# Stacks and Moduli

*working draft*

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# Abstract

These notes provide the foundations of moduli theory in algebraic geometry with the goal of providing self-contained proofs of the following theorems:

**Theorem A.** *The moduli space $\overline{\mathcal{M}}_g$ of stable curves of genus $g \geq 2$ is a smooth, proper, and irreducible Deligne–Mumford stack of dimension $3g-3$ which admits a projective coarse moduli space.*

**Theorem B.** *The moduli space $\mathrm{Bun}^{\mathrm{ss}}_{r,d}(C)$ of semistable vector bundles of rank $r$ and degree $d$ over a smooth, connected, and projective curve $C$ of genus $g$ is a smooth, universally closed, and irreducible algebraic stack of dimension $r^2(g-1)$ which admits a projective good moduli space.*

Along the way we develop the foundations of algebraic spaces and stacks, which provide a convenient language to discuss and establish geometric properties of moduli spaces. Introducing these foundations requires developing several themes at the same time including:

- using the functorial and groupoid perspective in algebraic geometry: we will introduce the new algebro-geometric structures of algebraic spaces and stacks,
- replacing the Zariski topology on a scheme with the étale topology: we will introduce Grothendieck topologies proving a generalization of topological spaces, and we will systematically use descent theory for étale morphisms, and
- relying on several advanced topics not typically seen in a first algebraic geometry course: properties of flat, étale and smooth morphisms of schemes, algebraic groups and their actions, deformation theory, and the birational geometry of surfaces.

Choosing a linear order in presenting the foundations is no easy task. We attempt to mitigate this challenge by relegating much of the background to appendices. We keep the main body of the notes always focused on developing moduli theory with the above two theorems in mind.

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# Chapter 0

# Introduction and motivation

### 0.0.1 Introduction to moduli spaces

Moduli spaces arise as solutions to one of the most fundamental problems in mathematics:

> **Classification problem**: Can we classify the isomorphism classes of all algebro-geometric objects of a certain type?

There are many types of objects that we may want to classify:

- subspaces $V \subset \mathbb{C}^n$ of dimension $k$;
- plane curves $C \subset \mathbb{P}^2$ of degree $d$;
- curves $C$ of genus $g$ together with a degree $d$ morphisms $C \to \mathbb{P}^1$;
- line bundles on a fixed projective variety $X$;
- representations of a group, e.g. an absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$, the fundamental group $\pi_1(\Sigma)$ of a topological surface of genus $g$, or the path algebra of quiver.

But the following two examples are our primary interest:

(1) smooth (or more generally stable) curves of genus $g$, and

(2) vector bundles (or more specifically semistable vector bundles) on a *fixed* smooth curve $C$.

They will be used throughout this book to illustrate the concepts of moduli.

> A **moduli space** is a *space* whose points are in *natural* bijection with isomorphism classes of the given algebro-geometric objects.

The keyword above is 'natural', and it is probably not clear to you what this could mean. Indeed, one of the main challenges in developing moduli theory is precisely formulating what this means. After all, any two complex manifolds or varieties of positive dimension are bijective as they both have the cardinality of the continuum. Our approach to clarify a 'natural bijection' will be to introduce the notion of a *family of objects* and require a certain relationship between maps to the moduli space and families of objects.

Moreover, the structure of the 'space' depends on the context: if we are classifying topological objects, we might hope for the structure of a topological space, while if we are classifying differential objects, we might hope for the structure of a manifold. In this book, we are mainly focused on classifying objects appearing in algebraic geometry, and we desire a moduli space with the structure of a variety, ideally a projective variety.

Once one starts viewing spaces through the lens of moduli, everything appears to be a moduli space: every space $M$ is the moduli space of its points. It is of course more interesting when there are alternative descriptions. Projective space $\mathbb{P}^1$ is the set of points in $\mathbb{P}^1$ (not so interesting) or the set of lines in the plane passing through the origin (more interesting). It is even more interesting when there are several descriptions, and even better when these viewpoints integrate several branches of mathematics. This is the case in both of our main examples in this text:

(1) a smooth algebraic curve of genus $g$ can also be viewed as a compact Riemann surface of genus $g$, and

(2) a semistable algebraic vector bundle on a fixed curve $C$ can be equivalently considered as a holomorphic vector bundle with flat unitary connection or as an irreducible unitary representation of $\pi_1(C)$.

This leads to a rich interplay between algebraic, analytic, and topological approaches. As Mumford writes in the preface of [Mum04]:

> Besides being a form of cartography, the theory of moduli spaces has the wonderful feature of having many doors, many techniques by which this theory can be developed. Of course, there is traditional algebraic geometry, but there is also invariant theory, complex-analytic techniques such as Teichmüller theory, global topological techniques, and purely characteristic $p$ methods such as counting objects over finite fields. This is another part of its charm.

**Discrete vs continuous moduli.** Depending on the types of objects, the moduli space could be discrete or continuous, or a combination of the two. We illustrate this with the following examples:

- The moduli space of line bundles on $\mathbb{P}^1$ is the discrete set $\mathbb{Z}$: every line bundle on $\mathbb{P}^1$ is isomorphic to $\mathcal{O}(n)$ for a unique integer $n \in \mathbb{Z}$.
- The moduli space parameterizing quadric plane curves $C \subset \mathbb{P}^2$ is the connected space $\mathbb{P}^5$: a curve defined by $a_0x^2 + a_1xy + a_2xz + a_3y^2 + a_4yz + a_5z^2$ is uniquely determined by the point $[a_0, \ldots, a_5] \in \mathbb{P}^5$, and as a plane curve varies continuously (i.e. by varying the coefficients $a_i$), the corresponding point in $\mathbb{P}^5$ does too.
- For smooth curves, the genus $g$ is a discrete invariant while for smooth curves of a fixed genus $g$, the moduli space $M_g$ is a variety (it is in fact an *irreducible* quasi-projective variety but we are now getting far ahead of ourselves). The moduli space of all smooth curves can thus be viewed as the disjoint union $\coprod_g M_g$.
- For vector bundles on a smooth curve $C$, the rank $r$ and degree $d$ are discrete invariants while the moduli space $\mathrm{Bun}^{\mathrm{ss}}_{C,r,d}$ of semistable bundles of rank $r$ and degree $d$ is an irreducible projective variety.

**Why study moduli spaces?** Properties of moduli spaces can inform us about the properties of the objects themselves. Many properties of objects are best formulated in terms of moduli spaces. For instance, to express the condition that a general genus 3 curve can be parameterized by an explicit coordinate system—namely a general genus 3 curve is canonically embedded into $\mathbb{P}^2$ as a plane quartic and thus parameterized by a point in the space $\mathbb{P}(\Gamma(\mathbb{P}^2, \mathcal{O}(4))) \cong \mathbb{P}^{14}$—we could say that the moduli space $M_3$ is *unirational*, i.e. there is a dominant rational map $\mathbb{P}^{14} \dashrightarrow M_3$.

**Do moduli spaces exist?** Before we can even begin to discuss the geometry of moduli spaces, we need to show that they exist. This is no easy task and is one of the major goals of this book. We develop the foundations of moduli theory in order to prove that there is a projective moduli space parameterizing stable curves of genus $g$ (Theorem A) and a projective moduli space parameterizing semistable vector bundles of rank $r$ and degree $d$ on a fixed smooth curve (Theorem B). It seems like a miraculous coincidence that these moduli spaces exist as varieties! Their existence is the starting point of moduli theory.

### 0.0.2 The trichotomy of moduli

A recurring theme in moduli is the impact of automorphism groups on both the properties of a moduli space and the techniques used to study its geometry. There is a trichotomy in moduli theory depending on the size of the automorphism groups: (1) no automorphisms, (2) finite automorphisms, and (3) infinite automorphisms. In (3), the moduli spaces are particularly well-behaved when the *closed* points of the moduli stack have *reductive* automorphisms.

| | No Auts | Finite Auts | Reductive Auts |
|---|---|---|---|
| Type of space | Scheme/algebraic space | Deligne–Mumford stack | algebraic stack |
| Defining property | Zariski/étale-locally an affine scheme | étale-locally an affine scheme | smooth-locally an affine scheme |
| Examples | $\mathbb{P}^n$, $\mathrm{Gr}(k,n)$, Hilb, Quot | $\mathcal{M}_g$ | $\mathrm{Bun}_{r,d}(C)$ |
| Quotient stacks $[X/G]$ | action is free | finite stabilizers | reductive stabilizers |
| Existence of moduli space | fine moduli space | coarse moduli space | good moduli space |

**Our approach.** In this chapter, we motivate the approach of this text by gradually adding more enriched structures to sets and groupoids. We first introduce families of objects and the functorial worldview (Section 0.3) and then develop the groupoid perspective (Section 0.4). After motivating the étale topology (Section 0.5), we combine these perspectives by introducing moduli stacks (Section 0.6) and sketch our main techniques to construct a projective moduli space (Section 0.7).

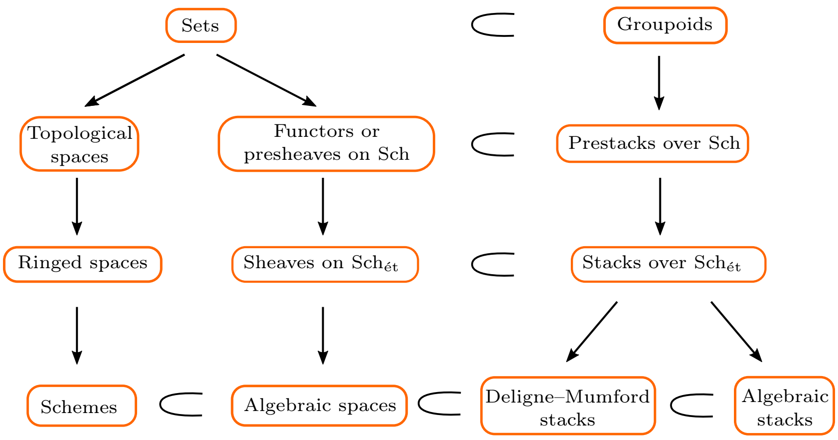



Figure 1: Schematic diagram algebro-geometric enrichments of sets and groupoids.

## 0.1 A brief history of moduli

We quickly discuss the historical development of moduli theory to provide a first glimpse of many themes in moduli.

### 0.1.1 Riemann and the origins of $M_g$

> *Die $3p-3$ übrigen Verzweigungswerthe in jenen Systemen gleichverzweigter $\mu$ werthiger Functionen können daher beliebige Werthe annehmen; und es hängt also eine Klasse von Systemen gleichverzweigter $\overline{2p+1}$ fach zusammenhängender Functionen und die zu ihr gehörende Klasse algebraischer Gleichungen von $3p-3$ stetig veränderlichen Grössen ab, welche die Moduln dieser Klasse genannt werden sollen.*
>
> *Translation: The remaining $3p-3$ branch points in these systems of similarly branching $\mu$-valued functions can therefore be assigned any given values; and thus a class of systems of similarly branching functions with connectivity $2p+1$, and the corresponding class of algebraic equations, depends on $3p-3$ continuous variables, which we shall call the moduli of the class.*
>
> Riemann [Rie1857, pg.33]

This is a remarkable sentence in a remarkable paper—Riemann both introduces the concept of 'moduli' and computes that the 'number of moduli' of $M_g$ is $3g-3$. Riemann's idea went something like this: instead of considering abstract smooth curves, let's view curves as branched covers over $\mathbb{P}^1$ and consider the moduli space

$$\mathrm{Hur}_{d,g} = \left\{ [C \to \mathbb{P}^1] \;\middle|\; \begin{array}{l} \bullet\ C \text{ is a smooth curve of genus } g \\ \bullet\ C \to \mathbb{P}^1 \text{ is a simply branched covering of degree } d \end{array} \right\}. \tag{0.1.1}$$

Formally studied later by Hurwitz [Hur1891], these moduli spaces—which are now referred to as *Hurwitz spaces*—also play an essential role in irreducibility arguments for $M_g$ (see §5.7).

A simply branched covering is a finite map of smooth curves where the ramification indices are at most two and every fiber has at most one ramification point

(Definition 5.7.1). By Riemann–Hurwitz (5.7.2), every simply branched covering $C \to \mathbb{P}^1$ is branched over $2g + 2d - 2$ distinct points of $\mathbb{P}^1$. This gives a commutative diagram

$$\begin{tikzcd}
 & \mathrm{Hur}_d(C) \arrow[r, hook] \arrow[dl] & \mathrm{Hur}_{d,g} \arrow[dl] \arrow[dr] & \\
\{[C]\} \arrow[r, hook] & M_g & & \mathrm{Sym}^{2d+2g-2}\,\mathbb{P}^1
\end{tikzcd} \tag{0.1.2}$$

where

- the map $\mathrm{Hur}_{d,g} \to \mathrm{Sym}^{2d+2g-2}\,\mathbb{P}^1$ sends a covering $[C \to \mathbb{P}^1]$ to the $2g + 2d - 2$ branched points; here $\mathrm{Sym}^N\,\mathbb{P}^1 = (\mathbb{P}^1)^N/S_N$ is the space classifying $N$ *unordered* points,
- the map $\mathrm{Hur}_{d,g} \to M_g$ is defined by $[C \to \mathbb{P}^1] \mapsto [C]$, and
- $\mathrm{Hur}_d(C)$ is the preimage of $[C] \in M_g$ under $\mathrm{Hur}_{d,g} \to M_g$, i.e. $\mathrm{Hur}_d(C)$ classifies simply branched coverings $C \to \mathbb{P}^1$ where $C$ is fixed.

