# Evolution of Stacks and Moduli

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*Illustrations by Elliot Kienzle*

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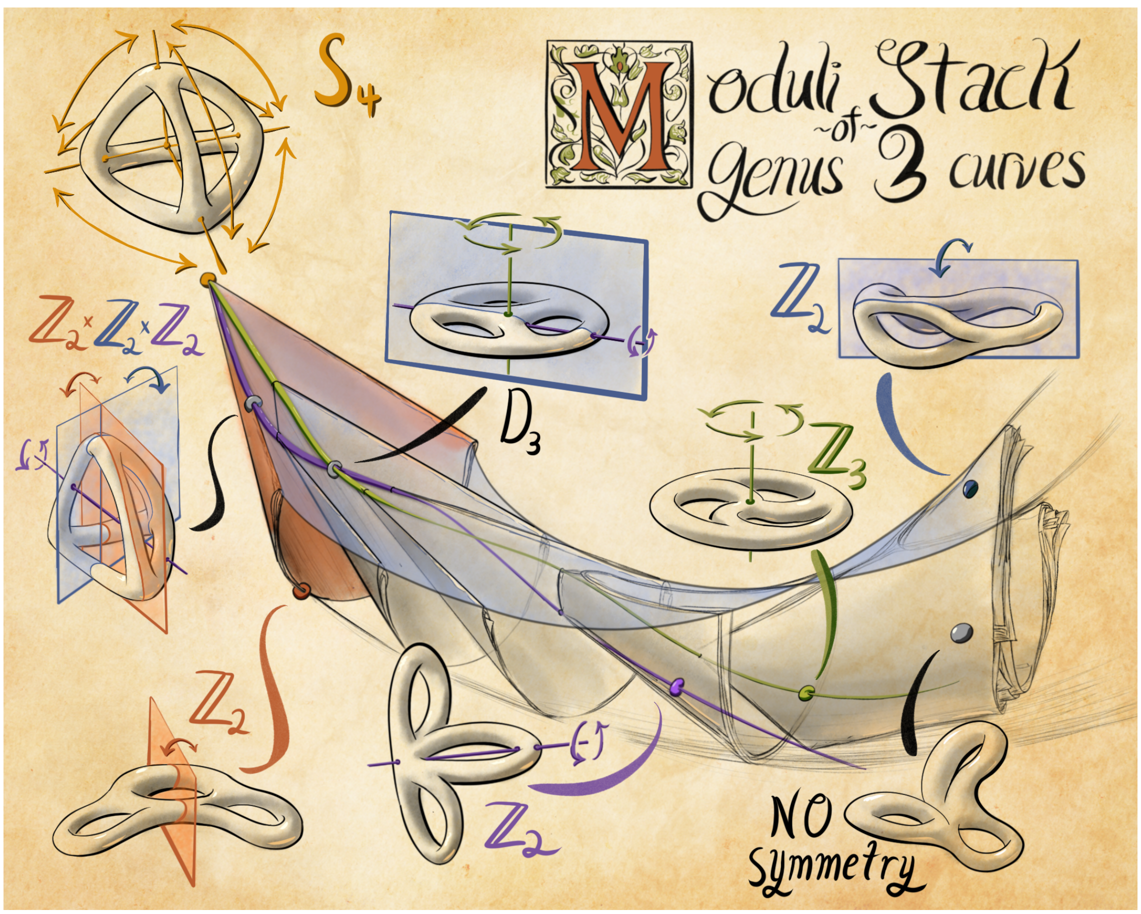

In the vast realm of algebraic varieties, moduli spaces stand out as some of the most enchanting varieties, capturing the imagination of algebraic geometers with their profound elegance and deep connections to other branches of mathematics. By a *moduli space*, we mean a geometric space whose points are in 'natural' bijection (more on what we mean by 'natural' in a moment) with isomorphism classes of your favorite mathematical objects, for example, Riemann surfaces or vector bundles on a fixed space. A moduli space is a solution to the classification problem: it packages all of the data of the geometric objects into a single space, a mathematical catalogue where any object can be located by selecting the corresponding point.

One example that might already be familiar is projective space $\mathbb{CP}^n$: a point in $\mathbb{CP}^n$ naturally corresponds to a complex line in $\mathbb{C}^{n+1}$ through the origin. Remarkably, many other types of objects in algebraic geometry can be naturally classified as a moduli space represented by an algebraic variety, which is often even a projective variety. Surprising, right? The existence of a *variety* $M_g$, whose points correspond to smooth curves of genus $g$, is so strikingly beautiful. Is this some sort of divine creation by the mathematical gods, or is there a a deeper reason why $M_g$ exists?

In the study of moduli spaces, many researchers simply take their existence for granted (as they probably should) and move on to explore their geometry. After all, one of the motivating principles is that the geometry of a moduli space reflects properties of the objects they represent. Over the past sixty years, mathematicians have unraveled charming geometric features of moduli spaces and their objects. Picking your favorite moduli problem and studying aspects of its geometry serves as an excellent way to introduce moduli spaces. This could be the basis for a fascinating *Notices* article and, indeed, some such articles have been written, e.g., [Vak03] and [Cha21], but this is not that article. We are singularly focused on the question:

| Why do moduli spaces exist as varieties? |
| --- |

By surveying how solutions to this question have evolved since Riemann's work in the 1850s, we will reveal many of the central ideas in modern moduli theory, and we will do so using the language of *stacks*. Stacks have unfortunately a formidable reputation (even in the algebraic geometry community), and while they are indeed abstract categorical gadgets, we will attempt to convince you that this reputation is undeserved by giving precise definitions and outlining their key properties. While we present this material primarily in the algebraic setting, we also try to highlight parallel constructions in topology and analytic geometry.

## Riemann's moduli problem

> The spirit of Riemann will move future generations as it has moved us.
>
> Lars Ahlfors (1953)

The study of moduli began with Bernhard Riemann: by viewing surfaces as branched covers over the projective line, Riemann determined that the "corresponding class of algebraic equations, depends on $3p-3$ continuous variables,[1] which we shall call the moduli of the class" [Rie57, p. 33]. In this extraordinary paper, Riemann both introduces the concept of 'moduli' and computes that the 'number of moduli' of $M_g$ is $3g-3$.

While Riemann's argument can be made completely rigorous with today's methods (as we try to summarize here), there were foundational issues—today, we would say that Riemann computed the dimension of a 'local deformation space'. Most notably, $M_g$ was not known to exist and it was not clear what type of space $M_g$ was supposed to be. Despite this, Riemann had an instinctive grasp of its geometry—in fact, in the same paper [Rie57], he introduced 'manifolds' to describe its geometry, a notion that was not formally defined until the 1940s.

| **Riemann's moduli problem**: Does $M_g$ exist as a complex analytic space? |
| --- |

Oswald Teichmüller was the first to give a precise formulation of Riemann's moduli problem in terms

[1] Here $p$ is the genus. We will use the symbol 'g'.

of a universal property. In the 1940s, Teichmüller constructed what is now referred to as the *Teichmüller space* $T_g$ parameterizing complex structures on a topological surface of genus $g$, showed that it is homeomorphic to a ball in $\mathbb{C}^{3g-3}$, and realized $M_g = T_g/\Gamma_g$ as the quotient of the action of the mapping class group $\Gamma_g$ of isotopy classes of automorphisms.

## The functorial worldview

A fundamental challenge in constructing a moduli space is formulating precisely what 'natural' means in the bijection between points of the moduli space and isomorphism classes of geometric objects. It is not enough to require only a bijective correspondence as, after all, any two complex manifolds or varieties of positive dimension are bijective (as they each have the cardinality of the continuum). The bijection should preserve a sense of proximity: curves that are not too different (e.g., the coordinates of the equations defining them are close in value) should be close in the moduli space.

Alexander Grothendieck's approach, which is the one we follow here, requires a psychological change of perspective. Instead of thinking of a space $X$ as a set with a topology endowed with additional structure, we think of a space $X$ by its relationship to all other spaces, i.e., by keeping track of the set $\mathrm{Mor}(T, X)$ of morphisms to $X$ from any other space $T$. For instance, one can recover the underlying set of a space $X$ as $\mathrm{Mor}(\mathrm{pt}, X)$.

This approach is justified by the *Yoneda Lemma*: for an object $X$ of a category $\mathcal{C}$, the contravariant functor

$$h_X : \mathcal{C} \to \mathrm{Sets}, \qquad S \mapsto \mathrm{Mor}(S, X)$$

recovers the object $X$ itself. This is formulated precisely as a functorial bijection $\mathrm{Mor}(h_X, G) \xrightarrow{\sim} G(X)$ for every other contravariant functor $G$. This is not a deep statement (the proof is shorter than the statement). Nevertheless, it is a giant leap of abstraction and, in the words of Dan Piponi, "is the hardest trivial thing in mathematics."

## Family matters

Suppose that we have a hypothetical moduli space $\mathcal{M}_g$ whose points are in bijective correspondence with isomorphism classes of smooth curves of genus $g$. If $T$ is a path, e.g., a smooth curve, in $\mathcal{M}_g$, then the map $T \to \mathcal{M}_g$ takes every point $t \in T$ to a curve $[C_t] \in \mathcal{M}_g$, where we expect the curves $C_t$ to vary continuously, even algebraically, for $t \in T$. Even better, one would hope that we could package these curves into a *family of curves*, i.e., a smooth proper morphism $\mathcal{C} \to T$ of varieties where the fiber of $t$ is $C_t$. In this dream world, there would be a universal family $U_g \to \mathcal{M}_g$, where the fiber over $[C] \in \mathcal{M}_g$ is $C$ and the family $\mathcal{C} \to T$ is the pullback of the universal family under $T \to \mathcal{M}_g$.

