonzero rows of $B$ form a basis for the row space of $A$.
   True

(d) If $A$ and $B$ are invertible $n$ by $n$ matrices, then $(AB)^{-1} = A^{-1}B^{-1}$.
   False

(e) There is a subspace $V$ of $\mathbb{R}^4$ with a basis $B$ for $V$ consisting of 5 vectors.
   False

(f) For any matrix $A$, the nullity of $A$ equals the nullity of $A^T$.
   False
(10) 2. Let $A = \begin{bmatrix} 3 & 5 \\ 1 & 1 \\ 2 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 4 & 5 \\ 2 & 0 & 3 \end{bmatrix}$.

(a) Compute $AB$ and $BA$. $AB = \begin{bmatrix} 13 & 12 & 30 \\ 3 & 4 & 8 \\ 6 & 8 & 16 \end{bmatrix}$, and $BA = \begin{bmatrix} 17 & 19 \\ 12 & 16 \end{bmatrix}$.

(b) Find a basis for null $B$. We see that the general solution to $Bx = 0$ is $x_1 = -3/2s_1$, $x_2 = -7/8s_1$, $x_3 = s_1$, so a basis for null $B$ is $\{ \begin{bmatrix} -12 \\ -7 \\ 8 \end{bmatrix} \}$.

(c) Find a basis for col $A$, and extend this basis to a basis for $\mathbb{R}^3$. The columns of $A$ are already linearly independent so they form a basis for the column base of $A$, and you can check that adding the vector $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ to this set gives a basis for $\mathbb{R}^3$. 
(10) 3. (a) Is the following a subspace of $\mathbb{R}^4$: the set of vectors in $\mathbb{R}^4$ of the form 

\[
\begin{bmatrix}
a \\ b \\ c \\ d
\end{bmatrix},
\]\n
where $abcd = 0$? Be sure to justify your answer.

No, it isn’t. The vectors 

\[
\begin{bmatrix}
1 \\ 1 \\ 0 \\ 0
\end{bmatrix}
\]

and 

\[
\begin{bmatrix}
0 \\ 0 \\ 1 \\ 1
\end{bmatrix}
\]

are in this set, but their sum isn’t.

(b) Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be the linear transformation defined by $T(\mathbf{v}) = \begin{bmatrix} 0 & 1 & 0 \\ 2 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix} \mathbf{v}$. Find a matrix $A$ so that the linear map $T^{-1}$ is given by $T^{-1}(\mathbf{v}) = A\mathbf{v}$.

This is asking you to find the inverse of this matrix. It is: 

\[
A = \begin{bmatrix}
-1 & 1/2 & 0 \\ 1 & 0 & 0 \\ 0 & -1/2 & 1
\end{bmatrix}.
\]

(c) Compute the determinant of the matrix 

\[
\begin{bmatrix}
0 & 1 & 0 \\ 2 & 2 & 0 \\ 1 & 1 & 1
\end{bmatrix}.
\]

The determinant is $(-1)^3(2 - 0) = -2$. 
(15) 4. (a) Let $S$ be the set vectors in $\mathbb{R}^4$ of the form
$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix},$$
where $a - b/2 + 3c - d = 0$.

Find a linear map $T : \mathbb{R}^4 \to \mathbb{R}$ so that the kernel of $T$ is $S$.

The map $T : \mathbb{R}^4 \to \mathbb{R}$ given by $T \left( \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \right) = a - b/2 + 3c - d$ works.

(b) Let $S$ be the set of vectors in $\mathbb{R}^4$ from part a. Suppose that $W : \mathbb{R}^5 \to \mathbb{R}^4$ is a linear map, and that the range of $W$ is equal to $S$. What is the dimension of the kernel of $W$? Justify your answer.

We know that $\dim \ker W + \dim \text{range } W = 5$, and $\dim \text{range } W = \dim S$. So, $\dim \ker W = 5 - \dim S$. But $\dim S = \dim \ker T$, where $T$ is the map from above, and $\dim \ker T + \dim \text{range } T = 4$. But $T$ is onto (linear maps to $\mathbb{R}$ are either 0 or onto), so $\dim S = 3$ and $\dim \ker W = 2$.

(c) Show that if $A$ is a $n \times n$ matrix satisfying $AA^T = I_n$, then for all vectors $v, w \in \mathbb{R}^n$, $v \cdot w = Av \cdot Aw$, where $v \cdot w$ indicates the dot product of vectors.

We have that for any vectors $v, w \in \mathbb{R}^n$, $v \cdot w = v^T w$. So, if $A$ is such a matrix, $(Av) \cdot (Aw) = (Av)^T(Aw) = v^T A^T A w$. But $AA^T = I$, so $A^T = A^{-1}$, so $A^T A = I$ as well. So, $v^T A^T A w = v^T I w = v^T w = v \cdot w$, as we wanted.