Some vignettes on sums-of-squares on varieties.

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Let $f \in \mathbb{R}[X_1, \ldots, X_n]$ be a multivariate polynomial with real coefficients.

**Definition.**

The polynomial $f$ is **nonnegative** ($f \in P$) if $f(\alpha) \geq 0$ for every $\alpha \in \mathbb{R}^n$.

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The polynomial $f$ is a **sum-of-squares** ($f \in \Sigma$) if there exist an integer $t > 0$ and polynomials $g_1, \ldots, g_t \in \mathbb{R}[X_1, \ldots, X_n]$ such that

$$f = g_1^2 + \cdots + g_t^2.$$
The cone of nonnegative polynomials is important because *it allows us formulate global optimization problems*:

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Such reformulations have many applications (see for instance J.B. Lasserre’s ”Moments, positive polynomials and their applications”)

Nonnegative polynomials (P)
Sums-of-squares (\(\Sigma\))

Sums of squares provide certificates of nonnegativity:

**Example:**

Is the following polynomial \( f \) nonnegative in \( \mathbb{R}^2 \)?

\[
f = 10x^6 - 4x^5y + 2x^4y^2 + 50x^4 - 14x^3y - 4x^3 + 4x^2y + 65x^2 - 14x + 2
\]
Sums-of-squares ($\Sigma$)

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**Example:**

Is the following polynomial $f$ nonnegative in $\mathbb{R}^2$?

$$f = (1 + x + x^3 + x^2y)^2 + (1 - 8x + 3x^3 + x^2y)^2.$$
Sums-of-squares ($\sum$)

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**Remark.**

*A polynomial $f$ is a sum-of-squares of elements of $V$ if and only if there exists a symmetric matrix $A \in \mathbb{R}^{e \times e}$ such that

$$A \succeq 0 \quad \text{and} \quad f = \tilde{m}^t A \tilde{m}$$

where $\tilde{m} = (h_1, \ldots, h_e)^t$ is a vector whose entries are a basis for $V$.  \*
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**Constructing SOS certificates reduces to semidefinite programming feasibility.**
Is every nonnegative polynomial a sum of squares?
Question.

Is every nonegative polynomial a sum of squares?

Question.

For which degrees $2d$ and number of variables $n$ is every nonnegative form (homogeneous polynomial) of degree $2d$ a sum-of-squares?
Theorem. (Hilbert 1888)

Every nonnegative form of degree $2d$ in $n$-variables is a sum-of-squares if and only if either,

1. $n = 2$ (bivariate forms) or 
2. $d = 1$ (quadratic forms) or 
3. $n = 3$ and $d = 2$ (ternary quartics).
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Question.

Can we find a natural context where we can understand and hopefully generalize Hilbert’s Theorem?
Let $X \subseteq \mathbb{P}^n$ be a real projective variety (reduced, not necessarily irreducible) and let $S := \mathbb{R}[X_0, \ldots, X_n]/I(X)$ be its homogeneous coordinate ring.

**Definition.**

The cone of nonnegative quadratic forms $P_X$ is given by

$$P_X = \{ f \in S_2 : \forall \alpha \in X(\mathbb{R}) (f(\alpha) \geq 0) \}$$

**Definition.**

The cone of sums-of-squares of linear forms

$$\Sigma_X = \left\{ f \in S_2 : \exists s_1, \ldots, s_t \in S_1 : f = \sum s_i^2 \right\}$$
Question.

*For which projective varieties does it happen that* $P_X = \Sigma_X$?
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In principle, restricting only to quadratic forms seems to be fairly restrictive. However, this is not the case since we are considering arbitrary varieties so quadratic forms in $\nu_d(X)$ correspond to $2d$-forms on $X$. 
A partial answer: irreducible varieties.

Let $X \subseteq \mathbb{P}^n$ be a real projective variety. Assume:

1. $X$ is non-degenerate and totally real.
2. $X$ is irreducible.

