ADVANCES IN MODULI THEORY

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ABSTRACT. The *ZAG Seminar* entitled 'Advances in Moduli Theory' presented on September 3, 2020 surveyed recent advances in constructing projective moduli spaces of objects with potentially non-finite automorphism groups, e.g., the moduli of sheaves or complexes on a fixed space, or the moduli of K-semistable Fano varieties.

1. BACKGROUND

In the 1960's, David Mumford developed Geometric Invariant Theory (GIT) as a tool for constructing projective moduli spaces [Mum65]. The central idea is as follows:

① *Rigidification:* Express the moduli problem $\mathcal{M}$ as a quotient U/G of the action of a quasi-projective variety U by a reductive group G and choose a G-equivariant compactification $\overline{U} \subseteq \mathbb{P}^n$, where G acts linearly on $\mathbb{P}^n$.

② *Stability analysis:* Show that a point $u \in \overline{U}$ is in U if and only if it is GIT semistable (i.e., there exists a non-constant G-invariant homogeneous $f \in \Gamma(\mathbb{P}^n, \mathcal{O}(d))^G$ with $f(u) \neq 0$). Realize the the moduli space $\mathcal{M} = U/G$ as the projective variety $\operatorname{Proj} \bigoplus_{d \geq 0} \Gamma(\mathbb{P}^n, \mathcal{O}(d))^G$.

Step 2 is the hardest: while he Hilbert–Mumford Criterion translates this problem into the enumerative and sometime combinatorial question of verifying that the non-negativity of the *Hilbert–Mumford index* $\mu(u, \lambda) \geq 0$ for every one-parameter subgroup $\lambda \colon \mathbb{G}_m \to G$, it can be difficult to verify this criterion.

Mumford [Mum77] and Gieseker [Gie82] constructed $\overline{M}_g$ as a projective variety using GIT following this procedure. For ①, for each integer $k \geq 3$, a choice of basis of the space $\Gamma(C, \Omega_C^{\otimes k})$ of k-pluricanonical sections defines an embedding into $\mathbb{P}^N$ with $N + 1 = \mathrm{h}^0(C, \Omega_C^{\otimes k}) = (2k-1)(g-1)$. The moduli of smooth k-pluricanonically embedded curves in $\mathbb{P}^N$ is a quasi-projective variety that can be compactified using either the Chow or Hilbert scheme. For ②, by reinterpreting the Hilbert–Mumford index in terms of geometric invariants on the surface $C \times \mathbb{A}^1$, they verified that smooth curves were GIT stable. With a trick using stable reduction, this reduces the construction of $\overline{M}_g$ to showing that non-stable curves are not GIT semistable. While this is the conceptually easier direction requiring one to construct a destabilizing one-parameter subgroup λ for each non-stable curve, it is the most technically and lengthy aspect of the argument.

The GIT approach also applies to constructing moduli spaces of objects with non-finite automorphisms. For a smooth, connected, and projective curve C, Mumford [Mum63] and Seshadri [Ses67] constructed a projective variety $M_{r,d}^{\mathrm{ps}}(C)$ parameterizing semistable vector bundles on C of rank r and degree d, up to S-equivalence, where two vector bundles E and F are *S-equivalent* if they admit filtrations $E^\bullet$ and $F^\bullet$ such that the associated graded bundles $\operatorname{gr} E^\bullet \cong \operatorname{gr} F^\bullet$ are isomorphic and semistable. The showed that for $n \gg 0$, every semistable vector bundle E can be expressed as a quotient $\mathcal{O}_C(-n)^{P(n)} \twoheadrightarrow E$, where $\mathrm{h}^0(C, E(n)) = P(n)$ and $\mathrm{h}^1(C, E(n)) = 0$, and that a quotient $[\mathcal{O}_C(-n)^{P(n)} \twoheadrightarrow Q] \in \operatorname{Quot}^P(\mathcal{O}_C(-n)^{P(n)})$ is GIT-semistable if and only if E is semistable. The GIT machinery has been widely applied to construct other moduli spaces.

