REDUCTIVITY OF THE AUTOMORPHISM GROUP OF K-POLYSTABLE FANO VARIETIES

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Abstract. We prove that K-polystable log Fano pairs have reductive automorphism groups. In fact, we deduce this statement by establishing more general results concerning the S-completeness and Θ-reductivity of the moduli of K-semistable log Fano pairs. Assuming the conjecture that K-semistability is an open condition, we prove that the Artin stack parametrizing K-semistable Fano varieties admits a separated good moduli space.

Contents

Throughout, we work over an algebraically closed field k of characteristic 0.

1. Introduction

The construction of moduli spaces parametrizing K-semistable and K-polystable Fano varieties is a profound goal in the study of Fano varieties. The *K-moduli Conjecture* predicts that the moduli functor $\mathfrak{X}^{\mathrm{Kss}}_{n,V}$ of K-semistable $\mathbb{Q}$-Fano varieties of dimension n and volume V, which sends a k-scheme S to

$$\mathfrak{X}^{\mathrm{Kss}}_{n,V}(S) = \left\{ \begin{array}{c} \text{Flat proper families } X \to S\text{, whose geometric fibers are} \\ \text{K-semistable } \mathbb{Q}\text{-Fano varieties of dimension } n \text{ and} \\ \text{volume } V\text{, satisfying Kollár's condition (see [BX18, §1])} \end{array} \right\},$$

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is represented by a finite type Artin stack $\mathfrak{X}^{\mathrm{Kss}}_{n,V}$ and it admits a projective good moduli space $\mathfrak{X}^{\mathrm{Kss}}_{n,V} \to X^{\mathrm{Kps}}_{n,V}$ (see Definition 2.1), whose closed points precisely parameterize n-dimensional K-polystable $\mathbb{Q}$-Fano varieties of volume V. The ingredients needed in the construction can be translated into deep properties of such Fano varieties. See [BX18, Introduction] for a more detailed discussion of the prior state of the art.

1.1. **Main theorems.** In this paper, we show that if the moduli functor $\mathfrak{X}^{\mathrm{Kss}}_{n,V}$ is represented by an Artin stack, then it admits a separated good moduli space (see Step (III) in [BX18, Introduction]). A prototype of the good moduli space of a stack is given by the morphism $[X^{\mathrm{ss}}/G] \to X/\!\!/G$ to the geometric invariant theory (GIT) quotient of a polarized projective variety (X, L) by a reductive group G. However, for the question of K-stability of Fano varieties, it is not clear how to realize it as a GIT question: on the one hand, we know there are K-polystable Fano varieties which are not asymptotically Chow semistable (see e.g. [OSY12, LLSW17]); on the other hand, the more natural CM line bundle is not positive on the Hilbert scheme (see [FR06]).

Roughly speaking, for moduli problems which are not known to be global GIT quotients, however, we still aim to find a quotient space, such that the quotient morphism behaves as well as the GIT quotient morphism $[X^{\mathrm{ss}}/G] \to X/\!\!/G$ from many perspectives (see Definition 2.1). In this note, we adapt the general framework developed in [AHLH18] to the case of K-semistable $\mathbb{Q}$-Fano varieties.

Theorem 1.1. *The functor $\mathfrak{X}^{\mathrm{Kss}}_{n,V}$ satisfies the valuative criterion for S-completeness (see Definition 2.3) and Θ-reductivity (see Definition 2.7) with respect to essentially of finite type DVRs.*

For an Artin stack of finite type with affine diagonal over a field of characteristic 0, [AHLH18, Theorem A] states that the conditions of S-completeness and Θ-reductivity are equivalent to the existence of a separated good moduli space. An immediate corollary is that

Corollary 1.2. *Let $\mathfrak{X} \subset \mathfrak{X}^{\mathrm{Kss}}_{n,V}$ be a subfunctor representable by an Artin stack of finite type, such that if $x \in \mathfrak{X}$ then $\overline{\{x\}} \subset \mathfrak{X}$. Then $\mathfrak{X}$ admits a separated good moduli space.*

Of course, we expect $\mathfrak{X}^{\mathrm{Kss}}_{n,V}$ itself is representable by an Artin stack of finite type. The stack $\mathfrak{X}_{n,V}$ is an Artin stack with affine diagonal, and it is known that the semistable locus is bounded (cf. [Jia17]), so it remains to show that $\mathfrak{X}^{\mathrm{ss}}_{n,V} \subset \mathfrak{X}_{n,V}$ is an open substack (see [BX18, Step II]).

In fact, we prove S-completeness and Θ-reductivity of the moduli functor parameterizing families of K-semistable log Fano pairs. Since S-completeness implies the reductivity of the automorphism group of any polystable point, we can conclude:

Theorem 1.3. *If (X, D) is a K-polystable log Fano pair, then $\mathrm{Aut}(X, D)$ is reductive.*

This theorem has a long history: it is a classical result for Kähler-Einstein Fano manifolds in [Mat57] (and even holds in the more general case of polarized manifolds with constant scalar curvature metrics). For log Fano pairs with a weak conical

Kähler-Einstein metric, this is a much harder result and it is a key step in the proofs of the Yau-Tian-Donaldson Conjecture for smooth Fano manifolds (see e.g. [CDS15, Tia15, BBE+19]). Our method is purely algebro-geometric. In [BX18], it was shown that if (X, D) is K-stable, then $\mathrm{Aut}(X, D)$ is finite. That paper also establishes a key ingredient in the proof of Theorem 1.3, the Finite Generation Condition 3.1.

1.2. **Sketch of the proof.** We sketch the main ideas in the proof of Theorem 1.1. The conditions of S-completeness and Θ-reductivity of $\mathfrak{X}^{\mathrm{Kss}}_{n,V}$ both involve extending certain families of K-semistable $\mathbb{Q}$-Fano varieties over the complement of a closed point in a certain regular surface to a families over the entire surface. We first show that the pushforward sheaves of m-th relative anti-pluri-canonical line bundles extends, then we prove that the direct sum of these sheaves is finitely generated. After taking Proj of this algebra, we argue that the central fiber is a K-semistable $\mathbb{Q}$-Fano, which gives the desired extension of the family of K-semistable $\mathbb{Q}$-Fano varieties. Of course, such finite generation results are highly nontrivial. Fortunately, for families of K-semistable Fano varieties, the finite generation needed for S-completeness was essentially settled in [BX18] and the case for Θ-reductivity is proved in Section 5, closely following similar arguments in [LWX18]. This general strategy could be conceivably applied to general K-semistable polarized varieties; however, the corresponding finite generation statements (see Conditions 3.1 and 5.1) appear to be very challenging.

We now explain in more detail the proof of S-completeness. The first extensive study of the geometry of K-semistable $\mathbb{Q}$-Fano varieties belonging to the same S-equivalence class was completed in [LWX18]. In particular, it was shown that there is a unique object, namely a K-polystable $\mathbb{Q}$-Fano variety, in each S-equivalence class.

Then in [BX18], the study of families of K-semistable Fano varieties is extended from test configurations to families over a curve. Namely, given two $\mathbb{Q}$-Gorenstein families of K-semistable $\mathbb{Q}$-Fano varieties $f\colon X \to C$ and $f'\colon X' \to C$ over the germ of a pointed smooth curve $(C = \mathrm{Spec}(R), 0)$ and an isomorphism $X \times_C (C \setminus 0) \cong X' \times_C (C \setminus 0)$, [BX18] established that X_0 and X'_0 are always S-equivalent. The argument for this fact can be divided into two parts: (1) one constructs filtrations $\mathcal{F}$ and $\mathcal{F}'$ of $V := \bigoplus_m V_m = \bigoplus_m H^0(X_0, -mrK_{X_0})$ and $V' = \bigoplus_m V'_m = \bigoplus_m H^0(X'_0, -mrK_{X'_0})$ for some fixed sufficiently divisible r such that $\mathrm{gr}_{\mathcal{F}}(V) = \bigoplus_m \mathrm{gr}_{\mathcal{F}}(V_m)$ is isomorphic to $\mathrm{gr}_{\mathcal{F}'}(V') = \bigoplus_m \mathrm{gr}_{\mathcal{F}'}(V'_m)$, and (2) one shows that the above graded rings are indeed finitely generated and moreover that their Proj give a common K-semistable degeneration of X_0 and X'_0.

Meanwhile, the property of S-completeness was introduced in [AHLH18] as part of a general criterion for the existence of good moduli space (see Theorem 2.9). The first key observation in this paper is that the construction of the filtration in [BX18] indeed can be put into this framework of S-completeness. More precisely, in the current note, we verify that for each fixed m, in the above construction from [BX18], the m-th graded module, $\mathrm{gr}_{\mathcal{F}}(V_m) \cong \mathrm{gr}_{\mathcal{F}'}(V'_m)$ is precisely the fiber over 0 of the pushforward along $\overline{\mathrm{ST}}_R \setminus 0 \subset \overline{\mathrm{ST}}_R$ (see (1) for the definition of $\overline{\mathrm{ST}}_R$) of the locally free sheaf over

$\overline{\mathrm{ST}}_R\setminus 0$ obtained by gluing $\mathcal{V}_m = f_*(-mrK_{X/C})$ and $\mathcal{V}'_m = f'_*(-mrK_{X'/C})$. Indeed, we show that the graded module in [BX18] is the same, up to a grading shift, as the one naturally arising from the module over $\overline{\mathrm{ST}}_R$. Hence by taking the direct sum over all m, we produce a graded algebra over $\overline{\mathrm{ST}}_R$, which is finitely generated exactly by the finite generation results proved in [BX18]. Finally, by taking the Proj, we construct the extended family of K-semistable $\mathbb{Q}$-Fano varieties over $\overline{\mathrm{ST}}_R$.

In some sense, the S-completeness criterion in [AHLH18] provides a conceptual framework for enhancing the 'pointwise' results in [LWX18, BX18] to results over families. Remarkably, this even yields new results for a single Fano variety, e.g. Theorem 1.3.

To prove the Θ-reductivity (see Definition 2.7), we need to show that given a family of K-semistable $\mathbb{Q}$-Fano varieties $f\colon X \to C$ over the germ of a pointed curve $(C = \mathrm{Spec}(R), 0)$, then any family of test configurations for $X \times_C (C \setminus 0)$ over $C \setminus 0$ with K-semistable central fibers can be extended to a family of test configurations for X over C with K-semistable central fibers. When X/C itself is a test configuration, the proof is contained in [LWX18]. To establish the Θ-reductivity, we need to generalize the argument in [LWX18] from the base curve being $\Theta = [\mathbb{A}^1/\mathbb{G}_m]$ to a more general base curve C. Nevertheless, the techniques are similar.

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2. Preliminaries

2.1. **Good moduli spaces.** In this section, we discuss some general facts about *good moduli spaces*. The following definition was introduced in [Alp13].

Definition 2.1 (Good moduli space)**.** If $\mathcal{X}$ is an Artin stack of finite type over k, a morphism $\phi\colon \mathcal{X} \to X$ to an algebraic space is called a *good moduli space* if (1) ϕ_* is exact on the category of coherent $\mathcal{O}_{\mathcal{X}}$-modules and (2) $\mathcal{O}_X \to \phi_*\mathcal{O}_{\mathcal{X}}$ is an isomorphism.

Remark 2.2. We note that X is unique as the map $\mathcal{X} \to X$ is initial for maps to algebraic spaces and X is necessarily of finite type over k. Moreover, two k-points of $\mathcal{X}$ are identified in X if and only if their closures intersect. In particular, there is a bijection between the closed k-points of $\mathcal{X}$ (i.e. *the polystable objects*) and the k-points of X.

The canonical example arises from GIT: if G is a reductive group acting on a closed G-invariant subscheme $X \subset \mathbb{P}(V)$, where V is a finite dimensional G-representation, then the morphism

$$[X^{\mathrm{ss}}/G] \to X^{\mathrm{ss}}/\!/G := \mathrm{Proj} \bigoplus_m H^0(X, \mathcal{O}_X(m))^G$$

to the GIT quotient is a good moduli space.

However, the K-stability moduli problem does not have a known GIT interpretation. So to prove the moduli stack $\mathfrak{X}^{\mathrm{Kss}}_{n,V}$ yields a good moduli space $X^{\mathrm{Kps}}_{n,V}$ is quite nontrivial.

2.1.1. *S-completeness.* If R is a DVR with fraction field K and uniformizing parameter π, we define the Artin stack

$$\overline{\mathrm{ST}}_R := [\mathrm{Spec}\big(R[s,t]/(st-\pi)\big)/\mathbb{G}_m], \tag{1}$$

where s and t have weight -1 and 1. This can be viewed as a local model of the quotient $[\mathbb{A}^2/\mathbb{G}_m]$ where $\mathbb{A}^2$ has coordinates s and t with weights -1 and 1; indeed, $\overline{\mathrm{ST}}_R$ is the base change of the good moduli space $[\mathbb{A}^2/\mathbb{G}_m] \to \mathrm{Spec}(k[st])$ along $\mathrm{Spec} R \to \mathrm{Spec}(k[st])$ defined by $st \mapsto \pi$. We denote by $0 \in \overline{\mathrm{ST}}_R$ the unique closed point defined by the vanishing of s and t. Observe that $\overline{\mathrm{ST}}_R \setminus 0$ is the non-separated union $\mathrm{Spec}(R) \bigcup_{\mathrm{Spec}(K)} \mathrm{Spec}(R)$.

Definition 2.3 (S-completeness)**.** A stack $\mathcal{X}$ over k is *S-complete* if for any DVR R and any diagram

$$\begin{array}{ccc} \overline{\mathrm{ST}}_R \setminus 0 & \longrightarrow & \mathcal{X} \\ \big\downarrow & \nearrow & \\ \overline{\mathrm{ST}}_R & & \end{array} \tag{2}$$

there exists a unique dotted arrow filling in the diagram.

Remark 2.4. This definition was introduced for Artin stacks in [AHLH18, §3.5]. Since it is still open whether the stack $\mathfrak{X}^{\mathrm{Kss}}_{n,V}$ of K-semistable $\mathbb{Q}$-Fanos is represented by an Artin stack, we have relaxed the condition that $\mathcal{X}$ be Artin in the above definition.

Remark 2.5. If $\mathcal{X}$ is Deligne-Mumford, then $\mathcal{X}$ is S-complete if and only if $\mathcal{X}$ is separated ([AHLH18, Prop. 3.43]). If $\mathcal{X}$ is an Artin stack with affine diagonal, then any lift is automatically unique ([AHLH18, Prop. 3.38]).

