# SECOND FLIP IN THE HASSETT-KEEL PROGRAM: PROJECTIVITY

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Abstract. For the moduli stacks of $\alpha$-stable curves introduced in [AFSv15], we prove nefness of natural log canonical divisors generalizing a well-known result of Cornalba and Harris for the moduli stack of Deligne-Mumford stable curves. We then deduce the projectivity of the good moduli spaces of $\alpha$-stable curves and identify these moduli spaces with the log canonical models of $\overline{\mathcal{M}}_{g,n}$ appearing in the Hassett-Keel program.

## 1. Introduction

This is the final part of the trilogy (see also [AFSv15, AFS16]) in which we construct the second flip in the Hassett-Keel program for $\overline{\mathcal{M}}_{g,n}$. In [AFSv15], we construct the moduli stacks $\overline{\mathcal{M}}_{g,n}(\alpha)$ of $\alpha$-stable curves (see Definition 2.6). In [AFS16], we prove that the moduli stacks $\overline{\mathcal{M}}_{g,n}(\alpha)$ admit proper good moduli spaces, as defined by the first author in [Alp13]. The main result of this paper is that the good moduli spaces of $\overline{\mathcal{M}}_{g,n}(\alpha)$ are projective and constitute steps in the Hassett-Keel program for $\overline{\mathcal{M}}_{g,n}$. Namely, for $\alpha > 2/3{-}\epsilon$, let $\overline{\mathbb{M}}_{g,n}(\alpha)$ be the good moduli space of $\overline{\mathcal{M}}_{g,n}(\alpha)$, which exists as a proper algebraic space by [AFS16, Theorem 1.1]. Consider the following log canonical models of $\overline{\mathcal{M}}_{g,n}$:

$$\overline{M}_{g,n}(\alpha) := \operatorname{Proj} \bigoplus_{m\geq 0} \mathrm{H}^0(\overline{\mathcal{M}}_{g,n}, \lfloor m(K_{\overline{\mathcal{M}}_{g,n}} + \alpha\delta + (1-\alpha)\psi)\rfloor). \tag{1.1}$$

We prove that the two independently defined objects, $\overline{\mathbb{M}}_{g,n}(\alpha)$ and $\overline{M}_{g,n}(\alpha)$, are in fact the same:

**Theorem 1.1.** *For $\alpha > 2/3{-}\epsilon$, the following statements hold:*

1. *The line bundle $K_{\overline{\mathcal{M}}_{g,n}(\alpha)} + \alpha\delta + (1-\alpha)\psi$ descends to an ample line bundle on $\overline{\mathbb{M}}_{g,n}(\alpha)$.*
2. *There is an isomorphism $\overline{\mathbb{M}}_{g,n}(\alpha) \simeq \overline{M}_{g,n}(\alpha)$.*

*In particular, the algebraic stack $\overline{\mathcal{M}}_{g,n}(\alpha)$ has a projective good moduli space for every $\alpha > 2/3{-}\epsilon$.*

The key ingredient in the proof of Theorem 1.1 is a positivity result for certain line bundles on $\overline{\mathcal{M}}_{g,n}(\alpha)$ generalizing the following well-known result of Cornalba and Harris (**N.B.** we use $\sim$ to denote numerical proportionality with a positive scalar).

**Theorem** ([CH88]). *The line bundle*

$$K_{\overline{\mathcal{M}}_{g,n}} + \frac{9}{11}\delta + \frac{2}{11}\psi \sim 11\lambda - \delta + \psi$$

*is nef on $\overline{\mathcal{M}}_{g,n}$ for all $(g, n)$, and has degree $0$ precisely on the families whose only non-isotrivial components are $A_1$-attached elliptic tails.*

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The first author was partially supported by the ARC grant DE140101519. The second author was partially supported by the NSF grant DMS-1259226 and the ANU MSRVP fund. The third author was partially supported by the ARC grant DE140100259.

By using the above result, Cornalba proved that $12\lambda-\delta+\psi$ is in fact ample on $\overline{\mathcal{M}}_{g,n}$ and thus obtained a direct intersection-theoretic proof of the projectivity of $\overline{M}_{g,n}$ [Cor93]. In the introduction to [Cor93], the author says that "... it is hard to see how [these techniques] could be extended to other situations." In what follows, we do precisely that in giving intersection-theoretic proofs of the projectivity for $\overline{\mathbb{M}}_{g,n}(7/10-\epsilon)$, $\overline{\mathbb{M}}_{g,n}(2/3)$, and $\overline{\mathbb{M}}_{g,n}(2/3-\epsilon)$ by proving the following positivity result:

**Theorem 1.2** (Positivity of log canonical divisors)**.**

(a) *The line bundle*

$$K_{\overline{\mathcal{M}}_{g,n}(9/11-\epsilon)}+\frac{7}{10}\delta+\frac{3}{10}\psi\sim 10\lambda-\delta+\psi$$

*is nef on* $\overline{\mathcal{M}}_{g,n}(9/11-\epsilon)$*, and, if* $(g,n)\neq(2,0)$*, has degree* 0 *precisely on the families whose only non-isotrivial components are* $A_1/A_1$*-attached elliptic bridges. It is trivial if* $(g,n)=(2,0)$*.*

(b) *The line bundle*

$$K_{\overline{\mathcal{M}}_{g,n}(7/10-\epsilon)}+\frac{2}{3}\delta+\frac{1}{3}\psi\sim\frac{39}{4}\lambda-\delta+\psi$$

*is nef on* $\overline{\mathcal{M}}_{g,n}(7/10-\epsilon)$*, and has degree* 0 *precisely on the families whose only non-isotrivial components are* $A_1$*-attached Weierstrass chains.*

Our proof of the above theorem is inspired by [Cor93]. We also refer the reader to [ACG11, Chapter 14] for an excellent exposition of Cornalba's original argument and a comprehensive treatment of intersection-theoretic approaches to the projectivity of $\overline{M}_{g,n}$, many of which make appearance in this paper.

1.1. **Connections with the earlier works.** In the case of unpointed curves, the stacks $\overline{\mathcal{M}}_g(7/10)$ and $\overline{\mathcal{M}}_g(7/10-\epsilon)$ were defined in a seminal paper of Hassett and Hyeon constructing the first flip of $\overline{\mathcal{M}}_g$ [HH13], which largely motivates the present trilogy of papers on the second flip. Hassett and Hyeon construct the good moduli space $\overline{\mathbb{M}}_g(7/10)$, respectively $\overline{\mathbb{M}}_g(7/10-\epsilon)$, as a GIT quotient parameterizing Chow, respectively Hilbert, semistable bicanonical curves. As we show in §5.4, Part (a) of Theorem 1.2 is readily deduced from Hassett and Hyeon's results. However, as explained in the introduction to [AFSv15], the present trilogy constructs the moduli stacks of $\alpha$-stable curves and establishes the projectivity of their moduli spaces for all $\alpha>2/3-\epsilon$ independently of the earlier GIT constructions for $\alpha>7/10-\epsilon$ and $n=0$ (namely those of [Sch91, HH09, HH13]). In particular, this work gives a new proof of one of the main results of [HH13], which is Theorem 1.1 for $\alpha>7/10-\epsilon$ and $n=0$. To preserve this independence, we include our intersection-theoretic proof of Theorem 1.2(a).

A major reason for our desire to avoid GIT as much as possible is the expectation that the GIT construction of $\overline{\mathcal{M}}_g(2/3)$ will necessitate the stability analysis of the $6^{th}$ Hilbert points of bicanonical curves [Mor09], something that seems completely out of reach at the moment. At the same time, our stack-theoretic methods have a hope of being pushed further for at least the next several steps in the Hassett-Keel program.

There appears no feasible way to deduce the results of Part (b) of Theorem 1.2, for a general $(g,n)$, from the existing GIT results. We do note that Theorem 1.2(b) was established in

[HL14, Section 3] for $g = 4$ using an assortment of GIT results. In particular, Hyeon and Lee's proof in genus 4 makes use of a complete analysis of Chow stability for canonical genus 4 curves obtained in [CMJL12, Kim13], which is greatly simplified by the fact that $\overline{\mathcal{M}}_4$ has only 3 boundary divisors. The Chow stability of higher genus canonical curves (especially for points corresponding to the generic points of the boundary divisors in $\overline{\mathcal{M}}_g$) has not been analyzed as of this writing. Nevertheless, our proof of Theorem 1.2(b) is fundamentally similar to that of [HL14]. Namely, we do use Chow stability of *smooth* canonical curves in the guise of Cornalba-Harris inequality (Proposition 4.1). For families with generically singular fibers, we are able to ultimately reduce to the generically smooth case. We refer to Section 6 for details.

1.2. **Roadmap.** Our proof of Theorems 1.1 and 1.2 is organized as follows. We recall the necessary notions and definitions in Section 2. In Section 3, we develop a theory of simultaneous normalization for families of at-worst tacnodal curves. By tracking how the relevant divisor classes change under normalization, we can reduce Theorem 1.2 to proving a (more complicated) positivity result for families of generically smooth curves. In Section 4, we collect several preliminary positivity results, stemming from three sources: the Cornalba-Harris inequality, the Hodge Index Theorem, and an effectivity result on $\overline{M}_{0,n}$. In Sections 5 and 6, we combine these ingredients to prove parts (a) and (b) of Theorem 1.2, respectively. Finally, in Section 7, we apply Theorem 1.2 to obtain Theorem 1.1.

## 2. Preliminaries on line bundles on $\overline{\mathcal{M}}_{g,n}(\alpha)$

The following terminology will be in force throughout the paper. We let $\widetilde{\mathcal{U}}_g(A_\infty)$ denote the stack of connected curves of arithmetic genus $g$ with only $A$-singularities, and let $\widetilde{\mathcal{U}}_g(A_\ell) \subset \widetilde{\mathcal{U}}_g(A_\infty)$ be the open substack parameterizing curves with at worst $A_k$, $k \leq \ell$, singularities. Since $\widetilde{\mathcal{U}}_g(A_\infty)$ is smooth, we may freely alternate between line bundles and divisor classes on $\widetilde{\mathcal{U}}_g(A_\infty)$. In addition, any relation between divisor classes on $\widetilde{\mathcal{U}}_g(A_\infty)$ that holds on the open substack of at-worst nodal curves extends to $\widetilde{\mathcal{U}}_g(A_\infty)$, because the locus of worse-than-nodal curves has codimension 2.

Let $\pi\colon \mathcal{X} \to \widetilde{\mathcal{U}}_g(A_\infty)$ be the universal family. We define the *Hodge class* as $\lambda := c_1(\pi_*\omega_\pi)$ and the *kappa class* as $\kappa := \pi_*(c_1(\omega_\pi)^2)$. The divisor parameterizing singular curves in $\widetilde{\mathcal{U}}_g(A_\infty)$ is denoted $\delta$. It can be further decomposed as $\delta = \delta_{\mathrm{irr}} + \delta_{\mathrm{red}}$, where $\delta_{\mathrm{red}}$ is the closed locus of curves with disconnecting nodes. (The fact that $\delta_{\mathrm{red}}$ is closed follows, for example, from [AFSv15, Corollary 2.11]).

By the preceding remarks, Mumford's relation $\kappa = 12\lambda - \delta$ holds on $\widetilde{\mathcal{U}}_g(A_\infty)$. Note that for $m \geq 2$, the higher Hodge bundles $\pi_*(\omega_\pi^m)$ are well-defined on the open locus in $\widetilde{\mathcal{U}}_g(A_\infty)$ of curves with nef dualizing sheaf, the complement of the closed substack of curves with rational tails. If we restrict to this open locus, the Grothendieck-Riemann-Roch formula gives

$$c_1(\pi_*(\omega_\pi^m)) = \lambda + \frac{m^2 - m}{2}\kappa. \tag{2.1}$$

Now let $\mathcal{C} \to B$ be a family of curves in $\widetilde{\mathcal{U}}_g(A_\infty)$. If $\sigma\colon B \to \mathcal{C}$ is any section of the family, we define $\psi_\sigma := \sigma^*\omega_{\mathcal{C}/B}$. We say that $\sigma$ is *smooth* if it avoids the relative singular locus of $\mathcal{C}/B$.

From now on, we work only with families $\mathcal{C} \to B$ over a smooth and proper curve $B$. If $\sigma\colon B \to \mathcal{C}$ is generically smooth and the only singularities of fibers that $\sigma(B)$ passes through are nodes, then $\sigma(B)$ is a $\mathbb{Q}$-Cartier divisor on $\mathcal{C}$, and we define the *index of* $\sigma$ to be

$$\iota(\sigma) := (\omega_{\mathcal{C}/B} + \sigma) \cdot \sigma. \tag{2.2}$$

Notice that the index $\iota(\sigma)$ is non-negative, and if $\sigma$ is smooth, then $\iota(\sigma) = 0$. We also have the following standard result:

**Lemma 2.1.** *Suppose $\mathcal{C} \to B$ is a generically smooth family of curves in $\widetilde{\mathcal{U}}_g(A_\infty)$.*

1. *Suppose $g \geq 1$, the family $\mathcal{C} \to B$ is not isotrivial, and $\sigma\colon B \to \mathcal{C}$ is a smooth section. Then $\sigma^2 < 0$.*
2. *If $g = 0$ and $\sigma, \sigma', \sigma''\colon B \to \mathcal{C}$ are 3 smooth sections such that $\sigma$ is disjoint from $\sigma'$ and $\sigma''$, then $\sigma^2 \leq 0$.*

*Proof.* In the positive genus case, the nodal reduction does not change $\sigma^2$ and passing to the relative minimal model only increases $\sigma^2$. The claim now follows from the well-known fact that any section of a non-isotrivial generically smooth family of Deligne-Mumford semistable curves has a negative self-intersection; see e.g., [Ara71, p.1291].

In the genus 0 case, we can consider the relative minimal model of $\mathcal{C}/B$ that is isomorphic to $\mathcal{C}$ around $\sigma$. The claim then follows from the fact that on a geometrically ruled surface any section that is disjoint from two other distinct sections must have a non-positive self-intersection. □

Let $\mathcal{C} \to B$ be a one-parameter family of curves in $\widetilde{\mathcal{U}}_g(A_\infty)$. If $p \in \mathcal{C}$ is a node of its fiber, then the local equation of $\mathcal{C}$ at $p$ is $xy = t^e$, for some $e \in \mathbb{Z}$ called *the index of* $p$ and denoted $\operatorname{index}(p)$. A *rational tail* (resp., a *rational bridge*) of a fiber is a $\mathbb{P}^1$ meeting the rest of the fiber in exactly one (resp., two) nodes. If $E \subset C_b$ is a rational tail and $p = E \cap \overline{(C_b \setminus E)}$, then *the index of* $E$ is defined to be $\operatorname{index}(p)$. Similarly, if $E \subset C_b$ is a rational bridge and $\{p, q\} = E \cap \overline{(C_b \setminus E)}$, then the index of $E$ is defined to be $\min\{\operatorname{index}(p), \operatorname{index}(q)\}$. We also denote the index of $E$ by $\operatorname{index}(E)$. We say that a rational bridge $E \subset C_b$ is *balanced* if $\operatorname{index}(p) = \operatorname{index}(q)$.

## 2.1. $\alpha$-stability.

For the reader's convenience, we recall the definition of $\alpha$-stability and the accompanying definitions of elliptic and Weierstrass tails and chains. The following definitions are taken almost verbatim from [AFSv15, Section 2.1].

**Definition 2.2.**

1. An *elliptic tail* is a 1-pointed curve $(E, q)$ of arithmetic genus 1 which admits a finite degree 2 map $\phi\colon E \to \mathbb{P}^1$ ramified at $q$.
2. An *elliptic bridge* is a 2-pointed curve $(E, q_1, q_2)$ of arithmetic genus 1 which admits a finite degree 2 map $\phi\colon E \to \mathbb{P}^1$ such that $\phi^{-1}(\{\infty\}) = \{q_1 + q_2\}$.
3. A *Weierstrass genus* 2 *tail* is a 1-pointed curve $(E, q)$ of arithmetic genus 2 which admits a finite degree 2 map $\phi\colon E \to \mathbb{P}^1$ ramified at $q$.

Recall that a *gluing morphism* $\gamma\colon (E, \{q_i\}_{i=1}^m) \to (C, \{p_i\}_{i=1}^n)$ between two pointed curves is a finite morphism $E \to C$, which is an open immersion when restricted to $E - \{q_1, \dots, q_m\}$. We do not require the points $\{\gamma(q_i)\}_{i=1}^m$ to be distinct, or to be marked points of $C$.

**Definition 2.3.** An *elliptic chain of length* $r$ is a 2-pointed curve $(E, p_1, p_2)$ which admits a surjective gluing morphism

$$\gamma\colon \coprod_{i=1}^{r}(E_i, q_{2i-1}, q_{2i}) \to (E, p_1, p_2)$$

such that:

1. $(E_i, q_{2i-1}, q_{2i})$ is an elliptic bridge for $i = 1, \dots, r$.
2. $\gamma(q_{2i}) = \gamma(q_{2i+1})$ is an $A_3$-singularity of $E$ for $i = 1, \dots, r-1$.
3. $\gamma(q_1) = p_1$ and $\gamma(q_{2r}) = p_2$.

A *Weierstrass chain of length* $r$ is a 1-pointed curve $(E, p)$ which admits a surjective gluing morphism

$$\gamma\colon \coprod_{i=1}^{r-1}(E_i, q_{2i-1}, q_{2i}) \coprod (E_r, q_{2r-1}) \to (E, p)$$

such that:

1. $(E_i, q_{2i-1}, q_{2i})$ is an elliptic bridge for $i = 1, \dots, r-1$, and $(E_r, q_{2r-1})$ is a Weierstrass tail.
2. $\gamma(q_{2i}) = \gamma(q_{2i+1})$ is an $A_3$-singularity of $E$ for $i = 1, \dots, r-1$.
3. $\gamma(q_1) = p$.

An elliptic (resp., Weierstrass) chain of length 1 is an elliptic bridge (resp., Weierstrass tail).

**Definition 2.4.** Let $(C, \{p_i\}_{i=1}^n)$ be an $n$-pointed curve. We say that $(C, \{p_i\}_{i=1}^n)$ has

1. *$A_k$-attached elliptic tail* if there is a gluing morphism $\gamma\colon (E, q) \to (C, \{p_i\}_{i=1}^n)$ such that
   (a) $(E, q)$ is an elliptic tail.
   (b) $\gamma(q)$ is an $A_k$-singularity of $C$, or if $k = 1$ we allow $\gamma(q)$ to be a marked point.
2. *$A_{k_1}/A_{k_2}$-attached elliptic chain* if there is a gluing morphism $\gamma\colon (E, q_1, q_2) \to (C, \{p_i\}_{i=1}^n)$ such that
   (a) $(E, q_1, q_2)$ is an elliptic chain.
   (b) $\gamma(q_i)$ is an $A_{k_i}$-singularity of $C$, or if $k_i = 1$ we allow $\gamma(q_i)$ to be a marked point $(i = 1, 2)$.
3. *$A_k$-attached Weierstrass chain* if there is a gluing morphism $\gamma\colon (E, q) \to (C, \{p_i\}_{i=1}^n)$ such that
   (a) $(E, q)$ is a Weierstrass chain.
   (b) $\gamma(q)$ is an $A_k$-singularity of $C$, or if $k = 1$ we allow $\gamma(q)$ to be a marked point.

