Rational points and derived categories

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Question (H. Esnault):

Let X and Y be smooth projective varieties over a non-closed field $k \neq \bar{k}$, with equivalent derived categories of coherent sheaves $D^b(X) \cong D^b(Y)$.

Does $X(k) \neq \varnothing$ imply $Y(k) \neq \varnothing$?

Today:

No. Abelian and hyperkähler counterexamples.

Why this question?

1. General curiosity: how much does $D^b(X)$ know about X?
   - $\dim X$.
   - Canonical ring, and anti-canonical ring.
   - $\bigoplus H^{2i}_{\mathrm{sing}}(X(\mathbb{C}), \mathbb{Q})$ and $\bigoplus H^{2i+1}_{\mathrm{sing}}(X(\mathbb{C}), \mathbb{Q})$ with their Hodge structures,

     $\bigoplus H^{2i}_{\text{ét}}(\bar{X}, \mathbb{Q}_\ell(i))$ and $\bigoplus H^{2i+1}_{\text{ét}}(\bar{X}, \mathbb{Q}_\ell(i))$ with their Galois action,

     $\bigoplus \mathrm{CH}^i(X) \otimes \mathbb{Q}$.

     But not the grading, integral structure, ring structure.
   - Not $\pi_1(X)$, $\mathrm{Br}(X)$, birational type...
2. Dreams of using $D^b(X)$ to study birational geometry.

What was known?

1. Honigs, Achter, Casalaina-Martin, and Vial, 2016:
   Yes over finite fields if $\dim X \leq 3$ or X is Abelian.

   Conjecture of Orlov $\Rightarrow$ yes over finite fields in general.

2. Antieau, Krashen, and Ward, 2014:
   Yes for genus-1 curves over any field.

3. Hassett and Tschinkel, 2014: K3 surfaces…
   - Over $\mathbb{R}$, if $D^b(X) \cong D^b(Y)$ then $X(\mathbb{R}) \cong Y(\mathbb{R})$.
   - Some results over local fields.
   - In general, if $X(k) \neq \varnothing$ then Y has a 0-cycle of degree 1.

What was known? (cont'd)

4. Ascher, Perry, Dasaratha, and Zhou, 2015:

   No for derived categories of *twisted* sheaves on K3 surfaces over $\mathbb{Q}$, $\mathbb{Q}_2$, or $\mathbb{R}$.

5. Auel and Bernardara, 2015:
   Studied geometrically rational surfaces...

   One result: A Del Pezzo surface S of degree ≥ 5 has a rational point iff $D^b(S)$ admits a full exceptional collection.

   Ballard and collaborators have also studied rational points and exceptional collections – different definition.

Abelian counterexamples

Theorem 1. For every $g \geq 2$, there is an Abelian g-fold X defined over $\mathbb{Q}$ and an X-torsor Y with $Y(\mathbb{Q}) = \varnothing$ such that

$$D^b(X) \cong D^b(Y).$$

The same holds over $\mathbb{F}_q(t)$ for any odd q.

Theorem 2. If C is a smooth, projective, geometrically connected curve of genus $g \geq 1$ over any field, then

$$D^b(\mathrm{Pic}^0_C) \cong D^b(\mathrm{Pic}^{g-1}_C).$$

To deduce Theorem 1 from Theorem 2, mine the literature for explicit curves such that Pic^{g-1}_C has no rational points:

Coray and Manoil, 1996. Poonen and Stoll, 1999.

Idea of Theorem 2

Consider the divisor

$$D = \{(L, M) : H^1(L \otimes M) \neq 0\} \subset \mathrm{Pic}^0_C \times \mathrm{Pic}^{g-1}_C \,.$$

Fiber of D over $0 \in \mathrm{Pic}^0_C$ is the canonical Θ-divisor in Pic^{g-1}.

Fibers over other points of Pic^0_C are translates of Θ.

The line bundle $\mathcal{O}(D)$ realizes Pic^0_C as a fine moduli space of line bundles on Pic^{g-1}_C, and vice versa.

Repackage Mukai's classic proof that $D^b(A) \cong D^b(\hat{A})$ to show that $\mathcal{O}(D)$ induces an equivalence $D^b(\mathrm{Pic}^0_C) \cong D^b(\mathrm{Pic}^{g-1}_C)$.

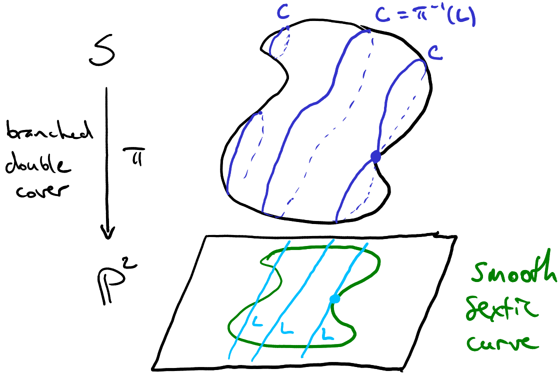
Hyperkähler counterexample

Theorem 3. There is an explicit K3 surface S, defined over $\mathbb{Q}$, and two smooth, projective, 4-dimensional moduli spaces X and Y of sheaves on S, such that

- X has infinitely many rational points,
- Y has no zero-cycle of degree 1, and
- $D^b(X) \cong D^b(Y)$.