Remark 2.6. If G is a linear algebraic group over k, then BG is S-complete (equivalently S-complete with respect to essentially of finite type DVRs) if and only if G is reductive ([AHLH18, Prop. 3.45 and Rem. 3.46]). Moreover, as S-completeness is preserved under closed substacks, it follows that every closed point (i.e. polystable object) in an Artin stack with affine diagonal, which is S-complete with respect to essentially of finite type DVRs, has *reductive stabilizer*.

2.1.2. Θ*-reductivity.* We define $\Theta = [\mathbb{A}^1/\mathbb{G}_m]$ with coordinate x on $\mathbb{A}^1$, and we set $\Theta_R = \Theta \times_k \mathrm{Spec}(R)$ for any DVR R. We let $0 \in \Theta_R$ be the unique closed point defined by the vanishing of x and a uniformizing parameter $\pi \in R$. Observe that $\Theta_R \setminus 0 = \Theta_K \bigcup_{\mathrm{Spec}(K)} \mathrm{Spec}(R)$.

Definition 2.7 (Θ-reductivity)**.** A stack $\mathcal{X}$ over k is Θ*-reductive* if for any DVR R and any diagram

$$\begin{array}{ccc} \Theta_R \setminus 0 & \longrightarrow & \mathcal{X} \\ \big\downarrow & \nearrow & \\ \Theta_R & & \end{array} \tag{3}$$

there exists a unique dotted arrow filling in the diagram.

Remark 2.8. This definition was introduced in [HL18]. As with S-completeness, if $\mathcal{X}$ is an Artin stack with affine diagonal, then any lift is automatically unique.

2.1.3. *The existence of good moduli spaces.* The following criterion is established in [AHLH18].

Theorem 2.9. [AHLH18, Thm. A] *Let $\mathcal{X}$ be an Artin stack of finite type with affine diagonal over k. Then $\mathcal{X}$ admits a good moduli space $\mathcal{X} \to X$ with X separated if and only if $\mathcal{X}$ is S-complete and Θ-reductive.*

Remark 2.10. In order to show that $\mathcal{X}$ admits a good moduli space, it suffices to show that the valuative criteria for S-completeness and Θ-reductivity for DVRs R essentially of finite type over k ([AHLH18, Rmk. 4.2 and Thm. 5.4]). In particular, it follows that the valuative criteria for S-completeness and Θ-reductivity hold for all DVRs R.

Remark 2.11 (Comparing with the previous criterion)**.** In [LWX19], a variant of the above theorem ([AFS17, Thm. 1.2]) was used to construct a good moduli space of $\mathbb{Q}$-Gorenstein smoothable, K-semistable Fano varieties. Specifically, [AFS17, Thm. 1.2] states that if $\mathcal{X}$ is an Artin stack of finite type with affine diagonal over k, then $\mathcal{X}$ admits a good moduli space $\mathcal{X} \to X$ if the following conditions hold:

1. for every closed point $x \in \mathcal{X}$, the stabilizer G_x is reductive and there exists an étale morphism $f\colon (\mathcal{W}, w) \to (\mathcal{X}, x)$ where $\mathcal{W} \cong [\mathrm{Spec}(A)/G_x]$ such that
   (a) f induces an isomorphism of stabilizer groups at all closed points and
   (b) f sends closed points to closed points, and
2. for any k-point $y \in \mathcal{X}$, the closure $\overline{\{y\}}$ admits a good moduli space.

Vaguely speaking, condition (1a) ensures that the two projections $\mathcal{R} := \mathcal{W} \times_{\mathcal{X}} \mathcal{W} \rightrightarrows \mathcal{W}$ induce isomorphism of stabilizer groups while conditions (1b) and (2) ensure that the projections send closed points to closed points. This is sufficient to imply that the two projections induce an étale equivalence relation $R \rightrightarrows W$ on good moduli spaces and that the algebraic space quotient W/R is a good moduli space of $\mathcal{X}$ Zariski-locally around x.

We would like to explain the general idea of why the properties of S-completeness and Θ-reductivity imply that the above conditions hold. First, S-completeness implies that G_x has a reductive stabilizer (Remark 2.6) and the existence of an étale morphism $f\colon (\mathcal{W} := [\mathrm{Spec}(A)/G_x], w) \to (\mathcal{X}, x)$ then follows from [AHR15, Thm. 1.2].

S-completeness implies that after shrinking $\mathrm{Spec}(A)$, we may arrange that (1a) holds. A complete argument is given in [AHLH18, Prop. 4.4] but we explain here only how S-completeness implies that f induces an isomorphism of stabilizer groups at any generization of w. Let $\xi\colon (\mathrm{Spec}(R), 0) \to (\mathcal{W}, w)$ be a morphism from a complete DVR R (with fraction field K). Then

$$\begin{aligned}\mathrm{Aut}_{\mathcal{W}}(\xi_K) &\cong \{\text{maps } g\colon \overline{\mathrm{ST}}_R \setminus 0 \to \mathcal{W} \text{ and isomorphisms } g|_{s\neq 0} \simeq \xi \simeq g|_{t\neq 0}\}\\ &\cong \{\text{maps } g\colon \overline{\mathrm{ST}}_R \to \mathcal{W} \text{ and isomorphisms } g|_{s\neq 0} \simeq \xi \simeq g|_{t\neq 0}\}\end{aligned}$$

where we have used S-completeness in the second line. There is an analogous description of $\mathrm{Aut}_{\mathcal{X}}(f(\xi_K))$. Since f is étale and R is complete, Tannaka duality implies that any map $(\overline{\mathrm{ST}}_R, 0) \to (\mathcal{X}, x)$ lifts uniquely to a map $(\overline{\mathrm{ST}}_R, 0) \to (\mathcal{W}, w)$. It follows that $\mathrm{Aut}_{\mathcal{W}}(\xi_K) \cong \mathrm{Aut}_{\mathcal{X}}(f(\xi_K))$.

Similarly, Θ-reductivity implies that after shrinking $\mathrm{Spec}(A)$ further, we may arrange that (1b) holds. A complete argument is given in [AHLH18, Prop. 4.4] but we show here that if $\xi \in \mathcal{W}$ is a generization of w such that $\xi \in \mathcal{W}_K$ is closed where $K = \overline{k(\xi)}$, then $\eta := f(\xi) \in \mathcal{X}_K$ is also closed. Indeed, suppose $\eta \rightsquigarrow \eta_0$ is a specialization to a closed point in $\mathcal{X}_K$; this can be realized by a map $\lambda\colon \Theta_K \to \mathcal{X}$. If $h\colon \mathrm{Spec}(R) \to \mathcal{W}$ is a map from a DVR with fraction field K realizing the specialization $\xi \rightsquigarrow w$, then λ and $f \circ h$ glue to form a map $\Theta_R \setminus 0 \to \mathcal{X}$ which can be extended (using Θ-reductivity) to a map $(\Theta_R, 0) \to (\mathcal{X}, x)$, and this in turn (using étaleness of f and completeness of R) lifts to a unique map $(\Theta_R, 0) \to (\mathcal{W}, w)$. But since $\xi \in \mathcal{W}_K$ is closed, the image of $\Theta_K \to \mathcal{W}$ consists of a single point, and thus the same is true for the image of λ. It follows that $f(\xi) = \eta_0 \in \mathcal{X}_K$ is closed.

Finally, both the S-completeness and Θ-reductivity imply that (2) holds. Let $y_0 \in \mathcal{Y} := \overline{\{y\}}$ be a closed point and $f\colon (\mathcal{W} := [\mathrm{Spec}(A)/G_{y_0}], w_0) \to (\mathcal{Y}, y_0)$ be an étale morphism in which we can arrange that w_0 is the unique preimage of y_0. By Zariski's main theorem, we may factor f as the composition of a dense open immersion $\mathcal{W} \hookrightarrow \widetilde{\mathcal{W}}$ and a finite morphism $\widetilde{\mathcal{W}} \to \mathcal{Y}$. Note that $w_0 \in \widetilde{\mathcal{W}}$ is necessarily closed and that any other closed point in $\widetilde{\mathcal{W}}$ is a specialization of a k-point in $\mathcal{W}$. As $\widetilde{\mathcal{W}}$ is also Θ-reductive, any k-point has a unique specialization to a closed point. It follows that w_0 is the unique closed point in $\widetilde{\mathcal{W}}$ and thus the complement $\widetilde{\mathcal{W}} \setminus \mathcal{W}$ is empty. This in turn implies that $f\colon \mathcal{W} \to \mathcal{Y}$ is finite étale of degree 1 and thus an isomorphism.

In [LWX19], by using the analytic results, a stronger result than (2) was obtained, and as a result that the good moduli space is a scheme instead of only being an algebraic space.

Lemma 2.12. *Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a finite type monomorphism of Artin stacks locally of finite type over k such that for every geometric point $x\colon \mathrm{Spec}(l) \to \mathcal{X}$, the image under $\mathcal{X}_l \to \mathcal{Y}_l$ of the closure $\overline{\{x\}} \subset \mathcal{X}_l$ is closed in $\mathcal{Y}_l$. If $\mathcal{Y}$ is Θ-reductive (resp., S-complete) with respect to essentially of finite type DVRs, then so is $\mathcal{X}$.*

Proof. Zariski's main theorem implies that there is a factorization $f\colon \mathcal{X} \hookrightarrow \widetilde{\mathcal{X}} \to \mathcal{Y}$ where $\mathcal{X} \hookrightarrow \widetilde{\mathcal{X}}$ is an open immersion and $\widetilde{\mathcal{X}} \to \mathcal{Y}$ is finite. By [AHLH18, Props. 3.18], $\widetilde{\mathcal{X}}$ is also Θ-reductive with respect to essentially of finite type DVRs, so may assume that f is an open immersion. Consider an essentially of finite type DVR R with residue field $l = R/\pi$ and a morphism $h\colon \Theta_R \setminus 0 \to \mathcal{X}$. Since $\mathcal{Y}$ is Θ-reductive, h extends to a diagram

$$\begin{array}{ccccc} \operatorname{Spec}(l) & \xrightarrow{\pi=0} & \Theta_R \setminus 0 & \xrightarrow{h} & \mathcal{X} \\ \big\downarrow & & \big\downarrow & & \big\downarrow f \\ \Theta_l & \xrightarrow{\pi=0} & \Theta_R & \xrightarrow{\widetilde{h}} & \mathcal{Y}. \end{array}$$

In particular, if x denotes the composition $\operatorname{Spec}(l) \to \Theta_R \setminus 0 \to \mathcal{X}$, we have a specialization $x \rightsquigarrow \widetilde{h}(0)$ in $\mathcal{Y}_l$. The hypotheses imply that $\widetilde{h}(0) \in \mathcal{X}_l$ so that $\widetilde{h}$ factors though $\mathcal{X}$. The argument for S-completeness is analogous. □

2.2. **Log Fano pairs and K-stability.** In this section, we introduce some basic notions concerning log Fano pairs and K-stability. For further background information, see [BX18, Sect. 2] and the references there.

A *pair* (X, D) is composed of a normal variety X and an effective $\mathbb{Q}$-divisor D on X such that $K_X + D$ is $\mathbb{Q}$-Cartier. See [KM98, 2.34] for the definitions of *klt*, *plt*, and *lc* pairs. A pair (X, D) is *log Fano* if X is projective, (X, D) is klt, and $-K_X - D$ is ample. A variety X is $\mathbb{Q}$-*Fano* if $(X, 0)$ is log Fano.

2.2.1. *Families of log Fano pairs.*

Definition 2.13. Let T be a normal scheme. A $\mathbb{Q}$-*Gorenstein family of log Fano pairs* $(X, D) \to T$ is composed of a flat projective morphism of normal schemes $X \to T$ and a $\mathbb{Q}$-divisor D on X satisfying:

1. $\operatorname{Supp}(D)$ does not contain a fiber,
2. $K_{X/T} + D$ is $\mathbb{Q}$-Cartier, and
3. $(X_{\bar{t}}, D_{\bar{t}})$ is a log Fano pair for all $t \in T$.

In (3), $D_{\bar{t}}$ denotes the *divisorial pullback* of D. More generally, if $S \to T$ is a morphism of normal schemes, we set $X_S := X \times_T S$ and write D_S for the $\mathbb{Q}$-divisor on X_S associated to $\operatorname{Cycle}(D \times_T S)$.

A *special test configuration* of a log Fano pair (X, D) is the data of a $\mathbb{G}_m$-equivariant $\mathbb{Q}$-Gorenstein family of log Fano pairs $(\mathcal{X}, \mathcal{D}) \to \mathbb{A}^1$ with an isomorphism $(\mathcal{X}_1, \mathcal{D}_1) \simeq (X, D)$ for $\{1\} \to \mathbb{A}^1$.

2.2.2. *K-stability.* Let (X, D) be an n-dimensional log Fano pair and E a prime divisor on a proper normal birational model $\mu : Y \to X$. Following [Fuj18], we set

$$\beta_{X,D}(E) = (-K_X - D)^n A_{X,D}(E) - \int_0^\infty \operatorname{vol}(\mu^*(-K_X - D) - tE)\, dt,$$

where $A_{X,D}(E) := 1 + \mathrm{coeff}_E(K_{Y/X})$ is the log discrepancy.

Definition 2.14. A log Fano pair (X, D) is

(1) *K-semistable* if $\beta_{X,D}(E) \geq 0$ for all divisors E over X;
(2) *K-stable* if $\beta_{X,D}(E) > 0$ for all divisors E over X;
(3) *K-polystable* if it is K-semistable and for any special test configuration of $(\mathcal{X}, \mathcal{D}) \to \mathbb{A}^1$ of (X, D) with $(\mathcal{X}_0, \mathcal{D}_0)$ K-semistable there is an isomorphism of $\mathbb{Q}$-Gorenstein families of log Fano pairs $(\mathcal{X}, \mathcal{D}) \simeq (X_{\mathbb{A}^1}, D_{\mathbb{A}^1}) := (X, D) \times \mathbb{A}^1$.

The equivalence of the above definition with the original definitions in [Tia97, Don02] was addressed in [Fuj19, Li17, LWX18, BX18].

Though the above notions of stability makes sense for log Fano pairs over characteristic zero fields that are not algebraically closed, we will not use them due to the following issue: Let (X_K, D_K) be a log Fano pair over a characteristic zero field K and K'/K a field extension. While it is expected that (X_K, D_K) is K-semistable if and only if $(X_{K'}, D_{K'})$ is K-semistable, the result is only known when both K and K' are algebraically closed (for example, see [BL18, Cor. 15]).

2.3. **Flat families of polarized schemes over a surface.** We will be considering S-completeness and Θ-reductivity stacks parameterizing polarized varieties. Both conditions are formulated in terms of the existence of extensions of equivariant flat families of polarized varieties over punctured regular surfaces.