This definition entails an essential, systematic abuse of notation: when we say that a curve has an $A_1$-attached tail or chain, we always allow the $A_1$-attachment points to be marked points.

**Definition 2.5.** We say that $(R, r_1, r_2)$ is a *rosary of length* $\ell$ if there exists a surjective gluing morphism

$$\gamma\colon \coprod_{i=1}^{\ell}(R_i, q_{2i-1}, q_{2i}) \hookrightarrow (R, r_1, r_2)$$

satisfying:

1. $(R_i, q_{2i-1}, q_{2i})$ is a 2-pointed smooth rational curve for $i = 1, \dots, \ell$.
2. $\gamma(q_{2i}) = \gamma(q_{2i+1})$ is an $A_3$-singularity of $R$ for $i = 1, \dots, \ell - 1$.

(3) $\gamma(q_1) = r_1$ and $\gamma(q_{2\ell}) = r_2$.

We say that $(C, \{p_i\}_{i=1}^n)$ has an *$A_{k_1}/A_{k_2}$-attached rosary of length $\ell$* if there exists a gluing morphism $\gamma\colon (R, r_1, r_2) \hookrightarrow (C, \{p_i\}_{i=1}^n)$ such that

(a) $(R, r_1, r_2)$ is a rosary of length $\ell$.
(b) For $j = 1, 2$, $\gamma(r_j)$ is an $A_{k_j}$-singularity of $C$, or if $k_j = 1$ we allow $\gamma(r_j)$ to be a marked point of $(C, \{p_i\}_{i=1}^n)$.

We say that $C$ is a *closed rosary of length $\ell$* if $C$ has $A_3/A_3$-attached rosary $\gamma\colon (R, r_1, r_2) \hookrightarrow C$ of length $\ell$ such that $\gamma(r_1) = \gamma(r_2)$ is an $A_3$-singularity of $C$.

**Definition 2.6** ($\alpha$-stability)**.** For $\alpha \in (2/3-\epsilon, 1]$, we say that an $n$-pointed curve $(C, \{p_i\}_{i=1}^n)$ is *$\alpha$-stable* if $\omega_C(\Sigma_{i=1}^n p_i)$ is ample and:

For $\alpha \in (9/11, 1)$: $C$ has only $A_1$-singularities.
For $\alpha = 9/11$: $C$ has only $A_1, A_2$-singularities.
For $\alpha \in (7/10, 9/11)$: $C$ has only $A_1, A_2$-singularities, and does not contain:
- $A_1$-attached elliptic tails.

For $\alpha = 7/10$: $C$ has only $A_1, A_2, A_3$-singularities, and does not contain:
- $A_1, A_3$-attached elliptic tails.

For $\alpha \in (2/3, 7/10)$: $C$ has only $A_1, A_2, A_3$-singularities, and does not contain:
- $A_1, A_3$-attached elliptic tails,
- $A_1/A_1$-attached elliptic chains.

For $\alpha = 2/3$: $C$ has only $A_1, A_2, A_3, A_4$-singularities, and does not contain:
- $A_1, A_3, A_4$-attached elliptic tails,
- $A_1/A_1, A_1/A_4, A_4/A_4$-attached elliptic chains.

For $\alpha \in (2/3-\epsilon, 2/3)$: $C$ has only $A_1, A_2, A_3, A_4$-singularities, and does not contain:
- $A_1, A_3, A_4$-attached elliptic tails,
- $A_1/A_1, A_1/A_4, A_4/A_4$-attached elliptic chains,
- $A_1$-attached Weierstrass chains.

A family of $\alpha$-stable curves is a flat and proper family whose geometric fibers are $\alpha$-stable. We let $\overline{\mathcal{M}}_{g,n}(\alpha)$ denote the stack of $n$-pointed $\alpha$-stable curves of arithmetic genus $g$. By [AFSv15, Theorem 2.7], we know that $\overline{\mathcal{M}}_{g,n}(\alpha)$ is an algebraic stack of finite type over $\mathbb{C}$. Since $\overline{\mathcal{M}}_{g,n}(\alpha)$ parameterizes unobstructed curves, it is a smooth algebraic stack and thus has a canonical divisor $K_{\overline{\mathcal{M}}_{g,n}(\alpha)}$. In addition, the universal sections $\{\sigma_i\}_{i=1}^n$ of the universal family $\pi\colon \mathcal{C} \to \overline{\mathcal{M}}_{g,n}(\alpha)$ give rise to the total $\psi$-class:

$$\psi = \sum_{i=1}^n \psi_{\sigma_i},$$

where the individual $\psi$-classes are defined as before by $\psi_{\sigma_i} = \sigma_i^*\big(\omega_{\mathcal{C}/\overline{\mathcal{M}}_{g,n}(\alpha)}\big)$. For brevity, we often write $\psi_i$ to denote $\psi_{\sigma_i}$.

Because non-nodal curves in $\overline{\mathcal{M}}_{g,n}(\alpha)$ form a closed substack of codimension 2, the standard relation on $\overline{\mathcal{M}}_{g,n}$ (cf. [Log03, Theorem 2.6]) gives

$$K_{\overline{\mathcal{M}}_{g,n}(\alpha)} = 13\lambda - 2\delta + \psi.$$

Since $\lambda, \delta$, and $\psi$ are defined on every $\overline{\mathcal{M}}_{g,n}(\alpha)$, we have the following formula

$$K_{\overline{\mathcal{M}}_{g,n}(\alpha_c \pm \epsilon)} = K_{\overline{\mathcal{M}}_{g,n}(\alpha_c)}|_{\overline{\mathcal{M}}_{g,n}(\alpha_c \pm \epsilon)}$$

for all $\alpha_c \in \{2/3, 7/10, 9/11\}$.

## 3. Degenerations and simultaneous normalization

Our first goal is to develop a theory of simultaneous (partial) normalization along generic singularities in families of at-worst tacnodal curves. In contrast to the situation for nodal curves, where normalization along a nodal section can always be performed because a node is not allowed to degenerate to a worse singularity, we must now deal with families where a node degenerates to a cusp or a tacnode, where two nodes degenerate to a tacnode, or where a cusp degenerates to a tacnode.

Recall the following definition from [AFSv15]:

**Definition 3.1** (Inner/Outer Singularities)**.** We say that an $A_k$-singularity $p \in C$ is *outer* if it lies on two distinct irreducible components of $C$, and *inner* if it lies on a single irreducible component. (N.B. If $k$ is even, then any $A_k$-singularity is necessarily inner.)

We begin by describing all possible degenerations of singularities in one-parameter families of tacnodal curves:

**Proposition 3.2.** *Suppose $\mathcal{C} \to \Delta$ is a family of at-worst tacnodal curves over $\Delta$, the spectrum of a DVR. Denote by $C_{\overline{\eta}}$ the geometric generic fiber and by $C_0$ the central fiber. Then the only possible limits in $C_0$ of the singularities of $C_{\overline{\eta}}$ are the following:*

1. *A limit of a tacnode of $C_{\overline{\eta}}$ is necessarily a tacnode of $C_0$. Moreover, a limit of an outer tacnode is necessarily an outer tacnode.*
2. *A limit of a cusp of $C_{\overline{\eta}}$ is either a cusp or a tacnode of $C_0$.*
3. *A limit of an inner node of $C_{\overline{\eta}}$ is either a node, a cusp, or a tacnode of $C_0$.*
4. *A limit of an outer node of $C_{\overline{\eta}}$ is either an outer node of $C_0$ or an outer tacnode of $C_0$. Moreover, if an outer tacnode of $C_0$ is a limit of an outer node, it must be a limit of two outer nodes, necessarily joining the same components.*

*Proof.* By deformation theory of $A$-singularities, a cusp deforms only to a node, a tacnode deforms only either to a cusp, or to a node, or to two nodes. Given this, the result follows directly from [AFSv15, Proposition 2.10]. □

We describe the operation of normalization along the generic singularities for each of the following degenerations:

(A) Inner nodes degenerate to cusps and tacnodes (see Proposition 3.4).
(B) Outer nodes degenerate to tacnodes (see Proposition 3.5).
(C) Cusps degenerate to tacnodes (see Proposition 3.6).

We begin with a preliminary result concerning normalization along a collection of generic nodes. Suppose $\pi \colon \mathcal{X} \to B$ is a family in $\widetilde{\mathcal{U}}_g(A_\infty)$ with sections $\{\sigma_i\}_{i=1}^k$ such that $\sigma_i(b)$ are distinct nodes of $\mathcal{X}_b$ for a generic $b \in B$ and such that $\{\sigma_i(B)\}_{i=1}^k$ do not meet any other generic singularities. (The last condition will be automatically satisfied when $\pi \colon \mathcal{X} \to B$ is a family in $\widetilde{\mathcal{U}}_g(A_3)$ and $\{\sigma_i\}_{i=1}^k$ is the collection of all inner or all outer nodes.) Let $\nu \colon \mathcal{Y} \to \mathcal{X}$ be the normalization of $\mathcal{X}$ along $\cup_{i=1}^{k}\sigma_i(B)$. Denote by $\{\eta_i^+, \eta_i^-\}$ the two preimages of $\sigma_i$ (which exist after a base change). Let $R_i^+\colon\ \nu_*\mathcal{O}_{\mathcal{Y}} \to \mathcal{O}_{\sigma_i(B)}$ (resp., $R_i^-\colon\ \nu_*\mathcal{O}_{\mathcal{Y}} \to \mathcal{O}_{\sigma_i(B)}$) be the morphisms of sheaves on $\mathcal{X}$ induced by pushing forward the restriction maps $\mathcal{O}_{\mathcal{Y}} \to \mathcal{O}_{\eta_i^{\pm}(B)}$ and composing with the natural isomorphisms $\nu_*(\mathcal{O}_{\eta_i^{\pm}(B)}) \simeq \mathcal{O}_{\sigma_i(B)}$. We let $R_i := R_i^+ - R_i^-$ be the difference map, and set

$$R := \oplus_{i=1}^{k} R_i\colon\ \nu_*\mathcal{O}_{\mathcal{Y}} \longrightarrow \oplus_{i=1}^{k}\mathcal{O}_{\sigma_i(B)}.$$

In this notation, we have the following result.

**Lemma 3.3.** *There is an exact sequence*

$$0 \to \mathcal{O}_{\mathcal{X}} \xrightarrow{\nu^{\#}} \nu_*\mathcal{O}_{\mathcal{Y}} \xrightarrow{R} \oplus_{i=1}^{k}\mathcal{O}_{\sigma_i(B)} \to \mathcal{K} \to 0, \tag{3.1}$$

*where $\mathcal{K}$ is supported on the finitely many points of $\mathcal{X}$ at which the generic nodes $\{\sigma_i(B)\}_{i=1}^{k}$ degenerate to worse singularities. Consequently,*

$$\lambda_{\mathcal{X}/B} = \lambda_{\mathcal{Y}/B} + \operatorname{length}(\pi_*\mathcal{K}).$$

*Proof.* Away from finitely many points on $\mathcal{X}$ where the generic nodes degenerate, we have $\operatorname{im}(\nu^{\#}) = \ker(R)$ and $R$ is surjective. Consider now a point $p \in \mathcal{X}$ where $a$ generic nodes coalesce to an $A_m$-singularity. A local chart of $\mathcal{X}$ around $p$ can be taken to be

$$\operatorname{Spec}\mathbb{C}[[x,y,t]]/\big(y^2 - (x-s_1(t))^2\cdots(x-s_a(t))^2 f(x,t)\big),$$

where $x = s_i(t)$ are the equations of generic nodes. By assumption on the generic nodes, $f(x,t)$ is a square-free polynomial. Hence

$$\mathcal{Y} = \operatorname{Spec}\mathbb{C}[[x,u,t]]/\big(u^2 - f(x,t)\big)$$

and the normalization map is given by

$$y \mapsto u\prod_{i=1}^{a}(x - s_i(t)).$$

Without loss of generality, the equation of $\eta_i^{\pm}$ is $u = \pm v_i(t)$, where $v_i(t)^2 = f(s_i(t),t)$. It follows that $R_i\colon\ \mathbb{C}[[x,u,t]]/(u^2 - f(x,t)) \to \mathbb{C}[[t]]$ is given by

$$R_i\big(g(x,u,t)\big) = g(s_i(t), v_i(t), t) - g(s_i(t), -v_i(t), t).$$

Write $\mathbb{C}[[x,u,t]]/(u^2 - f(x,t)) = \mathbb{C}[[x,t]] + u\mathbb{C}[[x,t]]$. Clearly, $\mathbb{C}[[x,t]] \subset \ker(R) \cap \operatorname{im}(\nu^{\#})$. Note that $ug(x,t) \in \ker(R)$ if and only if $R_i\big(ug(x,t)\big) = 2v_i(t)g(s_i(t),t) = 0$ for every $i = 1,\dots,a$ if and only if $g(x,t) \in (x - s_i(t))$ for every $i = 1,\dots,a$. Since the generic nodes are distinct, we conclude that $ug(x,t) \in \ker(R)$ if and only if $\prod_{i=1}^{a}(x - s_i(t)) \mid g(x,t)$ if and only if $ug(x,t) \in y\mathbb{C}[[x,t]] \subset \operatorname{im}(\nu^{\#})$. The exactness of (3.1) follows.

Pushing forward (3.1) to $B$ and noting that $c_1((\pi\circ\nu)_*\mathcal{O}_{\mathcal{Y}}) = c_1(\pi_*\mathcal{O}_X) = c_1(\pi_*\mathcal{O}_{\sigma_i(B)}) = 0$, we obtain

$$c_1(R^1(\pi\circ\nu)_*\mathcal{O}_{\mathcal{Y}}) = c_1(R^1\pi_*\mathcal{O}_{\mathcal{X}}) + c_1(\pi_*\mathcal{K}).$$

The formula relating Hodge classes now follows by relative Serre duality. □

**Proposition 3.4** (Type A degeneration). *Suppose $\mathcal{X}/B$ is a family in $\widetilde{\mathcal{U}}_g(A_3)$ with sections $\{\sigma_i\}_{i=1}^k$ such that $\sigma_i(b)$ are the inner nodes of $\mathcal{X}_b$ for a generic $b \in B$, degenerating to cusps and tacnodes over a finite set of points of $B$. Denote by $\mathcal{Y}$ the normalization of $\mathcal{X}$ along $\cup_{i=1}^k \sigma_i(B)$ and by $\{\eta_i^+, \eta_i^-\}$ the two preimages of $\sigma_i$. Then $\{\eta_i^{\pm}\}$ are sections of $\mathcal{Y}/B$ satisfying:*

1. *If $\sigma_i(b)$ is a cusp of $\mathcal{X}_b$, then $\eta_i^+(b) = \eta_i^-(b)$ is a smooth point of $\mathcal{Y}_b$.*
2. *If $\sigma_i(b)$ is a tacnode of $\mathcal{X}_b$ and $\sigma_j(b) \neq \sigma_i(b)$ for all $j \neq i$, then $\eta_i^+(b) = \eta_i^-(b)$ is a node of $\mathcal{Y}_b$ and $\eta_i^+ + \eta_i^-$ is Cartier at $b$.*
3. *If $\sigma_i(b) = \sigma_j(b)$ is a tacnode of $\mathcal{X}_b$ for some $i \neq j$, then (up to $\pm$) $\eta_i^+(b) = \eta_j^+(b)$ and $\eta_i^-(b) = \eta_j^-(b)$ are smooth and distinct points of $\mathcal{Y}_b$.*

*Set $\eta_i := \eta_i^+ + \eta_i^-$ and $\psi_{\eta_i} := \omega_{\mathcal{Y}/B} \cdot \eta_i = \psi_{\eta_i^+} + \psi_{\eta_i^-}$. Define*

$$\psi_{inner} := \sum_{i=1}^k \psi_{\eta_i}, \quad \delta_{tacn} := \sum_{i \neq j} \big(\eta_i \cdot \eta_j\big), \quad \textit{and} \quad \delta_{inner} = \sum_{i=1}^k (\eta_i^+ \cdot \eta_i^-).$$

*Then we have the following formulae:*

$$\lambda_{\mathcal{X}/B} = \lambda_{\mathcal{Y}/B} + \frac{1}{2}\delta_{tacn} + \delta_{inner} + \sum_{i=1}^k \iota(\eta_i^+),$$

$$\delta_{\mathcal{X}/B} = \delta_{\mathcal{Y}/B} - \psi_{inner} + 4\delta_{tacn} + 10\delta_{inner} + 10\sum_{i=1}^k \iota(\eta_i^+).$$

*A pair of sections $\{\eta_i^+, \eta_i^-\}$ arising from the normalization of a generic inner node will be called an* inner nodal pair *and $\eta_i^{\pm}$ will be called* inner nodal transforms.

*Proof.* The formula for the Hodge class follows from Lemma 3.3, whose notation we keep, once we analyze the torsion sheaf $\mathcal{K}$ on $\mathcal{X}$. Consider the following loci in $\mathcal{X}$:

(a) Cu is the locus of cusps in $\mathcal{X}/B$ which are limits of generic inner nodes.
(b) $\mathrm{Tn}_1$ is the locus of tacnodes in $\mathcal{X}/B$ which are limits of a single generic inner node.
(c) $\mathrm{Tn}_2$ is the locus of tacnodes in $\mathcal{X}/B$ which are limits of two generic inner nodes.

(a) A local chart of $\mathcal{X}$ around a point $p \in \mathrm{Cu}$ can be taken to be

$$\operatorname{Spec} \mathbb{C}[[x,y,t]]/\big(y^2 - (x-t^{2m})^2(x+2t^{2m})\big),$$

where $x = t^{2m}$ is the equation of the generic node $\sigma$ degenerating to the cusp $p$. Then $\mathcal{Y} = \operatorname{Spec} \mathbb{C}[[x,u,t]]/\big(u^2 - x - 2t^{2m}\big)$ and the normalization map is $y \mapsto u(x - t^{2m})$. The preimages $\eta^+$ and $\eta^-$ of the generic node $\sigma$ have equations $u = \sqrt{3}t^m$ and $u = -\sqrt{3}t^m$. Note that $\mathcal{Y}$ is smooth and the intersection multiplicity of $\eta^+$ and $\eta^-$ at the preimage of $p$ is $m$. It follows that the contribution of $p$ to $\delta_{inner}$ is $m$.

The elements of $\mathbb{C}[[x,u,t]]/\big(u^2 - x - 2t^{2m}\big)$ that do not lie in $\ker(R)$ are of the form $ug(x,t)$ and we have $R\big(ug(x,t)\big) = 2\sqrt{3}t^m g(t^{2m}, t)$. It follows that $\operatorname{im}(R) = (t^m) \subset \mathbb{C}[[t]]$. Hence $\mathcal{K}_p = \mathbb{C}[[t]]/\operatorname{im}(R)$ has length $m$.