If $d$ is sufficiently large, then for every general collection of $2d + 2g - 2$ points of $\mathbb{P}^1$, there exists a genus $g$ curve $C$ and a simply branched covering $C \to \mathbb{P}^1$ branched over precisely these points, and moreover there are at most finitely many such maps. In other words, $\mathrm{Hur}_{d,g} \to \mathrm{Sym}^{2d+2g-2}\,\mathbb{P}^1$ has dense image and finite fibers; see Lemma 5.7.7 for a precise statement. Therefore,

$$\begin{aligned} \dim M_g &= \dim \mathrm{Hur}_{d,g} - \dim \mathrm{Hur}_d(C) \\ &= 2d + 2g - 2 - \dim \mathrm{Hur}_d(C) \end{aligned} \tag{0.1.3}$$

To compute the dimension of $\mathrm{Hur}_d(C)$, we observe that a simply branched covering $C \to \mathbb{P}^1$ is the data of a degree $d$ line bundle $L$ and two base point free sections such that the induced map to $\mathbb{P}^1$ is simply branched. Since a general choice of two sections defines a simply branched covering (Lemma 5.7.5), we can compute

$$\dim \mathrm{Hur}_d(C) = \dim \mathrm{Pic}_d(C) + 2h^0(C, L) - 1,$$

where we subtract one since scaling any two sections will define the same map to $\mathbb{P}^1$. Riemann–Roch (5.1.4) tells us that $\mathrm{h}^0(C, L) = d + 1 - g$. On the other hand $\dim \mathrm{Pic}_d = \dim \mathrm{Pic}_0 = g$; this can be seen using the exponential sequence: $0 \to \mathbb{Z} \to \mathcal{O}_C \xrightarrow{\exp} \mathcal{O}_C^* \to 0$ yields a long exact sequence

$$\underbrace{\mathrm{H}^1(C, \mathbb{Z})}_{\mathbb{Z}^{2g}} \to \underbrace{\mathrm{H}^1(C, \mathcal{O}_C)}_{\mathbb{C}^g} \to \underbrace{\mathrm{H}^1(C, \mathcal{O}_C^*)}_{\mathrm{Pic}(C)} \xrightarrow{\deg} \underbrace{\mathrm{H}^2(C, \mathbb{Z})}_{\mathbb{Z}}, \tag{0.1.4}$$

and provides an identification $\mathrm{Pic}_0(C) \cong \mathbb{C}^g/\mathbb{Z}^{2g}$. We conclude that $\dim \mathrm{Pic}_0(C) = g$ and $\dim \mathrm{Hur}_d(C) = g + 2(d + 1 - g) - 1 = 2d - g + 1$. Plugging this into (0.1.3) yields

$$\dim M_g = (2d + 2g - 2) - (2d - g + 1) = 3g - 3.$$

Riemann in fact gave several other heuristic arguments computing the dimension of $M_g$. See [GH78, pg. 255-257] or [Mir95, pg. 211-215] for further discussion on the number of moduli of $M_g$, and see [AJP16] for historical background of Riemann's computations.

**Riemann's moduli problem**: Does $M_g$ exist as a complex analytic space?

While Riemann's argument can be made completely rigorous with today's methods (as we do ourselves later in this text), there are foundational issues with Riemann's method—today we would say that Riemann computed the dimension of a 'local deformation space.' Most notably, $M_g$ was not known to exist and it wasn't clear what type of space $M_g$ was supposed to be. Despite this, Riemann had an instinctive grasp of its geometry—in fact in the same paper [Rie1857], Riemann introduced the word 'Mannigfaltigkeit' (or 'manifold') to describe its geometry. Manifolds were not formally defined until much later in the 1940s following Teichmüller, Chern, and Weil. Riemann was aware of the foundational issues in geometry:

> *It is well known that geometry presupposes not only the concept of space but also the first fundamental notions for constructions in space as given in advance. It only gives nominal definitions for them,... while the relationship of these presumptions is left in the dark... From Euclid to Legendre, to name the most renowned of modern writers on geometry, this darkness has been lifted neither by the mathematicians nor the philosophers who have laboured upon it.* —Riemann 1854

Riemann's work has inspired countless mathematicians to lift us out of this darkness.

> *The spirit of Riemann will move future generations as it has moved us.* —Ahlfors [Ahl53, p. 53]

> *It is difficult to recall another example in the history of 19th century mathematics when a struggle for a rigorous proof led to such productive results.* —Monastyrsky [Mon87, p. 41]

### 0.1.2 Moduli of curves of low genus

**Genus 0.** For $n \geq 3$, the moduli space $M_{0,n}$ of smooth genus 0 curves with $n$ ordered distinct points can be described as

$$M_{0,n} = \left(\mathbb{P}^1 \setminus \{0, 1, \infty\}\right)^{n-3} \setminus (\text{all diagonals}).$$

Indeed, given $n$ ordered distinct points $p_1, \ldots, p_n$ on $\mathbb{P}^1$, there is a unique automorphism $g \in \mathrm{Aut}(\mathbb{P}^1) \cong \mathrm{PGL}_2$ taking $(p_1, p_2, p_3)$ to $(0, 1, \infty)$. When $n = 4$, we obtain that a bijection $M_{0,4} = \mathbb{P}^1 \setminus \{0, 1, \infty\}$ given by the classical cross-ratio of four points in $\mathbb{P}^1$ first discovered by Pappus of Alexandria [Ale86] in 300 AD (see also Example 6.7.8).

**Genus 1.** Every elliptic curve $(E, p)$, i.e. a smooth genus 1 curve $E$ with a marked point $p \in E$, can be described as a plane cubic in Weierstrass form

$$E_\lambda = V(y^2 z - x(x - z)(x - \lambda z)) \subset \mathbb{P}^2$$

for some $\lambda \neq 0, 1$, where $p = [0 : 1 : 0] \in E_\lambda$. However, the choice of $\lambda$ is not unique: the values $\lambda$, $1/\lambda$, $1 - \lambda$, $1/(1 - \lambda)$, $\lambda/(\lambda - 1)$, and $(\lambda - 1)/\lambda$ determine isomorphic elliptic curves. In other words, the map $\mathbb{A}^1 \setminus \{0,1\} \to M_{1,1}$ given by $\lambda \mapsto [E_\lambda]$ is a 6-to-1 surjective map. The $j$-invariant on the other hand

$$j = 2^8 \frac{(\lambda^2 - \lambda + 1)^3}{\lambda^2(\lambda - 1)^2}$$

uniquely determines the isomorphism class of the curve and thus gives a bijection $M_{1,1} \cong \mathbb{A}^1$. For a modern treatment, see [Har77, §4].

**Genus 2.** Every smooth genus 2 curve $C$ is hyperelliptic and can be written as a double cover $y^2 = (x - a_1) \cdots (x - a_6)$ over $\mathbb{P}^1$. This is a consequence of the sheaf of differentials $\Omega_C$ being a base point free line bundle of degree 2 with 2 global sections; the induced map $C \to \mathbb{P}^1$ is ramified over 6 points by Riemann–Hurwitz (5.7.2). We obtain the description that

$$M_2 = \big(\Gamma(\mathbb{P}^1, \mathcal{O}(6)) \setminus \Delta\big) / \operatorname{GL}_2,$$

where $\Delta \subset \Gamma(\mathbb{P}^1, \mathcal{O}(6))$ denotes the locus of binary sextics with a double root. After a projective change of coordinates on $\mathbb{P}^1$, we can arrange that the curve is ramified over 0, 1, $\infty$ and 3 other points $a_4, a_5, a_6 \in \mathbb{P}^1 \setminus (0, 1, \infty)$. In this way, we obtain a surjective map $(\mathbb{P}^1 \setminus \{0, 1, \infty\})^3 \setminus \Delta \to M_2$.

Invariant theory of binary sextics (see [Cle1870]) provides an even sharper description: the ring of invariant polynomials, i.e. polynomials in $a_0, \ldots, a_6$ that are invariant under automorphisms of $\mathbb{P}^1$, is generated by invariants $J_2$, $J_4$, $J_6$, $J_8$, $J_{10}$, and $J_{15}$, whose degree is indicated by the subscript, with a single relation $J_{15}^2 = G(J_2, J_4, J_6, J_{10})$ for some polynomial $G$. The invariant $J_{10}$ is the discriminant while $J_{15}$ does not affect the scheme structure. This yields that $M_2$ is an open subset of weighted projective space

$$M_2 = \operatorname{Proj} \mathbb{C}[J_2, J_4, J_6, J_{10}] \setminus \{J_{10} = 0\},$$

and thus $M_2$ is an affine variety embedded into $\mathbb{A}^8$ via

$$\frac{J_2^5}{J_{10}}, \frac{J_2^3 J_4}{J_{10}}, \frac{J_2^2 J_6}{J_{10}}, \frac{J_2 J_4^2}{J_{10}}, \frac{J_2 J_6^3}{J_{10}^2}, \frac{J_4^5}{J_{10}^2}, \frac{J_4 J_6}{J_{10}}, \frac{J_6^5}{J_{10}^3}.$$

We can identify this coordinate ring with the invariant ring of the action on $\mathbb{Z}/5$ on $\mathbb{A}^3$ where a generator $\zeta \in \mathbb{Z}/5$ acts via $\zeta \cdot (x, y, z) = (\zeta x, \zeta^2 y, \zeta^3 z)$; the above functions are identified with the invariants $x^5$, $x^3 y$, $x^2 z$, $xy^2$, $xz^3$, $y^5$, $yz$, $z^5$. This yields the rather elegant global description

$$M_2 = \mathbb{A}^3 / (\mathbb{Z}/5).$$

This was studied classically by Bolza [Bol1887] and more recently by Igusa [Igu60].

**Genus 3.** A non-hyperelliptic smooth genus 3 curve embeds as a quartic in $\mathbb{P}^2$ under the canonical embedding. Letting $\Delta \subset \Gamma(\mathbb{P}^1, \mathcal{O}(4))$ be the locus of singular quartics, we obtain a description of the open locus of non-hyperelliptic curves as the quotient $\big(\Gamma(\mathbb{P}^1, \mathcal{O}(4)) \setminus \Delta\big) / \operatorname{GL}_3$—this is the first time that our description only describes a *general* curve. On the other hand, a hyperelliptic genus 3 curve is double cover of $\mathbb{P}^1$ ramified over 8 points, and we obtain a *set-theoretic* decomposition

$$M_3 = \underbrace{\big(\Gamma(\mathbb{P}^1, \mathcal{O}(4)) \setminus \Delta\big) / \operatorname{GL}_3}_{\dim = 6} \coprod \underbrace{\big(\Gamma(\mathbb{P}^1, \mathcal{O}(8)) \setminus \Delta\big) / \operatorname{GL}_2}_{\dim = 5},$$

suggesting that the locus of hyperelliptic curves is a divisor in $M_3$.

**Genus 4.** A non-hyperelliptic smooth curve $C$ of genus 4 embeds into $\mathbb{P}^3$ under its canonical embedding, and can be realized as the intersection $C = Q \cap S$ of a quadric $Q$ and cubic $S$. This gives a rational map $\Gamma(\mathbb{P}^3, \mathcal{O}(2)) \times \Gamma(\mathbb{P}^3, \mathcal{O}(3)) \dashrightarrow M_4$ whose image is the locus of non-hyperelliptic curves; as above the hyperelliptic locus can be parameterized by $\big(\Gamma(\mathbb{P}^1, \mathcal{O}(10)) \setminus \Delta\big) / \operatorname{GL}_2$. Alternatively, a general non-hyperelliptic smooth genus 4 curve can be realized as the normalization of a plane quintic with precisely two nodes, or as a degree 3 cover of $\mathbb{P}^1$ branched over 12 points.

**Genera 5–10.** Classically, general curves were described either as plane curves with prescribed singularities via the image of a map $C \to \mathbb{P}^2$, or as branched covers $C \to \mathbb{P}^1$. For a general genus $g$ curve $C$, the smallest degree $d$ such that $C$ is realized as normalization of a singular plane curve is $d = \lfloor \frac{2g+8}{3} \rfloor$. If the plane curve had at worst nodal singularities, then the number of nodes is $\delta := (d-1)(d-2)/2 - g$. Meanwhile, the minimum degree of a map $C \to \mathbb{P}^1$ is $\lfloor \frac{g+3}{2} \rfloor$.

| $g$ | $d =$ min degree of $\operatorname{im}(C \to \mathbb{P}^2)$ | $\delta = \#$ of nodes | $\frac{(d+1)(d+2)}{2} - 3\delta$ | min degree of $C \to \mathbb{P}^1$ | # of branch pts |
|---|---|---|---|---|---|
| 0 | 1 | 0 | 3 | 1 | 0 |
| 1 | 3 | 0 | 10 | 2 | 4 |
| 2 | 4 | 1 | 12 | 2 | 6 |
| 3 | 4 | 0 | 15 | 3 | 10 |
| 4 | 5 | 2 | 15 | 3 | 12 |
| 5 | 6 | 5 | 13 | 4 | 16 |
| 6 | 6 | 4 | 16 | 4 | 18 |
| 7 | 7 | 8 | 12 | 5 | 22 |
| 8 | 8 | 13 | 6 | 5 | 24 |
| 9 | 8 | 12 | 9 | 6 | 28 |
| 10 | 9 | 18 | 1 | 6 | 30 |
| 11 | 10 | 25 | -9 | 7 | 34 |

Table 1: General curves of low genus

In [Sev15] and [Sev21], Severi used such descriptions to show that $M_g$ is unirational for $g \leq 10$. Like other mathematicians of his era, Severi did not precisely formulate what it meant for $M_g$ to be a moduli space.