Theorem. (Blekherman, Smith, - , 2016)

The equality $P_X = \Sigma_X$ occurs if and only if $X$ is a variety of minimal degree (i.e. if the equality $\deg(X) = 1 + \text{codim}(X)$ holds).
If $X \subseteq \mathbb{P}^n$ is a positive-dimensional, irreducible and non-degenerate variety then its general hyperplane section is non-degenerate. It follows that

$$\deg(X) \geq \text{codim}(X) + 1$$
Varieties of minimal degree

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*With equality if and only if the intersection of $X$ with a general $\mathbb{P}^{\text{codim}(X)}$ is a basis for this space.*
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Theorem. (Del Pezzo, Bertini, 1880)

Let $X \subseteq \mathbb{P}^n$ be irreducible and not contained in any hyperplane in $\mathbb{P}^n$. If $X$ is of minimal degree (i.e. $\deg(X) = \text{codim}(X) + 1$) then either:

1. $X = \mathbb{P}^n$ or
2. $X$ is a quadric hypersurface or
3. $X$ is a cone over the Veronese surface $\nu_2(\mathbb{P}^2) \subset \mathbb{P}^5$ or
4. $X$ is a rational normal scroll.
Idea of Proof:

1. \( P_X = \mathcal{X} \) is preserved under projections away from real points.
2. Project away from \( \text{codim}(X) \) points and reach the hypersurface case.
3. For hypersurfaces \( P_X = \mathcal{X} \) is a quadric hypersurface.
4. \( P_X = \mathcal{X} \) is preserved under generic hyperplane sections (By our Bertini-type theorem for separators, convex geometry + complex geometry).
5. Slice \( X \) with a complementary subspace to obtain a set of points with \( P_X = \mathcal{X} \).
6. For a set of points \( X \) equality holds i.e. \( X \) is a linearly independent set.
Idea of Proof:

1. $P_X = \Sigma_X$ is preserved under projections away from real points,
   1. Project away from $\text{codim}(X) - 1$ points and reach the hypersurface case.
   2. For hypersurfaces $P_X = \Sigma_X$ iff $X$ is a quadric hypersurface.
Idea of Proof:

1. $P_X = \Sigma_X$ is preserved under projections away from real points,
   1. Project away from $\text{codim}(X) - 1$ points and reach the hypersurface case.
   2. For hypersurfaces $P_X = \Sigma_X$ iff $X$ is a quadric hypersurface.

2. $P_X \neq \Sigma_X$ is preserved under generic hyperplane sections (By our Bertini-type theorem for separators convex geometry + complex geometry).
   1. Slice $X$ with a complementary subspace to obtain a set of points with $P_X \neq \Sigma_X$.
   2. For a set of points $X$ equality holds iff $X$ is a linearly independent set.
We could **unify** and **generalize** results scattered in the literature:

1. $X = \nu_d(\mathbb{P}^n)$ is minimal degree if and only if... (Hilbert’s Theorem 1888).
2. $X = V(Q)$... (Yakubovich’s Theorem 1971)
3. $X = \sigma_{d_1, d_2}(\mathbb{P}^{n_1} \times \mathbb{P}^{n_2})$ is minimal degree if and only if... (Choi-Lam-Reznick 1980)
4. New SOS results on nonnegative polynomials with special support from rational normal scrolls (2016).
Vignette 1: How about denominators?

In 1927 Artin showed (solving Hilbert 17th) that every nonnegative polynomial admits a representation as a sum-of-squares of rational functions (and in particular as a ratio of sums of-squares).

Given \( f \in P \) find \( g \in \Sigma : fg \in \Sigma \).

Question.

Do such representations exist on varieties?

Theorem. (Blekherman, Smith, -, 2019)

Let \( X \subseteq \mathbb{P}^n \) be a totally real, non-degenerate curve of degree \( d \) and arithmetic genus \( p_a \). If \( f \in P_{X,2j} \) and \( k \geq \frac{2p_a}{d} \) then there exists \( g \in \Sigma_{X,2k} \) such that \( fg \in \Sigma_{X,2(j+k)} \). These bounds are sharp.
In 1984 Pfister showed that every nonnegative form in $\mathbb{R}^n$ has a rational SOS representation involving at most $2^n$ squares.

**Definition.**

The **pythagoras number** $\Pi(X)$ of a projective variety $X \subseteq \mathbb{P}^n$ is the smallest number of squares that suffices to write ANY element of $\Sigma_X$.

**Theorem.** (Blekherman, Smith, Sinn, -, 2020)

If $X$ is totally real, irreducible, non-degenerate and arithmetically Cohen-Macaulay then the following conditions are equivalent:

1. $\Pi(X) = 2 + \dim(X)$ (next-to-minimal)
2. $\deg(X) = 2 + \codim(X)$ or $X$ is codimension one in a variety of minimal degree.
References


- *Sums of Squares and Quadratic Persistence on Real Projective Varieties*, G. Blekherman, R. Sinn, G. Smith, M. Velasco, to appear in JEMS.