(b) A local chart of $\mathcal{X}$ around a point $p \in \mathrm{Tn}_1$ can be taken to be

$$\operatorname{Spec} \mathbb{C}[[x,y,t]]/\big(y^2 - (x-t^m)^2(x^2+t^{2c})\big),$$

where $x = t^m$ is the equation of the generic node $\sigma$ degenerating to the tacnode $p$. Then

$$\mathcal{Y} = \operatorname{Spec} \mathbb{C}[[x,u,t]]/(u^2 - x^2 - t^{2c})$$

is a normal surface with $A_{2c-1}$-singularity at the preimage of $p$, and the normalization map is given by

$$y \mapsto u(x - t^m).$$

The preimage of $\sigma$ is the bi-section given by the equation $u^2 = t^{2m} + t^{2c}$, which splits into two sections given by the equations $u = \pm v(t)$, where the valuation of $v(t)$ is equal to $\min\{m, c\}$. The map $R\colon\ \mathbb{C}[[x,u,t]]/(u^2 - x^2 - t^{2c}) \to \mathbb{C}[[t]]$ sends an element of the form $ug(x,t)$ to $2v(t)g(t^m, t)$ and all elements of $\mathbb{C}[[x,t]]$ to 0. We conclude that $\mathcal{K}_p = \mathbb{C}[[t]]/\operatorname{im}(R)$ has length $\min\{m, c\}$.

It remains to show that the contribution of $p$ to $\big(\eta^+ \cdot \eta^- + \iota(\eta^+)\big)$ is $\min\{m, c\}$. There are two cases to consider. First, suppose $c \leq m$. Then the equations of $\eta^+$ and $\eta^-$ are $u = \alpha t^c$ and $u = -\alpha t^c$ where $\alpha$ is a unit in $\mathbb{C}[[t]]$. The minimal resolution $h\colon \widetilde{\mathcal{Y}} \to \mathcal{Y}$ has the exceptional divisor

$$E_1 \cup \cdots \cup E_{2c-1},$$

which is a chain of $(-2)$-curves. The strict transforms $\widetilde{\eta}^+$ and $\widetilde{\eta}^-$ meet the central $(-2)$-curve $E_c$ at two distinct points. Clearly, $h^*\omega_{\mathcal{Y}/B} = \omega_{\widetilde{\mathcal{Y}}/B}$ and a straightforward computation shows that

$$h^*(\eta^+ + \eta^-) = \widetilde{\eta}^+ + \widetilde{\eta}^- + \sum_{i=1}^{c-1} i(E_i + E_{2c-i}) + cE_c.$$

It follows that the contribution of $p$ to $\big(\eta^+ \cdot \eta^- + \iota(\eta^+)\big) = (\omega_{\mathcal{Y}/B} + \eta^+ + \eta^-)\cdot \eta^+$ is $c$.

Suppose now that $c > m$. Then the equations of $\eta^+$ and $\eta^-$ are $u = \alpha t^m$ and $u = -\alpha t^m$, respectively, where $\alpha$ is a unit in $\mathbb{C}[[t]]$. The exceptional divisor of the minimal resolution $h\colon \widetilde{\mathcal{Y}} \to \mathcal{Y}$ is still a chain of $(-2)$-curves of length $2c-1$. However, $\widetilde{\eta}^+$ and $\widetilde{\eta}^-$ now meet $E_m$ and $E_{2c-m}$, respectively. It follows that the contribution of $p$ to $\big(\eta^+ \cdot \eta^- + \iota(\eta^+)\big)$ is $m$.

(c) A local chart of $\mathcal{X}$ around a point $p \in \mathrm{Tn}_2$ can be taken to be

$$\operatorname{Spec} \mathbb{C}[[x,y,t]]/\big(y^2 - (x - t^m)^2(x + t^m)^2\big),$$

where $x = t^m$ and $x = -t^m$ are the equations of the generic nodes $\{\sigma_1, \sigma_2\}$ coalescing to the tacnode $p$. Then

$$\mathcal{Y} = \operatorname{Spec} \mathbb{C}[[x,u,t]]/(u^2 - 1)$$

is a union of two smooth sheets, and the normalization map is given by

$$y \mapsto u(x - t^m)(x + t^m).$$

The preimages $\eta_1^+$ and $\eta_1^-$ of the generic node $\sigma_1$ have equations $\{u = 1, x = t^m\}$ and $\{u = -1, x = t^m\}$. The preimages $\eta_2^+$ and $\eta_2^-$ of the generic node $\sigma_2$ have equations $\{u = 1, x = -t^m\}$ and $\{u = -1, x = -t^m\}$. In particular, $\eta_j^\pm$ are smooth sections, with $\eta_1^+$ meeting $\eta_2^+$, and $\eta_1^-$ meeting $\eta_2^-$, each with intersection multiplicity $m$. It follows that the contribution of $p$ to $\delta_{tacn}$ is $2m$.

The elements of $\mathbb{C}[[x,u,t]]/(u^2 - 1)$ that do not lie in $\ker(R)$ are of the form $ug(x,t)$ and we have $R(ug(x,t)) = (2g(t^m, t), 2g(-t^m, t)) \in \mathbb{C}[[t]] \times \mathbb{C}[[t]]$. It follows that

$$\operatorname{im}(R) = \langle (1,1), (t,t), \ldots, (t^{m-1}, t^{m-1})\rangle + (t^m) \times (t^m) \subset \mathbb{C}[[t]] \times \mathbb{C}[[t]].$$

Hence $\mathcal{K}_p = (\mathbb{C}[[t]] \times \mathbb{C}[[t]])/\mathrm{im}\,(R)$ has length $m$.

It remains to prove the formula for the boundary classes. To begin, note that $\nu^*\omega_{\mathcal{X}/B} = \omega_{\mathcal{Y}/B}\big(\sum_{i=1}^k(\eta_i^+ + \eta_i^-)\big)$. Therefore,

$$\begin{aligned}\kappa_{\mathcal{X}/B} = \kappa_{\mathcal{Y}/B} + 2\sum_{1\le i<j\le k}\big((\eta_i^+ + \eta_i^-)\cdot(\eta_j^+ + \eta_j^-)\big) + 2\omega_{\mathcal{Y}/B}\cdot\sum_{i=1}^k(\eta_i^+ + \eta_i^-) + \sum_{i=1}^k(\eta_i^+ + \eta_i^-)^2 \\ = \kappa_{\mathcal{Y}/B}+2\delta_{tacn}+\omega_{\mathcal{Y}/B}\cdot\sum_{i=1}^k(\eta_i^+ + \eta_i^-)+\sum_{i=1}^k\big(\omega_{\mathcal{Y}/B}\cdot\eta_i^+ + (\eta_i^+)^2 + \omega_{\mathcal{Y}/B}\cdot\eta_i^- + (\eta_i^-)^2\big)+2\sum_{i=1}^k\big(\eta_i^+\cdot\eta_i^-\big) \\ = \kappa_{\mathcal{Y}/B} + 2\delta_{tacn} + \psi_{inner} + 2\sum_{i=1}^k\iota(\eta_i^+) + 2\delta_{inner}.\end{aligned}$$

Using Mumford's relation $\kappa = 12\lambda - \delta$ and the already established relation between $\lambda_{\mathcal{X}/B}$ and $\lambda_{\mathcal{Y}/B}$, we obtain the desired relation between $\delta_{\mathcal{X}/B}$ and $\delta_{\mathcal{Y}/B}$. ☐

**Proposition 3.5** (Type B degeneration)**.** *Suppose $\mathcal{X}/B$ is a family in $\widetilde{\mathcal{U}}_g(A_3)$ with sections $\{\sigma_i\}_{i=1}^k$ such that $\sigma_i(b)$ are outer nodes of $\mathcal{X}_b$ for a generic $b \in B$, degenerating to outer tacnodes over a finite set of points of $B$. Denote by $\mathcal{Y}$ the normalization of $\mathcal{X}$ along $\cup_{i=1}^k\sigma_i(B)$ and by $\{\zeta_i^+, \zeta_i^-\}$ the two preimages of $\sigma_i$. Then $\{\zeta_i^\pm\}_{i=1}^k$ are smooth sections of $\mathcal{Y}$ such that $\zeta_i^+$ and $\zeta_i^-$ lie on different irreducible components of $\mathcal{Y}$. Setting*

$$\delta_{tacn} := \sum_{i\neq j}(\zeta_i^+ + \zeta_i^-)\cdot(\zeta_j^+ + \zeta_j^-),$$

*we have the following formulae:*

$$\begin{aligned}\lambda_{\mathcal{X}/B} &= \lambda_{\mathcal{Y}/B} + \frac{1}{2}\delta_{tacn},\\ \delta_{\mathcal{X}/B} &= \delta_{\mathcal{Y}/B} - \sum_{i=1}^k(\psi_{\zeta_i^+} + \psi_{\zeta_i^-}) + 4\delta_{tacn}.\end{aligned}$$

*The sections $\{\zeta_i^+, \zeta_i^-\}_{i=1}^k$ will be called* outer nodal transforms.

*Proof.* By Proposition 3.2, outer nodes can degenerate only to outer tacnodes. Moreover, an outer tacnode which is a limit of one outer node is a limit of two outer nodes. The statement now follows by repeating verbatim the proof of Proposition 3.4 (Part (c)), and using Lemma 3.3. ☐

**Proposition 3.6** (Type C degeneration)**.** *Suppose $\mathcal{X}/B$ is a family in $\widetilde{\mathcal{U}}_g(A_3)$ with sections $\{\sigma_i\}_{i=1}^k$ such that $\sigma_i(b)$ is a cusp of $\mathcal{X}_b$ for a generic $b \in B$, degenerating to a tacnode over a finite set of points in $B$. Denote by $\mathcal{Y}$ the normalization of $\mathcal{X}$ along $\cup_{i=1}^k\sigma_i(B)$ and by $\xi_i$ the preimage of $\sigma_i$. Then $\xi_i$ is a section of $\mathcal{Y}/B$ such that $\xi_i(b)$ is a node of $\mathcal{Y}_b$ whenever $\sigma_i(b)$ is a tacnode of $\mathcal{X}_b$ and $\xi_i(b)$ is a smooth point of $\mathcal{Y}_b$ otherwise. Moreover, $2\xi_i$ is Cartier and we have* *the following formulae:*

$$\lambda_{\mathcal{X}/B} = \lambda_{\mathcal{Y}/B} - \sum_{i=1}^{k} \psi_{\xi_i} + 2\sum_{i=1}^{k} \iota(\xi_i),$$
$$\delta_{\mathcal{X}/B} = \delta_{\mathcal{Y}/B} - 12\sum_{i=1}^{k} \psi_{\xi_i} + 20\sum_{i=1}^{k} \iota(\xi_i).$$

*The sections $\xi_i$ will be called* cuspidal transforms*.*

*Proof.* The proof of this proposition is easier than the previous two results because a generic cusp cannot collide with another generic singularity. In particular, we can consider the case of a single generic cusp $\sigma$. Let $\nu\colon \mathcal{Y} \to \mathcal{X}$ be the normalization along $\sigma$. Suppose $\sigma(b)$ is a tacnode. Then the local equation of $\mathcal{X}$ around $\sigma(b)$ is

$$y^2 = (x - a(t))^3(x + 3a(t)),$$

where $x = a(t)$ is the equation of the generic cusp. It follows that $\mathcal{Y}$ has local equation $u^2 = (x - a(t))(x + 3a(t))$ and $\nu$ is given by $y \mapsto u(x - a(t))$. The preimage of $\sigma$ is a section $\xi\colon B \to \mathcal{Y}$ given by $x - a(t) = u = 0$. Note that $\xi(b) = \{x = u = t = 0\}$ is a node of $\mathcal{Y}_b$, and consequently $\xi$ is not Cartier at $\xi(b)$.

Clearly, $\nu^*\omega_{\mathcal{X}/B} = \omega_{\mathcal{Y}/B}(2\xi)$ and by duality theory for singular curves

$$\pi_*\omega_{\mathcal{X}/B} = (\pi \circ \nu)_*(\omega_{\mathcal{Y}/B}(2\xi)).$$

Therefore,

$$\kappa_{\mathcal{X}/B} = (\omega_{\mathcal{Y}/B} + 2\xi)^2 = (\omega_{\mathcal{Y}/B})^2 + 4(\xi^2 + \xi \cdot \omega_{\mathcal{Y}/B}) = \kappa_{\mathcal{Y}/B} + 4\iota(\xi),$$

and by Grothendieck-Riemann-Roch formula

$$\lambda_{\mathcal{X}/B} = c_1\big((\pi \circ \nu)_*(\omega_{\mathcal{Y}/B}(2\xi))\big) = \lambda_{\mathcal{Y}/B} - \psi_\xi + 2\iota(\xi).$$

The claim follows. □

## 4. Assorted positivity results

4.1. **Cornalba-Harris inequality.** We generalize a well-known Cornalba-Harris result on positivity of divisor classes for generically smooth families of Deligne-Mumford curves to the case of tacnodal curves.

**Proposition 4.1** (Cornalba-Harris inequality)**.** *Let $g \geq 2$. Suppose $f\colon \mathcal{C} \to B$ is a generically smooth family in $\widetilde{\mathcal{U}}_g(A_3)$, over a smooth and proper curve $B$, with $\omega_{\mathcal{C}/B}$ relatively nef. Then*

$$\left(8 + \frac{4}{g}\right)\lambda_{\mathcal{C}/B} - \delta_{\mathcal{C}/B} \geq 0.$$

*Moreover, if the general fiber of $\mathcal{C}/B$ is non-hyperelliptic and $\mathcal{C}/B$ is non-isotrivial, then the inequality is strict.*

*Remark.* When the total space $\mathcal{C}$ is *smooth*, this result was proved in [Xia87] and [Sto08, Theorem 2.1], with no restrictions on fiber singularities.

*Proof of Proposition 4.1 in the non-hyperelliptic case:* As in [Sto08, Theorem 2.1], if the general fiber of $\mathcal{C}/B$ is non-hyperelliptic, the result is obtained by the original argument of Cornalba and Harris [CH88], which we now recall.

Suppose $C_b$ for some $b \in B$ is a non-hyperelliptic curve of genus $g \geq 3$. After a finite base change, we can assume that $\lambda \in \operatorname{Pic}(B)$ is $g$-divisible. Then the line bundle $\mathcal{L} := \omega_{\mathcal{C}/B} \otimes f^*(-\lambda/g)$ on $\mathcal{C}$ satisfies the following conditions:

1. $\det(f_*(\mathcal{L})) \simeq \mathcal{O}_B$.
2. $f_*(\mathcal{L}^m)$ is a vector bundle of rank $(2m-1)(g-1)$ for all $m \geq 2$.
3. $\operatorname{Sym}^m f_*(\mathcal{L}) \to f_*(\mathcal{L}^m)$ is generically surjective for all $m \geq 1$.

For $m \geq 2$ and general $b \in B$, the map $\operatorname{Sym}^m \mathrm{H}^0(C_b, \omega_{C_b}) \to \mathrm{H}^0(C_b, \omega_{C_b}^m)$ defines the $m^{th}$ Hilbert point of $C_b$. Since the canonical embedding of $C_b$ has a stable $m^{th}$ Hilbert point for some $m \gg 0$ by [Mor09, Lemma 14], the proof of [CH88, Theorem 1.1] gives $c_1(f_*(\mathcal{L}^m)) \geq 0$. Using (2.1), we obtain

$$\left(8 + \frac{4}{g} - \frac{2(g-1)}{gm} + \frac{2}{gm(m-1)}\right)\lambda - \delta = c_1(f_*(\mathcal{L}^m)) \geq 0. \tag{4.1}$$

To conclude we note that $\delta \geq 0$, and if $\delta = 0$, then $\lambda > 0$ for any non-isotrivial family by the existence of the Torelli morphism $\mathcal{M}_g \to \mathcal{A}_g$. We conclude that $(8 + 4/g)\lambda - \delta > 0$. □

4.1.1. *Families of $2g+2$-pointed rational curves.* Suppose that $(\mathcal{Y}/B, \{\sigma_i\}_{i=1}^{2g+2})$ is a family of $(2g+2)$-pointed at-worst nodal rational curves where $\sigma_i$ are smooth sections, $\omega_{\mathcal{Y}/B}(\sum_{i=1}^{2g+2}\sigma_i)$ is relatively nef, and no more than 4 sections meet at a point. We say that an irreducible component $E$ in the fiber $Y_b$ of $\mathcal{Y}/B$ is an odd bridge if the following conditions hold:

1. $E$ meets the rest of the fiber $\overline{Y_b \setminus E}$ in two nodes of equal index,
2. $E \cdot \sum_{i=1}^{2g+2}\sigma_i = 2$,
3. the degree of $\sum_{i=1}^{2g+2}\sigma_i$ on each of the two connected components of $\overline{Y_b \setminus E}$ is odd.

Suppose $h\colon \mathcal{Y} \to \mathcal{Z}$ is a blow-down of some collection of odd bridges. The image of $\sum_{i=1}^{2g+2}\sigma_i$ in $\mathcal{Z}$ will be denoted by $\Sigma$. Note that while the individual images of $\sigma_i$'s are not Cartier on $\mathcal{Z}$ along the image of blown-down odd bridges, the total class of $\Sigma$ is Cartier on $\mathcal{Z}$. We say that a node $p \in Z_b$ (resp., $p \in Y_b$) is an odd node if the degree of $\Sigma$ (resp., $\sum_{i=1}^{2g+2}\sigma_i$) on each of the connected component of the normalization of $Z_b$ (resp., $Y_b$) at $p$ is odd. We denote by $\delta_{\text{odd}}$ the Cartier divisor on $B$ associated to all odd nodes of $\mathcal{Z}/B$ (resp., $\mathcal{Y}/B$).