**What goes wrong for $g \geq 11$?** As the genus grows, it becomes more difficult to describe a general genus $g$ curve. To give an indication of the challenges for $g \geq 11$, let's try to describe a genus $g$ curve as a degree $d = \lfloor \frac{2g+8}{3} \rfloor$ planar curve with $\delta = (d-1)(d-2)/2 - g$ nodes at prescribed points $p_1, \ldots, p_\delta \in \mathbb{P}^2$. If the plane curve defined by $f \in \Gamma(\mathbb{P}^2, \mathcal{O}(d))$ has a node at each $p_i$, then the equations $f_x(p_i) = f_y(p_i) = f_z(p_i) = 0$ imposes $3\delta$ linear equations on $\Gamma(\mathbb{P}^2, \mathcal{O}(d))$. For such nodal plane curves to exist, we would need

$$\dim \Gamma(\mathbb{P}^2, \mathcal{O}(d)) - 3\delta = \frac{(d+1)(d+2)}{2} - 3\left(\frac{(d-1)(d-2)}{2} - g\right) > 0.$$

As illustrated by Table 1, $g = 11$ is the first case where this does not hold!

**Severi's conjecture.** Severi conjectured that $M_g$ is unirational for all $g$: "Ritengo probabile che la varietà sia razionale o quanto meno che sia riferibile ad un'involuzione di gruppi di punti in uno spazio lineare..." [Sev15, pg. 881]. While this conjecture turned out to be false, it motivated mathematicians for decades: "Whether more $M_g$'s, $g \geq 11$, are unirational or not is a very interesting problem, but one which looks very hard too, especially if $g$ is quite large" [Mum75, pg. 37]. In the 1980s, Eisenbud, Harris, and Mumford disproved this conjecture and showed that in some sense quite the opposite is true in large genus: $M_g$ is general type for $g \geq 24$ [HM82], [EH87].

**Petri's description of canonical curves.** While most 19th and early 20th century mathematicians described curves as either plane singular curves or as covers of $\mathbb{P}^1$, Petri's explicit description [Pet23] of canonically embedded curves was an exception and is more reminiscent of modern approaches. As Mumford writes in [Mum75, p.17], Petri's approach "is unavoidably a bit messy, but just to be able to brag, I think it is a good idea to be able to say 'I have seen *every* curve once.'" Building on M. Noether's result [Noe1880] that the canonical embedding $C \hookrightarrow \mathbb{P}^{g-1}$ of a non-hyperelliptic smooth curve $C$ is projectively normal—that is, $\varphi\colon \operatorname{Sym}^* \Gamma(C, \Omega_C) \to \bigoplus_{d \geq 0} \Gamma(C, \Omega_C^{\otimes d})$ is surjective—and also building on work of Enriques [Enr19] and Babbage [Bab39], Petri showed that the homogeneous ideal $I = \ker \varphi$ is generated by quadrics unless $C$ is a plane quintic ($g = 6$) or trigonal (i.e. a triple covering of $\mathbb{P}^1$) in which case $I$ is generated in degree 2 and 3. Petri's analysis was remarkably constructive leading to explicit equations in $\mathbb{P}^{g-1}$ cutting out $C$ along with explicit syzygies among the equations. Petri's work continues to inspire research in the theory of moduli and syzygies. We won't cover this perspective further in this text but we recommend [SD73], [Mum75, pgs. 17-21], [AS78], [Gre82], [Gre84], and [ACGH85, §III.3].

### 0.1.3 Analytic approaches and the Teichmüller space

In the late 19th and early 20th century, Riemann surfaces were described as quotients of the upper half plane by a discrete subgroup of $\mathrm{PSL}_2(\mathbb{R})$; such subgroups are named *Fuchsian groups* after Fuchs [Fuc1866]. Fricke and Klein classified Fuchsian groups using the theory of automorphic functions in their 1300 page volumes [FK1892], [FK12]. They constructed what's now known as the Teichmüller space, showed that it is a contractible space, and even exhibited complex structures. Torelli showed that a Riemann surface can be constructed from its Jacobian [Tor13], and Siegel constructed the moduli space $A_g$ of abelian varieties of dimension $g$ as an analytic space [Sie35].

Teichmüller[1] was the first to give a precise formulation of Riemann's moduli problem, to construct $M_g$ as a complex analytic space, and to interpret $3g - 3$ as its complex dimension [Tei40], [Tei44]. Teichmüller constructed the *Teichmüller space* $T_g$ parameterizing complex structure on a topological surface $\Sigma_g$ of genus $g$ up to homeomorphism. The space $T_g$ is homeomorphic to a ball in $\mathbb{C}^{3g-3}$ and inherits an action of the *mapping class group* $\Gamma_g$ of diffeomorphisms of $\Sigma_g$ modulo the subgroup of diffeomorphisms isotopic to the identity. This action is properly discontinuous, and $M_g$ is realized as the quotient $T_g/\Sigma_g$. Although largely forgotten for nearly 20 years, Teichmüller theory was later greatly expanded by Ahlfors, Bers,

[1] Bers' use of the famous quote of Plutarch (Perikles 2.2) to describe Teichmüller in [Ber60a] seems fitting: "It does not of necessity follow that, if the work delights you with its grace, the one who wrought it is worthy of your esteem."

and Weil among others; see [Wei57], [Wei58], [AB60], [Ber60b], and [Ahl61]. For modern expository treatments, see [Ber72], [Hub06], and [FM12].

Teichmüller also introduced the notion of families of Riemann surfaces and showed that the Teichmüller space satisfies a universal property. Grothendieck, in a series of lectures at Cartan's seminar [Gro61], developed a general theory of analytic moduli spaces in the language of categories and functors, reformulated Teichmüller theory in this setting, and showed that $T_g$ represents a functor parameterizing families of Riemann surfaces. This set the stage for Grothendieck's later work on algebraic moduli: "One can hope that we shall be able one day to eliminate analysis completely from the theory of Teichmüller space, which should be purely geometric" [Gro61, Lecture I].

### 0.1.4 The origins of algebraic moduli theory

> *As for $M_g$ there is virtually no doubt that it can be provided with the structure of an algebraic variety.*
>
> Weil [Wei58, pg. 383]

**Cayley, Gordan, and Hilbert.** The invariant theorists of the 19th century were interested in classifying homogeneous polynomials of degree $d$ in $n$ variables up to projective automorphisms, or in other words in the moduli space $\Gamma(\mathbb{P}^{n-1}, \mathcal{O}(d))/\operatorname{PGL}_n$. They attempted to describe this moduli space by exhibiting explicit invariant polynomials in the coefficients $a_I$ of a polynomial $f = \sum_I a_I x^I$. The origins of invariant theory lie in work by Boole [Boo1841] and Cayley [Cay1845], and were further developed by Gordan and Hilbert along with many others. Gordan exhibited explicit generators of the ring of invariants of binary forms ($n = 2$) [Gor1868] and Hilbert later proved that the ring of invariants is finitely generated for any ring [Hil1890], [Hil1893].

Cayley constructed the moduli space of curves in $\mathbb{P}^3$ [Cay1860], [Cay1862], which is now referred to as the *Chow variety*. His idea was to associate to a degree $d$ curve $C \subset \mathbb{P}^3$ the set of lines $L \subset \mathbb{P}^3$ meeting $C$; this is a hypersurface of degree $d$ in $\operatorname{Gr}(1, \mathbb{P}^3)$, and the Chow variety is the closure of all hypersurfaces in $\Gamma(\operatorname{Gr}(1, \mathbb{P}^3), \mathcal{O}(d))$ obtained this way. The general theory of Chow varieties parameterizing subschemes in $\mathbb{P}^n$ of any dimension was later developed by Chow and van der Waerdan [CW37].

**Weil.** In Weil's work on the Riemann hypothesis for curves over finite fields [Wei48], he needed to construct the Jacobian of a curve parameterizing degree 0 line bundles. At that point, varieties had only been considered as embedded in affine or projective space, and in his foundational work [Wei62] Weil enlarged the category to *abstract varieties*. This was enough to construct the Jacobian and give a proof—in fact his second proof—of the Riemann hypothesis for curves. Later Weil and Chow independently showed that the Jacobian was projective.

**Baily.** Baily constructed the moduli space $A_g$ of principally polarized abelian varieties as a quasi-projective variety [Bai60a], [Bai60b], showed that Satake's topological compactification [Sat56] is algebraic [Bai58], and together with Borel introduced what's now known as the *Baily–Borel compactification* [BB66]. Using the period map $M_g \to A_g$ associating a curve to its Jacobian and Torelli's theorem that this map is injective, Baily concluded that $M_g$ has the structure of a quasi-projective variety. However, he did not prove that this provided a 'natural' structure of a variety nor that it had any uniqueness properties, i.e. that $M_g$ is a coarse moduli space.

> *Thirdly, in order to call $\mathcal{E}$ the variety of moduli of Riemann surfaces of genus $n$, one should be able to state that it is unique and in some sense universal among normal parameter varieties of algebraic systems of curves of genus $n$. Namely, given any normal algebraic system of curves of genus $n$ there should exist a natural map of the parameter variety of the nonsingular members of this system into $\mathcal{E}$.* — Baily [Bai60b, pgs. 59-60]

Mumford credits Baily for the quasi-projectivity of $M_g$ in [Mum75, p. 98] just as Gieseker does in his commentary in [Mum04].

**Grothendieck.** After Grothendieck's formalization of analytic moduli theory, in his 'FGA series' [$\mathrm{FGA}_{\mathrm{I}}$]–[$\mathrm{FGA}_{\mathrm{VI}}$], he applied his functorial approach to algebraic geometry. He defined the Hilbert, Quot, and Picard functors, and showed that they are representable by projective schemes. Grothendieck of course later reformulated the entire foundations of algebraic geometry by developing scheme theory. His profound influence on algebraic geometry and more broadly mathematics helped shape the future of moduli theory.

Although he didn't publish on $M_g$, Grothendieck was nevertheless very much interested in moduli theory and the existence of $M_g$ as a quasi-projective variety—see his email correspondence with Mumford in the early 1960s [Mum10, §II]. Grothendieck was aware that the presence of automorphisms obstructed the representability of the functor parameterizing smooth families of curves. He rigidified the moduli problem by also parameterizing a level structure on a curve. While he could show that the functor of smooth curves with level structure $r \geq 3$ was representable by a scheme, he struggled to show that it was quasi-projective. The idea was to construct $M_g$ as a quotient of the rigidified moduli space by modding out by the finite group acting on the choice of level $n$ structure. The lack of quasi-projectivity impeded this approach as the quotient of a non-quasi-projective variety by a finite group need not exist as a variety (see Example 0.5.5).

Grothendieck was also interested in the connectedness of $M_g$ in positive characteristic. In an email to Mumford on April 25, 1961 [Mum10, p. 638], Grothendieck wrote: "Yet I have some hope to prove the connectedness of the $M_{g,n}$ (arbitrary levels) using the transcendental result in char. 0 and the connectedness theorem; but first one should get a natural "compactification" of $M_{g,n}$ which should be simple over $\mathbb{Z}$." Deligne and Mumford [DM69] later provided the compactification $\overline{\mathcal{M}}_{g,n}$ and applied it to prove the connectedness of $M_g$ in all characteristics—this argument is given in §5.7.

**Mumford.** Motivated by Riemann's moduli space as well as by constructions of Chow varieties, Picard varieties, and the moduli of abelian varieties in the early 20th century, Mumford made immense contributions to the foundations of moduli theory. He was the first to systematically study their geometry.

> *The point is, I love maps, that is "maps" in the sense of "maps of the world," "charts of the ocean," "atlases of the sky"! I think one of the key things that attracted me to the group of problems was the hope of making a* map *of some parts of the world of algebraic varieties. An algebraic*

*variety felt like a tangible thing in the lectures of Oscar Zariski, so why shouldn't you venture out, like Magellan, and uncover their geography?*
—Mumford, preface of [Mum04]

By integrating Grothendieck's formalism of scheme theory with 19th century invariant theory, Mumford developed a theory of quotients in algebraic geometry now known as *Geometric Invariant Theory* (or *GIT*), and then applied this theory to construct both $M_g$ and $A_g$. His theory was originally sketched in [Mum61] and fully worked out in the definitive text [GIT].

