EXISTENCE OF MODULI SPACES FOR ALGEBRAIC STACKS

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Abstract. We provide necessary and sufficient conditions for when an algebraic stack admits a good moduli space. This theorem provides a generalization of the Keel–Mori theorem to moduli problems whose objects have positive dimensional automorphism groups. We also prove a semistable reduction theorem for points of algebraic stacks equipped with a Θ-stratification. Using these results we find conditions for the good moduli space to be separated or proper. To illustrate our method, we apply these results to construct proper moduli spaces parameterizing semistable $\mathcal{G}$-bundles on curves and moduli spaces for objects in abelian categories.

Contents

1. Introduction

In the study of moduli problems in algebraic geometry the construction of moduli spaces is a recurring problem. Given a moduli problem, described by an algebraic stack $\mathcal{X}$, the ideal solution would be for $\mathcal{X}$ to be representable by a scheme or an algebraic space. This is never the case when objects parameterized by $\mathcal{X}$ have non-trivial automorphism groups. In this case one hopes for the existence of a universal map to an algebraic space $q\colon \mathcal{X} \to X$ with useful properties.

For algebraic stacks with finite automorphism groups the Keel–Mori theorem [KM97] gave a satisfactory existence result from the intrinsic perspective. It states that if $\mathcal{X}$ is an algebraic stack of finite type over a noetherian base whose inertia stack is finite over $\mathcal{X}$, then there is a *coarse moduli space* $q\colon \mathcal{X} \to X$, which in addition to being a universal map to an algebraic space is bijective on geometric points.

The restriction to the case of finite automorphism groups is not necessary for the construction of moduli spaces using GIT. Furthermore in many examples, such as the moduli of vector bundles or coherent sheaves on a projective variety, one must consider objects with positive dimensional automorphism groups in order to construct moduli spaces which are proper.

In [Alp13], the first author introduced the notion of a *good moduli space* for an algebraic stack $\mathcal{X}$ as an intrinsic formulation of many of the useful properties of the notion of a *good quotient* [Ses72], a specific type of GIT quotient including all GIT quotients in characteristic 0. By definition, a good moduli space is a map $q\colon \mathcal{X} \to X$ to an algebraic space such that the pushforward q_* of quasi-coherent sheaves is exact, and such that the canonical map $\mathcal{O}_X \to q_*(\mathcal{O}_\mathcal{X})$ is an isomorphism. This simple definition leads to many useful properties, including that q is universal for maps to an algebraic space, and that the fibers of q classify orbit-closure equivalence classes of points in $\mathcal{X}$.

Our main result gives necessary and sufficient conditions under which an algebraic stack admits a good moduli space, and can be seen as uniting the main theorem of geometric invariant theory with the intrinsic perspective of the Keel–Mori theorem.

Theorem A (Theorem 4.1, Proposition 3.47, Proposition 3.45, Theorem 5.4)**.** *Let $\mathcal{X}$ be an algebraic stack locally of finite type with affine diagonal over a quasi-separated and locally noetherian algebraic space S. Then $\mathcal{X}$ admits a good moduli space if and only if*

(1) $\mathcal{X}$ is locally linearly reductive (Definition 2.1);
(2) $\mathcal{X}$ is Θ-reductive (Definition 3.10); and
(3) $\mathcal{X}$ has unpunctured inertia (Definition 3.53).

The good moduli space X is separated if and only if $\mathcal{X}$ is S-complete (Definition 3.37), and proper if and only if $\mathcal{X}$ is S-complete and satisfies the existence part of the valuative criterion for properness.

Assume in addition that S is of characteristic 0 and $\mathcal{X}$ is quasi-compact. If $\mathcal{X}$ is S-complete, then (1) and (3) hold automatically. In particular, $\mathcal{X}$ admits a separated good moduli space if and only if $\mathcal{X}$ is Θ-reductive and S-complete.

Let us give an informal explanation of the above conditions. The first condition is that closed points of $\mathcal{X}$ have linearly reductive stabilizers. In the language of geometric invariant theory this would amount to the condition that the automorphism groups of polystable objects are (linearly) reductive. The second condition is the geometric analog of the statement that filtrations by semistable objects extend under specialization. This is formulated in terms of maps from the stack $\Theta := [\mathbb{A}^1/\mathbb{G}_m]$ into $\mathcal{X}$. The third condition is an analog of the condition in the Keel–Mori theorem, it roughly states that the connected components of stabilizer groups extend to closed points. In particular this condition is automatic if all stabilizer groups are connected (which happens for example for moduli of coherent sheaves). In Section 5 we provide several "valuative" criteria, in the sense that they involve only conditions on maps $\operatorname{Spec}(R) \to \mathcal{X}$ where R is a discrete valuation ring (DVR), which are equivalent to unpunctured inertia under the hypotheses (1) and (2) (see Theorem 5.2).

Finally, S-completeness, where the S stands for "Seshadri," is a geometric property that is reminiscent of classical methods of establishing separatedness of moduli spaces. More precisely we introduce a geometric notion of an *elementary modification* (Definition 3.35) which relates two families over a DVR which are isomorphic at the generic point, and S-completeness states that any two families over a DVR which are isomorphic at the generic point differ by an elementary modification. It turns out that S-completeness has many desirable consequences: namely, in characteristic 0, S-completeness implies both conditions (1) and (3) in Theorem A. This fact follows from the more general results of Proposition 3.45 and Theorem 5.4, which are characteristic independent. Ultimately, both S-completeness and Θ-reductivity are local criteria in the sense that each is equivalent to a filling condition for $\mathbb{G}_m$-equivariant families over a suitable punctured regular 2-dimensional scheme.

The condition of linear reductivity is very strong in positive characteristic and it arises here through the recent local structure theorems on algebraic stacks from [AHR15, AHR]. In positive characteristics we would expect that an analogue of Theorem A holds with "good moduli space" replaced with "adequate moduli space" and "locally linearly reductive" replaced with "locally geometrically reductive." The main obstacle to prove such a result is the lack of an analogue of the local structure theorem for such stacks. However, we are careful to prove intermediate results that do not require linear reductivity.

Our second main theorem is an analog of Langton's semistable reduction theorem [Lan75] for moduli of bundles, that works for a large class of algebraic stacks equipped with a notion of stability that induces a Θ-stratification, a geometric analog of the notion of Harder–Narasimhan–Shatz stratifications. As in Langton's theorem, the statement is that if a family of objects parametrized by a DVR specializes to a point that is more unstable than the generic fiber of the family, then one can modify the family along the closed point to get a family that has the same stability properties as the generic fiber. Surprisingly the existence of modifications can be obtained from the local geometry of Θ-stratifications. The formal statement is the following.

Theorem B (Theorem 6.3)**.** *Let $\mathcal{X}$ be an algebraic stack locally of finite type with affine diagonal over a noetherian algebraic space S, and let $\mathcal{S} \hookrightarrow \mathcal{X}$ be a Θ-stratum (Definition 6.1). Let R be a DVR with fraction field K and residue field κ. Let $\xi\colon \operatorname{Spec}(R) \to \mathcal{X}$ be an R-point such that the generic point ξ_K is not mapped to $\mathcal{S}$,* *but the special point ξ_κ is mapped to $\mathcal{S}$:*

$$\begin{array}{ccccc} \mathrm{Spec}(K) & \hookrightarrow & \mathrm{Spec}(R) & \hookleftarrow & \mathrm{Spec}(\kappa) \\ \downarrow{\scriptstyle \xi_K} & & \downarrow{\scriptstyle \xi_R} & & \downarrow{\scriptstyle \xi_\kappa} \\ \mathcal{X} - \mathcal{S} & \overset{j}{\hookrightarrow} & \mathcal{X} & \overset{\iota}{\hookleftarrow} & \mathcal{S}. \end{array}$$

Then there exists an extension $R \to R'$ of DVRs with $K \to K' = \mathrm{Frac}(R')$ finite and an elementary modification (Definition 3.35) ξ' of $\xi|_{R'}$ such that $\xi' \colon \mathrm{Spec}(R') \to \mathcal{X}$ lands in $\mathcal{X} - \mathcal{S}$.

We may apply the above results to the semistable locus $\mathcal{X}^{\mathrm{ss}} \subset \mathcal{X}$ defined by a class $\ell \in H^2(\mathcal{X}; \mathbb{R})$ via the Hilbert–Mumford criterion (see Definition 6.13). As many properties of $\mathcal{X}$ are inherited by the semistable locus, we can provide conditions on $\mathcal{X}$ ensuring that the semistable locus $\mathcal{X}^{\mathrm{ss}}$ admits a separated good moduli space and a further condition ensuring that the good moduli space is proper. To summarize, we have:

Theorem C (Corollary 6.12, Proposition 6.14, Corollary 6.18)**.** *Let $\mathcal{X}$ be an algebraic stack locally of finite type with affine diagonal over a quasi-separated and locally noetherian algebraic space S, and let $\ell \in H^2(\mathcal{X}; \mathbb{R})$ be a class defining a semistable locus $\mathcal{X}^{\mathrm{ss}} \subset \mathcal{X}$ which is part of a well-ordered Θ-stratification of $\mathcal{X}$ compatible with l.*[1] *Then if $\mathcal{X}$ is either Θ-reductive, S-complete, or satisfies the existence part of the valuative criterion for properness, then the same is true for $\mathcal{X}^{\mathrm{ss}}$.*

In particular, if in addition S has characteristic 0, $\mathcal{X} \to S$ is S-complete and Θ-reductive, and $\mathcal{X}^{\mathrm{ss}} \to S$ is quasi-compact, then there exists a good moduli space $\mathcal{X}^{\mathrm{ss}} \to X$ such that X is separated over S (and proper over S if $\mathcal{X} \to S$ satisfies existence part of the valuative criterion for properness).

We expect that in the semistable reduction theorem (Theorem B), weak Θ-strata (that only require canonical filtrations to exist after a purely inseparable extension) should be sufficient and these are available in greater generality in positive characteristic. Similarly, in positive characteristic, we expect that Theorem C holds with "good moduli space" replaced with "adequate moduli space." The main obstruction for these generalizations is a version of the local structure theorem where the embedding of a stratum is replaced with a radicial map.

Applications. To illustrate our results we give some applications that may be of independent interest. First, we use the semistable reduction theorem to give a proof that the Hitchin fibration for semistable G-Higgs bundles is proper if the characteristic of the ground field is not too small (Corollary 6.21). This result is of course expected, but it doesn't seem to appear in the literature.

Second we apply our existence theorem to construct some new good moduli spaces. Namely we construct proper good moduli spaces for semistable $\mathcal{G}$-bundles for a Bruhat–Tits groups scheme $\mathcal{G}$ over a smooth geometrically connected projective curve over a field of characteristic 0, generalizing work of Balaji and Seshadri [BS15] (Theorem 8.1). We also construct proper good moduli spaces for objects in abelian categories in Theorem 7.21 and Theorem 7.25. As a special case, we construct proper moduli spaces of semistable complexes with respect to a Bridgeland

[1] See Proposition 6.14 for the precise compatability condition between l and the Θ-stratification.

stability condition on a smooth projective variety X over a field of characteristic 0. Whereas in these examples the lack of a convenient global quotient description of the corresponding moduli problems seems to pose a serious obstruction to a construction using GIT, the verification of the conditions of our main theorems turns out to be surprisingly simple.

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2. Preliminaries

Throughout we will fix a base S that will be a quasi-separated algebraic space, but of course the most interesting case for most readers will be when $S = \operatorname{Spec}(k)$ is the spectrum of a field.

As our arguments build on the one hand on local structure theorems and on the other hand on notions that came up in the study of notions of stability on algebraic stacks, we briefly recall these results in this section.

2.1. Reminder on local structure theorems for algebraic stacks.

For ease of notation let us introduce the following terminology.

Definition 2.1. An algebraic stack $\mathcal{X}$ with affine stabilizers is *locally linearly reductive* if every point specializes to a closed point and every closed point of $\mathcal{X}$ has a linearly reductive automorphism group.

Note that in the case of a quasi-compact quotient stack $\mathcal{X} = [X/G]$ the closed points correspond to closed orbits of G on X, so in this case the above condition only requires that points contained in closed orbits have a linearly reductive stabilizer. In particular, a locally linearly reductive stack will often have geometric points with non-reductive stabilizers.

Definition 2.2. If $\mathcal{X}$ is an algebraic stack and $x \in |\mathcal{X}|$ is a point with residual gerbe $\mathcal{G}_x$, we call an étale and affine pointed morphism $f\colon (\mathcal{W}, w) \to (\mathcal{X}, x)$ of algebraic stacks a *local quotient presentation around* x if (1) $\mathcal{W} \cong [\operatorname{Spec}(A)/\operatorname{GL}_N]$ for some N and (2) $f|_{f^{-1}(\mathcal{G}_x)}$ is an isomorphism.

The following is the key result on the local structure of locally linearly reductive stacks.

Theorem 2.3. [AHR, Thm. 1.1] *Let S be a quasi-separated algebraic space. Let $\mathcal{X}$ be an algebraic stack locally of finite presentation with affine diagonal over S. If $x \in |\mathcal{X}|$ is a point with image $s \in |S|$ such that the residue field extension $\kappa(x)/\kappa(s)$ is finite and the stabilizer of x is linearly reductive, then there exists a local quotient presentation $f\colon (\mathcal{W}, w) \to (\mathcal{X}, x)$ around x.*

In particular, if in addition $\mathcal{X}$ is locally linearly reductive, then there exist local quotient presentations around any closed point.

Remark 2.4. If S is the spectrum of an algebraically closed field, the above theorem follows from [AHR15, Thm. 1.2]. In this case, one can arrange that there is a local quotient presentation $(\mathcal{W}, w) \to (\mathcal{X}, x)$ with $\mathcal{W} \cong [\mathrm{Spec}(A)/G_x]$, the quotient of an affine scheme by the stabilizer $G_x = \mathrm{Aut}_{\mathcal{X}}(x)$ of x.

Remark 2.5. While GL_N is linearly reductive in characteristic 0, it is not linearly reductive in positive or mixed characteristic. For the same reason, the morphism $[\mathrm{Spec}(A)/\mathrm{GL}_n] \to \mathrm{Spec}(A^{\mathrm{GL}_N})$ will only be an adequate moduli space (and not a good moduli space) in general.

To prove the semistable reduction theorem, we will need a relative version of the above local structure theorem where we fix a subgroup isomorphic to the multiplicative group $\mathbb{G}_m$ of the stabilizer $G_x = \mathrm{Aut}_{\mathcal{X}}(x)$, but do not assume G_x to be linearly reductive. A very general result of this form is the following theorem.

Theorem 2.6. [AHHR] *Let $\mathcal{X}$ be an algebraic stack locally of finite presentation with affine diagonal over a quasi-separated algebraic space S, and let $G \subset \mathrm{GL}_{N,S}$ be a closed subgroup which is linearly reductive over S. If $\mathcal{Y} \subset \mathcal{X}$ is a closed substack, then any representable and smooth (resp. étale) morphism $[Y/G] \to \mathcal{Y}$, with $Y \to S$ affine, extends to a representable and smooth (resp. étale) morphism $[X/G] \to \mathcal{X}$ with $X \to S$ affine, i.e. we have $[X/G] \times_{\mathcal{X}} \mathcal{Y} \cong [Y/G]$.*

Remark 2.7. As the proof of the result has not yet appeared let us recall a special case, which will be sufficient for us if $S = \mathrm{Spec}(k)$ is the spectrum of a field and all stabilizer groups of $\mathcal{X}$ are smooth (a condition that is automatic in characteristic 0). Namely, if $S = \mathrm{Spec}(k)$ is the spectrum of an algebraically closed field, $x \in \mathcal{X}(k)$ with smooth automorphism group G_x, $\mathcal{Y} = B_k G_x \subset \mathcal{X}$ is the canonical inclusion, $\mathbb{G}_m^r \subset G_x$ is a subgroup and $Y = [\mathrm{Spec}(k)/\mathbb{G}_m^r]$, the above result is a special case of [AHR15, Thm. 1.2].

2.2. **Reminder on mapping stacks and filtrations.** As in [Hal14] we will denote by $\Theta := [\mathbb{A}^1/\mathbb{G}_m]$ the quotient stack defined by the standard contracting action of the multiplicative group on the affine line and by $B\mathbb{G}_m = [\mathrm{pt}/\mathbb{G}_m]$, the classifying stack of the group $\mathbb{G}_m$. Both stacks are defined over $\mathrm{Spec}(\mathbb{Z})$ and therefore pull back to any base S. Note that since $\mathbb{G}_m$ is a linearly reductive group, the structure morphisms $\Theta \to \mathrm{Spec}(\mathbb{Z})$ and $B\mathbb{G}_m \to \mathrm{Spec}(\mathbb{Z})$ are good moduli spaces.

Maps from Θ into a stack are the key ingredient to define stability notions on algebraic stacks [Hal14, Hei17] and we need to recall some of their properties.

By definition for any stack $\mathcal{X}$ and point $\mathrm{Spec}(k) \to S$ a map $B\mathbb{G}_{m,k} \to \mathcal{X}$ is a point $x \in \mathcal{X}(k)$ together with a cocharacter $\mathbb{G}_{m,k} \to \mathrm{Aut}_{\mathcal{X}}(x)$. As the action of $\mathbb{G}_m$ on a vector space is the same as a grading on the vector space, we often think of a morphism $B\mathbb{G}_m \to \mathcal{X}$ as a point of $\mathcal{X}$ equipped with a grading.

Similarly, a vector bundle on $\Theta = [\mathbb{A}^1/\mathbb{G}_m]$ is the same as a $\mathbb{G}_m$ equivariant bundle on $\mathbb{A}^1$ and these are the same as vector spaces equipped with a filtration. So we think of morphisms $f\colon \Theta_k \to \mathcal{X}$ as an object of $x_1 \in \mathcal{X}(k)$ (the object $f(1)$) together with a filtration of x_1 and as $f(0) = x_0$ as the associated graded object.

In examples it is often easy to see that once one has found that some moduli problem is described by an algebraic stack, the stacks of filtered or graded objects are again algebraic. This turns out to be a general phenomenon, which we recall next. For algebraic stacks $\mathcal{X}$ and $\mathcal{Y}$ over S, we denote by

$$\underline{\mathrm{Map}}_S(\mathcal{Y}, \mathcal{X})$$

the stack over S parameterizing S-morphisms $\mathcal{Y} \to \mathcal{X}$. If $\mathcal{Y}$ is defined over $\mathrm{Spec}(\mathbb{Z})$, we will use the convention that $\underline{\mathrm{Map}}_S(\mathcal{Y}, \mathcal{X})$ denotes the mapping stack $\underline{\mathrm{Map}}_S(\mathcal{Y} \times S, \mathcal{X})$.

That these mapping stacks are again algebraic if $\mathcal{Y} = \Theta$ or $\mathcal{Y} = B\mathbb{G}_m$ for quite general $\mathcal{X}$ follows from a general result established in [AHR] and [HLP14, Thm. 1.6]: if $\mathcal{X}$ is locally of finite presentation and quasi-separated over an algebraic space S with affine stabilizers, and $\mathcal{Y}$ is of finite presentation and with affine diagonal over S such that $\mathcal{Y} \to S$ is flat and a good moduli space, then $\underline{\mathrm{Map}}_S(\mathcal{Y}, \mathcal{X})$ is an algebraic stack locally of finite presentation over S, with affine stabilizers and quasi-separated diagonal. Moreover, if $\mathcal{X} \to S$ has affine (resp. quasi-affine, resp. separated) diagonal, then so does $\underline{\mathrm{Map}}_S(\mathcal{Y}, \mathcal{X})$. The *stack of filtrations* $\underline{\mathrm{Map}}_S(\Theta, \mathcal{X})$ is denoted $\mathrm{Filt}(\mathcal{X})$ in [Hal14], and the *stack of graded objects* $\underline{\mathrm{Map}}_S(B\mathbb{G}_m, \mathcal{X})$ is denoted $\mathrm{Grad}(\mathcal{X})$.

2.3. **The example of quotient stacks.** To compute examples we recall that stacks of filtrations and graded objects have a concrete description for quotient stacks. If $\mathcal{X} = [X/G]$ is a quotient stack locally of finite type over a field k, where G is a smooth algebraic group acting on a quasi-separated algebraic space X, these mapping stacks have a classical interpretation [Hal14, Thm. 1.37]. To state this recall that given $\lambda\colon \mathbb{G}_m \to G$, one defines

$$L_\lambda = \{l \in G \,|\, l = \lambda(t) l \lambda(t)^{-1}\ \forall t\} \quad \text{and} \quad P_\lambda^+ = \{p \in G \,|\, \lim_{t\to 0} \lambda(t) p \lambda(t)^{-1} \text{ exists}\}.$$

If G is geometrically reductive, then $P_\lambda^+ \subset G$ is a parabolic subgroup. There is a surjective homomorphism $P_\lambda^+ \to L_\lambda$, defined by $p \mapsto \lim_{t\to 0} \lambda(t) p \lambda(t)^{-1}$.

Similarly, one defines the functors:

$$X_\lambda^0 := \underline{\mathrm{Map}}_k^{\mathbb{G}_m}(\mathrm{Spec}(k), X) \qquad \text{(the fixed locus)}$$
$$X_\lambda^+ := \underline{\mathrm{Map}}_k^{\mathbb{G}_m}(\mathbb{A}^1, X) \qquad \text{(the attractor)}$$

By [Dri13, Thm. 1.4.2], these functors are representable by algebraic spaces. Moreover, there are the following natural morphisms: a closed immersion $X_\lambda^0 \hookrightarrow X$, an unramified morphism $X_\lambda^+ \to X$ (given by evaluation at 1) and an affine [AHR15, Thm. 2.22] morphism $X_\lambda^+ \to X_\lambda^0$ (given by evaluation at 0). If X is separated, then $X_\lambda^+ \to X$ is a monomorphism.

The k-points of X_λ^0 are simply the λ-fixed points, and if X is separated, the k-points of X_λ^+ are the points $x \in X(k)$ such that $\lim_{t\to 0} \lambda(t) \cdot x$ exists. The algebraic space X_λ^0 inherits an action of L_λ and X_λ^+ inherits an action of P_λ^+ such that the evaluation map $\mathrm{ev}_0\colon X_\lambda^+ \to X_\lambda^0$ is equivariant with respect to the surjection $P_\lambda^+ \to L_\lambda$.

We can now recall the description of our mapping stacks for quotient stacks:

Proposition 2.8. [Hal14, Thm. 1.37] *Let X be a quasi-separated algebraic space locally of finite type over a field k equipped with an action of a smooth algebraic group G over k with a split maximal torus. Let Λ be a complete set of conjugacy classes of cocharacters $\mathbb{G}_m \to G$. Then there are isomorphisms*

$$\underline{\mathrm{Map}}_k(B\mathbb{G}_m, [X/G]) \cong \bigsqcup_{\lambda\in\Lambda} [X_\lambda^0 / L_\lambda];$$
$$\underline{\mathrm{Map}}_k(\Theta, [X/G]) \cong \bigsqcup_{\lambda\in\Lambda} [X_\lambda^+ / P_\lambda^+].$$

Moreover, the morphism $\mathrm{ev}_1\colon \underline{\mathrm{Map}}_k(\Theta,[X/G]) \to [X/G]$ *is induced by the* $(P_\lambda^+ \to G)$*-equivariant morphism* $X_\lambda^+ \to X$*. The morphism* $\mathrm{ev}_0\colon \underline{\mathrm{Map}}_k(\Theta,[X/G]) \to \underline{\mathrm{Map}}(B\mathbb{G}_m,[X/G])$ *is induced by the* $(P_\lambda^+ \to L_\lambda)$*-equivariant morphism* $X_\lambda^+ \to X_\lambda^0$*.*

3. Valuative criteria for stacks

In this section, we introduce and study three valuative criteria for algebraic stacks—Θ-reductive morphisms (Definition 3.10), S-complete morphisms (Definition 3.37), and unpunctured inertia (Definition 3.53)—which appear in the formulation of Theorem 4.1. Additionally, we introduce the notion of Θ-surjective morphisms in §3.4 which is fundamental in our proof of Theorem 4.1 and the notions of modifications and elementary modifications (Definition 3.35) which are critical to our discussion of the semistable reduction theorem in Section 6.

3.1. **Morphisms of stacks of filtrations.** It will be important to understand the behavior of the stacks $\underline{\mathrm{Map}}(\Theta,\mathcal{X})$ under morphisms $\mathcal{X} \to \mathcal{Y}$, i.e., study the behavior of filtrations on objects under morphisms.

Lemma 3.1. *Let* S *be a quasi-separated algebraic space. Let* $f\colon \mathcal{X} \to \mathcal{Y}$ *be a morphism of algebraic stacks, locally of finite presentation and quasi-separated over* S*, with affine stabilizers. Suppose* f *satisfies one of the following properties*

- *(a) representable;*
- *(b) monomorphism;*
- *(c) separated;*
- *(d) unramifed; or*
- *(e) étale,*
- *(f) étale, surjective and representable.*

then $\underline{\mathrm{Map}}_S(\Theta,\mathcal{X}) \to \underline{\mathrm{Map}}_S(\Theta,\mathcal{Y})$ *has the same property.*

Proof. Properties (a) and (b) are clear. Property (c) follows from the valuative criterion and descent. Properties (d) and (e) follow from the formal lifting criterion and descent. For (f), it remains to show that $\underline{\mathrm{Map}}_S(\Theta,\mathcal{X}) \to \underline{\mathrm{Map}}_S(\Theta,\mathcal{Y})$ is surjective. Let $h\colon \Theta_k \to \mathcal{Y}_s$ be a morphism over a geometric point $s\colon \mathrm{Spec}(k) \to S$. We will use Tannaka duality to construct a lift to $\mathcal{X}$. As any étale representable cover of $B\mathbb{G}_{m,k}$ admits a section, we may choose a lift $B\mathbb{G}_{m,k} \to \mathcal{X}_s$ of $B\mathbb{G}_{m,k} \hookrightarrow \Theta_k \xrightarrow{h} \mathcal{Y}_s$. Let $\Theta_k^{[n]} = [\mathrm{Spec}(k[x]/x^{n+1})/\mathbb{G}_m]$ be the nth nilpotent thickening of $B\mathbb{G}_m \hookrightarrow \Theta$. Since f is étale, there exist compatible lifts $\Theta_k^{[n]} \to \mathcal{X}_s$ of $\Theta_k^{[n]} \hookrightarrow \Theta_k \xrightarrow{h} \mathcal{Y}_s$. Since Θ_k is coherently complete along $B\mathbb{G}_{m,k}$, by [AHR15, Cor. 3.6], there is an equivalence of categories $\mathrm{Map}_k(\Theta_k,\mathcal{X}_s) = \varprojlim_n \mathrm{Map}_k(\Theta_k^{[n]},\mathcal{X}_s)$. This constructs the desired lift $\Theta_k \to \mathcal{X}_s$ of h. See also [Hal14, Lem. 4.33]. □

Property (f) is not preserved if the representability hypothesis is dropped. For instance, if $\mathcal{X} = B\mathbb{G}_m \to B\mathbb{G}_m = \mathcal{Y}$ is induced by $\mathbb{G}_m \to \mathbb{G}_m, t \to t^d$ for $d > 1$, then $\underline{\mathrm{Map}}_S(\Theta,\mathcal{X}) \to \underline{\mathrm{Map}}_S(\Theta,\mathcal{Y})$ is not surjective. However, let us recall the following useful lemma, whose proof relies on Theorem 2.6:

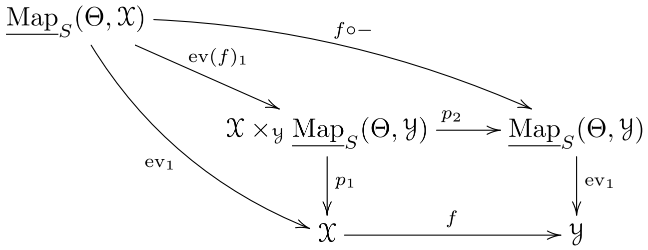
Lemma 3.2 ([Hal14, Lem. 4.34])**.** *Let* $\mathcal{X}$ *be an algebraic stack of finite type with affine diagonal over a noetherian algebraic space* S*. Then there is an algebraic space* X *over* S *with* $X \to S$ *affine, a* $\mathbb{G}_m^n$ *action on* X *for some* $n \geq 0$*, and a smooth, surjective and representable morphism* $[X/\mathbb{G}_m^n] \to \mathcal{X}$ *such that the morphism* $\underline{\mathrm{Map}}_S(\Theta,[X/\mathbb{G}_m^n]) \to \underline{\mathrm{Map}}_S(\Theta,\mathcal{X})$ *is smooth, surjective and representative.*

3.2. **Property Θ-$\mathcal{P}$.** If $f\colon \mathcal{X} \to \mathcal{Y}$ is a morphism of algebraic stacks over an algebraic space S, we denote by $\mathrm{ev}(f)_1$ the induced morphism of stacks

$$\mathrm{ev}(f)_1\colon \underline{\mathrm{Map}}_S(\Theta, \mathcal{X}) \to \mathcal{X} \times_{\mathcal{Y},\mathrm{ev}_1} \underline{\mathrm{Map}}_S(\Theta, \mathcal{Y}), \qquad \lambda \mapsto (\mathrm{ev}_1(\lambda), f \circ \lambda),$$

i.e., this morphism takes an object together with a filtration in $\mathcal{X}$ and remembers the object together with the induced filtration of the image in $\mathcal{Y}$. It sits in a commutative diagram:

(3.1)



Definition 3.3. Let $\mathcal{P}$ be a property of morphisms of algebraic stacks. We say that a morphism $f\colon \mathcal{X} \to \mathcal{Y}$ of algebraic stacks, locally of finite presentation and quasi-separated over an algebraic space S, with affine stabilizers, has *property* Θ-$\mathcal{P}$ if $\mathrm{ev}(f)_1\colon \underline{\mathrm{Map}}_S(\Theta, \mathcal{X}) \to \mathcal{X} \times_{\mathcal{Y},\mathrm{ev}_1} \underline{\mathrm{Map}}_S(\Theta, \mathcal{Y})$ has property $\mathcal{P}$. We say that $\mathcal{X}$ has *property* Θ-$\mathcal{P}$ if $\mathcal{X} \to \mathrm{Spec}(\mathbb{Z})$ does.

For example a morphisms $f\colon \mathcal{X} \to \mathcal{Y}$ is Θ-*surjective* if one can lift filtrations on any point $f(x)$ to filtrations on x.

The assignment $f \mapsto \mathrm{ev}(f)_1$ behaves well with respect to compositions and base change. Namely, given a composition $g \circ f\colon \mathcal{X} \xrightarrow{f} \mathcal{Y} \xrightarrow{g} \mathcal{Z}$ of morphisms of algebraic stacks over S, then $\mathrm{ev}(g \circ f)_1$ is naturally isomorphic to the composition

$$\begin{aligned} \underline{\mathrm{Map}}_S(\Theta, \mathcal{X}) &\xrightarrow{\mathrm{ev}(f)_1} \mathcal{X} \times_{\mathcal{Y}} \underline{\mathrm{Map}}_S(\Theta, \mathcal{Y}) \\ &\xrightarrow{\mathrm{id} \times \mathrm{ev}(g)_1} \mathcal{X} \times_{\mathcal{Y}} (\mathcal{Y} \times_{\mathcal{Z}} \underline{\mathrm{Map}}_S(\Theta, \mathcal{Z})) \cong \mathcal{X} \times_{\mathcal{Z}} \underline{\mathrm{Map}}_S(\Theta, \mathcal{Z}), \end{aligned}$$

and if

$$\begin{array}{ccc} \mathcal{X}' & \xrightarrow{f'} & \mathcal{Y}' \\ \downarrow & & \downarrow \\ \mathcal{X} & \xrightarrow{f} & \mathcal{Y} \end{array}$$

is a Cartesian diagram of algebraic stacks over S, then

$$\begin{array}{ccccc} \underline{\mathrm{Map}}_S(\Theta, \mathcal{X}') & \xrightarrow{\mathrm{ev}(f')_1} & \mathcal{X}' \times_{\mathcal{Y}'} \underline{\mathrm{Map}}_S(\Theta, \mathcal{Y}') & \longrightarrow & \underline{\mathrm{Map}}_S(\Theta, \mathcal{Y}') \\ \downarrow & & \downarrow & & \downarrow \\ \underline{\mathrm{Map}}_S(\Theta, \mathcal{X}) & \xrightarrow{\mathrm{ev}(f)_1} & \mathcal{X} \times_{\mathcal{Y}} \underline{\mathrm{Map}}_S(\Theta, \mathcal{Y}) & \longrightarrow & \underline{\mathrm{Map}}_S(\Theta, \mathcal{Y}) \end{array} \tag{3.2}$$

is Cartesian. We conclude:

Proposition 3.4. *Let $\mathcal{P}$ be a property of morphisms of algebraic stacks. If $\mathcal{P}$ is stable under composition and base change, then so is the property Θ-$\mathcal{P}$. If $\mathcal{P}$ is stable under fppf (resp. smooth, resp. étale) descent, then Θ-$\mathcal{P}$ is stable under descent by morphisms $\mathcal{Y}' \to \mathcal{Y}$ such that $\underline{\mathrm{Map}}_S(\Theta, \mathcal{Y}') \to \underline{\mathrm{Map}}_S(\Theta, \mathcal{Y})$ is fppf (resp. smooth and surjective, resp. étale and surjective).* □

Lemma 3.5. *Let $\mathcal{P}$ be a property of representable morphisms of algebraic stacks. If $\mathcal{P}$ is stable under étale descent, then Θ-$\mathcal{P}$ is stable under descent by representable, étale and surjective morphisms.*

Proof. This follows immediately from Proposition 3.4 and Lemma 3.1(f). $\square$

Lemma 3.6. *If $\mathcal{X}$ is an algebraic stack with quasi-finite and separated inertia and T is a locally noetherian algebraic space, any morphism $\Theta_T \to \mathcal{X}$ factors uniquely through $\Theta_T \to T$.*

Proof. This follows from [Hal14, Lem. 1.29]. $\square$

Lemma 3.7. *Let S be a quasi-separated algebraic space. Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks, locally of finite presentation and quasi-separated over S, with affine stabilizers. Assume that $\mathcal{X}$ and $\mathcal{Y}$ have separated diagonals.*

(1) The morphism $\mathrm{ev}(f)_1$ is representable.
(2) If f is separated, then so is $\mathrm{ev}(f)_1$.
(3) If f is representable and separated, then $\mathrm{ev}(f)_1$ is a monomorphism.
(4) If $\mathcal{X}$ and $\mathcal{Y}$ have quasi-finite inertia, then $\mathrm{ev}(f)_1$ is an isomorphism.
(5) If f is étale, then so is $\mathrm{ev}(f)_1$.
(6) If f is representable, étale, and separated, then $\mathrm{ev}(f)_1$ is an open immersion.

Proof. For (1), by diagram (3.1), it suffices to show that $\mathrm{ev}_1\colon \underline{\mathrm{Map}}_S(\Theta, \mathcal{X}) \to \mathcal{X}$ is representable, which is [Hal14, Lem. 1.10, Rem. 1.11].

Part (2) follows from Lemma 3.1(c).

For (3), to show that $\mathrm{ev}(f)_1$ is a monomorphism, we need to show that for every affine scheme $\mathrm{Spec}(R)$, any commutative diagram of solid arrows

$$\begin{array}{ccc} \mathrm{Spec}(R) & \longrightarrow & \mathcal{X} \\ \downarrow & \nearrow & \downarrow \\ \Theta_R & \longrightarrow & \mathcal{X} \times_{\mathcal{Y}} \mathcal{X} \end{array}$$

can be filled in with a dotted arrow. As f is representable and separated, the base change $\mathcal{X} \times_{\mathcal{X}\times_{\mathcal{Y}}\mathcal{X}} \Theta_R \to \Theta_R$ is a closed immersion containing the dense set $\mathrm{Spec}(R)$; it is therefore an isomorphism.

Part (5) follows directly from Lemma 3.1(e) using diagram (3.1). Part (6) follows directly from Parts (3) and (5) as étale monomorphisms are open immersions.

For (4), it suffices by diagram (3.1) to show that $\mathrm{ev}_1\colon \underline{\mathrm{Map}}_S(\Theta, \mathcal{X}) \to \mathcal{X}$ is an isomorphism if $\mathcal{X}$ has quasi-finite inertia which follows immediately from Lemma 3.6. $\square$

Remark 3.8. The morphism $\mathrm{ev}(f)_1$ is not in general quasi-compact. For an example, if $f\colon B\mathbb{G}_{m,k} \to \mathrm{Spec}(k)$, the morphism $\mathrm{ev}(f)_1$ is the evaluation morphism is $\mathrm{ev}_1\colon \underline{\mathrm{Map}}_S(\Theta, B\mathbb{G}_{m,k}) = \bigsqcup_{n\in\mathbb{Z}} B\mathbb{G}_{m,k} \to B\mathbb{G}_{m,k}$.

Remark 3.9. If f is representable but not separated, then $\mathrm{ev}(f)_1$ is not necessarily a monomorphism.

3.3. **Θ-reductive morphisms.** In this section, we study the class of Θ-reductive morphisms as introduced in [Hal14]. As before, we set $\Theta := [\mathbb{A}^1/\mathbb{G}_m]$ defined over $\mathrm{Spec}(\mathbb{Z})$. If R is a DVR with fraction field K, we set $0 \in \Theta_R := \Theta \times \mathrm{Spec}(R)$ to be the unique closed point. Observe that a morphism $\Theta_R \setminus 0 \to \mathcal{X}$ is the data of morphisms $\mathrm{Spec}(R) \to \mathcal{X}$ and $\Theta_K \to \mathcal{X}$ together with an isomorphism of their restrictions to $\mathrm{Spec}(K)$.

Definition 3.10. A morphism $f\colon \mathcal{X} \to \mathcal{Y}$ of locally noetherian algebraic stacks is Θ-*reductive* if for every DVR R, any commutative diagram

$$\begin{array}{ccc} \Theta_R \setminus 0 & \longrightarrow & \mathcal{X} \\ \downarrow & \nearrow & \downarrow f \\ \Theta_R & \longrightarrow & \mathcal{Y} \end{array} \tag{3.3}$$

of solid arrows can be uniquely filled in.

Remark 3.11. Let S be a noetherian algebraic space and $f\colon \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks, locally of finite type and quasi-separated over S, with affine stabilizers. Then f is Θ-reductive if and only if $\mathrm{ev}(f)_1\colon \underline{\mathrm{Map}}_S(\Theta, \mathcal{X}) \to \mathcal{X} \times_{\mathcal{Y},\mathrm{ev}_1} \underline{\mathrm{Map}}_S(\Theta, \mathcal{Y})$ satisfies the valuative criterion for properness with respect to DVR's, that is, for every DVR R with fraction field K, any diagram

$$\begin{array}{ccc} \mathrm{Spec}(K) & \longrightarrow & \underline{\mathrm{Map}}_S(\Theta, \mathcal{X}) \\ \downarrow & \nearrow & \downarrow \mathrm{ev}(f)_1 \\ \mathrm{Spec}(R) & \longrightarrow & \mathcal{X} \times_{\mathcal{Y},\mathrm{ev}_1} \underline{\mathrm{Map}}_S(\Theta, \mathcal{Y}) \end{array}$$

of solid arrows can be uniquely filled in. Note that the morphism $\mathrm{ev}(f)_1$ is always representable (Lemma 3.7(1)) and locally of finite type. However, the morphism $\mathrm{ev}(f)_1$ is not in general quasi-compact (see Remark 3.8) and therefore $\mathrm{ev}(f)_1$ is not in general proper.