*Proof of Proposition 4.1 in the hyperelliptic case:* Suppose now that $\mathcal{C} \to B$ is a family of at-worst tacnodal curves with a relatively nef $\omega_{\mathcal{C}/B}$ and a smooth hyperelliptic generic fiber. The hyperelliptic involution on the generic fiber of $f\colon \mathcal{C} \to B$ extends to all of $\mathcal{C}$ and realizes $\mathcal{C}/B$ as a double cover of a family $(\mathcal{Z}/B, \Sigma)$ described in §4.1.1 in such a way that $\mathcal{C} \to \mathcal{Z}$ ramifies over $\Sigma$. Let $\delta_{\text{odd}}$ be the divisor of odd nodes of $\mathcal{Z}/B$. We have the following standard formulae:

$$\begin{aligned}
\lambda_{\mathcal{C}/B} &= \frac{1}{8}\left(\Sigma^2 + 2\omega_{\mathcal{Z}/B}\cdot\Sigma - \delta_{odd}\right)_{\mathcal{Z}/B}, \\
\delta_{\mathcal{C}/B} &= \left(\Sigma^2 + \omega_{\mathcal{Z}/B}\cdot\Sigma + 2\omega_{\mathcal{Z}/B}^2 - \frac{3}{2}\delta_{odd}\right)_{\mathcal{Z}/B}.
\end{aligned}$$

Consider $h\colon \mathcal{Y}\to\mathcal{Z}$. Then $h^*(\Sigma)=\sum_{i=1}^{2g+2}\sigma_i+E$, where $E$ is a collection of odd bridges, and $h^*\omega_{\mathcal{Z}/B}=\omega_{\mathcal{Y}/B}$. Set $\psi_{\mathcal{Y}/B}:=\omega_{\mathcal{Y}/B}\cdot\sum_{i=1}^{2g+2}\sigma_i$, $\delta_{inner}:=\sum_{i\neq j}(\sigma_i\cdot\sigma_j)$, and $e:=-\frac{1}{2}E^2$. Then

$$\lambda_{\mathcal{C}/B}=\left(\frac{1}{8}(\psi_{\mathcal{Y}/B}+2\delta_{inner}-\delta_{odd})+\frac{1}{2}e\right)_{\mathcal{Y}/B},$$

$$\delta_{\mathcal{C}/B}=\left(2\delta_{inner}+2\delta_{even}+\frac{1}{2}\delta_{odd}+5e\right)_{\mathcal{Y}/B}.$$

We obtain

$$\left(8+\frac{4}{g}\right)\lambda_{\mathcal{C}/B}-\delta_{\mathcal{C}/B}=\left(\frac{2g+1}{2g}\psi+\frac{1}{g}\delta_{inner}+\left(\frac{2}{g}-1\right)e-2\delta_{even}-\left(\frac{3}{2}+\frac{1}{2g}\right)\delta_{odd}\right)_{\mathcal{Y}/B}.$$

Multiplying by $2g$, we reduce to proving that on $\mathcal{Y}/B$ we have

$$(2g+1)\psi+2\delta_{inner}-4g\delta_{even}-(3g+1)\delta_{odd}-(2g-4)e\geq 0.$$

Noting that for any family $\mathcal{Y}/B$ as in §4.1.1, we have

$$(2g+1)\psi+2\delta_{inner}=\sum_{i=1}^{g+1}i(2g+2-i)\delta_i,$$

and using the inequality $2e\leq\delta_{odd}$, we obtain the desired claim. □

**Hodge Index Theorem Inequalities.** We apply a method of Harris [Har84] to obtain inequalities between the $\psi$-classes, indices of cuspidal and inner nodal transforms, and the $\kappa$ class. In the following lemmas, we use the following variant of Hodge Index Theorem for singular surfaces.

**Lemma 4.2.** *Let $S$ be a proper integral algebraic space of dimension 2. Suppose $H$ is a $\mathbb{Q}$-Cartier divisor on $S$ such that $H^2>0$. Then the intersection pairing on any $d$-dimensional subspace of $\mathrm{NS}(S)$ containing $H$ has signature $(1,d-1)$.*

*Proof.* Let $\pi\colon\widetilde{S}\to S$ be the minimal desingularization of the normalization of $S$. Then $\widetilde{S}$ is a smooth projective surface. Note that $\pi^*\colon\mathrm{NS}(S)\to\mathrm{NS}(\widetilde{S})$ is an injection preserving the intersection pairing. The statement now follows from the Hodge Index Theorem for smooth projective surfaces. □

**Lemma 4.3.** *Suppose $\mathcal{X}/B$ is a family of Gorenstein curves of arithmetic genus $g\geq 2$ with a section $\xi$. Assume $\mathcal{X}$ is irreducible. Let $\iota(\xi)=(\xi+\omega_{\mathcal{X}/B})\cdot\xi$ be the index of $\xi$. Then*

$$\psi_\xi\geq\frac{(g-1)}{g}\iota(\xi)+\frac{\kappa}{4g(g-1)}.\tag{4.2}$$

*Proof.* Apply the Hodge Index Theorem to the three divisor classes $\langle F,\xi,\omega_{\mathcal{X}/B}\rangle$, where $F$ is the fiber class. Since $\xi+kF$ has positive self-intersection for $k\gg 0$, the determinant of the following intersection pairing matrix is non-negative:

$$\begin{pmatrix}0 & 1 & 2g-2\\ 1 & -\psi_\xi+\iota(\xi) & \psi_\xi\\ 2g-2 & \psi_\xi & \kappa\end{pmatrix}.$$

The claim follows by expanding the determinant. □

**Lemma 4.4.** *Suppose $\mathcal{X}/B$ is a family of Gorenstein curves of arithmetic genus $g \geq 2$ with a pair of sections $\eta^+, \eta^-$. Assume $\mathcal{X}$ is irreducible. Then*

$$\psi_{\eta^+} + \psi_{\eta^-} \geq \frac{2(g-1)}{g+1}\big((\eta^+ \cdot \eta^-) + \iota(\eta^+)\big) + \frac{\kappa}{g^2-1}. \tag{4.3}$$

*Proof.* Consider the three divisor classes $\langle F, \eta = \eta^+ + \eta^-, \omega_{\mathcal{X}/B}\rangle$, where $F$ is the fiber class. Since $\eta + kF$ has positive self-intersection for $k \gg 0$, the Hodge Index Theorem implies that the determinant of the following intersection pairing matrix is non-negative:

$$\begin{pmatrix} 0 & 2 & 2g-2 \\ 2 & -\psi_{\eta^+} - \psi_{\eta^-} + 2(\eta^+ \cdot \eta^-) + \iota(\eta^+) + \iota(\eta^-) & \psi_{\eta^+} + \psi_{\eta^-} \\ 2g-2 & \psi_{\eta^+} + \psi_{\eta^-} & \kappa \end{pmatrix}.$$

The claim follows by expanding the determinant. □

**Lemma 4.5.** *Suppose $\mathcal{X}/B$ is a family in $\widetilde{\mathcal{U}}_2(A_3)$ with a smooth section $\tau$. Assume $\mathcal{X}$ is irreducible. Then*

$$8\psi_\tau \geq \kappa. \tag{4.4}$$

*Moreover, if $\delta_{\mathrm{red}} = 0$, then the equality is satisfied if and only if $(\mathcal{X}/B, \tau)$ is a family of Weierstrass genus 2 tails in $\overline{\mathcal{M}}_{2,1}(7/10 - \epsilon)$.*

*Proof.* The inequality follows directly from Lemma 4.3 by taking $g = 2$. Moreover, the proof of Lemma 4.3 shows that equality holds if and only if the intersection pairing on $\langle F, \tau, \omega_{\mathcal{X}/B}\rangle$ is degenerate. Assuming $\delta_{\mathrm{red}} = 0$, there is a global hyperelliptic involution $h\colon \mathcal{X} \to \mathcal{X}$. Hence $\omega_{\mathcal{X}/B} \equiv \tau + h(\tau) + xF$, for some $x \in \mathbb{Z}$. Observe that $\omega_{\mathcal{X}/B} \cdot \tau = \omega_{\mathcal{X}/B} \cdot h(\tau)$ and $F \cdot \tau = F \cdot h(\tau)$. Since no combination of $\omega$ and $F$ is in the kernel of the intersection pairing, we conclude that

$$\tau^2 = \tau \cdot h(\tau).$$

However, the intersection number on the left is negative by Lemma 2.1 and the intersection number on the right is non-negative whenever $\tau \neq h(\tau)$. We conclude that equality holds if only if $h(\tau) = \tau$, that is $\tau$ is a Weierstrass section. □

We will need special variants of Lemmas 4.3 and 4.4 for the case of relative genus 1 and 0.

**Lemma 4.6.** *Let $\mathcal{X}/B$ be a family of Gorenstein curves of arithmetic genus 1 with a pair of sections $\eta^+, \eta^-$, and suppose that $\eta^+$ and $\eta^-$ are disjoint from $N$ smooth pairwise disjoint section of $\mathcal{X}/B$. Assume $\mathcal{X}$ is irreducible. Then*

$$(\eta^+ \cdot \eta^-) + \iota(\eta^+) \leq \frac{N+2}{2N}(\psi_{\eta^+} + \psi_{\eta^-}) + \frac{1}{2N^2}\delta_{\mathrm{red}}.$$

*Proof.* Let $\Sigma$ be the sum of $N$ pairwise disjoint smooth sections of $\mathcal{X}/B$ disjoint from $\{\eta^+, \eta^-\}$. Then $(\omega_{\mathcal{X}/B} + 2\Sigma)^2 = \omega_{\mathcal{X}/B}^2 = \kappa$. Apply the Hodge Index Theorem to $\langle F, \eta^+ + \eta^-, \omega_{\mathcal{X}/B} + 2\Sigma\rangle$, where $F$ is the fiber class. The determinant of the matrix

$$\begin{pmatrix} 0 & 2 & 2N \\ -\psi_{\eta^+} - \psi_{\eta^-} + 2(\eta^+ \cdot \eta^-) + \iota(\eta^+) + \iota(\eta^-) & \psi_{\eta^+} + \psi_{\eta^-} & \\ 2N & \psi_{\eta^+} + \psi_{\eta^-} & \kappa \end{pmatrix}$$

is non-negative. Therefore

$$-4\kappa + 8N(\psi_{\eta^+} + \psi_{\eta^-}) + 4N^2(\psi_{\eta^+} + \psi_{\eta^-}) \geq 8N^2\big((\eta^+ \cdot \eta^-) + \iota(\eta^+)\big),$$

which gives the desired inequality using $\kappa = -\delta_{\mathrm{red}}$. □

**Lemma 4.7.** *Let $\mathcal{X}/B$ be a family of Gorenstein curves of arithmetic genus 1 with a section $\xi$, and suppose that $\xi$ is disjoint from $N$ smooth pairwise disjoint sections of $\mathcal{X}$. Assume $\mathcal{X}$ is irreducible. Then*

$$\iota(\xi) \leq \frac{N+1}{N}\psi_\xi + \frac{1}{4N^2}\delta_{\mathrm{red}}.$$

*Furthermore, suppose $N = 1$, with $\tau$ being a smooth section disjoint from $\xi$, and $\delta_{\mathrm{red}} = 0$. Then equality holds if and only if $2\xi \sim 2\tau$.*

*Proof.* Let $\Sigma$ be the collection of smooth sections of $\mathcal{X}/B$ disjoint from $\xi$. By the Hodge Index Theorem applied to $\langle F, \xi, \omega_{\mathcal{X}/B} + 2\Sigma\rangle$, the determinant of the matrix

$$\begin{pmatrix} 0 & 1 & 2N \\ 1 & -\psi_\xi + \iota(\xi) & \psi_\xi \\ 2N & \psi_\xi & \kappa \end{pmatrix}$$

is non-negative. Therefore

$$\iota(\xi) \leq \psi_\xi + \frac{1}{N}\psi_\xi - \frac{1}{4N^2}\kappa. \tag{4.5}$$

This gives the desired inequality using $\kappa = -\delta_{\mathrm{red}}$.

To prove the last assertion observe that because $\delta_{\mathrm{red}} = 0$ all fibers of $\mathcal{X}/B$ are irreducible curves of genus 1. In particular, $\omega_{\mathcal{X}/B} \equiv \lambda F$ and it follows from the existence of the group law on the set of sections of $\mathcal{X}/B$ that there exists a section $\tau'$ such that $2\xi - \tau \equiv \tau'$. Since $\tau \cap \xi = \varnothing$, we have $\tau' \cap \xi = \varnothing$. If equality in (4.5) holds, then the intersection pairing matrix on the classes $\langle F, \xi, \tau\rangle$ is degenerate. Hence some linear combination $(x\xi + y\tau + zF)$ intersects $F, \xi, \tau$ trivially. Clearly, $y \neq 0$. Intersecting with $\tau$, we obtain $y(\tau \cdot \tau) + z = 0$; and intersecting with $\tau'$, we obtain $y(\tau \cdot \tau') + z = 0$. Hence $\tau^2 = \tau \cdot \tau'$. This leads to a contradiction if $\tau \neq \tau'$. □

## 4.2. An inequality between divisor classes on $\overline{M}_{0,N}$.

The proof of Theorem 1.2 will require the following effectivity result on $\overline{M}_{0,N}$.

**Lemma 4.8.** *Suppose $(\mathcal{C}/B; \{\sigma_i\}_{i=1}^N)$ is a one-parameter family in $\overline{M}_{0,N}$ with generically smooth fibers. Suppose $\{\eta_i^+, \eta_i^-\}_{i=1}^a$ is a subset of $\{\sigma_i\}_{i=1}^N$. Set $\psi_{inner} := \sum_{i=1}^a \left(\psi_{\eta_i^+} + \psi_{\eta_i^-}\right)$ and $\delta_{inner} := \sum_{i=1}^a \delta_{\{\eta_i^+, \eta_i^-\}}$. If $a \geq 2$, then we have*

$$\psi_{inner} \geq 4\delta_{inner} + 4\sum_{i=1}^a \sum_{\beta \notin \{\eta_i^+, \eta_i^-\}_{i=1}^a} \delta_{\{\eta_i^+, \eta_i^-, \beta\}} + 2\frac{a-2}{a-1}\sum_{i \neq j} \delta_{\{\eta_i^\pm, \eta_j^\pm\}} + \frac{5a-9}{a-1}\sum_{i=1}^a \sum_{j \neq i} \delta_{\{\eta_i^+, \eta_i^-, \eta_j^\pm\}}. \tag{4.6}$$

*Proof.* Since the generic point of $B$ maps to the interior $M_{0,N} \subset \overline{M}_{0,N}$, it suffices to show that the left-hand side of (4.6) is an effective combination of the boundary divisors on $\overline{M}_{0,N}$. To do so, we apply the Effective Boundary Lemma [Fed14, Lemma 2.3.3]. Let $V$ be the set of $N$ elements identified with the set $\{\sigma_i\}_{i=1}^N$. We define the following weighting on the edges of the complete graph with vertices in $V$:

(1) The edges joining $\eta_i^\pm$ and $\eta_j^\pm$ for $1 \leq i < j \leq a$ will have weight 1.
(2) The edge joining $\eta_i^+$ and $\eta_i^-$ will have weight $-(a-1)$.
(3) All other edges will have weight 0.

Using the terminology of [Fed14, Section 2.2], the flow through the vertex $\eta_i^\pm$ is $a-1$ for every $1 \leq i \leq a$, and the flow through all other vertices is 0.

Suppose $I \sqcup J = V$ is a partition with $|I|, |J| \geq 2$. Let $x$ be the number of pairs $\{\eta_i^+, \eta_i^-\}$ such that $\eta_i^+ \in I$ and $\eta_i^- \in J$, or vice versa, and let $y$ be the number of pairs $\{\eta_i^+, \eta_i^-\}$ contained entirely in $I$ or $J$. Set $z = a - x - y$. Then the flow across $I \sqcup J$ is $w(I \mid J) = ((x+2y)(x+2z) - x) - (a-1)x = x(y+z) + 4yz$. We see that

(1) $w(I \mid J) \geq 0$ for every $I \sqcup J$.
(2) If $I = \{\eta_i^+, \eta_i^-\}$ or $J = \{\eta_i^+, \eta_i^-, \beta\}$, where $\beta \notin \{\eta_i^+, \eta_i^-\}_{i=1}^a$, then $x = 0$ and $y = 1$, and so $w(I \mid J) = 4(a-1)$.
(3) If $I = \{\eta_i^\pm, \eta_j^\pm\}$ for $i \neq j$, then $x = 2$ and $y = 0$, and so $w(I \mid J) = 2(a-2)$.
(4) If $I = \{\eta_i^+, \eta_i^-, \eta_j^\pm\}$ for $j \neq i$, then $x = 1$ and $y = 1$, and so $w(I \mid J) = 5a - 9$.

It follows that the left-hand side of (4.6) is an effective combination of the boundary divisors by [Fed14, Lemma 2.3.3]. ☐

## 5. Proof of Theorem 1.2(a)

In this section, we give an intersection-theoretic proof of Theorem 1.2(a). At the end of the section, we also include a short argument deducing Theorem 1.2(a) from the results of [HH09, HH13].

To begin, notice that $10\lambda - \delta + \psi = 0$ on $\overline{\mathcal{M}}_{2,0}(9/11 - \epsilon)$ by the standard relation $10\lambda = \delta_{\mathrm{irr}} + 2\delta_{\mathrm{red}}$ that holds for all families in $\mathcal{U}_{2,0}(A_\infty)$.

We now prove that $10\lambda - \delta + \psi$ is nef on $\overline{\mathcal{M}}_{g,n}(9/11 - \epsilon)$ and has degree 0 precisely on families whose only non-isotrivial components are $A_1/A_1$-attached elliptic bridges, for all $(g, n) \neq (2, 0)$. Let $(\mathcal{C}/B, \{\sigma_i\}_{i=1}^n)$ be a $(9/11 - \epsilon)$-stable family. The proof proceeds by normalizing $\mathcal{C}$ along generic singularities to arrive at a family of generically smooth curves, where the Cornalba-Harris inequality holds, or at a family of low genus curves, where the requisite inequality is established by ad-hoc methods. Keeping in mind that generic outer nodes and generic cusps of $\mathcal{C}/B$ do not degenerate, but generic inner nodes of $\mathcal{C}/B$ can degenerate to cusps, we begin by normalizing generic outer nodes, then normalize generic cusps, and finally normalize generic inner nodes.

### 5.1. Reduction 1: Normalization along generic outer nodes.

Let $\mathcal{X}$ be the normalization of $\mathcal{C}$ along generic outer nodes, marked by nodal transforms. By [AFSv15, Lemma 2.17], every connected component of $\mathcal{X}/B$ is a family of generically irreducible $(9/11 - \epsilon)$-stable curves. By Proposition 3.5, we have

$$(10\lambda - \delta + \psi)_{\mathcal{C}/B} = (10\lambda - \delta + \psi)_{\mathcal{X}/B}.$$

We have reduced to proving $10\lambda - \delta + \psi \geq 0$ for a family with generically irreducible fibers.

5.2. **Reduction 2: Normalization along generic cusps.** Suppose $(\mathcal{X}/B, \{\sigma_i\}_{i=1}^n)$ is a family of $(9/11-\epsilon)$-stable curves with generically irreducible fibers. Let $\mathcal{Y}$ be the normalization of $\mathcal{X}$ along generic cusps. Denote by $\{\xi_i\}_{i=1}^c$ the cuspidal transforms on $\mathcal{Y}$. Set $\psi_{cusp} := \sum_{i=1}^c \psi_{\xi_i}$ and $\psi_{\mathcal{Y}/B} := \psi_{\mathcal{X}/B} + \psi_{cusp}$. Then by Proposition 3.6, we have

$$(10\lambda - \delta + \psi)_{\mathcal{X}/B} = (10\lambda - \delta + \psi)_{\mathcal{Y}/B} + \psi_{cusp}.$$

We have reduced to proving $10\lambda - \delta + \psi + \psi_{cusp} \geq 0$ for a family $(\mathcal{Y}/B, \{\sigma_i\}_{i=1}^n, \{\xi_i\}_{i=1}^c)$, where

1. The fibers are at-worst cuspidal and the generic fiber is irreducible and at-worst nodal.
2. $\{\sigma_i\}_{i=1}^n, \{\xi_i\}_{i=1}^c$ are smooth sections and $\omega_{\mathcal{Y}/B}(\sum_{i=1}^n \sigma_i + \sum_{i=1}^c \xi_i)$ is relatively ample. (**N.B.** *A priori* only $\omega_{\mathcal{Y}/B}(\sum_{i=1}^n \sigma_i + 2\sum_{i=1}^c \xi_i)$ is relatively ample. However, a rational tail cannot meet only a single cuspidal transform because the original family $\mathcal{X}/B$ cannot have cuspidal elliptic tails.)