Moreover $X_{\mathbb{C}}$ and $Y_{\mathbb{C}}$ are not isomorphic, or even birational.

Some geometry of degree-2 K3 surfaces



Given a line $L \subset \mathbb{P}^2$, get a genus-2 curve $C = \pi^{-1}(L) \subset S$.

As the line moves, get a family of curves parametrized by $\mathbb{P}^{2*}$.

Fine print: We'll choose our sextic so that $[C]$ generates $\mathrm{Pic}(\bar{S})$, and in particular every curve C is geometrically integral.

The two fourfolds

X and Y are the moduli spaces of stable sheaves on S of rank 0, $c_1 = [C]$, and $\chi = -1$ or 0.

More geometrically:
$X \to \mathbb{P}^{2*}$ is the relative $\overline{\mathrm{Pic}}^0$ of the family of curves.
$Y \to \mathbb{P}^{2*}$ is the relative $\overline{\mathrm{Pic}}^1$.

The equivalence $D^b(X) \cong D^b(Y)$ is a family version of our earlier equivalence.

Extension to singular curves is due to Arinkin, 2010.

Addington, Donovan, and Meachan, 2015 studied this example over $\mathbb{C}$ with a view toward autoequivalences.

Rational points

$X = \overline{\mathrm{Pic}}^0$ has lots of rational points, because every C has a degree-0 line bundle $\mathcal{O}_C$.

Fun: X is birational (and derived equivalent) to $\mathrm{Hilb}^2(S)$, which also has lots of rational points.

$X_{\mathbb{C}}$ and $Y_{\mathbb{C}}$ are not birational: Sawon, 2005.

$Y = \overline{\mathrm{Pic}}^1$ has $\mathbb{R}$-points and $\mathbb{Q}_p$-points for every p,
but we can choose our sextic so that Y has no $\mathbb{Q}$-points…

Let $\alpha \in \mathrm{Br}(Y)$ be Brauer class that obstructs existence of a universal sheaf on $S \times Y$.

Use α as a Brauer–Manin obstruction to rational points.

Brauer–Manin story

We have $\alpha \in \mathrm{Br}(Y)$ from the fact that Y is a moduli space of geometrically stable sheaves.

Strategy: for all $y \in Y(\mathbb{R})$ we want $\alpha|_y = \frac{1}{2} \in \mathrm{Br}(\mathbb{R})$,
and for all $y \in Y(\mathbb{Q}_p)$ we want $\alpha|_y = 0 \in \mathrm{Br}(\mathbb{Q}_p)$.

A point $y \in Y(\mathbb{Q})$ would give points in $Y(\mathbb{R})$ and $Y(\mathbb{Q}_p)$, but

$$\begin{array}{ccccccc} 0 \to \mathrm{Br}(\mathbb{Q}) & \to & \mathrm{Br}(\mathbb{R}) \oplus \bigoplus \mathrm{Br}(\mathbb{Q}_p) & \to & \mathbb{Q}/\mathbb{Z} \to 0 \\ \alpha|_y & \mapsto & (\frac{1}{2}, 0, 0, 0, \dots) & \mapsto & 1/2 \ \text{(not 0!)} \end{array}$$

For $\mathbb{R}$, we ask that $S(\mathbb{R}) = \varnothing$ and make a little argument.

For $\mathbb{Q}_p$, if we reduce to $\mathbb{F}_p$ and no semi-stable sheaves appear, then $\alpha|_y = 0$ for all $y \in Y(\mathbb{Q}_p)$ as desired.

Finitely many primes where we have to worry about semi-stables / the curves C can become reducible / the sextic has tritangent lines. Find conditions to control these.

Last Slide

Use Magma to search for an example that satisfies all our conditions.

$$
\begin{aligned}
w^2 = -x^6 - x^5 z - x^4 y^2 - x^4 z^2 - x^3 y z^2 - x^2 y^2 z^2 \\
- x y^5 - x y^4 z - x z^5 - y^6 - y^3 z^3 - y^2 z^4 - y z^5 - z^6
\end{aligned}
$$

Troublesome primes: 5, 31, 7517, 847160373981361103087\99, and

4424904772196959344085200612883251617292465803437757948
5992572698404066491363246248977477562371729031497984350
0902180031058767256453958545754450340721124283977338015
36646126422607590015238685542160768254044 19681.

Thanks!