FINITE HILBERT STABILITY OF (BI)CANONICAL CURVES

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To Joe Harris on his sixtieth birthday

ABSTRACT. We prove that a generic canonically or bicanonically embedded smooth curve has semistable m^{th} Hilbert points for all $m \geq 2$. We also prove that a generic bicanonically embedded smooth curve has stable m^{th} Hilbert points for all $m \geq 3$. In the canonical case, this is accomplished by proving finite Hilbert semistability of special singular curves with $\mathbb{G}_m$-action, namely the canonically embedded *balanced ribbon* and the canonically embedded *balanced double A_{2k+1}-curve*. In the bicanonical case, we prove finite Hilbert stability of special hyperelliptic curves, namely *Wiman curves*. Finally, we give examples of canonically embedded smooth curves whose m^{th} Hilbert points are non-semistable for low values of m, but become semistable past a definite threshold.

CONTENTS

1. Introduction

Geometric Invariant Theory (GIT) was developed by Mumford in order to construct quotients in algebraic geometry, and in particular to construct moduli spaces. To use GIT to construct a moduli space one must typically prove that a certain class of embedded varieties has stable or semistable Hilbert points. The prototypical example of a stability result is Gieseker and Mumford's asymptotic stability theorem for pluricanonically embedded curves [Mum77, Gie82, Gie83]:

Theorem 1.1 (Asymptotic Stability). *Suppose $C \subset \mathbb{P}\mathrm{H}^0\left(C, K_C^n\right)$ is a smooth curve embedded by the complete linear system $|K_C^n|$, where $n \geq 1$. Then the mth Hilbert point of C is stable for all $m \gg 0$.*

Gieseker and Mumford's arguments are non-effective, and there is no known bound on how large m must be in order to obtain the conclusion of the theorem. In light of this theorem, it is natural to ask: for which finite values of m do pluricanonically embedded smooth curves have stable or semistable Hilbert points? This has been a basic open problem in GIT since the pioneering work of Gieseker and Mumford, but has gained renewed interest from recent work of Hassett and Hyeon on the log minimal model program for $\overline{M}_g$. Indeed, Hassett and Hyeon observed that a stability result for finite Hilbert points of canonically and bicanonically embedded smooth curves would enable one to use GIT to construct a sequence of new projective birational models of M_g that would constitute steps of the log minimal model program for $\overline{M}_g$ [HH08]. In this paper, we prove the requisite stability result.

Theorem 1.2 (Main Result).

(1) *If C is a generic canonically or bicanonically embedded smooth curve, then the mth Hilbert point of C is semistable for every $m \geq 2$.*

(2) *If C is a generic bicanonically embedded smooth curve, then the mth Hilbert point of C is stable for every $m \geq 3$.*

Part (1) of the main result is proved in Corollaries 4.2 (odd genus canonical), 4.11 (even genus canonical), and Theorem 6.2 (bicanonical case). Part (2) of the main result is proved in Theorem 6.2. This is, to our knowledge, the first example of a result in which the (semi)stability of *all* Hilbert points of a given variety is established by a uniform method. In the case of canonically and bicanonically embedded curves, we recover a weak form of the asymptotic stability theorem by a much simpler proof. Furthermore, as a sidelight to our main result, we give an example of an embedded *smooth* curve whose m^{th} Hilbert point changes from semistable to non-semistable as m decreases (Theorem 5.2). We will explain our method of proof in the next section. First, however, let us conclude this introduction by describing a fascinating application of the main result, anticipated in the work of Hassett and Hyeon [HH08], and by considering prospects for future generalizations.

Fix $g \geq 2$, $n \geq 1$, $m \geq 2$, and set $r = (2n-1)(g-1) - 1$ if $n \geq 2$, and $r = g - 1$ if $n = 1$. To an n-canonically embedded smooth genus g curve C we associate its mth Hilbert point $[C]_m \in \mathbb{P}W_m$; these are defined in more detail in Section 2 below. We denote by $\overline{H}_{g,n}^m$ the closure in $\mathbb{P}W_m$ of the locus of mth Hilbert points of n-canonically embedded smooth curves of genus g. Then the $\mathrm{SL}(r+1)$-action on $\overline{H}_{g,n}^m$ admits a natural linearization $\mathcal{O}(1)$, which defines an open locus $(\overline{H}_{g,n}^m)^{ss} \subset \overline{H}_{g,n}^m$ of semistable points. *Assuming that $(\overline{H}_{g,n}^m)^{ss}$ is*

non-empty, one obtains a GIT quotient

$$(\overline{H}_{g,n}^m)^{ss} /\!\!/ \operatorname{SL}(r+1) := \operatorname{Proj} \bigoplus_{k \geq 0} \operatorname{H}^0 \left(\overline{H}_{g,n}^m, \mathcal{O}(k) \right)^{\operatorname{SL}(r+1)}$$

as a projective variety associated to the algebra of $\operatorname{SL}(r+1)$-invariant functions in the homogenous coordinate ring of $\overline{H}_{g,n}^m$.

When $m \gg 0$, the critical assumption $\left(\overline{H}_{g,n}^m \right)^{ss} \neq \varnothing$ is satisfied by Theorem 1.1, and the corresponding quotients have been analyzed using GIT [Gie82, Gie83, Sch91, HH09, HH08, HL10, HM10]. The results of this analysis can be summarized as follows:

$$\left(\overline{H}_{g,n}^m \right)^{ss} /\!\!/ \operatorname{SL}(r+1) \simeq \begin{cases} \overline{M}_g & \text{if } n \geq 5, m \gg 0, \\ \overline{M}_g^{ps} & \text{if } n = 3, 4, m \gg 0, \\ \overline{M}_g^{hs} & \text{if } n = 2, m \gg 0. \end{cases}$$

Here, $\overline{M}_g^{ps}$ is the moduli space of pseudostable curves, in which elliptic tails have been replaced by cusps, and $\overline{M}_g^{hs}$ is the moduli space of h-semistable curves, in which elliptic bridges have been replaced by tacnodes. Furthermore, the birational transformations $\overline{M}_g \to \overline{M}_g^{ps} \dashrightarrow \overline{M}_g^{hs}$ constitute the first two steps of the log minimal model program, namely the first divisorial contraction and the first flip [HH09, HH08].

The key point is that the next stage of the log minimal model program cannot be constructed using an asymptotic stability result. Indeed, an examination of the formula for the divisor class of the polarization on the GIT quotient $\left(\overline{H}_{g,n}^m \right)^{ss} /\!\!/ \operatorname{SL}(r+1)$ suggests that the next model occurring in the log minimal model program should be $\left(\overline{H}_{g,2}^6 \right)^{ss} /\!\!/ \operatorname{SL}(3g-3)$. Thus, in marked contrast to the cases $n \geq 3$, where finite Hilbert linearizations are not expected to yield new birational models of $\overline{M}_g$, it is widely anticipated that in the cases $n = 1, 2$, there will exist several values of m at which the corresponding GIT quotients undergo nontrivial birational modifications caused by the fact that curves with worse than nodal singularities become semistable for low values of m. For $n = 1$ we expect the number of threshold values of m at which $\left(\overline{H}_{g,n}^m \right)^{ss}$ changes to grow with g, while for $n = 2$ the only interesting values are $m \leq 6$, irrespectively of g; for a detailed analysis of the expected threshold values of m see [FS10] and [AFS10]. Until now, the main obstacle to verifying these expectations has been proving $\left(\overline{H}_{g,n}^m \right)^{ss} \neq \varnothing$ for explicit, finite values of m and arbitrary genus g. Theorem 1.2 removes this obstacle, and thus opens the door to analyzing a whole menagerie of new GIT quotients $\left(\overline{H}_{g,n}^m \right)^{ss} /\!\!/ \operatorname{SL}(r+1)$.

Finally, let us discuss a slight sharpening of our main result which follows naturally from the methods employed in this paper. We observe that the canonically embedded curve of even genus for which we establish finite Hilbert semistability in Section 4.2 is in fact trigonal, i.e. it lies in the closure of the locus of canonically embedded smooth trigonal curves. Similarly, in Section 4.3, we prove the finite Hilbert semistability of the bicanonically embedded curve of odd genus, which is easily seen to be in the closure of the locus of bicanonically embedded smooth bielliptic curves. From these observations, we obtain the following result:

Theorem 1.3 (Stability of trigonal and bielliptic curves)**.**

(1) *Suppose $C \subset \mathbb{P} \operatorname{H}^0 \left(C, K_C \right)$ is a generic canonically embedded smooth trigonal curve of even genus. Then the mth Hilbert point of C is semistable for every $m \geq 2$.*

(2) *Suppose $C \subset \mathbb{P}\mathrm{H}^0\big(C, K_C^2\big)$ is a generic bicanonically embedded smooth bielliptic curve of odd genus. Then the mth Hilbert point of C is semistable for every $m \geq 2$.*

This result naturally raises the questions: Is it true that *all* canonically embedded smooth trigonal curves have semistable m^{th} Hilbert points for $m \geq 2$? Similarly, do other curves with low Clifford index, such as canonical bielliptic curves, have this property? Surprisingly, the answer to both questions is no. In Section 5 of this paper, we prove that the mth Hilbert point of a canonically embedded smooth bielliptic curve is non-semistable below a certain definite threshold value of m (depending on g), while the mth Hilbert point of a generic canonically embedded bielliptic curve of odd genus is semistable for large values of m. As for trigonal curves, it is not difficult to see that the 2nd Hilbert point of a canonically embedded trigonal curve with positive Maroni invariant is non-semistable; see [FJ11, Corollary 3.2]. On the other hand, in Section 5 we give heuristic reasons for believing that a canonically embedded smooth trigonal curve should have semistable m^{th} Hilbert points for $m \geq 3$.

Notation and conventions. We work over the field of complex numbers $\mathbb{C}$. In particular, we denote $\mathbb{G}_m := \operatorname{Spec} \mathbb{C}[t, t^{-1}]$. In Section 6, we use the term *multiset* to denote a collection of elements with possibly repeating elements.

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2. GIT BACKGROUND

The proof of our main result is surprisingly simple. In the canonical (resp., bicanonical) case, we exhibit a curve C such that the action of $\operatorname{Aut}(C)$ on $V = \mathrm{H}^0\big(C, \omega_C\big)$ (resp., $V = \mathrm{H}^0\big(C, \omega_C^2\big)$) is multiplicity-free, i.e. no representation occurs more than once in the decomposition of V into irreducible $\operatorname{Aut}(C)$-representations. As Ian Morrison observed some thirty years ago, under this hypothesis, powerful results of Kempf imply that the mth Hilbert point of C is semistable if and only if it is semistable with respect to one-parameter subgroups of $\operatorname{SL}(V)$ which act diagonally on a *fixed basis* of V. Verifying stability with respect to the resulting fixed torus of $\operatorname{SL}(V)$ is a discrete combinatorial problem which we solve explicitly for every $m \geq 2$. We thus prove the semistability of all Hilbert points of C and deduce the semistability of a generic smooth curve by openness of the semistable locus. In Section 3 we will give a precise description of the (rather exotic) curves C appearing in our argument. In this section, we recall the relevant definitions from GIT and explain the general framework for proving semistability of Hilbert points due to Mumford, as well as the aforementioned refinements of Kempf.

Let us begin by recalling the definition of the mth Hilbert point of an embedded scheme. If $X \subset \mathbb{P}V$ is a closed subscheme such that the restriction map $\mathrm{H}^0\big(\mathbb{P}V, \mathcal{O}(m)\big) \to \mathrm{H}^0\big(X, \mathcal{O}_X(m)\big)$ is surjective (equivalently, $h^1\big(X, \mathcal{I}_X(m)\big) = 0$), set

$$W_m := \bigwedge^{h^0\big(X, \mathcal{O}_X(m)\big)} \mathrm{H}^0\big(\mathbb{P}V, \mathcal{O}(m)\big)^{\vee}.$$

The mth *Hilbert point of* $X \subset \mathbb{P}V$ is a point $[X]_m \in \mathbb{P}W_m$, defined as follows. First, consider the surjection

$$\mathrm{H}^0\big(\mathbb{P}V, \mathcal{O}(m)\big) \to \mathrm{H}^0\big(X, \mathcal{O}_X(m)\big) \to 0.$$

Taking the $h^0\big(X, \mathcal{O}_X(m)\big)$-fold wedge product and dualizing, we obtain the mth Hilbert point:

$$[X]_m := \left[\bigwedge^{h^0\big(X, \mathcal{O}_X(m)\big)} \mathrm{H}^0\big(\mathbb{P}V, \mathcal{O}(m)\big) \to \bigwedge^{h^0\big(X, \mathcal{O}_X(m)\big)} \mathrm{H}^0\big(X, \mathcal{O}_X(m)\big) \to 0 \right]^{\vee} \in \mathbb{P}(W_m).$$

Recall that if W is any linear representation of $\mathrm{SL}(V)$, a point $x \in \mathbb{P}(W)$ is *semistable* if the origin of W is not contained in the closure of the orbit of $\tilde{x} \in W$, where $\tilde{x}$ is any lift of x. Thus, to show that a Hilbert point $[X]_m \in \mathbb{P}(W_m)$ is semistable, we must prove that $0 \in W_m$ is not in the closure of $\mathrm{SL}(V) \cdot \widetilde{[X]}_m$, where $\widetilde{[X]}_m$ is any lift of $[X]_m$. An obvious necessary condition is that for any *one-parameter subgroup* $\rho \colon \mathrm{Spec}\,\mathbb{C}[t, t^{-1}] \to \mathrm{SL}(V)$, we have $\lim_{t \to 0} \rho(t) \cdot \widetilde{[X]}_m \neq 0$. A foundational theorem of Mumford asserts that this necessary condition is sufficient.

Proposition 2.1 (Hilbert-Mumford Numerical Criterion)**.** *Let* $X \subset \mathbb{P}V$ *be as above. The Hilbert point* $[X]_m$ *is semistable if and only if* $\lim_{t \to 0} \rho(t) \cdot \widetilde{[X]}_m \neq 0$ *for every one-parameter subgroup* $\rho \colon \mathrm{Spec}\,\mathbb{C}[t, t^{-1}] \to \mathrm{SL}(V)$.

Given a one-parameter subgroup $\rho \colon \mathrm{Spec}\,\mathbb{C}[t, t^{-1}] \to \mathrm{SL}(V)$, we may reformulate the condition $\lim_{t \to 0} \rho(t) \cdot \widetilde{[X]}_m \neq 0$ as follows. First, we may choose a basis $\{x_i\}_{i=0}^r$ of V which diagonalizes the action of ρ. Then $\rho(t) \cdot x_i = t^{\rho_i} x_i$ for some integers ρ_i satisfying $\sum_{i=0}^r \rho_i = 0$. We call $\{x_i\}_{i=0}^r$ a *ρ-weighted basis*. If we set $N_m := h^0\big(X, \mathcal{O}_X(m)\big)$, a basis for $W_m = \bigwedge^{N_m} \mathrm{H}^0\big(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(m)\big)$ diagonalizing the ρ-action consists of N_m-tuples $e_1 \wedge \ldots \wedge e_{N_m}$ of distinct monomials of degree m in the variables x_i's. If $e_\ell = \prod_{i=0}^r x_i^{a_{\ell i}}$, then ρ acts on $e_1 \wedge \ldots \wedge e_{N_m}$ with weight $\sum_{\ell=1}^{N_m} \sum_{i=0}^r a_{\ell i} \rho_i$. Now the condition that $\lim_{t \to 0} \rho(t) \cdot \widetilde{[X]}_m \neq 0$ is equivalent to the existence of one such coordinate which is non-vanishing on $[X]_m$ and on which ρ acts with non-positive weight. The condition that a coordinate $e_1 \wedge \ldots \wedge e_{N_m}$ is non-zero on $[X]_m$ is precisely the condition that the restrictions of $\{e_\ell\}_{\ell=1}^{N_m}$ to X form a basis of $\mathrm{H}^0\big(X, \mathcal{O}_X(m)\big)$. This discussion leads us to the following definition.

Definition 2.2. If $\{x_i\}_{i=0}^r$ is a ρ-weighted basis of V, a *monomial basis* of $\mathrm{H}^0\big(X, \mathcal{O}_X(m)\big)$ is a set $\mathcal{B} = \{e_\ell\}_{\ell=1}^{N_m}$ of degree m monomials in the variables $\{x_i\}_{i=0}^r$ such that $\mathcal{B}$ maps onto a basis of $\mathrm{H}^0\big(X, \mathcal{O}_X(m)\big)$ via the restriction map $\mathrm{H}^0\big(\mathbb{P}V, \mathcal{O}(m)\big) \to \mathrm{H}^0\big(X, \mathcal{O}_X(m)\big)$.

Moreover, if $e_\ell = \prod_{i=0}^r x_i^{a_{\ell i}}$, we define the *$\rho$-weight* of $\mathcal{B}$ to be $w_\rho(\mathcal{B}) := \sum_{\ell=1}^{N_m} \sum_{i=0}^r a_{\ell i} \rho_i$.

With this terminology, we have the following criterion.

Proposition 2.3 (Numerical Criterion for Hilbert points)**.** $[X]_m$ *is semistable (resp., stable) if and only if for every ρ-weighted basis of V, there exists a monomial basis of* $\mathrm{H}^0\big(X, \mathcal{O}_X(m)\big)$ *of non-positive (resp., negative) ρ-weight.*

The Hilbert-Mumford criterion reduces the problem of proving semistability of $[X]_m$ to a concrete algebro-combinatorial problem concerning the defining equations of $X \subset \mathbb{P}V$. However, this problem is not discretely computable since it requires checking *all* one-parameter subgroups of $\mathrm{SL}(V)$. A theorem of Kempf allows us, under certain hypotheses on $\mathrm{Aut}(X)$,

to check only those one-parameter subgroups of $\mathrm{SL}(V)$ which act diagonally on a fixed basis. This reduces the problem to one which is discretely computable.