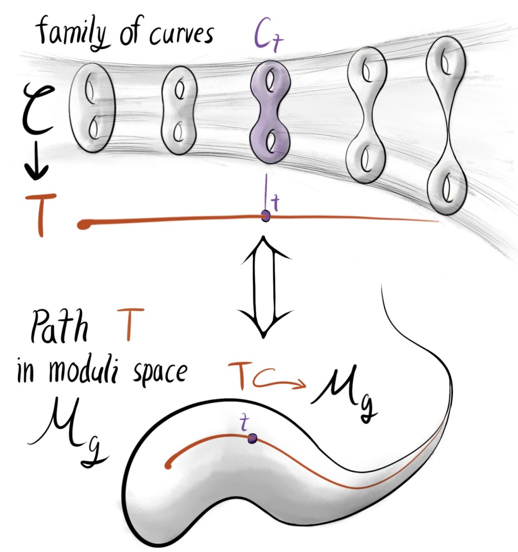


Figure 1: In addition to the correspondence between families of curves and paths in the moduli space, this image also demonstrates another property: the moduli space $\mathcal{M}_g$ of smooth curves is not compact! As the path approaches the boundary of the moduli space, the curve degenerates to a singular nodal curve. We will see later that the moduli space $\overline{\mathcal{M}}_g$ of *stable curves* is compact.

It may seem that we just made life more difficult as we must specify an immense amount of categorical data to define a space. In practice, however, it is usually (but not always) straightforward to define well-behaved notions of families. For example, projective space $\mathbb{CP}^n$ parameterizes *quotients*[2] $\mathbb{C}^{n+1} \twoheadrightarrow \mathbb{C}$, and a map $S \to \mathbb{CP}^n$ is classified by a quotient line bundle $\mathcal{O}_S^{\oplus n+1} \twoheadrightarrow L$ of the trivial rank $n+1$ vector bundle. For $M_g$, we consider the functor

$$
\begin{aligned}
F_{M_g} &\colon \mathrm{Sch}/\mathbb{C} \to \mathrm{Sets} \\
&S \mapsto \left\{ \begin{array}{l} \text{families } \mathcal{C} \to S \text{ of smooth} \\ \text{curves of genus } g \end{array} \right\} / \sim
\end{aligned}
$$

defined on the category of all $\mathbb{C}$-schemes.[3] In the topological (resp., analytic) setting, one instead defines the functor on the category of topological spaces (resp., analytic spaces) using families of Riemann surfaces.

A compelling feature of the functorial approach is that if a moduli functor $F \colon \mathcal{C} \to \mathrm{Sets}$ is represented by a scheme $M$, i.e., there is a functorial bijection $F(S) \cong \mathrm{Mor}(S, M)$, then there is a *universal family*. This is an object of the moduli functor $\xi_{\mathrm{univ}} \in F(M)$ over $M$ (corresponding to the identity map under $F(M) \cong \mathrm{Mor}(M, M)$) from which every other object is obtained: if $\eta \in F(S)$ is an object over $S$, there is a unique morphism $S \to M$ such that $\xi_{\mathrm{univ}}$ pulls back to $\eta$.

Unfortunately, many moduli functors of interest, including $F_{M_g}$, are unfortunately not representable. Why you ask? Automorphisms! A non-trivial automorphism $\alpha$ of a smooth complex curve $C$ can be used to glue a trivial family, e.g., $C \times \mathbb{R} \to \mathbb{R}$, to a non-trivial family, e.g., $\mathcal{C} \to S^1$, where every fiber is isomorphic to $C$.

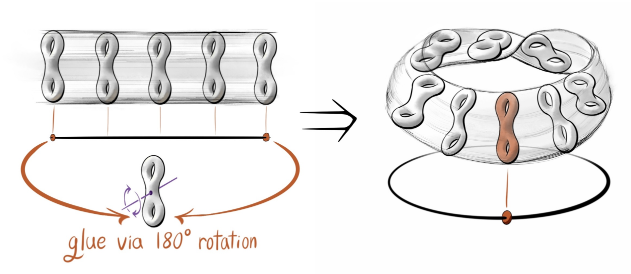


Figure 2: A non-trivial family obtained by gluing the trivial family via an automorphism.

If the functor $F_{M_g}$ were representable by a space $M_g$, this family would correspond to the *constant map* to $M_g$ sending every point to $[C] \in M_g$, but this is a contradiction as the constant map should correspond to the trivial family.

## The Grothendieck revolution

> One can hope that we shall be able one day to eliminate analysis completely from the theory of Teichmüller space, which should be purely geometric.
>
> Alexander Grothendieck (1960)

Grothendieck, in a series of ten lectures at Cartan's seminar [Gro60], developed a general theory of analytic moduli spaces in the language of categories and functors, reformulated Teichmüller theory in this setting, and showed that the Teichmüller space $T_g$ represents a functor parameterizing families of Riemann surfaces. He then applied his functorial approach to algebraic geometry in his FGA series [Gro59]: he introduced the Hilbert, Quot, and Picard functors, showed that they were representable by projective schemes, developed descent theory, and introduced the notions of prestacks and stacks. At the same time, Grothendieck also redeveloped the entire foundations of algebraic geometry in the language of schemes in nearly 9000 pages in his EGA and SGA series. His profound contributions to algebraic geometry and more broadly mathematics provided the tools with which we study moduli spaces today.

Grothendieck was fascinated with $M_g$. While he did not publish much, he generously shared his ideas,

[2] While one usually thinks of points in $\mathbb{CP}^n$ as lines $\mathbb{C} \subseteq \mathbb{C}^{n+1}$, it is sometimes convenient to to parameterize the dual notion of one-dimensional quotients.

[3] A *scheme* is a generalization of a variety where functions are allowed to be nilpotent. Moduli theory is, in fact, one subject where the power of scheme theory is especially evident.

as evidenced by his written correspondence with Mumford, Artin, Serre, and others. He recognized that the presence of automorphisms prevented the existence a scheme $M_g$ representing the functor $F_{M_g}$ of families of smooth curves. In an effort to circumvent this issue, he rigidified the moduli problem by parameterizing the data of a curve $C$ together with a *level $n$ structure*, i.e., a symplectic basis of $H_1(C, \mathbb{Z}/n\mathbb{Z})$. While he could show that the rigidified moduli problem was representable by a scheme, he struggled to show that it was quasi-projective. If he could have verified quasi-projectivity, he also would have been able to construct $M_g$ as a quasi-projective variety by taking the quotient of this space by the finite group of symplectic automorphisms, relying on the easy fact that *quasi-projective varieties* are closed under taking quotients by *finite* groups.

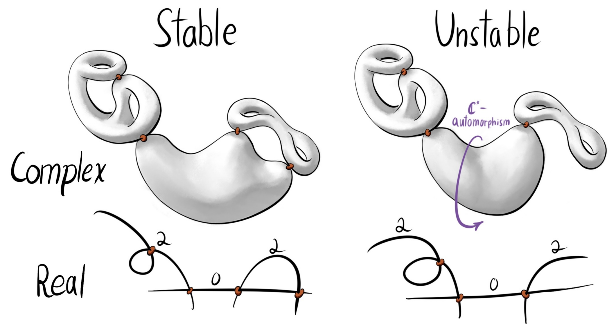


Figure 3: Stable and unstable curves of genera 6 and 5, respectively, illustrated both over the real and complex numbers.

We will now explain a general approach for constructing moduli spaces as quotients, which we will then apply it to construct $\overline{M}_g$ as a projective variety and therefore $M_g$ as a quasi-projective variety.

## Stable curves and compactifying $M_g$

The moduli space $M_g$ is not compact as illustrated by Figure 1. It turns out that the GIT construction of $M_g$ by Mumford and Gieseker (which we will outline shortly) also produces a compactification, namely a projective variety $\overline{M}_g$ containing $M_g$ as an open subset. Mayer and Mumford introduced stable curves in 1964 to give a modular compactification of $M_g$ where points on the boundary $\overline{M}_g \setminus M_g$ correspond to singular stable curves. A connected projective complex curve $C$ of genus $g \geq 2$ with at worst nodal singularities (locally given by $xy = 0$) is called *stable* if its automorphism group is finite, or equivalently if every component isomorphic to $\mathbb{CP}^1$ meets the rest of the curve in at least three nodes.

## Mumford's approach using GIT

> As for $M_g$ there is virtually no doubt that it can be provided with the structure of an algebraic variety.
>
> André Weil (1958)

By integrating Grothendieck's formalism of scheme theory with 19th century invariant theory, David Mumford developed a theory of quotients in algebraic geometry now known as *Geometric Invariant Theory* (or *GIT*), and applied this theory to construct moduli spaces [Mum65]. The central idea is to first construct a moduli space parameterizing additional data, e.g., a smooth curve $C$ *together* with a choice of embedding $C \hookrightarrow \mathbb{CP}^n$, or a vector bundle $F$ on a fixed projective variety $X$ *together* with a basis of the global sections of a fixed twist $F(n)$, and then to quotient out by a reductive group acting transitively on the set of choices.

<u>Outline of the GIT strategy</u>

① Express the moduli problem $\mathcal{M}$ as a quotient $U/G$ of the action of a quasi-projective variety $U$ by a reductive group $G$ and choose a $G$-equivariant compactification $\overline{U} \subseteq \mathbb{CP}^n$, where $G$ acts linearly on $\mathbb{CP}^n$.

② Show that a point $u \in \overline{U}$ is in $U$ if and only if it is GIT semistable, i.e., there exists a non-constant $G$-invariant homogeneous $f \in \Gamma(\overline{U}, \mathcal{O}(d))^G$ with $f(u) \neq 0$.