We thus consider a normal noetherian 2-dimensional scheme, and a closed point $0 \in S$. Let $j : S \setminus 0 \to S$ be the open immersion.

Lemma 2.15. *Let $q : \mathcal{X} \to S \setminus 0$ be a flat projective morphism of schemes, and let $\mathcal{L}$ be a relatively ample line bundle on $\mathcal{X}$. Then the following are equivalent:*

(1) there exists an extension of q to a flat projective family $\widetilde{\mathcal{X}} \to S$ with an ample $\mathbb{Q}$-line bundle $\widetilde{\mathcal{L}}$ extending $\mathcal{L}$;
(2) the algebra $\bigoplus_{m \geq 0} j_*(q_*(\mathcal{O}_{\mathcal{X}}(m\mathcal{L})))$ is finitely generated as an $\mathcal{O}_S$-algebra; and
(3) the restriction $\bigoplus_{m \geq 0} j_*(q_*(\mathcal{O}_{\mathcal{X}}(m\mathcal{L})))|_0$ is finitely generated as a $\kappa(0)$-algebra.

If these conditions hold, then

$$\widetilde{\mathcal{X}} = \mathrm{Proj}_S \left(\bigoplus_{m \geq 0} j_*\big(q_*\mathcal{O}_{\mathcal{X}}(m\mathcal{L})\big) \right)$$

is the unique extension, with the polarization $\mathcal{O}_{\widetilde{\mathcal{X}}}(1)$. If $\mathcal{X}$ is equivariant for the action of $\mathbb{G}_m$ on S, then so is $\widetilde{\mathcal{X}}$.

Proof. The key observation is that j_* induces an equivalence between the category of locally free sheaves on $S \setminus 0$ and locally free sheaves on S, and more generally between the categories of flat quasi-coherent sheaves, with inverse given by restriction.

(1) $\Leftrightarrow$ (2): Note that $q_*(\mathcal{O}_{\mathcal{X}}(m\mathcal{L}))$ is locally free for $m \gg 0$ because q is flat. It follows that $\widetilde{\mathcal{X}} = \mathrm{Proj}_S(\bigoplus_m j_*(q_*\mathcal{O}_{\mathcal{X}}(m\mathcal{L})))$ is a flat extension of $\mathcal{X}$ if this algebra is finitely generated, and conversely for any flat extension $\Gamma(\widetilde{\mathcal{X}}, \mathcal{O}_{\widetilde{\mathcal{X}}}(m\widetilde{\mathcal{L}})) = j_*(q_*(\mathcal{O}_{\mathcal{X}}(m\mathcal{L})))$ for $m \gg 0$.

(3) $\Leftrightarrow$ (2): Note that (2) $\Rightarrow$ (3) automatically, and finite generation is local over S by definition, so we may assume S is affine. Then we may lift a finite set of generators of $\bigoplus_{m\geq 0} j_*(q_*(\mathcal{O}_{\mathcal{X}}(m\mathcal{L}))) \otimes_{\mathcal{O}_S} \kappa(0)$ to $\bigoplus_{m\geq 0} j_*(q_*(\mathcal{O}_{\mathcal{X}}(m\mathcal{L})))$, and by assumption we may find elements in the latter which generate the algebra $\bigoplus_{m\geq 0} q_*(\mathcal{O}_{\mathcal{X}}(m\mathcal{L}))$ after restriction to $S \setminus 0$. Together these generate $\bigoplus_{m\geq 0} j_*(q_*(\mathcal{O}_{\mathcal{X}}(m\mathcal{L})))$ as an $\mathcal{O}_S$-algebra.

Note that if $\widetilde{\mathcal{X}}$ is equivariant for a $\mathbb{G}_m$-action on S, then $\bigoplus_m j_*(q_*(\mathcal{O}_{\mathcal{X}}(m\mathcal{L})))$ has an additional grading coming from the $\mathbb{G}_m$-action, and this grading induces a $\mathbb{G}_m$-action on $\widetilde{\mathcal{X}}$ extending the one on $\mathcal{X}$. $\square$

3. S-completeness

In this section, we will prove that the moduli of K-semistable log Fano pairs is S-complete (Theorem 3.3). We first study S-completeness for quasi-coherent sheaves in Section 3.1 and then S-completeness of polarized varieties in Section 3.2. Applying this to the direct sum of the pushforwards of the m-th tensor product of the polarization for a family of polarized varieties, this naturally leads to a finite generation condition on the graded algebra (see Condition 3.1). In Section 3.3, we confirm the Finite Generation Condition 3.1 for K-semistable log Fano pairs.

3.1. ***S*-completeness for coherent sheaves.** We first recall the correspondence between flat coherent sheaves on Θ and filtrations. A quasi-coherent sheaf F on $\Theta_k = [\mathrm{Spec}(k[x])/\mathbb{G}_m]$ corresponds to a $\mathbb{Z}$-graded $k[x]$-module $\bigoplus_{p\in\mathbb{Z}} F_p$, which in turn corresponds to diagram of k-vector spaces: $\cdots \to F_{p+1} \xrightarrow{x} F_p \xrightarrow{x} F_{p-1} \to \cdots$. The restriction of F along $\mathrm{Spec}(k) \xrightarrow{1} \Theta_k$ is $\mathrm{colim}(\cdots \to F_{p+1} \xrightarrow{x} F_p \to \cdots)$ and along $B_k\mathbb{G}_m \xrightarrow{0} \Theta_k$ is the associated graded $\bigoplus_p F_p/xF_{p+1}$. Moreover, F is flat and coherent if and only if each F_p is flat and coherent, the maps x are injective, $F_p = 0$ for $p \gg 0$ and $x\colon F_p \to F_{p-1}$ is an isomorphism for $p \ll 0$.

Similarly, if R is a DVR with fraction field K, residue field κ and uniformizing parameter π, then a quasi-coherent sheaf F on $\overline{\mathrm{ST}}_R$ corresponds to a $\mathbb{Z}$-graded $R[s,t]/(st-\pi)$-module $\bigoplus_{p\in\mathbb{Z}} F_p$, which in turn corresponds to a diagram of maps of R-modules

$$\cdots \underset{t}{\overset{s}{\rightleftarrows}} F_{p+1} \underset{t}{\overset{s}{\rightleftarrows}} F_p \underset{t}{\overset{s}{\rightleftarrows}} F_{p-1} \underset{t}{\overset{s}{\rightleftarrows}} \cdots,$$

such that $st = ts = \pi$. The restriction of F along

- along $\mathrm{Spec}(R) \xhookrightarrow{s\neq 0} \overline{\mathrm{ST}}_R$ is $\mathrm{colim}(\cdots \xrightarrow{s} F_p \xrightarrow{s} F_{p-1} \xrightarrow{s} \cdots)$,
- along $\mathrm{Spec}(R) \xhookrightarrow{t\neq 0} \overline{\mathrm{ST}}_R$ is $\mathrm{colim}(\cdots \xleftarrow{t} F_p \xleftarrow{t} F_{p-1} \xleftarrow{t} \cdots)$,

- along $\Theta_\kappa \xhookrightarrow{s=0} \overline{\mathrm{ST}}_R$ is the object corresponding to the sequence

$$(\cdots \xleftarrow{t} F_p/sF_{p+1} \xleftarrow{t} F_{p-1}/sF_p \xleftarrow{t} \cdots),$$

- along $\Theta_\kappa \xhookrightarrow{t=0} \overline{\mathrm{ST}}_R$ is $(\cdots \xrightarrow{s} F_{p+1}/tF_p \xrightarrow{s} F_p/tF_{p-1} \xrightarrow{s} \cdots)$, and
- along $B_\kappa\mathbb{G}_m \xhookrightarrow{s=t=0} \overline{\mathrm{ST}}_R$ is the $\mathbb{Z}$-graded κ-module $\bigoplus_{p\in\mathbb{Z}} F_p/(sF_{p+1}+tF_{p-1})$.

The sheaf F is a flat and coherent over $\overline{\mathrm{ST}}_R$ if and only if each F_p is flat and coherent, the maps s and t are injective, the induced maps $t\colon F_{p-1}/sF_p \to F_p/sF_{p+1}$ are injective, $s\colon F_p \to F_{p-1}$ is an isomorphism for $p \ll 0$ and $t\colon F_{p-1} \to F_p$ is an isomorphism for $p \gg 0$.

Let $j\colon \overline{\mathrm{ST}}_R \setminus 0 \hookrightarrow \overline{\mathrm{ST}}_R$ be the open immersion. We will show how to compute the pushforward of coherent sheaves under this open immersion. Let $j_t, j_s\colon \mathrm{Spec}(R) \to \overline{\mathrm{ST}}_R$ and $j_{st}\colon \mathrm{Spec}(K) \to \overline{\mathrm{ST}}_R$ be the open immersions corresponding to $t \neq 0$, $s \neq 0$ and $st \neq 0$. Let $\mathcal{E}$ be a flat coherent sheaf on $\overline{\mathrm{ST}}_R \setminus 0$; this corresponds to a pair of R-modules E and E' together with an isomorphism $\alpha\colon E_K \to E'_K$. Under α, we may identify both E and E' as submodules of E_K. Then $j_*\mathcal{E} \cong (j_t)_*E \cap (j_s)_*E' \subset (j_{st})_*E_K$. As graded $R[s,t]/(st-\pi)$-modules, j_t and j_s correspond to the graded inclusions $R[s,t]/(st-\pi) \subset R[t]_t$ and $R[s,t]/(st-\pi) \subset R[s]_s$, and j_{st} corresponds to $R[s,t]/(st-\pi) \subset K[t]_t$. We compute that

$$\begin{aligned}
(j_{st})_*E_K &\cong E_K \otimes_R R[t]_t \cong \bigoplus_{p\in\mathbb{Z}} E_K t^{-p},\\
(j_t)_*E &\cong E \otimes_R R[t]_t \cong \bigoplus_{p\in\mathbb{Z}} E t^{-p} \subset (j_{st})_*E_K,\\
(j_s)_*E' &\cong E' \otimes_R R[s]_s \cong \bigoplus_{p\in\mathbb{Z}} (\pi^p \cdot E') t^{-p} \subset (j_{st})_*E_K
\end{aligned}$$

where in the last line we have used the identification $s = t^{-1}\pi$. Finally, we compute that

$$j_*\mathcal{E} \cong \bigoplus_{p\in\mathbb{Z}} \big(E \cap (\pi^p \cdot E')\big) t^{-p} \subset \bigoplus_{p\in\mathbb{Z}} E_K t^{-p}. \tag{4}$$

If we define the filtration $\mathcal{G}^pE = E \cap (\pi^p \cdot E')$, then $j_*\mathcal{E}$ is the $\mathcal{O}_{\overline{\mathrm{ST}}_R}$-module given by the diagram

$$\cdots \underset{t}{\overset{s}{\rightleftarrows}} \mathcal{G}^{p+1}E \underset{t}{\overset{s}{\rightleftarrows}} \mathcal{G}^pE \underset{t}{\overset{s}{\rightleftarrows}} \mathcal{G}^{p-1}E \underset{t}{\overset{s}{\rightleftarrows}} \cdots,$$

of R-modules where $s\colon \mathcal{G}^{p+1}E \to \mathcal{G}^pE$ is inclusion and $t\colon \mathcal{G}^pE \to \mathcal{G}^{p+1}E$ is multiplication by π. Note that $j_*\mathcal{E}$ is necessarily a flat and coherent $\mathcal{O}_{\overline{\mathrm{ST}}_R}$-module, because non-equivariantly it is the pushforward of a vector bundle from the complement of a closed point in the regular surface $\mathrm{Spec}(R[s,t]/(st-\pi))$.

3.2. **S-completeness for polarized varieties.** Suppose (X, L) and (X', L') are flat families of polarized varieties over $\mathrm{Spec}(R)$ and $\alpha\colon (X_K, L_K) \to (X'_K, L'_K)$ is an isomorphism. Then (X, L) and (X', L') can be glued along the isomorphism α to a polarized family $(\mathcal{X}, \mathcal{L}) \to \overline{\mathrm{ST}}_R \setminus 0$. This yields a diagram

$$\begin{array}{ccc} \mathcal{X} & & \\ \downarrow{\scriptstyle q} & & \\ \overline{\mathrm{ST}}_R \setminus 0 & \xrightarrow{j} & \overline{\mathrm{ST}}_R\,. \end{array}$$

Now we state our key condition:

Condition 3.1 (Finite Generation Condition)**.** The $\mathcal{O}_{\overline{\mathrm{ST}}_R}$-algebra $\bigoplus_{m \geq 0} j_* q_* \mathcal{O}_{\mathcal{X}}(m\mathcal{L})$ is finitely generated.

By Lemma 2.15, this condition is equivalent to the existence of a flat extension of $\mathcal{X}$ to a polarized family $(\widetilde{\mathcal{X}}, \widetilde{\mathcal{L}}) \to \overline{\mathrm{ST}}_R$. To provide a more explicit description for this algebra, Equation (4) implies that for each $m \geq 0$,

$$\begin{aligned} j_* q_* \mathcal{O}_{\mathcal{X}}(m\mathcal{L}) &\cong \bigoplus_{p \in \mathbb{Z}} \big(H^0(X, \mathcal{O}_X(mL)) \cap \pi^p H^0(X', \mathcal{O}_{X'}(mL'))\big) t^{-p} \\ &\subset \bigoplus_{p \in \mathbb{Z}} H^0(X_K, \mathcal{O}_{X_K}(mL_K)) t^{-p}. \end{aligned}$$

Define a filtration of $V_m := H^0(X, \mathcal{O}_X(mL))$ by

$$\mathcal{G}^p V_m := H^0(X, \mathcal{O}_X(mL)) \cap \pi^p H^0(X', \mathcal{O}_{X'}(mL')),$$

which consists of sections $s \in V_m$ with at worst a pole of order p along X'_0. We have a diagram of R-modules

$$\cdots \underset{t}{\overset{s}{\rightleftarrows}} \mathcal{G}^{p+1} V_m \underset{t}{\overset{s}{\rightleftarrows}} \mathcal{G}^p V_m \underset{t}{\overset{s}{\rightleftarrows}} \mathcal{G}^{p-1} V_m \underset{t}{\overset{s}{\rightleftarrows}} \cdots,$$

where $s\colon \mathcal{G}^{p+1} V_m \to \mathcal{G}^p V_m$ is inclusion and $t\colon \mathcal{G}^p V_m \to \mathcal{G}^{p+1} V_m$ is multiplication by π. This gives the direct sum $\bigoplus_{p,m} \mathcal{G}^p V_m$ the structure of a bigraded $R[s,t]/(st - \pi)$-algebra. Assume the Finite Generation Condition 3.1 holds, then the grading in m defines a projective scheme

$$\mathcal{P} = \underline{\mathrm{Proj}}_{\mathrm{Spec}(R[s,t]/(st-\pi))} \bigoplus_{p,m} \mathcal{G}^p V_m$$

and the grading in p gives an action of $\mathbb{G}_m$ on $\mathcal{P}$ and a linearization of $\mathcal{O}_{\mathcal{P}}(1)$. Observe that $(\widetilde{\mathcal{X}}, \mathcal{O}_{\widetilde{\mathcal{X}}}(1)) = ([\mathcal{P}/\mathbb{G}_m], \mathcal{O}_{\mathcal{P}}(1))$.