In order to state the next proposition, let us establish a bit more terminology. Given an embedding $X \subset \mathbb{P}V$ by a complete linear system, there is a natural action of $\mathrm{Aut}(X)$ on $V = \mathrm{H}^0\big(X, \mathcal{O}_X(1)\big)$. Given a linearly reductive subgroup $G \subset \mathrm{Aut}(X)$, we say that V is a *multiplicity-free* G-representation (or simply *multiplicity-free* if G is understood) if it contains no irreducible G-representation more than once in its decomposition into irreducible G-representations. We say that a basis of V, say $\{x_i\}_{i=0}^r$, is *compatible with the irreducible decomposition of* V if each irreducible G-representation in V is spanned by a subset of the x_i's. We may now state the reformulation of Kempf's results that we will use. We keep the assumption that X is embedded by a complete linear system $|\mathcal{O}_X(1)|$ and that the restriction map $\mathrm{H}^0\big(\mathbb{P}V, \mathcal{O}(m)\big) \to \mathrm{H}^0\big(X, \mathcal{O}_X(m)\big)$ is surjective.

Proposition 2.4 (Kempf-Morrison Criterion). *Suppose $G \subset \mathrm{Aut}(X)$ is a linearly reductive subgroup, and that $V = \mathrm{H}^0\big(X, \mathcal{O}_X(1)\big)$ is a multiplicity-free representation of G. Let $\{x_i\}_{i=0}^r$ be a basis of V which is compatible with the irreducible decomposition of V. Then $[X]_m$ is semistable (resp., stable) if and only if for every one-parameter subgroup $\rho \colon \mathrm{Spec}\,\mathbb{C}[t, t^{-1}] \to \mathrm{SL}(V)$ acting diagonally on $\{x_i\}_{i=0}^r$, we have $\lim_{t \to 0} \rho(t) \cdot \widetilde{[X]}_m \neq 0$ (resp., $\lim_{t \to 0} \rho(t) \cdot \widetilde{[X]}_m$ does not exist). Equivalently, for every weighted basis $\{x_i\}_{i=0}^r$ of V, there exists a monomial basis of $\mathrm{H}^0\big(X, \mathcal{O}_X(m)\big)$ of non-positive (resp., negative) weight.*

Proof. If $[X]_m$ is not semistable, then [Kem78, Theorem 3.4 and Corollary 3.5] implies that there is a one-parameter subgroup $\rho_* \colon \mathrm{Spec}\,\mathbb{C}[t, t^{-1}] \to \mathrm{SL}(V)$ with $\lim_{t \to 0} \rho_*(t) \cdot \widetilde{[X]}_m = 0$ such that the parabolic subgroup $P \subseteq \mathrm{SL}(V)$ associated to the ρ_*-weight filtration

$$0 = U_0 \subseteq U_1 \subseteq \cdots \subseteq U_{k-1} \subseteq U_k = V$$

contains $\mathrm{Aut}(X)$. Let $V = \bigoplus_j V_j$ be the decomposition into irreducible G-representations. Since V is multiplicity-free, each U_i can be written as a direct sum of some of the V_j's. The maximal torus $T \subset \mathrm{SL}(V)$ associated to the basis $\{x_i\}_{i=0}^r$ fixes each V_j and thus the filtration. Therefore, $T \subset P$. By [Kem78, Theorem 3.4 (c)(4)], there exists a one-parameter subgroup $\rho \colon \mathrm{Spec}\,\mathbb{C}[t, t^{-1}] \to T$ such that $\lim_{t \to 0} \rho(t) \cdot \widetilde{[X]}_m = 0$. The statement for semistability follows.

The statement for *stability* follows by the same argument by replacing the concept of semistability (0-stability in Kempf's terminology) by a more general concept of S-stability; see [Kem78]. We are grateful to Ian Morrison for pointing this out. $\qquad\square$

For the sake of concreteness, let us reiterate the Kempf-Morrison criterion in the case of a canonically (resp., bicanonically) embedded curve $C \subset \mathbb{P}^r$. In order to prove that $[C]_m$ is semistable, we must first check that $V = \mathrm{H}^0\big(C, K_C\big)$ (resp., $V = \mathrm{H}^0\big(C, K_C^2\big)$) is a multiplicity-free representation of some linearly reductive $G \subset \mathrm{Aut}(C)$. Second, we fix a basis $\{x_i\}_{i=0}^r$ of V compatible with the irreducible decomposition of V. Now any one-parameter subgroup ρ acting diagonally on $\{x_i\}_{i=0}^r$ is given by an integer weight vector $(\rho_0, \ldots, \rho_r)$ satisfying $\sum_{i=0}^r \rho_i = 0$. To show that $[C]_m$ is semistable with respect to ρ, we must find a monomial basis $\mathcal{B}$ of $\mathrm{H}^0\big(C, \mathcal{O}_C(m)\big)$ such that $w_\rho(\mathcal{B}) \leq 0$. Note that for a fixed monomial basis $\mathcal{B}$, the ρ-weight function $w_\rho(\mathcal{B})$ is linear in $(\rho_0, \ldots, \rho_r)$. Therefore, each monomial basis determines a half-space of weight vectors for which $[C]_m$ is ρ-semistable, namely the half-space $w_\rho(\mathcal{B}) \leq 0$. It follows that as soon as one produces sufficiently many monomial bases such that the union

of these half-spaces contains all weight vectors $(\rho_0, \ldots, \rho_r)$ satisfying $\sum_{i=0}^{r} \rho_i = 0$, the proof of semistability for $[C]_m$ is completed. We summarize this discussion in the following lemma:

Lemma 2.5. *Let $G \subset \mathrm{Aut}(C)$ be a linearly reductive subgroup such that $V = \mathrm{H}^0\big(C, \mathcal{O}_C(1)\big)$ is a multiplicity-free representation of G, and let $\{x_i\}_{i=0}^r$ be a basis of V which is compatible with the irreducible decomposition of V. Suppose there exists a finite set $\{\mathcal{B}_j\}_{j \in J}$ of monomial bases of $\mathrm{H}^0\big(C, \mathcal{O}_C(m)\big)$ and $\{c_j\}_{j \in J} \subset \mathbb{Q} \cap (0, \infty)$ such that*

$$\sum_{j \in J} c_j w_\rho(\mathcal{B}_j) = 0$$

for every $\rho \colon \mathbb{G}_m \to \mathrm{SL}(V)$ acting on $\{x_i\}_{i=0}^r$ diagonally. Then $[C]_m$ is semistable.

The idea of applying these results of Kempf to the semistability of finite Hilbert points of curves is due to Morrison and Swinarski [MS11]. In their paper, they consider the so-called hyperelliptic *Wiman curve* C with its bicanonical embedding. They check that the automorphism group, which is cyclic of order $4g+2$, acts on $\mathrm{H}^0\big(C, K_C^2\big)$ with $3g-3$ distinct characters. They fix a basis $\mathrm{H}^0\big(C, K_C^2\big) = \{x_0, \ldots, x_r\}$ compatible with the decomposition of $\mathrm{H}^0\big(C, K_C^2\big)$ into characters, and then, for low values of g and m, use a computer to enumerate monomial bases of $\mathrm{H}^0\big(C, \mathcal{O}_C(m)\big)$ until the associated half-spaces cover the hyperplane $\sum_{i=0}^{r} \rho_i = 0$.

In this paper, we apply the Kempf-Morrison criterion to canonically embedded ribbons of odd genus (Section 4.1), canonically embedded balanced double A_{2k+1}-curves of even genus (Section 4.2), bicanonically embedded rosaries of odd genus (Section 4.3), and bicanonically embedded Wiman curves (Section 6.3). For each $m \geq 2$, we write down *by hand* sufficiently many monomial bases to establish the requisite (semi)stability result.

3. Curves with $\mathbb{G}_m$-action: Ribbons, A_{2k+1}-curves, and rosaries

As discussed in the previous section, the key to our proof is to find a singular Gorenstein curve C such that $\mathrm{H}^0\big(C, \omega_C\big)$ (resp., $\mathrm{H}^0\big(C, \omega_C^2\big)$) is a multiplicity-free representation of $\mathrm{Aut}(C)$ in the canonical case (resp., bicanonical case). In this section, we describe the curves we will use. In the odd genus canonical case, we will use a certain ribbon with $\mathbb{G}_m$-action, the so-called *balanced ribbon*. In the even genus canonical case, we will use the *balanced double A_{2k+1}-curve*, i.e. a curve comprised of three $\mathbb{P}^1$'s meeting in two higher tacnodes with trivial crimping. In the bicanonical case, we will use the so-called *rosary*, i.e. a cycle of $\mathbb{P}^1$'s attached by tacnodes, introduced by Hassett and Hyeon in their classification of asymptotically stable bicanonical curves [HH08].

A word of motivation as to where on earth these curves come from may be useful. That some class of canonically embedded ribbons should be GIT-semistable is intuitively plausible, since ribbons arise as flat limits of families of canonically embedded smooth curves degenerating abstractly to a hyperelliptic curve. The fact that the balanced ribbon of odd genus is the only ribbon with $\mathbb{G}_m$-action that has the potential to be Hilbert semistable was proved in [AFS10, Theorem 7.2]. Hence, it was natural to attempt to prove that this curve is, in fact, semistable. Our motivation for considering double A_{2k+1}-curves comes from the log minimal model program for $\overline{M}_{2k}$, where we expect the $2k-4$ dimensional locus of double A_{2k+1}-curves to replace the locus in the boundary divisor $\Delta_k \subset \overline{M}_{2k}$ consisting of nodal curves $C_1 \cup C_2$ such that each C_i is a hyperelliptic curve of genus k. Indeed, this prediction has already been verified in $g = 4$ by the second author who showed that the divisor $\Delta_2 \subset \overline{M}_4$ is contracted to the point corresponding to the unique genus 4 double A_5-curve in the final non-trivial log

canonical model of $\overline{M}_4$ [Fed12]. In the bicanonical case, we made use of the classification of asymptotically semistable curves in [HH08]. We simply looked through the curves on their list for one with a large enough symmetry group to satisfy the hypotheses of Proposition 2.4. The rosary was the first curve we checked, and it worked!

3.1. **Canonical case, odd genus: The balanced ribbon with $\mathbb{G}_m$-action.** In this section we will construct, for every odd $g \geq 3$, a special non-reduced curve C of arithmetic genus g whose canonical embedding satisfies the hypotheses of Proposition 2.4. Given a positive odd integer $g = 2k + 1$, where $k \geq 1$, set $U := \operatorname{Spec} \mathbb{C}[u, \epsilon]/(\epsilon^2)$, $V := \operatorname{Spec} \mathbb{C}[v, \eta]/(\eta^2)$, and identify $U - \{0\}$ and $V - \{0\}$ via the isomorphism

$$u \mapsto v^{-1} - v^{-k-2}\eta,$$
$$\epsilon \mapsto v^{-g-1}\eta.$$

The resulting scheme C is evidently a complete, locally planar curve of arithmetic genus g; see [BE95, Section 3] for more details on such curves. Note that C admits $\mathbb{G}_m$-action by the formulae

$$t \cdot u = tu,$$
$$t \cdot v = t^{-1}v,$$
$$t \cdot \epsilon = t^{k+1}\epsilon,$$
$$t \cdot \eta = t^{-k-1}\eta.$$

Since C is locally planar, it is Gorenstein and its dualizing sheaf ω_C is a line bundle. Using adjunction, we may identify global sections of ω_C with regular functions $f(u, \epsilon)$ on U. To be precise, the global sections of ω_C consist of all differentials

$$f(u, \epsilon)\frac{du \wedge d\epsilon}{\epsilon^2}$$

which transform to differentials $h(v, \eta)\frac{dv \wedge d\eta}{\eta^2}$ with $h(v, \eta)$ regular on V. One easily writes down a basis of g functions satisfying this condition to obtain the following lemma, which is a special case of a more general [BE95, Theorem 5.1].

Lemma 3.1. *A basis for* $\mathrm{H}^0\big(C, \omega_C\big)$ *is given by differentials* $f(u, \epsilon)\frac{du \wedge d\epsilon}{\epsilon^2}$ *where* $f(u, \epsilon)$ *runs over the following list of g functions:*

$$x_i := u^i, \quad 0 \leq i \leq k, \qquad y_{k+i} := u^{k+i} + iu^{i-1}\epsilon, \quad 1 \leq i \leq k.$$

Lemma 3.2. ω_C *is very ample.*

Proof. Using the basis of $\mathrm{H}^0\big(C, \omega_C\big)$ from Lemma 3.1, we see that $|\omega_C|$ separates points of $C_{\mathrm{red}} \simeq \mathbb{P}^1$ and defines a closed embedding when restricted to U and V. The claim follows. $\square$

Proposition 3.3. $\mathrm{H}^0\big(C, \omega_C\big)$ *is a multiplicity-free representation of* $\mathbb{G}_m \subset \operatorname{Aut}(C)$ *and* $\{x_0, \ldots, x_k, y_{k+1}, \ldots, y_{2k}\}$ *is compatible with its irreducible decomposition.*

Proof. The basis $\{x_0, \ldots, x_k, y_{k+1}, \ldots, y_{2k}\}$ diagonalizes the action of $\mathbb{G}_m$ on $\mathrm{H}^0\big(C, \omega_C\big)$ with the $2k + 1$ distinct weights $-k, \ldots, -1, 0, 1, \ldots, k$. $\square$

In order to apply Proposition 2.4, we will need an effective way of determining when a set of monomials in the g variables $\{x_0, \ldots, x_k, y_{k+1}, \ldots, y_{2k}\}$ forms a monomial basis of $\mathrm{H}^0(C, \omega_C^m)$. To do this, observe that the global sections of ω_C^m are easily identified with regular functions on U via $f(u, \epsilon) \mapsto f(u, \epsilon) \dfrac{(du \wedge d\epsilon)^m}{\epsilon^{2m}}$. With this convention, we record the following observation used throughout the paper.

Lemma 3.4 (Ribbon Product Lemma). *The expansion in u and ϵ of the degree m monomial $x_{i_1} \ldots x_{i_\ell} y_{i_{\ell+1}} \ldots y_{i_m}$ is $u^a + (a - b)u^{a-k-1}\epsilon$, where*

$$a = i_1 + \cdots + i_m,$$
$$b = i_1 + \cdots + i_\ell + k(m - \ell).$$

The following proposition determines a basis for $\mathrm{H}^0(C, \omega_C^m)$ under the above identification.

Proposition 3.5. *For $m \geq 2$, the product map $\mathrm{Sym}^m \mathrm{H}^0(C, \omega_C) \to \mathrm{H}^0(C, \omega_C^m)$ is surjective. A basis for $\mathrm{H}^0(C, \omega_C^m)$ is given by differentials $f(u, \epsilon)\frac{(du \wedge d\epsilon)^m}{\epsilon^{2m}}$ where $f(u, \epsilon)$ runs over the following $(2m - 1)(g - 1)$ functions on U:*

$$\{u^i\}_{i=0}^{2mk-(k+1)}, \quad \{u^i + (i - k)u^{i-k-1}\epsilon\}_{i=k+1}^{2mk}.$$

Proof. We will show that the image of the product map $\mathrm{Sym}^m \mathrm{H}^0(C, \omega_C) \to \mathrm{H}^0(C, \omega_C^m)$ contains the given functions. Since $h^0(C, \omega_C^m) = (2m - 1)(g - 1)$ by Riemann-Roch, and because the given functions are linearly independent, this will prove the proposition.

Lemma 3.4 gives $u^a = x_0^{m-1}x_a$ for $0 \leq a \leq k$, $u^{2mk-k} + (2mk - 2k)u^{2mk-2k-1}\epsilon = y_{2k}^{m-1}x_k$, and $u^{(2m-1)k+a} + \big((2m-2)k+a\big)u^{(2m-2)k+a-1}\epsilon = y_{2k}^{m-1}y_a$ for $1 \leq a \leq k$. For the intermediate u-degrees, note simply that since the dimension of the space $\{cu^i + du^{i-k-1}\epsilon : c, d \in \mathbb{C}\}$ is two, we need to exhibit two linearly independent functions of this form as degree m monomials in $\{x_0, \ldots, y_{2k}\}$. Using Lemma 3.4, this is an easy exercise which we leave to the reader. $\square$

This result gives a very simple way of checking whether a set $\mathcal{B}$ of degree m monomials in $\{x_0, \ldots, y_{2k}\}$ projects to a basis for $\mathrm{H}^0(C, \mathcal{O}_C(m))$. If we simply view the monomials in $\mathcal{B}$ as polynomials in $\mathbb{C}[u, \epsilon]/(\epsilon^2)$ via the identification preceding Lemma 3.4, then $\mathcal{B}$ is a monomial basis of $\mathrm{H}^0(C, \mathcal{O}_C(m))$ if and only if

(1) $\mathcal{B}$ contains one polynomial of each u-degree $0, \ldots, k$,
(2) $\mathcal{B}$ contains two linearly independent polynomials of each u-degree $k + 1, \ldots, (2m - 1)k - 1$,
(3) $\mathcal{B}$ contains one polynomial of each u-degree $2mk - k, \ldots, 2mk$.

We can rephrase this as follows.

Lemma 3.6. *A set of degree m monomials*

$$\{x_{i_1} \cdots x_{i_\ell} y_{i_{\ell+1}} \cdots y_{i_m}\}_{(i_1, \ldots, i_m) \in S}$$

forms a monomial basis of $\mathrm{H}^0(C, \mathcal{O}_C(m))$ if and only if the following two conditions hold:

(1) *For $0 \leq a \leq k$ and $(2m - 1)k \leq a \leq 2mk$, there is exactly one index vector $(i_1, \ldots, i_m) \in S$ with $i_1 + \cdots + i_m = a$.*
(2) *For $k < a < (2m - 1)k$, there are exactly two index vectors $(i_1, \ldots, i_m) \in S$ satisfying $i_1 + \cdots + i_m = a$. Furthermore, for these two index vectors, the associated integers $i_{\ell+1} + \cdots + i_m - k(m - \ell)$ are distinct.*

Proof. Immediate from the preceding observations and the Ribbon Product Lemma 3.4. $\square$

3.2. **Canonical case, even genus: The balanced double A_{2k+1}-curve with $\mathbb{G}_m$-action.**
In this section we will construct special singular curves of even genus, whose canonical embeddings satisfy the hypotheses of Proposition 2.4. We define a *double A_{2k+1}-curve* to be any curve obtained by gluing three copies of $\mathbb{P}^1$ along two A_{2k+1} singularities (Figure 1). The arithmetic genus of a double A_{2k+1}-curve is $g = 2k$, and double A_{2k+1}-curves have $2k - 4$ moduli corresponding to the crimping of the A_{2k+1}-singularities, i.e. deformations that preserve the analytic types of the singularities as well as the normalization of the curve (see [vdW10] for a comprehensive treatment of crimping moduli). Indeed, the moduli space of crimping for an A_{2k+1}-singularity with automorphism-free branches has dimension k, but the presence of automorphisms of the pointed $\mathbb{P}^1$'s in our situation reduces the dimension of crimping moduli by 4. Among double A_{2k+1}-curves, there is a unique double A_{2k+1}-curve with $\mathbb{G}_m$-action, corresponding to the trivial choice of crimping for both A_{2k+1}-singularities. We call this curve the *balanced double A_{2k+1}-curve*.