③ Realize the the moduli space $\mathcal{M} = U/G$ as the projective variety defined by the graded ring $\bigoplus_{d \geq 0} \Gamma(\overline{U}, \mathcal{O}(d))^G$.

Step ② is the hardest. While the Hilbert–Mumford Criterion translates this problem into the enumerative and sometime combinatorial question of verifying that the *Hilbert–Mumford index* $\mu(u, \lambda) \leq 0$ is nonpositive for every one-parameter subgroup $\lambda \colon \mathbb{C}^* \to G$, it can be difficult to verify this criterion.

We now outline the strategy of Mumford [Mum77] and Gieseker [Gie82] to construct $\overline{M}_g$ as a projective variety using GIT. Let $\Omega_C = (T_C)^\vee$ be the *canonical bundle* (also called the *cotangent bundle*) and $\Omega_C^{\otimes k}$ be its tensor powers. For each integer $k \geq 3$, a choice of basis of the space $\Gamma(C, \Omega_C^{\otimes k})$ of $k$-pluricanonical sections defines an embedding into $\mathbb{CP}^N$ with $N + 1 = \mathrm{h}^0(C, \Omega_C^{\otimes k}) = (2k-1)(g-1)$. The moduli of smooth $k$-pluricanonically embedded curves in $\mathbb{CP}^N$ is a quasi-projective variety that can be compactified using either the Hilbert scheme or, as in Mumford's original approach, the Chow scheme. Applying the Hilbert–Mumford Criterion, one needs to show for a curve $C$

$$C \text{ is stable} \iff \mu([C], \lambda) \leq 0 \text{ for all } \lambda,$$

where $[C]$ denotes the choice of a lift of $C$ to either the Hilbert or Chow scheme. Mumford ingeniously discovered a proof of the implication: by reinterpreting the Hilbert–Mumford index in terms of geometric invariants on the complex surface $C \times \mathbb{C}^1$, he verified the criterion for *smooth* curves. With a trick using GIT semistable reduction, this reduces the construction of $\overline{M}_g$ to the ($\Leftarrow$) implication. While ($\Leftarrow$) is the conceptually easier direction requiring one to construct a destabilizing one-parameter subgroup $\lambda$ for each non-stable curve $C$, it is the also most demanding step requiring pages and pages of calculations.

Grothendieck was at first skeptical of this approach. In a letter to Mumford on January 29, 1962, he writes: "I am afraid you will not convince me of the usefulness of Chow coordinates, in fact your example shows again that the wrong method will lead to prove statements under unnatural assumptions (such as normality)." Later the *same day* (perhaps after trying to construct $M_g$ himself), he writes: "I have been too rash in my reply to your last letter... I have no means of attacking the problem, which in fact meets with a few unsolved problems on equivalence relations I still had in store. Therefore I grant you that, for the time being, in your example Chow coordinates do give mathematical information about existence of quotients which is not obtained by other means. I do not expect this situation to hold for long still!" It did, however, take another 30 years until Kollár gave an alternative argument utilizing techniques in birational geometry.

Mumford also used GIT to construct a quasi-projective variety parameterizing *stable* vector bundles on a fixed smooth curve, and C.S. Seshadri exhibited a projective compactification parameterizing *semistable* vector bundles [Ses67]. The GIT machinery has been widely applied to construct other moduli spaces.

## Why stacks?

> The notion of stacks came up in the sixties. But to swallow schemes was already enough for one generation of mathematicians.
>
> Gerd Faltings

As we have seen above, automorphisms obstruct the functor $F_{M_g}$ from being representable. Rather than defining a moduli functor by specifying families of objects and *when* two families are isomorphic, we will define them as a *moduli stack* by specifying families of objects and *how* they are isomorphic. In other words, we will give an assignment

$$\mathrm{Sch}/\mathbb{C} \to \mathrm{Groupoids}, \quad S \mapsto \mathrm{Fam}_S$$

taking a $\mathbb{C}$-scheme $S$ to a *groupoid*[4] $\mathrm{Fam}_S$ of families of objects over $S$. But what exactly do we mean by this? As groupoids form a '2-category', some care is needed to make this precise.

The first definition of an algebraic stack—now referred to as *Deligne–Mumford stacks*—was introduced by Deligne and Mumford in a remarkable paper [DM69]. Unfortunately, Deligne and Mumford did not prove any of their assertions about stacks, and the lack of rigorous foundations contributed to the forbidding reputation of algebraic stacks over the following decades. It was not until the 2000s that algebraic stacks entered into mainstream algebraic geometry. There are now several textbooks covering algebraic stacks—see [LMB00], [Ols16], and [Alp24]. Notably, Johan de Jong's *Stacks Project* [Sta24] has provided an unquestionably solid foundation.

As we will see, the language of stacks offers an intrinsic approach to constructing moduli spaces that Grothendieck may have been happy with.

## Prestacks

> An absence of proof is a challenge; an absence of definition is deadly.
>
> Pierre Deligne

The categorical gadget that we will use is a *prestack*, conventionally called a *category fibered in groupoids*. Instead of trying to define an assignment $S \mapsto \mathrm{Fam}_S$, we will build one massive category $\mathcal{X}$ that encodes all of the groupoids $\mathrm{Fam}_S$ and that lives over another category $\mathcal{S}$, e.g., the category Sch of schemes. Loosely speaking, the objects of $\mathcal{X}$ will be a family $a$ of objects over a scheme $S$, i.e., $a \in \mathrm{Fam}_S$, and a morphism $a \to b$ between a family $a$ over $S$ and a family $b$ over $T$ will be the data of a morphism $f\colon S \to T$ together with an isomorphism $\alpha\colon a \xrightarrow{\sim} f^*b$ of $a$ will be the pullback of $b$. We visualize this data as

$$\begin{tikzcd} \mathcal{X} \arrow[d] & a \arrow[r, "\alpha"] \arrow[d, mapsto] & b \arrow[d, mapsto] \\ \mathcal{S} & S \arrow[r, "f"] & T \end{tikzcd}$$

where the lower case letters $a, b$ are objects in $\mathcal{X}$ and the upper case letters $S, T$ are objects in $\mathcal{S}$, e.g., schemes.

**Definition 1** (Prestacks)**.** A functor $\mathcal{X} \to \mathcal{S}$ is a *prestack over a category* $\mathcal{S}$ if

(1) (pullbacks exist) for every diagram

$$\begin{tikzcd} a \arrow[r, dashed] \arrow[d, mapsto, dashed] & b \arrow[d, mapsto] \\ S \arrow[r] & T \end{tikzcd}$$

of solid arrows, there exists a morphism $a \to b$ over $S \to T$; and

(2) (universal property for pullbacks) for every diagram

$$\begin{tikzcd} a' \arrow[r, dashed] \arrow[rr, bend left] \arrow[d, mapsto] & a \arrow[r] \arrow[d, mapsto] & b \arrow[d, mapsto] \\ S' \arrow[r] & S \arrow[r] & T \end{tikzcd}$$

of solid arrows, there exists a unique arrow $a' \to a$ over $S' \to S$ filling in the diagram.

A contravariant functor $F\colon \mathcal{S} \to \mathrm{Sets}$ (which we often think as a *presheaf*) can be viewed as a prestack: the objects are pairs $(S, a)$ consisting of $S \in \mathcal{S}$ and $a \in F(S)$, and a morphism $(S, a) \to (T, b)$ is a map $f\colon S \to T$ such that $F(f)(b) = a$. Likewise, an object $X \in \mathcal{S}$ can be viewed as the prestack $\mathcal{S}/X$ of objects in $\mathcal{S}$ over $X$, i.e., pairs $(S, S \to X)$ consisting of $S \in \mathcal{S}$ and a morphism $S \to X$ in $\mathcal{S}$. By abuse of notation, we will use $X$ also to denote this prestack.

There is a 'Yoneda 2-Lemma' giving an equivalence between objects of a prestack $\mathcal{X}$ over $T \in \mathcal{S}$ and a morphism $T \to \mathcal{X}$, and it is customary to abuse notation by using the same symbol to refer to both the object and morphism.

A prototypical example of a prestack over Sch is the category $\underline{\mathrm{QCoh}}$ of pairs $(S, F)$ where $S$ is a scheme

[4] A groupoid is a category where all morphisms are isomorphisms.

and $F$ is a quasi-coherent sheaf on $S$. As we will shortly see, this is also a *stack* because quasi-coherent sheaves and their objects glue uniquely. These two gluing properties should be familiar in the case of a Zariski covering $\{U_i\}$ of a scheme $S$: (1) given quasi-coherent sheaves $F$ and $G$ on $S$, morphisms $\phi_i \colon F_i \to G$ satisfying $\phi_i|_{U_i \cap U_j} = \phi_j|_{U_i \cap U_j}$ glue to a unique morphism $\phi \colon F \to G$; and (2) quasi-coherent sheaves $F_i$ on $U_i$, and isomorphisms $\alpha_{ij} \colon F_i|_{U_i \cap U_j} \xrightarrow{\sim} F_j|_{U_i \cap U_j}$ on the intersection $U_i \cap U_j$ that satisfy the cocycle condition $\alpha_{jk} \circ \alpha_{ij} = \alpha_{ik}$ on $U_i \cap U_j \cap U_k$, there exists a unique quasi-coherent sheaf $F$ on $S$ and isomorphisms $\phi_i \colon F|_{U_i} \xrightarrow{\sim} F_i$. These two gluing properties translate into $\underline{\mathrm{QCoh}}$ being a *stack in the big Zariski topology on* Sch.