Later, Mumford constructed a quasi-projective variety parameterizing *stable* vector bundles on a fixed smooth curve [Mum63], and Seshadri then showed the moduli space of *semistable* vector bundles provides a projective compactification [Ses67]. In the seminal joint work [DM69], Deligne and Mumford not only introduce the compactification $\overline{\mathcal{M}}_{g,n}$ and apply it (as noted above) to prove the connectedness of $M_g$ in all characteristics, they also first introduce the notion of an algebraic stack—now referred to as *Deligne–Mumford stacks*.

**Artin.** The theory of algebraic spaces and stacks was developed by M. Artin. Similar to Weil's enlargement of affine and projective varieties to abstract varieties, enlarging the category of schemes to algebraic spaces allows us to construct the quotient of finite group actions or more generally any étale equivalence relation.[2] Knutson, a student of Artin, was the first to write down the theory of algebraic spaces [Knu71].

In 1969, Artin proved two crucial results in moduli theory—Artin Approximation (Theorem A.10.9) and Artin Algebraization (Theorem D.6.6). In his groundbreaking paper [Art74], Artin not only introduced the concept of algebraic stacks broadening the definition of Deligne and Mumford, but he also provided a local deformation-theoretic characterization of algebraic stacks. This is known as 'Artin's Criteria' and can be used to verify the algebraicity of a moduli stack (see §D.7).

As Faltings said: "The notion of stack came up in the sixties. But to swallow schemes was already enough for one generation of mathematicians." The theory of stacks once had a formidable reputation and a somewhat questionable foundation. This could be in part due to the abstract categorical nonsense involved in its formulation and to 2-categorical subtleties, or perhaps due to shifting conditions on the diagonal of an algebraic stack in the literature. Or it could be partly due to that Deligne and Mumford did not give proofs for their results on algebraic stacks in [DM69, §4]—they write: "The proofs of the results stated in this section will be given elsewhere." Sure enough, future mathematicians worked out the details, and there are now excellent textbooks on stacks such as [LMB00] and [Ols16]. Of course, the *Stacks Project* [SP] has now provided an unquestionably solid foundation.

For further historical background, we recommend [Mum75], [Oor81], [Kle05], [JP13], [AJP16], and [Kol18].

## 0.2 Moduli sets of curves, vector bundles, and triangles

To define a moduli space as a set entails specifying two things:

[2] Matsusaka also built a theory of $\mathbb{Q}$-*varieties*by considering certain quotients of equivalence relations [Mat64] but it was not as robust as algebraic spaces.

(1) a class of certain types of objects, and

(2) an equivalence relation on objects.

Here's our first attempt at defining $M_g$:

**Example 0.2.1** (Moduli set of smooth curves)**.** The objects of the *moduli set of smooth curves*, denoted as $M_g$, are smooth, connected, and projective curves of genus $g$ over $\mathbb{C}$. Two curves are declared equivalent if they are isomorphic. There are many variants obtained by parameterizing additional structures or choosing different equivalence relations.

- We already saw the Hurwitz moduli set $\mathrm{Hur}_{d,g}$ in (0.1.1) parameterizing branched covers $C \to \mathbb{P}^1$ of degree $d$.
- The moduli set $M_{g,n}$ of $n$-pointed smooth genus $g$ curves parameterizes the data of a smooth curve $C$ together with $n$ ordered distinct points $p_1, \dots, p_n \in C$; two objects $(C, p_i) \sim (C', p'_i)$ are equivalent if there is an isomorphism $\alpha \colon C \to C'$ with $\alpha(p_i) = p'_i$.
- The moduli set $M_g[n]$ of smooth genus $g$ curves with level $n$ structure parameterizes smooth, connected, and projective curves $C$ of genus $g$ over $\mathbb{C}$ together with a basis $(\alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g)$ of $H_1(C, \mathbb{Z}/n\mathbb{Z})$ such that the intersection pairing is symplectic, while two objects $(C, \alpha_i, \beta_i) \sim (C', \alpha'_i, \beta'_i)$ are declared equivalent if there is an isomorphism $C \to C'$ taking $\alpha_i$ and $\beta_i$ to $\alpha'_i$ and $\beta'_i$.
- For the moduli set whose objects are plane curves $C \subset \mathbb{P}^2$, there are several choices for equivalence relations $C \sim C'$: (a) $C$ and $C'$ are equal as subschemes, (b) $C$ and $C'$ are projectively equivalent (i.e. there is an automorphism of $\mathbb{P}^2$ taking $C$ to $C'$), or (c) $C$ and $C'$ are abstractly isomorphic.

**Example 0.2.2** (Moduli set of vector bundles on a curve)**.** The moduli set $\mathrm{Bun}_{r,d}(C)$ parameterizes vector bundles of rank $r$ and degree $d$ on a fixed smooth, connected, and projective curve $C$; the equivalence relation here is isomorphism. The special case of $r = 1$ yields the set $\mathrm{Pic}^d(C)$ parameterizing degree $d$ line bundles on $C$. This is non-canonically identified with with the abelian variety $\mathrm{H}^1(C, \mathcal{O}_C)/\mathrm{H}^1(C, \mathbb{Z}) = \mathbb{C}^g/\mathbb{Z}^{2g}$ via the exponential exact sequence (0.1.4).

A recurring theme in moduli is the exhibition of moduli spaces as quotients of group actions.

**Example 0.2.3** (Moduli set of orbits)**.** Given a group action of a group $G$ on a set $X$, we define the *moduli set of orbits* by taking the objects to be all elements $x \in X$ and by declaring $x$ to be equivalent to $x'$ if they have the same orbit $Gx = Gx'$. In other words, the moduli set of orbits is the quotient set $X/G$.

Some examples to keep in mind are the $\mathbb{Z}/2$-action on $\mathbb{A}^1$ via $(-1) \cdot x = -x$ and the usual scaling action of $\mathbb{G}_m$ on $\mathbb{A}^n$ via $t \cdot (x_1, \dots, x_n) = (tx_1, \dots, tx_n)$. The quotient set $(\mathbb{A}^n \setminus 0)/\mathbb{G}_m$ is identified with $\mathbb{P}^{n-1}$. The quotient $\mathbb{A}^n/\mathbb{G}_m$ including the origin—and particularly the case of $\mathbb{A}^1/\mathbb{G}_m$—shows up repeatedly in this text. Another interesting example is the $\mathbb{G}_m$-action on $\mathbb{A}^2$ given by $t \cdot (x, y) = (tx, t^{-1}y)$.

### 0.2.1 Toy example: moduli of triangles

Before diving deeper into $M_g$ and $\mathrm{Bun}_{r,d}(C)$, let's study the simple yet surprisingly fruitful example of the moduli of triangles. These moduli spaces are easy to visualize and, as M. Artin has remarked, are useful to illustrate various themes of stacks and moduli.

**Example 0.2.4** (Labelled triangles)**.** A *labeled triangle* is a triangle in $\mathbb{R}^2$ where the vertices are labeled with '1', '2' and '3', and the distances of the edges are denoted as $a$, $b$, and $c$. We require that triangles have nonzero area or equivalently that their vertices are not collinear.

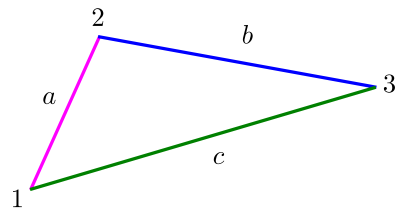



Figure 2: To keep track of the labeling, we color the edges.

We define the *moduli set of labeled triangles* $M$ as the set of labeled triangles where two triangles are said to be equivalent if they are the same triangle in $\mathbb{R}^2$ with the same vertices and same labeling. By writing $(x_1, y_1)$, $(x_2, y_2)$ and $(x_3, y_3)$ as the coordinates of the labeled vertices, we obtain a bijection

$$M \cong \left\{(x_1, y_1, x_2, y_2, x_3, y_3) \mid \det \begin{pmatrix} x_2 - x_1 & x_3 - x_1 \\ y_2 - y_1 & y_3 - y_1 \end{pmatrix} \neq 0\right\} \subset \mathbb{R}^6 \tag{0.2.1}$$

with the open subset of $\mathbb{R}^6$ whose complement is the codimension 1 closed subset defined by the condition that the vectors $(x_2, y_2) - (x_1, y_1)$ and $(x_3, y_3) - (x_1, y_1)$ are linearly dependent.

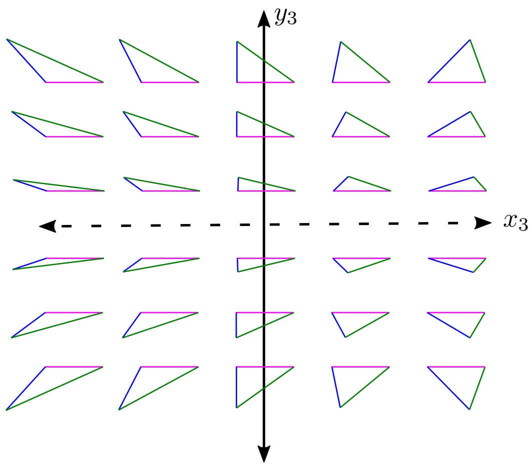



Figure 3: Picture of the slice of the moduli space $M$ where $(x_1, y_1) = (0, 0)$ and $(x_2, y_2) = (1, 0)$. Triangles are described by their third vertex $(x_3, y_3)$ with $y_3 \neq 0$. We've drawn representative triangles for a handful of points in the $x_3y_3-$ plane.

**Example 0.2.5** (Labelled triangles up to similarity)**.** We define the *moduli set of labeled triangles up to similarity*, denoted by $M^{\text{lab}}$, by taking the same class of objects as in the previous example—labeled triangles—but changing the equivalence relation to label-preserving similarity.

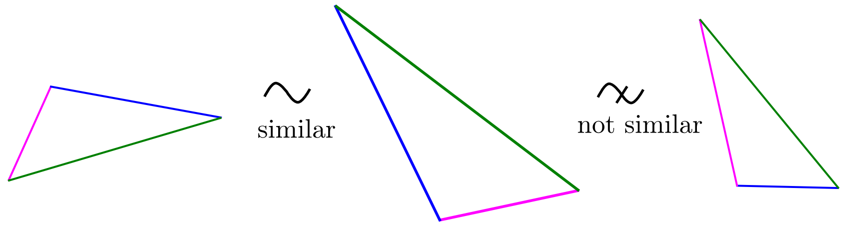


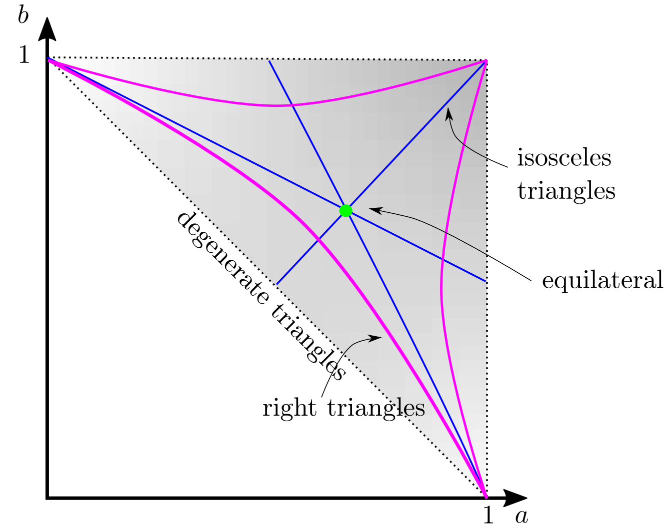

Figure 4: The two triangles on the left are similar, but the third is not.

Every labeled triangle is similar to a unique labeled triangle with perimeter $a+b+c=2$. We have the description

$$M^{\text{lab}} = \left\{ (a,b,c) \;\middle|\; \begin{array}{l} a+b+c=2 \\ 0<a<b+c \\ 0<b<a+c \\ 0<c<a+b \end{array} \right\}. \tag{0.2.2}$$

By setting $c = 2-a-b$, we may visualize $M^{\text{lab}}$ as the analytic open subset of $\mathbb{R}^2$ defined by pairs $(a,b)$ satisfying $0 < a, b < 1$ and $a+b > 1$.

Figure 5: $M^{\text{lab}}$ is the shaded triangle. The pink lines represent the right triangles defined by $a^2+b^2=c^2$, $a^2+c^2=b^2$ and $b^2+c^2=a^2$, the blue lines represent isosceles triangles defined by $a=b$, $b=c$ and $a=c$, and the green point is the unique equilateral triangle defined by $a=b=c$.




**Example 0.2.6** (Unlabeled triangles up to similarity)**.** We now turn to the moduli of unlabeled triangles up to similarity, which reveals a new feature not seen in to the two previous examples: symmetry!