Now let us give a more precise description of the balanced double A_{2k+1}-curve: Let C_0, C_1, C_2 denote three copies of $\mathbb{P}^1$, and label the uniformizers at 0 (resp., at ∞) by u_0, u_1, u_2 (resp., by v_0, v_1, v_2). Fix an integer $k \geq 2$, and let C be the arithmetic genus $g = 2k$ curve obtained by gluing three $\mathbb{P}^1$'s along two A_{2k+1} singularities with trivial crimping. More precisely, we impose an A_{2k+1} singularity at $(\infty \in C_0) \sim (0 \in C_1)$ by gluing $C_0 \setminus 0$ and $C_1 \setminus \infty$ into an affine singular curve

$$(3.1) \qquad \operatorname{Spec} \mathbb{C}[(v_0, u_1), (v_0^{k+1}, -u_1^{k+1})] \simeq \operatorname{Spec} \mathbb{C}[x, y]/(y^2 - x^{2k+2}).$$

Similarly, we impose an A_{2k+1} singularity at $(\infty \in C_1) \sim (0 \in C_2)$ by gluing $C_1 \setminus 0$ and $C_2 \setminus \infty$ into

$$(3.2) \qquad \operatorname{Spec} \mathbb{C}[(v_1, u_2), (v_1^{k+1}, -u_2^{k+1})] \simeq \operatorname{Spec} \mathbb{C}[x, y]/(y^2 - x^{2k+2}).$$
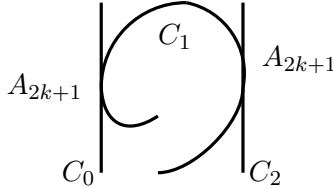

FIGURE 1. Double A_{2k+1}-curves

The automorphism group of C is given by $\operatorname{Aut}(C) = \mathbb{G}_m \rtimes \mathbb{Z}_2$ where $\mathbb{Z}_2$ acts via $u_i \leftrightarrow v_{2-i}$ and $\mathbb{G}_m = \operatorname{Spec} \mathbb{C}[t, t^{-1}]$ acts via

$$t \cdot u_0 = t u_0,$$
$$t \cdot u_1 = t^{-1} u_1,$$
$$t \cdot u_2 = t u_2.$$

Using the description of the dualizing sheaf on a singular curve as in [Ser88, Ch.IV] or [BHPVdV04, Ch.II.6], we can write down a basis of $\mathrm{H}^0(C, \omega_C)$ as follows:

$$(3.3) \qquad x_i = \left(u_0^i \frac{du_0}{u_0}, \, u_1^{-i} \frac{du_1}{u_1}, \, 0 \right), \qquad y_i = \left(0, \, u_1^i \frac{du_1}{u_1}, \, u_2^{-i} \frac{du_2}{u_2} \right), \qquad 1 \leq i \leq k.$$

It is straightforward to generalize this description to the spaces of pluricanonical differentials.

Lemma 3.7. *For $m \geq 2$, the product map $\mathrm{Sym}^m \mathrm{H}^0(C, \omega_C) \to \mathrm{H}^0(C, \omega_C^m)$ is surjective and a basis of $\mathrm{H}^0(C, \omega_C^m)$ consists of the following $(2m-1)(2k-1)$ differentials:*

$$\omega_j = \left(u_0^j \frac{(du_0)^m}{u_0^m}, \; u_1^{-j} \frac{(du_1)^m}{u_1^m}, \; 0 \right), \quad \eta_j = \left(0, \; u_1^j \frac{(du_1)^m}{u_1^m}, \; u_2^{-j} \frac{(du_2)^m}{u_2^m} \right), \quad m \leq j \leq mk.$$

and

$$\chi_\ell = \left(0, \; u_1^\ell \frac{(du_1)^m}{u_1^m}, \; 0 \right), \qquad -k(m-1)+1 \leq \ell \leq k(m-1)-1.$$

Proof. By Riemann-Roch formula, $h^0(C, \omega_C^m) = (2m-1)(2k-1)$. Thus, it suffices to observe that the given $(2m-1)(2k-1)$ differentials all lie in the image of the map $\mathrm{Sym}^m \mathrm{H}^0(C, \omega_C) \to \mathrm{H}^0(C, \omega_C^m)$. Using the basis of $\mathrm{H}^0(C, \omega_C)$ given by (3.3), one easily checks that the differentials $\{\omega_j\}_{j=m}^{mk}$ are precisely those arising as m-fold products of x_i's, the differentials $\{\eta_j\}_{j=m}^{mk}$ are those arising as m-fold products of y_i's, and the differentials $\{\chi_\ell\}_{\ell=-k(m-1)+1}^{k(m-1)+1}$ are those arising as mixed m-fold products of x_i's and y_i's. $\qquad \square$

Next, we show that $|\omega_C|$ is a very ample linear system, so that C admits a canonical embedding, and the corresponding Hilbert points are well defined.

Proposition 3.8. ω_C *is very ample. The complete linear system $|\omega_C|$ embeds C as a curve on a balanced rational normal scroll*

$$\mathbb{P}^1 \times \mathbb{P}^1 \xleftarrow{\quad |\mathcal{O}(1,k-1)| \quad} \mathbb{P}^{g-1}.$$

Moreover, C_0 and C_2 map to $(1,0)$-curves on $\mathbb{P}^1 \times \mathbb{P}^1$, and C_1 maps to a $(1, k+1)$ curve. In particular, C is a $(3, k+1)$ curve on $\mathbb{P}^1 \times \mathbb{P}^1$ and has a g_3^1 cut out by the $(0,1)$ ruling.

Proof. To see that the canonical embedding of C lies on a balanced rational normal scroll in $\mathbb{P}^{2k-1}$, recall that the scroll is the determinantal variety (see [Har92, Lecture 9]) defined by:

$$(3.4) \qquad \mathrm{rank} \begin{pmatrix} x_1 & x_2 & \cdots & x_{k-1} & y_k & y_{k-1} & \cdots & y_2 \\ x_2 & x_3 & \cdots & x_k & y_{k-1} & y_{k-2} & \cdots & y_1 \end{pmatrix} \leq 1.$$

From our explicit description of the basis of $\mathrm{H}^0(C, \omega_C)$ given by (3.3), one easily sees that the differentials x_i's and y_i's on C satisfy the determinantal condition (3.4). Moreover, we see that $|\omega_C|$ embeds C_0 and C_2 as degree $k-1$ rational normal curves in $\mathbb{P}^{2k-1}$ lying in the class $(1,0)$ on the scroll. Also, we see that $|\omega_C|$ embeds C_1 via the very ample linear system

$$\mathrm{span}\{1, u_1, \ldots, u_1^{k-1}, u_1^{k+1}, \ldots, u_1^{2k}\} \subset |\mathcal{O}_{\mathbb{P}^1}(2k)|$$

as a curve in the class $(1, k+1)$. It follows that $|\omega_C|$ separates points and tangent vectors on each component of C. We now prove that $|\omega_C|$ separates points of different components and tangent vectors at the A_{2k+1}-singularities. First, observe that C_0 and C_2 span disjoint subspaces. Therefore, being $(1,0)$ curves, they must be distinct and non-intersecting. Second, C_0 and C_1 are the images of the two branches of an A_{2k+1}-singularity and so have contact of order at least $k+1$. However, being $(1,0)$ and $(1, k+1)$ curves on the scroll, they have order of contact at most $k+1$. It follows that the images of C_0 and C_1 on the scroll meet precisely in an A_{2k+1}-singularity. We conclude that $|\omega_C|$ is a closed embedding at each A_{2k+1}-singularity.

We can also directly verify that $|\omega_C|$ separates tangent vectors at an A_{2k+1} singularity of C, say the one with uniformizers v_0 and u_1. The local generator of ω_C at this singularity is

$$x_k = \left(-\frac{dv_0}{v_0^{k+1}}, \frac{du_1}{u_1^{k+1}}, 0\right).$$

On the open affine chart $\operatorname{Spec}\mathbb{C}[(v_0, u_1), (v_0^{k+1}, -u_1^{k+1})]$ defined in Equation (3.1), we have $x_{k-1} = (v_0, u_1) \cdot x_k$ and $y_1 = (0, u_1^{k+1}) \cdot x_k$. Under the identification $\mathbb{C}[(v_0, u_1), (v_0^{k+1}, -u_1^{k+1})] \simeq \mathbb{C}[x, y]/(y^2 - x^{2k+2})$, we have $(v_0, u_1) = x$ and $(0, u_1^{k+1}) = (x^{k+1} - y)/2$. We conclude that x_{k-1} and y_1 span the cotangent space, and thus separate tangent vectors, at the singularity. $\square$

Finally, the following elementary observation is the key to analyzing the stability of Hilbert points of C.

Lemma 3.9. $\mathrm{H}^0(C, \omega_C)$ *is a multiplicity-free* $\operatorname{Aut}(C)$*-representation and the basis* $\{x_i, y_i\}_{i=1}^k$ *is compatible with its irreducible decomposition.*

Proof. Note that $\mathbb{G}_m \subset \operatorname{Aut}(C)$ acts on x_i with weight i and on y_i with weight $-i$. Thus $\mathrm{H}^0(C, \omega_C)$ decomposes into $g = 2k$ distinct characters of $\mathbb{G}_m$. $\square$

3.3. **Bicanonical case, odd genus: The rosary with $\mathbb{G}_m$-action.** In this section we will construct, in every odd genus, a singular curve C whose bicanonical embedding satisfies the hypotheses of Proposition 2.4. For any odd integer $g \geq 3$, we define C to be the curve, called a *rosary* in [HH08, Section 8.1], obtained from a set of $(g-1)$ $\mathbb{P}^1$'s indexed by $i \in \mathbb{Z}_{g-1}$ and having uniformizers u_i at 0 and v_i at ∞ (so that $u_i = 1/v_i$) by cyclically identifying v_i with u_{i+1} to specify $g-1$ tacnodes. Note that $\mathbb{G}_m \rtimes D_{g-1} \subset \operatorname{Aut}(C)$, where the dihedral group D_{g-1} permutes the components and $\mathbb{G}_m = \operatorname{Spec}\mathbb{C}[t, t^{-1}]$ acts by $u_i \mapsto t^{(-1)^i} u_i$. We should remark that in the case of even genus, one may still define the curve C, but C does not admit $\mathbb{G}_m$-action and does not satisfy the hypotheses of Proposition 2.4. Thus, in what follows, we always assume g odd.

Lemma 3.10. *(a) A basis for* $\mathrm{H}^0(C, \omega_C)$ *is given by the following differentials:*

$$\omega_i = \left(\ldots, 0, du_i, \frac{du_{i+1}}{u_{i+1}^2}, 0, \ldots\right), \quad i \in \mathbb{Z}_{g-1},$$

$$\eta = \left(\frac{du_0}{u_0}, \frac{du_1}{u_1}, \ldots, \frac{du_{g-2}}{u_{g-2}}\right).$$

(b) A basis for $\mathrm{H}^0(C, \omega_C^2)$ *is given by the following differentials:*

$$\begin{aligned}
x_i &= \omega_i^2, & i \in \mathbb{Z}_{g-1}, \\
y_i &= \omega_i \eta, & i \in \mathbb{Z}_{g-1}, \\
z_i &= \omega_{i-1} \omega_i, & i \in \mathbb{Z}_{g-1}.
\end{aligned}$$

Proof. Using duality on singular curves as in [Ser88, Ch.IV] or [BHPVdV04, Ch.II.6], it is straightforward to verify that each differential from (a) is a Rosenlicht differential and hence is an element of $\mathrm{H}^0(C, \omega_C)$. Since these g differentials are linearly independent, Part (a) is established. Part (b) follows immediately: The $(3g-3)$ differentials from (b) are products of elements in $\mathrm{H}^0(C, \omega_C)$ and are easily seen to be linearly independent. $\square$

Lemma 3.11. ω_C *is very ample for odd* $g \geq 5$ *and* ω_C^2 *is very ample for odd* $g \geq 3$.

Proof. We prove that C is canonically embedded for $g \geq 5$. First, observe that $|\omega_C|$ embeds each $\mathbb{P}^1$ as a conic in $\mathbb{P}^{g-1}$, and that the plane spanned by the ith conic meets only the planes spanned by the cyclically adjacent conics, and meets each of these only at the corresponding tacnode. This shows that $|\omega_C|$ separates points and tangent vectors at smooth points. To see that $|\omega_C|$ separates tangent vectors at the tacnode obtained by the identification $v_i = u_{i+1}$, note that the local generator of ω_C at this tacnode is ω_i. Locally around the tacnode, we have $\eta = (v_i, u_{i+1}) \cdot \omega_i$ and $\omega_{i+1} = (0, u_{i+1}^2) \cdot \omega_i$. Under the identification $\mathbb{C}[(v_i, u_{i+1}), (0, u_{i+1}^2)] \simeq \mathbb{C}[x, y]/\big(y(x^2 - y)\big)$, we have $(v_i, u_{i+1}) = x$ and $(0, u_{i+1}^2) = y$. We conclude that η and ω_{i+1} span the cotangent space, and thus separate tangent vectors, at the tacnode.

A straightforward computation shows that ω_C^2 is also very ample for $g = 3$. We finish by noting that C is hyperelliptic in genus 3 and thus is not canonically embedded. $\square$

The $\mathbb{G}_m$-action on $\mathrm{H}^0\big(C, \omega_C\big)$ is given by

$$t \cdot \omega_i = t^{(-1)^i} \omega_i,$$
$$t \cdot \eta = \eta.$$

The $\mathbb{G}_m$-action on $\mathrm{H}^0\big(C, \omega_C^2\big)$ is given by $x_i \mapsto (t^2)^{(-1)^i} x_i$, $y_i \mapsto t^{(-1)^i} y_i$, $z_i \mapsto z_i$. We define the weight of a monomial to be its $\mathbb{G}_m$-weight.

Proposition 3.12. *Both $\mathrm{H}^0\big(C, \omega_C\big)$ and $\mathrm{H}^0\big(C, \omega_C^2\big)$ are multiplicity-free representations of $\mathbb{G}_m \rtimes \mathbb{Z}_{g-1} \subset \mathrm{Aut}(C)$. Moreover, the basis $\{\omega_0, \ldots, \omega_{g-2}, \eta\}$ is compatible with the irreducible decomposition of $\mathrm{H}^0\big(C, \omega_C\big)$, and the basis $\{x_i, y_i, z_i \ : \ i \in \mathbb{Z}_{g-1}\}$ is compatible with the irreducible decomposition of $\mathrm{H}^0\big(C, \omega_C^2\big)$.*

Proof. The action of $\mathbb{Z}_{g-1} \subset D_{g-1}$ on the span of $\{\omega_i\}_{i=0}^{g-2}$ (resp., $\{x_i\}_{i=0}^{g-2}$, $\{y_i\}_{i=0}^{g-2}$, $\{z_i\}_{i=0}^{g-2}$) corresponds to the regular representation of $\mathbb{Z}_{g-1}$ and is thus multiplicity-free. Since the weight of ω_i is ± 1 and of η is 0 (resp., the weight of x_i is ± 2, of y_i is ± 1, and of z_i is 0), it follows that $\mathrm{H}^0\big(C, \omega_C\big)$ (resp., $\mathrm{H}^0\big(C, \omega_C^2\big)$) is a multiplicity-free representation of $\mathbb{G}_m \rtimes \mathbb{Z}_{g-1}$. $\square$

The following lemmas are elementary and so we omit the proofs.

Lemma 3.13. *The multiplication map $\mathrm{Sym}^m \mathrm{H}^0\big(C, \omega_C\big) \to \mathrm{H}^0\big(C, \omega_C^m\big)$ is surjective. A set $\mathcal{B}$ of degree m monomials in $\omega_0, \ldots, \omega_{g-2}, \eta$ forms a monomial basis of $\mathrm{H}^0\big(C, \omega_C^m\big)$ if and only if the following conditions are satisfied:*

(1) *$\mathcal{B}$ contains the $(g-1)$ monomials $\{\omega_i^m\}_{i=0}^{g-2}$ of weight $\pm m$,*
(2) *$\mathcal{B}$ contains the $(g-1)$ monomials $\{\omega_i^{m-1}\eta\}_{i=0}^{g-2}$ of weight $\pm(m-1)$,*
(3) *$\mathcal{B}$ contains $(g-1)$ linearly independent monomials of each weight $2 - m \leq j \leq m - 2$.*

The reader may wish to check, as an example, that $\{\omega_i^j \eta^{m-j}\}_{i=0}^{g-2}$ and $\{\omega_i^{j+1}\omega_{i-1}\eta^{m-j-2}\}_{i=0}^{g-2}$ give $2g - 2$ linearly independent monomials, with $(g-1)$ monomials of weights j and $-j$ each. Thus, taking the union of all these monomials, together with $\{\omega_i^m\}_{i=0}^{g-2}$ and $\{\omega_i^{m-1}\eta\}_{i=0}^{g-2}$ gives a monomial basis of $\mathrm{H}^0\big(C, \omega_C^m\big)$.

Lemma 3.14. *The multiplication map $\mathrm{Sym}^m \mathrm{H}^0\big(C, \omega_C^2\big) \to \mathrm{H}^0\big(C, \omega_C^{2m}\big)$ is surjective. A set $\mathcal{B}$ of degree m monomials in $\{x_i\}_{i=0}^{g-2}$, $\{y_i\}_{i=0}^{g-2}$, $\{z_i\}_{i=0}^{g-2}$ forms a monomial basis of $\mathrm{H}^0\big(C, \omega_C^{2m}\big)$ if and only if the following conditions are satisfied:*

(1) *$\mathcal{B}$ contains the $(g-1)$ monomials $\{x_i^m\}_{i=0}^{g-2}$ of weight $\pm 2m$,*
(2) *$\mathcal{B}$ contains the $(g-1)$ monomials $\{x_i^{m-1}y\}_{i=0}^{g-2}$ of weight $\pm(2m-1)$,*
(3) *$\mathcal{B}$ contains $(g-1)$ linearly independent monomials of each weight $2 - 2m \leq j \leq 2m - 2$.*

4. Monomial bases and semistability

4.1. **Canonically embedded ribbon.** Let C denote the balanced ribbon as defined in Section 3.1. In this section, we prove the odd genus case of the first part of our Main Result.