Remark 3.12. In the context of the previous remark, when $\mathcal{Y}$ has quasi-finite inertia the morphism $\mathrm{ev}_1 : \underline{\mathrm{Map}}_S(\Theta, \mathcal{Y}) \to \mathcal{Y}$ is an equivalence (Lemma 3.6), and $\mathrm{ev}(f)_1$ is isomorphic to $\mathrm{ev}_1 : \underline{\mathrm{Map}}_S(\Theta, \mathcal{X}) \to \mathcal{X}$. Therefore, f is Θ-reductive if and only if $\mathcal{X}$ is Θ-reductive in the absolute sense (i.e. $\mathcal{X} \to \mathrm{Spec}(\mathbb{Z})$ is Θ-reductive). In order to be consistent with the terminology of Θ-reductivity introduced in of [Hal14, Def. 4.16], we have deviated from the "property Θ-$\mathcal{P}$" naming convention.

3.3.1. *Examples illustrating* Θ-*reductivity.* In the following examples, we work over a field k. The following proposition gives a criterion using the notation from §2.3 for when a quotient stack $[X/G]$ is Θ-reductive.

Proposition 3.13. *Let* $\mathcal{X} = [X/G]$ *be a quotient stack, where* X *is a quasi-separated algebraic space locally of finite type over a field* k *and* G *is a (smooth but not necessarily connected) split reductive algebraic group over* k*. Then* $\mathcal{X}$ *is* Θ*-reductive if and only if for every cocharacter* $\lambda\colon \mathbb{G}_m \to G$*, the morphism* $X_\lambda^+ \to X$ *is proper.*

Remark 3.14. If X is separated, then $X_\lambda^+ \to X$ is proper if and only if it is a closed immersion.

Proof. This follows easily from the explicit description of the mapping stack $\underline{\mathrm{Map}}_S(\Theta, \mathcal{X})$ in Proposition 2.8. Indeed, there is a factorization

$$\mathrm{ev}_1\colon [X_\lambda^+/P_\lambda^+] \to [X/P_\lambda^+] \to [X/G]$$

and since G is reductive, each $P_\lambda^+ \subset G$ is a parabolic subgroup. Since the quotient G/P_λ^+ is projective, the morphism $[X/P_\lambda^+] \to [X/G]$ is proper. Thus properness of ev_1 is equivalent to properness of $X_\lambda^+ \to X$. □

In order to develop some intuition for Θ-reductivity, we use this result to provide some basic examples and counterexamples of Θ-reductivity. For an integer n, we denote by $\lambda_n \colon \mathbb{G}_m \to \mathbb{G}_m$ the cocharacter defined by $t \mapsto t^n$; in this way, the integers $\mathbb{Z}$ index the cocharacters of $\mathbb{G}_m$.

Example 3.15 (Affine quotients)**.** Consider the action of $\mathbb{G}_m$ on $X = \mathbb{A}^2$ via $t \cdot (x, y) = (tx, t^{-1}y)$. Then

$$X_{\lambda_n}^+ = \begin{cases} V(y) & \text{if } n > 0 \\ \mathbb{A}^2 & \text{if } n = 0 \\ V(x) & \text{if } n < 0 \end{cases}$$

The evaluation morphism restricted to the component indexed by λ_n is $[X_{\lambda_n}^+/\mathbb{G}_m] \to [X/\mathbb{G}_m]$ which is induced by the inclusion $X_{\lambda_n}^+ \to X$. We see directly that $[X/\mathbb{G}_m]$ is Θ-reductive.

More generally, if $X = \mathrm{Spec}(A)$ is an affine scheme of finite type over k with an action of a reductive algebraic group G, then $[X/G]$ is Θ-reductive. Indeed, if $\lambda \colon \mathbb{G}_m \to G$ is a cocharacter, then A inherits a $\mathbb{Z}$-grading $A = \bigoplus_{n\in\mathbb{Z}} A_n$. If I_λ^- denotes the ideal generated by homogeneous elements of strictly negative degree, then it is easy to see that $X_\lambda^+ = V(I_\lambda^-)$; see [Dri13, §1.3.4]. Thus, $X_\lambda^+ \to X$ is a closed immersion and the conclusion follows from the characterization in Proposition 3.13.

Example 3.16. In contrast, quotients of schemes that are not affine are not always Θ-reductive. Consider the action of $\mathbb{G}_m$ on $X = \mathbb{A}^2 \setminus 0$ via $t \cdot (x, y) = (tx, y)$. Then

$$X_{\lambda_n}^+ = \begin{cases} \{y \neq 0\} & \text{if } n > 0 \\ X & \text{if } n = 0 \\ V(x) & \text{if } n < 0 \end{cases}$$

and we see that $[X/\mathbb{G}_m]$ is *not* Θ-reductive as $X_{\lambda_n}^+ \to X$ is not proper for $n > 0$. Similarly, for a DVR R, the algebraic stack $\Theta_R \setminus 0$ is not Θ-reductive. These are the prototypical examples of non-Θ-reductive stacks.

Another example is given by projective quotients. Consider the multiplication action of $\mathbb{G}_m$ on $X = \mathbb{P}^1$ via $t \cdot [x, y] = [tx, y]$ and on the nodal cubic $C \subset \mathbb{P}^2$ such that the normalization $\mathbb{P}^1 \to X$ is $\mathbb{G}_m$-equivariant. Then

$$X_{\lambda_n}^+ = \begin{cases} \mathbb{P}^1 \setminus \{0\} \sqcup \{0\} & \text{if } n > 0 \\ \mathbb{P}^1 & \text{if } n = 0 \\ \mathbb{P}^1 \setminus \{\infty\} \sqcup \{\infty\} & \text{if } n < 0 \end{cases} \quad \text{and} \quad C_{\lambda_n}^+ = \begin{cases} \mathbb{P}^1 \setminus \{0\} & \text{if } n > 0 \\ C & \text{if } n = 0 \\ \mathbb{P}^1 \setminus \{\infty\} & \text{if } n < 0 \end{cases}$$

where the maps $C_{\lambda_n}^+ \to C$ for $n \neq 0$ are induced by the normalization. We see that $[\mathbb{P}^1/\mathbb{G}_m]$ and $[C/\mathbb{G}_m]$ are *not* Θ-reductive.

3.3.2. *Properties of Θ-reductive morphisms.* We now give a few properties of Θ-reductive morphisms. First observe that Θ-reductive morphisms are stable under composition and base change. We first show that one can check the lifting criterion of (3.3) after taking extensions of the DVR.

Proposition 3.17. *Let $\mathcal{X} \to \mathcal{Y}$ be a morphism of locally noetherian algebraic stacks, and consider a diagram of the form (3.3). There exists a unique dotted arrow filling in the diagram if either*

(1) *there exists a unique filling after passing to an unramifed extension $R \subset R'$ of DVR's which is an isomorphism on residue fields, such as the completion of R, or*
(2) *$\mathcal{X} \to \mathcal{Y}$ has affine diagonal, and there exists a filling after an arbitrary extension of DVR's $R \subset R'$.*

In particular, to verify that a morphism of locally noetherian algebraic stacks is Θ-reductive, it suffices to check the lifting criterion (3.3) for complete DVRs.

Proof. The first statement follows from an explicit descent argument similar to [Hei17, Rmk. 2.5]. Alternatively, if $R \subset R'$ is an unramified extension of DVRs with isomorphic residue fields, then

$$\begin{array}{ccc} \Theta_{R'} \setminus 0 & \longrightarrow & \Theta_R \setminus 0 \\ \downarrow & & \downarrow \\ \Theta_{R'} & \longrightarrow & \Theta_R \end{array}$$

is a flat Mayer–Vietoris square ([HR16, Defn. 1.2]) and thus by [HR16, Thm. A] is a pushout in the 2-category of algebraic stacks. This establishes the first statement.

For the second statement, we begin with the observation that if $\mathcal{X} \to \mathcal{Y}$ has affine diagonal and $j \colon \mathcal{U} \to \mathcal{T}$ is an open immersion of algebraic stacks over $\mathcal{Y}$ with $j_* \mathcal{O}_{\mathcal{U}} = \mathcal{O}_{\mathcal{T}}$, then any two extensions $f_1, f_2 \colon \mathcal{T} \to \mathcal{X}$ of a $\mathcal{Y}$-morphism $\mathcal{U} \to \mathcal{X}$ are canonically 2-isomorphic. Indeed, since $\mathrm{Isom}_{\mathcal{T}}(f_1, f_2) \to \mathcal{T}$ is affine, the section over $\mathcal{U}$ induced by the 2-isomorphism $f_1|_{\mathcal{U}} \xrightarrow{\sim} f_2|_{\mathcal{U}}$ extends uniquely to a section of $\mathcal{T}$.

Consider a diagram (3.3), an extension of DVRs $R \subset R'$ and a lifting $\Theta_{R'} \to \mathcal{X}$. The open immersion $j \colon \Theta_R \setminus 0 \to \Theta_R$ satisfies $j_* \mathcal{O}_{\Theta_R \setminus 0} = \mathcal{O}_{\Theta_R}$ and by flat base change, the same property holds for the morphisms obtained by base changing j along $\Theta_{R'} \to \Theta_R$, $\Theta_{R'} \times_{\Theta_R} \Theta_{R'} \to \Theta_R$, and $\Theta_{R'} \times_{\Theta_R} \Theta_{R'} \times_{\Theta_R} \Theta_{R'} \to \Theta_R$. By the above observation, there exists a canonical 2-isomorphism between the two extensions $\Theta_{R'} \times_{\Theta_R} \Theta_{R'} \rightrightarrows \Theta_{R'} \to \mathcal{X}$ which necessarily satsifies the cocycle condition. By fpqc descent, the lifting $\Theta_{R'} \to \mathcal{X}$ descends to a lifting $\Theta_R \to \mathcal{X}$. □

Θ-reductivity satisfies the following two descent properties. The second property is not used in this paper and is only included for thoroughness.

Proposition 3.18. *Let $\mathcal{X} \to \mathcal{Y}$ be a morphism of locally noetherian algebraic stacks.*

(1) *If $\mathcal{Y}' \to \mathcal{Y}$ is an étale, representable and surjective morphism, then $\mathcal{X} \to \mathcal{Y}$ is Θ-reductive if and only if $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}' \to \mathcal{Y}'$ is Θ-reductive.*
(2) *If $\mathcal{X}' \to \mathcal{X}$ is a finite, étale and surjective morphism, then $\mathcal{X} \to \mathcal{Y}$ is Θ-reductive if and only if $\mathcal{X}' \to \mathcal{Y}$ is Θ-reductive.*

Remark 3.19. If $\mathcal{X} \to \mathcal{Y}$ has affine diagonal, then (2) also holds, with a similar proof, if one replaces the words 'finite, étale' with 'quasi-compact, universally closed.'

Proof. For (1), to check Θ-reductivity of $\mathcal{X} \to \mathcal{Y}$, by Proposition 3.17 we may assume we have a diagram (3.3) where R is a complete DVR. As any étale, representable cover of Θ_R has a section after a finite étale extension $R \subset R'$, we may lift the composition $\Theta_{R'} \to \Theta_R \to \mathcal{Y}$ to $\Theta_{R'} \to \mathcal{Y}'$. The Θ-reductivity of $\mathcal{X}' := \mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}' \to \mathcal{Y}'$ shows that the lift $\Theta_{R'} \setminus 0 \to \mathcal{X}'$ extends uniquely to a morphism $\Theta_{R'} \to \mathcal{X}'$. This implies that the lift $\Theta_{R'} \setminus 0 \to \mathcal{X}$ extends uniquely to a morphism $\Theta_{R'} \to \mathcal{X}$ as well, because both extension problems can be rephrased in terms of sections of $\Theta_{R'} \times_{\mathcal{Y}'} \mathcal{X}' \simeq \Theta_{R'} \times_{\mathcal{Y}} \mathcal{X}$.

Finally, the first few levels of the Cech nerve for the étale cover $\Theta_{R'} \to \Theta_R$ have the form

$$\cdots \to \bigsqcup_j \Theta_{R'''_j} \rightrightarrows \bigsqcup_i \Theta_{R''_i} \rightrightarrows \Theta_{R'} ,$$

for some complete DVR's R''_i and R'''_j. The argument of the previous paragraph shows that for any of the DVR's $A = R', R''_i, R'''_j$ the lift $\Theta_A \setminus 0 \to \mathcal{X}$ extends uniquely to a lift $\Theta_A \to \mathcal{X}$ of the map $\Theta_A \to \mathcal{Y}$. Étale descent now implies that the original lift $\Theta_R \setminus 0 \to \mathcal{X}$ extends uniquely to a morphism $\Theta_R \to \mathcal{X}$.

For (2), the 'only if' direction follows since finite morphisms are Θ-reductive. Conversely, given a diagram (3.3), we may find a finite étale extension $R \subset R'$ with fraction field $K \subset K'$ such that the composition $\mathrm{Spec}(K) \to \Theta_R \setminus 0 \to \mathcal{X}$ lifts to a map $\mathrm{Spec}(K') \to \mathcal{X}'$. As $\mathcal{X}' \to \mathcal{X}$ is finite, we may extend this morphism uniquely to a map $\Theta_{R'} \setminus 0 \to \mathcal{X}'$ lifting $\Theta_{R'} \setminus 0 \to \Theta_R \setminus 0 \to \mathcal{X}$. By Θ-reductivity of $\mathcal{X}' \to \mathcal{Y}$, this map extends uniquely to a morphism $\Theta_{R'} \to \mathcal{X}'$. The composition $\Theta_{R'} \to \mathcal{X}' \to \mathcal{X}$ is an extension of $\Theta_{R'} \setminus 0 \to \Theta_R \setminus 0 \to \mathcal{X}$, and is unique since $\mathcal{X}' \to \mathcal{X}$ is Θ-reductive. By an étale descent argument similar to the one given in Part (1), this descends uniquely to the desired lift $\Theta_R \to \mathcal{X}$. □

We now provide some important classes of Θ-reductive morphisms.

Proposition 3.20.

(1) *An affine morphism of locally noetherian algebraic stacks is Θ-reductive.*
(2) *Let S be a locally noetherian scheme. Let $G \to S$ be a geometrically reductive and étale-locally embeddable group scheme (e.g. reductive) acting on a locally noetherian scheme X affine over S. Then the morphism $[X/G] \to S$ is Θ-reductive.*
(3) *A good moduli space $\mathcal{X} \to X$, where $\mathcal{X}$ is a locally noetherian algebraic stack with affine diagonal, is Θ-reductive.*

Remark 3.21. In the case that $S = \mathrm{Spec}(k)$ where k is an algebraically closed field, Part (2) implies that $[\mathrm{Spec}(A)/G]$, where G is a geometrically reductive algebraic group, is Θ-reductive. In the case that G is smooth, then this follows from the explicit calculation in Example 3.15.

Proof. For (1), since $0 \in \Theta_R$ has codimension 2 and Θ_R is regular for a DVR R, we have that $(\Theta_R \setminus 0 \to \Theta_R)_* \mathcal{O}_{\Theta_R \setminus 0} = \mathcal{O}_{\Theta_R}$. Given an affine morphism $f \colon \mathcal{X} \to \mathcal{Y}$, we have canonical isomorphisms

$$\begin{aligned} \mathrm{Map}_{\mathcal{Y}}(\Theta_R \setminus 0, \mathcal{X}) &\cong \mathrm{Map}_{\mathcal{O}_{\mathcal{Y}}-\mathrm{alg}}(f_* \mathcal{O}_{\mathcal{X}}, (\Theta_R \setminus 0 \to \mathcal{Y})_* \mathcal{O}_{\Theta_R \setminus 0}) \\ &\cong \mathrm{Map}_{\mathcal{O}_{\mathcal{Y}}-\mathrm{alg}}(f_* \mathcal{O}_{\mathcal{X}}, (\Theta_R \to \mathcal{Y})_* \mathcal{O}_{\Theta_R}) \\ &\cong \mathrm{Map}_{\mathcal{Y}}(\Theta_R, \mathcal{X}). \end{aligned}$$

See also [Hal14, Prop. 1.19], which shows that $\mathrm{ev}(f)_1$ is a closed immersion when f is affine.

For (2), since Θ-reductive morphisms descend under representable, étale and surjective morphisms (Proposition 3.18), we may assume that S is an affine noetherian scheme and that G is a closed subgroup of $\mathrm{GL}_{N,S}$ for some N. We first show that $B_{\mathbb{Z}} \mathrm{GL}_N = [\mathrm{Spec}(\mathbb{Z})/\mathrm{GL}_N]$ is Θ-reductive, which implies that $B_S \mathrm{GL}_N = [S/\mathrm{GL}_{N,S}]$ is also Θ-reductive. A morphism $\Theta_R \setminus 0 \to \mathcal{X}$ corresponds to a vector bundle $\mathcal{E}$ on $\Theta_R \setminus 0$. If $\widetilde{\mathcal{E}}$ is any coherent sheaf on Θ_R extending $\mathcal{E}$, then the double dual $\widetilde{\mathcal{E}}^{\vee\vee}$ is a vector bundle extending $\mathcal{E}$. This provides the desired extension $\Theta_R \to \mathcal{X}$. Since $\mathrm{GL}_{N,S}/G$ is affine [Alp14, Thm. 9.4.1], $B_S G \to B_S \mathrm{GL}_N$ is affine. By Part (1), $B_S G$ is Θ-reductive. Since X is affine over S, $[X/G] \to B_S G$ is affine which implies using again Part (1) that $[X/G]$ is Θ-reductive.

For (3), we may assume that X is quasi-compact. By [AHR], there exists an étale cover $\mathrm{Spec}(B) \to X$ such that $\mathcal{X} \times_X \mathrm{Spec}(B) \cong [\mathrm{Spec}(A)/\mathrm{GL}_N]$ for some N and $B = A^{\mathrm{GL}_N}$. Since Θ-reductive morphisms descend under representable, étale and surjective morphisms, this reduces the claim to the statement that $[\mathrm{Spec}(A)/\mathrm{GL}_N] \to \mathrm{Spec}(A^{\mathrm{GL}_N})$ is Θ-reductive which follows from Part (2). □

Proposition 3.22. *A morphism $f\colon \mathcal{X} \to \mathcal{Y}$ of locally noetherian algebraic stacks, such that $\mathcal{X}$ and $\mathcal{Y}$ both have quasi-finite and separated inertia, is Θ-reductive.*

Proof. This follows from Lemma 3.6. □

3.3.3. *Specialization of k-points.* Next we provide general criteria for when specialization of k-points can be realized by a morphism from Θ_k.

Lemma 3.23. *Let $\mathcal{X}$ be an algebraic stack locally of finite type over a perfect field k such that either (1) $\mathcal{X}$ is locally linearly reductive or (2) $\mathcal{X} \cong [\mathrm{Spec}(A)/\mathrm{GL}_N]$ for some N. Then any specialization $x \rightsquigarrow x_0$ of k-points where x_0 is a closed point is realized by a morphism $\Theta_k \to \mathcal{X}$.*

Proof. The first case follows from the second by Theorem 2.3 while the second case follows from the Hilbert–Mumford criterion [Kem78, Thm. 4.2]. □

The topology of k-points of Θ-reductive stacks is analogous to the topology of quotient stacks arising from GIT.

Lemma 3.24. *Let $\mathcal{X}$ be an algebraic stack locally of finite type over a field k such that either (1) $\mathcal{X}$ is locally linearly reductive or (2) $\mathcal{X} \cong [\mathrm{Spec}(A)/\mathrm{GL}_N]$ for some N. If $\mathcal{X}$ is Θ-reductive, then the closure of any k-point p contains a unique closed point x.*

Proof. Assume that x and x' are two closed points in the closure of p. After replacing k with an extension if necessary, we may assume that k is perfect, and that x and x' are k-rational. It follows from Lemma 3.23 that these specializations come from two filtrations $f, f' : \Theta_k \to \mathcal{X}$ with $f(1) \simeq f'(1) \simeq p$, $f(0) \simeq x$ and $f'(0) \simeq x'$. The maps f and f' glue to define a map $[\mathbb{A}^2_k - \{(0,0)\}/(\mathbb{G}_m^2)_k]$, and choosing one of the two $\mathbb{G}_m$ factors we can apply Θ-reductivity to extend this morphism to a map $[\mathbb{A}^2/\mathbb{G}_m] \to \mathcal{X}$. Then $\gamma(0,0)$ is a specialization of both $x \simeq \gamma(1,0)$ and $x' \simeq \gamma(0,1)$, which because x and x' are closed implies that $x \simeq \gamma(0,0) \simeq x'$. □

3.4. **Θ-surjective morphisms.** In this section, we study the class of Θ-surjective morphisms. We will observe that Θ-surjective morphisms between locally linearly reductive algebraic stacks necessarily map closed points to closed points (Lemma 3.27). This notion will play a fundamental role in our proof of Theorem 4.1; namely, we will use Θ-reductivity to ensure that we can find local quotient presentations which are Θ-surjective (Proposition 4.4(1)).

By Definition 3.3, a morphism $f\colon \mathcal{X} \to \mathcal{Y}$ (of algebraic stacks, locally of finite presentation and quasi-separated over a quasi-separated algebraic space S, with affine stabilizers) is Θ-*surjective* if

$$\mathrm{ev}(f)_1\colon \underline{\mathrm{Map}}(\Theta, \mathcal{X}) \to \mathcal{X} \times_{\mathcal{Y},\mathrm{ev}_1} \underline{\mathrm{Map}}(\Theta, \mathcal{Y})$$

is surjective. From Proposition 3.4 and Lemma 3.5, Θ-surjective morphisms are stable under composition and base change, and they descend under representable, étale and surjective morphisms.

Remark 3.25. The condition of Θ-surjectivity translates into the following lifting criterion: For a field k, denote by $i\colon \mathrm{Spec}(k) \hookrightarrow \Theta_k$ the open immersion. Then $f\colon \mathcal{X} \to \mathcal{Y}$ is Θ-surjective if and only if for any algebraically closed field k, any commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}(k) & \longrightarrow & \mathcal{X} \\ {\scriptstyle i}\big\downarrow & \nearrow & \big\downarrow{\scriptstyle f} \\ \Theta_k & \longrightarrow & \mathcal{Y} \end{array} \tag{3.4}$$

of solid arrows can be filled in with a dotted arrow.

Remark 3.26. If f is representable and separated, it follows from Lemma 3.7(3) that there is at most one lift in diagram (3.4), that is, f is Θ-universally injective (or equivalently Θ-radicial). This fails for non-separated morphisms.

We also note that if f is proper, then the valuative criterion for properness implies that there exists a unique lift in the above diagram. Therefore proper representable morphisms are Θ-universally bijective.

If $\mathcal{X}$ is an algebraic stack over a quasi-separated algebraic space S and $s \in |S|$, let $\mathcal{X}_s$ be the fiber product $\mathcal{X} \times_S \mathrm{Spec}(\kappa(s))$, where $\kappa(s)$ is the residue field of s.

Lemma 3.27. *Let S be a quasi-separated algebraic space and $f\colon \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks, locally of finite presentation over S with affine stabilizers. Suppose that $\mathcal{Y}$ is locally linearly reductive and f is Θ-surjective. If $x \in |\mathcal{X}|$ is a point with image $s \in |S|$ such that $x \in |\mathcal{X}_s|$ is closed, then $f(x) \in |\mathcal{Y}_s|$ is closed.*

Proof. We immediately reduce to the case when S is the spectrum of an algebraically closed field k and $x \in |\mathcal{X}|$ is a closed point. If $f(x)$ is not closed, then there exists a specialization $f(x) \rightsquigarrow y_0$ of k-points to a closed point. By Lemma 3.23, there exists a morphism $\Theta_k \to \mathcal{Y}$ realizing $f(x) \rightsquigarrow y_0$. As the diagram

$$\begin{array}{ccc} \mathrm{Spec}(k) & \xrightarrow{x} & \mathcal{X} \\ {\scriptstyle i}\big\downarrow & \overset{h}{\nearrow} & \big\downarrow{\scriptstyle f} \\ \Theta_k & \longrightarrow & \mathcal{Y} \end{array}$$

can be filled in with a morphism h and $x \in |\mathcal{X}|$ is closed, $h(0) = h(1)$. It follows that $f(x) = y_0$ is closed. ☐

Remark 3.28. The converse of Lemma 3.27 is not true; see Example 3.34.

For the construction of good moduli spaces we will need a variant of the above properties. Let $\mathcal{X}$ and $\mathcal{Y}$ be algebraic stacks of finite type with affine diagonal over a noetherian algebraic space S, and let $f\colon \mathcal{X} \to \mathcal{Y}$ be a morphism. Define

$\Sigma_f \subset |\mathcal{X}|$ be the set of points $x \in |\mathcal{X}|$ where f is *not* Θ-surjective at x, i.e., points $x \in |\mathcal{X}|$ where there exists a representative $\mathrm{Spec}(k) \to \mathcal{X}$ of x with k algebraically closed and a commutative diagram as in diagram (3.4) which cannot be filled in. By definition, Σ_f is the image under p_1 of the complement of the image of $\mathrm{ev}(f)_1$, i.e.,

$$\Sigma_f = p_1\Big(\big(\mathcal{X} \times_{\mathcal{Y}} \underline{\mathrm{Map}}_S(\Theta, \mathcal{Y})\big) \setminus \mathrm{ev}(f)_1\big(\underline{\mathrm{Map}}_S(\Theta, \mathcal{X})\big) \Big) \subset |\mathcal{X}|. \tag{3.5}$$

Lemma 3.29. *Let $\mathcal{X}$ and $\mathcal{Y}$ be algebraic stacks of finite type with affine diagonal over a noetherian algebraic space S, and let $f\colon \mathcal{X} \to \mathcal{Y}$ be a representable, quasi-finite, and separated morphism. Suppose that either*

(1) $\mathcal{Y}$ admits a good moduli space; or
(2) $\mathcal{Y} \cong [\mathrm{Spec}(A)/\mathrm{GL}_N]$ for some N.

Then the locus $\Sigma_f \subset |\mathcal{X}|$ is closed.

Proof. Zariski's Main Theorem [LMB, Thm. 16.5] provides a factorization $f\colon \mathcal{X} \xrightarrow{i} \widetilde{\mathcal{Y}} \xrightarrow{\nu} \mathcal{Y}$ where i is an open immersion and ν is a finite morphism. As ν is proper and therefore Θ-surjective we have $\Sigma_i = \Sigma_f$. Thus, it suffices to assume that f is an open immersion. Let $\mathcal{Z} \subset \mathcal{Y}$ be the reduced complement of $\mathcal{X}$ and let $\pi\colon \mathcal{Y} \to Y$ denote the adequate moduli space. We claim that $\Sigma_f = \pi^{-1}(\pi(|\mathcal{Z}|)) \cap |\mathcal{X}|$.

Indeed, the inclusion "$\subset$" is clear: the morphism $\mathcal{Y} \setminus \pi^{-1}(\pi(|\mathcal{Z}|)) \hookrightarrow \mathcal{Y}$ is the base change of the Θ-surjective morphism $Y \setminus \pi(|\mathcal{Z}|) \hookrightarrow Y$ of algebraic spaces. For the inclusion "$\supset$," let $x \in \pi^{-1}(\pi(|\mathcal{Z}|)) \cap |\mathcal{X}|$ and let $\overline{x}\colon \mathrm{Spec}(k) \to \mathcal{X}$ be a representative of x, where k is algebraically closed, with image $s\colon \mathrm{Spec}(k) \to S$. Let $x_s \in |\mathcal{X}_s|$ be the image of $\mathrm{Spec}(k) \to \mathcal{X}_s$ and $z \in |\mathcal{Z}_s|$ be the unique closed point in the closure of x_s. If $\mathcal{Y}$ admits a good moduli space, it is in particular locally linearly reductive. Therefore, in either case (1) or (2), we may apply Lemma 3.23 to obtain a morphism $\Theta_k \to \mathcal{Y}_s$ realizing the specialization $x_s \rightsquigarrow z$. Since the commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}(k) & \xrightarrow{\overline{x}} & \mathcal{X} \\ \downarrow & \nearrow & \downarrow{\scriptstyle f} \\ \Theta_k & \longrightarrow & \mathcal{Y} \end{array}$$

does not admit a lift, $x \in \Sigma_f$. As $\pi^{-1}(\pi(|\mathcal{Z}|)) \subset |\mathcal{Y}|$ is closed, the conclusion follows. □

Proposition 3.30. *Let $\mathcal{X}$ and $\mathcal{Y}$ be algebraic stacks, of finite type with affine diagonal over a noetherian algebraic space S, and let $f\colon \mathcal{X} \to \mathcal{Y}$ be a representable, quasi-finite and separated morphism. If $\mathcal{Y}$ is locally linearly reductive, then $\Sigma_f \subset \mathcal{X}$ is constructible.*

Proof. By Theorem 2.3, the hypotheses imply that there exists a representable, étale and surjective morphism $g\colon \mathcal{Y}' \to \mathcal{Y}$, where $\mathcal{Y}' \cong [\mathrm{Spec}(A)/\mathrm{GL}_N]$ for some N. Let $\mathcal{X}' = \mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}'$ with projections $g'\colon \mathcal{X}' \to \mathcal{X}$ and $f'\colon \mathcal{X}' \to \mathcal{Y}'$. By Lemma 3.1(f), the morphism $\underline{\mathrm{Map}}_S(\Theta, \mathcal{Y}') \to \underline{\mathrm{Map}}_S(\Theta, \mathcal{Y})$ is surjective. Therefore by Cartesian Diagram (3.2), the complement of $\underline{\mathrm{Map}}_S(\Theta, \mathcal{X}')$ in $\mathcal{X}' \times_{\mathcal{Y}'} \underline{\mathrm{Map}}_S(\Theta, \mathcal{Y}')$ surjects onto the complement of $\underline{\mathrm{Map}}_S(\Theta, \mathcal{X})$ in $\mathcal{X} \times_{\mathcal{Y}} \underline{\mathrm{Map}}_S(\Theta, \mathcal{Y})$. It follows that $\Sigma_f = g'(\Sigma_{f'})$. By Chevalley's Theorem and Lemma 3.29, the locus Σ_f is constructible. □

Let us give some simple examples and non-examples of Θ-surjectivity. In these examples, we work over a field k.

Example 3.31. If $\phi\colon \mathcal{X} \to X$ is an adequate moduli space and $\mathcal{U} \subset \mathcal{X}$ is an open substack, then $\mathcal{U}$ is saturated (i.e. $\phi^{-1}(\phi(\mathcal{U})) = \mathcal{U}$) if and only if $\mathcal{U} \hookrightarrow \mathcal{X}$ is Θ-surjective. In this case, $\mathcal{U} \hookrightarrow \mathcal{X}$ is even a Θ-isomorphism.

Example 3.32. The open immersion $\operatorname{Spec}(k) \hookrightarrow [\mathbb{A}^1/\mathbb{G}_m]$ is Θ-reductive but *not* Θ-surjective. Indeed, this is the prototypical example of a morphism that does not send closed points to closed points.

Example 3.33. Consider the action of $\mathbb{G}_m$ on $X = \mathbb{A}^2 \setminus 0$ via $t \cdot (x, y) = (tx, y)$ (as in Example 3.16)) and the open immersion $f\colon \mathbb{A}^1 \hookrightarrow [X/\mathbb{G}_m]$ of the locus where x is non-zero. Then

$$\operatorname{ev}(f)_1\colon \mathbb{A}^1 = \underline{\operatorname{Map}}(\Theta, \mathbb{A}^1) \to \underline{\operatorname{Map}}(\Theta, [X/\mathbb{G}_m]) = \mathbb{A}^1 \sqcup \Big(\bigsqcup_{n<0} \mathbb{A}^1 \setminus 0 \Big)$$

which is the inclusion onto the first factor. Again, f is affine and hence Θ-reductive but not Θ-surjective.

Example 3.34. Let $C \subset \mathbb{P}^2$ be the nodal cubic with a $\mathbb{G}_m$-action and consider the étale presentation $f\colon [W/\mathbb{G}_m] \to [C/\mathbb{G}_m]$ where $W = \operatorname{Spec}(k[x, y]/xy)$ and $\mathbb{G}_m$ acts with weights 1 and -1 on x and y, respectively. Then f clearly maps closed points to closed points but we claim it is *not* Θ-surjective. Indeed, there is no lift in the diagram

$$\begin{array}{ccc} \operatorname{Spec}(k) & \longrightarrow & [\operatorname{Spec}(k[x,y]/xy)/\mathbb{G}_m] \\ \downarrow & \nearrow & \downarrow{\scriptstyle f} \\ \Theta & \longrightarrow & [C/\mathbb{G}_m] \end{array}$$

where $\operatorname{Spec}(k) \to [\operatorname{Spec}(k[x, y]/xy)/\mathbb{G}_m]$ is defined by $y = 0$ and $x \neq 0$, and $\Theta \to [C/\mathbb{G}_m]$ is the composition of the morphism $\Theta \to [\operatorname{Spec}(k[x, y]/xy)/\mathbb{G}_m]$ defined by $x = 0$ and the morphism f.

3.5. Elementary modifications and S-complete morphisms.

3.5.1. *Modifications and elementary modifications.* As in [Hei17, §2.B] the following stack, which depends on a choice of DVR R, plays an important role in our analysis of criteria for separatedness of good moduli spaces.

$$\overline{\mathrm{ST}}_R := [\operatorname{Spec}\big(R[s, t]/(st - \pi)\big)/\mathbb{G}_m], \tag{3.6}$$

where s and t have $\mathbb{G}_m$-weights 1 and -1 respectively, and π is a choice of uniformizer for R. A different choice of π results in an isomorphic stack.

Observe that $\overline{\mathrm{ST}}_R \setminus 0 \cong \operatorname{Spec}(R) \cup_{\operatorname{Spec}(K)} \operatorname{Spec}(R)$, where K is the fraction field of R, because the locus where $s \neq 0$ in $\overline{\mathrm{ST}}_R$ is isomorphic to $[\operatorname{Spec}\big(R[s, t]_s/(t - \pi/s)\big)/\mathbb{G}_m] \cong [\operatorname{Spec}(R[s]_s)/\mathbb{G}_m] \cong \operatorname{Spec}(R)$ and the locus where $t \neq 0$ has a similar description. A morphism $h\colon \overline{\mathrm{ST}}_R \setminus 0 \to \mathcal{X}$ to an algebraic stack is the data of two morphisms $\xi, \xi'\colon \operatorname{Spec}(R) \to \mathcal{X}$, where $\xi := h|_{\{s\neq 0\}}$ and $\xi' := h|_{\{t\neq 0\}}$, together with an isomorphism $\xi_K \simeq \xi'_K$.

Definition 3.35. Let $\mathcal{X}$ be an algebraic stack and let $\xi\colon \operatorname{Spec}(R) \to \mathcal{X}$ be a morphism where R is a DVR with fraction field K.

(1) A *modification* of ξ is the data of a morphism $\xi'\colon \operatorname{Spec}(R) \to \mathcal{X}$ along with an isomorphism between the restrictions $\xi|_K \simeq \xi'|_K$.

(2) An *elementary modification* of ξ is the data of a morphism $h\colon \overline{\mathrm{ST}}_R \to \mathfrak{X}$ along with an isomorphism $\xi \simeq h|_{\{s\neq 0\}}$.

An elementary modification is clearly also a modification.

Remark 3.36. The terminology here is inspired by the terminology of [Lan75], but does not exactly coincide. Langton's notion of "elementary modifications" of families of vector bundles over a DVR are examples of the notion of elementary modification above which flip two-step filtrations. To see this, let X be a noetherian scheme and $\underline{\mathrm{Coh}}(X)$ the stack of coherent sheaves on X. Let R be a DVR with fraction field K and residue field κ. A quasi-coherent sheaf on $X\times\overline{\mathrm{ST}}_R$ corresponds to a $\mathbb{Z}$-graded coherent sheaf $\bigoplus_{n\in\mathbb{Z}} F_n$ on X_R together with a diagram of maps

$$\cdots \underset{t}{\overset{s}{\rightleftarrows}} F_{n-1} \underset{t}{\overset{s}{\rightleftarrows}} F_n \underset{t}{\overset{s}{\rightleftarrows}} F_{n+1} \underset{t}{\overset{s}{\rightleftarrows}} \cdots, \tag{3.7}$$

such that $st = ts = \pi$. Moreover, F is coherent if each F_n is coherent, $s\colon F_{n-1} \to F_n$ is an isomorphism for $n \gg 0$ and $t\colon F_n \to F_{n-1}$ is an isomorphism for $n \ll 0$. The sheaf F is a flat over $\overline{\mathrm{ST}}_R$ if and only if the maps s and t are injective, and the induced map $t\colon F_{n+1}/sF_n \to F_n/sF_{n-1}$ is injective. (See Corollary 7.13 for a proof of that these properties characterize coherence and flatness of F.)

Suppose that we have a coherent sheaf E on X_R which is flat over R whose restriction E_κ to X fits into a short exact sequence

$$0 \to B \to E_\kappa \to G \to 0. \tag{3.8}$$

(In Langton's algorithm, one takes $B \subset E_\kappa$ to be the maximal destabilizing subsheaf.) Let $E' = \ker(E \to E_\kappa \to G)$. Then E' is flat over R and $E_K = E'_K$. Moreover, we have that $\pi E \subset E' \subset E$ with $E/E' = G$ and $E'/\pi E = B$; this implies that E'_κ fits into a short exact sequence

$$0 \to G \to E'_\kappa \to B \to 0. \tag{3.9}$$

This data defines a coherent sheaf on $X \times \overline{\mathrm{ST}}_R$ flat over $\overline{\mathrm{ST}}_R$ as follows: set F_n to be E if $n \geq 0$ and E' if $n < 0$. Let s and t act via

$$\cdots \underset{1}{\overset{\pi}{\rightleftarrows}} E' \underset{1}{\overset{\pi}{\rightleftarrows}} E' \underset{\pi}{\overset{1}{\rightleftarrows}} E \underset{\pi}{\overset{1}{\rightleftarrows}} E \underset{\pi}{\overset{1}{\rightleftarrows}} \cdots$$

In general, the restriction of F to $\{s \neq 0\} = \mathrm{Spec}(R)$ is the colimit over the $\mathbb{Z}$-sequence of maps $s\colon F_n \to F_{n+1}$. In our case, this restriction is E. Likewise, the restriction of F to $\{t \neq 0\}$ is the colimit over the t maps so its E' in our case. In general, the restriction of F to $\{s = 0\} = \Theta_\kappa$ is the (generalized) $\mathbb{Z}$-filtration

$$\cdots \leftarrow F_n/sF_{n-1} \overset{t}{\leftarrow} F_{n+1}/sF_{n+2} \leftarrow \cdots$$

which in our case corresponds to $E'_\kappa \supseteq G$ of (3.8). Similarly, the restriction of E to $\{t = 0\}$ corresponds to the filtration $B \subset E_\kappa$ of (3.9).

An analogous construction shows that any finite sequence of steps in Langton's algorithm can be realized by a *single* elementary modification.

3.5.2. *S-complete morphisms.*

Definition 3.37. We say that a morphism $f\colon \mathcal{X} \to \mathcal{Y}$ of locally noetherian algebraic stacks is *S-complete* if for any DVR R and any commutative diagram

$$\begin{array}{ccc} \overline{\mathrm{ST}}_R \setminus 0 & \longrightarrow & \mathcal{X} \\ \downarrow & \nearrow & \downarrow{\scriptstyle f} \\ \overline{\mathrm{ST}}_R & \longrightarrow & \mathcal{Y} \end{array} \tag{3.10}$$

of solid arrows, there exists a unique dotted arrow filling in the diagram.