Example 3.2. Let (X, L) be a polarized κ-variety, and let $R = \kappa[\![t]\!]$ and $K = \kappa(\!(t)\!)$. Let $(X_K, L_K) \to (X_K, L_K)$ be an automorphism induced from a one-parameter subgroup $\alpha\colon \mathbb{G}_m \to \mathrm{Aut}(X, L)$. The above construction produces a flat family $(\widetilde{\mathcal{X}}, \widetilde{\mathcal{L}})$

over $\overline{\mathrm{ST}}_R$ which corresponds to the trivial flat family

$$\big(X \times \mathrm{Spec}(R[s,t]/(st-\pi)), p_1^*L\big)$$

over $\mathrm{Spec}(R[s,t]/(st-\pi))$ with the $\mathbb{G}_m$-action given by α on the first factor. Observe that if $\mathrm{Aut}(X,L)$ is reductive, then any $\alpha \in \mathrm{Aut}(X,L)(K)$ is in the same double coset as a one-parameter subgroup by the Iwahori decomposition, and it follows that any family over $\overline{\mathrm{ST}}_R \setminus 0$ obtained by gluing two trivial families over $\mathrm{Spec}(R)$ along an isomorphism $\alpha \in \mathrm{Aut}(X,L)(K)$ extends to a family over $\overline{\mathrm{ST}}_R$. On the other hand if $\mathrm{Aut}(X,L)$ is not reductive, such an extension need not exist.

3.3. **S-completeness for K-semistable log Fano pairs.** In this section, we will prove that Condition 3.1 holds for K-semistable log Fano pairs with anticanonical polarization (Theorem 3.3). This is obtained by showing that the filtration considered in [BX18] is equivalent to the filtration in Section 3.2 up to a grading shift. Hence, we can invoke finite generation results proved in [BX18] to verify Condition 3.1 is satisfied and then use a result in [LWX18] to show that corresponding special fiber of the flat extension over $\overline{\mathrm{ST}}_R$ is K-semistable.

Let R be a DVR essentially of finite type over k with uniformizer π, fraction field K, and residue field κ. Let

$$(X,D) \to \mathrm{Spec}(R) \quad \text{and} \quad (X',D') \to \mathrm{Spec}(R)$$

be $\mathbb{Q}$-Gorenstein families of log Fano pairs and assume there is a birational map $\alpha\colon X \dashrightarrow X'$ that induces an isomorphism $(X_K,D_K) \to (X'_K,D'_K)$. Following Section 3.2, the above data gives a $\mathbb{G}_m$-equivariant $\mathbb{Q}$-Gorenstein family of log Fano pairs

$$(\mathcal{X},\mathcal{D}) \to \mathrm{Spec}\left(R[s,t]/(st-\pi)\right) \setminus 0, \tag{5}$$

where $0 \in \mathrm{Spec}\left(R[s,t]/(st-\pi)\right)$ is the closed point defined by the vanishing of (s,t).

Theorem 3.3. *If $(X_{\overline{\kappa}},D_{\overline{\kappa}})$ and $(X'_{\overline{\kappa}},D'_{\overline{\kappa}})$ are K-semistable, then the map in (5) extends to a $\mathbb{G}_m$-equivariant $\mathbb{Q}$-Gorenstein family of log Fano pairs*

$$(\widetilde{\mathcal{X}},\widetilde{\mathcal{D}}) \to \mathrm{Spec}\left(R[s,t]/(st-\pi)\right).$$

Furthermore, the geometric fiber over 0 is K-semistable.

Remark 3.4. (1) The above theorem is an extension of [BX18, Thm 1.1.1], which states that if $(X_{\overline{\kappa}},D_{\overline{\kappa}})$ and $(X'_{\overline{\kappa}},D'_{\overline{\kappa}})$ are K-semistable, then they degenerate to a common K-semistable log Fano pair via special test configurations. Indeed, the restriction of $(\mathcal{X},\mathcal{D}) \to \mathrm{Spec}\left(R[s,t]/(st-\pi)\right)$ to $s=0$ and $t=0$ are naturally test configurations of (X_κ,D_κ) and (X'_κ,D'_κ) with special fiber $(\widetilde{\mathcal{X}}_0,\widetilde{\mathcal{D}}_0)$.

(2) The results in [BX18] are phrased in the setting of families over a smooth pointed curve, not the spectrum of a DVR. Fortunately, the proof in [BX18, Sect. 5] extends with little change to the more general setting of families over DVRs which are essentially of finite type over k.

However, the argument does not automatically generalize to families over the spectrum of $k[[t]]$, since a key ingredient in the proof relies on the MMP, specifically [BCHM10]. While the latter results hold for varieties (and, hence, have natural extensions to essentially of finite type k-schemes), they are not known for $k[[t]]$-schemes.

3.3.1. *Vanishing order filtration.* Consider a diagram

$$\begin{array}{ccc} & Y & \\ {\scriptstyle \rho}\swarrow & & \searrow{\scriptstyle \rho'} \\ X & \overset{\phi'}{\dashrightarrow} & X' \end{array},$$

where ρ and ρ' are proper birational morphisms and Y is normal. Write $\widetilde{X}_0$ and $\widetilde{X}'_0$ for the birational transforms of X_0 and X'_0 on Y.

Fix a positive integer r so that $L := -r(K_X + D)$ and $L' := -r(K_{X'} + D')$ are Cartier divisors. Let

$$V := \bigoplus_{m\in\mathbb{N}} V_m = \bigoplus_{m\in\mathbb{N}} H^0(X, \mathcal{O}_X(mL)) \quad \text{and} \quad V' := \bigoplus_{m\in\mathbb{N}} V'_m = \bigoplus_{m\in\mathbb{N}} H^0(X', \mathcal{O}_{X'}(mL'))$$

denote the section rings of X and X' with respect to L and L'. We write $V_\kappa = \bigoplus_m V_{\kappa,m}$ and $V_K = \bigoplus_m V_{K,m}$ for the restrictions of V to $\mathrm{Spec}(\kappa)$ and $\mathrm{Spec}(K)$. Note that each V_m is a flat R-module and satisfies cohomology and base change, since $H^i(X, \mathcal{O}_X(mL)) = 0$ for $i > 0$ and $m \geq 0$ by [Kol13, Thm. 10.37]. Therefore, V_κ and V_K are isomorphic to the section rings of L_κ and L_K.

Following [BX18, Sect. 5.1], for each $m \in \mathbb{N}$ and $p \in \mathbb{Z}$, we set

$$\mathcal{F}^p V_m := \{g \in V_m \,|\, \mathrm{ord}_{\widetilde{X}'_0}(g) \geq p\},$$

where $\mathrm{ord}_{\widetilde{X}'_0}(g)$ equals the coefficient of $\widetilde{X}'_0$ in $\{\rho^*(g) = 0\}$. Observe that

$$\pi \mathcal{F}^{p-1} V_m = \mathcal{F}^p V_m \cap \pi V_m \tag{6}$$

and setting

$$\mathcal{F}^p V_{\kappa,m} := \mathrm{im}(\mathcal{F}^p V_m \otimes_R \kappa \to V_{m,\kappa}) \subseteq V_{\kappa,m},$$

gives a filtration of the section ring V_κ. We state two results from [BX18, Section 5.2] concerning this filtration.

Proposition 3.5. *If $(X_{\overline{\kappa}}, D_{\overline{\kappa}})$ and $(X'_{\overline{\kappa}}, D'_{\overline{\kappa}})$ are K-semistable, then:*

(1) The $\kappa[t]$-algebra

$$\bigoplus_{m\in\mathbb{N}} \bigoplus_{p\in\mathbb{Z}} (\mathcal{F}^p V_{\kappa,m})\, t^{-p} \tag{7}$$

is finitely generated;

(2) The test configuration $(\mathcal{X}_\kappa, \mathcal{D}_\kappa) \to \mathbb{A}^1_\kappa$ of (X_κ, D_κ) induced by (7) is special and the geometric fiber over 0 is K-semistable.

Proof. The argument in [BX18, Sect. 5.2] implies (1) and that the induced test configuration $(\mathcal{X}_\kappa, \mathcal{D}_\kappa) \to \mathbb{A}^1_\kappa$ of (X_κ, D_κ) is a special test configuration with Futaki invariant zero. Since $\mathrm{Fut}(\mathcal{X}_{\overline{\kappa}}, \mathcal{D}_{\overline{\kappa}}) = \mathrm{Fut}(\mathcal{X}_\kappa, \mathcal{D}_\kappa)$ and the latter is zero, $(\mathcal{X}_{\overline{\kappa}}, \mathcal{D}_{\overline{\kappa}})_0$ must be K-semistable by [LWX18, Lem. 3.1]. □

In the proof of Theorem 3.3, we will need to show that the boundary divisor $\mathcal{D}$ in (5) extends to a well defined family of cycles over $\mathrm{Spec}\,(R[s,t]/(st-\pi))$. For this, let B be a prime divisor in $\mathrm{Supp}(D)$ and write $I_B \subseteq \bigoplus_m V_m$ for the homogenous ideal defining B. Consider the homogenous ideal

$$I := \bigoplus_{m\in\mathbb{N}} \bigoplus_{p\in\mathbb{Z}} \mathrm{im}\,(I_B \cap \mathcal{F}^p V_m \to \mathrm{gr}^p_{\mathcal{F}} V_{\kappa,m}) \subseteq \bigoplus_{m\in\mathbb{N}} \bigoplus_{p\in\mathbb{Z}} \mathrm{gr}^p_{\mathcal{F}} V_{\kappa,m}. \tag{8}$$

Proposition 3.6. *If $(X_{\overline{\kappa}}, D_{\overline{\kappa}})$ and $(X'_{\overline{\kappa}}, D'_{\overline{\kappa}})$ are K-semistable, then the subscheme defined by I is of codimension at least one.*

Proof. Let $(\mathcal{X}_\kappa, \mathcal{D}_\kappa) \to \mathbb{A}^1_\kappa$ denote the test configuration described in Proposition 3.5 and write $\mathcal{B}_\kappa$ for the closure of $B_\kappa \times (\mathbb{A}^1 \setminus 0)$ in $\mathcal{X}_\kappa$ under the imbedding $X_\kappa \times (\mathbb{A}^1 \setminus 0) \hookrightarrow \mathcal{X}_\kappa$. Clearly, the scheme theoretic fiber of $\mathcal{B}_\kappa$ over 0 is codimension one in $(\mathcal{X}_\kappa)_0 \simeq \mathrm{Proj}(\bigoplus_m \bigoplus_p \mathrm{gr}^p_{\mathcal{F}} V_{\kappa,m})$. Since $V(I)$ and the scheme theoretic fiber of $\mathcal{B}_\kappa$ over 0 agree away from a codimension 2 subset by [BX18, Prop. 5.13.1], $V(I)$ is also of codimension at least one. □

In light of the discussion in Section 3.2, observe that

$$\bigoplus_{m\in\mathbb{N}} \bigoplus_{p\in\mathbb{Z}} (\mathcal{F}^p V_m)\, t^{-p} \tag{9}$$

has the structure of a $\mathbb{Z}$-graded $R[s,t]/(st-\pi)$-module, where the map $(\mathcal{F}^p V_m)\, t^{-p} \xrightarrow{s} (\mathcal{F}^{p+1} V_m)\, t^{-p-1}$ is defined by $gt^{-p} \mapsto \pi g t^{-p-1}$. Additionally,

$$\Big(\bigoplus_{m\in\mathbb{N}} \bigoplus_{p\in\mathbb{Z}} (\mathcal{F}^p V_m)\, t^{-p}\Big) \bigotimes_{R[s,t]/(st-\pi)} \kappa[t] \simeq \bigoplus_{m\in\mathbb{N}} \bigoplus_{p\in\mathbb{Z}} (\mathcal{F}^p V_{\kappa,m})\, t^{-p}, \tag{10}$$

since (6) implies $\frac{\mathcal{F}^p V_m}{\pi \mathcal{F}^{p-1} V_m} = \frac{\mathcal{F}^p V_m}{\pi V_m \cap \mathcal{F}^p V_m} \simeq \mathrm{im}\left(\mathcal{F}^p V_m \to \frac{V_m}{\pi V_m}\right)$. Therefore,

$$\Big(\bigoplus_{m\in\mathbb{N}} \bigoplus_{p\in\mathbb{Z}} (\mathcal{F}^p V_m) t^{-p}\Big) \bigotimes_{R[s,t]/(st-\pi)} \kappa \simeq \bigoplus_{m\in\mathbb{N}} \bigoplus_{p\in\mathbb{Z}} \mathrm{gr}^p_{\mathcal{F}} V_{\kappa,m}, \tag{11}$$

where $R[s,t]/(st-\pi) \to \kappa$ is the morphism that that sends s and t to 0.

The following proposition states that the filtration from [BX18] coincides with the filtration from [AHLH18]. See [BHJ17, Section 2.5], [Fuj19, Claim 5.4] or [Li17, (64)] for related arguments applied to test configurations.

Proposition 3.7. *For each $p \in \mathbb{Z}$ and $m \in \mathbb{N}$,*

$$\mathcal{F}^{p-mra} V_m = V_m \cap \pi^p V'_m,$$

where $a := \mathrm{coeff}_{\widetilde{X}'_0}(K_Y - \rho^*(K_X + D))$.

The intersection in the above proposition is taken after using the isomorphism $\alpha^* : K(X') \to K(X)$ to view $V'_m := H^0(X', \mathcal{O}_{X'}(mL'))$ as a sub R-module of $K(X)$.