## Detour: the étale topology

Grothendieck showed that quasi-coherent sheaves can not only be glued in the Zariski topology, but also in the étale topology. An *étale morphism* between complex varieties is a covering space, except that we do not require that every fiber has the same cardinality.

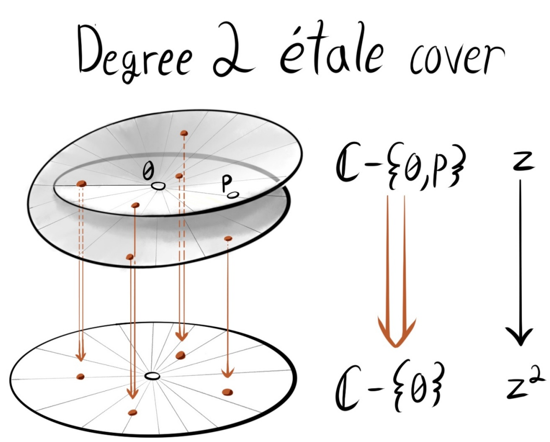


Figure 4: The map $\mathbb{C} \setminus \{0, p\} \to \mathbb{C} \setminus 0$, defined by $z \mapsto z^2$, is étale for any $p \in \mathbb{C}$. It is finite and étale, i.e., a covering space, for $p = 0$.

The Zariski topology unfortunately is not sufficient for many purposes. As a Zariski open subset is, by definition, the complement of a closed loci defined by finitely many polynomials, they are simply too big. The broader class of étale morphisms are more flexible for many constructions and play a similar role to analytic opens. For instance, any Zariski open neighborhood of a node on a irreducible curve remains irreducible, but there are reducible étale neighborhoods separating the branches.

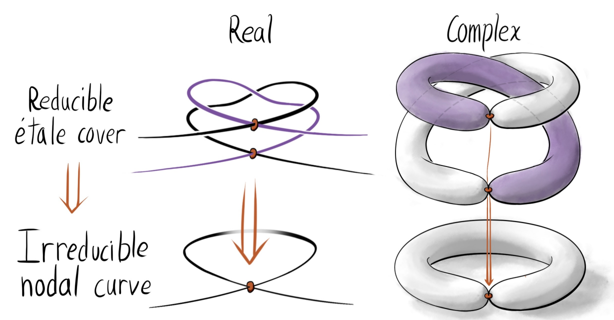


Figure 5: In the étale topology, nodal singularities can be separated into two branches.

In the arithmetic setting, conjugation on $\mathbb{C}$ induces a $\mathbb{Z}/2$-bundle $\operatorname{Spec} \mathbb{C} \to \operatorname{Spec} \mathbb{R}$. As $\operatorname{Spec} \mathbb{R}$ is just one point, this $\mathbb{Z}/2$-bundle is not trivializable in the Zariski topology, but it does become trivial after the *étale* cover $\operatorname{Spec} \mathbb{C} \to \operatorname{Spec} \mathbb{R}$.

## Sites

In order to formulate the descent condition of a stack, we must first make sense of what we mean by a topology where an 'open subset' is an étale morphism. That is, we need to modify the concept of a topological space replacing open subsets with a more abstract notion of *covering.* We use that the notation that $X_{ij} := X_i \times_X X_j$ for maps $X_i \to X$ and $X_j \to X$. When each $X_i \hookrightarrow X$ is the inclusion of an open subset, $X_{ij} = X_i \cap X_j$ is simply the intersection.

**Definition 2** (Sites)**.** A *site* is a category $\mathcal{S}$ together with the following data: for each object $X \in \mathcal{S}$, there is a set $\mathrm{Cov}(X)$ consisting of *coverings* of $X$, i.e., collections of morphisms $\{X_i \to X\}_{i \in I}$ in $\mathcal{S}$. We require that:

(1) (identity) If $X' \to X$ is an isomorphism, then $(X' \to X) \in \mathrm{Cov}(X)$.

(2) (restriction) If $\{X_i \to X\}_{i \in I} \in \mathrm{Cov}(X)$ and $Y \to X$ is a morphism, then the fiber products $X_i \times_X Y$ exist in $\mathcal{S}$ and the collection $\{X_i \times_X Y \to Y\}_{i \in I} \in \mathrm{Cov}(Y)$.

(3) (composition) If $\{X_i \to X\}_{i \in I} \in \mathrm{Cov}(X)$ and $\{X_{ij} \to X_i\}_{j \in J_i} \in \mathrm{Cov}(X_i)$ for each $i \in I$, then $\{X_{ij} \to X_i \to X\}_{i \in I, j \in J_i} \in \mathrm{Cov}(X)$.

The category Top of topological spaces, where a cover $\{X_i \to X\}$ is a usual open covering $\{X_i\}$, is a site. For us, the most important example is the *big étale site* $\mathrm{Sch}_{\text{ét}}$ on the category of schemes, where a covering $\{X_i \to X\}$ of a scheme $X$ is the data of étale morphisms $X_i \to X$ such that $\amalg_i X_i \to X$ is surjective.

A *presheaf* is a contravariant functor $F \colon \mathcal{S} \to \mathrm{Sets}$, while a *sheaf* is a presheaf such that for every covering $\{S_i \to S\}$ and objects $a_i \in F(S_i)$ such that $a_i|_{S_{ij}} = a_j|_{S_{ij}}$, there exists a unique object $a \in F(S)$ such that $a_i = a|_{S_i}$.

## Stacks

A stack is to a prestack as a sheaf is to a presheaf.

Grothendieck introduced stacks in [Gro59] in order to formulate properties of descent, and showed that the prestack $\underline{\mathrm{QCoh}}$ of quasi-coherent sheaves is a *stack over* $\mathrm{Sch}_{\text{ét}}$.

**Definition 3** (Stacks)**.** A prestack $\mathcal{X}$ over a site $\mathcal{S}$ is a *stack* if for every covering $\{S_i \to S\}$:

(1) (morphisms glue) For objects $a, b \in \mathcal{X}$ over $S \in \mathcal{S}$ and morphisms $\phi_i \colon a|_{S_i} \to b$ such that $\phi_i|_{S_{ij}} = \phi_j|_{S_{ij}}$

$$\begin{tikzcd} & a|_{S_i} \arrow[dr] \arrow[drr, bend left, "\phi_i"] & & \\ a|_{S_{ij}} \arrow[ur] \arrow[dr] & & a \arrow[r, dashed, "\phi"] & b \\ & a|_{S_j} \arrow[ur] \arrow[urr, bend right, "\phi_j"'] & & \end{tikzcd} \quad \text{over} \quad \begin{tikzcd} & S_i \arrow[dr] & \\ S_{ij} \arrow[ur] \arrow[dr] & & S, \\ & S_j \arrow[ur] & \end{tikzcd}$$

there exists a unique morphism $\phi \colon a \to b$ over $\mathrm{id}_S$ with $\phi|_{S_i} = \phi_i$.

(2) (objects glue) For objects $a_i$ over $S_i$ and isomorphisms $\alpha_{ij} \colon a_i|_{S_{ij}} \to a_j|_{S_{ij}}$

$$\begin{tikzcd} & a_i \arrow[dr] & \\ a_i|_{S_{ij}} \arrow[ur] \arrow[d, "\alpha_{ij}"] & & a \\ a_j|_{S_{ij}} \arrow[r] & a_j \arrow[ur] & \end{tikzcd} \quad \text{over} \quad \begin{tikzcd} & S_i \arrow[dr] & \\ S_{ij} \arrow[ur] \arrow[dr] & & S \\ & S_j \arrow[ur] & \end{tikzcd}$$

satisfying the cocycle condition $\alpha_{jk}|_{S_{ijk}} \circ \alpha_{ij}|_{S_{ijk}} = \alpha_{ik}|_{S_{ijk}}$ on $S_{ijk}$, there exists an object $a$ over $S$ and isomorphisms $\phi_i \colon a|_{S_i} \to a_i$ over $\mathrm{id}_{S_i}$ such that $\phi_i|_{S_{ij}} = \phi_j|_{S_{ij}} \circ \alpha_{ij}$ on $S_{ij}$.

## Algebraic spaces and stacks

Recall that a smooth manifold, by definition, has analytic local charts by $\mathbb{R}^n$, while a scheme has Zariski local charts by affine schemes. Algebraic stacks are defined in essentially the same way: by requiring a stack $\mathcal{X}$ over $\mathrm{Sch}_{\text{ét}}$ to have local charts in the étale topology (resp., smooth topology) by affine schemes, we obtain *Deligne–Mumford stacks* (resp., algebraic stacks).

To formulate precisely what a chart $U \to \mathcal{X}$ is, we need some terminology. A morphism $\mathcal{X} \to \mathcal{Y}$ of prestacks (or presheaves) is Sch is *representable by schemes* if for every morphism $T \to \mathcal{Y}$ from a scheme, the base change $\mathcal{X} \times_{\mathcal{Y}} T$ is a scheme. Similarly, a morphism is *representable* if each base change is an algebraic space (as defined below). Most properties of morphisms (e.g., surjective, étale, and smooth) are étale local, even smooth local on the base, and therefore extends to representable morphisms.

**Definition 4** (Algebraic spaces and stacks)**.**

(1) An *algebraic space* is a sheaf $X$ on $\mathrm{Sch}_{\text{ét}}$ such that there exists a scheme $U$ and a surjective étale morphism $U \to X$ representable by schemes.

(2) A *Deligne–Mumford stack* or simply *DM stack* is a stack $\mathcal{X}$ over $\mathrm{Sch}_{\text{ét}}$ such that there exists a scheme $U$ and a surjective, étale, and representable morphism $U \to \mathcal{X}$.