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2. Six steps toward projective moduli

There is an alternative, more intrinsic approach to constructing moduli spaces that relies on the theory of algebraic stacks:

① *Algebraicity:* Express the moduli problem $\mathcal{M}$ as a substack

$$\mathcal{M} \subseteq \mathcal{X}$$

of a larger moduli stack $\mathcal{X}$, and define an object $x \in \mathcal{X}$ to be *semistable* if it is in $\mathcal{M}$. Show that $\mathcal{X}$ is an algebraic stack locally of finite type.

② *Openness of semistability:* Show that semistability is an open condition, i.e., $\mathcal{M} = \mathcal{X}^{\mathrm{ss}} \subseteq \mathcal{X}$ is an open substack.

③ *Boundedness of semistability:* Show that semistability is bounded, i.e., $\mathcal{M} = \mathcal{X}^{\mathrm{ss}}$ is of finite type.

④ *Semistable reduction:* Show that $\mathcal{M}$ satisfies the existence part of the valuative criterion for properness.

⑤ *Existence of a moduli space:* Show that there is a fine/coarse/good moduli space $\mathcal{M} \to M$ where M is a proper algebraic space.

⑥ *Projectivity:* Show that a tautological line bundle on $\mathcal{M}$ descends to an ample line bundle on M, i.e., M is projective.

In Step ⑤, the trichotomy of moduli in terms of automorphisms controls properties of the moduli space. When there are no automorphisms, the moduli problem is often represented by an algebraic space, in which case we call it a *fine moduli space*. The remaining challenge is projectivity, and there are various ampleness criteria, e.g., the Nakai–Moishezon criterion, that can be applied to tautological line bundles. When there are finite automorphisms and the moduli stack $\mathcal{M}$ is proper (and in particular separated!), the Keel–Mori Theorem [KM97] guarantees the existence of a coarse moduli space $\mathcal{M} \to M$, where M is a proper algebraic space.

Stable curves have finite automorphisms, and Deligne and Mumford proved that the moduli stack $\overline{\mathcal{M}}_g$ of stable curves is proper [DM69]. Kollár gave a solution to the final and most difficult step in [Kol90]. By considering the multiplication map on pluricanonical bundles

$$\operatorname{Sym}^m \pi_*(\omega_{\mathcal{U}_g/\overline{\mathcal{M}}_g}^{\otimes k}) \to \pi_*(\omega_{\mathcal{U}_g/\overline{\mathcal{M}}_g}^{\otimes mk}),$$

where $\pi\colon \mathcal{U}_g \to \overline{\mathcal{M}}_g$ is the universal family, and using that the first vector bundle has a reduction of structure group to PGL_v where $v = \mathrm{rk}\pi_*(\omega_{\mathcal{U}_g/\overline{\mathcal{M}}_g}^{\otimes k})$, he constructed a quasi-finite morphism $\overline{\mathcal{M}}_g \to [\mathrm{Gr}(q, \operatorname{Sym}^m \Bbbk^v)/\operatorname{PGL}_v]$, and deduced the projectivity of $\overline{\mathcal{M}}_g$ from the projectivity of the Grassmannian. In the case of finite automorphisms, this six-step strategy has been applied to construct many other projective moduli spaces of interest, e.g., Hassett's moduli spaces of weighted stable curves, and moduli spaces of stable maps. Most notably, by using advances in the minimal model program and the finite generation of the canonical ring, Kollár, with the assistance of many others, constructed projective moduli spaces of canonically polarized varieties of any dimension [Kol23].

The case of infinite automorphisms is more subtle as one may need to identity non-isomorphic objects, e.g., S-equivalent vector bundles. When the automorphisms of polystable objects, i.e., closed points of the moduli stack, are reductive, one can hope to construct a good moduli space.