Proof. First, observe that there are natural isomorphisms

$$\begin{aligned}\pi^p H^0\left(X', \mathcal{O}_{X'}(mL')\right) &\simeq H^0\left(X', \mathcal{O}_{X'}(mL' - pX'_0)\right) \\ &\simeq H^0\left(Y, \mathcal{O}_Y\left(\rho'^*(mL' - pX'_0)\right)\right) \\ &= H^0\left(Y, \mathcal{O}_Y\left(m\rho^*L + m(\rho'^*L' - \rho^*L) - p\rho'^*X'_0\right)\right).\end{aligned}$$

Next, fix $g \in H^0(X, \mathcal{O}_X(mL))$ and set $G = \{g = 0\}$. By the above isomorphisms, $g \in \pi^p H^0(X', \mathcal{O}_{X'}(mL'))$ if and only if

$$G' := \rho^*G + \left(m(\rho'^*L' - \rho^*L) - p\rho'^*X'_0\right) \geq 0.$$

Note that G' is the pullback of a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X', since

$$G' \sim_{\mathbb{Q}} m\rho^*L + \left(m(\rho'^*L' - \rho^*L) - p\rho'^*X'_0\right) \sim_{\mathbb{Q}} \rho'^*(mL' - pX'_0).$$

Therefore, G' is effective if and only if ρ'_*G' is effective.

To understand whether or not ρ'_*G' is effective, observe

$$\rho'^*L' - \rho^*L = r((K_Y - \rho^*(K_X + D)) - (K_Y - \rho'^*(K_{X'} + D'))).$$

and, hence,

$$\rho'_*\left(m(\rho'^*L' - \rho^*L) - p\rho'^*X'_0\right) = m\left(r(D' + aX'_0) - rD'\right) - pX'_0 = (mra - p)X'_0.$$

Therefore, ρ'_*G' is effective if and only if $\operatorname{coeff}_{\widetilde{X}'_0}(\rho^*G) + (mra - p) \geq 0$. ☐

3.3.2. *Proof of S-completeness.* Finally, we are in position to prove Theorem 3.3 as a consequence of results in Sections 3.2 and 3.3.1.

Proof of Theorem 3.3. Following Section 3.2, we consider the $\mathbb{Z}$-graded $R[s,t]/(st-\pi)$-algebra

$$\bigoplus_{m\in\mathbb{N}}\bigoplus_{p\in\mathbb{Z}} \left(V_m \cap \pi^p V'_m\right) t^{-p}. \tag{12}$$

By Proposition 3.7, the algebra equals $\bigoplus_m\bigoplus_p \left(\mathcal{F}^{p-mra}V_m\right)t^{-p}$. Therefore, Equation (11) implies that its restriction to $0 \in \operatorname{Spec}(R[s,t]/(st-\pi))$ is isomorphic to $\bigoplus_m \bigoplus_p \operatorname{gr}_{\mathcal{F}}^{p-mra} V_{\kappa,m}$. Since the latter κ-algebra is of finite type by Proposition 3.5.1, we can apply Lemma 2.15 to conclude that the algebra in (12) is finitely generated as well.

Set $\widetilde{\mathcal{X}} := \underline{\operatorname{Proj}}_{R[s,t]/(st-\pi)}\left(\bigoplus_m\bigoplus_p (V_m \cap \pi V'_m)t^{-p}\right)$ and write $\widetilde{\mathcal{D}}$ for the componentwise closure of $D \times (\mathbb{A}^1 \setminus 0)$ under the embedding $X \times (\mathbb{A}^1 \setminus 0) \simeq \widetilde{\mathcal{X}}_{t\neq 0} \hookrightarrow \widetilde{\mathcal{X}}$. The grading with respect to p gives a $\mathbb{G}_m$-action on $\widetilde{\mathcal{X}}$ that fixes $\widetilde{\mathcal{D}}$.

We claim that $\widetilde{\mathcal{X}} \to \operatorname{Spec}(R[s,t]/(st-\pi))$ has normal fibers and no component of $\widetilde{\mathcal{D}}$ contains a fiber. The statement is clear away from the fiber over 0. Next, note that

$\widetilde{\mathcal{X}}_0 \simeq \operatorname{Proj}(\bigoplus_m \bigoplus_{p-mra} \operatorname{gr}^p_{\mathcal{F}} V_{\kappa,m})$, which is the fiber over $0 \in \mathbb{A}^1_\kappa$ of the special test configuration in Proposition 3.5. Hence, $\widetilde{\mathcal{X}}_0$ is normal.

To see $\widetilde{\mathcal{X}}_0 \not\subset \operatorname{Supp}(\widetilde{\mathcal{D}})$, fix a prime divisor B in the support of D and write $\widetilde{\mathcal{B}}$ for the closure of $B \times (\mathbb{A}^1 \setminus 0)$ in $\widetilde{\mathcal{X}}$. If $I_B \subseteq \bigoplus_m V_m$ is the homogenous ideal defining B, then $\widetilde{\mathcal{B}}$ is defined by the homogenous ideal

$$\bigoplus_m \bigoplus_p (I_B \cap \mathcal{F}^{p-mra} V_m) t^{-p} \subseteq \bigoplus_m \bigoplus_p \left(\mathcal{F}^{p-mra} V_m\right) t^{-p}.$$

Hence, the scheme theoretic fiber of $\widetilde{\mathcal{B}}$ over $0 \in \operatorname{Spec}(R[s,t]/(st-\pi))$ agrees with the vanishing of the ideal in (8). Since the latter idea defines a locus of codimension at least one in $\widetilde{\mathcal{X}}_0$ by Proposition 3.6, $\widetilde{\mathcal{X}}_0 \not\subset \mathcal{B}$.

We are left to show $K_{\widetilde{\mathcal{X}}} + \widetilde{\mathcal{D}}$ is $\mathbb{Q}$-Cartier and that $(\widetilde{\mathcal{X}}_{\overline{0}}, \widetilde{\mathcal{D}}_{\overline{0}})$ is a K-semistable log Fano pair. For the first statement, fix a $\mathbb{Q}$-divisor $\widetilde{\mathcal{L}}$ on $\widetilde{\mathcal{X}}$ such that $mr\widetilde{\mathcal{L}}$ is in the linear equivalence class of $\mathcal{O}_{\widetilde{\mathcal{X}}}(m)$ for a positive integer m. By construction $\widetilde{\mathcal{L}}|_{s\neq 0} \sim_{\mathbb{Q}} (-K_{\widetilde{\mathcal{X}}} - \widetilde{\mathcal{D}})|_{s\neq 0}$. Therefore, $\widetilde{\mathcal{L}} \sim_{\mathbb{Q}} -K_{\widetilde{\mathcal{X}}} - \widetilde{\mathcal{D}} + G$, for some $\mathbb{Q}$-divisor G supported on $\widetilde{\mathcal{X}}|_{s=0}$. Since $\widetilde{\mathcal{X}}|_{s=0}$ is an irreducible Cartier divisor, $-K_{\widetilde{\mathcal{X}}} - \widetilde{\mathcal{D}}$ must be $\mathbb{Q}$-Cartier. Next, note that $(\widetilde{\mathcal{X}}, \widetilde{\mathcal{D}})|_{s=0} \to \mathbb{A}^1_\kappa$ coincides with the special test configuration in Proposition 3.5 by (10). Therefore, $(\widetilde{\mathcal{X}}_{\overline{0}}, \widetilde{\mathcal{D}}_{\overline{0}})$ is a K-semistable log Fano pair. □

Remark 3.8. As in the introduction, let $\mathfrak{X}^{\mathrm{Kss}}_{n,V}$ be the stack parameterizing n-dimensional K-semistable $\mathbb{Q}$-Fano varieties with volume V. Theorem 3.3 immediately implies that $\mathfrak{X}^{\mathrm{Kss}}_{n,V}$ is S-complete with respect to essentially of finite type DVRs. We remind the reader that it is still unknown whether $\mathfrak{X}^{\mathrm{Kss}}_{n,V}$ is Artin since the property that K-semistability is an open condition is also unknown.

Similarly, Theorem 3.3 implies that any suitably defined stack parameterizing K-semistable log Fano pairs is S-complete. However, at the moment, it is not clear how one should define families of log Fano pairs (or more generally klt pairs) over non-reduced schemes.

4. Reductivity of the automorphism group

In this section, we prove that if (X, D) is a K-polystable log Fano pair, then the automorphism group

$$\operatorname{Aut}(X, D) := \{g \in \operatorname{Aut}(X) \,|\, g^*D = D\}.$$

is reductive (Theorem 1.3).

We note that this result would follow formally from results in the previous section if one could establish that a suitably defined stack parameterizing K-semistable log Fano pairs was represented by a finite type Artin stack. Indeed, Theorem 3.3 would show that this stack is S-complete with respect to essentially of finite type DVRs and therefore any closed point (i.e. a K-polystable log Fano pair) has reductive stabilizer (Remark 2.6). We will provide a direct alternative argument for the reductivity of

$\mathrm{Aut}(X,D)$ inspired by the property of S-completeness. Our argument has the advantage that it entirely avoids the language of stacks.

4.1. **Setup.** In this section, we fix a log Fano pair (X,D) and write $D=\sum_{i\in I}a_iD_i$ where the D_i are distinct prime divisors. For each a in the coefficient set $\{a_i\,|\,i\in I\}$, set $B_a:=\bigcup_{a=a_i}D_i$. Choose r sufficiently divisible and large so that $\mathcal{L}:=\mathcal{O}_X(-r(K_X+D))$ is a very ample line bundle.

We will now equip $\mathrm{Aut}(X,D)$ with the structure of a linear algebraic group. Since $\mathcal{L}$ is very ample, $\mathrm{Aut}(X,\mathcal{L}):=\{g\in\mathrm{Aut}(X)\mid g^*\mathcal{L}\simeq\mathcal{L}\}$ is a linear algebraic group as it is a closed subgroup of $\mathrm{PGL}(H^0(X,\mathcal{L}))$. For an element $g\in\mathrm{Aut}(X)$, observe that $g^*D=D$ if and only if $g^*(\mathcal{L})\simeq\mathcal{L}$ and for all a in the coefficient set, g fixes B_a. In other words,

$$\mathrm{Aut}(X,D)=\{g\in\mathrm{Aut}(X,\mathcal{L})\mid\forall a,\ g(B_a)=B_a\}$$

As the conditions that $g(B_a)=B_a$ are closed conditions, this shows that $\mathrm{Aut}(X,D)\subset\mathrm{Aut}(X,\mathcal{L})$ is a closed subgroup.

4.2. **Isotrivial families of K-polystable log Fano pairs.** We begin by stating a special case of Theorem 3.3 when the family is obtained by gluing two trivial families.

Let R be a DVR essentially of finite type over k with fraction field K and residue field κ. Fix a birational map $X_R\dashrightarrow X_R$ that induces an isomorphism $\alpha\colon(X_K,D_K)\to(X_K,D_K)$. As $\mathrm{Spec}(R[s,t]/(st-\pi))$ is the union of $\mathrm{Spec}(R[s]_s)$ and $\mathrm{Spec}(R[t]_t)$ along $\mathrm{Spec}(K[s]_s)=\mathbb{G}_{m,K}$, we may glue the two trivial families $X_{R[s]_s}\to\mathrm{Spec}(R[s]_s)$ and $X_{R[t]_t}\to\mathrm{Spec}(R[t]_t)$ along the $\mathbb{G}_m$-equivariant isomorphism induced by α to obtain a $\mathbb{G}_m$-equivariant $\mathbb{Q}$-Gorenstein family of log Fano pairs

$$(\mathcal{X},\mathcal{D})\to\mathrm{Spec}\left(R[s,t]/(st-\pi)\right)\setminus 0. \tag{13}$$

Note that if we write $\mathcal{B}_a$ for the closure of $B_a\times\mathrm{Spec}(R[t]_t)$ under the inclusion $X_{R[t]_t}\hookrightarrow\mathcal{X}$, then $\mathcal{D}=\sum a\mathcal{B}_a$.

Proposition 4.1. *If (X,D) is K-polystable, then*

$$(\mathcal{X},\mathcal{D})\to\mathrm{Spec}\left(R[s,t]/(st-\pi)\right)\setminus 0$$

extends to a $\mathbb{G}_m$-equivariant $\mathbb{Q}$-Gorenstein family of log Fano pairs

$$(\widetilde{\mathcal{X}},\widetilde{\mathcal{D}})\to\mathrm{Spec}\left(R[s,t]/(st-\pi)\right).$$

with $(\widetilde{\mathcal{X}}_{\overline{0}},\widetilde{\mathcal{D}}_{\overline{0}})\simeq(X_{\overline{\kappa}},D_{\overline{\kappa}})$. Furthermore, if we write $\widetilde{\mathcal{D}}=\sum a\widetilde{\mathcal{B}}_a$, where $\widetilde{\mathcal{B}}_a$ is the closure of $\mathcal{B}_a$, then each $\widetilde{\mathcal{B}}_a$ is flat over $\mathrm{Spec}(R[s,t]/(st-\pi))$ with pure fibers.

By *pure* fibers, we mean that the fibers are equidimensional and have no embedded components.

Proof. By Theorem 3.3, the map in (13) extends to a family $(\widetilde{\mathcal{X}},\widetilde{\mathcal{D}})$ with K-semistable geometric fiber over $0\in\mathrm{Spec}\left(R[s,t]/(st-\pi)\right)$. Hence, the restriction $(\widetilde{\mathcal{X}},\widetilde{\mathcal{D}})|_{s=0,\overline{\kappa}}$ is naturally a special test configuration of $(X_{\overline{\kappa}},D_{\overline{\kappa}})$ with K-semistable geometric fiber over $0 \in \mathbb{A}^1_\kappa$. Since (X, D) is K-polystable, this test configuration must be product type (i.e. it is isomorphic to $(X_{\mathbb{A}^1_{\overline{\kappa}}}, D_{\mathbb{A}^1_{\overline{\kappa}}})$). Therefore, $(\widetilde{\mathcal{X}}_{\overline{0}}, \widetilde{\mathcal{D}}_{\overline{0}}) \simeq (X_{\overline{\kappa}}, D_{\overline{\kappa}})$.

