Voisin - Schiffer variations of hypersurfaces and the generic Torelli theorem

Genus-Torelli theorem for hypersurfaces via Schiffer variation

Voisin

The meaning of generic Torelli then is $X \neq \tilde{X}$ in some
hypersurface & $\exists$ another hypersurface $\tilde{H}'(X, Q) \neq H'(\tilde{X}, Q) \Rightarrow X \neq \tilde{X}$?

- Rank: Weaker than Torelli if period map is generically one-to-one.

1. Stronger: From is with $Q$-coefficient not with $Z$.

2. Cogas - for curve, Torelli is wrong with $(Q)$.

3. Generically from period map one needs polarization to have no polarization needed.

    Except for $c_n$, $(3, 3)$ polarization of $H^{2n}(X, Q)$ is unique.

    So it has to preserve polarization.

4. Consider set of $S = \{(t, i) | \exists \tilde{x}, \tilde{y}, \tilde{z} \in C^2, H_i(x, y, z) = H_i(x, y, z)\}$

    set is a Hodge locus $\Rightarrow$ Deligne-Catanri-Kollár

    $S = \emptyset$ closed subset.

$\bar{U}_n$ = universal family.

1983: Thm (Donagi): The generic Torelli theorem holds if $(c_n) \neq (3, 3)$ (it doesn't very but we have a full moduli) with possible exclusion

1. $d | n+1$  
2. $d = 4 \Rightarrow 2 \equiv 2 \pmod{4}$
3. $d = 6, n+1 \equiv 3 \pmod{6}$

Voisin: Out of Donagi's leftover cases there are only finitely many exceptions to the generic Torelli theorem via hypersurfaces.

Where does Donagi fail? Suppose that $X' = X$, otherwise...
have \( H_{\pi}^c(X, \mathbb{Q}) = \prod_{x \in X'} H^{\pi}_{x, \mathbb{Q}} \) or \( \varphi U_{\pi} \chi X' \chi \).

Consider \( \Gamma \) local \( \text{then } (d'v) = (2, 3) \) \( x \in X \quad V \) \( \pi \) \( \text{period map is isom.} \)

\( \rightarrow \) \( \exists \varphi \in C(U \ast U) \) \( \text{a local holomorphic} \)

\( \Psi \circ U \cdot U \Psi : H^\pi_{\pi}(X, \mathbb{Q})_x \rightarrow H^\pi_{\pi}(X, \mathbb{Q})_x \) isom \( H^\pi_{\pi}(X, \mathbb{Q})_x \)

\( \rightarrow \) \( \Psi(U) \rightarrow V \) \( \Psi : T_{\pi, u} \rightarrow T_{\pi, v} \)

\( \frac{d}{dt} \left( \begin{array}{c} \Psi(t, x) \\ \text{for } (t, x) \rightarrow (t, H^\pi_{\pi}(X, \mathbb{Q})_x) \rightarrow \text{hom}(R_{\pi}^{\text{short}}, H^\pi_{\pi}(X, \mathbb{Q})_x) \end{array} \right) \)

\( S \in \mathbb{C}[x_0, \ldots, x_d] \)

\( x_i = (x_i) \quad R^*_y = S^*/f^*_y \quad \text{Griffiths residue} \)

\( \phi_{R^*_y} \quad R^*_y \rightarrow R^*_x \quad \text{Hodge residue} \)

\( T_{\pi, x_0} \quad \text{commute with} \quad R^*_x \quad \text{and} \quad R^*_y \)

\( \Rightarrow \text{isomorphism} \quad \text{of} \quad R^*_y \quad \text{and} \quad R^*_x \)

\( \text{except in the exceptional dim ordering, } f \text{ can be reconstructed from } \Psi \)

More precisely, the trick of Donagi is that in \( \Psi \) we don't have full Jacobian, but \( \Psi \) determines the whole Jacobian ring, therefore \( R^*_y \approx R^*_x \quad X' = (f' = 0) \quad X = (f = 0) \)

\( \Rightarrow f = f' \text{ under } \text{GL}(n+1) \quad R^*_y = S^* \quad R^*_x \rightarrow S^* \rightarrow R^*_x \)
\[ \text{Set } s \rightarrow J_f \rightarrow S^d \rightarrow R_f \rightarrow 0 \]
\[ o \rightarrow J_f \rightarrow S^d \rightarrow R_f \rightarrow 0 \]

Matter: Yani! If you have two hypersurfaces with \( J = J' \), then \( f < f' (\text{generic}) \). If \( f \) is generic
\[ f = 2f', \quad x + f \]

Why \( \text{dim}(t) \text{ is a problem?} \)

Dnagi uses symmetrization lemma to allow to recompute math. in \( R_f \) from
\[ R_f \xrightarrow{\text{Hodge theory}} \text{(IV+5 of } X_f) \]
when \( \text{dim}(t) \), symmetrization lemma doesn't bring new info.

Main idea: Schiffer variation for \( S^d \) is a one parameter family \( f_t \) with \( x \in S^d \) a linear form.

Why Schiffer? For curve \( C \) is a deformation of \( C \) supp at \( x \).
\[ \text{ie } p \left( H^*(C, T_C) \right) = p \left( H^*(C, 2K_C) \right) \]

True: \[ [Hp = H^*(C, 2K_C - p)] \]

Note: \( x \)-form are supp at \( x \).

First order Schiffer variation: \( x^\sigma \in C \cdot R_f \)

Lemma: \( f \) is generic, \( \sigma \rightarrow 0 \). Then the set of first order Schiffer variations determines \( f \).

Proof: First order:
\[ S^d \rightarrow R_f \]
\[ x^\sigma \rightarrow \]
\[
P(R^d_f) \rightarrow P(s^d)_{\text{Var}} \quad \text{Torreso}
\]

Get \( f \) by Mather-Yamasaki.

Define complete embedding.

\[H^0(\mathcal{H}, \mathcal{O}(1)) \rightarrow H^0(W, \mathcal{L}_k)\]

Proof of main: Characterize schiffer variation via Hodge theory.

Part(1): \( \mathcal{U} = \mathbb{R}^d \because R^d_f \) Artinian ring

Say \( I^d_k \subset \mathcal{R}^d_f \)

Ideal generated by \( X^k \).

\[\dim I^d_k = \dim R^d \quad \text{for} \quad x \leq 3 \]

\[\text{have, } I^d_k \cdot \mathcal{R}^d_f \subset \mathcal{R}^d_f\]

In part - \( I^d_1 = \langle x^2, + \rangle\)

Part(2): Among schiffer variation \( R^d / I^d \) is cont

Proof: \( f = f + t x^d \Rightarrow \frac{\partial f}{\partial x^i} = \frac{\partial f}{\partial x^j} (\text{mod} \ x^d) \)

Def. Deform of IV HS - (second order)

Prop: \( f \) is generic \& \( d \gg 0 \)

then \( (1) + (2) \) characterize schiffer variation

\( \square \) Where is \( d \gg 0 \) needed.
Ano. \( \mathbb{A}^d : \mathbb{R}^d \to \mathbb{R}^{2d} \) I want this to be injective for \( f \) generically (need this for part (ii)).

This needs \( d > \binom{n+1}{d+1} \) when

\[
\begin{align*}
f &= \text{Formal polynomial} = \sum x^d \\
R^f &= C(x, \ldots, x) \\
H^2(C(x, \ldots, x)) &= H^2(C(x, \ldots, x))
\end{align*}
\]

On Counterexample to Forelli for hypersurface.

No, not that's known. No idea.