Existence Theorem. [AHLH23, Thm. A] *If $\mathcal{X}$ is an algebraic stack of finite type over a characteristic zero field $\Bbbk$ with affine diagonal, then there exists a good moduli space $\mathcal{X} \to X$ where X is a separated algebraic space if and only if $\mathcal{X}$ is Θ-complete and S-complete.*

Θ- and S-completeness are valuative criteria that serve as replacements for the separatedness condition (as any algebraic stack with affine and non-finite automorphisms is necessarily not separated). These conditions require that $\mathbb{G}_m$-equivariant families over certain $\mathbb{G}_m$-equivariant punctured surfaces extend over the surface. For instance, defining

$\Theta = [\mathbb{A}^1/\mathbb{G}_m]$ and letting 0 denote the unique closed point of $\Theta_R := \Theta \times \operatorname{Spec} R$ for a DVR R, Θ-completeness requires that for every DVR R with fraction field K, every morphism from $\Theta_R \setminus 0 = \Theta_K \bigcup_{\operatorname{Spec} K} \operatorname{Spec} R \to \mathcal{X}$ extends uniquely to a morphism $\Theta_R \to \mathcal{X}$.

3. Application to the moduli of semistable vector bundles

The moduli stack $\mathcal{B}un_{r,d}(C)$, parameterizing all vector bundles of rank r and degree d on smooth, connected, and projective curve C, is unbounded (i.e., not finite type), horribly non-separated, and does not admit a reasonable moduli space. Remarkably, semistablility offers a solution to all three of these issues. A vector bundle F on C to be *semistable* if for every subsheaf $E \subseteq F$ satisfies $\frac{\deg E}{\operatorname{rk} E} \leq \frac{\deg F}{\operatorname{rk} F}$. We explain how the six-step approach can be applied to the moduli stack $\mathcal{B}un^{\mathrm{ss}}_{r,d}(C)$ of semistable vector bundles of rank r and degree d. A more complete exposition can be found in [ABB+22].

Step ①(Algebraicity): We can embed $\mathcal{B}un^{\mathrm{ss}}_{r,d}(C)$ as a substack of the algebraic stack $\underline{\mathrm{Coh}}_{r,d}(C)$ parameterizing coherent sheaves of rank r and degree d. (One can use the smaller stack $\mathcal{B}un_{r,d}(C)$ of vector bundles, but this stack is less convenient in Step ⑤below as it is neither Θ- nor S-complete.)

Step ②(Openness of stability): Given a family of vector bundles F on $C \times T$ over a finite type scheme T, there are only finitely many ranks r' and degrees d' of a destabilizing subsheaf $E \subseteq F_t$ of a fiber $F_t := F|_{C\times\{t\}}$ over $t \in T$, and for each pair (r', d'), there is a relative Quot scheme $Q_{r',d'}$ proper over T parameterizing such subsheaves. Each image of $Q_{r',d'}$ is closed in T, and removing each such locus gives an open subscheme of T consisting precisely of points $t \in T$ with F_t semistable.

Step ③(Boundedness of stability): Choosing an ample line bundle $\mathcal{O}_C(1)$ on C, one shows that if F is semistable and $n > 2g - 2 - d/r$, then $\mathrm{H}^1(C, F(n)) = 0$ (this is not hard: by Serre–Duality, $\mathrm{H}^1(C, F(n)) \cong \operatorname{Hom}_{\mathcal{O}_C}(F(n), \Omega_C)$, and this group is zero as there are no maps $F(n) \to \Omega_C$ as both are semistable and the slope of Ω_C is smaller). It follows that if $n > 2g - 1 - d/r$, then $F(n)$ is globally generated by $P(n) = \mathrm{h}^0(C, F(n))$ sections. Therefore, the finite type Quot scheme $\operatorname{Quot}^P(\mathcal{O}_C(-n)^{\oplus P(n)})$ shows that $\mathcal{B}un^{\mathrm{ss}}_{r,d}(C)$ is of finite type.

Step ④(Semistable reduction): The existence part of the valuative criterion was proven by Stacy Langton [Lan75]: for each DVR R with fraction field K, any semistable vector bundle F on C_K of rank r and degree d can be extended to a semistable vector bundle $\widetilde{F}$ on C_R Langton's strategy is to first extend F to any vector bundle $\widetilde{F}$ on C_R. If the central fiber $\widetilde{F}_0$ is not semistable, there is a destabilizing subsheaf $E_0 \subset \widetilde{F}_0$ and one replaces $\widetilde{F}$ with $\ker(\widetilde{F} \twoheadrightarrow \widetilde{F}_0 \twoheadrightarrow \widetilde{F}_0/E_0)$. The central fiber of $\widetilde{F}$ is now closer to being semistable, and she proved that after finitely many steps it becomes semistable.