We define the *moduli set of unlabeled triangles up to similarity*, denoted by $M^{\text{unl}}$, where the objects are unlabeled triangles in $\mathbb{R}^2$ and the equivalence relation is similarity. We can describe an unlabeled triangle uniquely by the ordered tuple $(a,b,c)$ of increasing side lengths as follows:

$$M^{\text{unl}} = \left\{ (a,b,c) \;\middle|\; \begin{array}{l} 0<a\le b\le c<a+b \\ a+b+c=2 \end{array} \right\}. \tag{0.2.3}$$

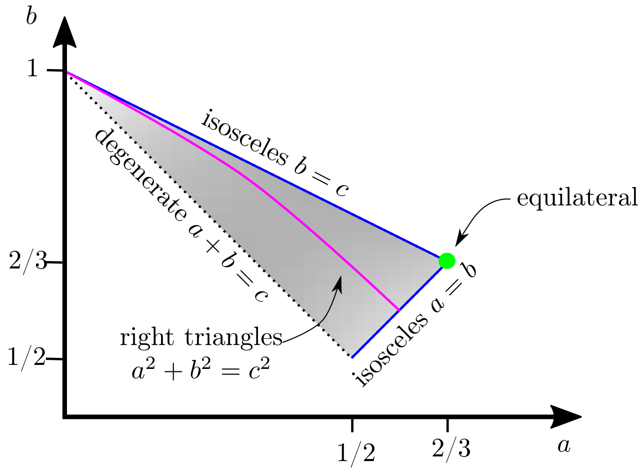



Figure 6: Picture of $M^{\text{unl}}$

The isosceles triangles with $a = b$ or $b = c$ and the equilateral triangle with $a = b = c$ have symmetry groups of $\mathbb{Z}/2$ and $S_3$, respectively. This is unfortunately not encoded into our description $M^{\text{unl}}$ above. Note that we can identify $M^{\text{unl}}$ as the quotient $M^{\text{lab}}/S_3$ under the natural action of $S_3$ on the labelings, and that the stabilizers of isosceles and equilateral triangles are precisely their symmetry groups $\mathbb{Z}/2$ and $S_3$. The action of $S_3$ on the locus of triangles that are not isosceles or equilateral is free.

## 0.3 The functorial worldview

Defining a moduli functor requires specifying:

(1) families of objects,

(2) when two families of objects are equivalent, and

(3) how families pull back under morphisms.

### 0.3.1 Family matters

Introducing families allows us to give a precise formulation of the moduli problem. A family $\mathcal{F} \to S$ of objects defines a set-theoretic map $S \to M$ to the moduli space by assigning a point $s \in S$ to the fiber $\mathcal{F}_s$ (or more precisely the pullback of $\mathcal{F} \to S$ under the inclusion $\{s\} \hookrightarrow S$). We will require that $S \to M$ is a morphism in whatever category we are working in (e.g. topological spaces or schemes).

We say that $M$ is a *fine moduli space* if there is a bijective correspondence between families over $S$ and morphisms $S \to M$, or in other words that the space $M$ *represents* the functor assigning a space $S$ to the set of all families over $S$.

Here are some advantages of defining families:

- We can endow the set $M$ of objects in the moduli problem with a topological space: an arbitrary subset $U \subset M$ is declared to be *open* if for every family of objects $\mathcal{F} \to S$, the locus $\{s \in S \mid [\mathcal{F}_s] \in U\}$ is an open subset of $S$. In a similar manner, we can introduce other structures, e.g. a global function on $M$ could be defined as the data of compatible global functions on $S$ for every family $\mathcal{F} \to S$.

- In the situation that $M$ is a fine moduli space, the identity map $\mathrm{id}\colon M \to M$ corresponds to a family of objects $\mathcal{U} \to M$ over the moduli space. This is the *universal family*: for any other family $\mathcal{F} \to S$ of objects, there is a unique morphism $S \to M$ such that the universal family $\mathcal{U} \to M$ pulls back to $\mathcal{F} \to S$.

This is certainly a giant leap in abstraction! And it may seem that we just made life more difficult: rather than introducing a space by specifying its points, its topology, and any other structures, we must specify an immense amount of categorical data. But as examples illustrate, it is usually quite straightforward to define good notions of families.

**Example 0.3.1** (Families of labeled triangles)**.** Revisiting the moduli of labeled triangles up to similarity introduced in Example 0.2.5, we define a *family of labeled triangles over a topological space* $S$ as a tuple $(\mathcal{T}, \sigma_1, \sigma_2, \sigma_3)$ where $\mathcal{T} \to S$ is a fiber bundle with three sections $\sigma_i\colon S \to \mathcal{T}$ equipped with a continuous distance function $d\colon \mathcal{T} \times_S \mathcal{T} \to \mathbb{R}_{\geq 0}$ such that for every point $s \in S$, the restriction $d_s\colon \mathcal{T}_s \times \mathcal{T}_s \to \mathbb{R}_{\geq 0}$ is a metric on the fiber $\mathcal{T}_s$ with $\mathcal{T}_s$ isometric to a triangle with vertices $\sigma_i(s)$.

We say two families $(\mathcal{T}, (\sigma_i))$ and $(\mathcal{T}', (\sigma_i'))$ of labeled triangles over $S \in \mathrm{Top}$ are *similar* if there is a homeomorphism $f\colon \mathcal{T} \to \mathcal{T}'$ over $S$ compatible with the sections (i.e. $f \circ \sigma_i = \sigma_i'$) such that for each $s \in S$, the induced map $\mathcal{T}_s \to \mathcal{T}_s'$ on fibers is a similarity of triangles, i.e. an isometry after rescaling. Given a family $\mathcal{T} \to S$ of labeled triangles and a continuous map $S' \to S$, the pullback family is defined as the fiber product $\mathcal{T} \times_S S'$ of sets together with the pullback sections $\sigma_i'\colon S' \to \mathcal{T}'$ and its inherited distance function.

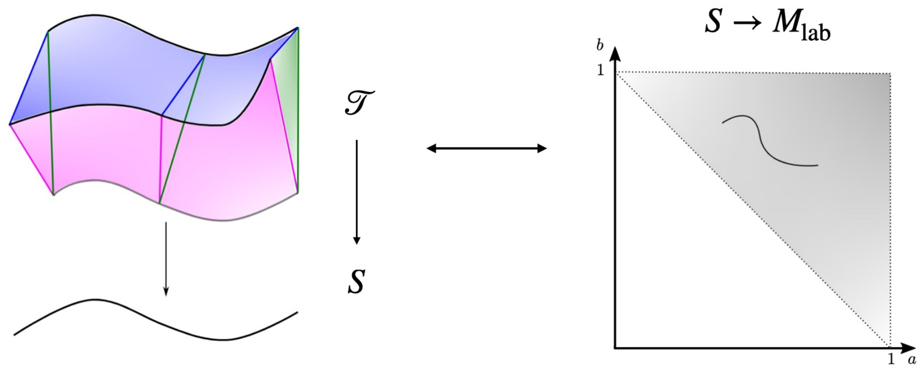



Figure 7: A family of labeled triangles over a curve corresponds to an arc in the moduli space.

We define the *moduli functor of labeled triangles* as

$$F_{M^{\mathrm{lab}}}\colon \mathrm{Top} \to \mathrm{Sets}, \quad S \mapsto \{\text{families } (\mathcal{T} \to S, \sigma_i) \text{ of labeled triangles}\}/(\text{similarity}).$$

Recall from (0.2.2) that the assignment of a triangle to its side lengths yields a bijection between $F_{M^{\mathrm{lab}}}$ and

$$M^{\mathrm{lab}} = \left\{ (a,b,c) \;\middle|\; \begin{array}{c} a+b+c=2 \\ 0<a<b+c \\ 0<b<a+c \\ 0<c<a+b \end{array} \right\}.$$

Since this extends to a compatible isomorphisms of $F_{M^{\mathrm{lab}}}(S) \to \mathrm{Mor}(S, M^{\mathrm{lab}})$ for every space $S$, the topological space $M^{\mathrm{lab}}$ represents the functor $F_{M^{\mathrm{lab}}}$. Consequently, there is a universal family $\mathcal{T}_{\mathrm{univ}} \subset M^{\mathrm{lab}} \times \mathbb{R}^2$ with $\sigma_i\colon M^{\mathrm{lab}} \to \mathcal{T}_{\mathrm{univ}}$.

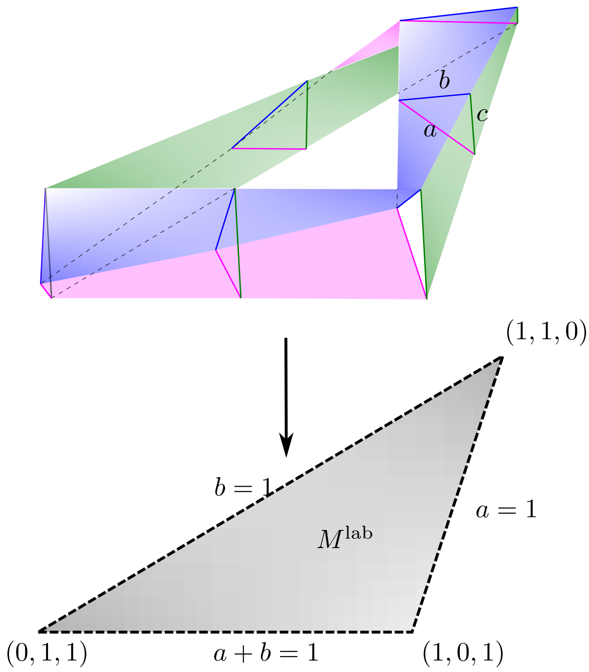



Figure 8: The universal family $U^{\text{lab}} \to M^{\text{lab}}$ of labeled triangles up to similarity.

**Example 0.3.2** (Families of unlabeled triangles)**.** Revisiting Example 0.2.6, we define a *family of unlabeled triangles* as a fiber bundle $\mathcal{T} \to S$ equipped with a continuous distance function $d \colon \mathcal{T} \times_S \mathcal{T} \to \mathbb{R}_{\geq 0}$ that restricts to a metric on every fiber and such that every fiber is isometric to a triangle. Two families $\mathcal{T} \to S$ and $\mathcal{T}' \to S$ are *similar* if there is a homeomorphism $f \colon \mathcal{T} \to \mathcal{T}'$ over $S$ compatible with the sections inducing similarities of triangles on fibers.

We define the functor

$$F \colon \text{Top} \to \text{Sets}, \quad S \mapsto \{\text{families } \mathcal{T} \subset S \times \mathbb{R}^2 \text{ of triangles}\}/\text{similarity}$$

but we can already see complications arising from the presence of symmetries of our objects—each equilateral triangle has symmetry group $S_3$ while the isosceles triangles have symmetry groups $\mathbb{Z}/2$. This functor is *not* representable as there are non-trivial families of triangles $\mathcal{T}$ such that all fibers are similar triangles (Proposition 0.3.19). For instance, we construct a non-trivial family of triangles over $S^1$ by gluing two trivial families via a symmetry of an equilateral triangle.

Figure 9: A trivial (left) and non-trivial (right) family of equilateral triangles. Image taken from a video produced by Jonathan Wise: see `http://math.colorado.edu/~jonathan.wise/visual/moduli/index.html`.

### 0.3.2 Moduli functors of curves, vector bundles, and orbits

Defining a moduli functor $F\colon \mathrm{Sch}/\mathbb{C} \to \mathrm{Sets}$ in the category of $\mathbb{C}$-schemes entails specifying for every $\mathbb{C}$-scheme $S$ a set $F(S)$ of families of objects over $S$, and pull back maps $F(S) \to F(S')$ for morphisms $S' \to S$ which are compatible under composition.

To gain intuition for a moduli functor, it is always useful to plug in special test schemes. For instance, by plugging in $S = \operatorname{Spec}\mathbb{C}$, we obtain the underlying moduli set $F(\operatorname{Spec}\mathbb{C})$ of objects. By plugging in $S = \mathbb{C}[\epsilon]/(\epsilon^2)$, we obtain a set of pairs consisting of a $\mathbb{C}$-point and a tangent vector, and plugging in a curve (or a DVR) gives families of objects over the curve.

**Example 0.3.3** (Moduli functor of smooth curves)**.** A *family of smooth curves of genus* $g$ is a smooth, proper morphism $\mathcal{C} \to S$ of schemes such that for every $s \in S$, the fiber $\mathcal{C}_s$ is a connected curve of genus $g$.

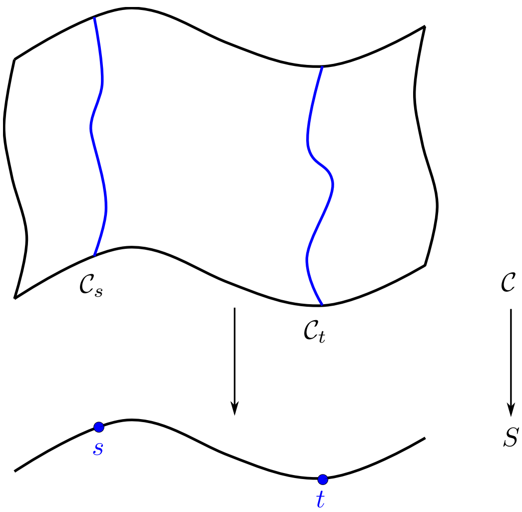



Figure 10: A family of curves over a curve $S$.