Remark 3.38. The motivation for the terminology "S-complete" comes from Seshadri's work on the S-equivalence of semistable vector bundles. Namely, if $\mathcal{X}$ is the moduli stack of semistable vector bundles over a smooth projective curve C over k, then $\mathcal{X}$ is S-complete (see e.g., Lemma 8.4). If R is a DVR with fraction field K and residue field k, and $\mathcal{E}, \mathcal{F}$ are two families of semistable vector bundles on C_R which are isomorphic over C_K, then S-completeness implies that the special fibers $\mathcal{E}_0$ and $\mathcal{F}_0$ on C are S-equivalent.

Remark 3.39. S-complete morphisms are stable under composition and base change. A morphism of quasi-separated and locally noetherian algebraic spaces is S-complete if and only if it is separated (Proposition 3.44). While affine morphisms are always S-complete (Proposition 3.42(1)), it is not true that separated, representable morphisms are S-complete. For instance, the open immersion $\overline{\mathrm{ST}}_R \setminus 0 \to \overline{\mathrm{ST}}_R$ is not S-complete. This example also shows that S-complete morphisms do not satisfy smooth descent; however, S-completeness does descend along representable, étale and surjective morphisms (Proposition 3.41).

We now state properties of S-completeness analogous to Proposition 3.17, Proposition 3.18, and Proposition 3.20. In each case, the proof is identical.

Proposition 3.40. *Let $\mathcal{X} \to \mathcal{Y}$ be a morphism of locally noetherian algebraic stacks, and consider a diagram of the form (3.10), then there exists a unique dotted arrow filling the diagram if either*

1. *there exists a unique filling after passing to an unramifed extension $R \subset R'$ of DVR's which is an isomorphism on residue fields, such as the completion of R, or*
2. *$\mathcal{X} \to \mathcal{Y}$ has affine diagonal, and there exists a filling after an arbitrary extension of DVR's $R \subset R'$.*

In particular, to verify that a morphism of locally noetherian algebraic stacks is S-complete, it suffices to check the lifting criterion (3.10) for complete DVRs. □

Proposition 3.41. *Let $\mathcal{X} \to \mathcal{Y}$ be a morphism of locally noetherian algebraic stacks.*

1. *If $\mathcal{Y}' \to \mathcal{Y}$ is an étale, representable and surjective morphism, then $\mathcal{X} \to \mathcal{Y}$ is S-complete if and only if $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}' \to \mathcal{Y}'$ is S-complete.*
2. *If $\mathcal{X}' \to \mathcal{X}$ is a finite, étale and surjective morphism, then $\mathcal{X} \to \mathcal{Y}$ is S-complete if and only if $\mathcal{X}' \to \mathcal{Y}$ is S-complete.* □

Proposition 3.42.

1. *An affine morphism of locally noetherian algebraic stacks is S-complete.*

(2) Let S be a locally noetherian scheme. Let $G \to S$ be a geometrically reductive and étale-locally embeddable group scheme (e.g. reductive) acting on a locally noetherian scheme X affine over S. Then the morphism $[X/G] \to S$ is S-complete

(3) A good moduli space $\mathcal{X} \to X$, where $\mathcal{X}$ is a locally noetherian algebraic stack with affine diagonal, is S-complete. □

We now detail additional important properties of S-completeness.

Lemma 3.43. *If $\mathcal{X}$ is an algebraic stack with quasi-finite and separated inertia and T is a locally noetherian algebraic space, any morphism $\overline{\mathrm{ST}}_T \to \mathcal{X}$ factors uniquely through $\overline{\mathrm{ST}}_T \to T$.*

Proof. This can be established with the same method as Lemma 3.6. □

Proposition 3.44. *Let $f: \mathcal{X} \to \mathcal{Y}$ be a morphism of quasi-separated and locally noetherian algebraic stacks such that $\mathcal{X}$ and $\mathcal{Y}$ both have quasi-finite and separated inertia. Then f is S-complete if and only if f is separated.*

Proof. Let R be a DVR with fraction field K. By Lemma 3.43, any morphism from $\overline{\mathrm{ST}}_R$ to $\mathcal{X}$ or $\mathcal{Y}$ factors through $\overline{\mathrm{ST}}_R \to \mathrm{Spec}(R)$. As $\overline{\mathrm{ST}}_R \setminus 0 = \mathrm{Spec}(R) \bigcup_{\mathrm{Spec}(K)} \mathrm{Spec}(R)$, we see that the valuative criterion of Diagram 3.10 is equivalent to the valuative criterion for separatedness. □

Proposition 3.45. *If G is an algebraic group over a field k, then G is geometrically reductive if and only if B_kG is S-complete. In particular, a closed point of an S-complete locally noetherian algebraic stack with affine stabilizers has a geometrically reductive stabilizer.*

Proof. From Proposition 3.42(2), we know that if G is geometrically reductive, then B_kG is S-complete. For the converse, we may assume that k is algebraically closed. Suppose that G is not geometrically reductive. Then by considering the unipotent radical $R_u(G)$ of the reduced group scheme G^{red}, the induced morphism $B_kR_u(G) \to B_kG$ is affine. Similarly, by taking a normal subgroup $\mathbb{G}_a \subset R_u(G)$, there is an affine morphism $B_k\mathbb{G}_a \to B_kR_u(G)$. The composition $B_k\mathbb{G}_a \to B_kR_u(G) \to B_kG$ is affine. Since B_kG is S-complete, by Proposition 3.42(1) so is $B_k\mathbb{G}_a$, a contradiction. □

Remark 3.46. The proof shows more generally that if $\mathcal{X}$ is a locally noetherian algebraic stack that is S-complete with respect to DVRs essentially of finite type over R, then any closed point of $\mathcal{X}$ has geometrically reductive stabilizer.

Expanding on Proposition 3.42(3), we have the following criterion for when a good moduli space is separated.

Proposition 3.47. *Let $\mathcal{X}$ be a locally noetherian algebraic stack with affine diagonal over an algebraic space S, and let $\mathcal{X} \to X$ be a good moduli space. Then*

(1) the morphism $\mathcal{X} \to X$ is S-complete;

(2) the morphism $X \to S$ is separated if and only if $\mathcal{X} \to S$ is S-complete; and

(3) the morphism $X \to S$ is proper if and only if $\mathcal{X} \to S$ is of finite type, universally closed and S-complete.

Remark 3.48. If in addition $\mathcal{X} \to S$ is of finite type, then in verifying that $\mathcal{X}$ is S-complete, it suffices to verify that for every DVR R essentially of finite type over S, any commutative diagram (3.3) has a unique lift after an extension of the DVR R. Likewise, in verifying that $\mathcal{X} \to S$ is universally closed, it suffices to verify the existence part of the valuative criterion for properness of $\mathcal{X} \to S$ with respect to DVR's which are essentially of finite type over S.

Proof. Part (1) is Proposition 3.42(3). The implication '$\Rightarrow$' in Part (2) follows from Part (1) and the fact that separated algebraic spaces are S-complete. Conversely, suppose $\mathcal{X}$ is S-complete. Suppose $f, g\colon \operatorname{Spec}(R) \to X$ are two maps such that $f|_K = g|_K$. After possibly an extension of R, we may choose a lift $\operatorname{Spec}(K) \to \mathcal{X}$ of $f|_K = g|_K$. Since $\mathcal{X} \to X$ is universally closed, after possibly further extensions of R, we may choose lifts $\widetilde{f}, \widetilde{g}\colon \operatorname{Spec}(R) \to \mathcal{X}$ of f, g such that $\widetilde{f}|_K \cong \widetilde{g}|_K$. By applying the S-completeness of $\mathcal{X}$, we can extend $\widetilde{f}, \widetilde{g}$ to a morphism $\overline{\mathrm{ST}}_R \to \mathcal{X}$. As $\overline{\mathrm{ST}}_R \to \operatorname{Spec}(R)$ is a good moduli space and hence universal for maps to algebraic spaces [Alp13, Thm. 6.6], the morphism $\overline{\mathrm{ST}}_R \to \mathcal{X}$ descends to a unique morphism $\operatorname{Spec}(R) \to X$ which necessarily must be equal to both f and g. We conclude that X is separated by the valuative criterion for separatedness. Part (3) follows from Part (1) using the fact that X is universally closed if and only if $\mathcal{X}$ is. ☐

Remark 3.49. Assume instead that $\mathcal{X} \to X$ is an adequate moduli space (rather than good moduli space) while keeping the other hypotheses on $\mathcal{X}$. The same argument as above shows that if $\mathcal{X}$ is S-complete (resp. universally closed and S-complete), then X is separated (resp. proper). We suspect that the conclusion of all parts of Proposition 3.47 hold but at the moment we cannot show this as we do not have a slice theorem to reduce to the case of $[\operatorname{Spec}(A)/G]$ with G geometrically reductive.

Corollary 3.50. *Let $\mathcal{X}$ be a locally noetherian algebraic stack with affine diagonal and let $\mathcal{X} \to X$ be a good moduli space. Let R be any DVR and consider two morphisms $\xi_0, \xi_1\colon \operatorname{Spec}(R) \to \mathcal{X}$ with $(\xi_0)|_K \cong (\xi_1)|_K$ Then the following are equivalent:*

(1) ξ_0 and ξ_1 differ by an elementary modification,
(2) ξ_0 and ξ_1 differ by a finite sequence of elementary modifications,
(3) the compositions $\xi_i : \operatorname{Spec}(R) \to \mathcal{X} \to X$ agree for $i = 0, 1$.

Proof. Clearly (1) $\Rightarrow$ (2). The projection $\overline{\mathrm{ST}}_R \to \operatorname{Spec}(R)$ is a good moduli space and hence universal for maps to algebraic spaces [Alp13, Thm. 6.6]. It follows that any two maps which differ by an elementary modification induce the same R-point of X, and thus (2) $\Rightarrow$ (3). The implication (3) $\Rightarrow$ (1) follows from part (1) of Proposition 3.47. ☐

Remark 3.51. The above conditions are not equivalent to saying that ξ_0 and ξ_1 are modifications such that the closures of $\xi_0(0)$ and $\xi_1(0)$ intersect. For instance, let X be the non-locally separated algebraic space obtained by taking the free $\mathbb{Z}/2$-quotient of the non-separated affine line, where the action of $\mathbb{Z}/2$ is via $x \mapsto -x$ and swaps the origins. Then there are two distinct maps $\xi_0, \xi_1\colon \operatorname{Spec}(R) \to X$ with $\xi_0|_K = \xi_1|_K$ and $\xi_0(0) = \xi_1(0)$.

Remark 3.52 (Hartog's principle)**.** Both Θ-reductivity and S-completeness are conditions asserting the existence and uniqueness of extending morphisms along a codimension two locus. One might be tempted to unify these two notions by defining that a morphism $f\colon \mathcal{X} \to \mathcal{Y}$ of locally noetherian algebraic stacks satisfies *Hartogs's principle* if for any regular local ring S of dimension 2 with closed point $0 \in \mathrm{Spec}(S)$, there exists a unique dotted arrow filling in any commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}(S) \setminus 0 & \longrightarrow & \mathcal{X} \\ \downarrow & \nearrow & \downarrow{\scriptstyle f} \\ \mathrm{Spec}(S) & \longrightarrow & \mathcal{Y} \end{array} \tag{3.11}$$

of solid arrows. Any such morphism is necessarily both Θ-reductive and S-complete. Moreover, the analogues of Proposition 3.20 and Proposition 3.42 hold for such morphisms. However, many algebraic stacks (e.g. the stack $\underline{\mathrm{Coh}}(X)$ of coherent sheaves on a proper scheme X over a field k) are both Θ-reductive and S-complete but do *not* satisfy Hartog's principle.

3.6. **Unpunctured inertia.** We now give the last of the properties that will turn out to be necessary for the existence of good moduli spaces.

Definition 3.53. We say that a noetherian algebraic stack has *unpunctured inertia* if for any closed point $x \in |\mathcal{X}|$ and versal deformation $p\colon (U, u) \to (\mathcal{X}, x)$, where U is the spectrum of a local ring with closed point u, each connected component of the inertia group scheme $\mathrm{Aut}_{\mathcal{X}}(p) \to U$ has non-empty intersection with the fiber over u.

Remark 3.54. The condition of unpuncturedness is related to the property of purity of the morphism $\mathrm{Aut}_{\mathcal{X}}(p) \to U$ as defined in [RG71, §3.3] and further studied in [Stacks, Tag 0CV5]. If U is the spectrum of a strictly henselian local ring, then purity requires that if $s \in U$ is *any* point and γ is an associated point in the fiber $\mathrm{Aut}_{\mathcal{X}}(p)_s$, then the closure of γ in $\mathrm{Aut}_{\mathcal{X}}(p)$ has non-empty intersection with the fiber over u.

In Section 5, we will provide valuative criteria which can be used to verify that a stack has unpunctured inertia. In this section though, we provide only a few situations in which this condition is easy to check.

Proposition 3.55. *If $\mathcal{X}$ is a noetherian algebraic stack with quasi-finite inertia, then $\mathcal{X}$ has unpunctured inertia if and only if $\mathcal{X}$ has finite inertia.*

Proof. If $\mathcal{X}$ has finite inertia, then $\mathrm{Aut}_{\mathcal{X}}(p) \to U$ is finite so clearly the image of each connected component contains the unique closed point $u \in U$. For the converse, we may assume that U is the spectrum of a Henselian local ring in which case $\mathrm{Aut}_{\mathcal{X}}(p) = G \sqcup H$ where $G \to U$ finite and the fiber of $H \to U$ over u is empty. If $\mathrm{Aut}_{\mathcal{X}}(p)$ is not finite, then H is non-empty and any connected component of H will have empty intersection with the fiber over u. □

Proposition 3.56. *Let $\mathcal{X}$ be a noetherian algebraic stack. If $\mathcal{X}$ has connected stabilizer groups, then $\mathcal{X}$ has unpunctured inertia.*

Proof. This is clear, by definition all fibers of $\mathrm{Aut}_{\mathcal{X}}(p) \to U$ are connected, so any connected component of $\mathrm{Aut}_{\mathcal{X}}$ intersects the component containing the identity section. □

The following example shows that unpuncturedness need not be preserved when passing to open substacks.

Example 3.57. Consider the action of $G = \mathbb{G}_m \rtimes \mathbb{Z}/2$ on $X = \mathbb{A}^2$ via $t \cdot (a, b) = (ta, t^{-1}b)$ and $-1 \cdot (a, b) = (b, a)$. Note that every point $(a, b) \in X$ with $ab \neq 0$ is fixed by the order 2 element $(a/b, -1) \in G$. The algebraic stack $[(X \setminus 0)/G]$ does not have unpunctured inertia by Proposition 3.55. However, it will follow from Proposition 5.7 that $[X/G]$ has unpunctured inertia.

4. Existence of good moduli spaces

The goal of this section is to prove the following theorem providing necessary and sufficient conditions for an algebraic stack to admit a good moduli space.

Theorem 4.1. *Let $\mathcal{X}$ be an algebraic stack, locally of finite type with affine diagonal over a quasi-separated and locally noetherian algebraic space S. Then $\mathcal{X}$ admits a good moduli space if and only if*

(1) $\mathcal{X}$ is locally linearly reductive (Definition 2.1);
(2) $\mathcal{X}$ is Θ-reductive (Definition 3.10); and
(3) $\mathcal{X}$ has unpunctured inertia (Definition 3.53).

Remark 4.2. The theorem also holds if one replaces the condition (2) with

($2'$) for every DVR R essentially of finite type over S, any commutative diagram (3.3) has a unique lift.

The idea of the proof is simple. We use the slice theorem (Theorem 2.3) to reduce to quotient stacks and glue the resulting moduli spaces. As this only works étale locally we need to apply the slice theorem carefully in such a way that preserves the stabilizer groups and the topology of finite type points in order to ensure that the étale covering of the stack induces an étale covering on the level of good moduli spaces.

4.1. Reminder on maps inducing étale maps on good moduli spaces.

If $f: \mathcal{X} \to \mathcal{Y}$ is a morphism of algebraic stacks and $x \in |\mathcal{X}|$, we say that f is *stabilizer preserving at x* if there exists a representative $\widetilde{x}: \operatorname{Spec}(l) \to \mathcal{X}$ of x (equivalently, for all representatives of x), the natural map $\operatorname{Aut}_{\mathcal{X}}(\widetilde{x}) \to \operatorname{Aut}_{\mathcal{Y}}(f \circ \widetilde{x})$ is an isomorphism.

Proposition 4.3. *Let $\mathcal{X}$ and $\mathcal{Y}$ be noetherian algebraic stacks with affine diagonal. Consider a commutative diagram*

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{f} & \mathcal{Y} \\ \downarrow{\scriptstyle \pi_{\mathcal{X}}} & & \downarrow{\scriptstyle \pi_{\mathcal{Y}}} \\ X & \xrightarrow{g} & Y \end{array} \tag{4.1}$$

where f is representable, étale and separated, and both $\pi_{\mathcal{X}}$ and $\pi_{\mathcal{Y}}$ are good moduli spaces (and in particular $\mathcal{X}$ and $\mathcal{Y}$ are locally linearly reductive). If f is Θ-surjective and f is stabilizer preserving at every closed point in $\mathcal{X}$, then g is étale and Diagram 4.1 is Cartesian.

Proof. This result is essentially a stack-theoretic reformation of Luna's fundamental lemma [Alp10, Thm. 6.10]. To see why this version holds, we first reduce by étale descent to the case that Y is affine. If $x \in |\mathcal{X}|$ is a closed point, then $y = \pi_{\mathcal{Y}}(f(x)) \in Y$ is necessarily locally closed so after replacing Y with an open subspace, we may assume that $y \in Y$ is closed. Since f is Θ-surjective, $f(x) \in |\mathcal{Y}|$ is a closed point by Lemma 3.27, and [Alp10, Thm. 6.10] implies that there is an open subspace $U \subset X$ containing $\pi_{\mathcal{X}}(x)$ such that $g|_U$ is étale and $\pi_{\mathcal{X}}^{-1}(U) = U \times_Y \mathcal{Y}$. □

4.2. **Proof of the existence result.** We first provide conditions on an algebraic stack ensuring that there are local quotient presentations which are Θ-surjective and stabilizer preserving. This is the key ingredient in the proof of Theorem 4.1.

Proposition 4.4. *Let $\mathcal{Y}$ be an algebraic stack, locally of finite type with affine diagonal over a quasi-separated and locally noetherian algebraic space S, and let $y \in |\mathcal{Y}|$ be a closed point. Let $f\colon (\mathcal{X}, x) \to (\mathcal{Y}, y)$ be a pointed étale and affine morphism such that there exists an adequate moduli space $\pi\colon \mathcal{X} \to X$ and f induces an isomorphism $f|_{f^{-1}(\mathcal{G}_y)}$ over the residual gerbe at y (e.g. f is a local quotient presentation).*

(1) If $\mathcal{Y}$ is Θ-reductive, then there exists an affine open subspace $U \subset X$ of $\pi(x)$ such that $f|_{\pi^{-1}(U)}$ is Θ-surjective.
(2) If $\mathcal{Y}$ has unpunctured inertia, then there exists an affine open subspace $U \subset X$ of $\pi(x)$ such that $f|_{\pi^{-1}(U)}$ which induces an isomorphism $I_{\pi^{-1}(U)} \to \pi^{-1}(U) \times_{\mathcal{Y}} I_{\mathcal{Y}}$.

In particular, if $\mathcal{Y}$ is locally linearly reductive, is Θ-reductive and has unpunctured inertia, then there exists a local quotient presentation $g\colon \mathcal{W} \to \mathcal{Y}$ around y which is Θ-surjective and induces an isomorphism $I_{\mathcal{W}} \to \mathcal{W} \times_{\mathcal{Y}} I_{\mathcal{Y}}$.

Proof. For (1) let us first show that the morphism $\mathrm{ev}(f)_1\colon \mathrm{Map}_S(\Theta, \mathcal{X}) \to \mathcal{X} \times_{\mathcal{Y}} \mathrm{Map}_S(\Theta, \mathcal{Y})$ is an open and closed embedding. As f is representable, étale and separated, the map is an open embedding by Lemma 3.7. As $\mathcal{X}$ admits a good moduli space, it is Θ-reductive by Proposition 3.20, as is $\mathcal{Y}$ by assumption. Thus $\mathrm{ev}(f)_1$ is proper and in particular closed. Let $\mathcal{Z} \subset \mathcal{X} \times_{\mathcal{Y}} \underline{\mathrm{Map}}_S(\Theta, \mathcal{Y})$ be the open and closed complement of $\underline{\mathrm{Map}}_S(\Theta, \mathcal{X})$. By Equation (3.5), the image $p_1(\mathcal{Z}) \subset |\mathcal{X}|$ consists of the points where f is not Θ-surjective. By Proposition 3.30, the image $p_1(\mathcal{Z}) \subset |\mathcal{X}|$ is constructible.[2] On the other hand, since $\mathcal{Y}$ is Θ-reductive, the image $p_1(\mathcal{Z})$ is closed under specializations.[3]

Consider an arbitrary diagram of solid arrows:

$$\begin{array}{ccc} \mathrm{Spec}(k) & \xrightarrow{x} & \mathcal{X} \\ \downarrow{\scriptstyle i} & \nearrow & \downarrow{\scriptstyle f} \\ \Theta_k & \xrightarrow{\lambda} & \mathcal{Y} \end{array} .$$

As $y = f(x) \in |\mathcal{Y}|$ is a closed point, λ factors through the residual gerbe $\mathcal{G}_y$ of y. The induced map $\mathcal{G}_x \to \mathcal{G}_y$ on residual gerbes is an isomorphism so λ lifts to a morphism $\Theta_k \to \mathcal{G}_x \to \mathcal{X}$ filling in the dotted arrow. It follows that f is Θ-surjective at x, i.e. $x \notin p_1(\mathcal{Z})$.

Let $U \subset X \setminus \pi(p_1(\mathcal{Z}))$ be an open affine neighborhood of $\pi(x)$, and let $\mathcal{U} = \pi^{-1}(U)$. We claim that $f|_{\mathcal{U}}\colon \mathcal{U} \to \mathcal{Y}$ is Θ-surjective. First, observe that the

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[2] Alternatively, one could invoke [Hal14, Lem. 4.36], which implies unconditionally that the image in $|\mathcal{X}|$ of any open and closed substack of $\mathcal{X} \times_{\mathcal{Y}} \underline{\mathrm{Map}}_S(\Theta, \mathcal{Y})$ is constructible.

[3] It is here where the Θ-reductivity hypothesis on $\mathcal{Y}$ is used in an essential way. Note that the implication that $p_1(\mathcal{Z})$ is closed would follow from the weaker condition of uniqueness of lifts in the valuative criterion (3.3) for DVR's R essentially of finite type over S. This justifies Remark 4.2.

inclusion $\iota\colon \mathcal{U} \hookrightarrow \mathcal{X}$ is a Θ-isomorphism (i.e. $\mathrm{ev}(\iota)_1$ is an isomorphism); see Example 3.31. The composition $\mathcal{U} \hookrightarrow \mathcal{X} \to \mathcal{Y}$ induces a commutative diagram

$$\begin{array}{ccccc}
\underline{\mathrm{Map}}_S(\Theta,\mathcal{U}) \xrightarrow{\mathrm{ev}(\iota)_1} \mathcal{U}\times_{\mathcal{X}} \underline{\mathrm{Map}}_S(\Theta,\mathcal{X}) & \longrightarrow & \underline{\mathrm{Map}}_S(\Theta,\mathcal{X}) & & \\
\searrow^{\mathrm{ev}(f\circ\iota)_1} \quad \downarrow & & \downarrow{\scriptstyle \mathrm{ev}(f)_1} & & \\
\mathcal{U}\times_{\mathcal{Y}} \underline{\mathrm{Map}}_S(\Theta,\mathcal{Y}) & \longrightarrow & \mathcal{X}\times_{\mathcal{Y}} \underline{\mathrm{Map}}_S(\Theta,\mathcal{Y}) & \longrightarrow & \underline{\mathrm{Map}}_S(\Theta,\mathcal{Y}) \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{U} & \longrightarrow & \mathcal{X} & \longrightarrow & \mathcal{Y}
\end{array}$$

where all squares are Cartesian. The substack $\mathcal{U}$ was chosen precisely such that $\mathcal{U}\times_{\mathcal{X}} \underline{\mathrm{Map}}_S(\Theta,\mathcal{X}) \to \mathcal{U}\times_{\mathcal{Y}} \underline{\mathrm{Map}}_S(\Theta,\mathcal{Y})$ is an isomorphism. It follows that $\mathrm{ev}(f\circ\iota)_1$ is an isomorphism.

For (2), it suffices to find an open neighborhood $\mathcal{U} \subset \mathcal{X}$ of x such that $f|_{\mathcal{U}}\colon \mathcal{U} \to \mathcal{Y}$ induces an isomorphism $I_{\mathcal{U}} \to \mathcal{U}\times_{\mathcal{Y}} I_{\mathcal{Y}}$. We have a Cartesian diagram

$$\begin{array}{ccc}
I_{\mathcal{X}} & \longrightarrow & \mathcal{X}\times_{\mathcal{Y}} I_{\mathcal{Y}} \\
\downarrow & & \downarrow \\
\mathcal{X} & \longrightarrow & \mathcal{X}\times_{\mathcal{Y}} \mathcal{X}.
\end{array}$$

Since f is étale and affine, the morphism $I_{\mathcal{X}} \to \mathcal{X}\times_{\mathcal{Y}} I_{\mathcal{Y}}$ is an open and closed immersion; let $\mathcal{Z} \subset \mathcal{X}\times_{\mathcal{Y}} I_{\mathcal{Y}}$ be the open and closed complement, so the fiber over a point $p : \mathrm{Spec}(k) \to \mathcal{X}$ consists of the complement of the subgroup $\mathrm{Aut}_{\mathcal{X}}(p) \subset \mathrm{Aut}_{\mathcal{Y}}(f\circ p)$. Denote $p_1\colon \mathcal{X}\times_{\mathcal{Y}} I_{\mathcal{Y}} \to \mathcal{X}$. We know that $x \notin p_1(\mathcal{Z})$ as f is stabilizer preserving at x. Moreover, if we choose a versal deformation $(U,u) \to (\mathcal{Y}, y)$ where U is the spectrum of a local ring, then using that $\mathcal{Y}$ has unpunctured inertia, we know that the preimage of $\mathcal{Z}$ in $\mathcal{X}\times_{\mathcal{Y}} I_{\mathcal{Y}} \times_{\mathcal{Y}} U$ is empty; indeed, if there were a non-empty connected component of this preimage, it must intersect the fiber over u non-trivially contradicting that $x \notin p_1(\mathcal{Z})$. This in turn implies that $x \notin \overline{p_1(\mathcal{Z})}$. Therefore, if we set $\mathcal{U} = \mathcal{X} \setminus \overline{p_1(\mathcal{Z})}$, the induced morphism $I_{\mathcal{U}} \to \mathcal{U}\times_{\mathcal{Y}} I_{\mathcal{Y}}$ is an isomorphism. □

Lemma 4.5. *Let $\mathcal{X}$ be a locally noetherian algebraic stack with affine diagonal. Suppose that $\{\mathcal{U}_i\}_{i\in I}$ is a Zariski-cover of $\mathcal{X}$ such that each $\mathcal{U}_i$ admits a good moduli space and each inclusion $\mathcal{U}_i \hookrightarrow \mathcal{X}$ is Θ-surjective. Then $\mathcal{X}$ admits a good moduli space.*

Proof. Let $\pi_i\colon \mathcal{U}_i \to U_i$ denote the good moduli space. Since each inclusion $\mathcal{U}_i \cap \mathcal{U}_j \hookrightarrow \mathcal{U}_i$ is Θ-surjective, there exist open subspaces $U_{i,j} \subset U_i$ with $\pi_i^{-1}(U_{i,j}) = \mathcal{U}_i \cap \mathcal{U}_j$ (see Example 3.31). By universality of good moduli spaces [Alp13, Thm. 6.6], there are isomorphisms $U_{i,j} \xrightarrow{\sim} U_{j,i}$ providing gluing data for an algebraic space U. The morphisms π_i glue to produce a good moduli space $\mathcal{U} \to U$. □

Using Proposition 4.4 and Lemma 4.5, we can now establish Theorem 4.1.

Proof of Theorem 4.1. For the sufficiency of these three conditions, we follow the proof of [AFS17, Thm. 2.1]. First observe that by Lemma 4.5 and Proposition 4.4(1), it suffices to show that every closed point $x \in |\mathcal{X}|$ has an open neighborhood $\mathcal{U}$ admitting a good moduli space. By Proposition 4.4, there exists a local quotient presentation $f\colon \mathcal{X}_1 = [\mathrm{Spec}(A)/\,\mathrm{GL}_N] \to \mathcal{X}$ around x such that f

is Θ-surjective and f induces an isomorphism $I_{\mathcal{X}_1} \to \mathcal{X}_1 \times_{\mathcal{X}} I_{\mathcal{X}}$. After replacing $\mathcal{X}$ with $f(\mathcal{X}_1)$, we may assume that f is surjective. Since $I_{\mathcal{X}_1} \to \mathcal{X}_1 \times_{\mathcal{X}} I_{\mathcal{X}}$ is an isomorphism, every closed point of $[\operatorname{Spec}(A)/\operatorname{GL}_n]$ has linearly reductive stabilizer. It follows from [AHR] that $[\operatorname{Spec}(A)/\operatorname{GL}_N]$ is cohomologically affine. We let $\pi_1 \colon \mathcal{X}_1 \to X_1 := \operatorname{Spec}(A^{\operatorname{GL}_N})$ be the induced good moduli space.

Set $\mathcal{X}_2 = \mathcal{X}_1 \times_{\mathcal{X}} \mathcal{X}_1$. The projections $p_1, p_2 \colon \mathcal{X}_2 \to \mathcal{X}_1$ are also étale, affine, surjective, and Θ-surjective morphisms that induce isomorphisms $I_{\mathcal{X}_2} \to \mathcal{X}_2 \times_{\mathcal{X}_1} I_{\mathcal{X}_1}$. Since f is affine, $\mathcal{X}_2$ is cohomologically affine and admits a good moduli space $\pi_2 \colon \mathcal{X}_2 \to X_2$. By Proposition 4.3, both commutative squares in the diagram

$$\begin{array}{ccccc} \mathcal{X}_2 & \overset{p_1}{\underset{p_2}{\rightrightarrows}} & \mathcal{X}_1 & \xrightarrow{f} & \mathcal{X} \\ \downarrow{\scriptstyle \pi_2} & & \downarrow{\scriptstyle \pi_1} & & \\ X_2 & \overset{q_1}{\underset{q_2}{\rightrightarrows}} & X_1 & & \end{array}$$

are Cartesian. Moreover, by the universality of good moduli spaces, the étale groupoid structure on $\mathcal{X}_2 \rightrightarrows \mathcal{X}_1$ induces a étale groupoid structure on $X_2 \rightrightarrows X_1$. The fact that f induces isomorphisms of stabilizer groups implies that $\Delta \colon X_2 \to X_1 \times X_1$ is a monomorphism (see the argument of [AFS17, Prop. 3.1]). Thus, $X_2 \rightrightarrows X_1$ is an étale equivalence relation and there exists an algebraic space quotient X. It follows from descent that there is an induced morphism $\pi \colon \mathcal{X} \to X$ which is a good moduli space.

Conversely suppose that $\mathcal{X}$ admits a good moduli space $\pi \colon \mathcal{X} \to X$. Then the closed points of $\mathcal{X}$ have linearly reductive stabilizers. If $x \in |\mathcal{X}|$ is a point and $U \subset X$ is a quasi-compact open containing $\pi(x)$, then x specializes to a closed point $x \in |\pi^{-1}(U)|$ which is necessarily also closed in $\mathcal{X}$. As x_0 has linearly reductive stabilizer, we see that $\mathcal{X}$ is locally linearly reductive. Moreover, Proposition 3.20(3) implies that $\mathcal{X}$ is Θ-reductive. Establishing that $\mathcal{X}$ has unpunctured inertia will take more effort, and we will prove this in Theorem 5.2.

□

We note another consequence of Proposition 4.4, which will be used in §5 below.

Proposition 4.6. *Let $\mathcal{X}$ be an algebraic stack which is of finite type with affine diagonal over a field k. Suppose that $\mathcal{X}$ is Θ-reductive and there exists a single closed point $x \in |\mathcal{X}|$ which has a linearly reductive stabilizer. Then $\mathcal{X}$ admits a good moduli space. If k is algebraically closed, then $\mathcal{X} \cong [\operatorname{Spec}(A)/G_x]$, and if in addition $\mathcal{X}$ is reduced, then $\mathcal{X} \to \operatorname{Spec}(k)$ is the good moduli space.*

Proof. Choose a local quotient presentation $f \colon (\mathcal{X}_1, x_1) \to (\mathcal{X}, x)$ with $\mathcal{X}_1 = [\operatorname{Spec}(B)/\operatorname{GL}_n]$ such that $x_1 \in |\mathcal{X}_1|$ is the unique point mapping to x. Since $\mathcal{X}$ is Θ-reductive, by Proposition 4.4(1), we can assume that f is Θ-surjective. This implies that f sends closed points to closed points and both projections $\mathcal{X}_2 = \mathcal{X}_1 \times_{\mathcal{X}} \mathcal{X}_1 \rightrightarrows \mathcal{X}_1$ send closed points to closed points. Since both $\mathcal{X}$ and $\mathcal{X}_1$ have a unique closed point and f induces an isomorphism of residual gerbes $\mathcal{G}_{x_1} \to \mathcal{G}_x$, it follows that $\mathcal{X}_2$ has a unique closed point and that both projections $\mathcal{X}_2 \rightrightarrows \mathcal{X}_1$ induce isomorphism of stabilizers at this point. Moreover, there are good moduli spaces $\mathcal{X}_1 \to X_1$ and $\mathcal{X}_2 \to X_2$. As in the proof of Theorem 4.1, Proposition 4.3 implies that the induced groupoid $X_2 \rightrightarrows X_1$ is an étale equivalence relation, and the quotient X_1/X_2 is a good moduli space for $\mathcal{X}$. The final statement follows from [AHR, Thm. 2.9] coupled with the observation that if $\mathcal{X}$ is reduced, so is its good moduli space. □

5. Criteria for unpunctured inertia

In this section we establish criteria which imply that a stack has unpunctured inertia. They are "valuative criteria" in the sense that they apply to families over discrete valuation rings. We will need the following notion:

Definition 5.1. Let $\mathcal{X}$ be an algebraic stack over an algebraic space S, let R be a DVR over S with fraction field K and residue field κ, and let $\xi\colon \operatorname{Spec}(R) \to \mathcal{X}$ be a morphism. A *nearby modification* of ξ is a morphism $\xi'\colon \operatorname{Spec}(R) \to \mathcal{X}$ along with an isomorphism $\xi'|_K \simeq \xi|_K$ such that the closures of $\xi'(0)$ and $\xi(0)$ in $|\mathcal{X} \times_S \operatorname{Spec}(\kappa)|$ have nonempty intersection.

This notion is stronger than that of a modification, and weaker than that of an elementary modification. Note however, that in an S-complete stack, the notions of "modification," "nearby modification," and "elementary modification" coincide. We can now state the main results of this section:

Theorem 5.2. *Let $\mathcal{X}$ be an algebraic stack locally of finite type and with affine diagonal over a quasi-separated and locally noetherian algebraic space S. If $\mathcal{X}$ has a good moduli space, then $\mathcal{X}$ has unpunctured inertia. Moreover, if $\mathcal{X}$ is locally linearly reductive and Θ-reductive, then the following conditions are equivalent:*

(1) *For any essentially finite type DVR R, any morphism $\xi\colon \operatorname{Spec}(R) \to \mathcal{X}$, and any $g \in \operatorname{Aut}_{\mathcal{X}}(\xi_K)$ of finite order, there is an extension of DVR's R'/R with fraction field K' and a nearby modification ξ' of $\xi|_{R'}$ such that $g|_{K'}$ extends to an automorphism of ξ';*
(2) *For any essentially finite type DVR R, any morphism $\xi\colon \operatorname{Spec}(R) \to \mathcal{X}$, and any geometrically connected component $H \subset \operatorname{Aut}_{\mathcal{X}}(\xi_K)$, there is an extension of DVR's R'/R with fraction field K' and a nearby modification ξ' of $\xi|_{R'}$ and some $g \in \operatorname{Aut}_{\mathcal{X}}(\xi')$ such that $g|_{K'}$ lies in H;*
(3) *$\mathcal{X}$ has unpunctured inertia; and*
(4) *$\mathcal{X}$ has a good moduli space.*

This elaborates on Theorem 4.1, which stated the equivalence of (3) and (4). Recall that only the implication of (3) ⇒ (4) was shown in the proof of Theorem 4.1, which quoted Theorem 5.2 for the implication (4) ⇒ (3).

Example 5.3. To illustrate the subtlety of the condition (1), let us exhibit in the context of Example 3.57 a map from a DVR to $[\mathbb{A}^2/G]$ (where $G = \mathbb{G}_m \rtimes (\mathbb{Z}/2)$) where performing an extension and elementary modification allows a generic automorphism to extend. Let $R = k[\![z]\!]$ and $K = k(\!(z)\!)$. Consider $\xi\colon \operatorname{Spec}(R) \to \mathbb{A}^2$ via $z \mapsto (z^2, z)$. Then $g = (z, -1) \in G(K)$ stabilizes ξ_K but does not extend to $G(R)$. Consider the degree 2 ramified extension $R \to R'$ with $R' = k[\![\sqrt{z}]\!]$ and $K' = k(\!(\sqrt{z})\!)$, and define $\xi'\colon \operatorname{Spec}(R') \to X$ by $\sqrt{z} \mapsto ((\sqrt{z})^3, (\sqrt{z})^3)$. Over the generic point, ξ' is isomorphic as a point in $[\mathbb{A}^2/G]$ to the restriction $\xi_{K'}$, because $(\sqrt{z}, -1) \cdot \xi'_{K'} = \xi_{K'}$. Under this isomorphism our generic automorphism g becomes $g' = (\sqrt{z}, -1)^{-1} \cdot g|_{K'} \cdot (\sqrt{z}, -1) = (1, -1)$ which clearly extends to an R'-point.

Our second main result states that when $\mathcal{X}$ is S-complete, which is the situation of most interest in applications, these conditions hold automatically.

Theorem 5.4. *If $\mathcal{X}$ is an algebraic stack which is locally linearly reductive and S-complete, then the conditions (1), (2), and (3) of Theorem 5.2 hold automatically.*

Recall from Proposition 3.45 that if a noetherian algebraic stack $\mathcal{X}$ with affine diagonal is S-complete and defined over $\mathbb{Q}$, then $\mathcal{X}$ is locally linearly reductive. It follows that Theorem 5.4 and Theorem 4.1 imply the second part of Theorem A.

Remark 5.5. As the proof will show, Theorem 5.4 holds more generally if $\mathcal{X}$ is only assumed to be S-complete with respect to DVRs essentially of finite type over R. Combining this observation with Remark 3.46 and Remark 4.2, we in fact obtain the following stronger version of the second part of Theorem A: if $\mathcal{X}$ is an algebraic stack of finite type with affine diagonal over a noetherian algebraic space S of characteristic 0, then $\mathcal{X}$ admits a separated good moduli space if and only if $\mathcal{X}$ is Θ-reductive and S-complete with respect to DVRs essentially of finite type over R.