Theorem 4.1. *If $C \subset \mathbb{P}\mathrm{H}^0(C, \omega_C)$ is a canonically embedded balanced ribbon, then the Hilbert points $[C]_m$ are semistable for all $m \geq 2$.*

Corollary 4.2. *Suppose $C \subset \mathbb{P}\mathrm{H}^0(C, K_C)$ is a canonically embedded generic smooth curve of odd genus. Then the mth Hilbert point of C is semistable for every $m \geq 2$.*

Proof of Corollary 4.2. Quite generally, the locus of semistable points $(\overline{H}_{g,1}^m)^{ss} \subset \overline{H}_{g,1}^m$ is open [MFK94]. Since $\overline{H}_{g,1}^m$ is an irreducible variety whose generic point is the mth Hilbert point of a canonically embedded smooth genus g curve, it remains to find a single semistable point in $\overline{H}_{g,1}^m$. The balanced ribbon C deforms to a smooth canonical curve by [Fon93] and Proposition 3.5 shows that $[C]_m \in \overline{H}_{g,1}^m$. Applying Theorem 4.1 finishes the proof. $\square$

We have already seen that there is a distinguished basis $\{x_0, \ldots, x_k, y_{k+1}, \ldots, y_{2k}\}$ of $\mathrm{H}^0(C, \omega_C)$ on which $\mathbb{G}_m \subset \mathrm{Aut}(C)$ acts with distinct weights (Proposition 3.3). According to Lemma 2.5, to prove Theorem 4.1 it suffices to find a set of monomial bases such that an effective linear combination of their ρ-weights is 0 with respect to every one-parameter subgroup $\rho \colon \mathbb{G}_m \to \mathrm{SL}(g)$. For ease of exposition, we will treat the cases $m = 2$ and $m \geq 3$ separately.

4.1.1. *Monomial bases of $\mathrm{H}^0(C, \omega_C^2)$.* First, we define two monomial bases, $\mathcal{B}^+$ and $\mathcal{B}^-$, of $\mathrm{H}^0(C, \omega_C^2)$ as follows. We define $\mathcal{B}^+$ to be the set of quadratic monomials divisible by one of x_0, x_k, or y_{2k}. More precisely,

$$(4.1) \quad \mathcal{B}^+ := \left\{ \{x_0 x_i\}_{i=0}^{k}, \{x_0 y_i\}_{i=k+1}^{2k}, \{x_k x_i\}_{i=1}^{k}, \{x_k y_i\}_{i=k+1}^{2k}, \{y_{2k} x_i\}_{i=1}^{k-1}, \{y_{2k} y_i\}_{i=k+1}^{2k} \right\}.$$

We define $\mathcal{B}^-$ as follows:

$$(4.2) \qquad \mathcal{B}^- := \left\{ \begin{array}{l} \{x_i^2\}_{i=0}^{k}, \{y_i^2\}_{i=k+1}^{2k}, \\ \{x_i x_{i+1}\}_{i=0}^{k-1}, x_k y_{k+1}, \{y_i y_{i+1}\}_{i=k+1}^{2k-1}, \\ \{x_i y_{i+k}\}_{i=1}^{k-1}, \{x_i y_{i+k+1}\}_{i=0}^{k-1} \end{array} \right\}.$$

Lemma 4.3. *$\mathcal{B}^+$ and $\mathcal{B}^-$ are monomial bases of $\mathrm{H}^0(C, \omega_C^2)$. For any one-parameter subgroup ρ acting on $(x_0, \ldots, y_{2k})$ diagonally with weights $(\rho_0, \ldots, \rho_{2k})$ the ρ-weights of $\mathcal{B}^+$ and $\mathcal{B}^-$ are:*

$$w_\rho(\mathcal{B}^+) = (g-2)(\rho_0 + \rho_k + \rho_{2k}),$$
$$w_\rho(\mathcal{B}^-) = -2(\rho_0 + \rho_k + \rho_{2k}).$$

Proof. Using Lemma 3.6, one easily checks that $\mathcal{B}^+$ and $\mathcal{B}^-$ are monomial bases. To compute the weight of $\mathcal{B}^+$ observe that variables $\{x_i, y_{k+i}\}_{i=1}^{k-1}$ each occur 3 times and variables $\{x_0, x_k, y_{2k}\}$ each occur $g+1$ times in Display (4.1). It follows that

$$w_\rho(\mathcal{B}^+) = 3 \sum_{i=1}^{k-1} (\rho_i + \rho_{k+i}) + (g+1)(\rho_0 + \rho_k + \rho_{2k}) = (g-2)(\rho_0 + \rho_k + \rho_{2k}),$$

where the last equality follows from the relation $\sum_{i=0}^{2k} \rho_i = 0$.

Similarly, variables $\{x_i, y_{k+i}\}_{i=1}^{k-1}$ each occur 6 times and variables $\{x_0, x_k, y_{2k}\}$ each occur 4 times in Display (4.2). It follows that

$$w_\rho(\mathcal{B}^-) = 6 \sum_{i=1}^{k-1} (\rho_i + \rho_{k+i}) + 4(\rho_0 + \rho_k + \rho_{2k}) = -2(\rho_0 + \rho_k + \rho_{2k}),$$

where the last equality again follows from the relation $\sum_{i=0}^{2k} \rho_i = 0$. $\qquad\square$

Corollary 4.4. *The 2nd Hilbert point of C is semistable.*

Proof. We have $2w_\rho(\mathcal{B}^+) + (g-2)w_\rho(\mathcal{B}^-) = 0$ for any $\rho \colon \mathbb{G}_m \to \mathrm{SL}(g)$ acting diagonally on the distinguished basis. The claim follows by Lemma 2.5. $\qquad\square$

4.1.2. *Monomial bases of* $\mathrm{H}^0(C, \omega_C^m)$ *for* $m \geq 3$. Finding monomial bases in higher degrees is slightly more cumbersome than in the case $m = 2$. First, we will need three monomial bases in every degree $m \geq 3$. Second, the precise form of one of these bases depends on the residue of $g = 2k + 1$ modulo 4. Nevertheless, the proof is conceptually no different than in the case $m = 2$. Finally, we work throughout with m fixed and each basis used in degree m is defined independently as a set of degree m monomials, though we have, for simplicity, suppressed the dependence on m in our notation.

We begin by defining two higher-degree analogues of the basis $\mathcal{B}^+$ from Section 4.1.1.

Definition 4.5. We define $\mathcal{B}_1^+$ to be the set of degree m monomials in the ideal

$$(x_0, x_k)^{m-1} \cdot (x_0, \ldots, x_{k-1}, y_{k+1}, \ldots, y_{2k}) + (x_k, y_{2k})^{m-1} \cdot (x_0, \ldots, x_{k-1}, y_{k+1}, \ldots, y_{2k}) + x_k^m.$$

We define $\mathcal{B}_2^+$ to be the set of degree m monomials in the ideal

$$(x_0, y_{2k})^{m-1} \cdot (x_1, \ldots, x_{k-1}, y_{k+1}, \ldots, y_{2k-1}) + x_k \cdot (x_0, y_{2k})^{m-2} \cdot (x_1, \ldots, x_{k-1}, y_{k+1}, \ldots, y_{2k-1})$$
$$+ (x_0, y_{2k})^m + x_k(x_0, y_{2k})^{m-1} + x_k^2(x_0, y_{2k})^{m-2} + x_k^3(x_0, y_{2k})^{m-3}$$

Lemma 4.6. $\mathcal{B}_1^+$ *and* $\mathcal{B}_2^+$ *are monomial bases of* $\mathrm{H}^0(C, \omega_C^m)$. *For any one-parameter subgroup ρ acting on $(x_0, \ldots, y_{2k})$ diagonally with weights $(\rho_0, \ldots, \rho_{2k})$ the ρ-weights of $\mathcal{B}_1^+$ and $\mathcal{B}_2^+$ are:*

$$w_\rho(\mathcal{B}_1^+) = \left((m-1)^2(g-1) - (2m-3)\right)\rho_k + \left(\frac{m(m-1)}{2}(g-1) - 1\right)(\rho_0 + \rho_{2k}),$$

$$w_\rho(\mathcal{B}_2^+) = \left((m-1)(g-1) + (2m-5)\right)\rho_k + \left((m-1)^2(g-1) - (2m-3)\right)(\rho_0 + \rho_{2k}).$$

Proof. Using Lemma 3.6, it is easy to see that $\mathcal{B}_1^+$ and $\mathcal{B}_2^+$ are monomial bases. Next, note that in $\mathcal{B}_1^+$ the variable x_k appears $(m-1)^2(g-1) + 2$ times, variables x_0 and y_{2k} each appear $\binom{m}{2}(g-1) + 2m - 2$ times, and variables $x_1, \ldots, x_{k-1}, y_{k+1}, \ldots, y_{2k-1}$ each appear $2m-1$ times. Recalling that $\sum_{i=0}^{2k} \rho_i = 0$, we deduce the formula for $w_\rho(\mathcal{B}_1^+)$.

The ρ-weight of $\mathcal{B}_2^+$ is computed analogously by observing that in $\mathcal{B}_2^+$ the variable x_k appears $(m-1)(g-1) + (4m-6)$ times, variables x_0 and y_{2k} each appear $(m-1)^2(g-1) + 2$ times, and variables $x_1, \ldots, x_{k-1}$ and $y_{k+1}, \ldots, y_{2k-1}$ each appear $2m-1$ times. $\qquad\square$

Next, we construct higher-degree analogues of the basis $\mathcal{B}^-$ from Section 4.1.1. Throughout the construction, we let ι be the involution exchanging x_i and y_{2k-i} and leaving x_k fixed.

Let $\ell = \lfloor k/2 \rfloor$. We introduce the following sets of monomials:

$$S_0 := \begin{cases} \{x_k^m, \; x_0 \, y_{2k} \, x_k^{m-2}\} & \text{if } m \text{ is odd,} \\ \{x_k^m, \; x_\ell \, y_{2k-\ell} \, x_k^{m-2}\} & \text{if } m \text{ is even.} \end{cases}$$

$$S_1 := \left\{ x_i^{m-d} \, x_{i+1}^d \; : \quad 0 \le i \le k-1, \; 0 \le d \le m-1 \right\}$$

$$S_2 := \begin{cases} x_i^{m-1-d} \, x_{i+1}^d \, y_{i+k+1} \; : \quad 0 \le i \le \ell-2, \; 0 \le d \le m-1 \\ x_{\ell-1}^{m-1-d} \, x_\ell^d \, y_{\ell+k} \; : \qquad 0 \le d \le m-2 \end{cases}$$

The definition of the next set of monomials depends on parity of k. If $k = 2\ell$, we define

$$S_3 := \left\{ x_i^{m-1-d} \, x_{i+1}^d \, y_{k+2\ell-1-i} \; : \quad \ell \le i \le k-2, \; 1 \le d \le m-2 \right\};$$

and if $k = 2\ell + 1$, we define

$$S_3 := \begin{cases} x_i^{m-1-d} \, x_{i+1}^d \, y_{k+2\ell-1-i} \; : \quad \ell \le i \le k-3, \; 1 \le d \le m-2 \\ x_{k-2}^{m-2-d} \, x_{k-1}^d \, x_\ell \, y_{3\ell+1} \; : \quad 0 \le d \le m-2 \end{cases}$$

We proceed to define

$$S_4 := \left\{ x_{k-1}^{m-2-d} \, x_k^d \, (x_0 y_{2k}) \; : \quad 0 \le d \le m-4 \right\}$$

$$S_5 := \begin{cases} \left\{ x_{k-1} \, x_\ell^{(m-1)/2} \, y_{2k-\ell}^{(m-1)/2} \right\} & \text{if } m \text{ is odd,} \\[2mm] \left\{ x_{k-1} \, x_k \, x_\ell^{(m-2)/2} \, y_{2k-\ell}^{(m-2)/2} \right\} & \text{if } m \text{ is even.} \end{cases}$$

Definition 4.7. We define a set $\mathcal{B}^-$ of degree m monomials by

$$\mathcal{B}^- := S_0 \cup \bigcup_{i=1}^5 \big(S_i \cup \iota(S_i) \big).$$

Lemma 4.8. *For $m \ge 3$, $\mathcal{B}^-$ is a monomial basis of $\mathrm{H}^0\big(C, \mathcal{O}_C(m)\big) = \mathrm{H}^0\big(C, \omega_C^m\big)$. For any $\rho \colon \mathbb{G}_m \to \mathrm{SL}(g)$ acting on $(x_0, \ldots, y_{2k})$ diagonally with weights $(\rho_0, \ldots, \rho_{2k})$ we have*

$$w_\rho(\mathcal{B}^-) = \begin{cases} -(m^2 - 3m + 5)(\rho_0 + \rho_{2k}) - (5m-10)\rho_k & \text{if } m \text{ is odd,} \\ -(m^2 - 3m + 6)(\rho_0 + \rho_{2k}) - (5m-12)\rho_k & \text{if } m \text{ is even.} \end{cases}$$

Proof. Although the precise definition of $\mathcal{B}^-$ depends on the parity of k, our proof of the lemma does not. Thus we suppress the parity of k in what follows. To prove that $\mathcal{B}^-$ is a monomial basis, we make use of the identification of $\mathrm{H}^0\big(C, \omega_C^m\big)$ with functions in $\mathbb{C}[u, \epsilon]/(\epsilon^2)$ made in Section 3.1. To begin, observe that $\mathcal{B}^-$ is invariant under ι. Since ι maps a monomial of u-degree d to a monomial of u-degree $2mk - d$, it suffices, in view of Lemma 3.6, to show that $\mathcal{B}^-$ contains one monomial of each u-degree $d = 0, \ldots, k$ and two linearly independent monomials of each u-degree $d = k+1, \ldots, mk$. To do this, note that S_0 consists of two linearly independent monomials of u-degree km; that S_1 consists by the Ribbon Product Lemma 3.4 of exactly pure powers of u of each u-degree $d = 0, \ldots, mk-1$; and that $S_2 \cup S_3 \cup S_4 \cup S_5$ contains exactly one monomial of each u-degree $d = k+1, \ldots, mk-1$ with a non-zero ϵ term. This finishes the proof that $\mathcal{B}^-$ is a monomial basis.

To compute the ρ-weight of $\mathcal{B}^-$, we observe that in $S_1 \cup \{x_k^m\} \cup \iota(S_1)$ all variables with the exception of x_0 and y_{2k} occur the same number of times, namely $2\sum_{d=1}^{m-1} d + m = m^2$

times, while x_0 and y_{2k} each occur $\sum_{d=1}^{m} d = m(m+1)/2$ times. It follows that the ρ-weight of $S_1 \cup \{x_k^m\} \cup \iota(S_1)$ is

$$m^2 \sum_{i=1}^{2k-1} \rho_i + \frac{m(m+1)}{2}(\rho_0 + \rho_{2k}) = -m(m-1)(\rho_0 + \rho_{2k})/2,$$

where the last equality follows from $\sum_{i=0}^{2k} \rho_i = 0$. Similarly, one can easily see that in the remaining monomials of $\mathcal{B}^-$ each of the variables $x_1, \ldots, x_{k-1}, y_{k+1}, \ldots, y_{2k}$ occurs exactly $m(m-1)$ times; each of the variables x_0 and y_{2k} occurs

$$\begin{cases} (m^2 + 3m - 10)/2 & (\text{if } m \text{ is odd}) \\ (m^2 + 3m - 12)/2 & (\text{if } m \text{ is even}) \end{cases} \quad \text{times;}$$

and x_k occurs

$$\begin{cases} m^2 - 6m + 10 & (\text{if } m \text{ is odd}) \\ m^2 - 6m + 12 & (\text{if } m \text{ is even}) \end{cases} \quad \text{times.}$$

Using $\sum_{i=0}^{2k} \rho_i = 0$, it follows that the total ρ-weight of these remaining monomials is

$$-\frac{(m^2 - 5m + 10)}{2}(\rho_0 + \rho_{2k}) - (5m - 10)\rho_k \quad \text{if } m \text{ is odd,}$$

$$-\frac{(m^2 - 5m + 12)}{2}(\rho_0 + \rho_{2k}) - (5m - 12)\rho_k \quad \text{if } m \text{ is even.}$$

The claim follows. $\square$

Lemma 4.9. *There exist $c_0, c_1, c_2 \in \mathbb{Q} \cap (0, \infty)$ such that*

$$c_0 w_\rho(\mathcal{B}^-) + c_1 w_\rho(\mathcal{B}_1^+) + c_2 w_\rho(\mathcal{B}_2^+) = 0$$

for all one-parameter subgroups of $\mathrm{SL}(g)$ acting on the basis $(x_0, \ldots, y_{2k})$ diagonally.