(3) An *algebraic stack* is a stack $\mathcal{X}$ over $\mathrm{Sch}_{\text{ét}}$ such that there exists a scheme $U$ and a surjective, smooth, and representable morphism $U \to \mathcal{X}$.

While not apparent from the definition, the diagonal $\mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is representable. It encodes the data of the automorphism groups, otherwise known as the stabilizers: given a point $x \in \mathcal{X}(\mathbb{C})$, the *stabilizer* $G_x$ is defined as the fiber product of the diagonal with $(x, x)\colon \operatorname{Spec}\mathbb{C} \to \mathcal{X} \times \mathcal{X}$. The group $G_x(\mathbb{C})$ of $\mathbb{C}$-points of $G_x$ is identified with automorphism $x \xrightarrow{\sim} x$ over the identity. The diagonal encodes the 'stackiness' of $\mathcal{X}$: $\mathcal{X}$ is an algebraic space if and only if all stabilizers are trivial or equivalently if the diagonal is a monomorphism, and $\mathcal{X}$ is DM if and only if all stabilizers are finite[5] groups or equivalently if the diagonal is unramified.

*Remark* 5 (Parallel theories). Modifying the underlying site and the properties of the covers $U \to \mathcal{X}$ leads to parallel theories of stacks. Using the site of topological spaces, smooth manifolds, or complex manifolds, and requiring that $U \to \mathcal{X}$ be a local homeomorphism (instead of étale), we obtain *topological*, *differential*, or *analytic Deligne–Mumford stacks*. Imposing instead that $U \to \mathcal{X}$ is a local Serre fibration (instead of smooth), submersion, or analytic submersion, we obtain general *topological*, *differential*, or *analytic stacks*.

An *orbifold* (resp., *orbispace*) is a differential (resp. topological) Deligne–Mumford stack $\mathcal{X}$ where the generic stabilizer is trivial, i.e., there is an open dense substack that is a manifold (resp., topological space).

*Remark* 6 (Groupoids). Groupoids provide an alternative framework to the language of stacks. If $U \to \mathcal{X}$ is a smooth presentation of an algebraic stack, then $s, t\colon R := U \times_{\mathcal{X}} U \rightrightarrows U$ is a *smooth groupoid* of schemes, where one views $R$ as a *scheme of relations on* $U$: a point $r \in R$ specifies a relation between the points $s(r), t(r) \in U$. There is a composition map $R \times_{s,U,t} R \to R$, inverse map $\iota\colon R \to R$, and identity map $e\colon U \to R$ subject to natural commutativity relations. A prototypical example is the smooth groupoid $G \times U \rightrightarrows U$ arising from the action of an algebraic group $G$ on a scheme $U$.

Likewise, there are *topological*, *differential* (or *Lie*), and *analytic groupoids*. In each setting, the quotient $\mathcal{X} := [U/R]$ of a groupoid $R \rightrightarrows U$ exists as a stack, and there is an equivalence between groupoids $R \rightrightarrows U$ and the data of a stack $\mathcal{X}$ together with a presentation $U \to \mathcal{X}$.

[5] This is true for quasi-separated stacks. Without this mild separation condition, infinite discrete groups such as $\mathbb{Z}$ may appear as stabilizer groups.

## Quotient stacks

An important example is a quotient stack $[X/G]$, arising from an action of an algebraic group $G$ on a scheme $X$ over $\mathbb{C}$, provide important examples of algebraic stacks. Objects of the quotient stack $[X/G]$ over a $\mathbb{C}$-scheme $S$ are diagrams $(S \leftarrow P \to X)$ where $P \to S$ is a principal $G$-bundle and $P \to X$ is a $G$-equivariant map.

Miraculously, the quotient stack $[X/G]$ behaves as if the action were free: the projection $X \to [X/G]$ is a principal $G$-bundle!

In particular, $X \to [X/G]$ is a surjective, smooth, and representable morphism, which verifies that $[X/G]$ is an algebraic stack.

Similar to how toric varieties provide concrete examples of schemes, quotient stacks are helpful for building geometric intuition for general algebraic stacks and, at the same time, serve as a fertile testing ground for conjectural results. On the other hand, it turns out that many algebraic stacks are quotient stacks or are at least locally quotient stacks. Thus many properties that hold for quotient stacks can also be established for many algebraic stacks. It is a delicate question whether a given algebraic stack is a global quotient stack, related to both arithmetic and geometric properties.

The geometry of $[X/G]$ could not be simpler: it is the $G$-equivariant geometry of $X$:

| Geometry of $[X/G]$ | $G$-equivariant geometry of $X$ |
|---|---|
| point $x \in [X/G]$ | orbit $Gx$ of $x \in X$ |
| aut. group $\mathrm{Aut}(x)$ | stabilizer $G_x$ |
| function $f \in \Gamma([X/G])$ | $G$-equiv. function $f \in \Gamma(X)^G$ |
| map $[X/G] \to Y$ | $G$-equiv. map $X \to Y$ |
| line bundle | $G$-equiv. line bundle |
| tangent space $T_{[X/G],x}$ | normal space $T_{X,x}/T_{Gx,x}$ |

## $\overline{\mathcal{M}}_g$ and $\mathcal{B}un_{r,d}(X)$ are algebraic

As a prestack over the big étale site $(\mathrm{Sch}/\mathbb{C})_{\text{ét}}$ of complex schemes, $\overline{\mathcal{M}}_g$ is the category of *families of stable curves of genus* $g$, i.e., flat[6] and proper morphisms $\mathcal{C} \to S$ of schemes such that every geometric fiber is a stable curve of genus $g$. A morphism $(\mathcal{C}' \to S') \to (\mathcal{C} \to S)$ is a cartesian diagram

$$\begin{array}{ccc} \mathcal{C}' & \longrightarrow & \mathcal{C} \\ \downarrow & \square & \downarrow \\ S' & \longrightarrow & S. \end{array}$$

For a stable curve $[C] \in \overline{\mathcal{M}}_g$ over $\operatorname{Spec} \mathbb{C}$, the group of isomorphisms $[C] \xrightarrow{\sim} [C]$ in the category $\overline{\mathcal{M}}_g$ over the identity $\mathrm{id}\colon \operatorname{Spec} \mathbb{C} \to \operatorname{Spec} \mathbb{C}$ is precisely the automorphism group $\operatorname{Aut}(C)$, as expected.

The key geometric fact needed to show both that $\overline{\mathcal{M}}_g$ is a stack and that it is algebraic is: every stable curve of genus $g$ is embedded into $\mathbb{CP}^{5g-6}$ by choosing a basis of $\Gamma(C, \omega_C^{\otimes 3})$, where $\omega_C$ is the dualizing sheaf (which is the cotangent bundle $\Omega_C$ when $C$ is smooth). Importantly, this also holds in families. While it is not hard to see that morphisms in $\overline{\mathcal{M}}_g$ glue uniquely (as an easy consequence of descent theory), this geometric fact allows one to glue objects by descending the closed subschemes of $\mathbb{CP}^{5g-6}$. The embedding arising from a choice of basis is unique up to the action of projective automorphisms $\operatorname{Aut}(\mathbb{CP}^{5g-6}) = \mathrm{PGL}_{5g-5}$. By considering the Hilbert scheme $H := \operatorname{Hilb}(\mathbb{CP}^{5g-6})$ parameterizing all (possibly very singular) subschemes of $\mathbb{CP}^{5g-6}$, one can show that there is locally closed[7] subscheme $H' \subseteq H$ parameterizing precisely stable curves of genus $g$ embedded by $\omega_C^{\otimes 3}$. This implies that $\overline{\mathcal{M}}_g \cong [H'/\,\mathrm{PGL}_{5g-6}]$ is an algebraic stack.

For the moduli of vector bundles on a fixed smooth projective curve $C$, we define $\mathcal{B}un_{r,d}(C)$ as the prestack over $\mathrm{Sch}/\mathbb{C}$ consisting of pairs $(S, F)$, where $S$ is a scheme and $F$ is a vector bundle $F$ on $C \times S$ such that for every $s \in S$ the restriction $F_s$ to $X_s = C \times_S \operatorname{Spec} \kappa(s)$ has rank $r$ and degree $d$. A morphism $(S', F') \to (S, F)$ is the data of a map $f\colon S' \to S$ and an isomorphism $f^*F \xrightarrow{\sim} F'$. Étale descent implies that $\mathcal{B}un_{r,d}(C)$ is a stack over $(\mathrm{Sch}/\mathbb{C})_{\text{ét}}$. As with $\overline{\mathcal{M}}_g$, algebraicity of $\mathcal{B}un_{r,d}(C)$ can be verified by rigidifying the moduli problem: for every vector bundle $F$ on $X$, there exists an integer $n$ such that the $n$th twist $F(n)$ is globally generated, which means that after choosing a basis $\Gamma(X, F(n))$ of rank $P(n)$, we can express $F(n)$ as a quotient $\mathcal{O}_X^{\oplus P(n)} \twoheadrightarrow F(n)$ and thus $F$ as a quotient $\mathcal{O}_X^{\oplus P(n)}(-n) \twoheadrightarrow F$. Utilizing Grothendieck's Quot scheme parameterizing all coherent quotient sheaves of $\mathcal{O}_X^{\oplus P(n)}(-n)$, one can show that there is an open subscheme $Q'_n$ parameterizing vector bundle quotients which induce an isomorphism on global sections after twisting by $n$. This expresses $\mathcal{B}un_{r,d}(C)$ as an infinite union of quotient stacks $[Q'_n/\,\mathrm{GL}_{P(n)}]$, and, in particular, verifies algebraicity.

[6]Flatness is defined as a rather abstract algebraic property, but it magically ensures that the fibers vary 'nicely' in families.