We prove Theorem 5.4 and Theorem 5.2 below, after establishing some preliminary results.

5.1. **A variant of the valuative criteria.** It will be convenient to introduce a variant of the valuative criterion (1) in Theorem 5.2 for an algebraic stack $\mathcal{X}$:

- (1′) For any DVR R with fraction field K, any morphism $\xi\colon \operatorname{Spec}(R) \to \mathcal{X}$, and any generic automorphism $g \in \operatorname{Aut}_{\mathcal{X}}(\xi_K)$ of finite order, there is an extension of DVR's R'/R with fraction field K' and a modification ξ' of $\xi|_{R'}$ such that $g|_{K'}$ extends to an automorphism of ξ'.

Unlike in the valuative criterion (1), criterion (1′) does not require that the modification ξ' is a nearby modification. Criterion (1′) is not a sufficient condition for $\mathcal{X}$ to have unpunctured inertia even when $\mathcal{X}$ is locally linearly reductive and Θ-reductive, but it has useful formal properties. Note in particular that in an S-complete stack, any modification is an elementary modification and in particular a nearby modification so condition (1′) is equivalent to condition (1) of Theorem 5.2.

Remark 5.6. In the case that $\mathcal{X} = [X/G]$ is a noetherian quotient stack defined over a field k, $\mathcal{X}$ satisfies (1′) if and only if for every map $\xi\colon \operatorname{Spec}(R) \to X$ and $g \in G_{\xi_K} \subset G(K)$ of finite order, there exists after an extension $R \subset R'$ (with $K' = \operatorname{Frac}(R')$) an element $h \in G(K')$ such that $h \cdot \xi_{K'}$ and $h^{-1} g|_{K'} h$ both extend to R'-points. Even in the case where $X = V$ is a linear representation of a linearly reductive group G, we are not aware of a completely elementary proof of this fact, despite the purely representation-theoretic nature of this property. This is the most challenging part of the proof of Theorem 5.2.

Proposition 5.7. *Let G be a geometrically reductive and étale-locally embeddable group scheme (e.g. reductive) over an algebraic space S and let $W \to S$ be an affine morphism of finite type with an action of G. Then $[W/G]$ satisfies the criterion (1′), and in particular $[W/G]$ satisfies condition (1) of Theorem 5.2 if S is separated.*

We will prove Proposition 5.7 at the end of this subsection, after establishing some preliminary results.

Lemma 5.8. *The stack $B_{\mathbb{Z}} \operatorname{GL}_N$ satisfies the condition (1′).*

Proof. In this case, because every vector bundle on $\operatorname{Spec}(R)$ is trivializable, the condition is equivalent to the claim that every element $g \in \operatorname{GL}_N(K)$ is conjugate to an element of $\mathrm{GL}_N(R)$ after passing to an extension of the DVR R. After an extension of R we may conjugate g to its Jordan canonical form. The fact that g has finite order implies that the diagonal entries of the resulting matrix are roots of unity. Because the group of k^{th} roots of unity is a finite group scheme, the entries of the Jordan canonical form must lie in R. $\square$

Lemma 5.9. *Let $p\colon \mathcal{X} \to \mathcal{Y}$ be a proper representable morphism of noetherian stacks. If $\mathcal{Y}$ satisfies the valuative criterion (1′), then so does $\mathcal{X}$.*

Proof. Since p is representable and separated, for any morphism $\xi\colon \mathrm{Spec}(R) \to \mathcal{X}$ from a DVR, we have a closed immersion $\mathrm{Aut}_{\mathcal{X}}(\xi) \hookrightarrow \mathrm{Aut}_{\mathcal{Y}}(p \circ \xi)$ of group schemes over $\mathrm{Spec}(R)$. Furthermore, because p is proper, any modification of $p \circ \xi$ lifts uniquely to a modification of ξ. Therefore, given a generic automorphism of ξ, we may pass to an extension R'/R and modify $p \circ \xi|_{R'}$ so that this generic automorphism extends, and then this lifts uniquely to a modification of $\xi|_{R'}$ such that the given generic automorphism extends. $\square$

Proof of Proposition 5.7. It suffices to show that $[\mathrm{Spec}(A)/G]$ satisfies the criterion (1′), where G and $\mathrm{Spec}(A)$ are defined over a DVR R and with A finitely generated over R. After passing to a finite extension of $K = \mathrm{Frac}(R)$ we may assume that G embeds as a closed subgroup $G \hookrightarrow \mathrm{GL}_{N,R}$ for some N. We may then replace G with $\mathrm{GL}_{N,R}$ and replace $\mathrm{Spec}(A)$ with $\mathrm{GL}_{N,R} \times^G \mathrm{Spec}(A)$, which will again be affine because G is geometrically reductive. Furthermore we can assume that A is reduced, because we are only considering maps from reduced schemes. So it suffices to prove the claim for $[\mathrm{Spec}(A)/\mathrm{GL}_{N,R}]$ for a reduced R-algebra A of finite type.

Now consider a GL_N-scheme X which is reduced and projective over $\mathrm{Spec}(A^{\mathrm{GL}_N})$ such that X contains $\mathrm{Spec}(A)$ as a dense GL_N-equivariant open subscheme and the complement $X \setminus \mathrm{Spec}(A)$ is the support of an ample GL_N-invariant Cartier divisor E. The construction in [Tel00, Lem. 6.1] for smooth schemes in characteristic 0 works here as well: We simply choose a closed G-equivariant embedding $\mathrm{Spec}(A) \hookrightarrow \mathbb{A}_{A^{\mathrm{GL}_N}}(\mathcal{E})$ for some locally free GL_n-module over A^{GL_N}, and then let X be the closure of $\mathrm{Spec}(A)$ in $\mathbb{P}_{A^{\mathrm{GL}_N}}(\mathcal{E} \oplus \mathcal{O})$. Thus we have a diagram:

$$\begin{array}{ccc} \mathrm{Spec}(A) & \hookrightarrow & X \\ & \searrow & \downarrow \\ & & \mathrm{Spec}(A^{\mathrm{GL}_N}) \end{array}$$

We claim that $\mathrm{Spec}(A)$ is precisely the semistable locus of X with respect to $\mathcal{O}_X(E)$ in the sense of [Ses77]. Indeed the tautological invariant section $s\colon \mathcal{O}_X \to \mathcal{O}_X(E)$ which restricts to an isomorphism over $\mathrm{Spec}(A)$ shows that $\mathrm{Spec}(A) \subset X^{\mathrm{ss}}$. Conversely s^n gives an isomorphism of A^{GL_N}-modules $\Gamma(\mathrm{Spec}(A), \mathcal{O}_X(nE))^{\mathrm{GL}_N} \simeq A^{\mathrm{GL}_N}$ for all $n > 0$. Under this isomorphism any invariant global section $f \in \Gamma(X, \mathcal{O}_X(nE))^{\mathrm{GL}_N}$, after restriction to the dense open subset $\mathrm{Spec}(A)$, agrees with a section of the form gs^n, where g is the pullback of a function under the map $X \to \mathrm{Spec}(A^{\mathrm{GL}_N})$. It follows that $f = g \cdot s^n$ because X is reduced. This shows that $X^{\mathrm{ss}} \subset \mathrm{Spec}(A)$.

Now Lemma 5.8 implies that the criterion (1′) holds for $\mathrm{Spec}(A^{\mathrm{GL}_N}) \times B\,\mathrm{GL}_N$, and hence Lemma 5.9 implies that the criterion holds for $[X/\mathrm{GL}_N]$. So in order to establish the criterion for $[\mathrm{Spec}(A)/\mathrm{GL}_N]$, it suffices to show that given a point $\xi\colon \mathrm{Spec}(R) \to [X/\mathrm{GL}_N]$ along with a finite order automorphism g of

ξ, if ξ_K lies in the open substack $[\operatorname{Spec}(A)/\operatorname{GL}_N]$, then after passing to an extension of R one can modify the pair (ξ, g) at the special point of $\operatorname{Spec}(R)$ so that the image of ξ lies in $[\operatorname{Spec}(A)/\operatorname{GL}_N]$. Note that the stabilizer group X-scheme $\operatorname{Stab}_{\operatorname{GL}_N}(X) \subset X \times \operatorname{GL}_N$ is equivariant for the GL_N action which acts by the given action on X and by conjugation on the GL_N factor. It suffices to show that given an R-point of $\operatorname{Stab}_{\operatorname{GL}_N}(X)$ whose generic point lies in $\operatorname{Spec}(A) \times \operatorname{GL}_N$, after passing an extension of R there is a modification of the resulting map $\xi\colon \operatorname{Spec}(R) \to [\operatorname{Stab}_{\operatorname{GL}_N}(X)/\operatorname{GL}_N]$ whose image lies in $[\operatorname{Stab}_{\operatorname{GL}_N}(\operatorname{Spec}(A))/\operatorname{GL}_N] = [\operatorname{Stab}_{\operatorname{GL}_N}(X) \cap (\operatorname{Spec}(A) \times \operatorname{GL}_N)/\operatorname{GL}_N]$.

Note that $\operatorname{Stab}_{\operatorname{GL}_N}(X)$ is projective over $\operatorname{Spec}(A^{\operatorname{GL}_N}) \times \operatorname{GL}_N$. We claim that the semistable locus of $\operatorname{Stab}_{\operatorname{GL}_N}(X)$ for the action of G with respect to the pullback of $\mathcal{O}_X(E)$ is precisely $\operatorname{Stab}_{\operatorname{GL}_N}(\operatorname{Spec}(A))$. Indeed, this follows from the Hilbert–Mumford criterion [Ses77]. Any destabilizing one parameter subgroup for $(x, g) \in \operatorname{Stab}_{\operatorname{GL}_N}(X)$ is destabilizing for $x \in X$. Conversely, for every point $(x, g) \in \operatorname{Stab}_{\operatorname{GL}_N}(X)$ whose underlying point $x \in X$ is unstable, Kempf's theorem on the existence of canonical destabilizing flags [Kem78] implies that there is a destabilizing one parameter subgroup λ for x which commutes with $\operatorname{Stab}_{\operatorname{GL}_N}(x)$, and this λ defines a destabilizing one parameter subgroup for the point (x, g). So the fact that any map $\operatorname{Spec}(R) \to [\operatorname{Stab}_{\operatorname{GL}_N}(X)/G]$ whose generic point lies in $[\operatorname{Stab}_{\operatorname{GL}_N}(\operatorname{Spec}(A))/G]$ admits a modification which lies in $[\operatorname{Stab}_{\operatorname{GL}_N}(\operatorname{Spec}(A))/G]$ after passing to an extension of R follows from the classical semistable reduction theorem in the setting of reductive group schemes [Ses77].[4] □

Remark 5.10. By appealing to Theorem A.8, the proof in fact shows that a slightly stronger version of (1′) holds in which the extension K'/K of fraction fields is *finite*. It follows that the statements of Theorem 5.2 and Theorem 5.4 remain true after replacing (1) and (2) with the stronger condition where the extension K'/K is required to be finite.

5.2. **The proof of Theorem 5.2.**

Lemma 5.11. *If $\mathcal{X}$ is a noetherian algebraic stack with affine automorphism groups, then the valuative criterion (1) implies the valuative criterion (2) in Theorem 5.2.*

Proof. It suffices to show that every connected component of the K-group $\operatorname{Aut}_{\mathcal{X}}(\xi_K)$ contains a finite type point of finite order. Let $g \in \operatorname{Aut}_{\mathcal{X}}(\xi_K)$ be a finite type point. After a finite field extension we can decompose $g = g_s g_u$ under the Jordan decomposition, where g_s is semisimple and g_u is unipotent. Now consider the reduced Zariski closed K-subgroup $H \subset \operatorname{Aut}_{\mathcal{X}}(\xi_K)$ generated by g_s. Because g_s is semisimple, H is a diagonalizable K-group and hence every component of H contains an element of finite order. We may thus replace g_s with a finite order element in the same connected component of $\operatorname{Aut}_{\mathcal{X}}(\xi_K)$ which still commutes with g_u. If $\operatorname{char}(K) > 0$, then g_u has finite order and we are finished. If $\operatorname{char}(K) = 0$, then g_u lies in the identity component of G, so g lies on the same component as the finite order element g_s. □

[4] This is not stated in this way in Seshadri's paper, but it follows from the results there: If X is projective over a finite type affine G-scheme and $\mathcal{L}$ is a G-ample bundle, then $[X^{\mathrm{ss}}/G]$ admits an adequate moduli space Y which is projective over $\operatorname{Spec}(\Gamma(X, \mathcal{O}_X)^G)$. So given a map $\operatorname{Spec}(R) \to [X/G]$ whose generic point lands in $[X^{\mathrm{ss}}/G]$, one can compose with the projection to get a map $\operatorname{Spec}(R) \to [X/G] \to \operatorname{Spec}(\Gamma(X, \mathcal{O}_X)^G)$. By construction one has a lift of the generic point along both maps $[X^{\mathrm{ss}}/G] \to Y \to \operatorname{Spec}(\Gamma(X, \mathcal{O}_X)^G)$. So because both maps are universally closed, one can lift this to a map $\operatorname{Spec}(R) \to [X^{\mathrm{ss}}/G]$ after an extension of R.

Lemma 5.12. *Suppose $\mathcal{X}$ is a noetherian algebraic stack with affine stabilizer groups, and suppose that the criterion (2) of Theorem 5.2 holds in such a way that one may always choose the nearby modification ξ' so that $\xi'(0)$ is a specialization of $\xi(0)$ in $|\mathcal{X}|$. Then $\mathcal{X}$ has unpunctured inertia.*

Proof. Let $x \in |\mathcal{X}|$ be a closed point, and let $p\colon (U,u) \to (\mathcal{X},x)$ be a versal deformation of x, and let $H \subset \mathrm{Aut}_{\mathcal{X}}(p)$ be a connected component. The image of the projection $H \to U$ is a constructible set whose closure contains u. It follows that we can find an essentially finite type DVR R and a map $\mathrm{Spec}(R) \to U$ whose special point maps to u and whose generic point lies in the image of $H \to U$. After an extension of the DVR R, we may assume that the generic point $\mathrm{Spec}(K) \to U$ lifts to H, and that the connected component $H' \subset H|_{\mathrm{Spec}(K)}$ containing this lift is geometrically connected. The hypotheses of the lemma imply that, after possibly further extending R, there exists a modification $\xi'\colon \mathrm{Spec}(R) \to \mathcal{X}$ of ξ such that the closure of H' in $\mathrm{Aut}_{\mathcal{X}}(\xi)$ meets the fiber over $0 \in \mathrm{Spec}(R)$ and $0 \in \mathrm{Spec}(R)$ still maps to u. By construction H' maps to H, which implies that $H \subset \mathrm{Aut}_{\mathcal{X}}(p)$ meets the fiber over u. □

Remark 5.13. The valuative criterion (2) does not imply the valuative criterion (1) without additional hypotheses. Consider the group $\mathbb{G}_m \ltimes \mathbb{G}_a$ given coordinates (z,y) and the product rule $(z_1,y_1)\cdot(z_2,y_2) = (z_1z_2, z_2y_1+y_2)$, and let $G \subset (\mathbb{G}_m \ltimes \mathbb{G}_a) \times \mathbb{A}^1_t$ be the hypersurface cut out by the equation $ty = 1-z$. Then G is in fact a smooth subgroup scheme over $\mathbb{A}^1$ whose fiber over 0 is $\mathbb{G}_a$ and whose fiber everywhere else is $\mathbb{G}_m$.

Let $\mathcal{X} = B_{\mathbb{A}^1}G$ and consider the map $\xi\colon \mathrm{Spec}(k[\![t]\!]) \to \mathcal{X}$ which is just the completion of the canonical map $\mathbb{A}^1_t \to \mathcal{X}$ at the origin. Then all modifications of ξ agree after composing with the projection $\mathcal{X} \to \mathbb{A}^1_t$, so after an extension of DVR's the automorphism group of ξ will be isomorphic to $G_{k[\![t]\!]}$. There is a generic automorphism of ξ given by the formula $(\alpha, (1-\alpha)/t)$, where α is a non-identity n^{th} root of unity. This automorphism does not extend to 0, and the generic automorphism group is abelian and hence acts trivially on itself by conjugation. It follows that no extension and modification of ξ will allow this generic automorphism to extend either.

Lemma 5.14. *Let $\mathcal{X}$ be an algebraic stack locally of finite type with affine diagonal over a quasi-separated and locally noetherian algebraic space S. Suppose that $\mathcal{X}$ is Θ-reductive, and that either (1) $\mathcal{X}$ is locally linearly reductive or (2) $\mathcal{X} \cong [\mathrm{Spec}(A)/\mathrm{GL}_N]$ for some N. Let R be a DVR and let $\xi\colon \mathrm{Spec}(R) \to \mathcal{X}$ be a morphism. If ξ' is a nearby modification of ξ and $g \in \mathrm{Aut}_{\mathcal{X}}(\xi')$, then there is a finite extension of DVR's R'/R with fraction field K' and a modification $\xi''\colon \mathrm{Spec}(R') \to \mathcal{X}$ of ξ such that $g|_{K'}$ extends to an automorphism of ξ'' and $\xi''(0)$ is a specialization of $\xi(0)$.*

Proof. Let us first reduce the case (1) to the case (2): Let κ be the residue field of R and let $\mathcal{Z} \subset \mathcal{X}_\kappa = \mathcal{X} \times_S \mathrm{Spec}(\kappa)$ be the closure of the point $p := \xi'(0) \in \mathcal{X}_\kappa$. By Lemma 3.24 we know that $\mathcal{Z}$ has a unique closed point $z \in |\mathcal{Z}|$, and in particular z is a specialization of both p and $\xi(0)$ because ξ' is a nearby modification of ξ. If necessary we pass to a finite extension of R so that we may assume that $z \in \mathcal{Z}(\kappa)$ as well. Under the hypothesis (1), Proposition 4.6 implies that $\mathcal{Z} \simeq [\mathrm{Spec}(A)/G_z]$ for some affine G_z-scheme $\mathrm{Spec}(A)$. Embedding $G_z \subset \mathrm{GL}_{N,\kappa}$ for some N, we may replace G_z with GL_N and $\mathrm{Spec}(A)$ with the affine scheme $\mathrm{GL}_N \times_{G_z} \mathrm{Spec}(A)$. It suffices to prove the claim for the stack $\mathcal{Z}$, so for the remainder of the proof we assume we are in the case (2).

Kempf's theorem [Kem78] implies that after passing to a finite purely inseparable extension of κ, which can be induced by a suitable finite extension of DVR's, there is a canonical filtration $f\colon \Theta_\kappa \to [\mathrm{Spec}(A)/\mathrm{GL}_N]$ with an isomorphism $f(1) \simeq p$ such that $f(0) = z$. The fact that f is canonical implies that any automorphism of $p = f(1)$ extends to an automorphism of the map f. In particular the restriction of $g \in \mathrm{Aut}_{\mathcal{X}}(\xi)$ to $p = \xi'(0)$ extends uniquely to an automorphism of f which we also denote g.

We now apply the strange gluing lemma (Corollary A.2), which states that after composing f with a suitable ramified cover $(-)^n\colon \Theta_\kappa \to \Theta_\kappa$, the data of the map $\xi'\colon \mathrm{Spec}(R) \to \mathcal{X}$ and the filtration $f\colon \Theta_\kappa \to \mathcal{X}$, comes from a *unique* map $\gamma\colon \overline{\mathrm{ST}}_R \to \mathcal{X}$, where f is the restriction of γ to the locus $\{s = 0\}$ and ξ' is the restriction of γ to the locus $\{t \neq 0\}$. The uniqueness of this extension guarantees that the automorphism g of ξ' and f extends uniquely to an automorphism of γ, which we again denote g. Finally we construct our modification ξ'' as the composition

$$\xi''\colon \mathrm{Spec}(R[\sqrt{\pi}]) \to \overline{\mathrm{ST}}_R \xrightarrow{\gamma} \mathcal{X},$$

where the first map is given in (s,t,π) coordinates by $(\sqrt{\pi},\sqrt{\pi},\pi)$, which maps the special point of $\mathrm{Spec}(R[\sqrt{\pi}])$ to the point $\{s = t = \pi = 0\}$ of $\overline{\mathrm{ST}}_R$. By construction the automorphism g of γ restricts to an automorphism of ξ'' extending $g|_{K[\sqrt{\pi}]}$, and the special point $\xi''(0)$ maps to the closed point z of $\mathcal{Z}$, which is a specialization of $\xi(0)$. □

Proof of Theorem 5.2. Lemma 5.11 shows that (1) ⇒ (2). Lemma 5.12 combined with Lemma 5.14 shows that under the locally linearly reductive and Θ-reductive hypotheses, (2) ⇒ (3). We have seen in Theorem 4.1 that under the locally linearly reductive and Θ-reductive hypotheses (3) ⇒ (4), so what remains is to show that (4) ⇒ (1).

Suppose that $\mathcal{X} \to X$ is a good moduli space, and let $\xi\colon \mathrm{Spec}(R) \to \mathcal{X}$ be a morphism, and let g be an automorphism of ξ_K of finite order. Then we may choose an étale map $U \to X$ whose image contains the image of $\mathrm{Spec}(R)$ and such that $\mathcal{U} := \mathcal{X} \times_X U \simeq [\mathrm{Spec}(A)/G]$ for a reductive group G [AHR]. After replacing R with an extension of DVR's we may assume that ξ' lifts to a map $\xi'\colon \mathrm{Spec}(R') \to \mathcal{U}$. Furthermore the map $\mathcal{U} \to \mathcal{X}$ is inertia preserving in the sense that $I_{\mathcal{U}} \simeq I_{\mathcal{X}} \times_{\mathcal{X}} \mathcal{U}$, which implies that g lifts to a finite order generic automorphism g' of $\xi'_{K'}$. By Proposition 5.7 the stack $\mathcal{U}$ satisfies condition (1) of Theorem 5.2. This provides a nearby modification of the map ξ' for which g' extends, and we can compose this with the map $\mathcal{U} \to \mathcal{X}$ to get a nearby modification of the original map for which $g|_{K'}$ extends. □

5.3. **The proof of Theorem 5.4.** Let $\mathcal{X}$ be a noetherian algebraic stack with affine stabilizers, let $p\colon \mathcal{Y} \to \mathcal{X}$ be an étale map with $\mathcal{Y} \cong [\mathrm{Spec}(A)/G]$, and let R be a complete DVR with fraction field K and residue field κ. Let $x \in |\mathcal{X}|$ be a closed point such that p induces an isomorphism $p^{-1}(\mathcal{G}_x) \simeq \mathcal{G}_x$, where $\mathcal{G}_x$ denotes the residual gerbe of x.

Lemma 5.15. *The functor* $\mathrm{Map}(\mathrm{Spec}(R), \mathcal{Y}) \to \mathrm{Map}(\mathrm{Spec}(R), \mathcal{X})$ *defined by composition with* p *induces an equivalence between the full subgroupoids of maps taking the special point of* $\mathrm{Spec}(R)$ *to* $p^{-1}(x)$ *and* x *respectively. The same is* *true for the functor* $\mathrm{Map}(\overline{\mathrm{ST}}_R, \mathcal{Y}) \to \mathrm{Map}(\overline{\mathrm{ST}}_R, \mathcal{X})$ *and the subgroupoid taking the point* $0 \in \overline{\mathrm{ST}}_R(\kappa)$ *to* $p^{-1}(x)$ *and* x *respectively.*

Proof. The map p is étale and induces an equivalence between the residual gerbe of $x \in |\mathcal{X}|$ and $p^{-1}(x) \in |\mathcal{Y}|$. It therefore induces an equivalence between the n^{th} order neighborhoods of these residual gerbes, so any map $\mathrm{Spec}(R) \to \mathcal{X}$ mapping 0 to x lifts uniquely along p over any nilpotent thickening of $0 \in \mathrm{Spec}(R)$. The result then follows from Tannaka duality and the fact that $\mathrm{Spec}(R)$ is coherently complete along its special point, so a compatible family of lifts over nilpotent thickenings of 0 corresponds to a unique lift of the map $\mathrm{Spec}(R) \to \mathcal{X}$ along p. The same argument applies to $\overline{\mathrm{ST}}_R$, which is coherently complete along the inclusion $(B\mathbb{G}_m)_\kappa \hookrightarrow \overline{\mathrm{ST}}_R$ at the point 0. □

Now let $\xi' \colon \mathrm{Spec}(R) \to \mathcal{Y}$ be an R-point mapping the special point to $p^{-1}(x)$, and let $\xi = p \circ \xi'$.

Lemma 5.16. *If* $\mathcal{X}$ *is* S*-complete, then* p *induces an isomorphism* $\mathrm{Aut}_{\mathcal{Y}}(\xi'_K) \to \mathrm{Aut}_{\mathcal{X}}(\xi_K)$.

Proof. For any map $f \colon \overline{\mathrm{ST}}_R \setminus 0 \to \mathcal{X}$, let f_1 and f_2 denote the two R-points resulting from f. For any stack $\mathcal{X}$ and $\xi \in \mathcal{X}(R)$, we have an equivalence of groupoids:

$$\{\text{automorphisms of } \xi_K\} \simeq \{\text{maps } \overline{\mathrm{ST}}_R \setminus 0 \to \mathcal{X} + \text{ equivalences } f_1 \simeq \xi \simeq f_2\}$$

Because both $\mathcal{Y}$ and $\mathcal{X}$ are S-complete, restriction gives an equivalence of groupoids $\mathrm{Map}(\overline{\mathrm{ST}}_R, -) \to \mathrm{Map}(\overline{\mathrm{ST}}_R \setminus 0, -)$ for both stacks. It follows that

$$\mathrm{Aut}_{\mathcal{X}}(\xi_K) \simeq \{\text{maps } \overline{\mathrm{ST}}_R \to \mathcal{X} + \text{ equivalences } f_1 \simeq \xi \simeq f_2\} \tag{5.1}$$

and likewise for $\mathcal{Y}$. If ξ maps the special point of $\mathrm{Spec}(R)$ to $x \in \mathcal{X}$, then any map $f \colon \overline{\mathrm{ST}}_R \to \mathcal{X}$ which admits an isomorphism $f_1 \simeq \xi$ must also map $(0,0)$ to x, because x is closed. Now the previous lemma implies that p induces a bijection of sets on the right hand side of (5.1) for $\mathcal{X}$ and $\mathcal{Y}$, and thus also on the left hand side.

□

Remark 5.17. Note that the conclusion of this lemma also applies without assuming that $\mathcal{Y} = [\mathrm{Spec}(A)/G]$—it suffices to assume $\mathcal{Y}$ is S-complete, or that the map p is affine (which implies that $\mathcal{Y}$ is S-complete).

Proof of Theorem 5.4. We first verify criterion (1) of Theorem 5.2. Consider a map from a DVR $\xi \colon \mathrm{Spec}(R) \to \mathcal{X}$. Let $x \in |\mathcal{X}|$ be a closed point in the closure of $\xi(0)$, and let $p : [\mathrm{Spec}(A)/\mathrm{GL}_N] \to \mathcal{X}$ be a local quotient presentation around x (see Definition 2.2). Then after an extension of DVR's R'/R we may lift ξ to a map $\xi' : \mathrm{Spec}(R') \to [\mathrm{Spec}(A)/\mathrm{GL}_N]$. Now Lemma 5.14 allows one to construct a modification of ξ', after replacing R' with a further finite extension, which maps the special point to the closed point $p^{-1}(x) \in [\mathrm{Spec}(A)/\mathrm{GL}_N]$. It follows from Lemma 5.16 that the map

$$\mathrm{Aut}_{[\mathrm{Spec}(A)/\mathrm{GL}_N]}(\xi'_{K'}) \to \mathrm{Aut}_{\mathcal{X}}(p \circ \xi'_{K'})$$

is an isomorphism of K'-groups. In particular given a finite order element $g \in \mathrm{Aut}_{\mathcal{X}}(\xi_K)$, one may lift this to ξ' after replacing R' with a further extension. We know that the criterion (1) of Theorem 5.2 holds for $[\mathrm{Spec}(A)/\mathrm{GL}_N]$ by Proposition 5.7, and after replacing R' with a further extension this produces a nearby modification for which $g|_{K'}$ extends. Composing with p gives a nearby modification of the original map ξ for which g extends. The same argument shows that $\mathcal{X}$ satisfies the criterion (2).

Finally, the previous paragraph shows that in verifying the criterion (2), we could choose the modification ξ' of ξ in such a way that $\xi'(0)$ is a specialization of $\xi(0)$. It follows from Lemma 5.12 that $\mathcal{X}$ has unpunctured inertia. □

6. Semistable reduction and Θ-stability

In this section we explain how completeness properties of stacks induce similar properties of the substack of semistable objects, if these are defined using the theory of Θ-stability. Our key result is Theorem 6.3 that is inspired by Langton's algorithm for semistable reduction for families of torsion-free sheaves on a projective variety. Recall from Remark 3.36 that this algorithm starts with a family of bundles parametrized by a DVR R such that the generic fiber is semistable and the special fiber is unstable, and then applies elementary modifications to arrive at a semistable family. Surprisingly, it turns out that his construction admits an analog that relies only on the geometry of the algebraic stack representing the moduli problem, not on the particular type of objects classified by the moduli problem. The structure we will need is that of a Θ-stratification from [Hal14, Def. 2.1] that formalizes the notion of canonical filtrations in geometric terms.

Definition 6.1. Let $\mathcal{X}$ be an algebraic stack locally of finite type over a noetherian algebraic space S.

(1) A Θ-*stratum in* $\mathcal{X}$ consists of a union of connected components $\mathcal{S} \subset \underline{\mathrm{Map}}_S(\Theta, \mathcal{X})$ such that $\mathrm{ev}_1 \colon \mathcal{S} \to \mathcal{X}$ is a closed immersion.
(2) A Θ-*stratification of* $\mathcal{X}$ indexed by a totally ordered set Γ is a cover of $\mathcal{X}$ by open substacks $\mathcal{X}_{\leq c}$ for $c \in \Gamma$ such that $\mathcal{X}_{\leq c} \subset \mathcal{X}_{\leq c'}$ for $c < c'$, along with a Θ-stratum $\mathcal{S}_c \subset \underline{\mathrm{Map}}_S(\Theta, \mathcal{X}_{\leq c})$ in each $\mathcal{X}_{\leq c}$ whose complement is $\bigcup_{c'<c} \mathcal{X}_{\leq c'} \subset \mathcal{X}_{\leq c}$. We require that $\forall x \in |\mathcal{X}|$ the subset $\{c \in \Gamma | x \in \mathcal{X}_{\leq c}\}$ has a minimal element. We assume for convenience that Γ has a minimal element $0 \in \Gamma$.
(3) We say that a Θ-stratification is *well-ordered* if for any point $x \in |\mathcal{X}|$, the totally ordered set $\{c \in \Gamma | \, \mathrm{ev}_1(\mathcal{S}_c) \cap \overline{\{x\}} \neq \emptyset\}$ is well-ordered.

Remark 6.2. It will be convenient for us to identify a Θ-stratum $\mathcal{S}$ with the closed substack it defines on $\mathcal{X}$, i.e., we will sometimes say that a closed substack $\mathcal{S} \subset \mathcal{X}$ is a Θ-stratum, if there exist a union of connected components $\mathcal{S}' \subset \underline{\mathrm{Map}}_S(\Theta, \mathcal{X})$ such that $\mathrm{ev}_1 \colon \mathcal{S}' \to \mathcal{S} \subset \mathcal{X}$ is an isomorphism.

Our notation differs slightly from [Hal14], which denotes the stack $\underline{\mathrm{Map}}_S(\Theta, \mathcal{X})$ by $\mathrm{Filt}(\mathcal{X})$ to promote the analogy of maps $\Theta_k \to \mathcal{X}$ as filtered objects in $\mathcal{X}$. In addition $\underline{\mathrm{Map}}_S(B\mathbb{G}_m, \mathcal{X})$ is denoted $\mathrm{Grad}(\mathcal{X})$ in order to promote the analogy of maps $B_k\mathbb{G}_m \to \mathcal{X}$ as graded objects in $\mathcal{X}$. Given a Θ-stratification, we denote the open substack $\mathcal{X}^{\mathrm{ss}} := \mathcal{X}_{\leq 0}$ as the semistable locus. For any unstable point $x \in \mathcal{X}(k) \setminus \mathcal{X}^{\mathrm{ss}}(k)$, the Θ-stratification determines a canonical filtration $f \colon \Theta_k \to \mathcal{X}$ with $f(1) \simeq x$, which we refer to as the *HN filtration*.

Restricting a map $f \colon \Theta \to \mathcal{X}$ to $B\mathbb{G}_m \hookrightarrow \Theta$ defines a map $\mathrm{ev}_0 \colon \underline{\mathrm{Map}}_S(\Theta, \mathcal{X}) \to \underline{\mathrm{Map}}_S(B\mathbb{G}_m, \mathcal{X})$ which corresponds to "passing to the associated graded object" of the filtration f. Composition with the projection $\Theta \to B\mathbb{G}_m$ defines a section $\sigma \colon \underline{\mathrm{Map}}_S(B\mathbb{G}_m, \mathcal{X}) \to \underline{\mathrm{Map}}_S(\Theta, \mathcal{X})$ of the map ev_0 which corresponds to the "canonical filtration of a graded object." These maps define a canonical $\mathbb{A}^1$

deformation retract of $\underline{\mathrm{Map}}_S(\Theta, \mathcal{X})$ onto $\underline{\mathrm{Map}}_S(B\mathbb{G}_m, \mathcal{X})$, and in particular induce bijections on connected components [Hal14, Lem. 1.24]. We refer to the union of connected components $\mathcal{Z} \subset \underline{\mathrm{Map}}_S(B\mathbb{G}_m, \mathcal{X})$ corresponding to $\mathcal{S}$ as the *center* of the Θ-stratum $\mathcal{S}$. The result is a diagram

$$\mathcal{Z} \underset{\mathrm{ev}_0}{\overset{\sigma}{\rightleftarrows}} \mathcal{S} \xhookrightarrow{\mathrm{ev}_1} \mathcal{X}.$$

6.1. **The semistable reduction theorem.**

Theorem 6.3 (Langton's algorithm)**.** *Let $\mathcal{X}$ be an algebraic stack locally of finite type with affine diagonal over a noetherian algebraic space S, and let $\mathcal{S} \hookrightarrow \mathcal{X}$ be a Θ-stratum. Let R be a DVR with fraction field K and residue field κ. Let $\xi_R \colon \mathrm{Spec}(R) \to \mathcal{X}$ be an R-point such that the generic point ξ_K is not mapped to $\mathcal{S}$, but the special point ξ_k is mapped to $\mathcal{S}$:*

$$\begin{array}{ccccc}
\mathrm{Spec}(K) & \hookrightarrow & \mathrm{Spec}(R) & \hookleftarrow & \mathrm{Spec}(\kappa) \\
\downarrow{\scriptstyle \xi_K} & & \downarrow{\scriptstyle \xi_R} & & \downarrow{\scriptstyle \xi_\kappa} \\
\mathcal{X} - \mathcal{S} & \xhookrightarrow{j} & \mathcal{X} & \xhookleftarrow{\iota} & \mathcal{S}.
\end{array}$$

Then there exists an extension $R \to R'$ of DVRs with $K \to K' = \mathrm{Frac}(R')$ finite and an elementary modification $\xi'_{R'}$ of $\xi_{R'}$ such that $\xi'_{R'} \colon \mathrm{Spec}(R') \to \mathcal{X}$ lands in $\mathcal{X} - \mathcal{S}$.

Remark 6.4. In the proof of the above result we will apply the non-local slice theorem (Theorem 2.6) for algebraic stacks. As the proof of this result has not appeared, we give an alternative argument using [AHR15, Thm. 1.2], which requires the additional hypothesis that S is the spectrum of an algebraically closed field and that for any $x \in \mathcal{X}(k)$, the automorphism group G_x is smooth – this suffices, in particular, for stacks over a field of characteristic 0.

This theorem is stated for a single stratum, but it immediately implies a version for a stack with a Θ-stratification:

Theorem 6.5 (Semistable reduction)**.** *Let $\mathcal{X}$ be an algebraic stack, locally of finite type with affine diagonal over a noetherian algebraic space S, with a well-ordered Θ-stratification. Then for any morphism $\mathrm{Spec}(R) \to \mathcal{X}$, after an extension $R \to R'$ of DVRs with $K \to K' = \mathrm{Frac}(R')$ finite there is a modification $\mathrm{Spec}(R') \to \mathcal{X}$, obtained by a finite sequence of elementary modifications, whose image lies in a single stratum of $\mathcal{X}$.*

Proof. Beginning with a map $\xi_R \colon \mathrm{Spec}(R) \to \mathcal{X}$ such that $\xi_K \in \mathcal{S}_c$ and $\xi_\kappa \in \mathcal{S}_{c_0}$ for $c_0 > c$, we may apply Theorem 6.3 iteratively to obtain a sequence of finite extensions of R and elementary modifications of ξ with special point in $\mathcal{S}_{c_i}$ for $c_0 > c_1 > \cdots$. Each $\mathcal{S}_{c_i}$ meets $\overline{\xi_K}$, so the well-orderness condition guarantees that this procedure terminates, and it can only terminate when $c_i = c$. □

Remark 6.6. In the relative situation when $\mathcal{X}$ is defined over a base algebraic stack S, one can base change the structure of a Θ-stratification along a smooth map $S' \to S$, so both Theorem 6.3 and Theorem 6.5 extend immediately to the case of a quasi-separated and locally noetherian base stack S.

6.1.1. *Langton's algorithm in the basic situation.* The main idea of the proof is to reduce to the situation where $\mathcal{X} = [\mathrm{Spec}(A)/\mathbb{G}_m]$ is the quotient of an affine scheme by an action of $\mathbb{G}_m$, $\mathcal{Z} = [(\mathrm{Spec}\, A)^{\mathbb{G}_m}/\mathbb{G}_m]$ is the substack defined by the fixed point locus of the action and $\mathcal{S} = [\mathrm{Spec}(A/I_+)/\mathbb{G}_m]$ is the attracting substack, where

$$I_+ := (\bigoplus_{n>0} A_n)$$

is the graded ideal generated by elements of positive weight. In this basic situation the theorem will then follow from an elementary calculation. We will first explain the proof of this special case and then show how to reduce to the basic situation.