Next, fix a in the coefficient set of D and consider the divisor $\widetilde{\mathcal{B}}_a$. By [Kol20, Thm. 4.33], there exists a locally closed decomposition $S \to \mathrm{Spec}(R[s,t]/(st-\pi))$ such that a morphism of schemes $T \to \mathrm{Spec}(R[s,t]/(st-\pi))$, with T reduced, factors through S if and only if $\widetilde{\mathcal{B}}_a|_T \to T$ is flat and has pure fibers. Now, the loci $\{s = 0\}$, $\{s \neq 0\}$ and $\{t \neq 0\}$ factor through S, since the divisorial restrictions $\widetilde{\mathcal{B}}_a|_{s=0}$, $\widetilde{\mathcal{B}}_a|_{s\neq 0}$ and $\widetilde{\mathcal{B}}_a|_{t\neq 0}$ are trivial families. It follows that $S = \mathrm{Spec}(R[s,t]/(st-\pi))$, so $\widetilde{\mathcal{B}}_a \to \mathrm{Spec}\,(R[s,t]/(st-\pi))$ is flat with pure fibers. □

4.3. **Reductivity via Iwahori decompositions.** Throughout this section, let $R = k[[\pi]]$ and $K = k((\pi))$. Given a linear algebraic group G, Iwahori's theorem (c.f. [MFK94, p.52]) states that if G is reductive, then for any element $g \in G(K)$, there exist $a, b \in G(R)$ and a one-parameter subgroup $\lambda \in \mathrm{Hom}(\mathbb{G}_m, G)$ such that $g = a \cdot \lambda|_K \cdot b$, where $\lambda|_K$ denotes the composition $\mathrm{Spec}(K) \to \mathbb{G}_m = \mathrm{Spec}(k[\pi]_\pi) \xrightarrow{\lambda} G$. If we denote by Λ_G the K-points induced by one-parameter subgroups of G, then Iwahori's theorem states that if G is reductive, then

$$G(K) = G(R)\Lambda_G G(R).$$

The following argument, which states that the converse also holds, was communicated to us by Jun Yu. See [AHLH19] for a proof using Artin stacks and S-completeness.

Proposition 4.2. *Let G be a linear algebraic group. If $G(K) = G(R)\Lambda_G G(R)$, then G is reductive.*

Proof. Arguing by a contradiction, assume G is not reductive. Write $G = G_u \rtimes G_s$ for the Levi decomposition of G. That means, G_u is the unipotent radical of G, which is a (connected) unipotent group over k, and G_s is a reductive group over k; the map $(x; y) \to xy$ where $(x \in G_u, y \in G_s)$ gives a bijection $G_u \times G_s \to G$.

For any one-parameter subgroup $\lambda\colon \mathbb{G}_m \to G$ defined over k, the image of λ consists of semisimple elements. Thus, it is contained in a conjugate of G_s. That means, there exists $g \in G(k)$ such that $\mathrm{Ad}(g)(\lambda)$ has image lying in G_s, or in other words $\mathrm{Ad}(g) \cdot \lambda \in \Lambda_{G_s}$. Therefore,

$$\Lambda_G = \mathrm{Ad}(G(k))(\Lambda_{G_s}) \subset G(k)\Lambda_{G_s}G(k).$$

Since $G = G_u \rtimes G_s$, we get

$$G(R) = G_u(R) \rtimes G_s(R) = G_u(R)G_s(R) = G_s(R)G_u(R).$$

Combining the above, we get

$$\begin{aligned} G(R)\Lambda_G G(R) &\subset G(R)G(k)\Lambda_{G_s}G(k)G(R) \\ &= G(R)\Lambda_{G_s}G(R) \\ &= G_u(R)G_s(R)\Lambda_{G_s}G_s(R)G_u(R) \\ &= G_u(R)G_s(K)G_u(R), \end{aligned}$$

where we apply Iwahori's theorem to G_s in the last equality. Thus, $G(R)\Lambda_G G(R) = G_u(R)G_s(K)G_u(R)$.

Suppose

$$G(K) = G(R)\Lambda_G G(R) = G_u(R)G_s(K)G_u(R).$$

Then,

$$G_u(K) = \big(G_u(R)G_s(K)G_u(R)\big) \cap G_u(K) = G_u(R)\big(G_s(K) \cap G_u(K)\big)G_u(R) = G_u(R).$$

Since G_u is a connected unipotent group, we have $G_u \cong \mathbb{A}^n$ as a variety over k, where $n = \dim G_u$. Then, $G_u(K) = G_u(R)$ implies that $K^n = R^n$, which is absurd. □

The following lemma will allow us to work with essentially finite type DVRs when checking that the hypotheses of Proposition 4.2 are satisfied.

Lemma 4.3. *If G is a linear algebraic group, then for any $g \in G(K)$, there is an algebraic point g_0 such that $g \cdot g_0^{-1} \in G(R)$, where algebraic means that $g_0 \in G(k(C))$ for the function field $k(C)$ of a smooth curve over k embedded in K via a dominant morphism* $\mathrm{Spec}(R) \to C$.

Proof. To prove the above claim, fix an embedding $G \subset \mathrm{GL}_m$ for some $m > 0$. Fix $N \gg 0$ so that $\pi^N \cdot g^{-1} \in M_{m\times m}(R)$. By Artin approximation ([Art69]), we can find an algebraic point $g_0 \in G(K)$ such that $g - g_0 \in \pi^{N+1} \cdot M_{m\times m}(R)$. Since $gg_0^{-1} = \left(1 - \frac{g-g_0}{g}\right)^{-1}$ and $\frac{g-g_0}{g} \in \pi \cdot M_{m\times m}(R)$, we know

$$g \cdot g_0^{-1} = 1 + \sum_{i=1}^{\infty} \left(\frac{g_0 - g}{g}\right)^i \in \mathrm{GL}_m(R) \cap G(K) = G(R).$$

□

4.4. **Proof of reductivity.** Theorem 1.3 is an immediate consequence of Lemma 4.2 and the following proposition.

Proposition 4.4. *If (X, D) is a K-polystable log Fano pair, then $G :=$ $\mathrm{Aut}(X, D)$ satisfies $G(K) = G(R)\Lambda_G G(R)$.*

Proof. Set $R = k[[\pi]]$ and $K = k((\pi))$. By Lemma 4.3, it suffices to show that all algebraic points of $G(K)$ are contained in $G(R)\Lambda_G G(R)$. To proceed, fix a smooth pointed curve $x \in C$ with local ring $R_0 := \mathcal{O}_{C,x}$, function field $K_0 := \mathrm{Frac}(\mathcal{O}_{C,x})$, and an extension of DVRs $R_0 \subset R$. We will show that if $g \in G(K_0)$, then $g|_K \in G(R)\Lambda_G G(R)$.

Consider the isomorphism $(X_{K_0}, D_{K_0}) \to (X_{K_0}, D_{K_0})$ of log Fano pairs induced by g. This data gives a $\mathbb{G}_m$-equivariant $\mathbb{Q}$-Gorenstein family of log Fano pairs

$$(\mathcal{X}, \mathcal{D}) \to \mathrm{Spec}\left(R_0[s,t]/(st - \pi)\right) \setminus 0$$

and we may write $\mathcal{D} = \sum a\mathcal{B}_a$. By Proposition 4.1, the above family extends to a $\mathbb{G}_m$-equivariant $\mathbb{Q}$-Gorenstein family of log Fano pairs $(\widetilde{\mathcal{X}}, \widetilde{\mathcal{D}}) \to \mathrm{Spec}(R_0[s,t]/(st - \pi))$ such that $(\widetilde{\mathcal{X}}_0, \widetilde{\mathcal{D}}_0) \simeq (X, D)$. Moreover, $\widetilde{\mathcal{D}} = \sum a\widetilde{\mathcal{B}}_a$ where each $\widetilde{\mathcal{B}}_a$ is flat over

$\mathrm{Spec}(R[s,t]/(st-\pi))$ with pure fibers. The $\mathbb{G}_m$-action on the fiber $(\widetilde{\mathcal{X}}_0, \widetilde{\mathcal{D}}_0)$ induces a 1-parameter subgroup $\lambda : \mathbb{G}_m \to G$.

Replace $(\widetilde{\mathcal{X}}, \widetilde{\mathcal{D}})$ with its base change by R to get a family over $\mathcal{S} := \mathrm{Spec}\,(R[s,t]/(st-\pi))$. As every geometric fiber of the family $(\widetilde{\mathcal{X}}, \widetilde{\mathcal{D}}) \to \mathrm{Spec}(R)$ is isomorphic to the base change of (X, D) and since each $\widetilde{\mathcal{B}}_a$ is flat over $\mathcal{S}$, the scheme

$$\widetilde{\mathcal{P}} := \mathrm{Isom}_{\mathcal{S}}((\widetilde{\mathcal{X}}, \widetilde{\mathcal{D}}), (X_{\mathcal{S}}, D_{\mathcal{S}}))$$

parameterizing isomorphisms is a $\mathbb{G}_m$-equivariant G-torsor over $\mathcal{S}$ (c.f. [SdJ10, Lemma 2.3.2]). We also consider the trivial G-torsor $\mathcal{P}_\lambda := \mathcal{S} \times G$ endowed with the $\mathbb{G}_m$-action induced by λ. We will show that there is a $\mathbb{G}_m$-equivariant isomorphism $\mathcal{P}_\lambda \simeq \widetilde{\mathcal{P}}$ of G-torsors.

The scheme $\mathcal{I} := \mathrm{Isom}_{\mathcal{S}}(\mathcal{P}_\lambda, \widetilde{\mathcal{P}})$, parameterizing isomorphisms of G-torsors, is smooth over $\mathcal{S}$ and since $\mathbb{G}_m$ is linearly reductive, the fixed locus $\mathcal{I}^{\mathbb{G}_m}$, parameterizing $\mathbb{G}_m$-equivariant isomorphisms, is also smooth over $\mathcal{S}$. Let $\mathcal{S}_n$ be the nth nilpotent thickening of $0 \in \mathcal{S}$. By construction, we have a $\mathbb{G}_m$-equivariant section $s_0 \colon \mathcal{S}_0 \to \mathcal{I}$. By the formal lifting criteria for smoothness, s_0 extends to a compatible family of $\mathbb{G}_m$-equivariant sections $s_n \colon \mathcal{S}_n \to \mathcal{I}$. Finally, we claim that the sections s_n algebraize to a $\mathbb{G}_m$-equivariant section $s \colon \mathcal{S} \to \mathcal{I}$. The $\mathbb{G}_m$-actions induce $\mathbb{Z}$-gradings $\Gamma(\mathcal{O}_{\mathcal{S}}) = \oplus_d \Gamma(\mathcal{O}_{\mathcal{S}})_d$, $\Gamma(\mathcal{O}_{\mathcal{S}_n}) = \oplus_d \Gamma(\mathcal{O}_{\mathcal{S}_n})_d$ and $\Gamma(\mathcal{O}_{\mathcal{I}}) = \oplus_d \Gamma(\mathcal{O}_{\mathcal{I}})_d$. To prove the existence of the desired section $s \colon \mathcal{S} \to \mathcal{I}$, it suffices to verify the existence of a graded homomorphism $\Gamma(\mathcal{O}_{\mathcal{I}}) \to \Gamma(\mathcal{O}_{\mathcal{S}})$ extending the given homomorphisms $\Gamma(\mathcal{O}_{\mathcal{I}}) \to \Gamma(\mathcal{O}_{\mathcal{S}_n})$. To see this, observe that for each d, the compatible maps $\Gamma(\mathcal{O}_{\mathcal{I}})_d \to \Gamma(\mathcal{O}_{\mathcal{S}_n})_d$ extend to a map $\Gamma(\mathcal{O}_{\mathcal{I}})_d \to \varprojlim_n \Gamma(\mathcal{O}_{\mathcal{S}_n})_d$. The latter R-module can be explicitly computed to be isomorphic to $\Gamma(\mathcal{O}_{\mathcal{S}})_d$ since R is complete.

To conclude, the $\mathbb{G}_m$-equivariant isomorphism $\mathcal{P}_\lambda \simeq \widetilde{\mathcal{P}}$ restricts to a $\mathbb{G}_m$-equivariant isomorphism $\mathcal{P}_\lambda|_{\mathcal{S}\setminus 0} \simeq \mathcal{P}$ of G-torsors. As the G-torsors $\mathcal{P}$ and $\mathcal{P}_\lambda|_{\mathcal{S}\setminus 0}$ are obtained by gluing two trivial G-torsors along the isomorphisms induced by $g|_K \in G(K)$ and $\lambda|_K \in G(K)$, respectively, we deduce that there are elements $a, b \in G(R)$ such that $a \cdot g|_K = \lambda|_K \cdot b$. □

5. Θ-reductivity

In this section, we will carry out an analysis similar to that in Section 3 for Θ-reductivity.

5.1. **Θ-reductivity for coherent sheaves.** Let R be a DVR with fraction field K and uniformizing parameter π. Recall that $\Theta = [\mathbb{A}^1/\mathbb{G}_m]$ and that $\Theta_R = \Theta \times \mathrm{Spec}(R) = [\mathrm{Spec}(R[x])/\mathbb{G}_m]$, where x has weight -1.

A quasi-coherent $\mathcal{O}_{\Theta_R}$-module F corresponds to a $\mathbb{Z}$-graded $R[x]$-module $\bigoplus_{p\in\mathbb{Z}} F_p$, which in turn corresponds to a diagram $\cdots \xrightarrow{x} F_{p+1} \xrightarrow{x} F_p \xrightarrow{x} F_{p-1} \xrightarrow{x} \cdots$ of R-modules. The restriction of F to $\mathrm{Spec}(R) \xhookrightarrow{x\neq 0} \Theta_R$ is the R-module $\mathrm{colim}\, F_p$ and the the restriction to $\Theta_\kappa \xhookrightarrow{x=0} \Theta_R$ is the graded R-module $\bigoplus_{p\in\mathbb{Z}} F_p$. The $\mathcal{O}_{\Theta_R}$-module F

is flat and coherent if and only if each F_p is flat and coherent, the maps $x\colon F_{p+1} \to F_p$ are injective, each F_p/F_{p+1} is flat, $F_p = 0$ for $p \gg 0$, and F_p stabilize for $p \ll 0$.

We will compute the pushforward along the open immersion $j\colon \Theta_R \setminus 0 \hookrightarrow \Theta_R$. Denote the open immersions by

$$j_x\colon \mathrm{Spec}(R) \xhookrightarrow{x\neq 0} \Theta_R,\ j_\pi\colon \Theta_K \xhookrightarrow{\pi\neq 0} \Theta_R \quad \text{and} \quad j_{x\pi}\colon \mathrm{Spec}(K) \xhookrightarrow{x\pi\neq 0} \Theta_R.$$

Let $\mathcal{E}$ be a flat coherent sheaf on $\Theta_R \setminus 0$; this corresponds to an R-module E and a $\mathbb{Z}$-filtration $\mathcal{G}^\bullet E_K\colon \cdots \subset \mathcal{G}^{p+1}E_K \subset \mathcal{G}^p E_K \subset \cdots$ of E_K. Then $j_*\mathcal{E} = (j_x)_*E \cap (j_\pi)_*\mathcal{G}^\bullet E_K \subset (j_{x\pi})_*E_K$. As graded $R[x]$-modules, j_x and j_π correspond to the graded inclusions $R[x] \subset R[x]_x$ and $R[x] \subset K[x]$, and $j_{x\pi}$ corresponds to $R[x] \subset K[x]_x$. We compute that

$$(j_{x\pi})_*E_K \cong K[x]_x \otimes_R E_K \cong \bigoplus_{p\in\mathbb{Z}} E_K x^{-p},$$

$$(j_x)_*E \cong E \otimes_R R[x]_x \cong \bigoplus_{p\in\mathbb{Z}} E x^{-p} \subset (j_{x\pi})_*E_K,$$

$$(j_\pi)_*\mathcal{G}^\bullet E_K \cong \bigoplus_{p\in\mathbb{Z}} (\mathcal{G}^p E_K) x^{-p} \subset (j_{x\pi})_*E_K$$

Therefore

$$j_*\mathcal{E} \cong \bigoplus_{p\in\mathbb{Z}} \big(E \cap \mathcal{G}^p E_K\big) x^{-p} \subset \bigoplus_{p\in\mathbb{Z}} E_K x^{-p}. \tag{14}$$

The $\mathcal{O}_{\Theta_R}$-module $j_*\mathcal{E}$ is flat and coherent, and is given by the filtration $\mathcal{G}^p E := E \cap \mathcal{G}^p E_K$ of E.