The *moduli functor of smooth curves of genus* $g$ is

$$F_{M_g}\colon \mathrm{Sch}/\mathbb{C} \to \mathrm{Sets}, \quad S \mapsto \{\text{families of smooth curves } \mathcal{C} \to S \text{ of genus } g\} \,/\sim,$$

where two families $\mathcal{C} \to S$ and $\mathcal{C}' \to S$ are equivalent if there is a isomorphism $\mathcal{C} \to \mathcal{C}'$ over $S$. If $S' \to S$ is a map of schemes and $\mathcal{C} \to S$ is a family of curves, the pullback is defined as the family $\mathcal{C} \times_S S' \to S'$.

**Example 0.3.4** (Moduli functor of vector bundles on a curve)**.** Let $C$ be a fixed smooth, connected, and projective curve over $\mathbb{C}$. A *family of vector bundles of rank* $r$ *and degree* $d$ over a scheme $S$ is a vector bundle $\mathcal{E}$ on $C \times S$ such that for every $s \in S$, the restriction $\mathcal{E}_s := \mathcal{E}|_{C_{\kappa(s)}}$ of $\mathcal{E}$ to $C_{\kappa(s)} := C \times_{\mathbb{C}} \kappa(s)$ has rank $r$ and degree $d$. The *moduli functor of vector bundles on* $C$ *of rank* $r$ *and degree* $d$ is

$$\mathrm{Sch}/\mathbb{C} \to \mathrm{Sets} \qquad S \mapsto \left\{ \begin{array}{l} \text{vector bundles } \mathcal{E} \text{ on } C \times S \\ \text{of rank } r \text{ and degree } d \end{array} \right\} /(\text{isomorphism}),$$

If $f\colon S' \to S$ is a map of schemes and $E$ is a vector bundle on $C \times S$, the pullback is defined as the vector bundle $(\mathrm{id} \times f)^*\mathcal{E}$ on $C \times S'$

We will see in Section 0.3.5 that these two functors are not representable, and correspondingly that there is no fine moduli space.

**Example 0.3.5** (Moduli functor of orbits)**.** Consider the action of an algebraic group $G$ over $\mathbb{C}$ acting on a scheme $X$. For every scheme $S$, the abstract group $G(S)$ acts on the set $X(S)$—in fact, giving such actions functorial in $S$ uniquely specifies the group action (Exercise C.1.10). We can consider the functor

$$\mathrm{Sch}/\mathbb{C} \to \mathrm{Sets} \qquad S \mapsto X(S)/G(S).$$

This is a very naive candidate for a moduli functor of a quotient, and very far from being representable even for free actions (see Exercise 0.3.25). We will modify this example in §0.6.5.

In some cases, you may know precisely which objects you want to parameterize, but it may not be straightforward to introduce a notion of families. Or there may be several candidate notions for a family of objects, which could translate to different scheme structures on the same topological space. This happens for instance for the moduli of higher dimensional varieties.

### 0.3.3 Yoneda's lemma and representable functors

Following Grothendieck, we study a scheme $X$ by studying all maps to it! That schemes are determined by maps into them is a completely formal fact that is true in every category. This is the Yoneda Lemma: for an object $X$ of a category $\mathcal{C}$, the contravariant functor

$$h_X \colon \mathcal{C} \to \mathrm{Sets}, \qquad S \mapsto \mathrm{Mor}(S, X)$$

recovers the object $X$ itself.

**Lemma 0.3.6** (Yoneda Lemma)**.** *Let $\mathcal{C}$ be a category and $X$ be an object. For every contravariant functor $G\colon \mathcal{C} \to \mathrm{Sets}$, the map*

$$\mathrm{Mor}(h_X, G) \to G(X), \qquad \alpha \mapsto \alpha_X(\mathrm{id}_X)$$

*is bijective and functorial with respect to both $X$ and $G$, where the left-hand side denotes the set of natural transformations $h_X \to G$ and $\alpha_X$ denotes the map $h_X(X) = \mathrm{Mor}(X, X) \to G(X)$.*

**Caution 0.3.7.** Throughout this book, we will consistently abuse notation by conflating an element $g \in G(X)$ and the corresponding morphism $h_X \to G$, which we will often write simply as $X \to G$.

**Exercise 0.3.8.**

(a) Spell out precisely what 'functorial with respect to both $X$ and $G$' means.

(b) Prove Yoneda's lemma.

**Definition 0.3.9** (Representable functors and fine moduli spaces)**.** We say that a functor $F\colon \mathrm{Sch} \to \mathrm{Sets}$ is *representable by a scheme* if there exists a scheme $X$ and an isomorphism of functors $F \xrightarrow{\sim} h_X$.

When $F$ is a moduli functor representable by a scheme $M$, we say that $M$ is a *fine moduli space.*

By the Yoneda Lemma (0.3.6), if a functor is representable, then it is representable by a *unique* scheme. One of our aims is to understand when a given moduli functor $F$ has a fine moduli space, i.e. is representable by a scheme.

**Example 0.3.10** (Projective space as a functor)**.** By [Har77, Thm. II.7.1], there is a functorial bijection

$$\mathrm{Mor}(S, \mathbb{P}^n_{\mathbb{Z}}) \cong \left\{ (L, s_0, \ldots, s_n) \;\middle|\; \begin{array}{l} L \text{ is a line bundle on } S \text{ globally} \\ \text{generated by } s_0, \ldots, s_n \in \Gamma(S, L) \end{array} \right\} / \sim,$$

where $(L, (s_i)) \sim (L', (s_i'))$ if there exists $t \in \Gamma(S, \mathcal{O}_S)^*$ such that $s_i' = ts_i$ for all $i$. In other words, the functor on the right is representable by the scheme $\mathbb{P}^n_{\mathbb{Z}}$. The condition that the sections $s_i$ are globally generated translates to the condition that for every $x \in S$, at least one section $s_i(x) \in L \otimes \kappa(t)$ is nonzero, or equivalently to the surjectivity of $(s_0, \ldots, s_n) \colon \mathcal{O}_S^{n+1} \to L$.

**Example 0.3.11** (The Grassmannian functor)**.** As a set, the Grassmannian $\mathrm{Gr}(k, n)$ parameterizing $k$-dimensional *quotients* of $n$-dimensional space.[3] But what are families of $k$-dimensional quotients over a scheme $S$? A naive guess might be quotients $q \colon \mathcal{O}_S^n \twoheadrightarrow \mathcal{O}_S^k$ but this has no chance to be representable (see Exercise 0.3.25). The case of projective space suggests we define the *Grassmannian functor* as

$$\begin{aligned} \mathrm{Gr}(k, n) \colon \mathrm{Sch} &\to \mathrm{Sets} \\ S &\mapsto \left\{ [\mathcal{O}_S^n \twoheadrightarrow Q] \;\middle|\; Q \text{ is a vector bundle of rank } k \right\} / \sim \end{aligned}$$

where $[\mathcal{O}_S^n \xrightarrow{q} Q] \sim [\mathcal{O}_S^n \xrightarrow{q'} Q']$ if there exists an isomorphism $\Psi \colon Q \xrightarrow{\sim} Q'$ such that

$$\begin{tikzcd} \mathcal{O}_S^n \arrow[r, "q"] \arrow[dr, "q'"'] & Q \arrow[d, "\Psi"] \\ & Q' \end{tikzcd}$$

commutes (i.e. $q' = \Psi \circ q$), or equivalently if $\ker(q) = \ker(q')$. Pullbacks are defined in the obvious manner.

We will later show that $\mathrm{Gr}(k, n)$ is representable by a scheme projective over $\mathbb{Z}$ (Theorem 1.1.1). The proof of this result is a good illustration of the utility of the functorial approach and a warmup for the representability of Hilb and Quot (Theorems 1.1.2 and 1.1.3).

These exercises will give you some practice.

**Exercise 0.3.12.**

(a) If $S$ is a scheme and $E$ is a vector bundle on $S$, show that the projectivization $\mathbb{P}_S(E) := \mathrm{Proj}_S \mathrm{Sym}^* E^\vee$ of $E$ represents the functor

$$\begin{aligned} \mathbb{P}_S(E) \colon \mathrm{Sch}/S &\to \mathrm{Sets} \\ (T \xrightarrow{f} S) &\mapsto \{\text{quotients } f^*E \overset{q}{\twoheadrightarrow} L \text{ where } L \text{ is a line bundle on } T\} / \sim \end{aligned}$$

where $[f^*E \overset{q}{\twoheadrightarrow} L] \sim [f^*E \overset{q'}{\twoheadrightarrow} L']$ if $\ker(q) = \ker(q')$ (or equivalently there is an isomorphism $\alpha \colon L \to L'$ with $q' = \alpha \circ q$).

[3] Alternatively, the points could be considered as $k$-dimensional *subspaces* but in these notes, we will follow Grothendieck's convention of quotients.

(b) Show that the same holds if $E$ is a finite type quasi-coherent sheaf on $S$ (e.g. a coherent sheaf if $S$ is noetherian).

Note that there is an isomorphism $\mathbb{P}^n_{\mathbb{Z}} \cong \mathbb{P}(\mathcal{O}^{n+1}_{\operatorname{Spec}\mathbb{Z}})$ of functors.

**Exercise 0.3.13.** Provide functorial descriptions of:

(a) $\mathbb{A}^n \setminus 0$;

(b) the blowup $\operatorname{Bl}_p \mathbb{P}^n$ of $\mathbb{P}^n$ at a point;

(c) the normalization $\widetilde{X}$ of a reduced scheme $X$;

(d) $\mathcal{S}\mathrm{pec}_S \mathcal{A}$ where $\mathcal{A}$ is a quasi-coherent sheaf of algebras on a scheme $S$; and

(e) $\operatorname{Proj} R$ where $R$ is a positively graded ring.

**Exercise 0.3.14.**

(a) Let $X$ be a scheme and $E, F$ be $\mathcal{O}_X$-modules. Show that the functor

$$\underline{\mathrm{Hom}}_{\mathcal{O}_X}(E,F)\colon \mathrm{Sch} \to \mathrm{Sets}, \quad T \mapsto \mathrm{Hom}_{\mathcal{O}_{X\times T}}(E_T, F_T),$$

where $E_T$ and $F_T$ denote the pullbacks of $E$ and $F$ under $X \times T \to T$, is representable by $\operatorname{Spec}\operatorname{Sym}\mathrm{Hom}_{\mathcal{O}_X}(E,F)^\vee$.

(b) Let $E \to F$ be a homomorphism of coherent sheaves on a noetherian scheme $X$. Show that the subfunctor of $X$ (or more precisely of $h_X = \mathrm{Mor}(-,X)$) defined by

$$\mathrm{Sch} \to \mathrm{Sets}, \quad T \mapsto \{\text{morphisms } T \to X \text{ such that } E_T \to F_T \text{ is zero}\}$$

is representable by a closed subscheme of $X$.

(c) Recall that the group $\mathrm{Ext}^1_{\mathcal{O}_X}(G,E)$ classifies extensions $0 \to E \to F \to G \to 0$ of $\mathcal{O}_X$-modules, where two extensions are identified if there is an isomorphism of short exact sequences inducing the identity map on $E$ and $G$ [Har77, Exer. III.6.1]. Show that the affine scheme $\underline{\mathrm{Ext}}^1_{\mathcal{O}_X}(G,E) := \operatorname{Spec}\operatorname{Sym}\mathrm{Ext}^1_{\mathcal{O}_X}(G,E)^\vee$ represents the functor

$$\mathrm{Sch} \to \mathrm{Sets}, \quad T \mapsto \mathrm{Ext}^1_{\mathcal{O}_{X\times T}}(p_1^*G, p_1^*E)$$

where $p_1\colon X \times T \to X$.

### 0.3.4 Universal families

**Definition 0.3.15.** If $F\colon \mathrm{Sch} \to \mathrm{Sets}$ is a moduli functor representable by a scheme $X$ via an isomorphism $\alpha\colon F \xrightarrow{\sim} h_X$ of functors, then the *universal family* of $F$ is the object $U \in F(X)$ corresponding under $\alpha$ to the identity morphism $\mathrm{id}_X \in h_X(X) = \mathrm{Mor}(X,X)$.

Suspend your skepticism for a moment and suppose that there actually exists a scheme $M_g$ representing the moduli functor of smooth curves of genus $g$ (Example 0.3.3). Then corresponding to the identity map $M_g \to M_g$ is a family of genus $g$ curves $U_g \to M_g$ satisfying the following universal property: for every smooth family of curves $\mathcal{C} \to S$ over a scheme $S$, there is a unique map $S \to M_g$ and cartesian diagram

$$\begin{tikzcd} \mathcal{C} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & U_g \arrow[d] \\ S \arrow[r] & M_g. \end{tikzcd}$$

The map $S \to M_g$ sends a point $s \in S$ to the curve $[\mathcal{C}_s] \in M_g$. The universal family might like something like:

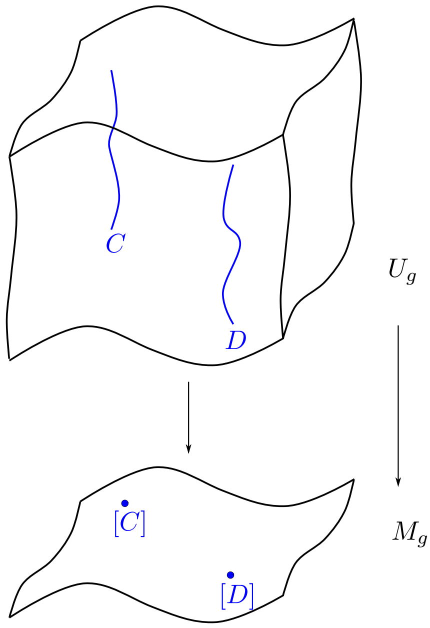



Figure 11: Visualization of a (non-existent) universal family over $M_g$.