Lemma 6.7. *In the setting of Theorem 6.3 suppose in addition that* $\mathcal{X} = [\mathrm{Spec}(A)/\mathbb{G}_m]$ *for a graded ring* $A = \bigoplus_{n\in\mathbb{Z}} A_n$ *and that* $\mathcal{S} = [\mathrm{Spec}(A/I_+)/\mathbb{G}_m]$. *Then the conclusion of Theorem 6.3 holds.*

Proof. Let us denote $X := \mathrm{Spec}(A)$ and $S := \mathrm{Spec}(A/I_+)$. As $X \to \mathcal{X}$ is a $\mathbb{G}_m$-torsor, we can lift ξ to a map $\xi'_R \colon \mathrm{Spec}(R) \to \mathrm{Spec}(A)$, obtaining a diagram

$$\begin{array}{ccccc} \mathrm{Spec}(K) & \hookrightarrow & \mathrm{Spec}(R) & \hookleftarrow & \mathrm{Spec}(\kappa) \\ \downarrow{\scriptstyle \xi'_K} & & \downarrow{\scriptstyle \xi'_R} & & \downarrow{\scriptstyle \xi'_\kappa} \\ X - S & \overset{j}{\hookrightarrow} & \mathrm{Spec}(A) & \overset{\iota}{\hookleftarrow} & \mathrm{Spec}(A/I_+). \end{array}$$

As $\xi'_\kappa \in S = \mathrm{Spec}(A/I_+)$ and A/I_+ is generated by elements of non-positive weight, the $\mathbb{G}_m$-orbit of ξ'_κ, corresponding to a map of graded algebras $A/I_+ \to k[t^{\pm 1}]$ where t has weight -1, extends to an equivariant morphism $\mathbb{A}^1_\kappa \to S$. Thus the $\mathbb{G}_m$-orbits of the points $\xi'_K, \xi'_R, \xi'_\kappa$ define a diagram:

$$\begin{array}{ccccc} \mathbb{G}_{m,K} & \hookrightarrow & \mathbb{G}_{m,R} & \hookleftarrow & \mathbb{G}_{m,\kappa} \\ & & & & \downarrow \\ \downarrow{\scriptstyle f_K} & & \downarrow{\scriptstyle f_R} & & \mathbb{A}^1_\kappa \\ & & & & \downarrow{\scriptstyle \overline{f}_\kappa} \\ X - S & \overset{j}{\hookrightarrow} & \mathrm{Spec}(A) & \overset{\iota}{\hookleftarrow} & \mathrm{Spec}(A/I_+). \end{array}$$

We know that $f_R^\#(I_+) \in \pi(R[t^{\pm 1}])$ since $f_\kappa^\#$ factors through A/I_+, and we have $K[t^{\pm 1}] \cdot f_R^\#(I_+) = K[t^{\pm 1}]$ since the image of f_K does not intersect S.

Let $a_i \in I_{d_i}$ be homogeneous generators of I_+. Then for all i we have $f_R^\#(a_i) = \epsilon_i \pi^{n_i} t^{-d_i}$ for some $n_i > 0$ and $\epsilon_i \in R^\times \cup \{0\}$. As $f_R^\#(I_+)$ is not 0 we can define

$$\frac{m}{d} := \min_i \left\{ \frac{n_i}{d_i} \,\middle|\, f_R^\#(a_i) \neq 0 \right\}$$

and let $R' := R[\pi^{\frac{1}{d}}]$. Since $n_i - \frac{d_i m}{d} \geq 0$ for each i, we can write

$$f_R^\#(a_i) = \epsilon_i (\pi^{n_i - \frac{d_i m}{d}})(\pi^{\frac{m}{d}} x^{-1})^{d_i} = \epsilon_i (\pi^{n_i - \frac{d_i m}{d}}) s^{d_i}.$$

Since $f_R^\#$ maps elements of negative weight to $R[t]$, we have a homomorphism of graded rings

$$f'^\#_{R'} \colon A \to R'[s,t]/(st - \pi^{\frac{m}{d}}) = R'[t, \pi^{\frac{m}{d}} t^{-1}] \subset R'[t, t^{-1}]$$

Furthermore, composing with the map setting $s = 1$ at least one $f'^\#_{R'}(a_i)$ is not mapped to 0 mod $\pi^{\frac{1}{d}}$, i.e. $f'_{R'}|_{\{s=1\}} \colon \mathrm{Spec}(R') \mapsto \mathrm{Spec}(A_{a_i}) \subset X - S$. The graded homomorphism $f'^{\#}_{R'}$ defines a morphism

$$[\mathrm{Spec}\left(R'[s,t]/(st-\pi^{\frac{m}{d}})\right)/\mathbb{G}_m] \to \mathcal{X} = [X/\mathbb{G}_m].$$

As $\pi^{\frac{m}{d}}$ is not a uniformizer for R', this is not quite an elementary modification. However, we can embed $R'[s,t]/(st-\pi^{m/d}) \subset R'[s^{1/m}, t^{1/m}]/(s^{1/m}t^{1/m} - \pi^{1/d})$. If we regard $s^{1/m}$ and $t^{1/m}$ as having weight 1 and -1 respectively, the map $\mathrm{Spec}(R'[s^{1/m}, t^{1/m}]/(s^{1/m}t^{1/m} - \pi^{1/d})) \to \mathrm{Spec}(R'[s,t]/(st-\pi^{m/d}))$ is equivariant with respect to the group homomorphism $\mathbb{G}_m \to \mathbb{G}_m$ given in coordinates by $z \mapsto z^m$. The resulting composition

$$\overline{\mathrm{ST}}_{R'} \to [\mathrm{Spec}\left(R'[s,t]/(st-\pi^{\frac{m}{d}})\right)/\mathbb{G}_m] \to \mathcal{X}$$

is the desired modification of ξ_R.

□

6.1.2. *Reduction to quasi-compact stacks.* We first show that by replacing $\mathcal{X}$ by a suitable open substack we may assume that $\mathcal{X}$ is quasi-compact.

Lemma 6.8. *In the setting of Theorem 6.3, let $\sigma \colon \mathcal{Z} \to \mathcal{S}$ be the center of the Θ-stratum $\mathrm{ev}_1 \colon \mathcal{S} \hookrightarrow \mathcal{X}$. Then for any point $x \in |\mathcal{Z}|$ and any open substack $\mathcal{U} \subset \mathcal{X}$ containing $\sigma(x)$, there is another open substack with $\sigma(x) \in \mathcal{V} \subset \mathcal{U}$ such that $\mathcal{S} \cap \mathcal{V}$ is a Θ-stratum in $\mathcal{V}$.*

Proof. We only need to find a substack $\mathcal{V} \subset \mathcal{X}$ containing $\sigma(x)$ such that for any $f \colon \Theta_k \to \mathcal{X}$, where k is a field, with $f \in \mathcal{S}$ and $f(1) \in \mathcal{V}$, we have $f(0) \in \mathcal{V}$ as well. Let $\mathcal{U}' = (\mathrm{ev}_1 \circ \sigma)^{-1}(\mathcal{U}) \subset \mathcal{Z}$, and let $\mathcal{Z}' = \mathcal{Z} \setminus \mathcal{U}'$ be its complement. Then the open substack

$$\mathcal{V} := \mathcal{U} \setminus (\mathcal{U} \cap \mathrm{ev}_1(\mathrm{ev}_0^{-1}(\mathcal{Z}'))) \subset \mathcal{X}$$

satisfies the condition. □

6.1.3. *Reminder on the normal cone to a Θ-stratum.* The main problem in finding a presentation of the form $[\mathrm{Spec}(A/I_+)/\mathbb{G}_m] \subset [\mathrm{Spec}(A)/\mathbb{G}_m]$ is that for an arbitrary morphism $[\mathrm{Spec}(A)/\mathbb{G}_m] \to \mathcal{X}$ the preimage of the Θ-stratum need not be defined by the ideal generated by the elements of positive weight. To find presentations for which this happens, we need to recall that the weights of the $\mathbb{G}_m$-action of the restriction of the conormal bundle of a Θ-stratum to its center $\mathcal{Z}$ are automatically positive. This property was already important in the work of Atiyah–Bott [AB83] and it appears in the language of spectral stacks in [Hal14, §1.2]. For completeness we provide a classical argument:

Lemma 6.9. *In the setting of Theorem 6.3, let $\sigma \colon \mathcal{Z} \to \mathcal{S}$ be the center of the Θ-stratum $\mathrm{ev}_1 \colon \mathcal{S} \hookrightarrow \mathcal{X}$ and $x \in \mathcal{Z}(k)$ be a k-point. By abuse of notation we will also denote $\sigma(x) \in \mathcal{X}(k)$ by x.*

1. *Let $T_{\mathcal{X},x} = \bigoplus_{n \in \mathbb{Z}} T_{\mathcal{X},x,n}$ be the decomposition of the tangent space at x into weight spaces with respect to the $\mathbb{G}_m$-action induced form the canonical cocharacter $\lambda_x \colon \mathbb{G}_m \to \mathrm{Aut}_{\mathcal{X}}(x)$. Then we have $T_{\mathcal{S},x} = \bigoplus_{n \geq 0} T_{\mathcal{X},x,n}$.*
2. *$\mathbb{G}_m$ acts with non-negative weights on $\mathrm{Lie}(Aut_{\mathcal{X}}(x))$.*

Proof. Let us first show that $\bigoplus_{n \geq 0} T_{\mathcal{X},x,n} \subseteq T_{\mathcal{S},x}$. Let $t \in \mathcal{X}(k[\epsilon]/\epsilon^2)$ be a tangent vector in $T_{\mathcal{X},x,n}$ for some $n \leq 0$, i.e. t comes equipped with an isomorphism $t \mod \epsilon \cong x$.

This means that we have a commutative diagram

$$\begin{array}{ccc} [\operatorname{Spec}(k[\epsilon]/\epsilon^2)/\mathbb{G}_m] & \xrightarrow{t} & \mathcal{X}, \\ \big\uparrow & \nearrow_{(x,\lambda_x)} & \\ [\operatorname{Spec}(k)/\mathbb{G}_m] & & \end{array}$$

where $\mathbb{G}_m$ acts on $\operatorname{Spec}(k[\epsilon]/\epsilon^2)$ via $(\lambda, \epsilon) \mapsto \lambda^n \epsilon$. In other words, we have a commutative diagram

$$\begin{array}{ccc} \mathbb{G}_m \times \operatorname{Spec}(k[\epsilon]/\epsilon^2) & \xrightarrow{(\lambda,\epsilon\mapsto\lambda^n\epsilon)} & \operatorname{Spec}(k[\epsilon]/\epsilon^2) \\ & \searrow & \big\downarrow t \\ & & \mathcal{X}. \end{array}$$

If $n \geq 0$ then the horizontal map extends to $\mathbb{A}^1$, i.e., we get an extension

$$\begin{array}{ccc} \mathbb{A}^1 \times \operatorname{Spec}(k[\epsilon]/\epsilon^2) & \xrightarrow{(\lambda,\epsilon\mapsto\lambda^n\epsilon)} & \operatorname{Spec}(k[\epsilon]/\epsilon^2) \\ & \searrow & \big\downarrow t \\ & & \mathcal{X} \end{array}$$

and this defines an extension of t to a $k[\epsilon]/\epsilon^2$-valued point of $\underline{\operatorname{Map}}_S(\Theta, \mathcal{X})$.

Conversely, an extension of the constant map $[\mathbb{A}^1/\mathbb{G}_m] \to [\operatorname{Spec}(k)/\mathbb{G}_m] \to \mathcal{X}$ to $[\mathbb{A}^1 \times \operatorname{Spec}(k[\epsilon]/(\epsilon^2))/\mathbb{G}_m] \to \mathcal{X}$ automatically factors through the first infinitesimal neighborhood of $x \in \mathcal{X}$. On a versal first order deformation this corresponds to a homomorphism of graded algebras $k[\epsilon_1, \dots, \epsilon_d]/(\epsilon_i)^2_{i=1,\dots d} \to k[\lambda, \epsilon]/(\epsilon^2)$, where we can choose ϵ_i to be homogeneous for the $\mathbb{G}_m$-action defined by λ_x. This has to vanish on those tangent directions ϵ_i on which λ_x acts with negative weights. This shows (1).

Similarly for (2), when we regard x as a k point of $\mathcal{Z} \hookrightarrow \mathcal{S} \subset \underline{\operatorname{Map}}_S(\Theta, \mathcal{X})$, it corresponds to a map which factors as $\Theta_k \to B_k G_x \hookrightarrow \mathcal{X}$, where we abbreviated $G_x = \operatorname{Aut}_{\mathcal{X}}(x)$. We know that $\operatorname{Aut}_{\mathcal{S}}(x) \to \operatorname{Aut}_{\mathcal{X}}(x)$ is an equivalence, so by the classification of G_x-bundles on $[\mathbb{A}^1/\mathbb{G}_m]$ (see [Hei17, Lem. 1.7] or [Hal14, Prop. A.1]) this implies that for the canonical cocharacter $\lambda_x \colon \mathbb{G}_m \to G_x$ we have $G_x = P(\lambda_x)$ as an algebraic group. In particular this means that $\mathbb{G}_m$ acts with non-negative weights on the Lie algebra of $G_x = P(\lambda_x)$. □

6.1.4. *Reduction to the basic situation - Case of smooth stabilizers over a field.*

Lemma 6.10. *Let $\mathcal{X}$ be an algebraic stack of finite type with affine diagonal over an algebraically closed field k. Let $\mathcal{S} \subset \mathcal{X}$ be a Θ-stratum with center $\sigma \colon \mathcal{Z} \to \mathcal{S}$, and let $x_0 \in \mathcal{Z}(k)$ be a point such that $x := \sigma(x_0)$ has a smooth automorphism group. Then there is a smooth representable morphism $p \colon [\operatorname{Spec}(A)/\mathbb{G}_m] \to \mathcal{X}$ whose image contains x and such that*

$$p^{-1}(\mathcal{S}) = [\operatorname{Spec}(A/I_+)/\mathbb{G}_m] \hookrightarrow [\operatorname{Spec}(A)/\mathbb{G}_m].$$

Proof. The point x_0 has a canonical non-constant homomorphism $\mathbb{G}_m \to \operatorname{Aut}_{\mathcal{Z}}(x_0)$, which induces a canonical homomorphism $\lambda : \mathbb{G}_m \to G_x := \operatorname{Aut}_{\mathcal{X}}(x)$. We may replace $\mathbb{G}_m$ with its image in G_x and thus assume that λ is injective. As we assumed that G_x is smooth the quotient $G_x/\lambda(\mathbb{G}_m)$ is smooth, so we may apply [AHR15, Thm. 1.2] to obtain a smooth representable morphism

$$p\colon [\mathrm{Spec}(A)/\mathbb{G}_m] \to \mathcal{X}$$

together with a point $w \in \mathrm{Spec}(A)(k)$ in $p^{-1}(x)$ which is fixed by $\mathbb{G}_m$ and such that $p^{-1}(B_k G_x) \cong B_k\mathbb{G}_m$. The isomorphism $p^{-1}(B_k G_x) \cong B_k\mathbb{G}_m$ implies that the relative tangent space to $\tilde{p}\colon \mathrm{Spec}(A) \to \mathcal{X}$ at w is naturally identified with $\mathrm{Lie}(G_x)/\mathrm{Lie}(\mathbb{G}_m)$ on which $\mathbb{G}_m$ acts with non-negative weights by part (2) of Lemma 6.9.

Note that connected components of $\mathrm{Spec}(A)^{\mathbb{G}_m}$ can be separated by invariant functions, so we may replace $\mathrm{Spec}(A)$ with a $\mathbb{G}_m$-equivariant affine open neighborhood of w so that $\mathrm{Spec}(A)^{\mathbb{G}_m}$ is connected. It follows that $\mathrm{Spec}(A/I_+)$ is connected as well.

This implies that $\mathcal{S}_A := [\mathrm{Spec}(A/I_+)/\mathbb{G}_m] \subseteq [\mathrm{Spec}(A)/\mathbb{G}_m]$ is isomorphic to a connected component of $\underline{\mathrm{Map}}(\Theta, [\mathrm{Spec}(A)/\mathbb{G}_m])$ and $\mathcal{Z}_A := [\mathrm{Spec}(A^{\mathbb{G}_m})/\mathbb{G}_m] \subset \mathcal{S}_A$ is the center of $\mathcal{S}_A$. As $p(x) \in \mathcal{Z}$ connectedness now implies that $p(\mathcal{Z}_A) \subset \mathcal{Z}_0$ and therefore we also have $[\mathrm{Spec}(A/I_+)/\mathbb{G}_m] \subset p^{-1}(\mathcal{S})$.

To conclude that $\mathcal{S}_A \cong p^{-1}(\mathcal{S})$ after possibly shrinking A, it suffices to check that the inclusion $[\mathrm{Spec}(A/I_+)/\mathbb{G}_m] \subseteq p^{-1}(\mathrm{ev}_1(\mathcal{S}))$ of closed substacks of $[\mathrm{Spec}(A)/\mathbb{G}_m]$ is an isomorphism locally at w. Consider the pull-back:

$$\begin{array}{ccc} p^{-1}(\mathrm{ev}_1(\mathcal{S})) = \mathrm{Spec}(B) & \hookrightarrow & \mathrm{Spec}(A) \\ \downarrow & & \downarrow{\scriptstyle p} \\ \mathcal{S} & \hookrightarrow & \mathcal{X} \end{array}$$

Then B is a graded ring and we still have an exact sequence

$$T_{p,w} \to T_{\mathrm{Spec}(B),w} \to \mathcal{T}_{\mathcal{S},x}.$$

As $\mathbb{G}_m$ acts with non-negative weight on the relative tangent bundle at w and also on $\mathcal{T}_{\mathcal{S},x}$ by Lemma 6.9, this shows that $\mathbb{G}_m$ acts with non-negative weights on $T_{\mathrm{Spec}(B),w}$. In particular the maximal ideal $\mathfrak{m}_w \subset B$ of w is generated by elements of non-positive weight locally at w.

Therefore, after possibly shrinking A we may assume that $B = \bigoplus_{n \leq 0} B_n$ is non-positively graded. As $\mathrm{Spec}(A/I_+) \subset \mathrm{Spec}(A)$ was the contracting subscheme for $\mathbb{G}_m$ we find that locally around w we thus have $p^{-1}(\mathcal{S}) \subset \mathrm{Spec}(A/I_+)$ locally around w. This proves our claim. □

6.1.5. *Reduction to the basic situation - general case.*

Lemma 6.11. *In the setting of Theorem 6.3 where $\mathcal{S} \hookrightarrow \mathcal{X}$ is a Θ-stratum and $\mathcal{X}$ is quasi-compact, there is a smooth representable morphism $p : [\mathrm{Spec}(A)/\mathbb{G}_m] \to \mathcal{X}$ such that $p^{-1}(\mathcal{S})$ is the Θ-stratum*

$$p^{-1}(\mathcal{S}) = [\mathrm{Spec}(A/I_+)/\mathbb{G}_m] \hookrightarrow [\mathrm{Spec}(A)/\mathbb{G}_m],$$

and $\mathcal{S}$ is contained in the image of p.

Proof. Because $\mathcal{X}$ is finite type over the base space S, we may apply Lemma 3.2 to obtain a smooth, surjective and representable map $p\colon [\mathrm{Spec}(A)/\mathbb{G}_m^n] \to \mathcal{X}$ such that $\mathrm{Map}(\Theta, [\mathrm{Spec}(A)/\mathbb{G}_m^n]) \to \mathrm{Map}(\Theta, \mathcal{X})$ is also smooth, surjective and representable. From Proposition 2.8, we know that $\mathrm{Map}(\Theta, [\mathrm{Spec}(A)/\mathbb{G}_m^n])$ is the disjoint union indexed by cocharacters $\mathbb{G}_m \to \mathbb{G}_m^n$ of stacks of the form $[\mathrm{Spec}(A/I_+)/\mathbb{G}_m^n]$, where I_+ is the ideal generated by positive weight elements with respect to a given cocharacter. Choosing different connected components if necessary and forgetting all but the relevant cocharacter in each component, we can construct a non-positively graded algebra $C = \bigoplus_{n \le 0} C_n$ along with a smooth surjective representable map $[\mathrm{Spec}(C)/\mathbb{G}_m] \to \mathcal{S}$.

We now discard the previously constructed $\mathrm{Spec}(A)$ and apply the relative slice theorem (Theorem 2.6) to the smooth surjective map $[\mathrm{Spec}(C)/\mathbb{G}_m] \to \mathcal{S}$, where we regard $\mathcal{S}$ as a closed substack of $\mathcal{X}$. This provides a map $p : [\mathrm{Spec}(A')/\mathbb{G}_m] \to \mathcal{X}$ along with an isomorphism $C \simeq A'/I_S$, where $I_S \subset A'$ is the ideal corresponding to $p^{-1}(\mathcal{S})$. By construction C has no positive weight elements, so the ideal I_+ generated by positive weight elements of A' is contained in I_S.

Because p is smooth, the relative cotangent complex of $\mathrm{Spec}(C) \hookrightarrow \mathrm{Spec}(A')$ is $p^*(\mathbb{L}_{\mathcal{S}/\mathcal{X}})$. In particular, the fiber of the conormal bundle of $\mathrm{Spec}(C) \hookrightarrow \mathrm{Spec}(A')$ has positive weights at every point of $\mathrm{Spec}(C)^{\mathbb{G}_m}$ by Lemma 6.9. One may therefore find a collection of positive weight elements of I_S which generate the fiber of I_S at every closed point of $\mathrm{Spec}(C)^{\mathbb{G}_m}$.

Moreover, as C is non-positively graded, the orbit closure of every point in $\mathrm{Spec}(C)$ meets the fixed locus $\mathrm{Spec}(C)^{\mathbb{G}_m}$. So by Nakayama's lemma we can actually find a collection of homogeneous elements of I_+ which generate the fiber of I_S at every point of $\mathrm{Spec}(C)$ and hence in a $\mathbb{G}_m$-equivariant open neighborhood of $\mathrm{Spec}(C) \hookrightarrow \mathrm{Spec}(A')$. We may thus invert a weight 0 element $a \in A'$ so that these elements of I_+ generate $(I_S)_a \subset A'_a$ and $C = A'/I_S = A'_a/(I_S)_a$ is unaffected.

In particular we have shown that after inverting a weight 0 element of A', we have a smooth map $p\colon [\mathrm{Spec}(A')/\mathbb{G}_m] \to \mathcal{X}$ such that $[\mathrm{Spec}(A'/I_+)/\mathbb{G}_m] = p^{-1}(\mathcal{S})$ and the map $\mathrm{Spec}(A'/I_+) \to \mathcal{S}$ is surjective.

□

We can now prove the semistable reduction theorem:

Proof of Theorem 6.3. Consider a map $\xi\colon \mathrm{Spec}(R) \to \mathcal{X}$ as in the statement of the theorem. Observe that for any smooth map $p\colon \mathcal{Y} \to \mathcal{X}$ such that $\mathcal{S}$ induces a Θ-stratum $p^{-1}(\mathcal{S})$ in $\mathcal{Y}$ and the image of p contains the image of ξ, if we know the conclusion of the theorem holds for $\mathcal{Y}$ then the conclusion holds for $\mathcal{X}$ as well: indeed after an extension of R we may lift ξ to a map $\xi'\colon \mathrm{Spec}(R') \to \mathcal{Y}$, construct an elementary modification in $\mathcal{Y}$ such that the new map $\xi''\colon \mathrm{Spec}(R') \to \mathcal{Y}$ lies in $\mathcal{Y} \setminus p^{-1}(\mathcal{S})$, and observe that the composition of this elementary modification with p gives an elementary modification of ξ such that the new map $p \circ \xi''\colon \mathrm{Spec}(R') \to \mathcal{X}$ lies in $\mathcal{X} \setminus \mathcal{S}$.

Using this observation and the fact that ξ_k lies in $\mathcal{S}$, we may use Lemma 6.8 to replace $\mathcal{X}$ with a quasi-compact open substack, then use Lemma 6.11 to construct a smooth map $p\colon [\mathrm{Spec}(A)/\mathbb{G}_m] \to \mathcal{X}$ whose image contains the image of ξ and for which $\mathcal{S}$ induces a Θ-stratum. Then we are finished by Lemma 6.7. □

6.2. **Comparison between a stack and its semistable locus.** As an immediate consequence of the semistable reduction theorem, we have the following:

Corollary 6.12. *Let $\mathcal{X}$ be an algebraic stack locally of finite type with affine diagonal over a noetherian algebraic space S. Let $\mathcal{X} = \bigcup_{c \in \Gamma} \mathcal{X}_{\le c}$ be a well-ordered Θ-stratification of $\mathcal{X}$. If $\mathcal{X} \to S$ satisfies the existence part of the valuative criterion for properness, then so does $\mathcal{X}_{\le c} \to S$ for every $c \in \Gamma$. In particular, if the semistable locus $\mathcal{X}^{\mathrm{ss}} := \mathcal{X}_{\le 0}$ is quasi-compact, then $\mathcal{X}^{\mathrm{ss}} \to S$ is universally closed.*

Proof. Consider a DVR R and a map $\mathrm{Spec}(R) \to S$ along with a lift $\mathrm{Spec}(K) \to \mathcal{X}_{\leq c}$. If $\mathcal{X} \to S$ satisfies the existence part of the valuative criterion, then after an extension of R one can extend this lift to a lift $\mathrm{Spec}(R') \to \mathcal{X}$ of $\mathrm{Spec}(R) \to S$. By hypothesis the generic point lies in $\mathcal{X}_{\leq c}$, so by Theorem 6.5 after passing to a further extension of R there is a sequence of elementary modifications resulting in a modification $\mathrm{Spec}(R') \to \mathcal{X}_{\leq c}$. Note that because $\mathrm{Spec}(R)$ is the good moduli space of $\overline{\mathrm{ST}}_R$, and good moduli spaces are universal for maps to an algebraic space [Alp13, Thm. 6.6], any elementary modification of a map $\mathrm{Spec}(R) \to S$ is trivial. It follows that our modified map $\mathrm{Spec}(R') \to \mathcal{X}_{\leq c}$ is a lift of the original map $\mathrm{Spec}(R) \to S$. □

Next let us briefly recall the notion of Θ-stability from [Hal14, Def. 4.1 & 4.4] and [Hei17, Def. 1.2].

Definition 6.13. Given a cohomology class $\ell \in H^2(\mathcal{X};\mathbb{R})$, we say that a point $p \in |\mathcal{X}|$ is *unstable* with respect to ℓ if there is a filtration $f\colon \Theta_k \to \mathcal{X}$ with $f(1) = p \in |\mathcal{X}|$ and such that $f^*(\ell) \in H^2(\Theta_k;\mathbb{R}) \simeq \mathbb{R}$ is positive. The Θ-*semistable locus* $\mathcal{X}^{\mathrm{ss}}$ is the set of points which are not unstable.

The above definition is simply an intrinsic formulation of the Hilbert–Mumford criterion for semistability in geometric invariant theory. We are somewhat flexible with what type of cohomology theory we use: if $\mathcal{X}$ is locally finite type over $\mathbb{C}$ we may use the Betti cohomology of the analytification of $\mathcal{X}$, if $\mathcal{X}$ is locally finite type over another field k, we can use Chow cohomology, and in general one may use the Neron–Severi group $NS(\mathcal{X})_{\mathbb{R}}$ for $H^2(\mathcal{X};\mathbb{R})$. In [Hal14, §3.7] we axiomatized the properties of the cohomology theory needed for the theory of Θ-stability.

Proposition 6.14. *Let $\mathcal{X}$ be an algebraic stack locally of finite type with affine diagonal over a noetherian algebraic space S, and let $\mathcal{X}^{\mathrm{ss}}$ be the Θ-semistable points with respect to a class $\ell \in H^2(\mathcal{X};\mathbb{R})$. Suppose that either*

- *(a) $\mathcal{X}^{\mathrm{ss}}$ is the open part of a Θ-stratification of $\mathcal{X}$, i.e. $\mathcal{X}^{\mathrm{ss}} = \mathcal{X}_{\leq 0}$, such that for each HN filtration $g\colon \Theta_k \to \mathcal{X}$ of an unstable point one has $g^*(\ell) > 0$ in $H^2(\Theta_k;\mathbb{R})$, or*
- *(b) $\mathcal{X}^{\mathrm{ss}} \subset \mathcal{X}$ is open and $\mathcal{X} \to S$ is Θ-reductive.*

Then

- *(1) if $\mathcal{X} \to S$ is S-complete, then so is $\mathcal{X}^{\mathrm{ss}} \to S$, and*
- *(2) if $\mathcal{X} \to S$ is Θ-reductive, then so is $\mathcal{X}^{\mathrm{ss}} \to S$.*

In the proof, we will need the following:

Lemma 6.15. *Under the hypotheses of Proposition 6.14, given a filtration $f\colon \Theta_k \to \mathcal{X}$ such that $f(1)$ is semistable with respect to ℓ, then $f^*(\ell) = 0$ if and only if $f(0)$ is semistable as well.*

Proof. The proof is a geometric reformulation of the corresponding argument for semistability for vector bundles. One direction is easy: for any semistable point $x \in \mathcal{X}(k)$ and any cocharacter $\lambda\colon \mathbb{G}_m \to G_x$, the restriction of ℓ to $H^2([\mathrm{Spec}(k)/\mathbb{G}_m];\mathbb{R}) \simeq \mathbb{R}$ along the resulting map $f_\lambda\colon \Theta_k \to [\mathrm{Spec}(k)/\mathbb{G}_m] \to \mathcal{X}$ must vanish, because the invariants for λ and λ^{-1} differ by sign and are both non-positive.

For the converse suppose that $f^*(\ell) = 0$ and $f(0) \notin \mathcal{X}^{\mathrm{ss}}$. We claim that there is a filtration $g\colon \Theta_k \to \mathcal{X}$ of $f(0)$ with $g^*(l) > 0$ which is invariant under the action of $\mathbb{G}_m$ on $f(0)$ induced from the filtration $f\colon \Theta_k \to \mathcal{X}$. This is automatic in case

(a) as HN filtrations are canonical. For case (b), since $\mathcal{X} \to S$ is Θ-reductive, the representable map $\underline{\mathrm{Map}}(\Theta, \mathcal{X}) \to \mathcal{X}$ satisfies the valuative criterion for properness, so the fiber of this map over $f(0) \in \mathcal{X}(k)$, which is denoted $\mathrm{Flag}(f(0))$ is an algebraic space of finite type over k which satisfies the valuative criterion for properness. The action of $\mathbb{G}_m$ by automorphisms of $f(0)$ gives a $\mathbb{G}_m$-action on $\mathrm{Flag}(f(0))$. Given some point $g_1 \in \mathrm{Flag}(f(0))(k)$ for which $g_1^*(\ell) > 0$, we can consider the orbit $\mathbb{G}_m \to \mathrm{Flag}(f(0))$ of g_1. Because $\mathrm{Flag}(f(0))$ satisfies the valuative criterion for properness, this map extends to an equivariant map $\mathbb{A}^1_k \to \mathrm{Flag}(f(0))$. This map sends $0 \in \mathbb{A}^1_k$ to a fixed point for the action of $\mathbb{G}_m$ on $\mathrm{Flag}(f(0))$, which corresponds to a $\mathbb{G}_m$-invariant filtration g of $f(0)$, and g is on the same connected component of $\mathrm{Flag}(f(0))$ as g_1, so $g^*(\ell) = g_1^*(\ell) > 0$.

Denote by $R = k[\![\pi]\!]$ the completion of the local ring of the affine line with coordinate π at 0. Then the map $f_R \colon \mathrm{Spec}(R) \to [\mathrm{Spec}(k[\pi])/\mathbb{G}_m] = \Theta_k \xrightarrow{f} \mathcal{X}$ and $g \colon \Theta_k \to \mathcal{X}$ define the datum needed to apply the gluing lemma Corollary A.2, which says that after restricting f_R to $R' = R[\pi^{1/n}]$ for $n \gg 0$ there is a unique extension $F_{R'} \colon \overline{ST}_{R'} \to \mathcal{X}$ such that $F|_{t \neq 0} \cong f_{R'}$ and $F|_{s=0} \cong g$. Let $\pi' = \pi^{1/n}$ denote the uniformizer in R'.

As $f_{R'}$ was the restriction of a map $f_{\mathbb{A}^1} \colon \mathbb{A}^1_k \to \Theta_k \to \mathcal{X}$ we find that this morphism extends canonically to

$$F \colon [\mathrm{Spec}(k[\pi', s, t]/(st - \pi'))/\mathbb{G}_m] = [\mathrm{Spec}(k[s,t])/\mathbb{G}_m] \to \mathcal{X}.$$

By uniqueness of the extension $F_{R'}$ and the fact that g is fixed by the $\mathbb{G}_m$-action on $f(0)$ induced by f, this morphism comes equipped with a descent datum for the standard $\mathbb{G}_m$-action on $\mathbb{A}^1 = \mathrm{Spec}(k[\pi'])$. We therefore obtain

$$\overline{F} \colon [\mathrm{Spec}(k[\pi', s, t]/(st - \pi'))/\mathbb{G}_m^2] = [\mathbb{A}^2_k/\mathbb{G}_m^2] \to \mathcal{X}.$$

where the action of the second copy of $\mathbb{G}_m$ is with weight -1 on s and trivial on t. Choosing a different basis for the cocharacter lattice of $\mathbb{G}_m^2$, we see that this is equivalent to the usual action of $\mathbb{G}_m^2$ on $\mathbb{A}^2_k$. In particular, every cocharacter $\lambda \colon \mathbb{G}_m \to \mathbb{G}_m^2$ has the form $\lambda(t) = (t^a, t^b)$ for some pair $\langle a, b \rangle$.

If $a, b \geq 0$, then the point $(1,1) \in \mathbb{A}^2_k$ has a limit under $\lambda(t)$ as $t \to 0$, and restricting $\overline{F}$ to the corresponding line we obtain a filtration $f_{\langle a,b \rangle} \colon \Theta_k \to \mathcal{X}$, with $f(1) \simeq f_{\langle a,b \rangle}(1)$. By construction the original filtration f corresponds to $f_{\langle 1,0 \rangle}$.

Note that $\overline{F}^*(\ell)|_{[0/\mathbb{G}_m^2]}$ defines a character χ of $\mathbb{G}_m^2$ such that for $a, b > 0$ the weight $f^*_{\langle a,b \rangle}(\ell)$ is given by the canonical pairing $(\langle a,b \rangle, \chi)$ between characters and cocharacters. As $F|_{\{s=0\}} = g$ was assumed to be destabilizing and F was defined by the subgroup $\langle -1, 1 \rangle$ we have $(\langle -1, 1 \rangle, \chi) = g^*(l) > 0$. We also have $(\langle 1, 0 \rangle, \chi) = f^*_{\langle 1,0 \rangle}(\ell) = 0$ by hypothesis. Thus for $a \gg b > 0$ we have $f^*_{\langle a,b \rangle}(\ell) > 0$, contradicting the assumption that $f(1)$ was semistable. □

Remark 6.16. In Theorem 7.25 below we will use a slightly more general notion of stability: we replace the weight of $f^*(\ell)$ with any function ℓ from the set of filtrations in $\mathcal{X}$ to a totally ordered real vector space V, and define $x \in \mathcal{X}$ to be semistable if $\ell(f) \leq 0$ for any filtration f with $f(1) = x$. Then the proof above applies verbatim, provided that 1) $\ell(f)$ is locally constant in algebraic families of filtrations, and 2) for any map $F \colon [\mathbb{A}^2_k/\mathbb{G}_m^2] \to \mathcal{X}$ the function on the cocharacter lattice $\lambda \mapsto \ell(f_\lambda) \in V$ is linear, where f_λ denotes the filtration associated to the cocharacter λ as in the proof of Lemma 6.15. The second condition is equivalent to requiring the function $\lambda \mapsto \ell(g_\lambda)$ is linear, where g_λ denotes the constant filtration of $F(0,0)$ induced from the $\mathbb{Z}$-grading of $F(0,0)$ associated to the cocharacter λ.

Remark 6.17. The proof of Lemma 6.15 is a special case of the technique used to prove the perturbation theorem on filtrations [Hal14, Thm. 3.60]. This theorem constructs a bijection between filtrations of a point $x \in \mathcal{X}$ which are "close" to a given filtration f and filtrations of the associated graded object $f(0) \in \mathrm{Map}_k(B\mathbb{G}_m, \mathcal{X})$ which are "close" to the canonical filtration defined by the action of $\mathbb{G}_m$ on $f(0)$. In this language, the proof that $f(0)$ is semistable if $f(1)$ is semistable and $f^*(\ell) = 0$ amounts to the observation that if $f(0)$ had a destabilizing filtration as a graded object, then because the canonical filtration of $f(0)$ has weight 0, one can find destabilizing filtrations of $f(0)$ which are arbitrarily close to the canonical filtration, and then one can identify these with destabilizing filtrations of $f(1)$ using the perturbation theorem.

Proof of Proposition 6.14. Consider a DVR R and a diagram

$$\begin{array}{ccc} \mathrm{Spec}(R) \cup_{\mathrm{Spec}(K)} \mathrm{Spec}(R) & \longrightarrow & \mathcal{X}^{\mathrm{ss}} \\ \downarrow & \nearrow & \downarrow \\ \overline{\mathrm{ST}}_R & \longrightarrow & S \end{array}.$$

By hypothesis we can fill the dotted arrow uniquely to a map $\overline{\mathrm{ST}}_R \to \mathcal{X}$. We claim that in fact the map $\overline{\mathrm{ST}}_R \to \mathcal{X}$ factors through $\mathcal{X}^{\mathrm{ss}}$. Because $\mathcal{X}^{\mathrm{ss}}$ is open, it suffices to check that the unique closed point maps to $\mathcal{X}^{\mathrm{ss}}$. By hypothesis the point $(\pi, s, t) = (0, 1, 0)$ and the point $(\pi, s, t) = (0, 0, 1)$ map to $\mathcal{X}^{\mathrm{ss}}$. Restricting the map $\overline{\mathrm{ST}}_R \to \mathcal{X}$ to the locus $\Theta_k \simeq \{s = 0\}$ and $\Theta_k \simeq \{t = 0\}$ give filtrations f_1 and f_2 in $\mathcal{X}$ of points in $\mathcal{X}^{\mathrm{ss}}$, and if one has $f_1^*(\ell) < 0$ then the other has $f_2^*(\ell) > 0$, which would contradict the fact that $f(1) \in \mathcal{X}^{\mathrm{ss}}$. Therefore $f^*(\ell) = 0$ for both filtrations, and it follows from Lemma 6.15 that $f(0) \in \mathcal{X}^{\mathrm{ss}}$ as well.

For the corresponding claim for Θ-reductivity is proved similarly. For the analogous filling diagram, we start with a map $f \colon \Theta_R \setminus \{(0,0)\} \to \mathcal{X}^{\mathrm{ss}}$ and fill it to a map $\tilde{f} \colon \Theta_R \to \mathcal{X}$. We claim that $(0,0)$ maps to $\mathcal{X}^{\mathrm{ss}}$ as well, and hence because $\mathcal{X}^{\mathrm{ss}} \subset \mathcal{X}$ is open it follows that $\tilde{f}$ lands in $\mathcal{X}^{\mathrm{ss}}$. Because the restriction f_K of f to $\Theta_K \subset \Theta_R \setminus \{(0,0)\}$ maps to $\mathcal{X}^{\mathrm{ss}}$, we know from Lemma 6.15 that $f_K^*(\ell) = 0$. The function $f \mapsto f^*(\ell) \in \mathbb{R}$, regarded as a function on $\mathrm{Map}(\Theta, \mathcal{X})$, is locally constant. It therefore follows that the restriction $\tilde{f}_k \colon \Theta_k \to \mathcal{X}$ of $\tilde{f}$ also has $\tilde{f}_k^*(\ell) = 0$. It follows that $\tilde{f}_k(0) \in \mathcal{X}^{\mathrm{ss}}$. □

Corollary 6.18. *Let $\mathcal{X}$ be an algebraic stack locally of finite type with affine diagonal over a noetherian algebraic space S defined over $\mathbb{Q}$. Assume that $\mathcal{X} \to S$ is S-complete and Θ-reductive. Let $\mathcal{X}^{\mathrm{ss}} \subset \mathcal{X}$ be the Θ-semistable locus with respect to some class $\ell \in H^2(\mathcal{X}; \mathbb{R})$. If $\mathcal{X}^{\mathrm{ss}} \subset \mathcal{X}$ is a quasi-compact open substack, then $\mathcal{X}^{\mathrm{ss}}$ admits a good moduli space which is separated over S. Furthermore if in addition $\mathcal{X} \to S$ satisfies the existence part of the valuative criterion for properness and $\mathcal{X}^{\mathrm{ss}}$ is the open part of a well-ordered Θ-stratification of $\mathcal{X}$, then the good moduli space for $\mathcal{X}$ is proper over S.*

Proof. The map $\mathcal{X}^{\mathrm{ss}} \to S$ is S-complete and Θ-reductive by Proposition 6.14. By Theorem A, $\mathcal{X}^{\mathrm{ss}}$ admits a separated good moduli space $\mathcal{X} \to X$. Under the additional hypotheses, $\mathcal{X}^{\mathrm{ss}} \to S$ satisfies the existence part of the valuative criterion for properness by Corollary 6.12, and hence $X \to S$ is proper by Proposition 3.47. □

6.3. **Application: Properness of the Hitchin fibration.** Let us illustrate how the semistable reduction theorem (Theorem 6.5) can be used to simplify and extend classical semistable reduction theorems for principal bundles and Higgs bundles on curves.