Proof. We need to show that $w_\rho(\mathcal{B}^-)$ given by Lemma 4.8 and considered as the linear function in $(\rho_0, \ldots, \rho_{2k})$ is the negative of an effective linear combination of $w_\rho(\mathcal{B}_1^+)$ and $w_\rho(\mathcal{B}_2^+)$ given by Lemma 4.6. In the case of odd m, the claim holds because the inequalities

$$\frac{m(m-1)(g-1) - 2}{2(m-1)^2(g-1) - 2(2m-3)} \leq \frac{(m^2 - 3m + 5)}{5m - 10} \leq \frac{(m-1)^2(g-1) - (2m-3)}{(m-1)(g-1) + (2m-5)},$$

are satisfied for all $g, m \geq 3$. In the case of even m, we require the same inequalities save that the middle term is replaced by $\frac{(m^2 - 3m + 6)}{5m - 12}$. $\square$

Proof of Theorem 4.1. The case of $m = 2$ was handled in Corollary 4.4. If $m \geq 3$, the claim follows from Lemma 4.9 and Lemma 2.5. $\square$

4.2. **Canonically embedded A_{2k+1}-curve.** Let C denote the balanced double A_{2k+1}-curve as defined in Section 3.2. In this section, we prove the even genus case of the first part of our Main Result. Since $\mathrm{H}^0(C, \omega_C)$ is a multiplicity-free representation of $\mathbb{G}_m \subset \mathrm{Aut}(C)$ by Lemma 3.9, we can apply the Kempf-Morrison Criterion (Proposition 2.4) to prove semistability of C. Namely, to prove that $[C]_m$ is semistable, it suffices to check that for every one-parameter subgroup $\rho \colon \mathbb{G}_m \to \mathrm{SL}(g)$ acting diagonally on the distinguished basis $\{x_1, \ldots, x_k, y_1, \ldots, y_k\}$ with integer weights $\lambda_1, \ldots, \lambda_k, \nu_1, \ldots, \nu_k$, there exists a monomial basis of $\mathrm{H}^0(C, \omega_C^m)$ of non-positive ρ-weight. Explicitly, this means that we must exhibit a

set $\mathcal{B}$ of $(2m-1)(2k-1)$ degree m monomials in the variables $\{x_i, y_i\}_{i=1}^k$ with the properties that:

(1) $\mathcal{B}$ maps to a basis of $\mathrm{H}^0(C, \omega_C^m)$ via $\mathrm{Sym}^m \mathrm{H}^0(C, \omega_C) \to \mathrm{H}^0(C, \omega_C^m)$.

(2) $\mathcal{B}$ has non-positive ρ-weight, that is, if $\mathcal{B} = \{e_\ell\}_{\ell=1}^{(2m-1)(2k-1)}$ and $e_\ell = \prod_{i=1}^k x_i^{a_{\ell i}} y_i^{b_{\ell i}}$, then

$$\sum_{\ell=1}^{(2m-1)(2k-1)} \sum_{i=1}^k (a_{\ell i} \lambda_i + b_{\ell i} \nu_i) \leq 0.$$

Theorem 4.10. *If $C \subset \mathbb{P}\mathrm{H}^0(C, \omega_C)$ is a canonically embedded balanced double A_{2k+1}-curve, then the Hilbert points $[C]_m$ are semistable for all $m \geq 2$.*

As an immediate corollary of this result, we obtain a proof of Theorem 1.3 (1) and hence of Theorem 1.2:

Corollary 4.11 (Theorem 1.3 (1)). *A generic canonically embedded smooth trigonal curve of even genus has semistable m^{th} Hilbert point for every $m \geq 2$.*

Proof of Corollary. Recall from Proposition 3.8 that the canonical embedding of the balanced double A_{2k+1}-curve C lies on a balanced surface scroll in $\mathbb{P}^{2k-1}$ in the divisor class $(3, k+1)$. It follows that C deforms flatly to a smooth curve in the class $(3, k+1)$ on the scroll. Such a curve is a canonically embedded smooth trigonal curve of genus $2k$. The semistability of a generic deformation of C follows from the openness of semistable locus. $\square$

Proof of Theorem 4.10. Recall from Lemma 3.7 that

$$\mathrm{H}^0(C, \omega_C^m) = \mathrm{span}\{\omega_j\}_{j=m}^{mk} \oplus \mathrm{span}\{\eta_j\}_{j=m}^{mk} \oplus \mathrm{span}\{\chi_\ell\}_{\ell=-k(m-1)+1}^{k(m-1)-1}.$$

Now, given a one-parameter subgroup ρ as above, we will construct the requisite monomial basis $\mathcal{B}$ as a union

$$\mathcal{B} = \mathcal{B}_\omega \cup \mathcal{B}_\eta \cup \mathcal{B}_\chi,$$

where $\mathcal{B}_\omega, \mathcal{B}_\eta$, and $\mathcal{B}_\chi$ are collections of degree m monomials which map onto the bases of the subspaces spanned by $\{\omega_j\}_{j=m}^{mk}, \{\eta_j\}_{j=m}^{mk}$, and $\{\chi_\ell\}_{\ell=-k(m-1)+1}^{k(m-1)-1}$, respectively.

To construct $\mathcal{B}_\omega$ and $\mathcal{B}_\eta$, we use Kempf's proof of the stability of Hilbert points of a rational normal curve. More precisely, consider the component C_0 of C with the uniformizer u_0 at 0. Clearly, $\omega_C|_{C_0} \simeq \mathcal{O}_{\mathbb{P}^1}(k-1)$. The restriction map $\mathrm{H}^0(C, \omega_C) \to \mathrm{H}^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(k-1))$ identifies $\{x_i\}_{i=1}^k$ with a basis of $\mathrm{H}^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(k-1))$ given by $\{1, u_0, \ldots, u_0^{k-1}\}$. Under this identification, the subspace $\mathrm{span}\{\omega_j\}_{j=m}^{mk}$ is identified with $\mathrm{H}^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(m(k-1)))$. Set $\lambda := \sum_{i=1}^k \lambda_i/k$. Given a one-parameter subgroup $\widetilde{\rho} \colon \mathbb{G}_m \to \mathrm{SL}(k)$ acting on $\{x_1, \ldots, x_k\}$ diagonally with weights $(\lambda_1 - \lambda, \ldots, \lambda_k - \lambda)$, Kempf's result on the semistability of a rational normal curve in $\mathbb{P}^{k-1}$ [Kem78, Corollary 5.3], implies the existence of a monomial basis $\mathcal{B}_\omega$ of $\mathrm{H}^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(m(k-1)))$ with non-positive $\widetilde{\rho}$-weight. Under the above identification, $\mathcal{B}_\omega$ is a monomial basis of $\mathrm{span}\{\omega_j\}_{j=m}^{mk}$ of ρ-weight at most $m(mk - m + 1)\lambda$. Similarly, if $\nu := \sum_{i=1}^k \nu_i/k$, we deduce the existence of a monomial basis $\mathcal{B}_\eta$ of $\mathrm{span}\{\eta_j\}_{j=m}^{mk}$ whose ρ-weight is at most $m(mk - m + 1)\nu$. Since $\lambda + \nu = 0$, it follows that the total ρ-weight of $\mathcal{B}_\omega \cup \mathcal{B}_\eta$ is non-positive.

Thus, to construct a monomial basis $\mathcal{B}$ of non-positive ρ-weight, it remains to construct a monomial basis $\mathcal{B}_\chi$ of non-positive ρ-weight for the subspace

$$\operatorname{span}\{\chi_\ell\}_{\ell=-(m-1)k-1}^{(m-1)k-1} \subset \mathrm{H}^0(C, \omega_C^m).$$

In Lemma 4.12, proved below, we show the existence of such a basis. Thus, we obtain the desired monomial basis $\mathcal{B}$ and finish the proof. $\qquad\square$

Note that if we define the *weighted degree* by $\deg(x_i) = i$ and $\deg(y_i) = -i$, then a set $\mathcal{B}_\chi$ of $2k(m-1) - 1$ degree m monomials in $\{x_1, \ldots, x_k, y_1, \ldots, y_k\}$ maps to a basis of $\operatorname{span}\{\chi_\ell\}_{\ell=-(m-1)k-1}^{(m-1)k-1}$ if and only if it satisfies the following two conditions:

(1) Each monomial has both x_i and y_i terms,
(2) Each weighted degree from $(m-1)k - 1$ to $-(m-1)k + 1$ occurs exactly once.

We call such a set of monomials a χ-*basis*. The following combinatorial lemma completes the proof of Theorem 4.10.

Lemma 4.12. *Suppose $\rho\colon \mathbb{G}_m \to \mathrm{SL}(2k)$ is a one-parameter subgroup which acts on $\{x_1, \ldots, x_k, y_1, \ldots, y_k\}$ diagonally with integer weights $\lambda_1, \ldots, \lambda_k, \nu_1, \ldots, \nu_k$ satisfying $\sum_{i=1}^k (\lambda_i + \nu_i) = 0$. Then there exists a χ-basis with non-positive ρ-weight.*

Proof of Lemma 4.12 for $m = 2$. Take the first χ-basis to be

$$\mathcal{B}_1 := \left\{ \{x_i\, y_{k-i}\}_{1 \le i \le k-1}, \{x_i\, y_{k-i+1}\}_{1 \le i \le k} \right\}.$$

In this basis, all variables except x_k and y_k occur twice and x_k, y_k occur once each. Thus

$$w_\rho(\mathcal{B}_1) = 2(\lambda_1 + \cdots + \lambda_{k-1}) + 2(\nu_1 + \cdots + \nu_{k-1}) + \lambda_k + \nu_k = -(\lambda_k + \nu_k).$$

Take the second χ-basis to be

$$\mathcal{B}_2 := \left\{ \{x_k\, y_i\}_{1 \le i \le k}, \{x_i\, y_k\}_{1 \le i \le k-1} \right\}.$$

We have

$$w_\rho(\mathcal{B}_2) = k(\lambda_k + \nu_k) + \sum_{i=1}^{k-1}(\lambda_i + \nu_i) = (k-1)(\lambda_k + \nu_k).$$

We conclude that for any one-parameter subgroup ρ, we have $(k-1)w_\rho(\mathcal{B}_1) + w_\rho(\mathcal{B}_2) = 0$. It follows that either $\mathcal{B}_1$ or $\mathcal{B}_2$ gives a χ-basis of non-positive weight. $\qquad\square$

Proof of Lemma 4.12 for $m \ge 3$. We will prove the Lemma by exhibiting one collection of χ-bases whose ρ-weights sum to a positive multiple of $\lambda_k + \nu_k$ and a collection of χ-bases whose ρ-weights sum to a negative multiple of $\lambda_k + \nu_k$. Since, for any given one-parameter subgroup ρ, we have either $\lambda_k + \nu_k \ge 0$ or $\lambda_k + \nu_k \le 0$, it follows at once that one of our χ-bases must have non-positive weight.

Throughout this section, we let ι be the involution exchanging x_i and y_i. We begin by writing down χ-bases maximizing the occurrences of x_k and y_k while balancing the occurrences of the other variables. Define T_1 as the set of all degree m monomials having both x_i and y_i terms that belong to the ideal

$$(x_k, y_k)^{m-1}(x_1, \ldots, x_k, y_1, \ldots, y_k).$$

The ρ-weight of T_1 is

$$\left(k(m-1) + (2k-1)\binom{m-1}{2}\right)(\lambda_k + \nu_k) + (m-1)\sum_{i=1}^{k-1}(\lambda_i + \nu_i).$$

Note that T_1 misses only the $m-2$ weighted degrees

$$k(m-3), k(m-5), \ldots, -k(m-5), -k(m-3).$$

For each $s = 1, \ldots, k-1$, define a set of $m-2$ monomials having exactly these missing degrees by

$$T_2(s) := \{x_k^{m-2-d} y_k^d (x_{k-s}x_s) : \quad 1 \le d \le m-2\}$$

For each s, the sets $T_1 \cup T_2(s)$ and $T_1 \cup \iota(T_2(s))$ are χ-bases. Using $\sum_{i=1}^{k}(\lambda_i + \nu_i) = 0$, one sees at once that the sum of the ρ-weights of such bases, as s ranges from 1 to $k-1$, is a *positive multiple* of $(\lambda_k + \nu_k)$.

We now write down bases minimizing the occurrences of x_k and y_k. We handle the case when k is even and odd separately.

Case of even k: If $k = 2\ell$, we define the following set of monomials where the weighted degrees range from $k(m-1)-1$ to m:

$$S_1 := \begin{cases} x_i^{m-1-d} x_{i-1}^d y_{k+1-i} : & \ell+2 \le i \le k, \ 0 \le d \le m-1 \\ x_i^{m-1-d} x_{i-1}^d y_{i-1} : & 2 \le i \le \ell+1, \ 0 \le d \le m-3 \end{cases}$$

In the set $S_1 \cup \iota(S_1)$, the variables x_k and y_k occur $(m^2 - m) - \binom{m}{2}$ times, $x_{\ell+1}$ and $y_{\ell+1}$ occur $(m^2 - m) - 1$ times, x_ℓ and y_ℓ occur $(m^2 - m) - m$ times, and x_1 and y_1 occur $m^2 - m - \left(\binom{m}{2} - 1\right)$ times while all of the other variables occur $m^2 - m$ times. To complete $S_1 \cup \iota(S_1)$ to a χ-basis, we define, for each $s = 1, \ldots, k-1$, the following set of monomials where the weighted degrees range from $m-1$ to $1-m$:

$$S_2(s) := \begin{cases} x_{\ell+1}\, y_\ell\, x_1^{m-2} \\ x_\ell\, y_\ell\, (x_s y_s)^i\, x_1^{m-2i-2} : & \text{for } 0 \le 2i \le m-2, \\ x_\ell\, y_\ell\, (x_s y_s)^i\, y_1^{m-2i-2} : & \text{for } 0 \le 2i < m-2, \\ (x_k\, y_s\, y_{k-s})\, (x_s y_s)^i\, x_1^{m-2i-3} : & \text{for } 0 \le 2i \le m-3, \\ (x_k\, y_s\, y_{k-s})\, (x_s y_s)^i\, y_1^{m-2i-3} : & \text{for } 0 \le 2i < m-3, \\ y_{\ell+1}\, x_\ell\, y_1^{m-2} \end{cases}$$

For each $s = 1, \ldots, k-1$, the sets $S_1 \cup \iota(S_1) \cup S_2(s)$ and $S_1 \cup \iota(S_1) \cup \iota(S_2(s))$ are χ-bases. We compute that in the union

$$\bigcup_{s=1}^{k-1}\left(S_1 \cup \iota(S_1) \cup S_2(s)\right) \cup \left(S_1 \cup \iota(S_1) \cup \iota(S_2(s))\right)$$

of $2(k-1)$ χ-bases the variables x_k and y_k each occurs

$$2(k-1)(m^2 - m) - (k-1)(m^2 - 2m + 2)$$

times while all of the other variables occur

$$2(k-1)(m^2 - m) + (m-2)(m-1)$$

times.

Using the relation $\sum_{i=1}^{k}(\lambda_i + \nu_i) = 0$, we conclude that the sum of the ρ-weights of all such χ-bases is a *negative multiple* of $(\lambda_k + \nu_k)$.

Case of odd k: If $k = 2\ell + 1$ is odd, χ-bases whose ρ-weight is a negative multiple of $(\lambda_k + \nu_k)$ can be constructed analogously to the case when k is even. For the reader's convenience, we spell out the details. We define the following set of monomials where the weighted degrees range from $k(m-1) - 1$ to $m - 1$:

$$S_1 := \begin{cases} x_i^{m-1-d}\, x_{i-1}^d\, y_{k+1-i} & : \quad \ell + 3 \leq i \leq k, \ 0 \leq d \leq m - 1 \\ x_i^{m-1-d}\, x_{i-1}^d\, y_{i-2} & : \quad 3 \leq i \leq \ell + 2, \ 0 \leq d \leq m - 3 \\ x_{\ell+2}\, y_\ell\, x_2^{m-2} & \\ x_{\ell+1}\, y_\ell\, x_2^{m-2-d}\, x_1^d & : \quad 0 \leq d \leq m - 2 \end{cases}$$

In the set of monomials $S_1 \cup \iota(S_1)$, the variables x_k and y_k occur $\binom{m}{2}$ times, $x_{\ell+1}$ and $y_{\ell+1}$ occur $m^2 - m - (m-1)$ times, and x_1 and y_1 occur $m^2 - m - \binom{m-1}{2}$ times, while all of the other variables occur $m^2 - m$ times. Finally, for each $s = 1, \ldots, k-1$, we define the following set of monomials where the weighted degrees range from $m - 2$ to $2 - m$:

$$S_2(s) := \begin{cases} x_{\ell+1}\, y_{\ell+1}\, (x_s y_s)^i\, x_1^{m-2-2i} & : \quad \text{for } 0 \leq 2i \leq m - 2, \\ x_{\ell+1}\, y_{\ell+1}\, (x_s y_s)^i\, y_1^{m-2-2i} & : \quad \text{for } 0 \leq 2i < m - 2, \\ (x_k\, y_s\, y_{k-s})\, (x_s y_s)^i\, x_1^{m-3-2i} & : \quad \text{for } 0 \leq 2i \leq m - 3, \\ (x_k\, y_s\, y_{k-s})\, (x_s y_s)^i\, y_1^{m-3-2i} & : \quad \text{for } 0 \leq 2i < m - 3 \end{cases}$$

For each $s = 1, \ldots, k-1$, the sets $S_1 \cup \iota(S_1) \cup S_2(s)$ and $S_1 \cup \iota(S_1) \cup \iota(S_2(s))$ are χ-bases. We compute that in the union

$$\bigcup_{s=1}^{k-1} \big(S_1 \cup \iota(S_1) \cup S_2(s)\big) \cup \big(S_1 \cup \iota(S_1) \cup \iota(S_2(s))\big)$$

of $2(k-1)$ χ-bases the variables x_k and y_k each occurs $2(k-1)\binom{m}{2} + (k-1)(m-2)$ times while all of the other variables occur $2(k-1)(m^2 - m) + (m-2)(m-1)$ times.

Using the relation $\sum_{i=1}^{k}(\lambda_i + \nu_i) = 0$, we conclude that the total ρ-weight of these χ-bases is a *negative multiple* of $(\lambda_k + \nu_k)$ and we are done. $\square$

4.3. **Bicanonically embedded rosary.** We continue our study of the rosary C defined in Section 3.3. In this section, we prove the Theorem 1.3 (2).

Theorem 4.13. *If $C \subset \mathbb{P}H^0\big(C, \omega_C^2\big)$ is a bicanonically embedded rosary, then the Hilbert points $[C]_m$ are semistable for all $m \geq 2$.*

Corollary 4.14 (Theorem 1.3 (2)). *Suppose $C \subset \mathbb{P}H^0\big(C, K_C^2\big)$ is a generic bicanonically embedded smooth bielliptic curve of odd genus. Then the mth Hilbert point of C is semistable for every $m \geq 2$.*

Proof of Corollary. This follows immediately from Theorem 4.13 and Lemma 5.3. $\square$

Proof of Theorem 4.13. We follow the notation of Lemma 3.10 (b). We need to show that for any one-parameter subgroup $\rho\colon \mathbb{G}_m \to \mathrm{SL}(3g-3)$ acting on the basis $\{x_i, y_i, z_i \ : \ i \in \mathbb{Z}_{g-1}\}$ of $\mathrm{H}^0\big(C, \mathcal{O}_C(1)\big) = \mathrm{H}^0\big(C, \omega_C^2\big)$ diagonally, there is a monomial basis of $\mathrm{H}^0\big(C, \mathcal{O}_C(m)\big) = \mathrm{H}^0\big(C, \omega_C^{2m}\big)$ of non-positive ρ-weight.