**Example 0.3.16.** The universal family of the moduli functor of projective space (Example 0.3.10) is the line bundle $\mathcal{O}(1)$ on $\mathbb{P}^n$ together with the sections $x_0, \ldots, x_n \in \Gamma(\mathbb{P}^n, \mathcal{O}(1))$.

**Example 0.3.17** (Universal extensions)**.** If $X$ is a scheme with vector bundles $E$ and $G$, the universal family for the moduli functor $\underline{\mathrm{Ext}}^1_{\mathcal{O}_X}(G, E)$ of extensions of Exercise 0.3.14(c) is an extension $0 \to p_1^* G \to \mathcal{F} \to p_1^* E \to 0$ of vector bundle on $X \times \underline{\mathrm{Ext}}^1_{\mathcal{O}_X}(G, E)$ whose restriction to $X \times \{t\}$ is the extension corresponding to $t \in \mathrm{Ext}^1_{\mathcal{O}_X}(G, E)$.

**Example 0.3.18** (Classifying spaces in algebraic topology)**.** Let $G$ be a topological group and $\mathrm{Top}^{\mathrm{para}}$ be the category of paracompact topological spaces where morphisms are defined up to homotopy. It is a theorem in algebraic topology that the functor

$$\mathrm{Top}^{\mathrm{para}} \to \mathrm{Sets}, \qquad S \mapsto \{\text{principal } G\text{-bundles } P \to S\}/\sim,$$

where $\sim$ denotes isomorphism, is represented by a topological space, which we denote by $BG$ and call the *classifying space*. The universal family is usually denoted by $EG \to BG$. For example, the classifying space $B\mathbb{C}^*$ is the infinite dimensional manifold $\mathbb{CP}^\infty$.

### 0.3.5 Examples of non-representable moduli functors

If $F\colon \mathrm{Sch}/\mathbb{C} \to \mathrm{Sets}$ is a moduli functor, then an object $E \in F(\mathbb{C})$ with a non-trivial automorphism can prevent the functor $F$ from being representable. This is because we may glue trivial families using the automorphism to construct a *non-trivial* family

$\mathcal{E}$ over a scheme $S$ such that every fiber $\mathcal{E}_s$ (i.e. the pullback of $\mathcal{E}$ along $\operatorname{Spec}\mathbb{C} \to S$) is isomorphic to $E$.

**Proposition 0.3.19.** *Let $F\colon \mathrm{Sch}/\mathbb{C} \to \mathrm{Sets}$ be a moduli functor. If there is a family of objects $\mathcal{E} \in F(S)$ over a variety $S$ such that*

*(a) the fibers $\mathcal{E}_s$ are isomorphic for $s \in S(\mathbb{C})$; and*

*(b) the family $\mathcal{E}$ is non-trivial, i.e. is not equal to the pullback of an object $E \in F(\mathbb{C})$ along the structure map $S \to \operatorname{Spec}\mathbb{C}$,*

*then $F$ is not representable.*

*Proof.* Suppose by way of contradiction that $F$ is represented by a scheme $X$. By condition (a), the restriction $E := \mathcal{E}_s$ is independent of $s \in S(\mathbb{C})$ and defines a unique point $x \in X(\mathbb{C})$. As $S$ is reduced, the map $S \to X$ factors as $S \to \operatorname{Spec}\mathbb{C} \xrightarrow{x} X$. This implies that the family $\mathcal{E}$ is the pullback under the constant map $S \to \operatorname{Spec}\mathbb{C} \xrightarrow{x} X$, i.e. $\mathcal{E}$ is a trivial family, which contradicts condition (b). $\square$

**Example 0.3.20** (Moduli of elliptic curves)**.** An *elliptic curve* is a pair $(E, p)$ where $E$ is a smooth, connected, and projective curve $E$ of genus 1 and $p \in E(\mathbb{C})$. A *family of elliptic curves* over a scheme $S$ is a pair $(\mathcal{E} \to S, \sigma)$ where $\mathcal{E} \to S$ is smooth proper morphism with a section $\sigma\colon S \to \mathcal{E}$ such that for every $s \in S$, the fiber $(\mathcal{E}_s, \sigma(s))$ is an elliptic curve over the residue field $\kappa(s)$. The *moduli functor of elliptic curves* is

$$\begin{aligned} F_{M_{1,1}}\colon \ \mathrm{Sch} &\to \mathrm{Sets} \\ S &\mapsto \{\text{families } (\mathcal{E} \to S, \sigma) \text{ of elliptic curves }\} / \sim, \end{aligned}$$

where $(\mathcal{E} \to S, \sigma) \sim (\mathcal{E}' \to S, \sigma')$ if there is a $S$-isomorphism $\alpha\colon \mathcal{E} \to \mathcal{E}'$ compatible with the sections (i.e. $\sigma' = \alpha \circ \sigma$).

**Exercise 0.3.21.** Consider the family of elliptic curves defined over $\mathbb{A}^1 \setminus 0$ (with coordinate $t$) by

$$\begin{tikzcd} \mathcal{E} := V(y^2 z - x^3 + tz^3) \arrow[r, hook] \arrow[d] & (\mathbb{A}^1 \setminus 0) \times \mathbb{P}^2 \\ \mathbb{A}^1 \setminus 0 & \end{tikzcd}$$

with section $\sigma\colon \mathbb{A}^1 \setminus 0 \to \mathcal{E}$ given by $t \mapsto [0, 1, 0]$. Show that $(\mathcal{E} \to \mathbb{A}^1 \setminus 0, \sigma)$ satisfies (a) and (b) in Proposition 0.3.19.

**Example 0.3.22** (Moduli functor of smooth curves)**.** Let $C$ be a curve with a non-trivial automorphism $\alpha \in \operatorname{Aut}(C)$ and let $N$ be the nodal cubic curve, which we can think of as $\mathbb{P}^1$ with the points 0 and $\infty$ glued together. We can construct a family $\mathcal{C} \to N$ by taking the trivial family $\pi\colon C \times \mathbb{P}^1 \to \mathbb{P}^1$ and gluing the fiber $\pi^{-1}(0)$ with $\pi^{-1}(\infty)$ via the automorphism $\alpha$. To show that the moduli functor of curves is not representable, it suffices to show that $\mathcal{C} \to N$ is non-trivial.

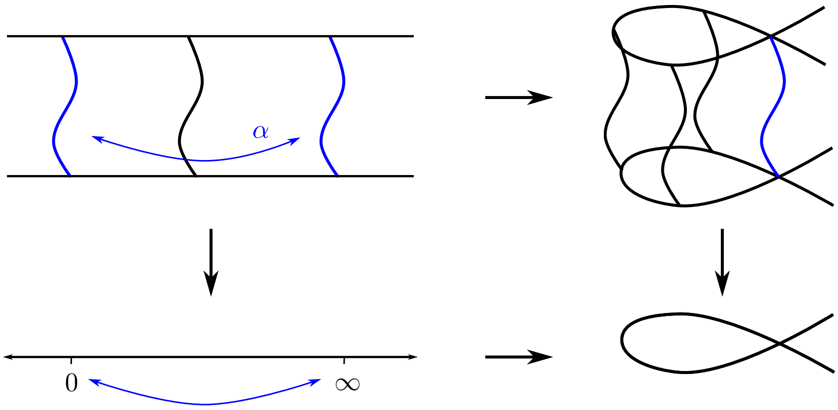



Figure 12: Family of curves over the nodal cubic obtaining by gluing the fibers over 0 and $\infty$ of the trivial family over $\mathbb{P}^1$ via $\alpha$. (It would be more illustrative to draw a Mobius band as the family of curves over the nodal cubic.)

**Exercise 0.3.23.** Show that $\mathcal{C} \to N$ is a non-trivial family.

**Exercise 0.3.24.** Show that the moduli functor of vector bundles over a curve $C$ is not representable.

### 0.3.6 Schemes are sheaves in the big Zariski topology

If $F\colon \mathrm{Sch} \to \mathrm{Sets}$ is representable by a scheme $X$, then $F$ is necessarily a *sheaf in the big Zariski topology*, that is, for every scheme $S$, the presheaf on the Zariski topology of $S$, defined by assigning to an open subset $U \subset S$ the set $F(U)$, is a sheaf on the Zariski topology of $S$. This is a restatement that morphisms into the fixed scheme $X$ glue uniquely. The failure to be a sheaf therefore provides another obstruction to the representability of a given moduli functor $F$.

**Exercise 0.3.25.**

(a) Show that the following naive Grassmannian functor

$$F\colon \mathrm{Sch} \to \mathrm{Sets}, \quad S \mapsto \{\text{quotients } q\colon \mathcal{O}_S^n \twoheadrightarrow \mathcal{O}_S^k\}/\sim$$

is not representable.

(b) Under the usual scaling action of $\mathbb{G}_m$ on $\mathbb{A}^{n+1} \setminus 0$ with the usual scaling action, show that the functor $S \mapsto (\mathbb{A}^{n+1} \setminus 0)(S)/\mathbb{G}_m(S)$ is not a sheaf.

The presence of non-trivial automorphisms often implies that a given moduli functor is not a sheaf in the big Zariski topology.

**Example 0.3.26.** Consider the moduli functor $F_{M_g}$ of smooth curves from Example 0.3.3. Let $\{S_i\}$ be a Zariski open covering of a scheme $S$, and suppose that $\mathcal{C}_i \to S_i$ are families of smooth curves $\mathcal{C}_i \to S_i$ with isomorphisms $\alpha_{ij}\colon \mathcal{C}_i|_{S_{ij}} \xrightarrow{\sim} \mathcal{C}_j|_{S_{ij}}$ on the intersection $S_{ij} := S_i \cap S_j$. The requirement that $F_{M_g}$ be a sheaf (when restricted to the Zariski topology on $S$) implies that the families $\mathcal{C}_i \to S_i$ glue uniquely to a family of curves $\mathcal{C} \to S$. However, we have not required the isomorphisms $\alpha_i$ to be compatible on the triple intersection (i.e. $\alpha_{ij}|_{S_{ijk}} \circ \alpha_{jk}|_{S_{ijk}} = \alpha_{ik}|_{S_{ijk}}$), which is necessary for gluing schemes [Har77, Exer II.2.12]. For this reason, $F_{M_g}$ fails to be a sheaf.

**Exercise 0.3.27.** Show that the moduli functors of smooth curves and elliptic curves are not sheaves by explicitly exhibiting a scheme $S$, an open cover $\{S_i\}$, and families of curves over $S_i$ that do not glue to a family over $S$.

### 0.3.7 The yoga of functors

Contravariant functors $F\colon \mathrm{Sch} \to \mathrm{Sets}$ form a category $\mathrm{Fun}(\mathrm{Sch}, \mathrm{Sets})$ where morphisms are natural transformations. This category has fiber products: given morphisms $F \xrightarrow{\alpha} G$ and $G' \xrightarrow{\beta} G$, we define

$$\begin{aligned} F \times_G G' \colon\ & \mathrm{Sch} \to \mathrm{Sets} \\ & S \mapsto \{(a,b) \in F(S) \times G'(S) \mid \alpha_S(a) = \beta_S(b)\} \end{aligned}$$

**Exercise 0.3.28.** Show that $F \times_G G'$ satisfies the universal property for fiber products in $\mathrm{Fun}(\mathrm{Sch}, \mathrm{Sets})$.

**Definition 0.3.29.**

(1) We say that a morphism $F \to G$ of contravariant functors is *representable by schemes* if for every map $S \to G$ from a scheme $S$, the fiber product $F \times_G S$ is representable by a scheme.

(2) We say that a morphism $F \to G$ is an *open immersion* or that a subfunctor $F \subset G$ is *open* if for every morphism $S \to G$ from a scheme $S$, $F \times_G S$ is representable by an open subscheme of $S$.

(3) We say that a set of open subfunctors $\{F_i\}$ of $F$ is a *Zariski open cover* if for every morphism $S \to F$ from a scheme $S$, $\{F_i \times_F S\}$ is a Zariski open cover of $S$ (and in particular each $F_i$ is an open subfunctor of $F$).

Each of these conditions can be checked on *affine* schemes.

Together with the following exercise, these definitions give a recipe for checking that a given functor $F$ is representable by a scheme: find a Zariski open cover $\{F_i\}$ where each $F_i$ is representable.

**Exercise 0.3.30.**

(a) Let $F\colon \mathrm{Sch} \to \mathrm{Sets}$ be a functor which is a sheaf in the big Zariski topology and $\{F_i\}$ be a Zariski open cover of $F$. Show that if each $F_i$ is representable by a scheme, then so is $F$.