[7]The local closedness is the most delicate detail.

## Verifying algebraicity intrinsically

> As a general fact, our knowledge of nonprojective existence theorems is exceedingly poor, and I hope this will change eventually.
>
> Grothendieck, letter to Murre, 1962

Until the late 1960s, moduli spaces such as Grothendieck's Hilbert and Quot schemes had mostly been constructed as projective varieties using tools of projective geometry. In the 1970s, Michael Artin established criteria—now known as *Artin's Axioms*—for a moduli problem to be an algebraic stack or a (possibly non-projective) algebraic space [Art74] in terms of the local properties of the moduli problem, i.e., its deformation and obstruction theory. Artin's Criteria can be applied to verify that $\mathcal{M}_g$, $\mathcal{B}un(C)$, and many other moduli problems are algebraic stacks.

## Constructing projective moduli

> However I do not claim at all that [GIT] should be avoided, but only that sometimes it may be good to have an alternative.
>
> Gerd Faltings [Fal93]

A pleasant byproduct of our definitions is that it is usually not hard to show that a given moduli problem is representable by an algebraic stack and equipped with a universal family. While many geometric questions can be studied (and arguably should be studied) on the moduli stack, it is often convenient to make a trade-off: by sacrificing the existence of a universal family, we can sometimes construct a more familiar geometric space, ideally a projective variety. This allows us to utilize the much larger toolkit of projective geometry (e.g., birational geometry, intersection theory, Hodge theory, ...) to study a moduli problem.

Trichotomy of moduli

| Auts | None | Finite | Reductive at closed points |
|---|---|---|---|
| Structure | Scheme/alg. space | DM stack | alg. stack |
| Defining property | Zariski/étale locally affine | étale locally affine | smooth locally affine |
| Examples | $\mathbb{CP}^n$, Hilb, Quot | $\mathcal{M}_g$ | $\mathcal{B}un^{\mathrm{ss}}_{r,d}(C)$ |
| Quotient stacks $[X/G]$ | action is free | finite stabilizers $G_x$ | reductive stabilizers $G_x$ at closed orbits |
| Local quotient structure | Zariski/étale locally $\operatorname{Spec} A$ | étale locally $[\operatorname{Spec} A/G_x]$ | étale locally $[\operatorname{Spec} A/G_x]$ |
| Type of moduli space | fine moduli space | coarse moduli space | good moduli space |

When there are no automorphisms, no tradeoff is necessary, and the moduli problem is often represented by an algebraic space and equipped with a universal family. In this case, we say that it is a *fine moduli space*. The remaining question is whether the moduli space is projective, and there are various ampleness criteria, e.g., the Nakai–Moishezon criterion, that can be applied.

When there are automorphisms, we try to approximate the moduli problem with an algebraic space that preserves many geometric features. As we will see shortly, when the automorphisms are finite (resp., reductive at closed points), there is often a *coarse moduli space* inducing a bijection on points (resp., a *good moduli space* inducing a bijection on *closed* points).

We have already summarized how Mumford's GIT can be used to construct projective moduli spaces. We now outline an alternative, more intrinsic approach, and then sketch how it can be applied to both the moduli of stable curves and semistable vector bundles.

Six steps toward projective moduli

① **Algebraicity:** Express the moduli problem $\mathcal{M}$ as a substack

$$\mathcal{M} \subseteq \mathcal{X}$$

of a larger moduli stack $\mathcal{X}$, and define an object $x \in \mathcal{X}$ to be *semistable* if it is in $\mathcal{M}$. Show that $\mathcal{X}$ is an algebraic stack locally of finite type.

② **Openness of semistability:** Show that semistability is an open condition, i.e., $\mathcal{M} = \mathcal{X}^{\mathrm{ss}} \subseteq \mathcal{X}$ is an open substack.

③ **Boundedness of semistability:** Show that semistability is bounded, i.e., $\mathcal{M} = \mathcal{X}^{\mathrm{ss}}$ is of finite type.

④ **Semistable reduction:** Show that $\mathcal{M}$ satisfies the existence part of the valuative criterion for properness.

⑤ **Existence of a moduli space:** Show that there is a fine/coarse/good moduli space $\mathcal{M} \to M$ where $M$ is a proper algebraic space.

⑥ **Projectivity:** Show that a tautological line bundle on $\mathcal{M}$ descends to an ample line bundle on $M$, i.e., $M$ is projective.

## Geometry of DM stacks

In order to explain how the above strategy applies to $\overline{\mathcal{M}}_g$, we first describe a few general properties of DM stacks. The first fundamental result is their local quotient structure.

**Theorem** (Local Structure)**.** *If $x \in \mathcal{X}$ is a point of a quasi-separated DM stack with stabilizer $G_x$ as a finite abstract group, there exists an étale morphism $([\operatorname{Spec} A/G_x], w) \to (\mathcal{X}, x)$ inducing an isomorphism of stabilizer groups at $w$* [LMB00, Thm. 6.2].

By definition, $x \in \mathcal{X}$ has an étale neighborhood $\operatorname{Spec} A \to \mathcal{X}$, but here we have an étale neighborhood $[\operatorname{Spec} A/G_x] \to \mathcal{X}$ that *preserves stabilizer group* at $x$. Analogously, Deligne–Mumford topological and differential stacks are also locally quotient stacks $[U/G]$ for a proper action of a finite group.

Quotient stacks $[\operatorname{Spec} A/G]$ by finite groups are particularly simple: the map

$$[\operatorname{Spec} A/G] \to \operatorname{Spec} A^G$$

induces a homeomorphism on topological spaces. The Local Structure Theorem reduces many properties of DM stacks to quotient stacks of affine schemes by finite groups.

Moreover, when the stack $\mathcal{X}$ is smooth, then there exists an étale morphism $([\operatorname{Spec} A/G_x], w) \to ([T_{\mathcal{X},x}/G_x], 0)$, where $T_{\mathcal{X},x}$ is the Zariski tangent space of $\mathcal{X}$ at $x$ or, in other words, the first order deformation space. For the moduli space $\mathcal{M}_g$ of smooth curves, the tangent space at a smooth curve $[C]$ is identified with $\mathrm{H}^1(C, T_C)$ and the stabilizer group is the automorphism group $\operatorname{Aut}(C)$. In this case, we have a roof diagram

$$\begin{tikzcd} & {[\operatorname{Spec} A/\operatorname{Aut}(C)]} \arrow[dl, "\text{étale}"'] \arrow[dr, "\text{étale}"] & \\ {[\mathrm{H}^1(C, T_C)/\operatorname{Aut}(C)]} & & \mathcal{M}_g \end{tikzcd}$$

and the local geometry at $[C] \in \mathcal{M}_g$ is the $\operatorname{Aut}(C)$-equivariant geometry of $\mathrm{H}^1(C, T_C)$. We can think of the quotient stack $[\mathrm{H}^1(C, T_C)/\operatorname{Aut}(C)]$ as the *stacky tangent space* at $[C]$. While the stacky tangent space is smooth, the quotient space $\mathrm{H}^1(C, T_C)/\operatorname{Aut}(C)$ may be singular.

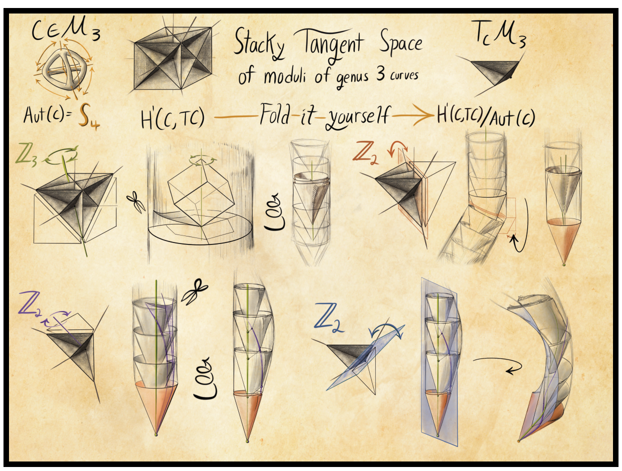


Figure 6: Visualization of the local structure of $\mathcal{M}_3$ at a curve $C$ with symmetry group $S_4$. By factoring $S_4$ into cyclic groups $\mathbb{Z}_2$ and $\mathbb{Z}_3$, we iteratively fold the vector space $\mathrm{H}^1(C, T_C)$ along the fixed loci of each cyclic group; the creases reflect the loci of curves with symmetry.

To approximate a DM stack $\mathcal{X}$ with an algebraic space, we will use the concept of a *coarse moduli space*, i.e., a map $\mathcal{X} \to X$ to an algebraic space such that every point of $X$ has an étale neighborhood of the form

$$\begin{tikzcd} {[\operatorname{Spec} A/G]} \arrow[r, "\text{étale}"] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{X} \arrow[d] \\ (\operatorname{Spec} A)/G := \operatorname{Spec} A^G \arrow[r, "\text{étale}"] & X, \end{tikzcd}$$

where $G$ is a finite group and the diagram is cartesian. Applying the Local Structure Theorem and gluing the quotients $(\operatorname{Spec} A)/G$ in the étale topology yields the *Keel–Mori Theorem*:

**Theorem** (Existence)**.** *If $\mathcal{X}$ is a separated DM stack, there exists a coarse moduli space $\mathcal{X} \to X$ where $X$ is a separated algebraic space* [KM97].

While the coarse moduli space gives a canonical map *to* an algebraic space, another fundamental result, sometimes referred to as *Le Lemma de Gabber*, asserts that any DM stack admits a finite morphism $V \to \mathcal{X}$ *from* a scheme. Most of the theory of schemes, e.g., quasi-coherent sheaves and their cohomology, carry over to DM stacks with little change.