5.3. **Reduction 3: Normalization along generic inner nodes.** Consider the family $(\mathcal{Y}/B, \{\sigma_i\}_{i=1}^n, \{\xi_i\}_{i=1}^c)$ as in Subsection 5.2. Let $a$ be the number of generic inner nodes of $\mathcal{Y}/B$. We let $\mathcal{Z} \to \mathcal{Y}$ be the normalization and denote by $\eta_i^+$ and $\eta_i^-$ the inner nodal transforms of the $i^{th}$ generic node. We obtain a family

$$\big(\mathcal{Z}/B, \{\sigma_i\}_{i=1}^n, \{\eta_i^{\pm}\}_{i=1}^a, \{\xi_i\}_{i=1}^c\big),$$

where

1. The fibers are at-worst cuspidal curves and the generic fiber is smooth.
2. The sections $\{\sigma_i\}_{i=1}^n, \{\eta_i^{\pm}\}_{i=1}^a, \{\xi_i\}_{i=1}^c$ are all smooth and pairwise disjoint, except that $\eta_i^+$ can intersect $\eta_i^-$ for each $i$.
3. $\omega_{\mathcal{Z}/B}\left(\sum_{i=1}^n \sigma_i + \sum_{i=1}^a(\eta_i^+ + \eta_i^-) + \sum_{i=1}^c \xi_i\right)$ is relatively ample.

By Proposition 3.4, we have that

$$(10\lambda - \delta + \psi + \psi_{cusp})_{\mathcal{Y}/B} = (10\lambda - \delta + \psi + \psi_{cusp})_{\mathcal{Z}/B},$$

where $\psi_{cusp} = \sum_{i=1}^c \psi_{\xi_i}$ and $\psi_{\mathcal{Z}/B} = \psi_{\mathcal{Y}/B} + \sum_{i=1}^a(\psi_{\eta_i^+} + \psi_{\eta_i^-})$.

We let $N = n + 2a + c$ be the total number of sections of $\mathcal{Z}/B$, including cuspidal and inner nodal transforms. Our proof that $(10\lambda - \delta + \psi + \psi_{cusp})_{\mathcal{Z}/B} \geq 0$ will depend on the relative genus $h$ of $\mathcal{Z}/B$.

5.3.1. *Suppose $h \geq 2$.* Passing to the relative minimal model of $\mathcal{Z}/B$ only decreases the degree of $(10\lambda - \delta + \psi + \psi_{cusp})$. Hence we will assume that $\omega_{\mathcal{Z}/B}$ is relatively nef. We still have $N$ smooth and distinct sections (which can now intersect pairwise). With $\omega_{\mathcal{Z}/B}$ relatively nef, we can apply the Cornalba-Harris inequality:

- If $h \geq 3$, then $10 > 8 + 4/h$ and so $10\lambda - \delta > 0$ by Proposition 4.1. We also have $\psi + \psi_{cusp} \geq 0$, so we are done.
- If $h = 2$, then Proposition 4.1 gives $10\lambda - \delta \geq 0$. Lemma 2.1 gives $\psi + \psi_{cusp} > 0$ since we must have $N \geq 1$ (if $N = 0$, then $\mathcal{C}/B$ was a family in $\overline{\mathcal{M}}_{2,0}(9/11 - \epsilon)$). We are done.

5.3.2. *Suppose* $h = 1$. Using relations on the stack on $N$-pointed Gorenstein genus 1 curves inherited from standard relations in $\mathrm{Pic}(\overline{\mathcal{M}}_{1,N})$ given by [AC98, Theorem 2.2], we have $\lambda = \delta_{\mathrm{irr}}/12$, and $\psi = N\delta_{irr}/12 + \sum_S |S|\delta_{0,S} \geq N\delta_{\mathrm{irr}}/12 + 2\delta_{\mathrm{red}}$. If $N \geq 3$, we obtain

$$10\lambda + \psi - \delta \geq 10\delta_{\mathrm{irr}}/12 + N\delta_{\mathrm{irr}}/12 + 2\delta_{\mathrm{red}} - (\delta_{\mathrm{irr}} + \delta_{\mathrm{red}}) > 0.$$

If $N = 2$, we obtain $10\lambda - \delta + \psi \geq \delta_{\mathrm{red}} \geq 0$ and $\psi_{cusp} \geq 0$. We conclude that $10\lambda - \delta + \psi + \psi_{cusp} \geq 0$ with the equality holding if only if $\psi_{cusp} = \delta_{\mathrm{red}} = 0$. This is possible if and only if all fibers are irreducible and there are no cuspidal transforms (by Lemma 2.1), which implies that $\mathcal{X}/B = \mathcal{Y}/B$ is a family of $A_1/A_1$-attached elliptic bridges.

5.3.3. *Suppose* $h = 0$. Then all fibers of $\mathcal{Z}/B$ are in fact at-worst nodal. Because $\lambda = 0$, we can write $(10\lambda - \delta + \psi + \psi_{cusp})_{\mathcal{Z}/B} = (\psi - \delta + \psi_{cusp})_{\mathcal{Z}/B}$. Blow-up the points of intersection of $\eta_i^+$ and $\eta_i^-$ for each $i$. We obtain a family $\big(\mathcal{W}/B, \{\sigma_i\}_{i=1}^n, \{\eta_i^\pm\}_{i=1}^a, \{\xi_i\}_{i=1}^c\big)$ in $\overline{\mathcal{M}}_{0,N}$. Setting $\delta_{inner} := \sum_{i=1}^a \delta_{\{\eta_i^+, \eta_i^-\}}$, we have

$$(\psi - \delta + \psi_{cusp})_{\mathcal{Z}/B} = (\psi - \delta - \delta_{inner} + \psi_{cusp})_{\mathcal{W}/B}.$$

If $a = 0$, then $\delta_{inner} = 0$ and we are done because $\psi - \delta > 0$ for any family of Deligne-Mumford stable rational curves, for example by [KM13, Lemma 3.6]. If $a \geq 2$, then by Lemma 4.8, $\sum_{i=1}^a (\psi_{\eta_i^+} + \psi_{\eta_i^-}) \geq 4\delta_{inner}$. In addition, $3\psi \geq 4\delta$ by a similar argument. It follows that $\psi > \delta + \delta_{inner}$ and so we are done.

Finally, if $a = 1$, then $\big(\mathcal{Y}/B, \{\sigma_i\}_{i=1}^n, \{\xi_i\}_{i=1}^c\big)$ obtained in Subsection 5.2 is a family of arithmetic genus 1 (generically nodal) curves. The proof in §5.3.2 goes through without any modifications to show that $(10\lambda - \delta + \psi + \psi_{cusp})_{\mathcal{Y}/B} \geq 0$ with the equality if and only if $\mathcal{X}/B = \mathcal{Y}/B$ is a (generically nodal) elliptic bridge.

5.4. **A different proof.** Theorem 1.2(a) also follows immediately from the results of Hassett and Hyeon [HH09, HH13]. Namely, in the case of $n = 0$, [HH09, Lemma 4.1 and Proposition 4.2] imply that $K_{\overline{\mathcal{M}}_g(9/11-\epsilon)} + \frac{7}{10}\delta$ is nef. Alternatively, Hassett and Hyeon show that $K_{\overline{\mathcal{M}}_g(9/11-\epsilon)} + \frac{7}{10}\delta$ is a pullback of an ample line bundle via a morphism to the GIT quotient of the Chow variety of bicanonical curves [HH13, p.944]. The case of $n \geq 1$ follows from the observation that $K_{\overline{\mathcal{M}}_{g+hn}(9/11-\epsilon)} + \frac{7}{10}\delta$ pulls back to $K_{\overline{\mathcal{M}}_{g,n}(9/11-\epsilon)} + \frac{7}{10}\delta + \frac{3}{10}\psi$ under the morphism $\overline{\mathcal{M}}_{g,n}(9/11 - \epsilon) \to \overline{\mathcal{M}}_{g+nh}(9/11-\epsilon)$ defined by attaching a fixed general curve of genus $h \geq 3$ to every marked point.

## 6. Proof of Theorem 1.2(b)

In the remaining part of the paper, we prove Theorem 1.2(b). Let $(\mathcal{C}/B, \{\sigma_i\}_{i=1}^n)$ be a $(7/10-\epsilon)$-stable generically non-isotrivial family of curves. We begin by dealing with the case when $\mathcal{C}/B$ has a generic rosary, or a generic $A_1/A_3$ or $A_3/A_3$-attached elliptic bridge. In both cases, generic tacnodes come into play and we will repeatedly use the following result that explains what happens under normalization of a generic tacnode:

**Proposition 6.1.** *Suppose $\mathcal{X}/B$ is a family in $\widetilde{\mathcal{U}}_g(A_\infty)$ with a section $\tau$ such that $\tau(b)$ is a tacnode of $\mathcal{X}_b$ for all $b \in B$. Denote by $\mathcal{Y}$ the normalization of $\mathcal{X}$ along $\tau$ and by $\tau^+$ and $\tau^-$ the* *preimages of* $\tau$*. Then* $\tau^{\pm}$ *are smooth sections satisfying* $\psi_{\tau^+}=\psi_{\tau^-}$ *and we have the following formulae:*

$$\lambda_{\mathcal{X}/B}=\lambda_{\mathcal{Y}/B}-\frac{1}{2}(\psi_{\tau^+}+\psi_{\tau^-}),$$
$$\delta_{\mathcal{X}/B}=\delta_{\mathcal{Y}/B}-6(\psi_{\tau^+}+\psi_{\tau^-}).$$

*Proof.* This is [Smy11, Proposition 3.4] (although it is stated there only in the case of $g=1$). □

6.1. **Reduction 1: The case of generic rosaries.** Let $C$ be the geometric generic fiber of $\mathcal{C}/B$ and consider a maximal length rosary $R=R_1\cup\cdots\cup R_\ell$ of $C$ (see Definition 2.5). Since $\mathcal{C}/B$ is non-isotrivial, the rosary cannot be closed. Let $T:=\overline{C\setminus R}$. The point $T\cap R_1$ (resp., $T\cap R_\ell$) is either an outer node or an outer tacnode, so its limit in every fiber is the same singularity by [AFSv15, Proposition 2.10]. Similarly, the limits of the tacnodes $R_i\cap R_{i+1}$, for $i=1,\dots,\ell-1$, remain tacnodes in every fiber. We then have that $\mathcal{C}=\mathcal{T}\cup\mathcal{R}_1\cup\cdots\cup\mathcal{R}_\ell$, where the geometric generic fiber of $\mathcal{R}_i$ and $\mathcal{T}$ is $R_i$ and $T$ respectively. Let $\chi_1$ (resp., $\chi_2$) be the nodal or tacnodal section along which $\mathcal{T}$ and $\mathcal{R}_1$ (resp., $\mathcal{R}_\ell$) meet. Let $\tau_i$, for $i=1,\dots,\ell-1$, be the tacnodal section along which $\mathcal{R}_i$ and $\mathcal{R}_{i+1}$ meet. In the rest of the proof we use the fact that self-intersections of 2 disjoint smooth sections on a $\mathbb{P}^1$-bundle over $B$ are equal of opposite signs. Together with Proposition 6.1, this gives

$$(\psi_{\chi_1})_{\mathcal{R}_1/B}=-(\psi_{\tau_1})_{\mathcal{R}_1/B}=-(\psi_{\tau_1})_{\mathcal{R}_2/B}=(\psi_{\tau_2})_{\mathcal{R}_2/B}=(-1)^{\ell-1}(\psi_{\tau_{\ell-1}})_{\mathcal{R}_\ell/B}=(-1)^{\ell}(\psi_{\chi_2})_{\mathcal{R}_\ell/B}.$$

In what follows, we set $\psi_{\mathcal{T}/B}=\sum_{i=1}^n\psi_{\sigma_i}+\psi_{\chi_1}+\psi_{\chi_2}=\psi_{\mathcal{C}/B}+\psi_{\chi_1}+\psi_{\chi_2}$.

*Case 1: $R$ is $A_1/A_1$-attached rosary.* An $A_1/A_1$-attached rosary of even length is an elliptic chain and thus cannot appear in an $\alpha$-stable curve for $\alpha<7/10-\epsilon$. Hence $\ell$ must be odd. By Proposition 6.1, we obtain

$$\left(\frac{39}{4}\lambda-\delta+\psi\right)_{\mathcal{C}/B}=\left(\frac{39}{4}\lambda-\delta+\psi\right)_{\mathcal{T}/B}.$$

Since $(\mathcal{T},\{\sigma_i\}_{i=1}^n,\chi_1,\chi_2)$ is $(7/10-\epsilon)$-stable and $\mathcal{R}/B$ is isotrivial, we reduce to proving Theorem 1.2(b) for $(\mathcal{T},\{\sigma_i\}_{i=1}^n,\chi_1,\chi_2)$, which has one less generic rosary than $(\mathcal{C}/B,\{\sigma_i\}_{i=1}^n)$.

*Case 2: $R$ is $A_1/A_3$-attached rosary.* Suppose $\chi_1$ is a nodal section and $\chi_2$ is a tacnodal section. By the maximality assumption on $R$, the irreducible component of $T$ meeting $R_\ell$ is not a 2-pointed smooth rational curve. It follows by Lemma 2.1 that $(\psi_{\chi_2})_{\mathcal{T}}\geq 0$. By Proposition 6.1, we have

$$\left(\frac{39}{4}\lambda-\delta+\psi\right)_{\mathcal{C}/B}=\left(\frac{39}{4}\lambda-\delta+\psi\right)_{\mathcal{T}/B}+(\psi_{\chi_1})_{\mathcal{R}_1/B}+\frac{5}{4}(\psi_{\chi_2})_{\mathcal{R}_\ell/B}+\frac{9}{4}\sum_{i=1}^{\ell-1}(\psi_{\tau_i})_{\mathcal{R}_i}.$$

If $\ell$ is odd, then $\sum_{i=1}^{\ell-1}(\psi_{\tau_i})_{\mathcal{R}_i}=0$ and $\psi_{\chi_1}=-\psi_{\chi_2}$. We thus obtain:

$$\left(\frac{39}{4}\lambda-\delta+\psi\right)_{\mathcal{C}/B}=\left(\frac{39}{4}\lambda-\delta+\psi\right)_{\mathcal{T}/B}+\frac{1}{4}(\psi_{\chi_2})_{\mathcal{T}/B}\geq\left(\frac{39}{4}\lambda-\delta+\psi\right)_{\mathcal{T}/B}.$$

Noting that $\psi_{\chi_2}=0$ only if $\mathcal{R}/B$ is isotrivial, we again reduce to proving Theorem 1.2(b) for $(\mathcal{T},\{\sigma_i\}_{i=1}^n,\chi_1,\chi_2)$.

If $\ell$ is even, then $\psi_{\chi_1} = \psi_{\chi_2}$ and $\sum_{i=1}^{\ell-1}(\psi_{\tau_i})_{\mathcal{R}_i} + \psi_{\chi_2} = 0$, so that

$$\left(\frac{39}{4}\lambda - \delta + \psi\right)_{\mathcal{C}/B} = \left(\frac{39}{4}\lambda - \delta + \psi\right)_{\mathcal{T}/B}.$$

Furthermore, we observe that $\mathcal{R}/B$ is isotrivial and we reduce to proving Theorem 1.2(b) for $(\mathcal{T}, \{\sigma_i\}_{i=1}^n, \chi_1, \chi_2)$.

*Case 3: $R$ is $A_3/A_3$-attached rosary.* By the maximality assumption on $R$, neither $T \cap R_1$ nor $T \cap R_2$ lies on a 2-pointed rational component of $T$. It follows by Lemma 2.1 that $(\psi_{\chi_1})_{\mathcal{T}}, (\psi_{\chi_2})_{\mathcal{T}} \geq 0$. However, $\psi_{\chi_1} = (-1)^{\ell}\psi_{\chi_2}$. Therefore, either $\psi_{\chi_1} = \psi_{\chi_2} = 0$, in which case $\mathcal{R}/B$ is an isotrivial family, or $\ell$ is even and $\psi_{\chi_1} = \psi_{\chi_2} > 0$. In either case, Proposition 6.1 gives

$$\left(\frac{39}{4}\lambda - \delta + \psi\right)_{\mathcal{C}/B} = \left(\frac{39}{4}\lambda - \delta + \psi\right)_{\mathcal{T}/B} + \frac{1}{4}(\psi_{\chi_2})_{\mathcal{T}/B} \geq \left(\frac{39}{4}\lambda - \delta + \psi\right)_{\mathcal{T}/B},$$

and the inequality is strict if $\mathcal{R}$ is not isotrivial. Thus we reduce to proving Theorem 1.2(b) for $(\mathcal{T}, \{\sigma_i\}_{i=1}^n, \chi_1, \chi_2)$.

6.2. **Reduction 2: The case of generic $A_1/A_3$ or $A_3/A_3$-attached elliptic bridges.** Suppose the geometric generic fiber of $\mathcal{C}/B$ can be written as $C = T_1 \cup E \cup T_2$, where $E$ is an $A_1/A_3$-attached elliptic bridge. Let $q_1 = T_1 \cap E$ be a node and $q_2 = T_2 \cap E$ be a tacnode. By [AFSv15, Proposition 2.10], the limit of $q_1$ (resp., $q_2$) remains a node (resp., a tacnode) in every fiber. Thus we can write $\mathcal{C} = (\mathcal{T}_1, \tau_0) \cup (\mathcal{E}, \tau_1, \tau_2) \cup (\mathcal{T}_2, \tau_3)$, where $\tau_0 \sim \tau_1$ are glued nodally and $\tau_2 \sim \tau_3$ are glued tacnodally. Since $A_1/A_1$-attached elliptic bridges are disallowed, fibers of $\mathcal{E}$ have no separating nodes and so $(\mathcal{E}, \tau_1, \tau_2)$ is a family of elliptic bridges. By [AFSv15, Lemma 2.16], $(\mathcal{T}_1, \tau_0)$ is $(7/10 - \epsilon)$-stable. Also, $(\mathcal{T}_2, \tau_3)$ is $(7/10 - \epsilon)$-stable because $\tau_3$ cannot lie on an $A_1$-attached elliptic tail in $\mathcal{T}_2$.