The setup for these results is the following (see e.g., [Ngô06, §2]). Let C be a smooth projective, geometrically connected curve over a field k and G a reductive algebraic group. As the notions are slightly easier to formulate over algebraically closed fields and the valuative criteria allow for extensions of the ground field, we will assume that k is algebraically closed in this section.

We denote by Bun_G the stack of principal G-bundles on C, i.e., for a k-scheme S we have that $\mathrm{Bun}_G(S)$ is the groupoid of principal G-bundles on $C \times S$. Fix a line bundle $\mathcal{L}$ on C. A G-Higgs bundle with coefficients in $\mathcal{L}$ on C is a pair $(\mathcal{P}, \phi)$ where $\mathcal{P}$ is a G-bundle on C and $\phi \in H^0(C, (\mathcal{P} \times^G \mathrm{Lie}(G)) \otimes \mathcal{L})$. We denote by Higgs_G the stack of G-Higgs bundles with coefficients in $\mathcal{L}$.

The stack Higgs_G comes equipped with the forgetful morphism $\mathrm{Higgs}_G \to \mathrm{Bun}_G$ and the Hitchin morphism $h\colon \mathrm{Higgs}_G \to \mathcal{A}_G$. Here $\mathcal{A}_G \cong \bigoplus_{i=1}^r H^0(C, \mathcal{L}^{d_i})$, where $d_1, \ldots, d_r$ are the degrees of homogeneous generators of $k[\mathrm{Lie}(G)^*]^G$ and h is defined by mapping $(\mathcal{P}, \phi)$ to the characteristic polynomial of ϕ.

On both Bun_G and Higgs_G there is a classical notion of stability, which is defined in terms of reductions to parabolic subgroups.

Let us recall how this notion is related to Θ-stability. For vector bundles there is an equivalence (Proposition 2.8, [Hei17, Lem. 1.10])

$$\underline{\mathrm{Map}}(\Theta, \mathrm{Bun}_{\mathrm{GL}_n}) \cong \left\langle (\mathcal{E}, \mathcal{E}^i)_{i \in \mathbb{Z}} \;\middle|\; \begin{array}{l} \mathcal{E} \in \mathrm{Bun}_{\mathrm{GL}_n}, \mathcal{E}^i \subseteq \mathcal{E}^{i+1} \subseteq \mathcal{E} \text{ subbundles} \\ \mathcal{E}^i = \mathcal{E} \text{ for } i \gg 0, \mathcal{E}^i = 0 \text{ for } i \ll 0 \end{array} \right\rangle$$

which is given by assigning to a weighted filtration of a vector bundle $\mathcal{E}$ the canonical $\mathbb{G}_m$-equivariant degeneration of $\mathcal{E}$ to the associated graded bundle.

This construction has an analog for principal bundles. To state this we fix (as in Proposition 2.8) a complete set of conjugacy classes of cocharacters $\Lambda \subset \mathrm{Hom}(\mathbb{G}_m, G)$. As in §2.3 we denote by $P_\lambda^+ \subseteq G$ the parabolic subgroup defined by λ and by $L_\lambda \subset P_\lambda^+$ the Levi subgroup defined by λ which is isomorphic to the quotient of P_λ^+ by its unipotent radical $U_\lambda^+ \subset P_\lambda^+$. Then there is an equivalence (see e.g., [Hei17, Lem. 1.13])

$$\underline{\mathrm{Map}}(\Theta, \mathrm{Bun}_G) \cong \coprod_{\lambda \in \Lambda} \mathrm{Bun}_{P_\lambda^+}.$$

For Higgs bundles note that the forgetful map $\mathrm{Higgs}_G \to \mathrm{Bun}_G$ is representable and therefore $\underline{\mathrm{Map}}(\Theta, \mathrm{Higgs}_G) \subset \underline{\mathrm{Map}}(\Theta, \mathrm{Bun}_G) \times_{\mathrm{Bun}_G} \mathrm{Higgs}_G$, i.e., a filtration of a Higgs bundle is the same as a filtration of the underlying principal bundle that preserves the Higgs field ϕ.

Recall that a G-bundle $\mathcal{E}$ is called semistable, if for all λ and all $\mathcal{E}_\lambda \in \mathrm{Bun}_{P_\lambda^+}$ with $\mathcal{E}_\lambda \times^{P_\lambda^+} G \cong \mathcal{E}$ we have $\deg(\mathcal{P}_\lambda \times^{P_\lambda^+} \mathrm{Lie}(P_\lambda^+)) \leq 0$. Similarly a Higgs bundle is called semistable if the same condition holds for all reductions that respect the Higgs field ϕ. This stability notion can be viewed as Θ-stability induced from the so called determinant line bundle $\mathcal{L}_{\det}$ on the stack Bun_G whose fiber at a point $\mathcal{P}$ (resp. a point $(\mathcal{P}, \phi)$) is given by the one dimensional vector space $\det(H^1(C, \mathcal{P} \times^G \mathrm{Lie}(G))) \otimes \det(H^0(C, \mathcal{P} \times^G \mathrm{Lie}(G)))^{-1}$ (see [Hei17, §1.F], [Hal14]).

As usual we denote by $\mathrm{Bun}_G^{\mathrm{ss}} \subseteq \mathrm{Bun}_G$ the open substack of semistable bundles and for by $\mathrm{Bun}_{P_\lambda^+}^{\mathrm{ss}} \subseteq \mathrm{Bun}_{P_\lambda^+}$ the open substack of bundles such that the associated L_λ-bundle is semistable.

Finally let us recall how the notion of Harder–Narasimhan reduction can be used to equip the stacks Bun_G and Higgs_G with a (well-ordered) Θ-stratification if the characteristic of k is not too small, i.e., such that Behrend's conjecture holds for G (see [Hei08a, Thm. 1] for explicit bounds depending on G; note that char 2 has to be excluded for groups of type B_n, D_n as well).

For any unstable G-bundle $\mathcal{P}$ there exists a canonical Harder–Narasimhan reduction $\mathcal{P}^{HN}$ to a parabolic subgroup P_λ^+, where λ is uniquely determined up to a positive integral multiple. We denote by

$$\underline{d} := \underline{\deg}(\mathcal{P}^{HN})\colon \mathrm{Hom}(P_\lambda, \mathbb{G}_m) \to \mathbb{Z}$$
$$\chi \mapsto \deg(\mathcal{P}_\lambda^{HN} \times^\chi \mathbb{G}_m)$$

the degree of $\mathcal{P}^{HN}$ and by $\mathrm{Bun}_{P_\lambda^+}^{\underline{d},\mathrm{ss}} \subset \mathrm{Bun}_{P_\lambda^+}^{\mathrm{ss}}$ the connected component defined by $\underline{d}$. The instability degree of $\mathcal{P}_\lambda$ is defined as

$$\mathrm{ideg}(\mathcal{P}) := \deg(\mathcal{P}^{HN} \times^{P_\lambda^+} \mathrm{Lie}(P)).$$

Behrend showed that the morphism $\mathrm{Bun}_{P_\lambda^+}^{\underline{d},ss} \to \mathrm{Bun}_G$ defined by the inclusion $P_\lambda^+ \subset G$ is radicial if the degree $\underline{d}$ is the degree of a canonical reduction [Beh] and the map is an embedding if Behrend's conjecture holds for G [Hei08b, Lem. 2.3] this condition is satisfied if the characteristic of k is not too small with respect to G (e.g., > 31).

Moreover the instability degree ideg is upper semicontinuous in families and if this invariant is constant on a family, then the family admits a global Harder–Narasimhan reduction [Beh, Prop. 7.1.3][Hei08b, Prop. 2.2]. Thus the Harder–Narasimhan reduction of bundles defines a Θ-stratification on Bun_G if the characteristic of k is not too small. The same arguments apply for Higgs bundles and this shows the following lemma.

Lemma 6.19. *If the characteristic of k is large enough so that Behrend's conjecture holds for G, then the Harder–Narasimhan stratifications of* Bun_G *and* Higgs_G *form a well-ordered Θ-stratification.*

To apply the semistable reduction theorem to $\mathrm{Higgs}_G \to \mathcal{A}_G$ we need to show that this morphism satisfies the existence part of the valuative criterion for properness. The existence result is probably well known (see e.g. [CL10, §8.4] for an argument over the regular locus) but we could not find a general reference.

Lemma 6.20. *Suppose that the characteristic of k is not a torsion prime for G and very good for G. Let R be a DVR with fraction field K, and let $(\mathcal{E}_K, \phi_K) \in$* $\mathrm{Higgs}_G(K)$ *be a Higgs bundle such that $h(\mathcal{E}_K, \phi_K) \in \mathcal{A}_G(K)$ extends to $\mathcal{A}_G(R)$. Then there exists an extension $R \to R'$ of DVRs with $K \to K' = \mathrm{Frac}(R')$ finite and a point $(\mathcal{E}'_R, \phi'_R) \in \mathrm{Higgs}_G(R')$ extending $(\mathcal{E}_K, \phi_K)$.*

Proof. First let us assume that the derived group of G is simply connected. The generic point of C will be denoted by η, $\mathfrak{g} = \mathrm{Lie}(G)$ and $\mathrm{car} := \mathfrak{g}/\!/G$ is the space of characteristic polynomials of elements of $\mathfrak{g}$.

Let $(\mathcal{E}_K, \phi_K) \in \mathrm{Higgs}_G(K)$ be a Higgs bundle such that $h(\mathcal{E}_K, \phi_K) \in \mathcal{A}_G(R) \subset \mathcal{A}_G(K)$.

We argue as in [CL10, §8.4]. After a finite extension of K we may assume that $\mathcal{E}_K$ is trivial at the generic point $K(\eta)$ of C_K. Choosing trivializations of $\mathcal{E}|_{K(\eta)}$ and $\mathcal{L}|_\eta$ identifies ϕ_K with an element in $X_K \in \mathfrak{g}(K(\eta))$. To conclude the argument as in loc.cit., it is sufficient to show that after passing to a finite extension of K we can conjugate X_K to an element of $\mathfrak{g}(R(\eta))$, because this allows one to extend ϕ_K to the trivial bundle over $R(\eta)$ and as sections of affine bundles extend canonically in codimension 2, the Higgs field ϕ_K will then define a Higgs field for any extension $\mathcal{E}_R$ of $\mathcal{E}_K$ that is trivial over $R(\eta)$.

We denote by $X_K = X_K^s + X_K^n$ the Jordan decomposition of X_K into the semisimple and nilpotent part.

As $h(\mathcal{E}, \phi)$ extends to R we know that the image of X_K in $\mathrm{car} = \mathfrak{g}//G$ defines an $R(\eta)$-valued point. We can use the Kostant section $\mathrm{car} \to \mathfrak{g}$ to obtain $Y_R \in \mathfrak{g}(R(\eta))$ with $h(Y_R) = h(X_K)$.

We claim that we can modify Y_R such that its generic fiber Y_K is semisimple. To see this let us consider the Jordan decomposition $Y_K = Y_K^s + Y_K^n$. By our assumptions on the characteristic of K the main result of [McN05] shows that there exists a parabolic subgroup $P_\lambda^+ \subset G$ defined by a cocharacter $\lambda\colon \mathbb{G}_m \to G_K$ such that Y_K^n is contained in the Lie algebra of the unipotent radical of P_λ^+ and as Y_K^s is in the centralizer of Y_K^n the element Y_K also lies in $\mathrm{Lie}(P_\lambda^+)$. As parabolic subgroups extend over valuation rings, we find that Y_R is contained in a parabolic subgroup $P_R \subset G_R$ and we can choose λ to be a cocharacter defined over R as well.

As $P(\lambda)$ is defined to be the set of points $p \in G$ such that $\lim_{t\to 0} \lambda(t)p\lambda(t)^{-1}$ exists, the limit $\lim_{t\to 0}\lambda(t)\cdot Y_R$ will be an R-valued point Y_R' such that $Y_K' = Y_K^s$ is semisimple.

As the semi-simple part of X_K is the unique closed orbit in the conjugacy class of X_K we know that X_K^s and Y_K^s lie in the same closed orbit. As we assumed that the derived group of G is simply connected and that p is not a torsion prime for G the centralizer $Z_G(X_s)$ is a connected reductive group [Ste75, Thm. 0.1]. By Steinberg's theorem, any $Z_G(X_s)$ torsor over $K(\eta)$ splits after a finite extension of K, so after possibly extending K the elements X_K^s and Y_K^s are conjugate. Thus after conjugating X_K we may assume that $X_K^s = Y_K^s$, i.e. we may assume that the semisimple part of X_K extends to R.

Now we can apply the previous argument to X_K, namely the element X_K is contained in a parabolic subalgebra defined by a cocharacter λ, such that X_K^n is contained in its unipotent radical, so that for some $a \in K$ the element $\lambda(a).X_K^n$ will extend to R as well.

Finally for any group G we can consider a z-extension

$$0 \to Z \to \widetilde{G} \to G \to 1$$

where Z is a central torus and the derived group G' of $\widetilde{G}$ is simply connected. Then the map $\mathrm{Bun}_{\widetilde{G}} \to \mathrm{Bun}_G$ is a smooth surjection. Moreover the covering $G' \to G$ is separable, because we assumed that the fundamental group of G has no p-torsion. Therefore $\mathrm{Lie}(\widetilde{G}) \cong \mathrm{Lie}(Z) \oplus \mathrm{Lie}(G)$ and therefore the map $\mathrm{Higgs}_{\widetilde{G}} \to \mathrm{Higgs}_G$ also admits local sections. Thus it suffices to prove the result for $\widetilde{G}$.

□

The semistable reduction theorem (Theorem 6.5) now allows us to deduce:

Corollary 6.21. *Suppose that the characteristic p of k is large enough such that Behrend's conjecture holds for G, such that p is not a torsion prime for G and such that p is very good for G, then the Hitchin morphism*

$$h\colon \mathrm{Higgs}_G^{\mathrm{ss}} \to \mathcal{A}_G$$

satisfies the existence part of the valuative criterion for properness, i.e. if R is a DVR with fraction field K and $x_K\colon \operatorname{Spec}(K) \to \mathrm{Higgs}_G^{\mathrm{ss}}$ is a morphism such that $h(x_K)\colon \operatorname{Spec}(K) \to \mathcal{A}_G$ extends to R, then there exists an extension $R \to R'$ of DVRs with $K \to K' = \operatorname{Frac}(R')$ finite and a morphism $x_{R'}\colon \operatorname{Spec}(R') \to \mathrm{Higgs}_G^{\mathrm{ss}}(R')$ extending x_K.

Note that in characteristic 0 this result is due to Faltings [Fal93, Thm. II.4] and for the regular part of the Hitchin fibration this is due to Chaudouard–Laumon [CL10, Thm. 8.1.1]. Over the complex numbers, the result can also be deduced from results of Simpson as explained in [Cat18].

Remark 6.22. Since $\mathrm{Higgs}_G^{\mathrm{ss}}$ is quasi-compact, the conclusion is equivalent to saying that the Hitchin morphism is universally closed. In particular the induced morphism on an adequate moduli space will be proper.

Proof. By Lemma 6.20 we can find an extension of x_K to x_R. As Higgs_G admits a well ordered Θ-stratification by Harder–Narasimhan reductions we can therefore apply the semistable reduction Theorem 6.5 to conclude. □

7. Good moduli spaces for objects in abelian categories

In this section we study the moduli functor for objects in a k-linear abelian category $\mathcal{A}$, following the foundational work of Artin and Zhang [AZ01], who explained that many of the results known for categories of quasi-coherent sheaves on a scheme can be carried out in an abstract setting. This construction has been studied more recently in [Gai05, AP06, CG13]. The general setup is very useful as it for example gives an easy way to formulate moduli problems for objects in the heart of different t-structures in the derived category of coherent sheaves on a scheme. It turns out that this setup leads to moduli problems in which the conditions of Θ-reductivity, S-completeness, and unpunctured inertia can be checked rather easily. Following the convention of [AZ01], throughout this chapter we exceptionally fix a base ring k, which is allowed to be any commutative ring.

7.1. **Formulation of the moduli problem.** Let us start by recalling the setup of [AZ01]. Let $\mathcal{A}$ be a k-linear abelian category that is assumed to be cocomplete, i.e. arbitrary small colimits exist in $\mathcal{A}$. To formulate a reasonable moduli problem we first need to recall some finiteness conditions on objects. Recall that an object $E \in \mathcal{A}$ is

- *finitely presentable* (also known as *compact*) if the canonical map

$$\operatorname{colim}_{\alpha\in I} \operatorname{Hom}(E, F_\alpha) \to \operatorname{Hom}(E, \operatorname{colim}_{\alpha\in I} F_\alpha) \tag{7.1}$$

  is an isomorphism for any (small) filtered system $\{F_\alpha\}_{\alpha\in I}$ in $\mathcal{A}$;
- *finitely generated* if (7.1) is an isomorphism for any filtered system of monomorphisms in $\mathcal{A}$, or equivalently, if $E = \bigcup_\alpha E_\alpha$ for a filtered system of subobjects, then $E = E_\alpha$ for some α [Pop73, Prop. 3.5.6]; and
- *noetherian* if every ascending chain of subobjects of E terminates, or equivalently, if every subobject of E is finitely generated.

We denote by $\mathcal{A}^{\mathrm{fp}}$ the full subcategory of $\mathcal{A}$ consisting of finitely presentable objects.

Example 7.1. If $\mathcal{A} = \mathrm{Mod}_R$ for a (possibly non-commutative) ring R, an object $E \in \mathcal{A}$ is finitely presentable (resp. finitely generated, resp. noetherian) if and only if the corresponding module is. The analogous statement holds if $\mathcal{A} = \mathrm{QCoh}(X)$ for a scheme X.

We say that $\mathcal{A}$ is

- *locally of finite type* if every object in $\mathcal{A}$ is the union of its finitely generated subobjects;
- *locally finitely presented* if every object in $\mathcal{A}$ can be written as the filtered colimit of finitely presentable objects, and $\mathcal{A}^{\mathrm{fp}}$ is essentially small;
- *locally noetherian* if it has a set of noetherian generators.

If $\mathcal{A}$ is locally noetherian, then finitely generated, finitely presentable, and noetherian objects coincide [AZ01, Prop. B1.3], and the category $\mathcal{A}^{\mathrm{fp}}$ is closed under kernels and hence abelian. Our main results will assume that $\mathcal{A}$ is locally noetherian.

The next ingredient to the formulation of moduli problems is the observation that the existence of colimits allows one to define a tensor product, which in turn provides a notion of base change.

More precisely, there is a canonical k-bilinear functor

$$(-) \otimes_k (-) \colon \operatorname{Mod}_k \times \mathcal{A} \to \mathcal{A} \tag{7.2}$$

which is characterized by the formula

$$\operatorname{Hom}_{\mathcal{A}}(M \otimes_k E, F) = \operatorname{Hom}_{\operatorname{Mod}_k}(M, \operatorname{Hom}_{\mathcal{A}}(E, F))$$

for objects $E, F \in \mathcal{A}$ and a k-module M. Explicitly, if one presents M as the cokernel of a morphism $k^I \to k^J$ for index sets I and J, then $M \otimes_k E$ can be computed as $\operatorname{coker}(E^I \to E^J)$ where the morphism $E^I \to E^J$ is induced by the matrix defining $k^I \to k^J$.

The functor $(-) \otimes_k (-)$ commutes with filtered colimits and is right exact in each variable. If $M \in \operatorname{Mod}_k$ is flat and $\mathcal{A}$ is locally noetherian then $M \otimes_k (-)$ is exact [AZ01, Lem. C1.1].

Definition 7.2. [AZ01, §C1] We say that an object $E \in \mathcal{A}$ is *flat* if $(-) \otimes_k E \colon \operatorname{Mod}_k \to \mathcal{A}$ is exact.

This tensor product leads to a base change formalism as follows.

Definition 7.3 (Base change categories)**.** [AZ01, §B2] For a commutative k-algebra R, let $\mathcal{A}_R$ denote the category of R-module objects in $\mathcal{A}$, i.e., pairs (E, ξ_E) where $E \in \mathcal{A}$ and $\xi_E \colon R \to \operatorname{End}_{\mathcal{A}}(E)$ is a morphism of k-algebras, and a morphism $(E, \xi_E) \to (E', \xi_{E'})$ in $\mathcal{A}_R$ is a morphism $E \to E'$ in $\mathcal{A}$ compatible with the actions of ξ_E and $\xi_{E'}$.

For a commutative k-algebra R, $\mathcal{A}_R$ is an R-linear abelian category [AZ01, Prop. B2.2], and $\mathcal{A}_k = \mathcal{A}$. Given a homomorphism of commutative rings $\phi \colon R_1 \to R_2$, the forgetful functor

$$\phi_* \colon \mathcal{A}_{R_2} \to \mathcal{A}_{R_1}$$

is faithfully exact, commutes with filtered colimits and faithful, and ϕ_* is fully faithful if ϕ is surjective [AZ01, Prop. B2.3]. Moreover, ϕ_* admits a left adjoint

$$\phi^* = R_2 \otimes_{R_1} (-) \colon \mathcal{A}_{R_1} \to \mathcal{A}_{R_2}$$

by [AZ01, Prop. B3.16].

Remark 7.4. The property of being locally noetherian is not stable under base change, but if $\mathcal{A}$ is locally noetherian and R is an essentially finite type k algebra, then $\mathcal{A}_R$ is locally noetherian [AZ01, Cor. B6.3].

The above constructions allow one to prove descent if $\mathcal{A}$ is locally noetherian [AZ01, Thm. C8.6], i.e., if $R \to S$ is a faithfully flat map of commutative k-algebras then $\mathcal{A}_R$ is equivalent to the category of objects in $\mathcal{A}_S$ equipped with a descent datum.

Remark 7.5 ($\mathcal{A}_{\mathfrak{X}}$ for stacks over k)**.** As the assignment $R \mapsto \mathcal{A}_R$ satisfies descent if $\mathcal{A}$ is locally noetherian, it defines a stack $\underline{\mathcal{A}}$ in the fppf topology on k-Alg. As in [LMB] this extends the category $\mathcal{A}_R$ naturally not only to schemes but also to algebraic stacks, i.e. for any algebraic stack $\mathfrak{X}$ over k we can define

$$\mathcal{A}_{\mathfrak{X}} := \mathrm{Map}_{\text{Fibered Cat}/k\text{- Alg}}(\mathfrak{X}, \underline{\mathcal{A}}).$$

If $\mathfrak{X}$ is the quotient stack for a groupoid of affine schemes $\mathfrak{X} = [X_1 \rightrightarrows X_0]$ with $X_i = \mathrm{Spec}(R_i)$, then descent implies that the category $\mathcal{A}_{\mathfrak{X}}$ is naturally equivalent to the category of objects of $\mathcal{A}_{X_0}$ equipped with a descent datum. We will use this description for the stacks Θ and $\overline{\mathrm{ST}}_R$.

Faithfully flat descent also allows one to extend the functor $R_2 \otimes_{R_1} (-) \colon \mathcal{A}_{R_1} \to \mathcal{A}_{R_2}$ above to a functor $f^* \colon \mathcal{A}_{\mathfrak{Y}} \to \mathcal{A}_{\mathfrak{X}}$ for any morphism of stacks $f \colon \mathfrak{X} \to \mathfrak{Y}$. To prove extension theorems, we will need the following construction.

Lemma 7.6 (Push-forward in $\mathcal{A}$)**.** *Suppose that $\mathcal{A}$ is locally noetherian. If $f \colon \mathfrak{X} \to \mathfrak{Y}$ is a quasi-compact morphism with affine diagonal of algebraic stacks then the restriction functor $f^* \colon \mathcal{A}_{\mathfrak{Y}} \to \mathcal{A}_{\mathfrak{X}}$ admits a right adjoint f_* which commutes with filtered colimits and flat base change.*

Proof. Let us first prove the claim when $\mathfrak{Y} = \mathrm{Spec}(A)$ is affine. In this case we can choose a presentation of $\mathfrak{X}$ by a groupoid in affine schemes $\mathfrak{X} \simeq [\mathrm{Spec}(R_1) \rightrightarrows \mathrm{Spec}(R_0)]$, and an object $E \in \mathcal{A}_{\mathfrak{X}}$ is described by an object $E_0 \in \mathcal{A}_{R_0}$ along with a descent datum, i.e. denoting by $d_i \colon R_0 \to R_1$ the structure maps of the presentation this is an isomorphism $\phi \colon R_1 \otimes_{d_0, R_0} E_0 \simeq R_1 \otimes_{d_1, R_0} E_0$, satisfying a cocycle condition. Then using the fact that homomorphisms in $\mathcal{A}_{[\mathrm{Spec}(R_1) \rightrightarrows \mathrm{Spec}(R_0)]}$ are homomorphisms in $\mathcal{A}_{R_0}$ which commute with the respective cocycles, one may verify directly that

$$f_*(E) = \ker \left(E_0 \xrightarrow{\phi \circ \eta_0 - \eta_1} B_1 \otimes_{d_1, B_0} E_0 \right) \in \mathcal{A}_A, \tag{7.3}$$

where $\eta_i \colon E_0 \to B_1 \otimes_{d_i, B_0} E_0$ for $i = 0, 1$ is the unit of the adjunction between $B_1 \otimes_{d_i, B_0} (-)$ and the forgetful functor $(d_i)_* \colon \mathcal{A}_{B_1} \to \mathcal{A}_{B_0}$. Both objects in (7.3) are regarded as objects of $\mathcal{A}_A$ via the canonical forgetful functor. The fact that f_* commutes with filtered colimits can be deduced from the formula (7.3) and the fact that filtered colimits are exact in $\mathcal{A}_A$[AZ01, Prop. B2.2]. The fact that f_* commutes with flat base change can be deduced from the fact that if $A \to A'$ is a flat ring map, then $A' \otimes_A (-)$ is exact and hence commutes with the formation of the kernel in (7.3).

Now let $\mathfrak{Y}$ be an algebraic stack with affine diagonal. For any $E \in \mathcal{A}_{\mathfrak{X}}$ and any morphism $\mathrm{Spec}(R) \to \mathfrak{Y}$ we have shown that the object $(f_R)_*(E|_{\mathfrak{X}_R}) \in \mathcal{A}_R$ is defined and its formation commutes with flat base change. Faithfully flat descent implies that these objects descend to a unique object $f_*(E) \in \mathcal{A}_{\mathfrak{Y}}$. Faithfully flat descent also shows that the resulting functor $f_* \colon \mathcal{A}_{\mathfrak{X}} \to \mathcal{A}_{\mathfrak{Y}}$ is right adjoint to f^*. □

Remark 7.7. If X is a separated scheme and $f \colon X \to \mathrm{Spec}(R)$ a morphism then the above construction reproduces the usual Cech description of the push forward, i.e. given $E \in \mathcal{A}_X$ choose a covering $X_0 = \bigsqcup_i U_i \to X$ by open affines then (7.3) reduces to

$$f_*(E) = \ker \left(\bigoplus_i (f_i)_*(E|_{U_i}) \to \bigoplus_{i<j} (f_{ij})_*(E|_{U_{ij}}) \right),$$

where $U_{ij} := U_i \cap U_j$.

Definition 7.8 (Moduli functor)**.** Let k be a commutative ring and let $\mathcal{A}$ be a locally noetherian, cocomplete, and k-linear abelian category. Then we define the category $\mathcal{M}_{\mathcal{A}}$ fibered in groupoids over k-alg by assigning the groupoid

$$\mathcal{M}_{\mathcal{A}}(R) := \langle \text{objects } E \in \mathcal{A}_R \text{ which are flat and finitely presented} \rangle$$

for a k-algebra R.

Lemma 7.9. *The category fibered in groupoids $\mathcal{M}_{\mathcal{A}}$ is a stack in the big fppf topology on k-alg and extends naturally to a stack on the big fppf topology on schemes over k.*

Proof. As we already quoted the result [AZ01, Thm. C8.6] that the categories $\mathcal{A}_R$ satisfy flat descent, we only need to check that the conditions of flatness and finite presentation are preserved by descent and pull-back. Given a ring map $R_1 \to R_2$, the pullback functor $R_2 \otimes_{R_1} (-)$ preserves flat objects by [AZ01, Lem. C1.2], and it preserves finitely presentable objects because its right adjoint commutes with filtered colimits. If $R_1 \to R_2$ is a faithfully flat map of k-algebras we have $R_2 \otimes_{R_1} (M \otimes_{R_1} (-)) \simeq (R_2 \otimes_{R_1} M) \otimes_{R_2} (R_2 \otimes_{R_1} (-))$, and the exactness of a sequence in $\mathcal{A}_{R_1}$ can be checked after applying the functor $R_2 \otimes_{R_1} (-)$, so $E \in \mathcal{A}_{R_1}$ is flat if $R_2 \otimes_{R_1} E$ is. Also, one can directly verify from the description of Hom in the category of descent data for the map $R_1 \to R_2$ that any descent data for a finitely presentable object $E \in \mathcal{A}_{R_2}$ is a finitely presentable object in the category of descent data. □

Warning 7.10 (Passing to subcategories changes the moduli functor)**.** For the above definition to make sense, the assumption that the category $\mathcal{A}$ is cocomplete and thus rather big is essential, as we defined R-modules to be elements of $\mathcal{A}$ equipped with additional structure. One might be tempted to replace this large category by a smaller ind-category generated by some of some subclass of finitely presented objects, but this will change the moduli problem $\mathcal{M}_{\mathcal{A}}$.

For example if we take $\mathcal{A}$ to be the category of representations of the fundamental group of a projective variety X, vector bundles with flat connections and finite dimensional representations of the fundamental group are equivalent categories, but as this equivalence is not algebraic it does not extend to an algebraic equivalence that identifies families over X_R for k-algebras R. Indeed families of representations of $\pi_1(X)$ are defined using finitely presented modules over the group algebra $R[\pi_1(X)]$, whereas families of sheaves with connection are defined in terms of quasi-coherent sheaves of $R \otimes_k \mathcal{D}_X$-modules, and these larger categories are not equivalent. If one instead considered $\mathcal{M}_{\mathcal{A}}$ where $\mathcal{A}$ is the ind-completion of the category of vector bundles with connection, then this moduli functor differs from both of these, because a finitely presented $R[\pi_1(X)]$ module need not be a filtered colimit of finite ones.

7.2. **Verification of the valuative criteria for the stack $\mathcal{M}_{\mathcal{A}}$.** To apply our existence results we will check Θ-reductivity and S-completeness for the stack $\mathcal{M}_{\mathcal{A}}$ with respect to discrete valuation rings which are essentially of finite type over k. The first step is to show that, as for module categories, $\mathbb{G}_m$-equivariant objects can be interpreted as graded objects. Let us recall these notions.

Recall from [AZ01, §B7] that the category of $\mathbb{Z}$-graded objects $\mathcal{A}^{\mathbb{Z}}$ consists of functors $\mathbb{Z} \to \mathcal{A}$, where $\mathbb{Z}$ is regarded as a category with only identity morphisms. Concretely all objects in $\mathcal{A}^{\mathbb{Z}}$ are of the form $\bigoplus_{n\in\mathbb{Z}} E_n$. Given a $\mathbb{Z}$-graded k-algebra A, *a $\mathbb{Z}$-graded A-module object* is a $\mathbb{Z}$-graded object $E = \bigoplus_{n\in\mathbb{Z}} E_n \in \mathcal{A}^{\mathbb{Z}}$ with the structure of an A-object such that multiplication $A \otimes_k E \to E$ maps $A_n \otimes_k E_m$ to E_{n+m}. The category $\mathcal{A}^{\mathbb{Z}}_A$ of $\mathbb{Z}$-graded A-module objects is abelian and locally noetherian if $\mathcal{A}_A$ is [AZ01, Prop. B7.5].

Now let A be a $\mathbb{Z}$-graded k-algebra, then objects of $\mathcal{A}_{[\mathrm{Spec}(A)/\mathbb{G}_m]}$ are by definition objects $E \in \mathcal{A}_A$ together with a descent datum, i.e., a coaction $\sigma\colon E \to A[t]_t \otimes_A E$ compatible with the coaction $\sigma_A\colon A \to k[t]_t \otimes_k A \cong A[t]_t$ (induced from the $\mathbb{Z}$-grading of A), i.e., σ is a morphism in $\mathcal{A}_A$, where $A[t]_t$ has the A-module given by σ_A, such that the diagrams of objects in $\mathcal{A}_A$

$$\begin{array}{ccc} E & \xrightarrow{\ \sigma\ } & A[t]_t \otimes_A E \\ \downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle t\mapsto tt'} \\ A[t]_t \otimes_A E & \xrightarrow{\ \mathrm{id}\otimes\sigma\ } & A[t,t']_{tt'} \otimes_A E \end{array} \qquad \begin{array}{ccc} E & \xrightarrow{\ \sigma\ } & A[t]_t \otimes_A E \\ & {\scriptstyle \mathrm{id}}\searrow & \downarrow{\scriptstyle t\mapsto 1} \\ & & E \end{array}$$

commute.

An object $E = \bigoplus_{n\in\mathbb{Z}} E_n \in \mathcal{A}^{\mathbb{Z}}_A$ induces a coaction σ_E on E where $\sigma|_{E_n}$ is the inclusion of E_n into the summand $A\langle t^n\rangle \otimes_A E \subset A[t]_t \otimes_A E$. The assignment $E \mapsto (E, \sigma_E)$ defines a functor $\mathrm{Can}\colon \mathcal{A}^{\mathbb{Z}}_A \to \mathcal{A}_{[\mathrm{Spec}(A)/\mathbb{G}_m]}$.

Proposition 7.11. *Let k be a commutative ring and let $\mathcal{A}$ be a locally noetherian k-linear abelian category. Let A be a $\mathbb{Z}$-graded k-algebra. Then the functor* $\mathrm{Can}\colon \mathcal{A}^{\mathbb{Z}}_A \to \mathcal{A}_{[\mathrm{Spec}(A)/\mathbb{G}_m]}$ *is an equivalence of categories between $\mathcal{A}_{[\mathrm{Spec}(A)/\mathbb{G}_m]}$ and the category $\mathcal{A}^{\mathbb{Z}}_A$ of $\mathbb{Z}$-graded A-module objects in $\mathcal{A}$ which restricts to an equivalence between $\mathcal{M}_{\mathcal{A}}([\mathrm{Spec}(A)/\mathbb{G}_m])$ and the groupoid of objects $\bigoplus_{n\in\mathbb{Z}} E_n$ in $\mathcal{A}^{\mathbb{Z}}_A$ such that each $E_n \in \mathcal{A}_A$ is flat and $E_n = 0$ for $n \ll 0$ and $n \gg 0$.*

Proof. As for graded modules, we can give an inverse to the functor Can. Let $E \in \mathcal{A}_A$ and $\sigma\colon E \to A[t]_t \otimes_A E$ be a coaction. For each integer d, we define

$$E_n := \ker(E \xrightarrow{\sigma - t^n\otimes \mathrm{id}} A[t]_t \otimes_A E),$$

where $t^n \otimes \mathrm{id}\colon E \cong A \otimes_A E \to A[t]_t \otimes_A E$ is induced by the A-module homomorphism $A \to A[t]_t$ defined by $1 \mapsto t^n$. It remains to show that the natural map $\bigoplus_{n\in\mathbb{Z}} E_n \to E$ is an isomorphism. Since $\mathcal{A}_A$ is locally finitely presentable, it suffices to show that $\mathrm{Hom}_{\mathcal{A}_A}(X, \bigoplus_{n\in\mathbb{Z}} E_n) \to \mathrm{Hom}_{\mathcal{A}_A}(X, E)$ is an isomorphism for all $X \in \mathcal{A}^{\mathrm{fp}}_A$. As X is finitely presentable, the coaction σ induces a coaction of A-modules

$$\sigma_X\colon \mathrm{Hom}_{\mathcal{A}_A}(X, E) \to \mathrm{Hom}_{\mathcal{A}_A}(X, A[t]_t \otimes_A E) \cong A[t]_t \otimes_A \mathrm{Hom}_{\mathcal{A}_A}(X, E).$$

compatible with σ_A. As $\mathrm{Hom}_{\mathcal{A}_A}(X, -)$ is left exact, we have an identification of $\mathrm{Hom}_{\mathcal{A}_A}(X, E_n)$ with

$$\mathrm{Hom}_{\mathcal{A}_A}(X, E)_n := \ker\left(\mathrm{Hom}_{\mathcal{A}_A}(X, E) \xrightarrow{\sigma_X - t^n\otimes\mathrm{id}} A[t]_t \otimes_A \mathrm{Hom}_{\mathcal{A}_A}(X, E)\right)$$

$\mathrm{Hom}_{\mathcal{A}_A}(X, \bigoplus_{d\in\mathbb{Z}} E_n) = \bigoplus_{d\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{A}_A}(X, E)_n \to \mathrm{Hom}_{\mathcal{A}_A}(X, E)$ is an isomorphism by the usual argument: clearly $\mathrm{Hom}_{\mathcal{A}_A}(X, E)_n \cap \mathrm{Hom}_{\mathcal{A}_A}(X, E)_{n'} = 0$ if $n \neq n'$ and if $\alpha \in \mathrm{Hom}_{\mathcal{A}_A}(X, E)$, we can write $\sigma_X(\alpha)$ as a finite sum $\sum_n t^n \otimes \alpha_n$

where all but finitely many of the α_n are zero and the coaction axioms of σ_X imply that $\alpha_n \in \mathrm{Hom}_{\mathcal{A}_A}(X,E)_n$ and that $\alpha = \sum_n \alpha_n$. $\square$

We can use the above result to describe objects of $\mathcal{A}$ over $\Theta_R = [\mathrm{Spec}(R[x])/\mathbb{G}_m]$ for any k-algebra R and over $\overline{\mathrm{ST}}_R := [\mathrm{Spec}\,(R[s,t]/(st-\pi))/\mathbb{G}_m]$ for any DVR R over k with uniformizing parameter π. Both descriptions are in terms of filtered objects.

Corollary 7.12. *Suppose that $\mathcal{A}$ is locally noetherian. Let R be a k-algebra then the category $\mathcal{A}_{\Theta_R}$ is equivalent to the category of sequences of morphisms*

$$E\colon \quad \cdots \to E_{n+1} \xrightarrow{x} E_n \to \cdots$$

in $\mathcal{A}_R$, such that the restriction of E

- *along $\mathrm{Spec}(R) \hookrightarrow \Theta_R$ is $\mathrm{colim}\, E_i$, and*
- *along $B\mathbb{G}_{m,R} \hookrightarrow \Theta_R$ is $\bigoplus_{n\in\mathbb{Z}} E_n/x(E_{n+1})$.*

This equivalence restricts to an equivalence between $\mathcal{M}_{\mathcal{A}}(\Theta_R)$ and the groupoid of $\mathbb{Z}$-weighted filtrations $\cdots \subset E_{n+1} \subset E_n \subset \cdots$ of an object E_∞ in $\mathcal{A}_R$ such that $E_n/E_{n+1} \in \mathcal{A}_R$ is flat and finitely presented, $E_n = E_\infty$ for $n \ll 0$ and $E_n = 0$ for $n \gg 0$.