Step ⑤(Existence of a moduli space): We claim that there is a good moduli space $\mathcal{B}un^{\mathrm{ss}}_{r,d}(C) \to M^{\mathrm{ps}}_{r,d}$ to a proper algebraic space whose points are in bijection with the closed points of $\mathcal{B}un^{\mathrm{ss}}_{r,d}(C)$, i.e., the *polystable* vector bundles. To apply the Existence Theorem, we need to verify that $\mathcal{B}un^{\mathrm{ss}}_{r,d}(C)$ is Θ- and S-complete. It is not hard to see that the larger stack $\underline{\mathrm{Coh}}_{r,d}(C)$ is both Θ- and S-complete. For instance, since a map from $\Theta = [\mathbb{A}^1/\mathbb{G}_m]$ to $\underline{\mathrm{Coh}}_{r,d}(C)$ corresponds to a vector bundle with a filtration, Θ-completeness reduces to the following: given a coherent sheaf F on C_R flat over a DVR R and a filtration of the generic fiber F_K on C_K (this data corresponds to a map $\Theta_R \setminus 0 \to \underline{\mathrm{Coh}}_{r,d}(C)$), the filtration of F_K extends to filtration of F. This follows easily from the properness of the Quot scheme. To conclude that $\mathcal{B}un^{\mathrm{ss}}_{r,d}(C)$ is Θ-complete, we can first use Θ-completeness of $\underline{\mathrm{Coh}}_{r,d}(C)$ to find a filtration of F extending the filtration of F_K, and then use properties of semistability to show that the central fiber of the unique extension is semistable.

Step ⑥(Projectivity): Faltings gave an explicit construction of $M^{\mathrm{ps}}_{r,d}$ as a projective variety [Fal93]. The central idea is to show that a determinantal line bundle of the form $\mathcal{L}_V := \det \mathrm{R}p_{2,*}(\mathcal{E}_{\mathrm{univ}} \otimes p_1^* V)^\vee$ on $\mathcal{B}un^{\mathrm{ss}}_{r,d}(C)$, where $\mathcal{E}_{\mathrm{univ}}$ is the universal vector bundle on $C \times$

$\mathcal{B}un^{\mathrm{ss}}_{r,d}(C)$ and V is a vector bundle on C, descends to an ample line bundle L_V on $M^{\mathrm{ps}}_{r,d}$ by explicitly constructing enough sections. For this, Faltings first proved an interesting characterization of semistability: a vector bundle E on C of rank r and degree d is semistable if and only if there exist a vector bundle V such that $\mathrm{H}^0(C, F\otimes V) = \mathrm{H}^1(C, F\otimes V) = 0$. This allows us to construct a section of $\mathcal{L}_V$ that doesn't vanish at F: the derived pushforward $\mathrm{R}p_{2,*}(\mathcal{E}_{\mathrm{univ}} \otimes p_1^* V)$ is represented by a two-term complex $\alpha\colon K^0 \to K^1$ of vector bundles of the same rank on $\mathcal{B}un^{\mathrm{ss}}_{r,d}(C)$, and the induced section $\mathcal{O}_{\mathcal{B}un^{\mathrm{ss}}_{r,d}(C)} \to \mathcal{L}_V = \det(K^1) \otimes \det(K^0)^\vee$ is nonzero at F because $\det(\alpha|_F)$ is an isomorphism, or in other words because $\mathrm{H}^0(C, F\otimes V) = \mathrm{H}^1(C, F\otimes V) = 0$. This shows that a positive tensor power of the descended line bundle L_V on $M^{\mathrm{ps}}_{r,d}$ is globally generated, i.e., defines a map to projective space, and a similar but more involved line of reasoning shows that this map doesn't contract any curves, which is sufficient to guarantee that L_V is ample.

This six-step approach has also been applied in the moduli of semistable objects with respect to a Bridgeland stability condition (see [AHLH23, §7] and [Taj23]), and to the moduli of K-semistable Fano varieties (see [ABHLX20] and [Xu24]).

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