We now define several monomial bases of $\mathrm{H}^0\big(C, \omega_C^{2m}\big)$. To begin, set

$$S_0 := \big\{x_i^m, \quad x_i^{m-1} y_i : \qquad i \in \mathbb{Z}_{g-1}\big\},$$

$$S_1 := \begin{cases} x_i^d \, z_i^{m-d}, & x_i^d \, z_{i+1}^{m-d} : & i \in \mathbb{Z}_{g-1}, & 1 \le d \le m-1 \\ x_i^d \, y_i z_i^{m-d-1}, & x_i^d \, y_i \, z_{i+1}^{m-d-1} : & i \in \mathbb{Z}_{g-1}, & 0 \le d \le m-2 \end{cases}\bigg\},$$

$$S_2 := \begin{cases} \big\{(y_{i-1}\, y_i)^\ell \, z_i : & i \in \mathbb{Z}_{g-1}\big\} & \text{if } m = 2\ell+1 \text{ is odd,} \\ \big\{(y_{i-1}\, y_i)^\ell \, z_i^2 : & i \in \mathbb{Z}_{g-1}\big\} & \text{if } m = 2\ell+2 \text{ is even.} \end{cases}$$

$$S_2' := \begin{cases} \big\{(y_{i-1}\, y_i)^\ell \, z_i : & i \in \mathbb{Z}_{g-1}\big\} & \text{if } m = 2\ell+1 \text{ is odd,} \\ \big\{(y_{i-1}\, y_i)^{\ell+1} : & i \in \mathbb{Z}_{g-1}\big\} & \text{if } m = 2\ell+2 \text{ is even.} \end{cases}$$

Note that the choice of S_0 is prescribed by Lemma 3.14 (1–2) and that there are $(g-1)$ linearly independent monomials of weight j in S_1, for each $1 \le |j| \le 2m-2$, and our choice of these monomials minimizes the occurrences of y_i's. Also, S_2 and S_2' each contains $(g-1)$ linearly independent monomials of weight 0. It follows that the following are monomial bases of $\mathrm{H}^0\big(C, \omega_C^{2m}\big)$

$$\mathcal{B}_1^+ := S_0 \cup S_1 \cup S_2,$$
$$\mathcal{B}_2^+ := S_0 \cup S_1 \cup S_2'.$$

Remark 4.15. When $g = 3$ and m is even, S_2' contains only one element. In this case, we take $\mathcal{B}_1^+ := S_0 \cup S_1 \cup \{(y_0 y_1)^\ell z_0^2, (y_0 y_1)^{\ell+1}\}$ and $\mathcal{B}_2^+ := S_0 \cup S_1 \cup \{(y_0 y_1)^\ell z_1^2, (y_0 y_1)^{\ell+1}\}$.

Let X^ρ, Y^ρ, Z^ρ denote the sum of the ρ-weights of the x_i's, y_i's, z_i's, respectively. In order to balance the occurrences of x_i's and z_i's, we consider the average of the ρ-weights of $\mathcal{B}_1^+$ and $\mathcal{B}_2^+$ and obtain

$$\frac{1}{2}\big(w_\rho(\mathcal{B}_1^+) + w_\rho(\mathcal{B}_1^+)\big) = (2m^2 - 2m + 1)X^\rho + (3m-2)Y^\rho + (2m^2 - 2m + 1)Z^\rho.$$

Next we define an alternate pair of monomial bases maximizing the occurrences of y_i's. To do so, we set

$$T_1 := \begin{cases} x_i^d \, y_i^{m-d}, & x_i^{d+1} y_i^{m-d-2} z_i & : i \in \mathbb{Z}_{g-1}, & 0 \le d \le m-2 \\ y_i^d \, z_i^{m-d}, & y_i^d \, z_{i+1}^{m-d} & : i \in \mathbb{Z}_{g-1}, & 2 \le d \le m-1 \\ y_i \, z_i^{m-1}, & y_i \, z_{i+1}^{m-1} & : i \in \mathbb{Z}_{g-1}. \end{cases}\bigg\}$$

and define

$$\mathcal{B}_1^- := S_0 \cup T_1 \cup S_2,$$
$$\mathcal{B}_2^- := S_0 \cup T_1 \cup S_2'.$$

One easily checks that $\mathcal{B}_1^-$ and $\mathcal{B}_2^-$ are monomial bases of $\mathrm{H}^0\big(C, \omega_C^{2m}\big)$ and that the average of their ρ-weights is

$$m^2 X^\rho + (2m^2 - m)Y^\rho + m^2 Z^\rho.$$

Using $X^\rho + Y^\rho + Z^\rho = 0$, we obtain

$$(m^2 - m)\big(w_\rho(\mathcal{B}_1^+) + w_\rho(\mathcal{B}_1^+)\big) + (2m^2 - 5m + 3)\big(w_\rho(\mathcal{B}_1^-) + w_\rho(\mathcal{B}_2^-)\big) = 0$$

for any one-parameter subgroup ρ. Lemma 2.5 now finishes the proof of the theorem. $\square$

5. Non-semistability results

5.1. **Canonically embedded rosary.** Let C denote the rosary defined in Section 3.3. In this section, we analyze finite Hilbert stability of the *canonical embedding* of C. We find that C is the first known example of a canonical curve in arbitrary (odd) genus such that stability of its Hilbert points depends on m: $[C]_m$ is semistable for large m but becomes non-semistable for small m. More precisely, we have the following result.

Theorem 5.1. *Let $C \subset \mathbb{P}\mathrm{H}^0\big(C, \omega_C\big)$ be the canonically embedded rosary of odd genus $g \geq 5$. Then $[C]_m$ is semistable if and only if $g \leq 2m + 1$.*

Proof. We follow the notation of Lemma 3.10 (a). First, we show that $[C]_m$ is semistable for $g \leq 2m + 1$. This is accomplished by the same technique as in the previous sections, namely by using Lemma 3.13 to find non-positive monomial bases of $\mathrm{H}^0\big(C, \omega_C^m\big)$. Let $\rho\colon \mathbb{G}_m \to \mathrm{SL}(g)$ be a one-parameter subgroup acting on the basis $(\omega_0, \ldots, \omega_{g-2}, \eta)$ diagonally with weights $(\rho_0, \ldots, \rho_{g-2}, \rho_{g-1})$. Set $W := \sum_{i=0}^{g-2} \rho_i = -\rho_{g-1}$. We will construct bases in which all the ω_i appear equally often and hence these bases have ρ-weights that are multiples of W:

First, we find a basis in which η appears as seldom as possible. We define a basis $\mathcal{B}^+$ to be the following set of monomials:

$$\mathcal{B}^+ := \begin{cases} \omega_i^m, \ \omega_i^{m-1}\eta & : i \in \mathbb{Z}_{g-1}, \\ \omega_i^{m-d}\omega_{i-1}^d, \ \omega_i^d\omega_{i-1}^{m-d} & : i \in \mathbb{Z}_{g-1}, \quad 1 \leq m - 2d \leq m - 2, \\ \omega_i^{m-d-1}\omega_{i-1}^d\eta, \ \omega_i^d\omega_{i-1}^{m-d-1}\eta & : i \in \mathbb{Z}_{g-1}, \quad 2 \leq m - 2d \leq m - 2, \\ \begin{cases} \omega_i^\ell\omega_{i-1}^\ell & \text{if } m = 2\ell \\ \omega_i^{\ell-1}\omega_{i-1}^{\ell-1}\eta & \text{if } m = 2\ell - 1 \end{cases} & : i \in \mathbb{Z}_{g-1}. \end{cases}$$

The ρ-weight of $\mathcal{B}^+$ is

$$(2m^2 - 2m + 1)W + (m-1)(g-1)\rho_{g-1} = \big(2m^2 - 2m + 1 - (m-1)(g-1)\big)W.$$

We now find a basis in which η appears as often as possible. Namely, we set

$$\mathcal{B}^- := \begin{cases} \omega_i^m, & \omega_i^{m-1}\eta & : i \in \mathbb{Z}_{g-1}, \\ \omega_i^d\eta^{m-d}, & \omega_i\omega_{i-1}^{d+1}\eta^{m-d-2} & : i \in \mathbb{Z}_{g-1}, \quad 1 \leq d \leq m - 2, \\ \omega_i\omega_{i-1}\eta^{m-2} & & : i \in \mathbb{Z}_{g-1}. \end{cases}$$

Then the ρ-weight of the basis $\mathcal{B}^-$ is

$$(m^2 + m - 1)W + (m-1)^2(g-1)\rho_{g-1} = \big(m^2 + m - 1 - (m-1)^2(g-1)\big)W.$$

If $(g, m) \neq (5, 2)$ and $g \leq 2m + 1$, then either $\mathcal{B}^+$ or $\mathcal{B}^-$ has non-positive weight with respect to ρ. If $(g, m) = (5, 2)$, then it is easy to find three explicit monomial bases that accomplish the same result. This finishes the proof of semistability.

Conversely, suppose $g \geq 2m + 3$. Consider the one-parameter subgroup ρ acting with weight (-1) on ω_i's and weight $g - 1$ on η. If $\mathcal{B}$ is a monomial basis of $\mathrm{H}^0\big(C, \mathcal{O}_C(m)\big) = \mathrm{H}^0\big(C, \omega_C^m\big)$, then for each odd ℓ each monomial of weight $\pm(m - \ell)$ with respect to $\mathbb{G}_m \subset$

$\mathrm{Aut}(C)$ necessarily has an η term (see Lemma 3.13). It follows that the variable η of weight $(g-1)$ occurs at least $(m-1)(g-1)$ times among monomials of $\mathcal{B}$. The remaining at most $m(2m-1)(g-1) - (m-1)(g-1)$ variables occurring in $\mathcal{B}$ all have weight (-1). It follows that the total ρ-weight of $\mathcal{B}$ is at least

$$(g-1)(m-1)(g-1) - \big(m(2m-1)(g-1) - (m-1)(g-1)\big)$$
$$= (g-1)\big((m-1)(g-1) - (2m^2 - 2m + 1)\big) \geq (g-1)(2m-3) > 0.$$

Thus ρ destabilizes C. $\qquad\square$

5.2. **Canonically embedded bielliptic curves.** Our main result raises a natural question of whether Hilbert points of smooth canonically embedded curves can at all be non-semistable. An indirect way to see that the answer is affirmative is as follows. By [HH08, Section 5], there is an open locus in $\big(\overline{H}_{g,1}^m\big)^{ss}$ over whose $\mathrm{SL}(g)$-quotient, the tautological GIT polarization is a positive multiple of $s_g^m \lambda - \delta$, where λ and δ are the Hodge and boundary classes and

$$(5.1) \qquad s_g^m := 8 + \frac{4}{g} - \frac{2(g-1)}{gm} + \frac{2}{gm(m-1)}.$$

By generalizing the proof of [CH88, Proposition 4.3], we see that if $B \to \overline{\mathcal{M}}_g$ is a family of stable curves whose generic fiber is canonically embedded and the slope $(\delta \cdot B)/(\lambda \cdot B)$ is greater than s_g^m, then every curve in B with a well-defined m^{th} Hilbert point must have non-semistable m^{th} Hilbert point.

Two observations now lead to a candidate for a non-semistable canonically embedded *smooth* curve. The first is that $s_g^m \leq 8$ for $g \geq 2m + 1 + 1/(m-1)$. The second is that families of bielliptic curves of slope 8 can be constructed by taking a double cover of a constant family of elliptic curves (e.g. [Xia87, Bar01]). In the following result, we establish that canonical bielliptic curves indeed become non-semistable for small values of m, and show that a generic canonical bielliptic curve is semistable for m large enough.

Theorem 5.2. *A canonically embedded smooth bielliptic curve of genus g has non-semistable m^{th} Hilbert point for all $m \leq (g-3)/2$. A generic canonically embedded bielliptic curve of odd genus has semistable m^{th} Hilbert point for all $m \geq (g-1)/2$.*

Proof. Let C be a bielliptic canonical curve. Then C is a quadric section of a projective cone over an elliptic curve $E \subset \mathbb{P}^{g-2}$ embedded by a complete linear system of degree $g-1$. Choose projective coordinates $[x_0 : \ldots : x_{g-1}]$ such that the vertex of the cone has coordinates $[0 : 0 : \ldots : 0 : 1]$. Let ρ be the one-parameter subgroup of $\mathrm{SL}(g)$ acting with weights $(-1, -1, \ldots, -1, g-1)$. For every monomial basis of $\mathrm{H}^0\big(C, \mathcal{O}_C(m)\big)$, the number of monomials of ρ-weight $-m$, that is degree m monomials in the variables $x_0, \ldots, x_{g-2}$, is bounded above by $h^0\big(E, \mathcal{O}_E(m)\big) = m(g-1)$. The remaining at least $(m-1)(g-1)$ elements of the monomial basis have ρ-weight at least $g-m$. Thus the ρ-weight of any monomial basis of $\mathrm{H}^0\big(C, \mathcal{O}_C(m)\big)$ is at least

$$(5.2) \qquad (m-1)(g-1)(g-m) - m^2(g-1) = (g-1)\big(m(g+1) - 2m^2 - g\big).$$

If $m \leq (g-3)/2$, then (5.2) is positive, and thus ρ destabilizes $[C]_m$.

To prove the generic semistability of bielliptic curves in the range $m \geq (g-1)/2$, note that we have already seen that the canonically embedded rosary of odd genus $g \geq 5$ has semistable m^{th} Hilbert point if and only if $g \leq 2m + 1$ (Theorem 5.1). It remains to observe

that the rosary deforms flatly to a smooth bielliptic curve. This is accomplished in Lemma 5.3 below. □

Lemma 5.3. *The rosary of genus $g \geq 4$ deforms flatly to a smooth bielliptic curve.*

Proof. Let C be the rosary considered in Section 3.3. Consider $\mathbb{P}^{g-2}$ with projective coordinates $[x_0 : \ldots : x_{g-2}]$ and define $E \subset \mathbb{P}^{g-2}$ to be the union of $g-1$ lines $L_i : \{x_{i+1} = \cdots = x_{i+g-3} = 0\}$ for $i \in \mathbb{Z}_{g-1}$. Then E is a nodal curve of arithmetic genus 1. Since $\mathrm{H}^1\big(E, \mathcal{O}_E(1)\big) = 0$, we can deform E flatly inside $\mathbb{P}^{g-2}$ to a smooth elliptic curve by [Kol96, p.83]. Using the basis $(\omega_0, \ldots, \omega_{g-2}, \eta)$ of $\mathrm{H}^0\big(C, \omega_C\big)$ described in Lemma 3.10 (a), we observe that the canonical embedding of C is cut out by the quadric

$$x_0 x_1 + x_1 x_2 + \cdots + x_{g-2} x_0 = x_{g-1}^2$$

on the projective cone over E in $\mathbb{P}^{g-1}$. Since E deforms to a smooth elliptic curve, it follows that C deforms to a smooth bielliptic curve. □

Remark 5.4 (Trigonal curves of higher Maroni invariant)**.** Theorem 1.3 (1) shows that a generic trigonal curve with Maroni invariant 0 has semistable m^{th} Hilbert point for all $m \geq 2$. In joint work of the second author with Jensen, it is shown that every trigonal curve with Maroni invariant 0 has semistable 2nd Hilbert point and every trigonal curve with a positive Maroni invariant has non-semistable 2nd Hilbert point [FJ11]. In view of the asymptotic stability of canonically embedded curves (Theorem 1.1), this result suggests that *every* smooth trigonal curve of Maroni invariant 0 has semistable m^{th} Hilbert point for every $m \geq 2$. One also expects that for a generic smooth trigonal curve of positive Maroni invariant already the third Hilbert point is semistable. Indeed, Equation 5.1 shows that the polarization on an open subset of $\big(\overline{H}_{g,1}^3\big)^{ss} /\!\!/ \mathrm{SL}(g)$ is a multiple of

$$\left(\frac{22}{3} + \frac{5}{g}\right) \lambda - \delta.$$

On the other hand, the maximal possible slope for a family of generically smooth trigonal curves of genus g is $36(g+1)/(5g+1)$ by [SF00]. We note that

$$36(g+1)/(5g+1) \leq \left(\frac{22}{3} + \frac{5}{g}\right)$$

whenever $(g-3)(2g-5) \geq 0$. Thus we expect that the 3rd Hilbert point of every canonically embedded smooth trigonal curve of genus $g \geq 4$ is stable.

6. Stability of bicanonical curves

While the major theme of this paper is establishing finite Hilbert semistability of very singular curves, our methods can be used to establish stability of smooth curves as well. In fact, the original motivation for our work is the problem of stability of low degree Hilbert points of smooth bicanonical curves.

Conjecture 6.1 (I. Morrison)**.** A smooth bicanonical curve of genus $g \geq 3$ has stable m^{th} Hilbert point whenever $(g, m) \neq (3, 2)$.

This problem was implicitly stated by Morrison [Mor09] in the wider context of GIT approaches to the log minimal model program for $\overline{M}_g$. In fact, it follows from the conjectural description, due to Hassett and Hyeon, of the second flip of $\overline{M}_g$ as the GIT quotient of

the variety of 6th Hilbert points of bicanonical genus g curves that almost all bicanonically embedded Deligne-Mumford stable curves should have stable m^{th} Hilbert points for every $m \geq 6$ [Mor09, Section 7.5].

Here, we make a step toward Conjecture 6.1 by establishing the following result.

Theorem 6.2 (Stability of generic bicanonical curves). *A generic bicanonically embedded smooth curve of genus $g \geq 3$ has stable m^{th} Hilbert point for every $m \geq 3$. In addition, a generic bicanonically embedded smooth curve of genus $g \geq 4$ has semistable 2nd Hilbert point.*

Our proof of Theorem 6.2 begins with the original idea of Morrison and Swinarski [MS11] in that we also consider the *Wiman hyperelliptic curves* and apply Kempf's instability results [Kem78]. Our strategy is however different in that instead of using symbolic computations with the ideal of the Wiman curve as in [MS11], we exploit the high degree of symmetry of the Wiman curve, together with the fact that it is defined by a single equation, to construct monomial bases by hand. We establish stability of the Wiman curve in Theorem 6.6, which immediately implies Theorem 6.2 by openness of semistability.