Proof. By Proposition 7.11, an object E in $\mathcal{A}_{\Theta_R}$ is a $\mathbb{Z}$-graded $R[x]$-module object of $\mathcal{A}_R$. This corresponds to a $\mathbb{Z}$-graded object $E = \bigoplus_{n\in\mathbb{Z}} E_n$ in $\mathcal{A}_R$ together with a multiplication $x\colon E \to E$ mapping E_{n+1} to E_n. The restriction of E along the open immersion $\mathrm{Spec}(R) \hookrightarrow \Theta_R$ is the $\mathbb{G}_m$-invariants (i.e. the degree 0 component) of the $\mathbb{Z}$-graded $R[x]_x$-module object $E \otimes_{R[x]} R[x]_x \in \mathcal{A}_{R[x]_x}$. We compute that

$$\begin{aligned} E \otimes_{R[x]} R[x]_x &= E \otimes_{R[x]} (\mathrm{colim}(\cdots \xrightarrow{x} R \xrightarrow{x} R \xrightarrow{x} \cdots) \\ &= \mathrm{colim}(\cdots \xrightarrow{x} E \xrightarrow{x} E \xrightarrow{x} \cdots) \\ &= \bigoplus_{n\in\mathbb{Z}} \mathrm{colim}(\cdots \xrightarrow{x} E_{n+1} \xrightarrow{x} E_n \xrightarrow{x} \cdots) \end{aligned}$$

whose $\mathbb{G}_m$-invariants is $\mathrm{colim}\, E_n$. The restriction of E along the closed immersion $B\mathbb{G}_{m,R} \hookrightarrow \Theta_R$ is the object $E \otimes_{R[x]} (R[x]/x) \cong E/xE$ in $\mathcal{A}_R$ with the $\mathbb{Z}$-grading $E/xE = \bigoplus_{n\in\mathbb{Z}} E_n/x(E_{n+1})$. This proves the first claim.

If $E \in \mathcal{A}_{\Theta_R}$ is flat and finitely presented, then so is the corresponding object in $\mathcal{A}_{R[x]}$ (also denoted by E). Since $E \in \mathcal{A}_{R[x]}$ is flat, it is torsion free and thus $x\colon E \to E$ is injective. The base change $E \otimes_{R[x]} (R[x]/x) = \bigoplus E_n/E_{n+1}$ is a flat and finitely presented object in $\mathcal{A}_R$ which implies that the sum is finite and the summands are finitely presentable and flat. This implies that E_n stabilizes for $n \ll 0$. Also, if $E \in \mathcal{A}_{R[x]}$ is finitely presentable, it must admit a surjection $R[x] \otimes_R F \to E$ for some $F \in \mathcal{A}_R^{\mathrm{fp}}$, corresponding to a map $F \to E$ in $\mathcal{A}_R$. Because F is finitely presentable it must factor through $\bigoplus_{n\le N} E_n \subset E$ for some N, and hence the image of $R[x] \otimes_R F \to E$ lies in $\bigoplus_{n\le N} E_n$ as well, which implies that $E_n = 0$ for $n \gg 0$.

Conversely, if $\cdots \subset E_{n+1} \subset E_n \subset \cdots$ satisfies the conditions above, then each E_n is constructed as a finite sequence of extensions of flat and finitely presentable objects in $\mathcal{A}_R$ and is thus flat and finitely presentable. It follows that the graded $R[x]$-module objects $R[x] \otimes_R E_n$ are flat and finitely presentable as objects of $\mathcal{A}_{R[x]}$. Furthermore, the object $E \in \mathcal{A}_{R[x]}^{\mathbb{Z}}$ corresponding to this $\mathbb{Z}$-weighted filtration can be constructed as a finite sequence of extensions of objects of the form $R[x] \otimes_R E_n\langle -n\rangle$, where the $\langle -n\rangle$ denotes a grading shift so that the resulting object is homogeneous of degree n. Hence $E \in \mathcal{A}_{\Theta_R}$ is finitely presentable and flat. $\square$

Corollary 7.13. *Suppose that $\mathcal{A}$ is locally noetherian. Let R be a DVR over k with uniformizing parameter π and residue field κ. The category $\mathcal{A}_{\overline{\mathrm{ST}}_R}$ is equivalent to the category of diagrams in $\mathcal{A}_R$*

$$(7.4) \qquad E\colon \qquad \cdots \underset{t}{\overset{s}{\rightleftarrows}} E_{n-1} \underset{t}{\overset{s}{\rightleftarrows}} E_n \underset{t}{\overset{s}{\rightleftarrows}} E_{n+1} \underset{t}{\overset{s}{\rightleftarrows}} \cdots,$$

satisfying $st = ts = \pi$, such that the restriction of E

- *along* $\mathrm{Spec}(R) \xhookrightarrow{s\neq 0} \overline{\mathrm{ST}}_R$ *is* $\mathrm{colim}(\cdots \xrightarrow{s} E_{n-1} \xrightarrow{s} E_n \xrightarrow{s} \cdots)$,
- *along* $\mathrm{Spec}(R) \xhookrightarrow{t\neq 0} \overline{\mathrm{ST}}_R$ *is* $\mathrm{colim}(\cdots \xleftarrow{t} E_{n-1} \xleftarrow{t} E_n \xleftarrow{t} \cdots)$,
- *along* $\Theta_\kappa \xhookrightarrow{s=0} \overline{\mathrm{ST}}_R$ *is the object corresponding to the sequence* $(\cdots \xleftarrow{t} E_n/sE_{n-1} \xleftarrow{t} E_{n+1}/sE_n \xleftarrow{t} \cdots)$, *and*
- *along* $\Theta_\kappa \xhookrightarrow{t=0} \overline{\mathrm{ST}}_R$ *is* $(\cdots \xrightarrow{s} E_{n-1}/tE_n \xrightarrow{s} E_n/tE_{n+1} \xrightarrow{s} \cdots)$.

This equivalence restricts to an equivalence between $\mathcal{M}_{\mathcal{A}}(\overline{\mathrm{ST}}_R)$ and the groupoid consisting of objects E such that: (a) s and t are injective, (b) $s\colon E_{n-1}/tE_n \to E_n/tE_{n+1}$ is injective for all n, (c) each $E_n \in \mathcal{A}_R$ is finitely presentable, (d) $s\colon E_{n-1} \to E_n$ is an isomorphism for $n \gg 0$, and (e) $t\colon E_n \to E_{n-1}$ is an isomorphism for $n \ll 0$.

Proof. The equivalence between $\mathcal{A}_{\overline{\mathrm{ST}}_R}$ and the category of diagrams as in (7.4) is argued as before (Corollary 7.12). Let us first show that flatness is characterized by conditions (a) and (b). Suppose $E \in \mathcal{A}_{\overline{\mathrm{ST}}_R}$ is flat, then the pullbacks of E to $R[s,t]/(st-\pi)$ and $\kappa[s]$ (by setting $t=0$) are both flat and in particular torsion free which gives conditions (a) and (b). Conversely if E is given by the diagram (7.4) flatness is a local condition and (a) implies that the restriction of E to $s \neq 0$ (or $t \neq 0$) is torsion free and thus flat. Condition (b) implies that the restriction to $\kappa[s]$ is $s-$torsion free and so E is also flat at the origin $s = 0 = t$ by applying [AZ01, Lem. C1.12].

To check that a finitely presentable, flat object E satisfies conditions (c)–(e) note that these are closed under cokernels, so we only need to check these for a generating class of objects. Now if $M \in \mathcal{A}_R$ is an R-module, then

$$R[s,t]/(st-\pi) \otimes_R M \simeq \bigoplus_{n<0} M \cdot t^{-n} \oplus M \oplus \bigoplus_{n>0} M \cdot s^n,$$

is a $\mathbb{Z}$-graded $R[s,t]/(st-\pi)$-module for which t is an isomorphism on negatively graded pieces, and s is an isomorphism on positively graded pieces. Therefore for any finitely presentable $M \in \mathcal{A}_R$, $R[s,t]/(st-\pi) \otimes_R M \in \mathcal{A}_R^{\mathbb{Z}}$ satisfies the conditions (c)–(e) of the lemma, and the same holds if $M \in \mathcal{A}_R^{\mathbb{Z}}$ is graded object with finitely many non-trivial graded pieces which are each finitely presentable. As any finitely generated object of $\mathcal{A}^{\mathbb{Z}}_{R[s,t]/(st-\pi)}$ admits a surjection from an object of this form, any $E \in (\mathcal{A}^{\mathbb{Z}}_{R[s,t]/(st-\pi)})^{\mathrm{fp}}$ admits a presentation of the form $E \simeq \mathrm{coker}(R[s,t]/(st-\pi) \otimes M_1 \to R[s,t]/(st-\pi) \otimes M_0)$ for some $M_0, M_1 \in (\mathcal{A}_R^{\mathbb{Z}})^{\mathrm{fp}}$, which proves (c)–(e) for E.

Conversely, suppose that the diagram (7.4) satisfies the conditions (c)–(e) of the lemma. Then (d) and (e) imply that for $N \gg 0$, the canonical homomorphism of graded $R[s,t]/(st-\pi)$-modules $R[s,t]/(st-\pi) \otimes_R \bigoplus_{-N \le n \le N} E_n \to E$ is surjective. Let $K = \bigoplus_n K_n$ be the kernel of this homomorphism. Because $R[s,t]/(st-\pi)\otimes_R \bigoplus_{-N\le n\le N} E_n$ satisfies conditions (d) and (e), it follows that K satisfies these conditions as well. Therefore $R[s,t]/(st-\pi)\otimes_R\bigoplus_{-M\le n\le M} K_n \to K$ is surjective as well for $M \gg 0$. By (c) each K_n is the kernel of a surjection of finitely presented objects, and it is thus finitely generated. We have thus expressed E as the cokernel

$$E = \operatorname{coker}\big(R[s,t]/(st-\pi)\otimes_R \bigoplus_{|n|\le M} K_n \to R[s,t]/(st-\pi)\otimes_R \bigoplus_{|n|\le N} E_n\big)$$

of a homomorphism from a finitely generated object to a finitely presented object, so E is finitely presented. □

Now our extension results will follow from a basic result about extensions in codimension 2 that again carries over from quasi-coherent modules.

Lemma 7.14. *Let $j\colon U\to X$ be an open subscheme of a regular noetherian scheme of dimension 2 whose complement is 0-dimensional. Then $j_*\colon \mathcal{A}_U\to\mathcal{A}_X$ maps flat objects to flat objects, and induces an equivalence between the full subcategory of flat objects over X and over U, with inverse given by $j^*\colon \mathcal{A}_X\to\mathcal{A}_U$.*

Proof. It suffices to show that j_* preserves flat objects, and that both the unit and counit of the adjunction between j^* and j_* are equivalences on flat objects. The property of being an isomorphism is local by descent, so we may assume that $X = \operatorname{Spec}(R)$ is affine and U is the complement of a single closed point. Localizing further it suffices to consider the case of $X = \operatorname{Spec}(R)$ for a regular ring R of dimension 2 and $U\subset X$ the complement of the closed point p whose maximal ideal is generated by a regular sequence $x, y\in R$. In particular $U = \operatorname{Spec}(R_x)\cup\operatorname{Spec}(R_y)$.

By construction (Lemma 7.6) we then have

$$j_*(E) = \ker(E|_{R_x}\oplus E|_{R_y}\to E|_{R_{xy}}).$$

So the natural map $j^*(j_*(E))\to E$ is an equivalence for any E as $\mathcal{A}_U$ was defined by descent from affine schemes.

Conversely if $E\in\mathcal{A}_R$ is flat we can tensor E with the left exact sequence

$$0\to R\to R_x\oplus R_y\to R_{xy}$$

to find that the sequence

$$0\to E\to E|_{R_x}\oplus E|_{R_y}\to E|_{R_{xy}}$$

is still exact, so $E\cong j_*(E|_U)$.

Finally we must show that j_* preserves flat objects. By [AZ01, Lem. C1.12] we only need to show that

$$\operatorname{Tor}_1^R(R/\mathfrak{p}, j_*E) = 0$$

for all prime ideals $\mathfrak{p}$ of R. (Here Tor is defined as usual by choosing projective resolutions of R-modules.) For prime ideals in U this follows from flatness of E, so it suffices to show that $\operatorname{Tor}_1(\kappa, j_*(E)) = 0$ for any flat $E\in\mathcal{A}_U$, where $\kappa = R/(x,y)$ is the residue field of the missing point, i.e. we need to show that tensoring j_*E with the Koszul complex

$$0\to R\to R\oplus R\to R\to\kappa\to 0$$

gives an exact sequence

$$0\to j_*(E)\xrightarrow{x\oplus(-y)} j_*(E)\oplus j_*(E)\xrightarrow{y\oplus x} j_*(E).$$

This is the pushforward of the tensor product of E with the short exact sequence of flat objects on U, $0 \to \mathcal{O}_U \to \mathcal{O}_U \oplus \mathcal{O}_U \to \mathcal{O}_U \to 0$ and therefore exact, because j_* is left exact. $\square$

This construction now allows us to check our conditions for the existence of good moduli spaces for $\mathcal{M}_{\mathcal{A}}$. In the following we will assume that $\mathcal{A}$ is locally noetherian and use the result [AZ01, Cor. B6.3] stating that then for any k-algebra that is essentially of finite type (i.e. a localization of a finitely generated k-algebra) the category $\mathcal{A}_R$ is again locally noetherian and thus finitely presentable and noetherian are equivalent.

We start with S-completeness.

Lemma 7.15. *If $\mathcal{A}$ is locally noetherian, then $\mathcal{M}_{\mathcal{A}}$ is S-complete with respect to any DVR R that is essentially of finite type over k.*

Proof. Let us denote by $j\colon \overline{\mathrm{ST}}_R \setminus 0 \subset \overline{\mathrm{ST}}_R$ the inclusion and take any $E \in \mathcal{M}_{\mathcal{A}}(\overline{\mathrm{ST}}_R \setminus 0)$. By Lemma 7.14, $j_*(E)$ is flat, so we only need to show that it is finitely presentable, i.e. we have to check conditions (c)–(e) of Corollary 7.13.

We shall compute $j_*(E)$ explicitly. Let us denote by K the fraction field of R. As E is flat, it is defined by an object $F \in \mathcal{A}_K$ and two R-module subobjects $E_1, E_2 \subset F$ such that $K \otimes_R E_i \to F$ is an isomorphism.

Let $j_i\colon \mathrm{Spec}(R) \to \overline{\mathrm{ST}}_R$ for $i = 1, 2$ denote the two open immersions and $j_{12}\colon \mathrm{Spec}(K) \to \overline{\mathrm{ST}}_R$ their intersection. Then by

$$j_*(E) = \ker\left((j_1)_*(E_1) \oplus (j_2)_*(E_2) \to (j_{12})_*(F)\right)$$

To compute this, we describe it as graded $R[s,t]/(st-\pi)$-module. Under this description j_1 is given by the graded inclusion $R[s,t]/(st-\pi) \subset R[t^{\pm 1}]$ and j_2 by the graded inclusion $R[s,t]/(st-\pi) \subset R[s^{\pm 1}]$. Thus

As the maps $E_i \to F$ are injective, we may thus identify $j_*(E) \subset (j_{12})_*(F)$ with the intersection of two subobjects $(j_1)_*(E_1)$ and $(j_2)_*(E_2)$ under the equivalence of Proposition 7.11, we compute

$$(j_{12})_*(F) = R[t^{\pm 1}] \otimes_R F = \bigoplus_n F t^n,$$

$$(j_1)_*(E_1) = E_1 \otimes_R R[t^{\pm 1}] = \bigoplus_n E_1 t^n,$$

$$(j_2)_*(E_2) = E_2 \otimes_R R[s^{\pm 1}] \simeq \bigoplus_{n \in \mathbb{Z}} (\pi^{-n} \cdot E_2) t^n \subset (j_{12})_*(F)$$

where in the third line we have used the identification $s = t^{-1}\pi$. We compute that:

$$j_*(E) \simeq \bigoplus_{n \in \mathbb{Z}} \left(E_1 \cap (\pi^{-n} \cdot E_2)\right) t^n \subset \bigoplus_{n \in \mathbb{Z}} F t^n.$$

Now each graded piece of $j_*(E)$ is finitely presentable because they are subobjects of the noetherian object E_1 (using that $\mathcal{A}_R$ is locally noetherian). The union of the ascending sequence $\cdots \subset E_1 \cap (\pi^{-n} \cdot E_2) \subset E_1 \cap (\pi^{-n-1} \cdot E_2) \subset \cdots$ is E_1 because $K \otimes_R E_2 \simeq F$, and because E_1 is finitely generated, this union must stabilize. By symmetry the same argument applies E_2 thus $j_*(E)$ is finitely presentable. $\square$

Next we show Θ-reductivity.

Lemma 7.16. *If $\mathcal{A}$ is locally noetherian, then $\mathcal{M}_{\mathcal{A}}$ is Θ-reductive with respect to any DVR R that is essentially of finite type over k.*

Proof. As in the proof of Lemma 7.15 let us denote by K the fraction field of R, $j\colon \mathcal{U} \hookrightarrow \Theta_R$ the complement of the closed point and we take $E \in \mathcal{M}_{\mathcal{A}}(\mathcal{U})$. Then by Lemma 7.14, $j_*(E)$ is flat and so we need to show that it is finitely presentable.

We pass to the presentation $\mathbb{A}^1_R \to \Theta_R$, where the open subset $U \subset \mathbb{A}^1_R$ corresponding to $\mathcal{U}$ is covered by the two affine subschemes defined by $R[x] \subset K[x]$ and $R[x] \subset R[x^{\pm 1}]$. Now $E \in \mathcal{A}_{\mathcal{U}}$ corresponds to an object $F \in \mathcal{A}(K)$, a R-submodule object $E_1 \subset F$ such that $K \otimes_R E_1 \to F$ is an isomorphism, and a weighted descending filtration $\cdots F_{n+1} \subset F_n \subset \cdots \subset F$ satisfying the hypotheses of Corollary 7.12, then $j_*(E)$ corresponds to the graded $R[x]$-module object

$$j_*(E) = \bigoplus_{n\in\mathbb{Z}} (F_n \cap E_1)x^{-n} \subset \bigoplus_{n\in\mathbb{Z}} Fx^{-n} = (\operatorname{Spec}(K) \to \Theta_R)_*(F).$$

Because $\mathcal{A}_R$ is locally noetherian each graded piece $G_n := F_n \cap E_1$ of $j_*(E)$ will be finitely presentable, the maps $G_{n+1} \to G_n$ are injective, $G_n = 0$ for $n \gg 0$, and $G_n = E_1$ for $n \ll 0$. Thus $j_*(E)$ is finitely presentable by Corollary 7.13. □

The valuative criteria for universal closedness turn out to be satisfied as well:

Lemma 7.17. *If $\mathcal{A}$ is a locally noetherian abelian category, the stack $\mathcal{M}_{\mathcal{A}}$ satisfies the valuative criterion for universal closedness, i.e. the existence part of the valuative criterion for properness, with respect to DVR's which are essentially of finite type over k.*

Proof. If R is a DVR, the statement that an object $E \in \mathcal{A}_R$ is flat if and only if it is torsion free follows from [AZ01, Lem. C1.12] as the condition is equivalent to the vanishing of Tor_1. If $j\colon \operatorname{Spec}(K) \to \operatorname{Spec}(R)$ denotes the inclusion of the generic point, then for any $E \in \mathcal{A}_K$, we can write $j_*(E) = \bigcup_\alpha F_\alpha$ as a directed union of finitely generated (hence finitely presentable) subobjects which must be torsion free. If E is finitely generated then $E = \bigcup_\alpha K \otimes_R F_\alpha$ must stabilize, so there is some flat and finitely presentable object F_α extending E. □

We will also use the following below:

Lemma 7.18. *Suppose that $\mathcal{A}$ is locally noetherian. If $\mathcal{M}_{\mathcal{A}}$ is an algebraic stack with affine stabilizers, κ is a field of finite type over k, and $E \in \mathcal{M}_{\mathcal{A}}(\kappa)$ represents a closed point, then E is a semisimple object in $\mathcal{A}_\kappa$.*

Proof. Because E is finitely presented, it can not be expressed as an infinite sum of non-zero objects. Therefore, we only have to show that every finite filtration of E splits. Now by Corollary 7.13 any finite filtration of E corresponds to a map $\Theta_\kappa \to \mathcal{M}_{\mathcal{A}}$ mapping $1 \mapsto E$. Because E is a closed point, the resulting map must factor through a map $\Theta_\kappa \to B_\kappa \operatorname{Aut}_{\mathcal{M}_{\mathcal{A}}}(E)$. We know from the classification of torsors on Θ_κ ([Hei17, Lem. 1.7] or [Hal14, Prop. A.1]) that any such map factors through the projection $\Theta_\kappa \to B_\kappa \mathbb{G}_m$, and thus the corresponding filtration of E_κ is split. □

Lemma 7.19. *If $\mathcal{M}_{\mathcal{A}}$ is an algebraic stack locally of finite presentation over k, then the diagonal of $\mathcal{M}_{\mathcal{A}}$ is affine.*

Proof. If R is a valuation ring over k with fraction field K and $E, F \in \mathcal{M}_{\mathcal{A}}(R)$, then $F \to K \otimes_R F$ is injective and hence so is the restriction map $\operatorname{Hom}_R(E,F) \to \operatorname{Hom}_K(K \otimes_R E, K \otimes_R F) \simeq \operatorname{Hom}_R(E, K \otimes_R F)$. This implies the valuative criterion for separatedness of the diagonal of $\mathcal{M}_{\mathcal{A}}$.

For any ring R over k and $E, F \in \mathcal{M}_{\mathcal{A}}(R)$, we claim that the functor $R'/R \mapsto \mathrm{Hom}_{R'}(R' \otimes_R E, R' \otimes_R F)$ is represented by a separated algebraic space $\underline{\mathrm{Hom}}_R(E, F)$ locally of finite presentation over R. First, observe that the subfunctor $P \subset \mathrm{Aut}_R(E \oplus F)$ classifying automorphisms of the form $\left[\begin{smallmatrix} A & 0 \\ C & D \end{smallmatrix}\right]$ is representable by a closed subspace, because it is the preimage of the (closed) identity section under the map of separated R-spaces $\mathrm{Aut}_R(E \oplus F) \to \mathrm{Aut}_R(E \oplus F)$ given by

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \mapsto \begin{bmatrix} 1 & B \\ 0 & 1 \end{bmatrix}.$$

Next observe that we have a group homomorphism $P \to \mathrm{Aut}_R(E) \times \mathrm{Aut}_R(F)$ over R given by

$$\begin{bmatrix} A & 0 \\ C & D \end{bmatrix} \mapsto (A, D).$$

The preimage of the (closed) identity section is the subgroup classifying automorphisms of the form $\left[\begin{smallmatrix} 1 & 0 \\ C & 1 \end{smallmatrix}\right]$, which is canonically identified with the functor $\underline{\mathrm{Hom}}(E, F)$. Thus $\underline{\mathrm{Hom}}(E, F)$ is a closed subgroup of $\mathrm{Aut}_R(E \oplus F)$, and it is representable, separated, and locally finitely presented over R.

From the functor of points definition of the algebraic space $X := \underline{\mathrm{Hom}}_R(E, F)$, there is a natural action of $\mathbb{G}_m$ which scales the homomorphism. Furthermore, the resulting map $\mathbb{G}_m \times X \to X$ extends (uniquely) to $\mathbb{A}^1 \times X$, i.e. $X = X^+$ in the terminology of Section 2.3. Under these hypotheses, the morphism $X^+ \to X^{\mathbb{G}_m}$ is affine [AHR]. Also, the fixed locus $X^{\mathbb{G}_m}$ is the zero section $X^{\mathbb{G}_m} \simeq \mathrm{Spec}(R) \hookrightarrow X$, and hence $X = X^+ = \underline{\mathrm{Hom}}_R(E, F)$ is affine as well.

The algebraic R-space $\mathrm{Isom}_R(E, F)$ is the closed subspace of $\underline{\mathrm{Hom}}_R(E, F) \times \underline{\mathrm{Hom}}_R(F, E)$ obtained as the preimage of the identity section under the map of separated R-schemes $\underline{\mathrm{Hom}}_R(E, F) \times_R \underline{\mathrm{Hom}}_R(F, E) \to \underline{\mathrm{Hom}}_R(E, E) \times \underline{\mathrm{Hom}}_R(F, F)$. Hence $\mathrm{Isom}_R(E, F)$ is affine, i.e. $\mathcal{M}_{\mathcal{A}}$ has affine diagonal. □

7.3. **Construction of good moduli spaces.** We now apply the previous discussion to construct moduli good moduli spaces for objects in a k-linear abelian category $\mathcal{A}$. As our results require linear reductivity we will now need to assume that k is a noetherian commutative ring over a field of characteristic 0.

Furthermore we assume that $\mathcal{M}_{\mathcal{A}}$, the moduli functor of flat families of finitely presentable objects of Definition 7.8, is an algebraic stack locally of finite type and with affine diagonal over k.

Example 7.20. Let X be a projective scheme over an algebraically closed field k, and consider the heart of a t-structure $\mathcal{C} \subset \mathrm{D}^b(X)$ which is noetherian and satisfies the "generic flatness property" [AP06, Prop. 3.5.1]. Then if we consider the ind-completion $\mathcal{A} := \mathrm{Ind}(\mathcal{C})$, $\mathcal{M}_{\mathcal{A}}$ is an open sub-functor of the moduli functor $\mathcal{D}^b_{pug}(X)$ of universally glueable relatively perfect complexes on X and is hence an algebraic stack locally of finite type with affine diagonal over k [Lie06], [Stacks, Tag 0DPW]. By [Pol07, Prop. 3.3.7], $\mathcal{M}_{\mathcal{A}}$ as defined above agrees with the moduli functor constructed in [AP06], which is the most commonly studied moduli functor for flat families of objects in the heart of a t-structure.

The first result concerns a situation in which no additional stability condition is required.

Theorem 7.21. *Let k be a noetherian ring of characteristic 0 and $\mathcal{A}$ be a locally noetherian, cocomplete and k-linear abelian category. Assume that $\mathcal{M}_{\mathcal{A}}$ is an algebraic stack locally of finite type over k. Then any quasi-compact closed substack*

$\mathcal{X} \subset \mathcal{M}_{\mathcal{A}}$ admits a proper good moduli space, and in this case points of $\mathcal{X}$ must parameterize objects of $\mathcal{A}$ of finite length.

Proof. The stack $\mathcal{X}$ is Θ-reductive (Lemma 7.16) and S-complete (Lemma 7.15) with respect to essentially finite type DVR's because both properties pass to closed substacks, and it has affine diagonal by Lemma 7.19. Therefore Theorem A and Remark 5.5 imply the existence of a separated good moduli space $\mathcal{X} \to X$. Since $\mathcal{X}$ satisfies the existence part of the valuative criterion for properness with respect to essentially finite type DVR's (Lemma 7.15), X is proper by Proposition 3.47 and Remark 3.48. The fact that closed points of $\mathcal{X}$ are represented by semisimple objects in $\mathcal{A}_\kappa$ for fields κ of finite type over k is Lemma 7.18. □

In general we will need a notion of "semistable" objects in $\mathcal{A}$. As in applications different notions of stability are used, we will use an abstract setup that includes many of these. We will illustrate how classical notions of stability fit into this context below.

For any connected component $\nu \in \pi_0(\mathcal{M}_{\mathcal{A}})$, we let $\mathcal{M}^\nu_{\mathcal{A}} \subset \mathcal{M}$ be the corresponding open and closed substack. Our notion of semistability on $\mathcal{M}^\nu_{\mathcal{A}}$ will be encoded by a locally constant function on $|\mathcal{M}_{\mathcal{A}}|$

$$p_\nu \colon \pi_0(\mathcal{M}_{\mathcal{A}}) \to V$$

where V is a totally ordered abelian group, $p_\nu(E) = 0$ for any $E \in \mathcal{M}^\nu_{\mathcal{A}}$, and p_ν is additive in the sense that $p_\nu(E \oplus F) = p_\nu(E) + p_\nu(F)$. We will say that a point of $\mathcal{M}^\nu_{\mathcal{A}}$ represented by $E \in \mathcal{A}_\kappa$ for some algebraically closed field κ over k, is *semistable* if for any subobject $F \subset E$, $p_\nu(F) \leq 0$ and unstable otherwise.[5] Note that this definition is unaffected by embedding V in a larger totally ordered group, so we may assume that V is a totally order vector space over $\mathbb{R}$ by the Hahn embedding theorem.

Using Corollary 7.12 to identify maps $f \colon \Theta_\kappa \to \mathcal{M}_{\mathcal{A}}$ with $\mathbb{Z}$-weighted descending filtrations $\cdots \subset E_{w+1} \subset E_w \subset \cdots$ in $\mathcal{A}_\kappa$, we define a locally constant function $\ell \colon |\underline{\mathrm{Map}}_k(\Theta, \mathcal{M}^\nu_{\mathcal{A}})| \to V$ by the formula

$$\ell(\cdots \subset E_{w+1} \subset E_w \subset \cdots) := \sum_w w p_\nu(E_w/E_{w+1}).$$

Lemma 7.22. *A point $x \in |\mathcal{M}^\nu_{\mathcal{A}}|$ is unstable if and only if there is some $f \in |\underline{\mathrm{Map}}_k(\Theta, \mathcal{M}^\nu_{\mathcal{A}})|$ with $f(1) = x$ and $\ell(f) > 0$.*

Proof. If $F \subset E$ is a destabilizing subobject, then we can simply consider the filtration where $E_2 = 0$, $E_1 = F$, and $E_w = E$ for $w \leq 0$. This filtration has $\ell(E_\bullet) = \deg_\nu(F) > 0$.

The converse is a linear algebra statement: Given a $\mathbb{Z}$-weighted filtration such that $\ell(\cdots \subset E_{w+1} \subset E_w \subset \cdots) := \sum_w w p_\nu(E_w/E_{w+1}) > 0$ and $p_v(E) = \sum_w p_\nu(E_w/E_{w+1}) = 0$ it follows that for some index i we have

$$p_v(E_i) = \sum_{w \geq i} p_\nu(E_w/E_{w+1}) > 0$$

so one of the filtration steps will be destabilizing. □

[5]Note that because the flag space $\underline{\mathrm{Map}}(\Theta, \mathcal{M}^\nu_{\mathcal{A}}) \times_{\mathrm{ev}_1, \mathcal{M}^\nu_{\mathcal{A}}, [E]} \mathrm{Spec}(\kappa)$ of $[E] \colon \mathrm{Spec}(\kappa) \to \mathcal{M}^\nu_{\mathcal{A}}$ is an algebraic space locally of finite type over κ, if there is a destabilizing subobject of E after base change to an arbitrary field extension $\kappa \subset \kappa'$, then there is a destabilizing subobject for E over κ, so this definition does not depend on the choice of representative.

Example 7.23 (Gieseker stability)**.** Let $\mathcal{A} = \mathrm{QCoh}(X)$ denote the category of quasi-coherent sheaves on a scheme X which is projective over a field k of characteristic 0. Then $\mathcal{M}_\mathcal{A}$ is the usual stack of flat families of coherent sheaves on X. Fix a numerical K-theory class $\gamma \in K_0^{\mathrm{num}}(X)$ corresponding to sheaves whose support has dimension d, fix an ample line bundle $\mathcal{O}_X(1)$ on X, and let $\mathcal{M}_\mathcal{A}^\gamma$ denote the open and closed substack of objects of class γ. For any $E \in \mathrm{Coh}(X)$, let $P(E) = \alpha_n(E)t^n + \cdots + \alpha_0(E)$ denote its Hilbert polynomial with respect to $\mathcal{O}_X(1)$.

We say that a coherent sheaf F of class γ is semistable with respect to $\mathcal{O}_X(1)$ if for all nonzero subsheaves $E \subset F$, the asymptotic inequality

$$p_\gamma(E) := \alpha_d(F)P(E) - \alpha_d(E)P(F) \leq 0$$

holds for all $t \gg 0$. If $E \subset F$ is a nonzero subsheaf with $\dim(\mathrm{supp}(E)) < d$, then $\alpha_d(E) = 0$ but $\alpha_d(F) > 0$, so E destabilizes F. Therefore a semistable sheaf must be pure, and in this case $P(E)/\alpha_d(E)$ and $P(F)/\alpha_d(F)$ are the reduced Hilbert polynomials, so our definition is equivalent to the classical definition of Gieseker semistability [HL10, Def. 1.2.4]. Note also that $P(F) = P(\gamma)$ only depends on γ, so p_γ defines a function $p_\gamma \colon \mathcal{M}_\mathcal{A} \to V_d$, where V_d denotes the vector space of polynomials of degree $\leq d$ totally ordered by asymptotic inequality as $t \to \infty$. The function p_γ is locally constant in flat families and additive in short exact sequences.

Example 7.24 (Bridgeland stability)**.** Consider a projective scheme X over an algebraically closed field k of characteristic 0. A Bridgeland stability condition is determined by the heart of a t-structure $\mathcal{C} \subset \mathrm{D}^b(X)$, and a *central charge* homomorphism $Z \colon K_0(\mathrm{D}^b(X)) \to \mathbb{C}$, which we assume factors through the numerical k-theory $K_0^{\mathrm{num}}(\mathrm{D}^b(X))$, i.e. the quotient $K_0(\mathrm{D}^b(X))$ by the kernel of the Euler pairing $\chi(-,-)$. The central charge Z is required to be a stability function on $\mathcal{C}$ with the Harder–Narasimhan property [Bri07, Prop. 5.3]. The simplest example is when X is a projective curve, $\mathcal{C} = \mathrm{Coh}(X)$ is the usual t-structure, and

$$Z(E) = -\deg(E) + i\,\mathrm{rank}(E).$$

For a stability condition $(\mathcal{C}, Z)$ on a general X, we take the formula above as the definition of rank and degree of objects in $\mathcal{C}$. As before we let $\mathcal{A} := \mathrm{Ind}(\mathcal{C})$. Given a class $\gamma \in K_0^{\mathrm{num}}(\mathrm{D}^b(X))$, there is an open and closed substack $\mathcal{M}_\mathcal{A}^\gamma \subset \mathcal{M}_\mathcal{A}$ whose k-points classify objects $E \in \mathcal{A}$ of numerical class γ. Then Bridgeland semistability on the stack $\mathcal{M}_\mathcal{A}^\gamma$ is determined by the degree function

$$p_\gamma(E) := \deg(E)\,\mathrm{rank}(\gamma) - \deg(\gamma)\,\mathrm{rank}(E),$$

so $E \in \mathcal{M}_\mathcal{A}^\gamma$ is unstable if and only if there is a subobject $F \subset E$ such that $p_\gamma(F) > p_\gamma(E) = 0$.

Theorem 7.25. *Let k be a noetherian ring of characteristic 0 and $\mathcal{A}$ be a locally noetherian, cocomplete and k-linear abelian category. Assume that $\mathcal{M}_\mathcal{A}$ is an algebraic stack locally of finite type over k. Let $\nu \in \pi_0(\mathcal{M}_\mathcal{A})$ be a connected component, and let $p_\nu \colon \pi_0(\mathcal{M}_\mathcal{A}) \to V$ be an additive function defining a notion of semistability on $\mathcal{M}_\mathcal{A}^\nu$, as above.*

If the substack of semistable points $\mathcal{M}_\mathcal{A}^{\nu,\mathrm{ss}} \subset \mathcal{M}_\mathcal{A}^\nu$ is open and quasi-compact then $\mathcal{M}_\mathcal{A}^{\nu,\mathrm{ss}}$ admits a separated good moduli space. If in addition $\mathcal{M}_\mathcal{A}^{\nu,\mathrm{ss}}$ is the open piece of a Θ-stratification of $\mathcal{M}_\mathcal{A}^\nu$, then $\mathcal{M}_\mathcal{A}^{\nu,\mathrm{ss}}$ admits a proper good moduli space.

Proof. We have already seen that $\mathcal{M}^{\nu}_{\mathcal{A}}$ has affine diagonal (Lemma 7.19), and with respect to essentially finite type DVR's $\mathcal{M}^{\nu}_{\mathcal{A}}$ is Θ-reductive (Lemma 7.16), S-complete (Lemma 7.15), and satisfies the existence part of the valuative criterion for properness (Lemma 7.17). It follows from Proposition 6.14 and Remark 6.16 that $\mathcal{M}^{\nu,\mathrm{ss}}_{\mathcal{A}}$ is Θ-reductive and S-complete with respect to essentially finite type DVR's as well[6], so it has a separated good moduli space $\mathcal{M}^{\nu,\mathrm{ss}}_{\mathcal{A}} \to M$ space by Theorem A and Remark 5.5. For the final statement, Lemma 7.17 and Corollary 6.12 applied to the Θ-stratification of $\mathcal{M}^{\nu}_{\mathcal{A}}$ imply that $\mathcal{M}^{\nu,\mathrm{ss}}_{\mathcal{A}}$ satisfies the existence part of the valuative criterion for properness with respect to essentially finite type DVR's and hence M is proper over $\mathrm{Spec}(k)$ by Proposition 3.47 and Remark 3.48. □

Example 7.26 (Gieseker stability, continued)**.** The Harder–Narasimhan stratification with respect to Gieseker semistability defines a Θ-stratification of the stack $\underline{\mathrm{Coh}}(X)$ (this is essentially the content of [Nit11], using different language), and the semistable locus is open and bounded. Therefore Theorem 7.25 provides an alternate construction of a proper good moduli space for Gieseker semistable sheaves.

Example 7.27 (Bridgeland stability, continued)**.** We consider Bridgeland stability conditions whose central charge factors through a fixed surjective homomorphism $\mathrm{cl}\colon K_0(\mathrm{D}^b(X)) \to K_0^{\mathrm{num}}(\mathrm{D}^b(X)) \to \Gamma$, where Γ is a finitely generated free abelian group, and we let $\mathrm{Stab}_\Gamma(X)$ denote the space of Bridgeland stability conditions whose central charge factors through cl and which satisfy the support property with respect to Γ. This is given the structure of a complex manifold in such a way that the map $\mathrm{Stab}_\Gamma(X) \to \mathrm{Hom}(\Gamma, \mathbb{C})$ which forgets the t-structure is a local isomorphism [Bri07, Thm. 1.2].

A stability condition $(\mathcal{C}, Z)$ is algebraic if $Z(\Gamma) \subset \mathbb{Q} + i\mathbb{Q}$. If $(\mathcal{C}, Z)$ is algebraic, then $\mathcal{C}$ is noetherian [AP06, Prop. 5.0.2]. Let $\mathrm{Stab}^*_\Gamma(X) \subset \mathrm{Stab}_\Gamma(X)$ be a connected component which contains an algebraic stability condition $\sigma_0 = (\mathcal{C}_0, Z_0)$ for which:

- the heart $\mathcal{C}_0$ satisfies the generic flatness condition, and
- $\mathcal{M}^{\gamma,\mathrm{ss}}_{\mathrm{Ind}(\mathcal{C}_0)}$ is bounded for every $\gamma \in \Gamma$.