Set $\mathcal{C}' = (\mathcal{T}_1, \tau_0) \cup (\mathcal{T}_2, \tau_3)$, where we glue by $\tau_0 \sim \tau_3$ nodally. Then $(\mathcal{C}'/B, \{\sigma_i\}_{i=1}^n)$ is a $(7/10 - \epsilon)$-stable family by [AFSv15, Lemma 2.17]. By Proposition 6.1, we have

$$\begin{aligned}\left(\frac{39}{4}\lambda - \delta + \psi\right)_{\mathcal{C}/B} &= \left(\frac{39}{4}\lambda - \delta + \psi\right)_{\mathcal{T}_1/B} + \left(\frac{39}{4}\lambda - \delta + \psi_{\tau_1} + \frac{5}{4}\psi_{\tau_2}\right)_{\mathcal{E}/B} + \left(\frac{39}{4}\lambda - \delta + \psi\right)_{\mathcal{T}_2/B} \\ &= \left(\frac{39}{4}\lambda - \delta + \psi\right)_{\mathcal{T}_1/B} + \left(\frac{39}{4}\lambda - \delta + \psi\right)_{\mathcal{T}_2/B} = \left(\frac{39}{4}\lambda - \delta + \psi\right)_{\mathcal{C}'/B},\end{aligned}$$

where we have used relations $(\psi_{\tau_1})_{\mathcal{E}/B} = (\psi_{\tau_2})_{\mathcal{E}/B} = \lambda_{\mathcal{E}/B}$ and $\delta_{\mathcal{E}/B} = 12\lambda_{\mathcal{E}/B}$, both of which hold because $(\delta_{\text{red}})_{\mathcal{E}/B} = 0$.

Note that $(\mathcal{E}/B, \tau_1, \tau_2)$ is trivial if and only if $\psi_{\tau_2} = \psi_{\tau_3} = 0$. Thus we have reduced to proving Theorem 1.2(b) for the family $\mathcal{C}'/B$ with one less generic $A_1/A_3$-attached elliptic bridge. Moreover, the equality for $\mathcal{C}'/B$ holds if and only if the equality for $\mathcal{C}/B$ holds and $\mathcal{C}'/B$ is obtained by replacing a generic node of $\mathcal{C}'$ by a family of elliptic bridges $A_1/A_3$-attached along the nodal transforms.

Similarly, if the generic fiber of $\mathcal{C}/B$ has an $A_3/A_3$-attached elliptic bridge, then we can remove the bridge and recrimp the two remaining components of $\mathcal{C}$ along a generic tacnode. The calculation similar to the above shows that the degree of $\left(\frac{39}{4}\lambda - \delta + \psi\right)$ does not change under this operation.

Observe that replacing an attaching node of a Weierstrass chain of length $\ell$ by an $A_1/A_3$-attached elliptic bridge in a way that preserves $(7/10-\epsilon)$-stability gives a Weierstrass chain of length $\ell+1$. Similarly, replacing a tacnode in a Weierstrass chain of length $\ell$ by an $A_3/A_3$-attached elliptic bridge gives a Weierstrass chain of length $\ell+1$. In what follows, we will prove that for a non-isotrivial $(7/10-\epsilon)$-stable family $(\mathcal{C}/B, \{\sigma_i\}_{i=1}^n)$ *with no generic $A_1/A_3$ or $A_3/A_3$-attached elliptic bridges*, we have $\left(\frac{39}{4}\lambda-\delta+\psi\right)_{\mathcal{C}/B}\geq 0$ and equality holds if and only if $\mathcal{C}/B$ is a family of *Weierstrass genus* 2 *tails*. By the preceding observation, this implies that for every non-isotrivial $(7/10-\epsilon)$-stable family $(\mathcal{C}/B, \{\sigma_i\}_{i=1}^n)$, we have $\left(\frac{39}{4}\lambda-\delta+\psi\right)_{\mathcal{C}/B}\geq 0$ and equality holds if and only if $\mathcal{C}/B$ is a family of *Weierstrass chains*!

6.3. **Reduction 3: Normalization along generic tacnodes.** Consider a family $(\mathcal{C}/B, \{\sigma_i\}_{i=1}^n)$ of $(7/10-\epsilon)$-stable curves with no generic rosaries and no generic $A_1/A_3$ or $A_3/A_3$-attached elliptic bridges. Let $\mathcal{X}$ be the normalization of $\mathcal{C}$ along generic tacnodes. Denote by $\{\tau_i^{\pm}\}_{i=1}^d$ the preimages of the generic tacnodes, and call them *tacnodal transforms*. Set $\psi_{tacn} := \sum_{i=1}^d(\psi_{\tau_i^+}+\psi_{\tau_i^-})$ and $\psi_{\mathcal{X}/B} := \psi_{\mathcal{C}/B}+\psi_{tacn}$. Applying Proposition 6.1 we have

$$\left(\frac{39}{4}\lambda-\delta+\psi\right)_{\mathcal{C}/B}=\left(\frac{39}{4}\lambda-\delta+\psi+\frac{1}{8}\psi_{tacn}\right)_{\mathcal{X}/B}.$$

If we now treat each tacnodal transform $\tau_i^{\pm}$ as a marked section, then every connected component of $\mathcal{X}$ is a generically $(7/10-\epsilon)$-stable family (because there are no generic $A_1/A_3$ or $A_3/A_3$-attached elliptic bridges). Blowing-down all rational tails meeting a single tacnodal transform and no other marked sections does not change $\left(\frac{39}{4}\lambda-\delta+\psi\right)_{\mathcal{X}/B}$ but makes $(\mathcal{X}/B, \{\sigma_i\}_{i=1}^n, \{\tau_i^{\pm}\}_{i=1}^d)$ into a $(7/10-\epsilon)$-stable family. We still have $\psi_{tacn}\geq 0$ by Lemma 2.1, with strict inequality if $d\geq 1$. Thus, we have reduced to proving Theorem 1.2(b) for a $(7/10-\epsilon)$-stable family with no generic tacnodes.

6.4. **Reduction 4: Normalization along generic outer nodes.** Consider a $(7/10-\epsilon)$-stable family $(\mathcal{X}/B, \{\sigma_i\}_{i=1}^n)$ with no generic tacnodes. Let $\mathcal{Y}$ be the normalization of $\mathcal{X}$ along the generic outer nodes and let $\{\zeta_i^+, \zeta_i^-\}_{i=1}^b$ be the transforms of the generic outer nodes. Set $\delta_{tacn} := \sum_{i\neq j}(\zeta_i^{\pm}\cdot\zeta_j^{\pm})$ and $\psi_{\mathcal{Y}/B} := \psi_{\mathcal{X}/B}+\sum_{i=1}^b(\psi_{\zeta_i^+}+\psi_{\zeta_i^-})$. Then by Proposition 3.5, we have

$$\left(\frac{39}{4}\lambda-\delta+\psi\right)_{\mathcal{X}/B}=\left(\frac{39}{4}\lambda-\delta+\psi+\frac{7}{8}\delta_{tacn}\right)_{\mathcal{Y}/B}.$$

6.5. **Reduction 5: Normalization along generic cusps.** Let $\mathcal{Y}$ be as in Subsection 6.4 and let $\mathcal{Z}$ be the normalization of (a connected component of) $\mathcal{Y}$ along generic cusps and let $\{\xi_i\}_{i=1}^c$ be the cuspidal transforms on $\mathcal{Z}$. Then the family $(\mathcal{Z}/B, \{\sigma_i\}_{i=1}^n, \{\zeta_i\}_{i=1}^b, \{\xi_i\}_{i=1}^c)$ satisfies the following properties:

1. The generic fiber is irreducible and at-worst nodal.
2. The sections $\{\sigma_i\}_{i=1}^n$ are smooth, pairwise non-intersecting and disjoint from $\{\zeta_i\}_{i=1}^b$.
3. The sections $\{\zeta_i\}_{i=1}^b$ are smooth and at most two of them can meet at any given point of $\mathcal{Z}$.
4. The sections $\{\xi_i\}_{i=1}^c$ are pairwise non-intersecting and disjoint from $\{\zeta_i\}_{i=1}^b$ and $\{\sigma_i\}_{i=1}^n$.

Set $c(B) := 2\sum_{i=1}^{c} \iota(\xi_i)$, where $\iota(\xi_i)$ is the index of the cuspidal transform $\xi_i$, and $\psi_{cusp} := \sum_{i=1}^{c} \psi_{\xi_i}$. Then we have by Proposition 3.6

$$\left(\frac{39}{4}\lambda - \delta + \psi + \frac{7}{8}\delta_{tacn}\right)_{\mathcal{Y}/B} = \left(\frac{39}{4}\lambda - \delta + \psi + \frac{5}{4}\psi_{cusp} - \frac{1}{4}c(B) + \frac{7}{8}\delta_{tacn}\right)_{\mathcal{Z}/B}. \tag{6.1}$$

Our goal for the rest of the section is to prove that the expression on the right-hand side of (6.1) is non-negative and equals 0 if and only if the only non-isotrivial components of the family $\mathcal{X}/B$ from Subsection 6.4 are $A_1$-attached Weierstrass genus 2 tails.

Let $h$ be the geometric genus of the generic fiber of $\mathcal{Z}$ and let $a$ be the number of generic inner nodes of $\mathcal{Z}$. Our further analysis breaks down according to the following possibilities:

(A) $h \geq 3$; see §6.5.1.
(B) $h = 2$, or $(h, a) = (1, 1)$, or $(h, a) = (0, 2)$; see §6.5.2.
(C) $h = 1$ and $a \neq 1$, or $(h, a) = (0, 1)$; see §6.5.3.
(D) $h = 0$ and $a \geq 3$, or $(h, a) = (0, 0)$; see §6.5.4.

6.5.1. *Case A: Relative geometric genus $h \geq 3$.* Suppose $\mathcal{Z}/B$ is a family as in Subsection 6.5. Let $\mathcal{W}$ be the normalization of $\mathcal{Z}$ along the generic inner nodes. Let $\{\eta_i^+, \eta_i^-\}_{i=1}^{a}$ be the inner nodal transforms on $\mathcal{W}$. Then $(\mathcal{W}/B, \{\sigma_i\}_{i=1}^{n}, \{\eta_i^{\pm}\}_{i=1}^{a}, \{\zeta_i\}_{i=1}^{b}, \{\xi_i\}_{i=1}^{c})$ satisfies the following properties:

(1) The generic fiber is a smooth curve of genus $h \geq 3$.
(2) Sections $\{\sigma_i\}_{i=1}^{n}$ are smooth, non-intersecting, and disjoint from $\{\eta_i^{\pm}\}_{i=1}^{a}$, $\{\zeta_i\}_{i=1}^{b}$, and $\{\xi_i\}_{i=1}^{c}$.
(3) Inner nodal transforms $\{\eta_i^{\pm}\}_{i=1}^{a}$ are disjoint from $\{\zeta_i\}_{i=1}^{b}$ and $\{\xi_i\}_{i=1}^{c}$. Their properties are described by Proposition 3.4.
(4) Outer nodal transforms $\{\zeta_i\}_{i=1}^{b}$ are disjoint from $\{\xi_i\}_{i=1}^{c}$. Their properties are described by Proposition 3.5.
(5) Cuspidal transforms $\{\xi_i\}_{i=1}^{c}$ have properties described by Proposition 3.6.

We let $\psi_{\mathcal{W}/B} := \psi_{\mathcal{Z}/B} + \sum_{i=1}^{a}(\psi_{\eta_i^+} + \psi_{\eta_i^-})$ and $(\delta_{tacn})_{\mathcal{W}/B} := (\delta_{tacn})_{\mathcal{Z}/B} + \sum_{i \neq j}(\eta_i^{\pm} \cdot \eta_j^{\pm})$. We set $\delta_{inner} := \sum_{i=1}^{a}(\eta_i^+ \cdot \eta_i^-)$ and $n(B) := \sum_{i=1}^{a} \iota(\eta_i^+)$, where $\iota(\eta_i^+)$ is the index of the inner nodal transform $\eta_i^+$. Then by Proposition 3.4:

$$\begin{aligned} &\left(\frac{39}{4}\lambda - \delta + \psi + \frac{5}{4}\psi_{cusp} - \frac{1}{4}c(B) + \frac{7}{8}\delta_{tacn}\right)_{\mathcal{Z}/B} \\ &\qquad = \left(\frac{39}{4}\lambda - \delta + \psi - \frac{1}{4}\delta_{inner} - \frac{1}{4}n(B) - \frac{1}{4}c(B) + \frac{5}{4}\psi_{cusp} + \frac{7}{8}\delta_{tacn}\right)_{\mathcal{W}/B}. \end{aligned} \tag{6.2}$$

Passing to the relative minimal model of $\mathcal{W}/B$ does not increase the degree of the divisor on the right-hand side of (6.2). Hence we will assume that $\omega_{\mathcal{W}/B}$ is relatively nef. Then by Proposition 4.1, we have $(8 + 4/h)\lambda - \delta \geq 0$. Since $h \geq 3$ and $\delta \geq 0$, we obtain $\frac{39}{4}\lambda - \delta > 0$ (when $\delta = 0$, we have $\lambda > 0$ by the existence of the Torelli morphism). We proceed to estimate the remaining terms of the right-hand side of (6.2). Clearly, $\delta_{tacn} \geq 0$. Since $h \geq 3$ and

$\kappa = 12\lambda - \delta > 0$, the inequalities of Lemmas 4.3 and 4.4 give

$$\psi_{cusp} = \sum_{i=1}^{c} \psi_{\xi_i} \geq \frac{(h-1)}{h} \sum_{i=1}^{c} \iota(\xi_i) + c\frac{\kappa}{4h(h-1)} = \frac{h-1}{2h} c(B) + c\frac{\kappa}{4h(h-1)} > \frac{1}{3}c(B),$$

$$\sum_{i=1}^{a} (\psi_{\eta_i^+} + \psi_{\eta_i^-}) \geq \frac{2(h-1)}{h+1} \sum_{i=1}^{a} ((\eta_i^+ \cdot \eta_i^-) + \iota(\eta_i^+)) + a\frac{\kappa}{h^2-1} > \delta_{inner} + n(B).$$

Summarizing, we conclude that the right-hand side of (6.2) is strictly positive.

6.5.2. *Case B: Relative genus* 2. Suppose $\mathcal{Z}/B$ is a family as in 6.5 with relative geometric genus $h = 2$. Let $\mathcal{W}$ be the normalization of $\mathcal{Z}$ along the generic inner nodes. As in §6.5.1, we reduce to proving that

$$\left(\frac{39}{4}\lambda - \delta + \psi - \frac{1}{4}\delta_{inner} - \frac{1}{4}n(B) - \frac{1}{4}c(B) + \frac{5}{4}\psi_{cusp} + \frac{7}{8}\delta_{tacn}\right)_{\mathcal{W}/B} \geq 0, \tag{6.3}$$

under the assumption that $\omega_{\mathcal{W}/B}$ is relatively nef.

For any family $\mathcal{W}/B$ of arithmetic genus 2 curves with a relatively nef $\omega_{\mathcal{W}/B}$, we have

$$10\lambda = \delta_{\mathrm{irr}} + 2\delta_{\mathrm{red}}. \tag{6.4}$$

This relation implies that $\delta \leq 10\lambda$ for any generically irreducible family and, consequently, $\kappa = 12\lambda - \delta \geq 2\lambda$, with the equality achieved only if $\delta_{\mathrm{red}} = 0$, i.e., if there are no fibers where two genus 1 components meet at a node. It follows that $\frac{39}{4}\lambda - \delta \geq -\lambda/4$, with the equality only if $\delta_{\mathrm{red}} = 0$.

By Lemma 4.4, we have

$$\sum_{i=1}^{a} (\psi_{\eta_i^+} + \psi_{\eta_i^-}) \geq \frac{2}{3}(\delta_{inner} + n(B)) + a\frac{\kappa}{3}.$$

By Lemma 4.3, we have

$$\psi_{cusp} \geq \frac{1}{4}c(B) + c\frac{\kappa}{8}.$$

Putting these inequalities together and using $\kappa \geq 2\lambda$, we obtain

$$\frac{9}{4}\psi_{cusp} + \sum_{i=1}^{a} (\psi_{\eta_i^+} + \psi_{\eta_i^-}) \geq \frac{1}{4}\delta_{inner} + \frac{1}{4}n(B) + \frac{1}{4}c(B) + \left(\frac{2a}{3} + \frac{9c}{16}\right)\lambda.$$

If $a + c \geq 1$, we obtain a strict inequality in (6.3) at once. Suppose $a = c = 0$. So far, we have that

$$\left(\frac{39}{4}\lambda - \delta + \psi\right)_{\mathcal{W}/B} \geq \sum_{i=1}^{n} \psi_{\sigma_i} + \sum_{i=1}^{b} \psi_{\zeta_i} - \frac{1}{4}\lambda.$$

We now invoke Lemma 4.5 that gives

$$\sum_{i=1}^{n} \psi_{\sigma_i} + \sum_{i=1}^{b} \psi_{\zeta_i} \geq \frac{(n+b)}{8}\kappa \geq \frac{(n+b)}{4}\lambda.$$

Since $n + b \geq 1$ (otherwise, $\mathcal{W}/B$ is an unpointed family of genus 2 curves, which is impossible), we conclude that $\sum_{i=1}^{n} \psi_{\sigma_i} + \sum_{i=1}^{b} \psi_{\zeta_i} - \lambda/4 \geq 0$ and that equality is achieved if and only if $n + b = 1$, $\delta_{\mathrm{red}} = 0$, and equality is achieved in Lemma 4.5. This is precisely the situation when $\mathcal{Y}/B = \mathcal{W}/B$ is a family of $A_1$-attached Weierstrass genus 2 tails.

Finally, if $(h, a) = (1, 1)$ or $(h, a) = (0, 2)$, we proceed exactly as above but without normalizing the inner nodes: For a family $(\mathcal{Z}/B, \{\sigma_i\}_{i=1}^n, \{\zeta_i\}_{i=1}^b, \{\xi_i\}_{i=1}^c)$ as in Subsection 6.5, where the relative arithmetic genus of $\mathcal{Z}/B$ is 2, we need to prove

$$\left(\frac{39}{4}\lambda - \delta + \psi + \frac{5}{4}\psi_{cusp} - \frac{1}{4}c(B) + \frac{7}{8}\delta_{tacn}\right)_{\mathcal{Z}/B} \geq 0.$$

Applying (6.4) to estimate $\delta$, Lemma 4.3 to estimate $\psi_{cusp}$, and Lemma 4.5 to estimate

$$\sum_{i=1}^n \psi_{\sigma_i} + \sum_{i=1}^b \psi_{\zeta_i}$$

(all of which apply even if the total space $\mathcal{Z}$ is not normal), we obtain

$$\frac{39}{4}\lambda - \delta + \psi + \frac{5}{4}\psi_{cusp} - \frac{1}{4}c(B) + \frac{7}{8}\delta_{tacn} \geq -\frac{1}{4}\lambda + \frac{4n+4b+9c}{16}\lambda + \frac{5}{16}c(B) + \frac{7}{8}\delta_{tacn} \geq 0.$$

Moreover, equality is achieved if and only if $\delta_{\text{red}} = 0$, $c = 0$, and $n + b = 1$, which is precisely the situation when $\mathcal{Y}/B = \mathcal{Z}/B$ is a family of $A_1$-attached (generically nodal) Weierstrass genus 2 tails.