6.1. **Wiman curves.** Recall that a genus g curve C is a *Wiman curve* if it is defined by the equation

$$(6.1) \qquad\qquad\qquad w^2 = z^{2g+1} + 1.$$

By [MS11, Section 6], we have

$$(6.2) \qquad \mathrm{H}^0\big(C, K_C^2\big) = \mathbb{C}\left\langle z^i \frac{(dz)^2}{w^2} \right\rangle_{0 \leq i \leq 2g-2} \bigoplus \mathbb{C}\left\langle z^j w \frac{(dz)^2}{w^2} \right\rangle_{0 \leq j \leq g-3}.$$

Since C is a smooth curve, $|K_C^2|$ defines a closed embedding $C \hookrightarrow \mathbb{P}^{3g-4}$ for $g \geq 3$. From now on, we let $\mathcal{O}_C(1) = K_C^2$. When discussing global sections of $\mathrm{H}^0\big(C, \mathcal{O}_C(m)\big) = \mathrm{H}^0\big(C, K_C^{2m}\big)$, we simply write $f(z, w)$ to denote an element $f(z, w)(dz)^{2m}/w^{2m}$. We also fix once and for all a distinguished basis of $\mathrm{H}^0\big(C, \mathcal{O}_C(1)\big)$ given by the following $3g - 3$ functions:

$$x_i := z^i, \quad 0 \leq i \leq 2g - 2,$$
$$y_j := z^j w, \quad 0 \leq j \leq g - 3.$$

For $m \geq 1$ and $k \leq m$, a monomial of the form $\prod_{a=1}^{k} x_{i_a} \prod_{b=1}^{m-k} y_{j_b}$ will be called a $(k, m-k)$-*monomial*.

The space of $(k, m-k)$-monomials in $\mathrm{Sym}^m \mathrm{H}^0\big(C, \mathcal{O}_C(1)\big)$ maps injectively into $\mathrm{H}^0\big(C, \mathcal{O}_C(m)\big)$ and we denote its image by $W(k, m-k)$. We note that

$$W(k, m-k) = \bigoplus_{d=0}^{(2g-2)k+(g-3)(m-k)} \mathbb{C}\left\langle z^d w^{m-k} \right\rangle.$$

For every $k \leq m - 2$, Equation (6.1) gives rise to an injective linear map

$$r \colon W(k, m-k) \to W(k+2, m-k-2),$$

defined by $r(z^d w^{m-k}) = z^d(z^{2g+1} + 1)w^{m-k-2}$, that realizes $W(k, m-k)$ as the subspace of $W(k+2, m-k-2)$. We record that

$$\dim_{\mathbb{C}} W(k+2, m-k-2)/r\big(W(k, m-k)\big) = 2g + 2,$$

and that there are isomorphisms

$$\text{(6.3)} \qquad \mathrm{H}^0\big(C, \mathcal{O}_C(m)\big) \simeq W(m,0) \oplus W(m-1,1),$$

$$\text{(6.4)} \qquad \mathrm{H}^0\big(C, \mathcal{O}_C(m)\big) \simeq \bigoplus_{k=0}^{m} W(k, m-k)/r\big(W(k-2, m-k+2)\big).$$

Definition 6.3. If $V \subset \mathrm{H}^0\big(C, \mathcal{O}_C(m)\big)$ is a linear subspace, a monomial basis of V composed of $(k, m-k)$-monomials is called a $(k, m-k)$-*monomial basis*.

Lemma 6.4. *The mth Hilbert point of $C \hookrightarrow \mathbb{P}\mathrm{H}^0\big(C, K_C^2\big)$ is well-defined.*

Proof. We need to show that $\mathrm{Sym}^m \mathrm{H}^0\big(C, \mathcal{O}_C(1)\big) \to \mathrm{H}^0\big(C, \mathcal{O}_C(m)\big)$ is surjective. This follows immediately from the identification $\mathrm{H}^0\big(C, \mathcal{O}_C(m)\big) \simeq W(m,0) \oplus W(m-1,1)$. $\qquad \square$

We recall that $\mathrm{Aut}(C) \simeq \mu_{4g+2}$, the cyclic group of order $4g+2$ [Wim95]. The action of the generator is given by

$$\zeta \cdot z = \zeta^2 z, \qquad \zeta \cdot w = \zeta^{2g+1} w.$$

We immediately obtain the following observation.

Lemma 6.5. $\mathrm{H}^0\big(C, K_C^2\big)$ *is a multiplicity-free representation of* $\mathrm{Aut}(C) \simeq \mu_{4g+2}$ *and the basis* $\{x_0, \ldots, x_{2g-2}, y_0, \ldots, y_{g-3}\}$ *is compatible with the irreducible decomposition of* $\mathrm{H}^0\big(C, K_C^2\big)$.

Proof. Consulting Equation (6.2), we see that the weights of the μ_{4g+2}-action on the listed generators are $2i - 4g + 2$, where $0 \leq i \leq 2g - 2$, and $2j - 2g + 3$, where $0 \leq j \leq g - 3$. $\quad \square$

Theorem 6.6. *The bicanonically embedded Wiman curve* $C \subset \mathbb{P}\mathrm{H}^0\big(C, K_C^2\big)$ *has stable m^{th} Hilbert point for every $m \geq 3$ if $g \geq 3$, and has semistable 2nd Hilbert point if $g \geq 4$.*

Proof of Theorem 6.6. Lemma 6.5 and Proposition 2.4 imply that it suffices to check stability of C with respect to one-parameter subgroups acting diagonally on the distinguished basis $\{x_0, \ldots, x_{2g-2}, y_0, \ldots, y_{g-3}\}$ of $\mathrm{H}^0\big(C, \mathcal{O}_C(1)\big) = \mathrm{H}^0\big(C, K_C^2\big)$. Suppose $\rho \colon \mathbb{G}_m \to \mathrm{SL}(3g - 3)$ is a one-parameter subgroup acting diagonally on this basis. We need to show that there is a monomial basis of $\mathrm{H}^0\big(C, \mathcal{O}_C(m)\big)$ whose ρ-weight is negative if $m \geq 3$ (resp., non-positive if $m = 2$). We do this in Corollary 6.13 for $m = 2$ and Corollary 6.18 for $m \geq 3$. $\qquad \square$

Let $\{\lambda_i\}_{i=0}^{2g-2}$ be the weights with which ρ acts on $\{x_i\}_{i=0}^{2g-2}$ and let $\{\nu_j\}_{j=0}^{g-3}$ be the weights with which ρ acts on $\{y_j\}_{j=0}^{g-3}$. We also set $\Lambda := \sum_{i=0}^{2g-2} \lambda_i$ and $N := \sum_{j=0}^{g-3} \nu_j$. Note that $\Lambda + N = 0$.

Before proceeding to the construction of the requisite monomial bases, we introduce additional terminology. A multiset $\mathbb{S} = \{\mathcal{B}_1, \ldots, \mathcal{B}_s\}$ of (monomial) bases of a subspace $V \subset \mathrm{H}^0\big(C, \mathcal{O}_C(m)\big)$ will be called a *(monomial) multibasis* of V. If $\mathbb{S} = \{\mathcal{B}_k\}_{k=1}^s$ and $\mathbb{T} = \{\mathcal{R}_\ell\}_{\ell=1}^t$, we will write $\mathbb{S} \cup \mathbb{T}$ to denote their concatenation. We will simply write $d \cdot \mathbb{S}$ to denote $\cup_{r=1}^d \mathbb{S}$.

If ρ is a one-parameter subgroup of $\mathrm{SL}(3g - 3)$, we define the ρ-weight of $\mathbb{S} = \{\mathcal{B}_k\}_{k=1}^s$ to be

$$w_\rho(\mathbb{S}) := \frac{1}{s} \sum_{k=1}^{s} w_\rho(\mathcal{B}_k).$$

Our motivation for considering multibases comes from an elementary observation that existence of a monomial multibasis of non-positive (negative) ρ-weight implies existence of

a monomial basis of non-positive (negative) ρ-weight. Multibases have the following useful property: If $\mathbb{S}_1 = \{\mathcal{B}_k\}_{k=1}^s$ and $\mathbb{S}_2 = \{\mathcal{R}_\ell\}_{\ell=1}^t$ are multibases of subspaces $V_1, V_2 \subset \mathrm{H}^0\big(C, \mathcal{O}_C(m)\big)$ and $V_1 \cap V_2 = \{0\}$, then we can form the multibasis

$$\mathbb{S}_1 + \mathbb{S}_2 := \{\mathcal{B}_k \cup \mathcal{R}_\ell\}_{1 \le k \le s, \ 1 \le \ell \le t}$$

of $V_1 + V_2$. Evidently, $w_\rho(\mathbb{S}_1 + \mathbb{S}_2) = w_\rho(\mathbb{S}_1) + w_\rho(\mathbb{S}_2)$.

We say that a monomial multibasis $\mathbb{S}$ is X-balanced if the variables $\{x_i\}_{i=0}^{2g-2}$ occur the same number of times in $\mathbb{S}$. Similarly, we define Y-balanced monomial multibases. Finally, $\mathbb{S}$ will be called *balanced* if it is both X- and Y-balanced. The ρ-weight of a balanced monomial multibasis is a linear combination of Λ and N.

6.2. **Key combinatorial lemmas.**

Lemma 6.7. *Suppose $x_0, \ldots, x_n$, $y_0, \ldots, y_m$ are weighted variables such that $\deg x_i = i$ for $0 \le i \le n$, and $\deg y_j = j$ for $0 \le j \le m$. Then there exists a multiset of quadratic monomials $S = \{x_i y_j\}_{(i,j) \in I}$ satisfying the following conditions:*

(1) *Every degree in the range $[0, n+m]$ occurs $|S|/(n+m+1)$ times in S.*
(2) *Each variable x_i occurs $|S|/(n+1)$ times in S.*
(3) *Each variable y_j occurs $|S|/(m+1)$ times in S.*

Proof. Let $c_{ij} = \binom{i+j}{i}\binom{n+m-i-j}{n-i}$. Then c_{ij}'s satisfy the following:

(i) $\sum\limits_{i+j=d} c_{ij}$ is the same for all d in the range $[0, n+m]$.
(ii) $\sum\limits_{j=0}^{m} c_{ij}$ is the same for all $0 \le i \le n$.
(iii) $\sum\limits_{i=0}^{n} c_{ij}$ is the same for all $0 \le j \le m$.

The multiset S in which the monomial $x_i y_j$ occurs c_{ij} times satisfies all requisite conditions. $\square$

Using preceding lemmas, we prove several results that enable our proof of Theorem 6.6.

Proposition 6.8. *Let $x_i := z^i$ for $0 \le i \le n$. For every $0 \le k \le n$, there exists an X-balanced quadratic monomial multibasis $\mathbb{H}_k^n$ of $Z_k := \mathrm{span}\{z^i \ : \ k \le i \le 2n-k\}$.*

Proof. To keep track of the number of appearances of variables x_i's in multibases, we assume that a one-parameter subgroup $\rho\colon \mathbb{G}_m \to \mathrm{GL}(n+1)$ acts on $\{x_i\}_{i=0}^n$ with weights $\{\lambda_i\}_{i=0}^n$. If $\mathbb{S}$ is a fixed multibasis of Z_k, then $w_\rho(\mathbb{S})$ is a linear function in λ_i's. Denote $\Lambda := \sum_{i=0}^n \lambda_i$. Evidently, $\mathbb{S}$ is X-balanced if and only if $w_\rho(\mathbb{S}) = \frac{2(2n-2k+1)}{n+1}\Lambda$ for every ρ.

We proceed by descending induction on k. If $k = n$, then $\mathbb{H}_n^n := \{x_i x_{n-i}\}_{i=0}^n$ is an X-balanced quadratic multibasis of $Z_n = \mathbb{C}\langle z^n \rangle$.

Suppose now $k \le n - 1$. Consider the following monomial bases of Z_k:

$$\mathcal{B}^- := \{x_i\, x_{k+i}, \ x_{i+1}\, x_{k+i} \ : \ 0 \le i \le n-k-1\} \cup \{x_{n-k}\, x_n\},$$
$$\mathcal{B}^+ := \{x_0\, x_i \ : k \le i \le n\} \cup \{x_n\, x_i \ : \ 1 \le i \le n-k\}.$$

Their weights are

$$w_\rho(\mathcal{B}^-) = \lambda_0 + 2\sum_{i=1}^{n-k} \lambda_i + 2\sum_{i=k}^{n-1} \lambda_i + \lambda_n, \qquad w_\rho(\mathcal{B}^+) = (n-k)(\lambda_0 + \lambda_n) + \sum_{i=k}^{n} \lambda_i + \sum_{i=0}^{n-k} \lambda_i.$$

If $k = 0$, then $\mathbb{H}_0^n := n \cdot \mathcal{B}^- \cup \mathcal{B}^+$ is an X-balanced monomial basis of Z_0. If $k \geq 1$, then let $\mathbb{H}_{k+1}^n$ be a balanced monomial multibasis of Z_{k+1}, which exists by the induction assumption. Let $\mathbb{T}_0 := \mathbb{H}_{k+1}^n + \{x_i x_{k-i} : 1 \leq i \leq k-1\} + \{x_{n-i} x_{n-k+i} : 1 \leq i \leq k-1\}$ be a multibasis of $Z_k = Z_{k+1} + \mathbb{C}\langle z^k \rangle + \mathbb{C}\langle z^{n-k} \rangle$. Then

$$w_\rho(\mathbb{T}_0) = 2 \frac{(2n - 2k - 1)}{n+1} \Lambda + \frac{2}{k-1} \left(\sum_{i=1}^{k-1} \lambda_i + \sum_{i=n-k+1}^{n-1} \lambda_i \right).$$

It follows that the weight of $\mathbb{T}^- := (k-1) \cdot \mathbb{T}_0 \cup \mathcal{B}^-$ is

$$\frac{1}{k} \left(\frac{2(k-1)(2n-2k-1)}{(n+1)} \Lambda + 4\Lambda - 3(\lambda_0 + \lambda_n) \right)$$

and the weight of $\mathbb{T}^+ := (k-1) \cdot \mathbb{T}_0 \cup 2 \cdot \mathcal{B}^+$ is

$$\frac{1}{k+1} \left(\frac{2(k-1)(2n-2k-1)}{(n+1)} \Lambda + 4\Lambda + (2n - 2k - 2)(\lambda_0 + \lambda_n) \right).$$

It follows that the multibasis $\mathbb{H}_k^n := k(2n - 2k - 2) \cdot \mathbb{T}^- \cup 3(k+1) \cdot \mathbb{T}^+$ is a well-defined X-balanced monomial multibasis of Z_k. $\qquad\square$

Remark 6.9. The statement of Proposition 6.8 for $k = 0$ is equivalent to semistability of the 2nd Hilbert point of a rational normal curve of degree n, proved by Kempf in [Kem78, Corollary 5.3]. A geometric interpretation of the remaining cases is more elusive.

Proposition 6.10. *There exists a balanced $(k, m - k)$-monomial multibasis $\mathbb{S}(k, m - k)$ of the space $W(k, m - k) \subset \mathrm{H}^0\big(C, \mathcal{O}_C(m)\big)$.*

Proof. We proceed by induction on m. The base case is $m = 1$. Here, we can even find a balanced basis: If $k = 0$, then $\{y_0, \ldots, y_{g-3}\}$ is a balanced $(0, 1)$-monomial basis of $W(0, 1)$; if $k = 1$, then $\{x_0, \ldots, x_{2g-3}\}$ is a balanced $(1, 0)$-monomial basis of $W(1, 0)$.

Suppose now that $m \geq 2$ and $k \geq 1$. Then $\mathbb{S}(k - 1, m - k)$ exists by the induction assumption. Write $\mathbb{S}(k - 1, m - k) = \{\mathcal{B}_\ell\}_{\ell=1}^r$, where each $\mathcal{B}_\ell = \{e_d^\ell\}_{d=0}^{(k-1)(2g-2)+(m-k)(g-3)}$ is a $(k-1, m-k)$-monomial basis of $W(k-1, m-k)$, and where we choose the indexing so that the monomial e_d^ℓ maps to $z^d w^{m-k}$ in $\mathrm{H}^0\big(C, \mathcal{O}_C(m)\big)$. Next, let $\deg(e_d^\ell) = d$ and $\deg(x_i) = i$, so that the degree corresponds to the power of z occurring in a monomial. Consider the multiset $S_\ell = \{x_i e_d^\ell\}_{(i,d) \in I}$ satisfying Lemma 6.7:

(1) If we write $x_i e_d^\ell = z^{d+i} w^{m-k}$, then each power of z occurs the same number of times.
(2) Each index $0 \leq i \leq 2g - 2$ occurs the same number of times in S_ℓ.
(3) Each index $0 \leq d \leq (k-1)(2g-2) + (m-k)(g-3)$ occurs the same number of times in S_ℓ.

Condition (1) implies that we can arrange the elements of S_ℓ into a $(k, m - k)$-monomial multibasis $\mathbb{T}_\ell$ of $W(k, m - k)$. Next, we set $\mathbb{S}(k, m - k) := \cup_{\ell=1}^r \mathbb{T}_\ell$. Then conditions (2–3) and the assumption that $\mathbb{S}(k - 1, m - k)$ is balanced imply that $\mathbb{S}(k, m - k)$ is a balanced $(k, m - k)$-monomial multibasis of $W(k, m - k)$.

If $k = 0$, then an analogous argument, with $\{x_i\}_{i=0}^{2g-2}$ replaced by $\{y_j\}_{j=0}^{g-3}$, constructs $\mathbb{S}(0, m)$ from $\mathbb{S}(0, m - 1)$. $\qquad\square$

Next, we record an application of the preceding combinatorial lemmas, which will be used in the proof of semistability of the 2nd Hilbert point of the Wiman curve.