## Moduli of stable curves

The variety $\overline{M}_g$ is perhaps the single most studied variety over the last sixty years. As an alternative to the GIT construction, we outline how to apply the six-step strategy.

*Step ① (Algebraicity):* We can view $\overline{\mathcal{M}}_g$ as a substack of the stack $\mathcal{M}_g^{\mathrm{all}}$ of all curves, and Artin's Axioms can be applied to verify that $\mathcal{M}_g^{\mathrm{all}}$ is an algebraic stack locally of finite type.

*Step ② (Openness of stability):* Since any deformation of a nodal singularity is either smooth or nodal, $\overline{\mathcal{M}}_g \subseteq \mathcal{M}_g^{\mathrm{all}}$ is an open substack.

*Step ③ (Boundedness of stability):* The third power $\omega_C^{\otimes 3}$ of the dualizing sheaf of a stable curve is very ample. Therefore, the representability of the Hilbert scheme as a *finite type* scheme implies that $\overline{\mathcal{M}}_g$ is of finite type. (It seems that every boundedness argument in algebraic geometry ultimately relies on the boundedness of a Hilbert or Quot scheme.)

*Step ④ (Stable reduction):* Properness of $\overline{\mathcal{M}}_g$ is established using the valuative criterion, which translates into an explicit geometric property: given a family $\mathcal{C} \to T \setminus \{t\}$ of smooth curves over a punctured smooth curve, there exists (after possibly replacing $T$ with a finite cover) a family $\widetilde{\mathcal{C}} \to T$ of stable curves extending $\mathcal{C}$. This can be demonstrated by first finding some extension (e.g., by using a Hilbert scheme) where the central fiber is possibly horribly singular, taking a resolution of singularities so that the reduced central fiber is nodal, choosing a ramified extension of $T$ so that the central fiber of the normalized base change is reduced, and then contracting any rational components meeting the rest of the curve at fewer than three points.

*Step ⑤ (Existence of a moduli space):* As $\overline{\mathcal{M}}_g$ is proper, the Keel–Mori Theorem yields a coarse moduli space $\overline{M}_g$ as a proper algebraic space.

*Step ⑥ (Projectivity):* Kollár gave a solution to the final and most difficult step in [Kol90]. By considering the multiplication map on pluricanonical bundles

$$\operatorname{Sym}^m \pi_*(\omega^{\otimes k}_{\mathcal{U}_g/\overline{\mathcal{M}}_g}) \to \pi_*(\omega^{\otimes mk}_{\mathcal{U}_g/\overline{\mathcal{M}}_g}),$$

where $\pi\colon \mathcal{U}_g \to \overline{\mathcal{M}}_g$ is the universal family, and using that the first vector bundle has a reduction of structure group to $\mathrm{PGL}_v$ where $v = \mathrm{rk}\pi_*(\omega^{\otimes k}_{\mathcal{U}_g/\overline{\mathcal{M}}_g})$, he constructed a quasi-finite morphism $\overline{\mathcal{M}}_g \to [\mathrm{Gr}(q, \operatorname{Sym}^m \mathbb{C}^v)/\operatorname{PGL}_v]$, and deduced the projectivity of $\overline{M}_g$ from the projectivity of the Grassmannian.

The projectivity of $\overline{M}_g$ is just the starting point to its geometry. It turns out that $\overline{M}_g$ is irreducible. Over $\mathbb{C}$, this follows from Teichmüller's description, but it is even true in positive characteristic as first shown by Deligne and Mumford [DM69]. We recommend [HM98] for a survey of the geometry of $\overline{M}_g$.

This six-step strategy has been applied to construct many other projective moduli spaces of interest, e.g., stable maps and Hassett's weighted stable curves. Most notably, by using advances in the minimal model program and the finite generation of the canonical ring, Kollár, with the assistance of many others, constructed projective moduli spaces of canonically polarized varieties of any dimension [Kol23].

## Geometry of algebraic stacks

Mirroring our discussion of DM stacks, we will focus on two foundational results: the local quotient structure and the existence of moduli spaces.

**Theorem** (Local Structure II)**.** *For an quasi-separated algebraic stack $\mathcal{X}$ of finite type over $\mathbb{C}$ with affine stabilizers and a point $x \in \mathcal{X}(\mathbb{C})$ with reductive*[8] *stabilizer $G_x$, there exists an étale morphism $([\operatorname{Spec} A/G_x], w) \to (\mathcal{X}, x)$ inducing an isomorphism of stabilizer groups* [AHR20, Thm. 1.1].

The upshot is that quotients stacks $[\operatorname{Spec} A/G]$ by a reductive group $G$ can be viewed as the basic building block of algebraic stacks near points with reductive stabilizer. In the differential setting, this is analogous to Weinstein's conjecture——now Zung's Theorem—that differential stacks are locally quotient stacks near points whose stabilizers are proper Lie groups.

Quotient stacks $[\operatorname{Spec} A/G]$ by a reductive group are particularly well-understood. For instance, Hilbert's 14th question—whether the finite generation of $A$ implies the finite generation of $A^G$—has a positive answer in this case, and thus $\operatorname{Spec} A^G$ is a variety. The affine case of GIT asserts that the map

$$[\operatorname{Spec} A/G] \to \operatorname{Spec} A^G$$

has desirable geometric properties of a quotient, e.g., it identifies *closed* $G$-orbits of $\operatorname{Spec} A$ with closed points of $\operatorname{Spec} A^G$. Examples such as $[\mathbb{C}^n/\mathbb{C}^*] \to \operatorname{Spec} \mathbb{C}$ (where $\mathbb{C}^*$-acts with weights $1, \dots, 1$) and $[\mathbb{C}^2/\mathbb{C}^*] \to \mathbb{C}^1$ (with weights 1 and $-1$) illustrate how two distinct orbits can be identified in the quotient if their orbit closures intersect.

[8] As we are in characteristic 0, the reductivity of an algebraic group $G$ translates into the complete reducibility of $G$-representations. Finite groups, tori, and the matrix groups $\mathrm{SL}_n$ and $\mathrm{GL}_n$ are all reductive.

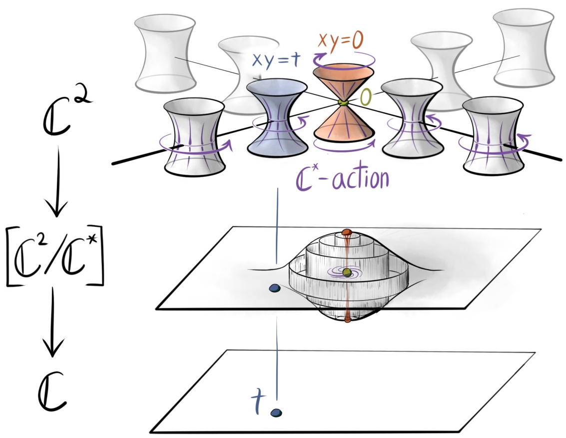


Figure 7: A visualization of $[\mathbb{C}^2/\mathbb{C}^*]$: it looks like the non-Hausdorff complex plane $\mathbb{C}\bigcup_{\mathbb{C}\setminus 0}\mathbb{C}$ where both origins specialize to an extra stacky point with $\mathbb{C}^*$ stabilizer. The map $[\mathbb{C}^2/\mathbb{C}^*] \to \mathbb{C}$, given by $(x, y) \mapsto xy$, is an example of a *good moduli space*. The fiber over $t \neq 0 \in \mathbb{C}$ in $\mathbb{C}^2$ is the orbit $xy = t$ and corresponds to a single point of $[\mathbb{C}^2/\mathbb{C}^*]$, while the fiber over $t = 0$ consists of three orbits.

The Local Structure Theorem allows us to formulate a generalization of a coarse moduli space well-suited for moduli problems that may have non-finite automorphisms, e.g., vector bundles on a curve. A morphism $\mathcal{X} \to X$, from an algebraic stack $\mathcal{X}$ of finite type over $\mathbb{C}$ with affine diagonal to an algebraic space $X$, is a *good moduli space* if every point of $X$ has an étale neighborhood of the form

$$\begin{tikzcd} {[\operatorname{Spec} A/G]} \arrow[r, "\text{étale}"] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{X} \arrow[d] \\ \operatorname{Spec} A^G \arrow[r, "\text{étale}"] & X, \end{tikzcd}$$

where $G$ is a reductive group and the diagram is cartesian.

While this formulates the properties of the desired moduli space, it does not address their existence. Similar to the construction of the coarse moduli space, the Local Structure Theorem allows us to attempt to glue the affine GIT quotients $\operatorname{Spec} A^{G_x}$ of étale neighborhoods $[\operatorname{Spec} A/G_x] \to \mathcal{X}$ into a good moduli space for $\mathcal{X}$. However, the presence of non-closed $\mathbb{C}$-points introduces an an additional complexity in the gluing in this case, e.g., $[\mathbb{C}^n/\mathbb{C}^*]$ has a point as its good moduli space but also contains $\mathbb{CP}^{n-1}$ as an open substack. In addition, $\mathcal{X}$ is necessarily non-separated once it contains points with affine but not finite stabilizers. By replacing the separatedness hypothesis with the conditions of $\Theta$-*completeness* and $\mathsf{S}$-*completeness* allows us to formulate an alternative to the Keel–Mori Theorem.

**Theorem** (Existence II)**.** *If $\mathcal{X}$ is an algebraic stack of finite type over $\mathbb{C}$ with affine diagonal, then there exists a good moduli space $\mathcal{X} \to X$ where $X$ is a separated algebraic space if and only if $\mathcal{X}$ is $\Theta$-complete and $\mathsf{S}$-complete* [AHLH23, Thm. A].