5.2. **Θ-reductivity for polarized families.** A polarized family $(\mathcal{X}, \mathcal{L})$ over $\Theta_R \setminus 0$ corresponds to a polarized family (X, L) over $\mathrm{Spec}(R)$ and a polarized family $(\mathcal{X}_K, \mathcal{L}_K)$ over Θ_K together with an isomorphism of (X_K, L_K) with the fiber of $(\mathcal{X}_K, \mathcal{L}_K)$ over 1. The polarized family $(\mathcal{X}_K, \mathcal{L}_K)$ over Θ_K corresponds to a test configuration over $\mathbb{A}^1_K$.

Consider the composition $\mathcal{X} \xrightarrow{q} \Theta_R \setminus 0 \xrightarrow{j} \Theta_R$.

Condition 5.1 (Finite Generation Condition)**.** The $\mathcal{O}_{\Theta_R}$-algebra $\bigoplus_{m\geq 0} j_*q_*\mathcal{O}_{\mathcal{X}}(m\mathcal{L})$ is finitely generated.

If Condition 5.1 holds, then

$$\widetilde{\mathcal{X}} := \underline{\mathrm{Proj}}_{\Theta_R} \bigoplus_{m\geq 0} j_*q_*\mathcal{O}_{\mathcal{X}}(m\mathcal{L}),$$

is a flat family of polarized schemes over Θ_R.

For each $m \geq 0$, set $V_m := H^0(X, \mathcal{O}_X(mL))$. For each $m \geq 0$, the vector space $V_{K,m} := H^0(\mathcal{X}_K, \mathcal{O}_{\mathcal{X}_K}(m\mathcal{L}_K))$ inherits a $\mathbb{Z}$-filtration $\mathcal{G}^\bullet V_{K,m}$. Equation (14) yields

$$j_*q_*\mathcal{O}_{\mathcal{X}}(m\mathcal{L}) \cong \bigoplus_{p\in\mathbb{Z}} \big(V_m \cap \mathcal{G}^p V_{K,m}\big) x^{-p} \subset \bigoplus_{p\in\mathbb{Z}} V_{K,m} x^{-p}.$$

If we set $\mathcal{G}^p V_m = V_m \cap \mathcal{G}^p V_{K,m}$, then the direct sum $\bigoplus_{p,m} \mathcal{G}^p V_m$ is a bigraded $R[x]$-module, where multiplication by x is given by the inclusions $\mathcal{G}^p V_m \to \mathcal{G}^{p-1} V_m$. The grading in m defines a projective scheme $\mathcal{P} = \underline{\mathrm{Proj}}_{\mathrm{Spec}(R[x])} \bigoplus_{p,m} \mathcal{G}^p V_m$ and the grading in p gives an action of $\mathbb{G}_m$ on $\mathcal{P}$ and a linearization of $\mathcal{O}_{\mathcal{P}}(1)$. Observe that $(\widetilde{\mathcal{X}}, \mathcal{O}_{\widetilde{\mathcal{X}}}(1)) = ([\mathcal{P}/\mathbb{G}_m], \mathcal{O}_{\mathcal{P}}(1))$.

5.3. **Θ-reductivity for K-semistable log Fano pairs.** In this section, we will verify that $\mathfrak{X}^{\mathrm{Kss}}_{V,n}$ satisfies the valuative criterion for Θ-reductivity over any essentially finite type DVR. The result follows from modifying an argument in [LWX18, Sect. 3].

Fix the following notation: Let R be a DVR essentially of finite type over k with fraction field K and residue field κ. We will write x for the parameter of $\mathbb{A}^1$. To avoid confusion, we write $0_K \in \mathbb{A}^1_K$ for the closed point defined by the vanishing of x and $0 \in \mathbb{A}^1_R$ for the one defined by the vanishing of x and a uniformizing parameter $\pi \in R$.

Fix a $\mathbb{Q}$-Gorenstein family of log Fano pairs $(X, D) \to \mathrm{Spec}(R)$ and a special test configuration $(\mathcal{X}_K, \mathcal{D}_K) \to \mathbb{A}^1_K$ of (X_K, D_K). Following Section 5.2, this data gives a $\mathbb{G}_m$-equivariant family of log Fano pairs

$$(\mathcal{X}, \mathcal{D}) \to \mathbb{A}^1_R \setminus 0\,.$$

Theorem 5.2. *If the geometric fibers of $(X, D) \to \mathrm{Spec}(R)$ and $(\mathcal{X}_K, \mathcal{D}_K) \to \mathbb{A}^1_K$ are K-semistable, then $(\mathcal{X}, \mathcal{D}) \to \mathbb{A}^1_R \setminus 0$ extends to a $\mathbb{G}_m$-equivariant $\mathbb{Q}$-Gorenstein family of log Fano pairs*

$$(\widetilde{\mathcal{X}}, \widetilde{\mathcal{D}}) \to \mathbb{A}^1_R.$$

Furthermore, the geometric fiber over 0 is K-semistable.

Throughout the proof, we will use notation similar to that in Section 3.3.1. Specifically, fix a positive integer r so that $L := -r(K_X + D)$ is a Cartier divisor. Let $V := \bigoplus_m V_m$ denote the section ring of X with respect to L. Recall that each V_m is a flat R-module and the restrictions of V to $\mathrm{Spec}(K)$ and $\mathrm{Spec}(\kappa)$, which we denote by $V_K := \bigoplus_m V_{K,m}$ and $V_\kappa := \bigoplus_m V_{\kappa,m}$ are isomorphic to the section rings of L_K and L_κ.

5.3.1. *Extending filtrations defined by a divisor.* Let E_K be a divisor over X_K and write $A := A_{X_K,D_K}(E_K)$. Setting

$$\mathcal{F}^p_K V_{K,m} := \{f \in V_{K,m} \,|\, \mathrm{ord}_{E_K}(f) \geq p\},$$

for each $p \in \mathbb{Z}$ and $m \in \mathbb{N}$, gives a filtration of V_K. The filtration $\mathcal{F}_K$ of $V_{K,m}$ extends to a filtration $\mathcal{F}$ of V_m by subbundles by setting

$$\mathcal{F}^p V_m := \mathcal{F}^p_K V_{K,m} \cap V_m.$$

Note that $\bigoplus_m \bigoplus_p (\mathcal{F}^p V_m)\, x^{-p}$ is a graded $R[x]$-algebra.

If the above algebra is finitely generated, we set $\mathcal{X} := \underline{\mathrm{Proj}}_{\mathbb{A}^1_R}\left(\bigoplus_m \bigoplus_p (\mathcal{F}^p V_m)\, x^{-p}\right)$. Since

$$\bigoplus_{m\in\mathbb{N}} \bigoplus_{p\in\mathbb{Z}} (\mathcal{F}^p V_m)\, x^{-p} \otimes_{R[x]} R[x,x^{-1}] \simeq V \bigotimes_R R[x,x^{-1}]$$

there is an isomorphism $X \times_R (\mathbb{A}^1_R \setminus 0) \simeq \mathcal{X}|_{\mathbb{A}^1_R\setminus 0}$. We write $\mathcal{D}$ for the closure of $D \times_R (\mathbb{A}^1_R \setminus 0)$ under the previous embedding $X \times_R (\mathbb{A}^1_R \setminus 0) \hookrightarrow \mathcal{X}$.

Proposition 5.3. *If the geometric fibers of $(X,D) \to \mathrm{Spec}(R)$ are K-semistable and $\beta_{X_K,D_K}(E_K) = 0$, then:*

(1) The $R[x]$-algebra $\bigoplus_m \bigoplus_p (\mathcal{F}^p V_m)\, x^{-p}$ is of finite type.
(2) The induced family $(\mathcal{X},\mathcal{D}) \to \mathbb{A}^1_R$ is a $\mathbb{Q}$-Gorenstein family of log Fano pairs and $(\mathcal{X}_{\overline{0}}, \mathcal{D}_{\overline{0}})$ is K-semistable.

The proof is a modification of an argument in [LWX18, Sect. 3]. Similar arguments are also used to prove main theorems in [BX18]. Throughout, we will use notation and background material from [BX18, Sect. 2] on valuations, log canonical thresholds, and the normalized volume function.

Proof. Let $(Y,\Gamma) \to \mathrm{Spec}(R)$ denote the relative cone over $(X,D) \to \mathrm{Spec}(R)$ with respect to the polarization L. Hence, $Y = \underline{\mathrm{Spec}}_R(V)$ and Γ is defined via pulling back D. Note that (Y_K,Γ_K) and (Y_κ,Γ_κ) are the cones over (X_K,D_K) and (X_κ,D_κ).

Following [BX18, Sect. 2.5.1], the divisor E_K over X_K induces a ray of quasimonomial valuations

$$\{v_t \,|\, t \in [0,+\infty)\} \subset \mathrm{Val}_{Y_K}$$

satisfying

$$A_{Y_K,\Gamma_K}(v_t) = 1/r + tA \quad \text{and} \quad \mathfrak{a}_p(v_t) = \bigoplus_m \mathcal{F}_K^{(p-m)/t} V_{K,m}.$$

For each $q \in \mathbb{N}$, there is a divisor $E_{K,q}$ over Y_K such that $q \cdot v_{1/q} = \mathrm{ord}_{E_{K,q}}$. The divisor $E_{K,q}$ over Y_K extends to a divisor E_q over Y. Note that $A_{Y,\Gamma}(E_q) = A_{Y_K,\Gamma_K}(E_{K,q})$ and

$$\mathfrak{a}_p(\mathrm{ord}_{E_q}) = \bigoplus_{m\in\mathbb{N}} \mathcal{F}^{p-mq} V_m. \tag{15}$$

To see Equation (15) holds, observe that the order of vanishing of $f \in \mathcal{O}_Y$ along E_q equals the order of vanishing of $f \cdot \mathcal{O}_{Y_K}$ along $E_{K,q}$. Hence, the statement follows from the definition of $\mathcal{F}$ and the formula $\mathfrak{a}_p(\mathrm{ord}_{E_{K,q}}) = \bigoplus_m \mathcal{F}_K^{p-mk} V_{K,m}$.

Claim 1. The following holds:

$$\lim_{q\to\infty} \left(A_{Y,\Gamma+Y_\kappa}(E_q) - \mathrm{lct}(Y, \Gamma + Y_\kappa; \mathfrak{a}_\bullet(\mathrm{ord}_{E_q})) \right) = 0.$$

For each positive integer q, consider the graded sequence of ideals on Y_κ given by

$$\mathfrak{b}_{q,\bullet} := \mathfrak{a}_\bullet(\mathrm{ord}_{E_q}) \cdot \mathcal{O}_{Y_\kappa}.$$

Note that $\mathfrak{a}_\bullet(\mathrm{ord}_{E_{K,q}}) = \mathfrak{a}_\bullet(\mathrm{ord}_{E_q}) \cdot \mathcal{O}_{Y_K}$ by (15). Therefore, the lower semicontinuity of the log canonical threshold and [BX18, Eq. (3)] imply

$$\mathrm{lct}\left(Y_\kappa, \Gamma_\kappa; \mathfrak{b}_{q,\bullet}\right) \leq \mathrm{lct}\left(Y_K, \Gamma_K; \mathfrak{a}_\bullet(\mathrm{ord}_{E_{K,q}})\right) \leq A_{Y_K,\Gamma_K}(E_{K,q}) \tag{16}$$

Additionally,

$$\mathrm{mult}(\mathfrak{b}_{q,\bullet}) = \lim_{p\to\infty} \frac{\dim_\kappa\left(\mathcal{O}_{Y_\kappa}/\mathfrak{b}_{q,p}\right)}{p^{n+1}/(n+1)!} = \lim_{p\to\infty} \frac{\dim_K\left(\mathcal{O}_{Y_K}/\mathfrak{a}_p(\mathrm{ord}_{E_{K,q}})\right)}{p^{n+1}/(n+1)!} = \mathrm{mult}(\mathfrak{a}_\bullet(\mathrm{ord}_{E_{K,q}})), \tag{17}$$

where the left and right equalities is the formula for multiplicity in [LM09, Thm 3.8] and the center equality follows from (15) and the fact that each $\mathcal{F}^p V_m \subset V_m$ is a subbundle.

We aim to show the inequalities:

$$\frac{Q}{r} \leq \mathrm{lct}(Y_\kappa, \Gamma_\kappa; \mathfrak{b}_{k,\bullet})^{n+1}\mathrm{mult}(\mathfrak{b}_{k,\bullet}) \leq A_{Y_K,\Gamma_K}(E_{K,q})^{n+1}\mathrm{mult}(\mathfrak{a}_\bullet(\mathrm{ord}_{E_{K,q}})) \leq \frac{Q}{r} + O\left(\frac{1}{q^2}\right), \tag{18}$$

with $Q := (-K_{X_K} - D_K)^n = (-K_{X_\kappa} - D_\kappa)^n$. The first inequality follows from [Liu18, Thm 7] and the assumption that $(X_{\overline{\kappa}}, D_{\overline{\kappa}})$ is K-semistable and the second from (16) and (17). For the remaining equality, Li's derivative formula (for example, see [BX18, Prop. 2.12]) gives

$$\left.\frac{d\,\widehat{\mathrm{vol}}(v_t)}{dt}\right|_{t=0^+} = (n+1)\beta_{X,D}(E).$$

Since the latter is zero, a Taylor expansion implies

$$\widehat{\mathrm{vol}}(v_{1/q}) = \mathrm{vol}(v_0) + O\left(\frac{1}{q^2}\right) = \frac{Q}{r} + O\left(\frac{1}{q^2}\right).$$

Using that $\widehat{\mathrm{vol}}$ is scaling invariant, we observe

$$\widehat{\mathrm{vol}}(v_{1/q}) = \widehat{\mathrm{vol}}(\mathrm{ord}_{E_{K,q}}) := A_{Y_K,\Gamma_K}(E_{K,q})^{n+1}\mathrm{mult}(\mathfrak{a}_\bullet(\mathrm{ord}_{E_{K,q}}))$$

and (18) is complete.