Example 6.11. Let $g \geq 3$. Consider the $(2g+2)$-dimensional linear space

$$V := \operatorname{span}\left\{z^i : 0 \leq i \leq 4g-4\right\} / \operatorname{span}\left\{z^i + z^{2g+1+i} : 0 \leq i \leq 2g-6\right\}.$$

We construct an X-balanced $(2,0)$-monomial multibasis of V in variables $\{x_i = z^i\}_{i=0}^{2g-2}$ as follows: Let $\mathbb{H}_{g-3}^{2g-3}$ be the balanced $(2,0)$-monomial multibasis of $\operatorname{span}\{x^i : g-3 \leq i \leq 3g-3\}$ in variables $\{x_i\}_{i=0}^{2g-3}$, which exists by Proposition 6.8. Set $\mathbb{T}_1 := \mathbb{H}_{g-3}^{2g-3} + \{x_{2g-2}^2\}$. Then $\mathbb{T}_1$ is a multibasis of V of weight

$$w_\rho(\mathbb{T}_1) = \frac{2(2g+1)}{2g-2} \sum_{i=0}^{2g-3} \lambda_i + 2\lambda_{2g-2}.$$

Let $\mathbb{H}_{g-2}^{2g-2}$ be the balanced $(2,0)$-monomial multibasis of $\operatorname{span}\{x^i : g-2 \leq i \leq 3g-2\}$ in variables $\{x_i\}_{i=0}^{2g-2}$, which exists by Proposition 6.8. Set $\mathbb{T}_2 := \mathbb{H}_{g-2}^{2g-2} + \{x_{2g-2}^2\}$. Then $\mathbb{T}_2$ is a multibasis of V of weight

$$w_\rho(\mathbb{T}_2) = \frac{2(2g+1)}{2g-1} \sum_{i=0}^{2g-2} \lambda_i + 2\lambda_{2g-2} = \frac{2(2g+1)}{2g-1} \sum_{i=0}^{2g-3} \lambda_i + \frac{8g}{2g-1}\lambda_{2g-2}.$$

Evidently, a suitable combination of $\mathbb{T}_1$ and $\mathbb{T}_2$ gives an X-balanced multibasis of V of weight $\frac{2(2g+2)}{2g-1} \sum_{i=0}^{2g-2} \lambda_i$.

6.3. Monomial multibases and stability.

The monomial (multi)bases of $\mathrm{H}^0\big(C, \mathcal{O}_C(m)\big)$ that we use will be of the following two types.

(1) A *Type I basis* consists of:
- a $(m,0)$-monomial basis of $W(m,0)$; that is, of $(2g-2)m+1$ linearly independent degree m monomials in the variables x_i's.
- a $(m-1,1)$-monomial basis of $W(m-1,1)$; that is, of $(2g-2)(m-1)+g-2$ linearly independent monomials that are products of a degree $m-1$ monomial in the variables x_i's and a y_j term.

That a set of such monomials is a basis of $\mathrm{H}^0\big(C, \mathcal{O}_C(m)\big)$ follows from Equation (6.3).

A *Type I multibasis* is a multibasis whose every element is a Type I basis.

(2) A *Type II basis* consists of:
- a $(0,m)$-monomial basis of $W(0,m)$,
- a $(1,m-1)$-monomial basis of $W(1,m-1)$,
- For $2 \leq k \leq m$, a $(k,m-k)$-monomial basis of $W(k,m-k)/r\big(W(k-2,m-k+2)\big)$.

That a set of such monomials is a basis follows from Equation (6.4).

A *Type II multibasis* is a multibasis whose every element is a Type II basis.

We pause for a moment to explain these definitions in the case of $m = 2$.

(1) A *Type I basis* of $\mathrm{H}^0\big(C, \mathcal{O}_C(2)\big)$ consists of $4g-3$ quadratic $(2,0)$-monomials spanning $W(2,0) = \operatorname{span}\{1,\ldots,z^{4g-4}\} \subset \mathrm{H}^0\big(C, \mathcal{O}_C(2)\big)$ and of $3g-4$ quadratic $(1,1)$-monomials spanning $W(1,1) = \operatorname{span}\{w, zw, \ldots, z^{3g-5}w\} \subset \mathrm{H}^0\big(C, \mathcal{O}_C(2)\big)$.

(2) A *Type II basis* of $\mathrm{H}^0\big(C, \mathcal{O}_C(2)\big)$ consists of $2g-5$ quadratic $(0,2)$-monomials spanning $W(0,2) = \operatorname{span}\{w^2, \ldots, z^{2g-6}w^2\} \subset \mathrm{H}^0\big(C, \mathcal{O}_C(2)\big)$; of $3g-4$ quadratic $(1,1)$-monomials spanning $W(1,1)$; and of $2g+2$ quadratic $(2,0)$-monomials that are linearly independent modulo $r\big(W(0,2)\big)$, that is, $2g+2$ monomials with exactly one from each pair

$$(z^d, z^{d+2g+1}), \quad 0 \leq d \leq 2g-6,$$

and with the remaining 7 being $z^{2g-5}, z^{2g-4}, z^{2g-3}, z^{2g-2}, z^{2g-1}, z^{2g}$, and z^{4g-4}.

Before proceeding with our construction of monomial bases of both types for every m, we illustrate our approach by considering the case of $m = 2$, thus establishing semistability of the 2nd Hilbert point of the bicanonically embedded Wiman curve for every $g \geq 4$.

Proposition 6.12. *There exist balanced Type I and Type II monomial multibases $\mathbb{B}_1$ and $\mathbb{B}_2$ of $\mathrm{H}^0(C, \mathcal{O}_C(2))$. Their weights are, respectively,*

$$w_\rho(\mathbb{B}_1) = \frac{11g - 10}{2g - 1}\Lambda + \frac{3g - 4}{g - 2}N, \qquad and \qquad w_\rho(\mathbb{B}_2) = \frac{7g}{2g - 1}\Lambda + 7N.$$

Proof. The existence of a balanced Type I multibasis follows from Proposition 6.10. The existence of a balanced Type II multibasis follows from Proposition 6.10 and Example 6.11. $\square$

Corollary 6.13. *The 2nd Hilbert point of C is semistable for $g \geq 4$.*

Proof. For $g \geq 4$, we have $\frac{11g-10}{2g-1} > \frac{3g-4}{g-2}$ and $\frac{7g}{2g-1} < 7$. Since $\Lambda + N = 0$, some positive linear combination of $w_\rho(\mathbb{B}_1)$ and $w_\rho(\mathbb{B}_2)$ is 0 for every ρ acting diagonally on the distinguished basis $\{x_0, \ldots, x_{2g-2}, y_0, \ldots, y_{g-3}\}$ of $\mathrm{H}^0(C, \mathcal{O}_C(1))$. Semistability now follows from Lemma 2.5. $\square$

6.3.1. *Construction of a balanced Type I multibasis.* A Type I basis is obtained by concatenating a $(m, 0)$-monomial multibasis of $W(m, 0)$ and a $(m - 1, 1)$-monomial multibasis of $W(m - 1, 1)$. By Proposition 6.10, there exists a balanced $(m, 0)$-monomial multibasis of $W(m, 0)$, whose weight is

$$m\frac{(2m(g - 1) + 1)}{(2g - 1)}\Lambda,$$

and a balanced $(m - 1, 1)$-monomial multibasis of $W(m - 1, 1)$, whose weight is

$$(m - 1)\frac{(2g - 2)(m - 1) + g - 2}{(2g - 1)}\Lambda + \frac{((2g - 2)(m - 1) + (g - 2))}{(g - 2)}N.$$

Summarizing, we obtain the following result.

Proposition 6.14. *There is a Type I multibasis of $\mathrm{H}^0(C, \mathcal{O}_C(m))$ of weight*

(6.5) $$\frac{((4g - 4)m^2 - (3g - 3)m + g)}{(2g - 1)}\Lambda + \frac{((2g - 2)m - g)}{(g - 2)}N.$$

Remark 6.15. We note that in Equation (6.5), the coefficient of Λ is greater than the coefficient of N for all values of $g \geq 3$ and all values of $m \geq 2$, with the sole exception of $(g, m) = (3, 2)$ for which we get $\frac{23}{5}\Lambda + 5N$. It is easy to see that in this exceptional case, the 2nd Hilbert point of the bicanonically embedded genus 3 Wiman curve is, in fact, *non-semistable*.

6.3.2. *Construction of a Type II basis.* In this section we construct a (balanced) Type II multibasis of $\mathrm{H}^0(C, \mathcal{O}_C(m))$. We begin with a preliminary result.

Lemma 6.16. (a) *Suppose $k \geq 3$. Then for $0 \leq i \leq 2g - 2$ and $0 \leq \epsilon \ll 1$, there is a $(k, m - k)$-monomial multibasis of $W(k, m - k)/r(W(k - 2, m - k + 2))$ whose weight is*

(6.6) $$\left(k\frac{(2g + 2)}{(2g - 1)} - \frac{\epsilon}{2g - 1}\right)\Lambda + (m - k)\frac{(2g + 2)}{(g - 2)}N + \epsilon\lambda_i.$$

(b) *Suppose $m - k \geq 1$. Then for $0 \leq j \leq g-3$ and $0 \leq \delta \ll 1$, there is a $(k, m-k)$-monomial multibasis of $W(k, m-k)/r\big(W(k-2, m-k+2)\big)$ whose weight is*

$$(6.7) \qquad k\frac{(2g+2)}{(2g-1)}\Lambda + \left((m-k)\frac{(2g+2)}{(g-2)} - \frac{\delta}{g-2}\right)N + \delta\nu_j.$$

Proof. We identify $W(k, m-k)/r\big(W(k-2, m-k+2)\big)$ with the vector space

$$\mathrm{span}\{z^d w^{m-k} \ : \ 0 \leq d \leq (2g-2)k + (g-3)(m-k)\}$$

modulo the relations

$$z^{d+2g+1}w^{m-k} + z^d w^{m-k} = 0, \quad 0 \leq d \leq (2g-2)k + (g-3)(m-k) - (2g+2).$$

We define a set of $(k, m-k)$-monomials that form a basis $W(k, m-k)/r\big(W(k-2, m-k+2)\big)$, and which depends on three parameters: $i \in \{0, \ldots, 2g-2\}$, $j \in \{0, \ldots, g-3\}$, and $u \in \{0, 1\}$:

$$(6.8) \qquad \mathbb{S}_u(i, j) := \{x_{2g-2}^k y_{g-3}^{m-k}\} \cup \left(x_i^{k-2} y_j^{m-k} \times \mathbb{T}_u\right),$$

where

- $\mathbb{T}_0 := \mathbb{H}_{g-3}^{2g-3}$ is the *quadratic* monomial multibasis of $\mathrm{span}\{z^i \ : \ g-3 \leq i \leq 3g-3\}$ in variables $\{x_0, \ldots, x_{2g-3}\}$, which exists by Proposition 6.8 and has weight

$$w_\rho(\mathbb{T}_0) = \frac{2(2g+1)}{2g-2}(\Lambda - \lambda_{2g-2}).$$

- $\mathbb{T}_1 := \mathbb{H}_{g-2}^{2g-2}$ is the *quadratic* monomial multibasis of $\mathrm{span}\{z^i \ : \ g-2 \leq i \leq 3g-2\}$ in variables $\{x_0, \ldots, x_{2g-2}\}$, which exists by Proposition 6.8 and has weight

$$w_\rho(\mathbb{T}_1) = \frac{2(2g+1)}{2g-1}\Lambda.$$

Setting $\nu(k, j) := (2g+1)(m-k)\nu_j + (m-k)\nu_{g-3}$, we deduce that

$$(6.9) \qquad w_\rho\big(\mathbb{S}_0(i, j)\big) = \frac{2(2g+1)}{2g-2}(\Lambda - \lambda_{2g-2}) + (2g+1)(k-2)\lambda_i + k\lambda_{2g-2} + \nu(k, j),$$

$$(6.10) \qquad w_\rho\big(\mathbb{S}_1(i, j)\big) = \frac{2(2g+1)}{2g-1}\Lambda + (2g+1)(k-2)\lambda_i + k\lambda_{2g-2} + \nu(k, j).$$

Since $\sum_{i=0}^{2g-3}\lambda_i = \Lambda - \lambda_{2g-2}$, the mutibasis $\mathbb{S}_0 := \cup_{i=0}^{2g-3}\mathbb{S}_0(i, j)$ has weight

$$k\frac{(2g+1)}{2g-2}(\Lambda - \lambda_{2g-2}) + k\lambda_{2g-2} + \nu(k, j).$$

If $a + b + c = 1$, then $\mathbb{S}_1 := a \cdot \mathbb{S}_0 \cup b \cdot \mathbb{S}_1(2g-2, j) \cup c \cdot \mathbb{S}_1(i, j)$ has weight

$$(6.11) \quad \left[ak\frac{2g+1}{2g-2} + 2(b+c)\frac{2g+1}{2g-1}\right]\Lambda + \left[ck + b\big((2g+2)k - 2(2g+1)\big) - \frac{3ak}{2g-2}\right]\lambda_{2g-2}$$
$$+ \left[c(2g+1)(k-2)\right]\lambda_i + \nu(k, j).$$

For any small non-negative c, we can find a and b in $[0, 1]$ satisfying $a + b + c = 1$ and such that the coefficient of λ_{2g-2} in (6.11) equals 0. If we additionally require that $c = 0$, which

then also determines a and b, the Λ coefficient in (6.11) simplifies to $k\dfrac{(2g+2)}{(2g-1)}$. For $c = \epsilon$, it follows that $\mathbb{S}_1$ has weight

$$\left(k\frac{(2g+2)}{(2g-1)} - \frac{\epsilon}{2g-1}\right)\Lambda + \epsilon\lambda_i + \nu(k,j).$$

Recall that $\nu(k,j) = (2g+1)(m-k)\nu_j + (m-k)\nu_{g-3}$. Since $(2g+1)(m-k) > (m-k)$, an averaging argument with ν's, analogous to the one given above for λ's, shows that there exist multibases of weights given by Equations (6.6) and (6.7). $\qquad\square$

Proposition 6.17. *Let $m \geq 3$. For $0 \leq i \leq 2g-2$, $0 \leq j \leq g-3$, and $0 \leq \epsilon, \delta \ll 1$, there exists a Type II multibasis of $\mathrm{H}^0\big(C, \mathcal{O}_C(m)\big)$ of weight $a\Lambda + bN + \epsilon\lambda_i + \delta\nu_j$, where*

$$a = \frac{1}{2g-1}\left((g+1)m^2 + (2g-2)m - g\right) - \frac{\epsilon}{2g-1},$$
$$b = \frac{1}{g-2}\left((3g-5)m^2 - (3g-3)m + g\right) - \frac{\delta}{g-2};$$

in particular $a < b$.

Proof. We begin with a balanced $(0,m)$-monomial multibasis of $W(0,m)$ which exists by Proposition 6.10 and whose weight is

$$(6.12) \qquad \frac{m\big((g-3)m+1\big)}{(g-2)}N.$$

Next, we take a balanced $(1, m-1)$-monomial multibasis of $W(1, m-1)$, which again exists by Proposition 6.10. Its weight is

$$(6.13) \qquad (m-1)\frac{\big((g-3)(m-1)+2g-1\big)}{(g-2)}N + \frac{\big((g-3)(m-1)+2g-1\big)}{(2g-1)}\Lambda.$$

By Lemma 6.16 there exists a multibasis of $W(k, m-k)/r\big(W(k-2, m-k+2)\big)$ of weight

$$\omega_k := k\frac{(2g+2)}{(2g-1)}\Lambda + (m-k)\frac{(2g+2)}{(g-2)}N,$$

for $2 \leq k \leq m$. Moreover, by the same lemma, there exists a $(3, m-3)$-monomial multibasis of $W(3, m-3)$ of weight

$$\omega_3' := \left(3\frac{(2g+2)}{(2g-1)} - \frac{\epsilon}{(2g-1)}\right)\Lambda + (m-3)\frac{(2g+2)}{(g-2)}N + \epsilon\lambda_i,$$

for any small non-negative ϵ and any i, and there also exists a multibasis of $W(2, m-2)$ of weight

$$\omega_2' := 2\frac{(2g+2)}{(2g-1)}\Lambda + \left((m-2)\frac{(2g+2)}{(g-2)} - \frac{\delta}{(g-2)}\right)N + \delta\nu_j,$$

again for any small non-negative δ and any j. Concatenating the above bases, we obtain a Type II multibasis. If we set $\epsilon = \delta = 0$, the weight of the resulting multibasis is

$$\frac{m\big((g-3)m+1\big)}{(g-2)}N + (m-1)\frac{\big((g-3)(m-1)+2g-1\big)}{(g-2)}N + \frac{\big((g-3)(m-1)+2g-1\big)}{(2g-1)}\Lambda$$
$$+\sum_{k=2}^{m}\omega_k = \frac{1}{2g-1}\left((g+1)m^2 + (2g-2)m - g\right)\Lambda + \frac{1}{g-2}\left((3g-5)m^2 - (3g-3)m + g\right)N.$$

The result follows. $\square$

We are now ready to prove Theorem 6.6 in the case of $m \geq 3$.

Corollary 6.18 (Stability of Wiman curves). *The mth Hilbert point of the bicanonically embedded Wiman curve C of genus $g \geq 3$ is stable for every $m \geq 3$.*

Proof. Lemma 6.5 and Proposition 2.4 reduce the verification of stability of C to verifying stability with respect to one-parameter subgroups acting diagonally on the distinguished basis $\{x_0, \ldots, x_{2g-2}, y_0, \ldots, y_{g-3}\}$ of $\mathrm{H}^0\big(C, \mathcal{O}_C(1)\big)$. To prove stability with respect to such one-parameter subgroup ρ, we need to find a monomial basis of $\mathrm{H}^0\big(C, \mathcal{O}_C(m)\big)$ of negative ρ-weight. By taking a suitable linear combination of the Type I monomial multibasis of Proposition 6.14 and the Type II monomial multibasis of Proposition 6.17, we can now construct a monomial multibasis of weight $\epsilon\lambda_i + \delta\nu_j$, where $0 \leq \epsilon, \delta \ll 1$ are arbitrary and indices i, j can be chosen freely. Since at least one of the weights $\{\lambda_0, \ldots, \lambda_{2g-2}, \nu_0, \ldots, \nu_{g-3}\}$ is negative, the claim follows. $\square$

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