6.5.3. *Case C: Relative genus* 1. Suppose $\mathcal{Z}/B$ is a family as in Subsection 6.5 of relative genus 1 and with $a$ generic inner nodes, where $a \neq 1$. We consider the case $a \geq 2$ first. Let $\mathcal{W}$ be the family obtained from $\mathcal{Z}$ by the following operations:

1. Normalize $\mathcal{Z}$ along all generic inner nodes to obtain inner nodal pairs $\{\eta_i^+, \eta_i^-\}_{i=1}^a$.
2. Blow-up all cuspidal and inner nodal transforms to make them Cartier divisors.
3. Blow-up points of $\eta_i^\pm \cap \eta_j^\pm$ for all $i \neq j$.
4. Blow-up points of $\zeta_i \cap \zeta_j$ for all $i \neq j$.

As a result, the sections of $\mathcal{W}/B$ do not intersect pairwise with the only possible exception that $\eta_i^+$ is allowed to meet $\eta_i^-$. A node of $\mathcal{Z}$ through which $\xi_i$ passes is replaced in $\mathcal{W}$ by a balanced rational bridge meeting the strict transform of $\xi_i$, which we continue to denote by $\xi_i$. We say that such a bridge is a *cuspidal bridge associated to* $\xi_i$. Moreover, if we let $c(\xi_i)$ be the sum of the indices of all bridges associated to $\xi_i$, then

$$2\iota(\xi_i)_{\mathcal{Z}/B} = c(\xi_i)_{\mathcal{W}/B}.$$

Suppose $\{\eta_i^+, \eta_i^-\}$ is an inner nodal pair of $\mathcal{Z}/B$. Then a node of $\mathcal{Z}$ through which $\eta_i^+$ and $\eta_i^-$ both pass is replaced in $\mathcal{W}$ by a balanced rational bridge meeting the strict transforms of $\eta_i^+$ and $\eta_i^-$, which we continue to denote by $\eta_i^+$ and $\eta_i^-$. We say that such a bridge is an *inner nodal bridge associated to* $\{\eta_i^+, \eta_i^-\}$. Moreover, if we let $n(\eta_i)$ be the sum of the indices of all bridges associated to $\{\eta_i^+, \eta_i^-\}$, then

$$((\eta_i^+ \cdot \eta_i^-) + \iota(\eta_i^+))_{\mathcal{Z}/B} = ((\eta_i^+ \cdot \eta_i^-) + n(\eta_i))_{\mathcal{W}/B}.$$

On $\mathcal{W}/B$, we define

$$\delta_{inner} := \sum_{i=1}^a (\eta_i^+ \cdot \eta_i^-), \quad \delta_{tacn} := \sum_{i \neq j} \delta_{0,\{\eta_i^\pm, \eta_j^\pm\}} + \sum_{i \neq j} \delta_{0,\{\zeta_i, \zeta_j\}},$$

and let $n(B)$ (resp., $c(B)$) be the sum of the indices of all inner nodal (resp., cuspidal) bridges. We reduce to proving that

$$\left(\frac{39}{4}\lambda-\delta+\psi-\frac{1}{4}\delta_{inner}-\frac{1}{4}n(B)-\frac{1}{4}c(B)+\frac{5}{4}\psi_{cusp}-\frac{1}{8}\delta_{tacn}\right)_{\mathcal{W}/B}\geq 0.$$

We will make use of the standard relations for pointed families of genus 1 curves and Lemmas 4.6 and 4.7. Let $N=n+2a+b+c$ be the total number of marked sections of $\mathcal{W}/B$. Clearly, $N\geq 2$. We consider first the case when $N\geq 3$. Then by Lemma 4.6, we have

$$\delta_{inner}+n(B)\leq\frac{N}{2(N-2)}\sum_{i=1}^{a}(\psi_{\eta_i^+}+\psi_{\eta_i^-})+\frac{a}{2(N-2)^2}\delta_{\mathrm{red}}.$$

Applying Lemma 4.7, we obtain

$$c(B)\leq\frac{2N}{N-1}\psi_{cusp}+\frac{c}{4(N-1)^2}\delta_{\mathrm{red}}.$$

Using the above two inequalities and rewriting $\delta=12\lambda+\delta_{\mathrm{red}}$, we see that

$$\begin{aligned}\frac{39}{4}\lambda-\delta+\psi-\frac{1}{4}\left(\delta_{inner}+n(B)+c(B)\right)+\frac{5}{4}\psi_{cusp}-\frac{1}{8}\delta_{tacn}\\ \geq-\frac{9}{4}\lambda+\left(\frac{5}{4}-\frac{N}{2(N-1)}\right)\psi_{cusp}+\psi-\frac{N}{8(N-2)}\sum_{i=1}^{a}(\psi_{\eta_i^+}+\psi_{\eta_i^-})\\ -\left(1+\frac{a}{8(N-2)^2}+\frac{c}{16(N-1)^2}\right)\delta_{\mathrm{red}}-\frac{1}{8}\delta_{tacn}.\end{aligned}\tag{6.5}$$

We rewrite each $\psi$-class on the right-hand side of (6.5) using the standard relation on families of arithmetic genus 1 curves:

$$\psi_\sigma=\lambda+\sum_{\sigma\in S}\delta_{0,S}.$$

The coefficient of $\lambda$ in the resulting expression for the right-hand side of (6.5) is

$$-\frac{9}{4}+c\left(\frac{5}{4}-\frac{N}{2(N-1)}\right)+N-\frac{aN}{4(N-2)}.\tag{6.6}$$

Using $N\geq 2a+c$ and the assumption $N\geq 3$, it is easy to check that (6.6) is always positive.

A similarly straightforward but tedious calculation shows that each boundary divisor $\delta_{0,S}$ appears in the resulting expression for the right-hand side of (6.5) with a positive coefficient. Thus we have shown that the right-hand side of (6.5) is positive for every non-isotrivial family with $N\geq 3$.

We consider now the case of $N=2$. Since $\mathcal{C}/B$ in Subsection 6.3 has no generic elliptic bridges (nodally or tacnodally attached), we must have $c=1$ and $n+b=1$. Let $\xi$ be the corresponding cuspidal transform and $\tau$ be either a marked smooth section (if $n=1$) or an outer nodal transform (if $b=1$). We trivially have $\delta_{inner}=n(B)=\delta_{tacn}=\delta_{\mathrm{red}}=0$. Using

$\delta_{\mathrm{irr}} = 12\lambda$ and the inequality $c(B) \leq 4\psi_{cusp}$ from Lemma 4.7, we obtain:

$$\begin{aligned}\frac{39}{4}\lambda - \delta + \psi - \frac{1}{4}\delta_{inner} - \frac{1}{4}n(B) - \frac{1}{4}c(B) + \frac{5}{4}\psi_{cusp} - \frac{1}{8}\delta_{tacn}&\\ = \frac{39}{4}\lambda - \delta + \psi - \frac{1}{4}c(B) + \frac{5}{4}\psi_{cusp} \geq \frac{39}{4}\lambda - 12\lambda + \psi + \frac{1}{4}\psi_{cusp}&\\ = \frac{39}{4}\lambda - 12\lambda + 2\lambda + \frac{1}{4}\lambda = 0.&\end{aligned}$$

Moreover, equality holds only if equality holds in Lemma 4.7. This happens if and only if $2\xi \sim 2\tau$ and implies that $\mathcal{Y}/B$ in 6.4 is a generically cuspidal family of $A_1$-attached Weierstrass genus 2 tails. We are done with the analysis in the case $h = 1$ and $a \neq 1$.

If $(h, a) = (0, 1)$, we proceed exactly as above, but without normalizing the inner node.

6.5.4. *Case D: Relative geometric genus* 0. Suppose $\mathcal{Z}/B$ is a family as in Subsection 6.5 of relative geometric genus 0 and with $a$ generic inner nodes, where either $a \geq 3$ or $a = 0$. We consider the case $a \geq 3$ first. Let $\mathcal{W}$ be the family obtained from $\mathcal{Z}$ by the following operations:

1. Normalize $\mathcal{Z}$ along all generic inner nodes to obtain inner nodal pairs $\{\eta_i^+, \eta_i^-\}_{i=1}^{a}$.
2. Blow-up all cuspidal and inner nodal transforms to make them Cartier divisors. This operation introduces cuspidal or nodal bridges as in §6.5.3.
3. Blow-up points of $\eta_i^{\pm} \cap \eta_j^{\pm}$ for all $1 \leq i < j \leq a$.
4. Blow-up points of $\zeta_i \cap \zeta_j$ for all $1 \leq i < j \leq b$.
5. Blow-up points of $\eta_i^+ \cap \eta_i^-$ for all $1 \leq i \leq a$.
6. Blow-down all rational tails marked by a single section (such tails are necessarily adjacent either to cuspidal or inner nodal bridges).

As a result, $\mathcal{W}/B$ is a family in $\overline{\mathcal{M}}_{0,N}$, where $N = n + 2a + b + c$ and $a \geq 3$. On $\mathcal{W}/B$, we define

$$\delta_{inner} := \sum_{i=1}^{a} \delta_{\{\eta_i^+, \eta_i^-\}}, \qquad \delta_{tacn} := \sum_{i \neq j} \delta_{\{\eta_i^{\pm}, \eta_j^{\pm}\}} + \sum_{i \neq j} \delta_{\{\zeta_i, \zeta_j\}},$$

$$\delta_3^{NB} := \sum_{i=1}^{a} \sum_{\beta \neq \eta_i^+, \eta_i^-} \delta_{\{\eta_i^+, \eta_i^-, \beta\}}, \qquad \delta_2^{CB} := \sum_{i=1}^{c} \sum_{\beta \neq \xi_i} \delta_{\{\xi_i, \beta\}},$$

and let $n(B)$ (resp., $c(B)$) be the sum of the indices of all inner nodal (resp., cuspidal) bridges. Then

$$\begin{aligned}&\left(\frac{39}{4}\lambda - \delta + \psi + \frac{5}{4}\psi_{cusp} - \frac{1}{4}c(B) + \frac{7}{8}\delta_{tacn}\right)_{\mathcal{Z}/B}\\ &\qquad = \left(\psi + \frac{5}{4}\psi_{cusp} - \delta - \frac{5}{4}\delta_{inner} - \frac{1}{4}(n(B) + c(B) + \delta_3^{NB} + \delta_2^{CB}) - \frac{1}{8}\delta_{tacn}\right)_{\mathcal{W}/B}.\end{aligned}$$

We are going to prove that a (strict!) inequality

$$\psi + \frac{5}{4}\psi_{cusp} - \delta - \frac{5}{4}\delta_{inner} - \frac{1}{4}(n(B) + c(B) + \delta_3^{NB} + \delta_2^{CB}) - \frac{1}{8}\delta_{tacn} > 0$$

always holds on $\mathcal{W}/B$. In doing so, we will use the following standard relation on $\overline{\mathcal{M}}_{0,N}$:

$$\psi = \sum_{r \geq 2} \frac{r(N-r)}{N-1}\delta_r. \tag{6.7}$$

First we deal with the case of a family with 3 inner nodal pairs and no other marked sections, i.e., $a = 3$ and $N = 6$. The desired inequality in this case simplifies to

$$\psi - \delta - \frac{5}{4}\delta_{inner} - \frac{1}{4}(n(B) + \delta_3^{NB}) - \frac{1}{8}(\delta_2 - \delta_{inner}) > 0.$$

We have an obvious inequality $2n(B) \leq \delta_2$. Thus we reduce to proving

$$\psi > \frac{5}{4}\delta_2 + \frac{9}{8}\delta_{inner} + \delta_3 + \frac{1}{4}\delta_3^{NB}. \tag{6.8}$$

For $a \geq 3$, Lemma 4.8 gives

$$\psi \geq 4\delta_{inner} + \delta_{tacn} + 3\delta_3^{NB} = 3\delta_{inner} + \delta_2 + 3\delta_3^{NB}.$$

Combining this with the standard relation $5\psi = 8\delta_2 + 9\delta_3$ gives

$$8\psi \geq 9\delta_{inner} + 11\delta_2 + 9\delta_3 + 9\delta_3^{NB}.$$

This clearly implies (6.8) as desired.

Next, we consider the case of $N \geq 7$. In this case, every inner nodal or cuspidal bridge is adjacent to a node from $\sum_{r\geq 3}\delta_r$. As a result, we have $n(B) + c(B) \leq 2\sum_{r\geq 3}\delta_r$. Furthermore, $\frac{1}{4}\delta_2^{CB} + \frac{1}{8}\delta_{tacn} + \frac{1}{4}\delta_{inner} \leq \frac{1}{4}\delta_2$ (because a node from $\delta_2$ can contribute only to one of the $\delta_{inner}$, $\delta_{tacn}$, or $\delta_2^{CB}$). Hence we reduce to proving

$$\psi + \frac{5}{4}\psi_{cusp} - \frac{5}{4}\delta_2 - \delta_{inner} - \frac{3}{2}\sum_{r\geq 3}\delta_r - \frac{1}{4}\delta_3^{NB} > 0 \tag{6.9}$$

We combine the inequality of Lemma 4.8 with the standard relation (6.7), and the obvious $\psi \geq \psi_{inner}$ to obtain

$$3\left(\psi - \sum_{r\geq 2}\frac{r(N-r)}{N-1}\delta_r\right) + \left(\psi_{inner} - 4\delta_{inner} - 3\delta_3^{NB}\right) + \left(\psi - \psi_{inner}\right) \geq 0.$$

This gives the estimate

$$4\psi \geq 4\delta_{inner} + \frac{6(N-2)}{N-1}\delta_2 + \frac{9(N-3)}{N-1}\sum_{r\geq 3}\delta_r + 3\delta_3^{NB}.$$

Using $N \geq 7$ and $\psi_{cusp} \geq 0$, we finally get

$$\psi + \frac{5}{4}\psi_{cusp} \geq \delta_{inner} + \frac{5}{4}\delta_2 + \frac{3}{2}\sum_{r\geq 3}\delta_r + \frac{3}{4}\delta_3^{NB}.$$

Moreover, the equality could be achieved only if $N = 7$ and $\psi - \psi_{inner} = 0$ which is impossible because $\psi = \psi_{inner}$ implies that all sections are inner nodal transforms and so $N$ must be even. Hence we have established (6.9) as desired.

At last, we consider the case of $a = 0$. Because the family $\mathcal{W}/B$ is non-isotrivial, we must have $N \geq 4$. In addition, if $N = 4$, then there exists a unique family of 4-pointed Deligne-Mumford stable rational curves. The requisite inequality is easily verified for this family by hand. If $N \geq 5$, then using the inequality $2c(B) \leq \delta$, we reduce to proving

$$\psi + \frac{5}{4}\psi_{cusp} - \frac{9}{8}\delta - \frac{1}{4}\delta_2^{CB} - \frac{1}{8}\delta_{tacn} > 0.$$

The standard relation (6.7) gives

$$\psi \geq \frac{3}{2}\sum_{r\geq 2}\delta_r > \frac{11}{8}\delta_2 + \frac{9}{8}\sum_{r\geq 3}\delta_r > \frac{9}{8}\delta + \frac{1}{4}\delta_2.$$

Finally, the inequality $\delta_2 \geq \delta_2^{CB} + \delta_{tacn}$ gives the desired result.

This completes the proof of Theorem 1.2 (b).

## 7. Projectivity from positivity

Throughout this section, we make use of the following standard abuse of notation: Whenever $\mathcal{L}$ is a line bundle on $\overline{\mathcal{M}}_{g,n}(\alpha)$ that descends to the good moduli space, we denote the corresponding line bundle on $\overline{\mathbb{M}}_{g,n}(\alpha)$ also by $\mathcal{L}$. In this situation, pullback defines a natural isomorphism $\mathrm{H}^0\big(\overline{\mathbb{M}}_{g,n}(\alpha),\mathcal{L}\big) \simeq \mathrm{H}^0\big(\overline{\mathcal{M}}_{g,n}(\alpha),\mathcal{L}\big)$.

We begin with a result that allows us to descend a multiple of $K_{\overline{\mathcal{M}}_{g,n}(\alpha)} + \alpha\delta + (1-\alpha)\psi$ from the stack to the good moduli space.

**Proposition 7.1.** *Let $\alpha_c \in \{2/3, 7/10, 9/11\}$. For any closed $\mathbb{C}$-point $[(C,\{p_i\}_{i=1}^n)] \in \overline{\mathcal{M}}_{g,n}(\alpha_c)$, the action of $\mathrm{Aut}(C,\{p_i\}_{i=1}^n)^\circ$ on the fiber of $K_{\overline{\mathcal{M}}_{g,n}(\alpha_c)}+\alpha_c\delta+(1-\alpha_c)\psi$ is trivial. Consequently, a positive multiple of $K_{\overline{\mathcal{M}}_{g,n}(\alpha_c)} + \alpha_c\delta + (1-\alpha_c)\psi$ descends to $\overline{\mathbb{M}}_{g,n}(\alpha_c)$.*

*Proof.* We recall that by [AFSv15, Theorem 2.22], an $\alpha_c$-stable curve $(C,\{p_i\}_{i=1}^n)$ is closed $\overline{\mathcal{M}}_{g,n}(\alpha_c)$ if and only if it is $\alpha_c$-closed (see [AFSv15, Definition 2.21]). Furthermore, the combinatorial types of $\alpha_c$-closed curves are described in [AFSv15, Definition 2.31]. We prove the proposition for $\alpha_c = 2/3$ and an $\alpha_c$-closed curve $(C,\{p_i\}_{i=1}^n)$ of Type A with just one $\alpha_c$-atom; the other cases being similar. Let

$$C = K' \cup L = K' \cup R_1 \cup \cdots \cup R_{\ell-1} \cup E$$

be the decomposition of $C$ as in [AFSv15, Definition 2.31], where $E$ is the $\alpha_c$-atom and $R_j$ are length 3 rosaries. Suppose that the rank of $\mathrm{Aut}(K')$ is $d$. By [AFSv15, Remark 2.28], there exist length 3 rosaries $R_1',\dots,R_d'$ in $K'$ such that $\mathrm{Aut}(K')^\circ \simeq \prod_{k=1}^d \mathrm{Aut}(R_k')$. Thus, we have

$$\mathrm{Aut}(C)^\circ = \mathrm{Aut}(K')^\circ \times \mathrm{Aut}(L) = \prod_{k=1}^{d}\mathrm{Aut}(R_k') \times \left[\prod_{j=1}^{\ell-1}\mathrm{Aut}(R_j) \times \mathrm{Aut}(E)\right].$$

Given a line bundle $\mathcal{L}$ on $\overline{\mathcal{M}}_{g,n}(\alpha_c)$ and any one-parameter subgroup $\rho\colon \mathbb{G}_m \to \mathrm{Aut}(C)$, we let $\langle \chi_{\mathcal{L}}, \rho\rangle$ be the character of the induced action of $\mathbb{G}_m$ on the fiber of the line bundle $\mathcal{L}$ over the point $[C]$. Let $\rho_k'\colon \mathbb{G}_m \to \mathrm{Aut}(C)$ (resp., $\rho_j$, $\varphi$) be the one-parameter subgroup corresponding to $\mathrm{Aut}(R_k') \subset \mathrm{Aut}(C)$ (resp., $\mathrm{Aut}(R_j), \mathrm{Aut}(E) \subset \mathrm{Aut}(C)$). By [AFS14, Sections 3.1.2–3.1.3], we have

$$\begin{array}{lll} \langle \chi_\lambda, \rho_k'\rangle = 0 & \langle \chi_\lambda, \rho_j\rangle = 0 & \langle \chi_\lambda, \varphi\rangle = 4 \\ \langle \chi_{\delta-\psi}, \rho_k'\rangle = 0 & \langle \chi_{\delta-\psi}, \rho_j\rangle = 0 & \langle \chi_{\delta-\psi}, \varphi\rangle = 39. \end{array}$$

We provide details only for one character calculation: By definition, $\varphi$ acts non-trivially only on the $\alpha_c$-atom $E$, which is an $A_4$-atom in the terminology of [AFS14]. Now, [AFS14, Corollary 3.3] says that the character of the induced action of $\mathbb{G}_m$ on the fiber of the line bundle $\lambda$ over the point $[C]$ is the same as the analogous character of $[E] \in \overline{\mathcal{M}}_{2,1}(\alpha_c)$. It follows by [AFS14, Section 3.1.2] that $\langle \chi_\lambda, \varphi\rangle = 4$.