Comparing (17) and (18), we see

$$\frac{1}{1 + O\left(\frac{1}{q^2}\right)} \leq \left(\frac{\mathrm{lct}(Y_\kappa, \Gamma_\kappa; \mathfrak{b}_{q,\bullet})}{A_{Y_K,\Gamma_K}(E_{K,q})}\right)^{n+1} \leq 1.$$

Since $A_{Y_K,\Gamma_K}(E_{K,q}) = A_{Y,\Gamma}(E_q) = A_{Y,\Gamma+Y_\kappa}(E_q)$,

$$\mathrm{lct}(Y_\kappa, \Gamma_\kappa; \mathfrak{b}_{q,\bullet}) = \mathrm{lct}(Y, \Gamma + Y_\kappa; \mathfrak{a}_\bullet(\mathrm{ord}_{E_q}))$$

by inversion of adjunction, and $(1 + O(\frac{1}{q^2}))^{1/(n+1)} = 1 + O(\frac{1}{q^2})$, it follows that

$$1 - O\left(\frac{1}{q^2}\right) \leq \frac{\mathrm{lct}(Y, \Gamma + Y_\kappa; \mathfrak{a}_\bullet(\mathrm{ord}_{E_q}))}{A_{Y,\Gamma+Y_\kappa}(E_q)} \leq 1.$$

Recall, $A_{Y,\Gamma+Y_\kappa}(E_q) = A_{Y,\Gamma}(E_q) = q/r + A$ is of order $O(q)$. Therefore,

$$A_{Y,\Gamma+Y_\kappa}(E_q) - \mathrm{lct}(Y, \Gamma + Y_\kappa; \mathfrak{a}_\bullet(\mathrm{ord}_{E_q})) = A_{Y,\Gamma+Y_\kappa}(E_q)\left(1 - \frac{\mathrm{lct}(Y, \Gamma + Y_\kappa; \mathfrak{a}_\bullet(\mathrm{ord}_{E_q}))}{A_{Y,\Gamma+Y_\kappa}(E_q)}\right)$$

is of order $O(1/q)$ and the desired limit is 0.

Claim 2: For $q \gg 0$, there exists an extraction $E_q \subset Y_q \xrightarrow{\mu} Y$ such that

$$(Y_q, \mu_*^{-1}(\Gamma + Y_\kappa) + E_q)$$

is lc. (By an extraction, we mean μ is a proper birational morphism, Y_q is normal, E_q appears as a divisor on Y_q, and $-E_q$ is μ-ample.)

Set $\varepsilon_k := A_{Y,\Gamma+Y_\kappa}(E_q) - \mathrm{lct}(Y, \Gamma + Y_\kappa(\mathfrak{a}_\bullet(\mathrm{ord}_{E_q})))$. Since $\lim_{q\to\infty} \varepsilon_q = 0$ by Claim 1, we may fix $q \gg 0$ so that $\varepsilon_q < 1$. Hence, [BX18, Prop. 2.2] may be applied to get an extraction $\mu : Y_q \to Y$ of E_q with

$$(Y_q, \mu_*^{-1}(\Gamma + Y_\kappa) + (1 - \varepsilon_q)E_q)$$

lc. Since $\lim_{q\to\infty} \varepsilon_q = 0$, the ACC for log canonical thresholds [HMX14] implies $(Y_q, \mu_*^{-1}(\Gamma + Y_\kappa) + E_q)$ is lc for $q \gg 0$ and the claim is complete.

Since $-E_q$ is μ-ample, $\bigoplus_{p\in\mathbb{N}} \mu_*\mathcal{O}_{Y_q}(-pE_q)$ is a finitely generated $\mathcal{O}_Y$-algebra. Using that

$$\mu_*\mathcal{O}_{Y_k}(-pE_q) = \mathfrak{a}_p(\mathrm{ord}_{E_q}) = \bigoplus_m \mathcal{F}^{p-mq} V_m,$$

we see

$$\bigoplus_{p\in\mathbb{N}} \bigoplus_{m\in\mathbb{N}} \mathcal{F}^{p-mq} V_m = \bigoplus_{m\in\mathbb{N}} \bigoplus_{p\geq -mq} \mathcal{F}^p V_m$$

is a finitely generated V-algebra. Using that V is a finitely generated R-algebra, we conclude that $\bigoplus_m \bigoplus_p (\mathcal{F}^p V_m)x^{-p}$ is finitely generated $R[x]$-algebra and we may consider the degeneration $(\widetilde{\mathcal{X}}, \widetilde{\mathcal{D}}) \to \mathbb{A}^1_R$ by taking $\mathrm{Proj}_{\mathbb{A}^1_R}$.

We also consider the degeneration $(\mathcal{Y}, \widetilde{\Gamma})$ of (Y, Γ) defined by

$$\mathcal{Y} := \underline{\mathrm{Spec}}_{\mathbb{A}^1_R}\Big(\bigoplus_{p\in\mathbb{Z}} \mathfrak{a}_p x^{-p}\Big), \qquad \text{where } \mathfrak{a}_p := \mu_*\mathcal{O}_{Y_q}(-pE_q) \subseteq \mathcal{O}_Y$$

and $\widetilde{\Gamma}$ is the degeneration of Γ as in [LWX18, Defn. 2.19]. Since $(Y_q, \mu_*^{-1}(\Gamma + Y_\kappa) + E_q)$ is lc, a relative version of [LWX18, Lem. 2.20] implies $(\mathcal{Y}, \widetilde{\Gamma} + \mathcal{Y}_{x=0} + \mathcal{Y}_\kappa)$ is lc. Using that that $(\widetilde{\mathcal{X}}, \widetilde{\mathcal{D}})$ is a $\mathbb{G}_m$-quotient of an open set of $(\mathcal{Y}, \widetilde{\Gamma})$, we see $(\widetilde{\mathcal{X}}, \widetilde{\mathcal{D}} + \widetilde{\mathcal{X}}_\kappa + \widetilde{\mathcal{X}}_{x=0})$ is lc as well.

Observe that $(\widetilde{\mathcal{X}}_K, \widetilde{\mathcal{D}}_K) \to \mathbb{A}^1_K$ is the test configuration induced by the filtration $\mathcal{F}_K$. By [Fuj17, Section 3.2], this test configuration is normal and its Futaki invariant is a multiple of $\beta_{X_K,D_K}(E_K)$, which is zero. Since $\mathrm{Fut}(\widetilde{\mathcal{X}}_{\overline{K}}, \widetilde{\mathcal{D}}_{\overline{K}}) = \mathrm{Fut}(\widetilde{\mathcal{X}}_K, \widetilde{\mathcal{D}}_K)$ and the latter is zero. Now, $(\widetilde{\mathcal{X}}_{\overline{K}}, \widetilde{\mathcal{D}}_{\overline{K}}) \to \mathbb{A}^1_{\overline{K}}$ must be special, since otherwise [LX14, Thm. 1] would imply there exists a test configuration of $(X_{\overline{K}}, D_{\overline{K}})$ with negative Futaki invariant. Applying [LWX18, Lem. 3.1], gives that the geometric fiber over 0_K is K-semistable.

We proceed to show $(\widetilde{\mathcal{X}}, \widetilde{\mathcal{D}}) \to \mathbb{A}^1_R$ is family of log Fano pairs with geometrically K-semistable fibers. The statement holds away from the fiber over $0 \in \mathbb{A}^1_R$, which equals $\mathcal{X}_\kappa \cap \mathcal{X}_{x=0}$. Recall, $(\widetilde{\mathcal{X}}, \widetilde{\mathcal{D}} + \widetilde{\mathcal{X}}_\kappa + \widetilde{\mathcal{X}}_{x=0})$ is lc. Therefore, $K_{\widetilde{\mathcal{X}}} + \widetilde{\mathcal{D}}$ is $\mathbb{Q}$-Cartier and, by [Kol13, Prop. 2.31.2], $\widetilde{\mathcal{D}}$ does not contain an irreducible component of $\widetilde{\mathcal{X}}_\kappa \cap \widetilde{\mathcal{X}}_{x=0}$.

To complete the proof, it suffices to show that the geometric fiber of $(\widetilde{\mathcal{X}}_\kappa, \widetilde{\mathcal{D}}_\kappa) \to \mathbb{A}^1_\kappa$ over 0 is log Fano and K-semistable. For this, note that $(\widetilde{\mathcal{X}}_\kappa, \widetilde{\mathcal{D}}_\kappa + \widetilde{\mathcal{X}}_0)$ is log canonical by adjunction. Therefore, $(\widetilde{\mathcal{X}}_\kappa, \widetilde{\mathcal{D}}_\kappa) \to \mathbb{A}^1_\kappa$ is a weakly special test configuration [LWX18, Defn. 2.16]. Now, recall that the Futaki invariant may be written as a combination of intersection numbers of line bundles and intersections number are locally constant in flat projective families. Therefore, $\mathrm{Fut}(\widetilde{\mathcal{X}}_{\overline{K}}, \widetilde{\mathcal{D}}_{\overline{K}}) = \mathrm{Fut}(\widetilde{\mathcal{X}}_{\overline{\kappa}}, \widetilde{\mathcal{D}}_{\overline{\kappa}})$ and the latter is also zero. Since $(X_{\overline{\kappa}}, D_{\overline{\kappa}})$ is K-semistable and $\mathrm{Fut}(\widetilde{\mathcal{X}}_{\overline{\kappa}}, \widetilde{\mathcal{D}}_{\overline{\kappa}}) = 0$, the test configuration $(\widetilde{\mathcal{X}}_{\overline{\kappa}}, \widetilde{\mathcal{D}}_{\overline{\kappa}}) \to \mathbb{A}^1_{\overline{\kappa}}$ must be special (otherwise [LX14, Theorem 1] would imply there exists a test configuration with negative Futaki invariant). Applying [LWX18, Prop. 3.11] gives that the fiber over 0 is K-semistable. ☐

5.3.2. *Proof of* Θ*-reductivity result.* The proof of Theorem 5.2 is a consequence of Proposition 5.3.

Proof of Theorem 5.2. Following Section 5.2, the test configuration $(\mathcal{X}_K, \mathcal{D}_K) \to \mathbb{A}^1_K$, corresponds to a $\mathbb{Z}$-filtration $\mathcal{G}_K$ of V_K. By setting

$$\mathcal{G}^p V_m := \mathcal{G}^p_K V_{K,m} \cap V_m \qquad \text{for each } p \in \mathbb{Z}\ ,$$

we get a filtration $\mathcal{G}$ of V_m by subbundles, which restricts to the filtration $\mathcal{G}_K$ over $\mathrm{Spec}(K)$. We consider the graded $R[x]$-algebra $\bigoplus_{m\in\mathbb{N}} \bigoplus_{p\in\mathbb{Z}} (\mathcal{G}^p V_m)\, x^{-p}$.

Since the test configuration $(\mathcal{X}_K, \mathcal{D}_K)$ is special, the filtration $\mathcal{G}_K$ is induced by a divisor over X_K [Fuj19, Claim 5.4]. Specifically, there is a divisor E_K over X_K and $b \in \mathbb{Z}_{>0}$ so that

$$\mathcal{G}^p_K V_{K,m} = \mathcal{F}_K^{mrA+\lceil p/b\rceil} V_{K,m},$$

where $\mathcal{F}_K$ is the filtration of V_K defined by E_K and $A := A_{X_K,D_K}(E_K)$.

Observe that $\mathrm{Fut}(\mathcal{X}_K, \mathcal{D}_K) = 0$. Indeed, $\mathrm{Fut}(\mathcal{X}_K, \mathcal{D}_K) = \mathrm{Fut}(\mathcal{X}_{\overline{K}}, \mathcal{D}_{\overline{K}})$ and the latter is the same as the Futaki invariant associated to the $\mathbb{G}_m$-action on $(\mathcal{X}_{\overline{K}}, \mathcal{D}_{\overline{K}})_0$. Since the Futaki invariants associated to a $\mathbb{G}_m$-action and its inverse add to zero [LWX18, Lem. 2.23] and $(\mathcal{X}_{\overline{K}}, \mathcal{D}_{\overline{K}})_0$ is K-semistable, they must both be zero.

Now, $\beta_{X_K,D_K}(E_K)$ is a multiple of $\mathrm{Fut}(\mathcal{X}_K, \mathcal{D}_K)$ by [Fuj19, Thm. 5.1]. Therefore, the value is zero and we may apply Proposition 5.3.1 to see that $\bigoplus_m \bigoplus_p (\mathcal{G}^p V_m)\, x^{-p}$ is a finitely generated $R[x]$-module. Furthermore, if we set

$$\widetilde{\mathcal{X}} := \underline{\mathrm{Proj}}_{\mathbb{A}^1_R}\Big(\bigoplus_{m\in\mathbb{N}} \bigoplus_{p\in\mathbb{N}} (\mathcal{G}^p V_m)\, x^{-p}\Big)$$

and $\mathcal{D}$ equal to the closure of $D \times (\mathbb{A}^1 \setminus 0)$ under the embedding

$$X \times (\mathbb{A}^1 \setminus 0) \simeq \widetilde{\mathcal{X}}|_{x\neq 0} \hookrightarrow \widetilde{\mathcal{X}},$$

then $(\widetilde{\mathcal{X}}, \widetilde{\mathcal{D}}) \to \mathbb{A}^1_R$ is a finite base change of the family considered in Proposition 5.3. Hence, $(\widetilde{\mathcal{X}}, \widetilde{\mathcal{D}}) \to \mathbb{A}^1_R$ is a $\mathbb{Q}$-Gorenstein family of log Fano pairs and the geometric fiber over $0 \in \mathbb{A}^1_R$ is K-semistable. ☐

Proof of Theorem 1.1. The S-completeness and Θ-reductivity of $\mathfrak{X}^{\mathrm{Kss}}_{n,V}$ statements follow immediately from Theorem 3.3 and Theorem 5.2. ☐

Proof of Corollary 1.2. It follows fromTheorem 1.1 and Lemma 2.12 that $\mathfrak{X}$ is S-complete and Θ-reductive with respect to essentially of finite type DVRs. Theorem 2.9 and Remark 2.10 imply that $\mathfrak{X}$ has a separated good moduli space. ☐

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