Then by [PT15, Prop. 4.12] the same is true for *any* algebraic stability condition in the connected component $\mathrm{Stab}^*_\Gamma(X)$. Furthermore, if $\sigma = (\mathcal{C}, Z) \in \mathrm{Stab}_\Gamma(X)$ is algebraic and satisfies the generic flatness and boundedness conditions, then the Harder–Narasimhan filtration defines a Θ-stratification of $\mathcal{M}^{\gamma}_{\mathrm{Ind}(\mathcal{C})}$ whose open piece is $\mathcal{M}^{\gamma,\mathrm{ss}}_{\mathrm{Ind}(\mathcal{C})}$ [Hal14, Prop. 5.40, §5.4(1)]. Therefore Theorem 7.25 implies that $\mathcal{M}^{\gamma,\mathrm{ss}}_{\mathrm{Ind}(\mathcal{C})}$ has a proper good moduli space for any algebraic stability condition in $\mathrm{Stab}^*(X)$.

Finally, we claim that for an arbitrary stability condition $\sigma \in \mathrm{Stab}^*(X)$, one can find an algebraic stability condition σ' which defines the same moduli functor $\mathcal{M}^{\gamma,\mathrm{ss}}_{\mathrm{Ind}(\mathcal{C})}$, so $\mathcal{M}^{\gamma,\mathrm{ss}}_{\mathrm{Ind}(\mathcal{C})}$ has a proper good moduli space for any stability condition in $\mathrm{Stab}^*(X)$.

To establish this claim, fix a class $\gamma \in \Gamma$. For any $\gamma' \in \Gamma$ which is linearly independent of γ over $\mathbb{Q}$, consider the real codimension 1 subset

$$\mathcal{W}_{\gamma'} := \{\sigma = (\mathcal{C}, Z) \in \mathrm{Stab}^*_\Gamma(X) \,|\, Z(\gamma') \in \mathbb{R}_{>0} \cdot Z(\gamma)\}.$$

[6]Technically it follows from the proof of Proposition 6.14 as we are only know here that $\mathcal{M}^{\nu}_{\mathcal{A}}$ is Θ-reductive and S-complete with respect to essentially of finite type DVR's.

If one restricts to a small compact neighborhood $\mathfrak{B} \subset \mathrm{Stab}^*_\Gamma(X)$ containing σ, then there is a finite subset $S \subset \Gamma$ such that for any $S' \subset S$ the moduli functor $\mathcal{M}^{\gamma,\mathrm{ss}}_{\mathrm{Ind}(\mathcal{C})}$ is constant for all $\sigma \in \mathfrak{C}_{S'} \cap \mathfrak{B}$ [Tod08, Prop. 2.8], where

$$\mathfrak{C}_{S'} := \left(\bigcup_{\gamma' \in S'} \mathcal{W}_{\gamma'} \right) \setminus \bigcup_{\gamma' \notin S'} \mathcal{W}_{\gamma'}.$$

$\mathcal{M}^{\gamma,\mathrm{ss}}_{\mathrm{Ind}(\mathcal{C})}$ is constant for $\sigma \in \mathfrak{B} \cap \mathfrak{C}_{S'}$ because $\mathcal{M}^{\gamma,\mathrm{ss}}_{\mathrm{Ind}(\mathcal{C})}$ can only change if the set of classes $\gamma' \in \Gamma$ with $Z(\gamma') \in \mathbb{R} \cdot Z(\gamma)$ changes, and the set of such γ' is constant for $\sigma \in \mathfrak{C}_{S'} \cap \mathfrak{B}$.

The condition that $\sigma \in \bigcup_{\gamma' \in S'} \mathcal{W}_{\gamma'}$ amounts to the claim that if $W \subset \Gamma_\mathbb{Q}$ is the span of γ and the $\gamma' \in S'$, then $\dim_\mathbb{Q}(Z(W)) = 1$. We may write $Z = Z_1 \oplus Z_2$ under a choice of splitting $\Gamma_\mathbb{Q} \simeq W \oplus U$, and the condition now amounts to $\mathrm{rank}(Z_1) = 1$. As rational points are dense in the space of rank 1 real matrices, we may find an arbitrarily small perturbation Z' of Z which is rational, and this perturbed central charge defines our new algebraic stability condition $\sigma' \in \mathfrak{B} \cap \mathfrak{C}_{S'}$.

8. Good moduli spaces for moduli of $\mathcal{G}$-torsors

To illustrate our general theorems we now construct good moduli spaces for semistable torsors under Bruhat–Tits group schemes. This generalizes the results obtained by Balaji and Seshadri who constructed such moduli spaces for generically split groups over the complex numbers. In [Hei17], the third author analyzed the coarse moduli space for the moduli of stable bundles, whose existence is guaranteed by the Keel–Mori theorem. Here we extend this analysis to include semistable bundles which are not stable. As in this article we are interested in existence theorems for good moduli spaces (instead of adequate moduli) we will have to assume that we work over a base field k of characteristic 0 in this section.

Let us briefly introduce the setup from [Hei17]. Let C be a smooth geometrically connected, projective curve over a field k and $\mathcal{G}/C$ a smooth Bruhat–Tits group scheme over C, i.e., $\mathcal{G}$ is smooth a affine group scheme over C that has geometrically connected fibers, such that over some dense open subset $U \subset C$ the group scheme is reductive and over all local rings at points p in $\mathrm{Ram}(\mathcal{G}) := C \smallsetminus U$ the group scheme $\mathcal{G}|_{\mathrm{Spec}(\mathcal{O}_{C,p})}$ is a connected parahoric Bruhat–Tits group. The simplest examples are of course reductive groups $G \times C$.

The stack of $\mathcal{G}$-torsors is denoted by $\mathrm{Bun}_\mathcal{G}$ and this is a smooth algebraic stack. To define stability one usually chooses a line bundle on $\mathrm{Bun}_\mathcal{G}$. As explained in [Hei17, §3.B] there are natural choices in our situation. First there is the determinant line bundle $\mathcal{L}_{\det}$ given by the adjoint representation, i.e., the fiber at a bundle $\mathcal{E} \in \mathrm{Bun}_{\mathcal{G}_p}$ is $\mathcal{L}_{\det,\mathcal{E}} = \det(H^*(C, \mathrm{ad}(\mathcal{E})))^\vee$, where $\mathrm{ad}(\mathcal{E}) = \mathcal{E} \times^\mathcal{G} \mathrm{Lie}(\mathcal{G}_p/C)$ is the adjoint bundle of $\mathcal{E}$.

Next any collection of characters $\underline{\chi} \in \prod_{p \in \mathrm{Ram}(\mathcal{G})} \mathrm{Hom}(\mathcal{G}_p, \mathbb{G}_m)$ defines line bundles on the classifying stacks $B\mathcal{G}_p$ and one obtains a line bundle $\mathcal{L}_{\underline{\chi}}$ on $\mathrm{Bun}_\mathcal{G}$, by pull back via the map $\mathrm{Bun}_\mathcal{G} \to B\mathcal{G}_p$ defined by restriction of $\mathcal{G}$ torsors on C to the point p. We will denote by $\mathcal{L}_{\det,\underline{\chi}} := \mathcal{L}_{\det} \otimes \mathcal{L}_{\underline{\chi}}$, call the corresponding notion of stability $\underline{\chi}$-stability and denote by $\mathrm{Bun}_\mathcal{G}^{\underline{\chi}-\mathrm{ss}} \subset \mathrm{Bun}_\mathcal{G}$ the substack of $\underline{\chi}$-semistable torsors.

Under explicit numerical conditions on $\underline{\chi}$ this satisfies the positivity assumption of loc.cit. [Hei17, Prop. 3.3], i.e., the restriction of $\mathcal{L}_{\det}$ to the affine Grassmannian $\mathrm{Gr}_{\mathcal{G},p}$ classifying $\mathcal{G}$ bundles together with a trivialization on $C \smallsetminus p$ is nef. The parameter $\underline{\chi}$ will be called positive if $\mathcal{L}_{\det,\underline{\chi}}$ is ample on $\mathrm{Gr}_{\mathcal{G},p}$ for all p. It is called admissible if $\underline{\chi}$ furthermore satisfies the numerical condition of [Hei17, Sec. 3.F].

Theorem 8.1 (Good moduli for semistable $\mathcal{G}$-torsors)**.** *Assume k is a field of characteristic 0, C is a smooth, projective, geometrically connected curve over k, $\mathcal{G}$ is a parahoric Bruhat–Tits group scheme over k and $\underline{\chi}$ is a admissible stability parameter. Then $\mathrm{Bun}_{\mathcal{G}}^{\underline{\chi}-\mathrm{ss}}$ admits a proper good moduli space $M_{\mathcal{G}}$.*

As remarked before, in the case that $\mathcal{G}$ is a generically split group scheme, the space $M_{\mathcal{G}}$ was constructed by Balaji and Seshadri [BS15].

To prove the theorem we only need to check that $\mathrm{Bun}_{\mathcal{G}}$ satisfies the assumptions of our main Theorems A, B and C, i.e., we need to show that the line bundle $\mathcal{L}_{\det,\chi}$ defines a well-ordered Θ-stratification of $\mathrm{Bun}_{\mathcal{G}}$, that this stack satisfies the existence part of the valuative criterion for properness and that $\mathrm{Bun}_{\mathcal{G}}^{\underline{\chi}-\mathrm{ss}}$ is Θ-reductive and S-complete. This will be done in a series of Lemmas.

Lemma 8.2. *The canonical reduction of $\mathcal{G}$-torsors defines a Θ-stratification on $\mathrm{Bun}_{\mathcal{G}}$ with semistable locus $\mathrm{Bun}_{\mathcal{G}}^{\underline{\chi}-\mathrm{ss}}$. This stratification admits a well-ordering.*

Let us briefly recall the context of $\underline{\chi}$-stability. To simplify the presentation will assume that our base field is algebraically closed. Then by [Hei17, Lem. 3.9] any map $f\colon \Theta \to \mathrm{Bun}_{\mathcal{G}}$ arises as a Rees construction $\mathrm{Rees}(\mathcal{E}_{\mathcal{P}}, \lambda)$ from a $\mathcal{G}$-bundle $\mathcal{E}$, together with a reduction $\mathcal{E}_{\mathcal{P}}$ to a parabolic subgroup $\mathcal{P} \subset \mathcal{G}$ and a generic cocharacter $\lambda\colon \mathbb{G}_{m,k(C)} \to \mathcal{G}_{k(C)}$ that is dominant for $\mathcal{P}_\eta$. The argument of the proof implies that all components of $\underline{\mathrm{Map}}(\Theta, \mathrm{Bun}_{\mathcal{G}})$ can be identified with components of $\mathrm{Bun}_{\mathcal{P}}$ for parabolics $\mathcal{P}$ equipped with a dominant λ. As reductions to parabolics are determined by the induced filtration of the adjoint bundle, this implies in particular that the forgetful morphism from any component to $\mathrm{Bun}_{\mathcal{G}}$ is quasi-compact. We denote by $\mathrm{wt}_{\mathcal{E}_{\mathcal{P}}}(\lambda)$ the weight of $f^*(\mathcal{L}_\chi)$.

Proof of Lemma 8.2. In [Hal14, Thm. 2.7, Simplif. 2.8, Simplif. 2.9] a list of necessary and sufficient conditions for a stratification to be a Θ-stratification is given. We will first verify these criteria (which we number as in loc.cit.), then briefly recall how the general proof works in the specific context of $\mathcal{G}$-bundles in order to illustrate the relation to classical arguments of Behrend. It suffices to assume the ground field k is algebraically closed [Hal14, Lem. 2.14].

Existence and uniqueness of HN filtrations (1)*:*

HN filtrations are maps $f\colon \Theta \to \mathrm{Bun}_{\mathcal{G}}$ corresponding to the canonical parabolic reductions for unstable $\mathcal{G}$-bundles constructed in [Hei17, Sec. 3.F]. To define canonical reductions in this setup we fixed an invariant inner form $(\,,)$ on the generic cocharacters of $\mathcal{G}$ defining the norm $||\cdot||$ we showed [Hei17, Prop. 3.6] that for any admissible $\underline{\chi}$ and any $\mathcal{G}$-torsor $\mathcal{E}$ there exists a canonical reduction $\mathcal{E}_{\mathcal{P}}$ to a parabolic subgroup $\mathcal{P} \subset \mathcal{G}$ defined by a dominant cocharacter $\lambda\colon \mathbb{G}_{m,\eta} \to \mathcal{G}_\eta$ maximizing $\mu_{\max}(\mathcal{E}) := \max(\frac{\mathrm{wt}_{\mathcal{E}_{\mathcal{P}}}(\lambda)}{||\lambda||})$.

The uniqueness of the reduction followed by checking that for any choice of a generic maximal torus $\mathcal{T}_\eta \subset \mathcal{G}_\eta^{\mathcal{E}} \cong \mathcal{G}_\eta$ that contains a maximal split torus the map sending a Borel subgroup $\mathcal{B}_\eta \subset \mathcal{G}_\eta$ containing $\mathcal{T}_\eta$ to the cocharacter defined by $-\mathrm{wt}_{\mathcal{E}_{\mathcal{B}}}(\cdot)$ defines a complementary polyhedron in the sense of Behrend [Beh95, Def. 2.1].

Consistency of HN filtrations (5)*:*

Any canonical reduction $f\colon \Theta \to \mathrm{Bun}_{\mathcal{G}}$ of $f(1)$ also defines the canonical reduction of the associated graded bundle $f(0)$. Indeed, the perspective of complementary polyhedra shows that a canonical reduction $\mathcal{E}_{\mathcal{P}}$ can also be characterized by the property that (1) $\mathrm{wt}_{\mathcal{E}_{\mathcal{P}}}(\lambda') > 0$ for any non-zero λ' that is non-negative on any root of $\mathcal{P}$, and (2) such that for any reductions $\mathcal{E}_{\mathcal{B}}$ of $\mathcal{E}_{\mathcal{P}}$ to a Borel subgroups $\mathcal{B} \subset \mathcal{P}$ we have $\mathrm{wt}_{\mathcal{E}_{\mathcal{B}}}(\check{\alpha}) < 0$ for all simple roots coroots $\check{\alpha}$ in the centralizer of λ [Beh95, Sec. 3] [HS10, Thm 4.3.2].

Specialization of HN filtrations $(2')$*:*

For any family $\mathcal{E}_R \in \mathrm{Bun}_{\mathcal{G}}(R)$ defined over a DVR R with fraction field K and residue field κ we have $\mu_{\max}(\mathcal{E}_K) \leq \mu_{\max}(\mathcal{E}_\kappa)$ and equality holds only if the canonical filtration over K extends to the family, by [Hei17, Lem. 3.17].

Local finiteness (4)*:*

For any family of $\mathcal{G}$-bundles over a finite type scheme U, only finitely many connected components of $\underline{\mathrm{Map}}(\Theta, \mathrm{Bun}_{\mathcal{G}})$ are necessary to realize the HN filtration of every fiber $\mathcal{E}_u$ for $u \in U$. This follows from the fact that there are only finitely many λ such the canonical reduction of $\mathcal{E}_u$ for some $u \in U$ has type $\mathcal{P}_\lambda$, which is established in the proof of [Hei17, Prop. 3.18].

Completing the proof:

The function $\frac{\mathrm{wt}_{\mathcal{E}_{\mathcal{P}}}(\lambda)}{||\lambda||}$ defines a locally constant real valued function μ on the components of $\underline{\mathrm{Map}}(\Theta, \mathrm{Bun}_{\mathcal{G}})$ containing a HN filtration. We have verified that μ satisfies the conditions of [Hal14, Thm. 2.7, Simplif. 2.8, Simplif. 2.9] and thus defines a weak Θ-stratification on $\mathrm{Bun}_{\mathcal{G}}$ in which the HN filtrations of unstable points correspond to canonical reductions of $\mathcal{G}$-bundles. In our context, the argument goes as follows:

"Local finiteness" and quasi-compactness of $\mathrm{Bun}_{\mathcal{P}_\lambda} \to \mathrm{Bun}_{\mathcal{G}}$ implies that the stratification of $\mathrm{Bun}_{\mathcal{G}}$ defined by $\mu_{\max}$ is constructible (See also the proof of [Hei17, Prop. 3.18]). Since the invariant $\mu_{\max}$ is semicontinuous, this implies that for any constant c the substacks $\mathrm{Bun}_{\mathcal{G}}^{\mu_{\max} \leq c}$ defined by the condition $\mu_{\max}(\mathcal{E}) \leq c$ are open. To show that this defines a Θ-stratification we are therefore left to show that the closed substacks $\mathrm{Bun}_{\mathcal{G}}^{\mu_{\max} \leq c} \setminus \mathrm{Bun}_{\mathcal{G}}^{\mu_{\max} < c}$ are unions of connected components of $\underline{\mathrm{Map}}(\Theta, \mathrm{Bun}_{\mathcal{G}}^{\mu_{\max} \leq c})$.

Let us fix an unstable bundle $\mathcal{E}$ with $\mu_{\max}(\mathcal{E}) = c$ and canonical reduction given as a reduction to $\mathcal{P}_\lambda$. Let us denote by $p_\lambda\colon \mathrm{Bun}_{\mathcal{P}_\lambda}^{\mathrm{HN}} \to \mathrm{Bun}_{\mathcal{G}}^{\mu_{\max} \leq c}$ the restriction of the canonical map $\mathrm{Bun}_{\mathcal{P}_\lambda} \to \mathrm{Bun}_{\mathcal{G}}$. The "consistency of HN filtrations," i.e. the fact that passing to the associated graded of the canonical filtration preserves the strata, shows that $\mathrm{Bun}_{\mathcal{P}_\lambda}^{\mathrm{HN}}$ is indeed a component of $\underline{\mathrm{Map}}(\Theta, \mathrm{Bun}_{\mathcal{G}}^{\mu_{\max} \leq c})$, and by uniqueness of the filtration and the "specialization property," the map p_λ is proper and universally injective. Hence the stratification induced by $\mu_{\max}$ is a weak Θ-stratification.

Any weak Θ-stratification is a Θ-stratification in characteristic 0 [Hal14, Cor. 2.6.1]. Alternatively, it is not hard to check directly that p_λ is injective on tangent spaces at any HN filtration, as in the case of Behrend's conjecture, so p_λ is a closed immersion whose image is $\mathrm{Bun}_{\mathcal{G}}^{\mu_{\max} \leq c} \setminus \mathrm{Bun}_{\mathcal{G}}^{\mu_{\max} < c}$. Finally, this Θ-stratification that admits a well-ordering, because for any c and any connected component of $\mathrm{Bun}_{\mathcal{G}}$ the open substack $\mathrm{Bun}_{\mathcal{G}}^{\mu_{\max} \leq c}$ are of finite type [Hei17, Prop. 3.18], so $\mathrm{Bun}_{\mathcal{G}}^{\mu_{\max}\leq c} \setminus \mathrm{Bun}_{\mathcal{G}}^{\mu_{\max}<c}$ can only contribute finitely many strata on each component of $\mathrm{Bun}_{\mathcal{G}}$.

□

Remark 8.3. Assume for simplicity that the ground field k is algebraically closed. The notion of $\underline{\chi}$-stability is controlled by the class $\ell = c_1(\mathcal{L}_{\underline{\chi}}) \in H^2(\mathrm{Bun}_{\mathcal{G}}; \mathbb{Q})$. If $p \in C$ is a regular point for $\mathcal{G}$, then generic cocharacters of $\mathcal{G}$ induce cocharacters in $\mathcal{G}_p$, and under this map the norm $||\lambda||$ on generic cocharacters can be induced from a conjugation invariant norm on cocharacters of $\mathcal{G}_p$. It follows that $||\lambda||$ is induced by a class in $H^4(\mathrm{Bun}_{\mathcal{G}}; \mathbb{Q})$ defined via pullback

$$(\mathrm{Sym}^2(N_{\mathbb{Q}}^*))^W \simeq H^4(B\mathcal{G}_p; \mathbb{Q}) \to H^4(\mathrm{Bun}_{\mathcal{G}}; \mathbb{Q}),$$

along the restriction morphism $\mathrm{Bun}_{\mathcal{G}} \to B\mathcal{G}_p$, where N denotes the coweight lattice of $\mathcal{G}_p$ and W denotes the Weyl group.

Therefore the function $\mu(\mathcal{E}_{\mathcal{P}}, \lambda) = \frac{\mathrm{wt}_{\mathcal{E}_{\mathcal{P}}}(\lambda)}{||\lambda||}$ is a standard numerical invariant in the sense of [Hal14, Def. 4.7] satisfying condition (R) by [Hal14, Lem. 4.10,Lem. 4.12]. It is also not difficult to check this directly (See [Hal14, Rem. 4.11]). So in the proof of Lemma 8.2 one could also apply [Hal14, Thm. 4.38], which provides a shorter list of criteria for a numerical invariant to define a weak Θ-stratification. In particular this implies that condition (5) above is automatic here.

Lemma 8.4. *The stack* $\mathrm{Bun}_{\mathcal{G}}^{\underline{\chi}-\mathrm{ss}}$ *is* S*-complete and locally linearly reductive.*

Proof. S-completeness holds because of the existence of a blow up of ST_R to linking two specializations. $\mathcal{L}_{\det,\underline{\chi}}$ is positive on the exceptional lines, so if the blow-up was necessary, one of the bundles was unstable.

In particular every closed substack of $\mathrm{Bun}_{\mathcal{G}}^{\mathrm{Bun}_{\mathcal{G}}^{\underline{\chi}-\mathrm{ss}}}$ is again S-complete, so that by Proposition 3.45 the automorphism groups of closed points are geometrically reductive. As we assumed our base field to be of characteristic 0 in this section, these groups are linearly reductive. □

Lemma 8.5. *The stack* $\mathrm{Bun}_{\mathcal{G}}^{\underline{\chi}-\mathrm{ss}}$ *is* Θ*-reductive.*

Proof. This again follows from semi-continuity of the numerical invariant as in [Hei17, Lem. 3.17]. We briefly recall the argument: Let R be a DVR with fraction field K and residue field κ. Let $f\colon \Theta_R \setminus 0 \to \mathrm{Bun}_{\mathcal{G}}^{\underline{\chi}-\mathrm{ss}}$ be a morphism. The restriction of f to $\mathrm{Spec}(R)$ defines a family $\mathcal{E}_R$ of $\mathcal{G}$-torsors over C_R.

The restriction of $f|_{\Theta_K}$ defines a filtrations on $\mathcal{E}_K$ and after possibly passing to a finite extension of R we may by [Hei17, Rem. 3.10] assume that this filtration is given by a reduction $\mathcal{E}_K^{\mathcal{P}}$ of $\mathcal{E}_K$ to a parabolic subgroup $\mathcal{P} \subset \mathcal{G}$ which is defined by a generic cocharacter $\lambda\colon \mathbb{G}_m \to \mathcal{G}_{k(\eta)}$.

As $\mathcal{G}_{k(\eta)}/\mathcal{P}_{k(\eta)}$ is projective, any reduction of $\mathcal{E}_{K(\eta)}$ to $\mathcal{P}$ extends to an open subset $U \subset C_R$ whose the complement consists of finitely many closed points of the special fiber. Also the reduction over U_κ extends canonically to a reduction $\mathcal{E}_\kappa^{\mathcal{P}'}$ to a parabolic subgroup $\mathcal{P}' \subset \mathcal{G}_\kappa$ over C_κ. Now, as in loc. cit. at any point p the adjoint bundle $\mathrm{ad}(\mathcal{E}_\kappa^{\mathcal{P}'}) \subset \mathrm{ad}(\mathcal{E}_\kappa)$ is the saturation of the adjoint bundle at the generic fiber. As $\underline{\chi}$ is admissible this implies that either the reduction $\mathcal{E}_K^{\mathcal{P}}$ extends over C_R or the weight of the reduction $\mathcal{E}_\kappa^{\mathcal{P}'}$ is strictly larger than the weight of $f|_{\Theta_K}$, which is 0 as f is a reduction in $\mathrm{Bun}_{\mathcal{G}}^{\underline{\chi}-\mathrm{ss}}$. As by assumption $\mathcal{E}_\kappa$ is $\underline{\chi}$-semistable the weight cannot increase, so the reduction extends to C_R. Now we can apply Lemma 6.15 to find that this filtration also lies in $\mathrm{Bun}_{\mathcal{G}}^{\chi,ss}$ and thus defines an extension of f to Θ_R. □

Proof of Theorem 8.1. We just proved that $\mathrm{Bun}_{\mathcal{G}}^{\chi-\mathrm{ss}}$ is S-complete, locally linearly reductive and Θ-reductive.

By [Hei17, Prop. 3.3] the stack $\mathrm{Bun}_{\mathcal{G}}$ satisfies the existence criterion for properness, i.e., if R is a DVR with fraction field K and $\mathcal{E}_K \in \mathrm{Bun}_{\mathcal{G}}(K)$ is a $\mathcal{G}$-torsor over C_K then there exists a finite extension R' of R such that $\mathcal{E}_K$ extends to a torsor over C_R. Therefore we can apply Theorem C to deduce the existence of a proper good moduli space. □

Appendix A. Strange gluing lemma

In this section, we establish two gluing results: Theorem A.1 and Theorem A.5. Additionally, we apply both results to give a refinement of the classical semistable reduction theorem in GIT (Theorem A.8).

A.1. Gluing results.

Let R be a DVR with fraction field K and residue field κ, and let $\pi \in R$ be a uniformizer parameter. For $n > 0$ we will consider the following quotient stack

$$\overline{\mathrm{ST}}_R^{n,1} = [\mathrm{Spec}(R[s,t]/(st^n - \pi))/\mathbb{G}_m]$$

where the $\mathbb{G}_m$-action is encoded by giving s weight $n > 0$ and giving t weight -1. We have a closed immersion $\Theta_\kappa \hookrightarrow \overline{\mathrm{ST}}_R^{n,1}$ defined by $s = 0$ and an open immersion $\mathrm{Spec}(R) \hookrightarrow \overline{\mathrm{ST}}_R^{n,1}$ defined by $t \neq 0$.

We will denote $0 \in \mathrm{Spec}(R)$ as the closed point and $1 \in \Theta_k$ as the open point. Observe that any morphism $\overline{\mathrm{ST}}_R^{n,1} \to \mathcal{X}$ restricts to morphisms $f \colon \Theta_\kappa \to \mathcal{X}$ and $\xi \colon \mathrm{Spec}(R) \to \mathcal{X}$ along with an isomorphism $\phi \colon \xi(0) \simeq f(1)$ in $\mathcal{X}(\kappa)$.

Theorem A.1. *Let $\mathcal{X}$ be an algebraic stack with affine diagonal and locally of finite presentation over an algebraic space S. Let R be a DVR with residue field κ and consider morphisms $f \colon \Theta_\kappa \to \mathcal{X}$ and $\xi \colon \mathrm{Spec}(R) \to \mathcal{X}$ over S together with an isomorphism $\phi \colon \xi(0) \simeq f(1)$. For all $n \gg 0$, there is a morphism $\overline{\mathrm{ST}}_R^{n,1} \to \mathcal{X}$ unique up to unique isomorphism extending the triple (f, ξ, ϕ).*

This theorem is inspired by the perturbation theorem [Hal14, Prop. 3.53], which is an analogous result for constructing map $[\mathbb{A}^2_\kappa/\mathbb{G}^2_{m,\kappa}] \to \mathcal{X}$ from maps from the loci $\{s = 0\}$ and $\{t \neq 0\}$.

Corollary A.2. *In the context of Theorem A.1, for $n \gg 0$ the data of the morphisms f and $\xi|_{\mathrm{Spec}(R[\pi^{1/n}])}$ with isomorphism ϕ extends canonically to a morphism $\overline{\mathrm{ST}}^{1,1}_{R[\pi^{1/n}]} \to \mathcal{X}$.*

Proof. Compose the uniquely defined map $\overline{\mathrm{ST}}_R^{n,1} \to \mathcal{X}$ of Theorem A.1 with the canonical map $\overline{\mathrm{ST}}^{1,1}_{R[\pi^{1/n}]} \to \mathcal{X}$ induced by the map of graded algebras $R[s,t]/(st^n - \pi) \to R[\pi^{1/n}][s^{1/n}, t]/(s^{1/n}t - \pi)$, where $s^{1/n}$ has weight 1. □

In order to establish Theorem A.1, we will need to recall the following fact concerning pushouts.

Lemma A.3. *If* $\operatorname{Spec}(A) \to \operatorname{Spec}(B)$ *is a closed immersion and* $\operatorname{Spec}(A) \to \operatorname{Spec}(C)$ *is a morphism, then*

$$\begin{array}{ccc} \operatorname{Spec}(A) & \longrightarrow & \operatorname{Spec}(C) \\ \downarrow & & \downarrow \\ \operatorname{Spec}(B) & \longrightarrow & \operatorname{Spec}(B \times_A C) \end{array}$$

is a pushout diagram in the category of algebraic stacks with affine diagonal.

Proof. Ferrand established that the diagram is a pushout in the category of ringed spaces [Fer03, Thm. 5.1]. Temkin and Tyomkin establish that it is a pushout in the category of algebraic spaces [TT16, Thm. 4.2.4]. The lemma follows by applying [TT16, Thm 4.4.1] or [Hal17, Lem. A.4] to the pullback of the diagram under an affine presentation of an algebraic stack $\mathcal{X}$ with affine diagonal. □

Lemma A.4. *Let* $C = R[t, \pi/t, \pi/t^2, \ldots] \subset R[t]_t$*. The commutative diagram*

$$\begin{array}{ccc} \operatorname{Spec}(\kappa) & \longrightarrow & \Theta_\kappa \\ \downarrow & & \downarrow \\ \operatorname{Spec}(R) & \longrightarrow & [\operatorname{Spec}(C)/\mathbb{G}_m] \end{array} \tag{A.1}$$

is cartesian and a pushout diagram in the category of algebraic stacks with affine diagonal.

Proof. Lemma A.3 implies that both diagrams

$$\begin{array}{ccc} \operatorname{Spec}(\kappa[t]_t) & \longrightarrow & \operatorname{Spec}(\kappa[t]) \\ \downarrow & & \downarrow \\ \operatorname{Spec}(R[t]_t) & \longrightarrow & \operatorname{Spec}(C) \end{array} \qquad \begin{array}{ccc} \mathbb{G}_m \times \operatorname{Spec}(\kappa[t]_t) & \longrightarrow & \mathbb{G}_m \times \operatorname{Spec}(\kappa[t]) \\ \downarrow & & \downarrow \\ \mathbb{G}_m \times \operatorname{Spec}(R[t]_t) & \longrightarrow & \mathbb{G}_m \times \operatorname{Spec}(C) \end{array}$$

are pushout diagrams in the category of algebraic stacks with affine diagonal. Since (A.1) is the $\mathbb{G}_m$-quotient of the left diagram above, the statement follows from descent. □

Proof of Theorem A.1. By Lemma A.4, the triple (f, ξ, ϕ) glues to a morphism $[\operatorname{Spec}(C)/\mathbb{G}_m] \to \mathcal{X}$ unique up to unique isomorphism. Write C as a union $C = \bigcup C_n$, where $C_n := R[t, \pi/t^n] \subset R[t]_t$. Note that $C_n \cong R[s,t]/(st^n - \pi)$ so in particular $[\operatorname{Spec}(C_n)/\mathbb{G}_m] \cong \overline{\mathrm{ST}}_R^{n,1}$ and that $\operatorname{Spec}(C) \to \operatorname{Spec}(C_n)$ is $\mathbb{G}_m$-equivariant. As $\mathcal{X} \to S$ is locally of finite presentation, for $n \gg 0$ the morphism $[\operatorname{Spec}(C)/\mathbb{G}_m] \to \mathcal{X}$ factors uniquely as $[\operatorname{Spec}(C)/\mathbb{G}_m] \to [\operatorname{Spec}(C_n)/\mathbb{G}_m] \to \mathcal{X}$. □

To setup the second gluing result, for $n > 0$ consider the subalgebra $R[t/\pi^n] \subset K[t]$ and the quotient stack

$$\Theta_{R,n} = [\operatorname{Spec}(R[t/\pi^n])/\mathbb{G}_m]$$

where t has weight -1. We have a closed immersion $B_R\mathbb{G}_m \hookrightarrow \Theta_{R,n}$ defined by $t/\pi^n = 0$ and an open immersion $\Theta_K \hookrightarrow \Theta_{R,n}$ defined by $\pi \neq 0$. Observe that any morphism $\Theta_{R,n} \to \mathcal{X}$ restricts to morphisms $g \colon B_R\mathbb{G}_m \to \mathcal{X}$ and $\lambda \colon \Theta_K \to \mathcal{X}$ along with an isomorphism $\phi \colon g|_{B_K\mathbb{G}_m} \simeq \lambda|_{B_K\mathbb{G}_m}$.

Theorem A.5. *Let $\mathcal{X}$ be an algebraic stack with affine diagonal and locally of finite presentation over an algebraic space S. Let R be a DVR with fraction field K and consider morphisms $g\colon B_R\mathbb{G}_m \to \mathcal{X}$ and $\lambda\colon \Theta_K \to \mathcal{X}$ over S together with isomorphism $\phi\colon g|_{B_K\mathbb{G}_m} \simeq \lambda|_{B_K\mathbb{G}_m}$. For all $n \gg 0$, there is a morphism $\Theta_{R,n} \to \mathcal{X}$ unique up to unique isomorphism extending the triple (g, λ, ϕ).*

Remark A.6. Observe that $\Theta_{R,n} \cong \Theta_R$ and that the above theorem states that any triple (g, λ, ϕ) extends uniquely to a morphism $\Theta_R \to \mathcal{X}$ after precomposing $\lambda\colon \Theta_K \to \mathcal{X}$ with the isomorphism $\Theta_K \to \Theta_K$, defined by $t \mapsto \pi^n t$, for $n \gg 0$.

We will prove this theorem by using the following pushout result.

Lemma A.7. *Let $D = R[t, t/\pi, t/\pi^2, \ldots] \subset K[t]$. The commutative diagram*

$$\begin{array}{ccc} B_K\mathbb{G}_m & \longrightarrow & B_R\mathbb{G}_m \\ \downarrow & & \downarrow \\ \Theta_K & \longrightarrow & [\operatorname{Spec}(D)/\mathbb{G}_m] \end{array} \tag{A.2}$$

is cartesian and a pushout diagram in the category of algebraic stacks with affine diagonal.

Proof. The proof is identical to the proof of Lemma A.4 using that $D = R[t]_t \times_{K[t]_t} K[t]$. □

Proof of Theorem A.5. By Lemma A.4, the triple (g, λ, ϕ) glues to a morphism $[\operatorname{Spec}(D)/\mathbb{G}_m] \to \mathcal{X}$ unique up to unique isomorphism. Writing $D = \bigcup_n R[t/\pi^n]$ and using that $\mathcal{X} \to S$ is locally of finitely presention, we have that for $n \gg 0$ the map $[\operatorname{Spec}(D)/\mathbb{G}_m] \to \mathcal{X}$ factors uniquely through $[\operatorname{Spec}(R[t/\pi^n])/\mathbb{G}_m]$ to yield a map $[\operatorname{Spec}(R[t/\pi^n])/\mathbb{G}_m] \to \mathcal{X}$. □

A.2. **Semistable reduction in GIT.**

Theorem A.8. *Let $\mathcal{X}$ be a noetherian algebraic stack with affine diagonal. Assume that either (1) there is a good moduli space $\pi\colon \mathcal{X} \to X$ or (2) $\mathcal{X} \cong [\operatorname{Spec}(A)/\operatorname{GL}_n]$ and that the adequate moduli space $\pi\colon \mathcal{X} \to X = \operatorname{Spec}(A^{\operatorname{GL}_N})$ is of finite type. Given a DVR R with fraction field K and a commutative diagram*

$$\begin{array}{ccc} \operatorname{Spec}(K) & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow{\scriptstyle \pi} \\ \operatorname{Spec}(R) & \longrightarrow & X \end{array}$$

there exists an extension of DVRs $R \to R'$ with $K \to K' := \operatorname{Frac}(R')$ finite together with a morphism $h\colon \operatorname{Spec}(R') \to \mathcal{X}$ fitting into a commutative diagram

$$\begin{array}{ccccc} \operatorname{Spec}(K') & \longrightarrow & \operatorname{Spec}(K) & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow & \overset{h}{\nearrow} & \downarrow{\scriptstyle \pi} \\ \operatorname{Spec}(R') & \longrightarrow & \operatorname{Spec}(R) & \longrightarrow & X \end{array}$$

such that $h(0) \in |\mathcal{X} \times_X \operatorname{Spec}(\kappa')|$ is a closed point, where κ' is the residue field of R'.

Remark A.9. If R is universally Japanese (e.g. excellent), then it can be arranged that $R \to R'$ is finite.

Remark A.10. It follows from the valuative criterion for universally closedness ([LMB, Thm. 7.3]) that there exists a lift $\operatorname{Spec}(R') \to \mathcal{X}$ with $K \to K'$ a *finitely generated* field extension. Even in the case that $\mathcal{X} = [\operatorname{Spec}(A)/G] \to \operatorname{Spec}(A^G) = X$ with G linearly reductive and A finitely generated over a field, the above result does not seem to appear in the literature.

Proof of Theorem A.8. Base changing by $\operatorname{Spec}(R) \to X$, we reduce to the case that $X = \operatorname{Spec}(R)$. We are given a K-point $x_K \in \mathcal{X}(K)$ and after possibly a finite extension of K (and a corresponding extension of R), the unique closed point in $\pi^{-1}(\pi(x_K))$ is represented by a K-point x'_K. The closure $\overline{\{x'_K\}}$ is flat over $\operatorname{Spec}(R)$ and it follows from [EGA, IV.17.16.2] that after a finite extension of K, the morphism $\overline{\{x'_K\}} \to \operatorname{Spec}(R)$ has a section z_R. Note that K-points z_K and x'_K are isomorphic. By the Hilbert–Mumford criterion (Lemma 3.23), the specialization $x_K \rightsquigarrow z_K$ can be realized by a morphism $\lambda\colon \Theta_K \to \mathcal{X}$ after possibly a further finite extension of K.

Restricting λ to 0 yields a map $B_K\mathbb{G}_m \to \mathcal{X}$. Since $\Psi\colon \underline{\operatorname{Map}}_R(B_R\mathbb{G}_m, \mathcal{X}) \to \mathcal{X}$, induced by precomposing with $\operatorname{Spec}(R) \to B_R\mathbb{G}_m$, satisfies the valuative criteria for properness, $B_K\mathbb{G}_m \to \mathcal{X}$ extends to a map $g\colon B_R\mathbb{G}_m \to \mathcal{X}$ such that z_R is isomorphic to $\Psi(g)$. Since $\mathcal{X} \to \operatorname{Spec}(R)$ is finitely presented ([AHR15, Thm. A.1]), Theorem A.5 implies that the maps $\lambda\colon \Theta_K \to \mathcal{X}$ and $g\colon B_R\mathbb{G}_m \to \mathcal{X}$ glue to a map $[\operatorname{Spec}(R[t/\pi^n])/\mathbb{G}_m] \to \mathcal{X}$. Restricting this map to $t/\pi^n - \pi$ yields a lift $h\colon \operatorname{Spec}(R) \to \mathcal{X}$ of x_K.

Finally, if $h(0) \in |\mathcal{X}_\kappa|$ is not closed, then after possibly a finite extension of the residue field κ, we may use the Hilbert–Mumford criterion to construct a map $\eta\colon \Theta_\kappa \to \mathcal{X}_\kappa$ which realizes the specialization of $h(0)$ to a closed point. Corollary A.2 implies that after a finite extension of R, h and η extends to a map $\overline{\mathrm{ST}}_R \to \mathcal{X}$. Restricting this map to $s = t = \sqrt{\pi}$ yields the desired extension $\operatorname{Spec}(R[\sqrt{\pi}]) \to \mathcal{X}$. □

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