Using the identity

$$K_{\overline{\mathcal{M}}_{g,n}(\alpha_c)} + \alpha_c\delta + (1-\alpha_c)\psi = 13\lambda + (\alpha_c - 2)(\delta - \psi)$$

one easily computes

$$\langle \chi_{K+\alpha_c\delta+(1-\alpha_c)\psi}, \rho'_k\rangle = \langle \chi_{K+\alpha_c\delta+(1-\alpha_c)\psi}, \rho_j\rangle = \langle \chi_{K+\alpha_c\delta+(1-\alpha_c)\psi}, \varphi\rangle = 0,$$

and the claim follows. The descent statement follows by [Alp13, Theorem 10.3]. ☐

**Proposition 7.2.** *Let $\alpha > 2/3 - \epsilon$. Suppose that $K_{\overline{\mathcal{M}}_{g,n}(\alpha)} + \beta\delta + (1-\beta)\psi$ descends to $\overline{\mathbb{M}}_{g,n}(\alpha)$ for some $\beta \leq \alpha$. Then we have*

$$\text{Proj } R\big(\overline{\mathbb{M}}_{g,n}(\alpha), K_{\overline{\mathcal{M}}_{g,n}(\alpha)} + \beta\delta + (1-\beta)\psi\big) \simeq \overline{M}_{g,n}(\beta).$$

*Proof.* Consider the rational map $f_\alpha \colon \overline{M}_{g,n} \dashrightarrow \overline{\mathbb{M}}_{g,n}(\alpha)$. If $\alpha > 9/11$, then $f_\alpha$ is an isomorphism. If $7/10 < \alpha \leq 9/11$, then $f_\alpha|_{\overline{M}_{g,n}\setminus\delta_{1,0}}$ is an isomorphism onto the complement of the codimension 2 locus of cuspidal curves in $\overline{\mathbb{M}}_{g,n}(\alpha)$. If $\alpha \leq 7/10$, then $f_\alpha|_{\overline{M}_{g,n}\setminus(\delta_{1,0}\cup\delta_{1,1})}$ is an isomorphism onto the complement of the codimension 2 locus of cuspidal and tacnodal curves in $\overline{\mathbb{M}}_{g,n}(\alpha)$. (If $n = 0$, then $\delta_{1,1} = \varnothing$). It follows that we have a discrepancy equation

$$f^*_\alpha\big(K_{\overline{\mathcal{M}}_{g,n}(\alpha)} + \beta\delta + (1-\beta)\psi\big) \simeq K_{\overline{\mathcal{M}}_{g,n}} + \beta\delta + (1-\beta)\psi + c_0\delta_{1,0} + c_1\delta_{1,1}, \tag{7.1}$$

where $c_0 = 0$ if $\alpha > 9/11$ and $c_1 = 0$ if $\alpha > 7/10$.

Let $T_1 \subset \overline{\mathcal{M}}_{g,n}$ be a non-trivial family of elliptic tails and $T_2 \subset \overline{\mathcal{M}}_{g,n} \setminus \delta_{1,0}$ be a non-trivial family of 1-pointed elliptic tails. Then $f_\alpha$ is regular along $T_1$, and for $\alpha \leq 9/11$ contracts $T_1$ to a point. Similarly, $f_\alpha$ is regular along $T_2$, and for $\alpha \leq 7/10$ contracts $T_2$ to a point. By intersecting both sides of (7.1) with $T_1$ and $T_2$, we obtain $c_0 = 11\beta - 9 \leq 0$ if $\alpha \leq 9/11$, and $c_1 = 10\beta - 7 \leq 0$ if $\alpha \leq 7/10$. It follows that

$$\text{Proj } R\big(\overline{\mathbb{M}}_{g,n}(\alpha), K_{\overline{\mathcal{M}}_{g,n}(\alpha)} + \beta\delta + (1-\beta)\psi\big) \simeq \text{Proj } R\big(\overline{M}_{g,n}, K_{\overline{\mathcal{M}}_{g,n}} + \beta\delta + (1-\beta)\psi\big).$$

☐

**Proposition 7.3.** *Fix $\alpha_c \in \{\alpha_1 = 9/11, \alpha_2 = 7/10, \alpha_3 = 2/3\}$ and take $\alpha_0 = 1$. Suppose that for all $0 < \epsilon \ll 1$,*

$$K_{\overline{\mathcal{M}}_{g,n}(\alpha_{c-1}-\epsilon)} + (\alpha_{c-1} - \epsilon)\delta + (1 - \alpha_{c-1} + \epsilon)\psi$$

*descends to an ample line bundle on $\overline{\mathbb{M}}_{g,n}(\alpha_{c-1} - \epsilon)$. In addition, suppose that*

$$K_{\overline{\mathcal{M}}_{g,n}(\alpha_{c-1}-\epsilon)} + \alpha_c\delta + (1-\alpha_c)\psi$$

*is nef on $\overline{\mathcal{M}}_{g,n}(\alpha_{c-1} - \epsilon)$ and all curves on which it has degree 0 are contracted by $\overline{\mathbb{M}}_{g,n}(\alpha_{c-1} - \epsilon) \to \overline{\mathbb{M}}_{g,n}(\alpha_c)$. Then $K_{\overline{\mathcal{M}}_{g,n}(\alpha)} + \alpha\delta + (1-\alpha)\psi$ descends to an ample line bundle on $\overline{\mathbb{M}}_{g,n}(\alpha)$ for all $\alpha \in [\alpha_c, \alpha_{c-1})$.*

*Proof.* By Proposition 7.1, $K_{\overline{\mathcal{M}}_{g,n}(\alpha_c)} + \alpha_c\delta + (1-\alpha_c)\psi$ descends to $\overline{\mathbb{M}}_{g,n}(\alpha_c)$. Consider the open immersion of stacks $\overline{\mathcal{M}}_{g,n}(\alpha_{c-1} - \epsilon) \hookrightarrow \overline{\mathcal{M}}_{g,n}(\alpha_c)$ and the induced map on the good moduli spaces $j \colon \overline{\mathbb{M}}_{g,n}(\alpha_{c-1} - \epsilon) \to \overline{\mathbb{M}}_{g,n}(\alpha_c)$. We have that

$$j^*\big(K_{\overline{\mathcal{M}}_{g,n}(\alpha_c)} + \alpha_c\delta + (1-\alpha_c)\psi\big) = K_{\overline{\mathcal{M}}_{g,n}(\alpha_{c-1}-\epsilon)} + \alpha_c\delta + (1-\alpha_c)\psi.$$

By our assumptions, $K_{\overline{\mathcal{M}}_{g,n}(\alpha_{c-1}-\epsilon)} + \alpha_c\delta + (1-\alpha_c)\psi$ descends to a nef line bundle on the projective variety $\overline{\mathbb{M}}_{g,n}(\alpha_{c-1} - \epsilon)$. First, we show that $K_{\overline{\mathcal{M}}_{g,n}(\alpha_{c-1}-\epsilon)} + \alpha_c\delta + (1-\alpha_c)\psi$ is semiample on $\overline{\mathbb{M}}_{g,n}(\alpha_{c-1}-\epsilon)$. To bootstrap from nefness to semiampleness, we first consider the case $n=0$ and $g\geq 3$. By Proposition 7.2, the section ring of $K_{\overline{\mathcal{M}}_g(\alpha_{c-1}-\epsilon)}+\alpha_c\delta$ on $\overline{\mathbb{M}}_g(\alpha_{c-1}-\epsilon)$ is identified with the section ring of $K_{\overline{\mathcal{M}}_g}+\alpha_c\delta$ on $\overline{M}_g$. The latter line bundle is big, by standard bounds on the effective cone of $\overline{M}_g$, and finitely generated by [BCHM10, Corollary 1.2.1]. We conclude that $K_{\overline{\mathcal{M}}_g(\alpha_{c-1}-\epsilon)}+\alpha_c\delta$ is big, nef, and finitely generated, and so is semiample by [Laz04, Theorem 2.3.15]. When $n\geq 1$, simply note that $K_{\overline{\mathcal{M}}_{g+hn}(\alpha_{c-1}-\epsilon)}+\alpha_c\delta$ pulls back to $K_{\overline{\mathcal{M}}_{g,n}(\alpha_{c-1}-\epsilon)}+\alpha_c\delta+(1-\alpha_c)\psi$ under the morphism $\overline{\mathcal{M}}_{g,n}(\alpha_{c-1}-\epsilon)\to\overline{\mathcal{M}}_{g+nh}(\alpha_{c-1}-\epsilon)$ defined by attaching a fixed general curve of genus $h\geq 3$ to every marked point.

We have established that

$$j^*\big(K_{\overline{\mathcal{M}}_{g,n}(\alpha_c)}+\alpha_c\delta+(1-\alpha_c)\psi\big)=K_{\overline{\mathcal{M}}_{g,n}(\alpha_{c-1}-\epsilon)}+\alpha_c\delta+(1-\alpha_c)\psi$$

is semiample on $\overline{\mathbb{M}}_{g,n}(\alpha_{c-1}-\epsilon)$. By assumption, it has degree 0 only on curves contracted by $\overline{\mathbb{M}}_{g,n}(\alpha_{c-1}-\epsilon)\to\overline{\mathbb{M}}_{g,n}(\alpha_c)$. We conclude that $K_{\overline{\mathcal{M}}_{g,n}(\alpha_c)}+\alpha_c\delta+(1-\alpha_c)\psi$ is semiample and is positive on all curves in $\overline{\mathbb{M}}_{g,n}(\alpha_c)$. Therefore, $K_{\overline{\mathcal{M}}_{g,n}(\alpha_c)}+\alpha_c\delta+(1-\alpha_c)\psi$ is ample on $\overline{\mathbb{M}}_{g,n}(\alpha_c)$.

The statement for $\alpha\in(\alpha_c,\alpha_{c-1})$ follows by interpolation. □

**Proposition 7.4.** *Fix $\alpha_c\in\{9/11,7/10,2/3\}$. Suppose that $K_{\overline{\mathcal{M}}_{g,n}(\alpha_c)}+\alpha_c\delta+(1-\alpha_c)\psi$ descends to an ample line bundle on $\overline{\mathbb{M}}_{g,n}(\alpha_c)$. Then for all $0<\epsilon\ll 1$,*

$$K_{\overline{\mathcal{M}}_{g,n}(\alpha_c-\epsilon)}+(\alpha_c-\epsilon)\delta_c+(1-\alpha_c+\epsilon)\psi$$

*descends to an ample line bundle on $\overline{\mathbb{M}}_{g,n}(\alpha_c-\epsilon)$.*

*Proof.* Consider the proper morphism $\pi\colon\overline{\mathbb{M}}_{g,n}(\alpha_c-\epsilon)\to\overline{\mathbb{M}}_{g,n}(\alpha_c)$ given by [AFS16, Theorem 4.25]. Our assumption implies that $K_{\overline{\mathcal{M}}_{g,n}(\alpha_c-\epsilon)}+\alpha_c\delta+(1-\alpha_c)\psi$ descends to a line bundle on $\overline{\mathbb{M}}_{g,n}(\alpha_c-\epsilon)$ which is a pullback of an ample line bundle on $\overline{\mathbb{M}}_{g,n}(\alpha_c)$ via $\pi$. To establish the proposition, it suffices to show that a positive multiple of $\psi-\delta$ on $\overline{\mathcal{M}}_{g,n}(\alpha_c-\epsilon)$ descends to a $\pi$-ample line bundle on $\overline{\mathbb{M}}_{g,n}(\alpha_c-\epsilon)$.

For every $(\alpha_c-\epsilon)$-stable curve $(C,\{p_i\}_{i=1}^n)$, Proposition 7.1 implies that the induced character of $\operatorname{Aut}(C,\{p_i\}_{i=1}^n)^\circ$ on both $K+\alpha_c\delta+(1-\alpha_c)\psi$ and $K+\alpha_{c+1}\delta+(1-\alpha_{c+1})\psi$ is trivial. It follows that the induced character of $\operatorname{Aut}(C,\{p_i\}_{i=1}^n)^\circ$ on $\delta-\psi$ is trivial. Applying [Alp13, Theorem 10.3], we obtain that a positive multiple of $\delta-\psi$ descends to a line bundle $\mathcal{N}$ on $\overline{\mathbb{M}}_{g,n}(\alpha_c-\epsilon)$.

To show that $\mathcal{N}^\vee$ is relatively ample over $\overline{\mathbb{M}}_{g,n}(\alpha_c)$, consider the commutative cube

$$\begin{tikzcd}
 & \mathcal{W} \arrow[dl, "f"'] \arrow[dd] & & \mathcal{W}_\chi^- \arrow[ll, hook'] \arrow[dl] \arrow[dd] \\
\overline{\mathcal{M}}_{g,n}(\alpha_c) \arrow[dd, "\phi_{\alpha_c}"] & & \overline{\mathcal{M}}_{g,n}(\alpha_c-\epsilon) \arrow[ll, hook'] \arrow[dd, "\phi_{\alpha_c-\epsilon}"] & \\
 & W/\!/G \arrow[dl] & & W_\chi^-/\!/G \arrow[ll] \arrow[dl] \\
\overline{\mathbb{M}}_{g,n}(\alpha_c) & & \overline{\mathbb{M}}_{g,n}(\alpha_c-\epsilon) \arrow[ll, "\pi"'] &
\end{tikzcd} \tag{7.2}$$

where $\mathcal{W}=[\operatorname{Spec}A/G]\to W/\!/G=\operatorname{Spec}A^G$ and $\mathcal{W}^-_{\chi_{\delta-\psi}}\to W^-_{\chi_{\delta-\psi}}/\!/G=\operatorname{Proj}\bigoplus_{d\geq 0}A_{-d}$ are the good moduli spaces as in [AFSv15, Proposition 3.3]. Since the vertical arrows are good moduli spaces, [AFS16, Proposition 2.7, Lemmas 2.8 and 3.3] imply that after shrinking $\mathcal{W}$ by a saturated open substack we can assume that $f$ sends closed points to closed points and is stabilizer preserving at closed points, and that the left and right faces of Diagram (7.2) are Cartesian. The argument in the proof of [AFS16, Theorem 1.3] concerning Diagram (7.2) shows that the bottom face is Cartesian.

The restriction of $\mathcal{N}^\vee$ to $\mathcal{W}^-_{\chi_{\delta-\psi}}$ descends to the relative $\mathcal{O}(1)$ on $W^-_{\chi_{\delta-\psi}}/\!\!/G$. Therefore, the pullback of $\mathcal{N}^\vee$ on $\overline{\mathbb{M}}_{g,n}(\alpha_c-\epsilon)$ to $W^-_{\chi_{\delta-\psi}}/\!\!/G$ is $\mathcal{O}(1)$ and, in particular, is relatively ample over $W/\!\!/G$. Since the bottom face is Cartesian, it follows by descent that $\mathcal{N}^\vee$ is relatively ample over $\overline{\mathbb{M}}_{g,n}(\alpha_c)$. The proposition follows. ☐

We proceed to prove Theorem 1.1 using Propositions 7.2, 7.3, 7.4 and Theorem 1.2.

*Proof of Theorem 1.1.* First, we show that Part (2) follows from Part (1). Indeed, suppose $K_{\overline{\mathcal{M}}_{g,n}(\alpha)} + \alpha\delta + (1-\alpha)\psi$ descends to an ample line bundle on $\overline{\mathbb{M}}_{g,n}(\alpha)$. Then

$$\overline{\mathbb{M}}_{g,n}(\alpha) \simeq \operatorname{Proj} R\big(\overline{\mathbb{M}}_{g,n}(\alpha), K_{\overline{\mathcal{M}}_{g,n}(\alpha)} + \alpha\delta + (1-\alpha)\psi\big) \simeq \overline{M}_{g,n}(\alpha),$$

where the second isomorphism is given by Proposition 7.2.

The proof of Part (1) proceeds by descending induction on $\alpha$ beginning with the known case $\alpha > 9/11$, when $\overline{\mathcal{M}}_{g,n}(\alpha) = \overline{\mathcal{M}}_{g,n}$. Let $\alpha_c \in \{\alpha_1 = 9/11, \alpha_2 = 7/10, \alpha_3 = 2/3\}$ and take $\alpha_0 = 1$. Suppose we know Part (1) for all $\alpha \geq \alpha_{c-1} - \epsilon$. By Theorem 1.2, the line bundle $K_{\overline{\mathcal{M}}_{g,n}(\alpha_{c-1}-\epsilon)} + \alpha_c\delta + (1-\alpha_c)\psi$ is nef on $\overline{\mathcal{M}}_{g,n}(\alpha_{c-1}-\epsilon)$ and all curves on which it has degree 0 are contracted by $\overline{\mathbb{M}}_{g,n}(\alpha_{c-1}-\epsilon) \to \overline{\mathbb{M}}_{g,n}(\alpha_c)$. It follows by Proposition 7.3 that the statement of Part (1) holds for all $\alpha \geq \alpha_c$. Finally, Proposition 7.4 gives the statement of Part (1) for $\alpha \geq \alpha_c - \epsilon$. ☐

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