Both $\Theta$- and $\mathsf{S}$-completeness are valuative criteria requiring that $\mathbb{C}^*$-equivariant families over certain $\mathbb{C}^*$-equivariant punctured surfaces extend over the surface. For instance, $\mathcal{X}$ is $\Theta$-*complete* if for every pointed curve $(T, t)$, every $\mathbb{G}_m$-equivariant morphism $\big(\mathbb{C} \times T\big) \setminus \{(0, t)\} \to \mathcal{X}$ extends uniquely to a $\mathbb{G}_m$-equivariant morphism $\mathbb{C} \times T \to \mathcal{X}$. Using the notation $\Theta := [\mathbb{C}^1/\mathbb{C}^*]$ and $\Theta_T := \Theta \times T$, this translated to requiring that every map $\Theta_T \setminus \{(0, t)\} \to \mathcal{X}$ extends uniquely to a map $\Theta_T \to \mathcal{X}$.

## Moduli of semistable vector bundles

Fix a connected, smooth, and projective complex curve $C$. The moduli stack $\mathcal{B}un_{r,d}(C)$, parameterizing vector bundles on $C$ of rank $r$ and degree $d$, is unbounded (i.e., not finite type), horribly non-separated, and does not admit a reasonable moduli space. Remarkably, semistability offers a solution to all three of these issues. Extending Mumford's definition of a stable vector bundle, Seshadri [Ses67] defined a vector bundle $F$ on $C$ to be *semistable* if for every subsheaf $E \subseteq F$ satisfies

$$\frac{\deg E}{\operatorname{rk} E} \leq \frac{\deg F}{\operatorname{rk} F}.$$

We define $\mathcal{B}un^{\mathrm{ss}}_{r,d}(C)$ as the full subcategory of $\mathcal{B}un_{r,d}(C)$ whose objects are pairs $(T, E)$ where $E$

is a vector bundle on $C \times T$ such that for every point $t \in T$, the restriction $E_t := E|_{C\times\{t\}}$ is semistable.

Seshadri constructed a projective moduli space of $\mathcal{B}un^{\mathrm{ss}}_{r,d}(C)$ using GIT [Ses67], but we can also apply the intrinsic six-step approach outlined above.

*Step ① (Algebraicity):* By construction, $\mathcal{B}un^{\mathrm{ss}}_{r,d}(C)$ is a substack of $\mathcal{B}un_{r,d}(C)$, which (as we sketched above) is an algebraic stack locally of finite type. In fact, it is convenient to view $\mathcal{B}un^{\mathrm{ss}}_{r,d}(C)$ as substack of the even larger algebraic stack $\underline{\mathrm{Coh}}_{r,d}(C)$ parameterizing coherent sheaves of rank $r$ and degree $d$.

*Step ② (Openness of stability):* Given a family of vector bundles $F$ on $C \times T$ over a finite type scheme $T$, there are only finitely many ranks $r'$ and degrees $d'$ of a destabilizing subsheaf $E \subseteq F_t$ of a fiber $F_t := F|_{C\times\{t\}}$. For each pair $(r', d')$, there is a relative Quot scheme $Q_{r',d'}$ proper over $T$ parameterizing such subsheaves whose image in $T$ is closed. Removing each such locus gives an open subscheme of $T$ consisting precisely of points $t \in T$ with $F_t$ semistable.

*Step ③ (Boundedness of stability):* Choosing an ample line bundle $\mathcal{O}_C(1)$ on $C$, one shows that if $F$ is semistable and $n > 2g-2-d/r$, then $\mathrm{H}^1(C, F(n)) = 0$ (this is not hard: by Serre–Duality, $\mathrm{H}^1(C, F(n)) \cong \mathrm{Hom}_{\mathcal{O}_C}(F(n), \Omega_C)$, and this group is zero as there are no maps $F(n) \to \Omega_C$ between semistable bundles where the slope of the target is smaller). It follows that if $n > 2g - 1 - d/r$, then $F(n)$ is globally generated by $P(n) = \mathrm{h}^0(C, F(n))$ sections. Therefore, the finite type Quot scheme $\mathrm{Quot}^P(\mathcal{O}_C(-n)^{\oplus P(n)})$ shows that $\mathcal{B}un^{\mathrm{ss}}_{r,d}(C)$ is of finite type.

*Step ④ (Semistable reduction):* The existence part of the valuative criterion was proven by Stacy Langton [Lan75]: for each punctured curve $T \setminus \{t\}$, any semistable vector bundle $F$ on $C \times (T \setminus \{t\})$ can be extended to a semistable vector bundle $\widetilde{F}$ on $C \times T$. Langton's strategy is to first find some extension $\widetilde{F}$ to $C \times T$, e.g., by using the properness of the Quot scheme. If the central fiber $\widetilde{F}_t := \widetilde{F}|_{C\times\{t\}}$ is not semistable, there is a destabilizing subsheaf $E_t \subset \widetilde{F}_t$ and one replaces $\widetilde{F}$ with $\ker(\widetilde{F} \twoheadrightarrow \widetilde{F}_t \twoheadrightarrow \widetilde{F}_t/E_t)$. The central fiber of $\widetilde{F}$ is now closer to being semistable, and she proved that the central fiber becomes semistable after finitely many steps.

*Step ⑤ (Existence of a moduli space):* We claim that there is a good moduli space $\mathcal{B}un^{\mathrm{ss}}_{r,d}(C) \to M^{\mathrm{ps}}_{r,d}(C)$ to a proper algebraic space whose points are in bijection with the closed points of $\mathcal{B}un^{\mathrm{ss}}_{r,d}(C)$, i.e., the *polystable* vector bundles. By applying the Existence Theorem, one must verify that $\mathcal{B}un^{\mathrm{ss}}_{r,d}(C)$ is $\Theta$- and $\mathsf{S}$-complete. It is not hard to see that the larger stack $\underline{\mathrm{Coh}}_{r,d}(C)$ is both $\Theta$- and $\mathsf{S}$-complete. For instance, we can use the correspondence between maps from $\Theta = [\mathbb{C}^1/\mathbb{C}^*]$ to $\underline{\mathrm{Coh}}_{r,d}(C)$ and filtrations of a vector bundle to reinterpret $\Theta$-completeness: since a map $(\Theta \times T) \setminus \{0, t\} \to \underline{\mathrm{Coh}}_{r,d}(C)$ corresponds to a family of coherent sheaves $F$ on $X \times T$ together with a filtration of the restriction $F|_{X\times(T\setminus\{t\})}$, $\Theta$-completeness is the requirement that the filtration extends to a filtration of $F$. This is an easy consequence of the properness of the Quot scheme. Therefore, to show that $\mathcal{B}un^{\mathrm{ss}}_{r,d}(C)$ is $\Theta$-complete, we can first use $\Theta$-completeness of $\underline{\mathrm{Coh}}_{r,d}(C)$ to find a filtration of $F$ extending the filtration of $F|_{X\times(T\setminus\{t\})}$, and then use properties of semistability to show that the central fiber of the unique extension is semistable.

*Step ⑥ (Projectivity):* Gerd Faltings gave an explicit construction of $M^{\mathrm{ps}}_{r,d}(C)$ as a projective variety [Fal93]. The central idea is to show that there is a vector bundle $V$ on $C$ such that the determinantal line bundle $\mathcal{L}_V := \det \mathrm{R}p_{2,*}(\mathcal{E}_{\mathrm{univ}} \otimes p_1^* V)^\vee$ on $\mathcal{B}un^{\mathrm{ss}}_{r,d}(C)$, where $\mathcal{E}_{\mathrm{univ}}$ is the universal vector bundle on $C \times \mathcal{B}un^{\mathrm{ss}}_{r,d}(C)$, descends to an ample line bundle $L_V$ on $M^{\mathrm{ps}}_{r,d}(C)$. Ampleness is verified by explicitly constructing sections. To achieve, this Faltings first proved an interesting characterization of semistability: a vector bundle $E$ on $C$ of rank $r$ and degree $d$ is semistable if and only if there exist a vector bundle $V$ such that $\mathrm{H}^0(C, F \otimes V) = \mathrm{H}^1(C, F \otimes V) = 0$. This allows us to construct a section of $\mathcal{L}_V$ that does not vanish at $F$: the derived pushforward $\mathrm{R}p_{2,*}(\mathcal{E}_{\mathrm{univ}} \otimes p_1^* V)$ is represented by a two-term complex $\alpha\colon K^0 \to K^1$ of vector bundles of the same rank on $\mathcal{B}un^{\mathrm{ss}}_{r,d}(C)$, and the induced section $\mathcal{O}_{\mathcal{B}un^{\mathrm{ss}}_{r,d}(C)} \to \mathcal{L}_V = \det(K^1) \otimes \det(K^0)^\vee$ is nonzero at $F$ because $\det(\alpha|_F)$ is an isomorphism, or in other words because $\mathrm{H}^0(C, F \otimes V) = \mathrm{H}^1(C, F \otimes V) = 0$. This implies that a positive tensor power of the descended line bundle $L_V$ on $M^{\mathrm{ps}}_{r,d}(C)$ is globally generated, i.e., defines a map to projective space. A similar but more involved line of reasoning shows that this map does not contract any curves, which is sufficient to guarantee that $L_V$ is ample.

For geometric properties of $M^{\mathrm{ps}}_{r,d}(C)$, we direct the reader to [HL10]. The above approach can be adapted to construct projective moduli spaces of $K$-polystable log Fano varieties and of semistable complexes with respect to certain Bridgeland stability conditions.

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*A greatly expanded version of this article can be found in the forthcoming book* [Alp24].

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