(b) Show that a collection of open subfunctors $\{F_i\}$ of $F$ is a Zariski open cover if and only if the map $\coprod_i F_i(\Bbbk) \to F(\Bbbk)$ is surjective for each algebraically closed field $\Bbbk$.

(c) Given morphisms of schemes $X \to Y$ and $Y' \to Y$, reprove the existence of the fiber product $X \times_Y Y'$ in the category of schemes by exhibiting a Zariski open cover $\{F_i\}$ of $X \times_Y Y'$ where each $F_i$ is representable by an affine scheme.

**Exercise 0.3.31.** Show that a scheme can be equivalently defined as a contravariant functor $F\colon \mathrm{AffSch} \to \mathrm{Sets}$ on the category of affine schemes (or covariant functor on the category of rings) as follows. Let $\mathcal{C}$ be a full subcategory of the category $\mathrm{Fun}(\mathrm{AffSch}, \mathrm{Sets})$ of contravariant functors. Extending Definitions 0.3.9 and 0.3.29, we define a functor $F\colon \mathrm{AffSch} \to \mathrm{Sets}$ to be *representable in* $\mathcal{C}$ if there exists an object $X \in \mathcal{C}$ and a functorial equivalence $F(S) = \mathrm{Mor}(S, X)$ for every $S \in \mathrm{AffSch}$. We say that a map $F \to G$ of functors from AffSch to Sets is *representable by open immersions in* $\mathcal{C}$ if for every morphism $\operatorname{Spec} B \to G$, the fiber product $F \times_G \operatorname{Spec} B$

is representable by an object $X \in \mathcal{C}$ which is an open subscheme of $\operatorname{Spec} B$. Finally, we say that a collection $\{F_i\}$ of subfunctors of $F$ is a *Zariski open $\mathcal{C}$-cover* if each $F_i \to F$ is representable by open immersions in $\mathcal{C}$ and for each algebraically closed field $\Bbbk$, the map $\coprod_i F_i(\Bbbk) \to F(\Bbbk)$ is surjective.

(a) Show that a scheme with affine diagonal can be equivalently defined as a functor $F\colon \mathrm{AffSch} \to \mathrm{Sets}$ such that there exists a Zariski open AffSch-cover $\{F_i\}$ of $F$ with each $F_i$ representable in AffSch.

(b) Give a similar characterization of separated schemes.

(c) Letting $\mathcal{C}$ be the category of schemes with affine diagonal, show that a scheme can be equivalently defined as a functor $F\colon \mathrm{AffSch} \to \mathrm{Sets}$ such that there exists a Zariski open $\mathcal{C}$-cover $\{F_i\}$ with each $F_i$ representable in $\mathcal{C}$.

Replacing Zariski opens with étale morphisms leads to the definition of an algebraic space.

## 0.4 Moduli groupoids

We now change our perspective: rather than specifying *when* two objects are identified, we specify *how*!

One of the most desirable properties of a moduli space is the existence of a universal family (see §0.3.4) and the presence of automorphisms obstructs its existence (see §0.3.5). Encoding automorphisms into our descriptions will allow us to get around this problem.

To define a moduli groupoid, we need to specify

(1) objects; and

(2) a set of equivalences (possibly empty) between any two objects.

Shortly we will combine the functorial worldview of the last section with this groupoid perspective to define moduli stacks.

### 0.4.1 Groupoids

A convenient mathematical structure to encode objects and their identifications is a *groupoid*.

**Definition 0.4.1.** A *groupoid* is a category $\mathcal{C}$ where every morphism is an isomorphism.

Two groupoids $\mathcal{C}_1$ and $\mathcal{C}_2$ are *equivalent* if there is an equivalence of categories $F\colon \mathcal{C}_1 \xrightarrow{\sim} \mathcal{C}_2$, i.e. there is a functor $G\colon \mathcal{C}_2 \to \mathcal{C}_1$ such that $F \circ G$ and $G \circ F$ are isomorphic to the identity functors, or equivalently $F$ is fully faithful and essentially surjective.

**Example 0.4.2** (Sets are groupoids)**.** If $\Sigma$ is a set, the category $\mathcal{C}_\Sigma$, whose objects are elements of $\Sigma$ and whose morphisms consist of only the identity morphism, is a groupoid.

We say that a groupoid $\mathcal{C}$ *is equivalent to a set* $\Sigma$ if there is an equivalence of categories $\mathcal{C} \to \mathcal{C}_\Sigma$.

**Example 0.4.3** (Classifying groupoid)**.** If $G$ is a group, the *classifying groupoid* $BG$ of $G$ is defined as the category with one object $\star$ such that $\operatorname{Aut}(\star) = \operatorname{Mor}(\star, \star) = G$.

**Example 0.4.4.** The category FB of finite sets where morphisms are bijections is a groupoid. The isomorphism classes of FB are in bijection with $\mathbb{N}$ while $\mathrm{Aut}(\{1,\dots,n\}) = S_n$ is the permutation group.

**Example 0.4.5** (Projective space)**.** Projective space can be with the moduli groupoid of lines $L \subset \mathbb{A}^{n+1}$ through the origin where the only morphisms are the identity maps. Alternatively, the objects are nonzero linear maps $x = (x_0,\dots,x_n)\colon \mathbb{C} \to \mathbb{C}^{n+1}$ and there is a unique morphism $x \to x'$ if and only if $\mathrm{im}(x) = \mathrm{im}(x') \subset \mathbb{C}^{n+1}$ (i.e. there exists a $\lambda \in \mathbb{C}^*$ such that $x' = \lambda x$).

### 0.4.2 Moduli groupoid of orbits

**Example 0.4.6** (Moduli groupoid of orbits)**.** Given an action of a group $G$ on a set $X$, we define the *moduli groupoid of orbits* $[X/G]$[4] by taking the objects to be all elements $x \in X$ and by declaring $\mathrm{Mor}(x,x') = \{g \in G \mid x' = gx\}$.

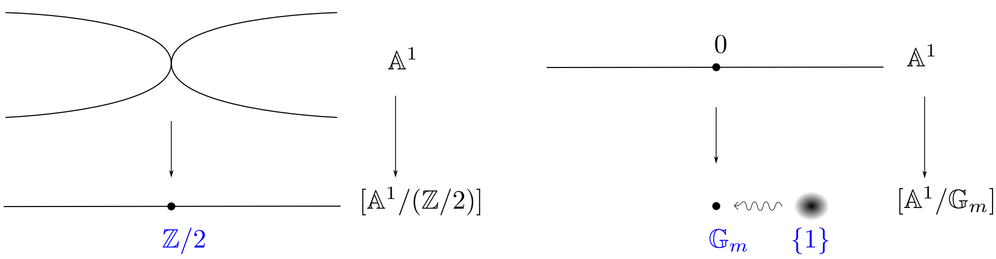



Figure 13: Pictures of the scaling actions of $\mathbb{Z}/2 = \{\pm 1\}$ and $\mathbb{G}_m$ on $\mathbb{A}^1$ over $\mathbb{C}$ with the automorphism groups listed in blue. Note that $[\mathbb{A}^1/\mathbb{G}_m]$ has two isomorphism classes of objects—0 and 1—corresponding to the two orbits—0 and $\mathbb{A}^1 \setminus 0$—such that $0 \in \overline{\{1\}}$ if the set $\mathbb{A}^1/\mathbb{G}_m$ is endowed with the quotient topology.

**Exercise 0.4.7.**

(a) Show that the moduli groupoid of orbits $[X/G]$ in Example 0.4.6 is equivalent to a set if and only if the action of $G$ on $X$ is free.

(b) Show that a groupoid $\mathcal{C}$ is equivalent to a set if and only if $\mathcal{C} \to \mathcal{C} \times \mathcal{C}$ is fully faithful.

**Example 0.4.8.** Consider the category $\mathcal{C}$ with two objects $x_1$ and $x_2$ such that $\mathrm{Mor}(x_i, x_j) = \{\pm 1\}$ for $i,j = 1,2$ where composition of morphisms is given by multiplication. Then $\mathcal{C}$ is equivalent $B\mathbb{Z}/2$.

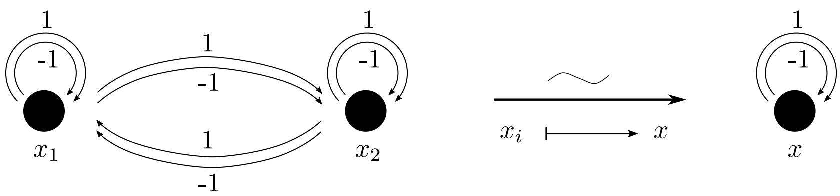



Figure 14: An equivalence of groupoids

**Exercise 0.4.9.** In Example 0.4.8, show that there is an equivalence of categories inducing a bijection on objects between $\mathcal{C}$ and either $[(\mathbb{Z}/2)/(\mathbb{Z}/4)]$ or $[(\mathbb{Z}/2)/(\mathbb{Z}/2 \times \mathbb{Z}/2)]$ where the action is given by surjections $\mathbb{Z}/4 \to \mathbb{Z}/2$ or $\mathbb{Z}/2 \times \mathbb{Z}/2 \to \mathbb{Z}/2$.

[4] We use brackets to distinguish the groupoid quotient $[X/G]$ from the set quotient $X/G$. Later when $G$ is an algebraic group and $X$ is a scheme, $[X/G]$ will denote the quotient stack.

**Example 0.4.10** (Projective space as a quotient)**.** The moduli groupoid of projective space (Example 0.4.5) can also be described as the moduli groupoid of orbits $[(\mathbb{A}^{n+1} \setminus 0)/\mathbb{G}_m]$.

We can also consider the quotient groupoid $[\mathbb{A}^{n+1}/\mathbb{G}_m]$, which is equivalent to the groupoid whose objects are (possibly zero) linear maps $x = (x_0, \ldots, x_n)\colon \mathbb{C} \to \mathbb{C}^{n+1}$ such that $\mathrm{Mor}(x, x') = \{t \in \mathbb{C}^* \mid x_i' = tx_i \text{ for all } i\}$. In this way, $\mathbb{P}^n$ is a subgroupoid of $[\mathbb{A}^{n+1}/\mathbb{G}_m]$.

**Exercise 0.4.11.** If a group $G$ acts on a set $X$ and $x \in X$ is a point, show that there is a fully faithful functor $BG_x \to [X/G]$. If the action is transitive, show that it is an equivalence.

A *morphisms of groupoids* $\mathcal{C}_1 \to \mathcal{C}_2$ is by definition a functor. The category $\mathrm{MOR}(\mathcal{C}_1, \mathcal{C}_2)$ has functors as objects and natural transformations as morphisms.

**Exercise 0.4.12.** If $\mathcal{C}_1$ and $\mathcal{C}_2$ are groupoids, show that $\mathrm{MOR}(\mathcal{C}_1, \mathcal{C}_2)$ is a groupoid.

**Exercise 0.4.13.** If $H$ and $G$ are groups, show that there is an equivalence

$$\mathrm{MOR}(BH, BG) = \coprod_{\phi \in \mathrm{Hom}(H,G)/G} B\big(C^G(\mathrm{im}\,\phi)\big)$$

where $\mathrm{Hom}(H, G)/G$ denotes equivalence classes of homomorphisms $H \to G$ up to conjugation by $G$, and $C^G(\mathrm{im}\,\phi)$ denotes the centralizer of $\mathrm{im}\,\phi$ in $G$.

**Exercise 0.4.14.** Provide an example of group actions of $H$ and $G$ on sets $X$ and $Y$ and a map $[X/H] \to [Y/G]$ of groupoids that *does not* arise from a group homomorphism $\phi\colon H \to G$ and a $\phi$-equivariant map $X \to Y$.

### 0.4.3 Examples of moduli groupoids

**Example 0.4.15** (Moduli groupoid of smooth curves)**.** In this case, the objects are smooth, connected, and projective curves of genus $g$ over $\mathbb{C}$ and for two curves $C, C'$, the set of morphisms is defined as the set of isomorphisms

$$\mathrm{Mor}(C, C') = \{\text{isomorphisms } \alpha\colon C \xrightarrow{\sim} C'\}.$$

**Example 0.4.16** (Moduli groupoid of vector bundles on a curve)**.** The objects are vector bundles $E$ of rank $r$ and degree $d$, and the morphisms are isomorphisms of vector bundles.

**Example 0.4.17** (Moduli groupoid of unlabeled triangles)**.** Let's revisit the moduli $M^{\mathrm{unl}}$ of unlabeled triangles up to similarity from Example 0.2.6. Recall that we have already introduced families of unlabeled triangles and shown that this functor is not representable (Example 0.3.2).

We define the *moduli groupoid of unlabeled triangles up to similarity*, denoted by $\mathcal{M}^{\mathrm{unl}}$ (note the calligraphic font), where the objects are unlabeled triangles and the morphisms are similarities. For example, an isosceles triangle and an equilateral triangle have automorphism groups $\mathbb{Z}/2$ and $S_3$.

We can draw essentially the same picture as Figure 6 except we record the automorphisms.

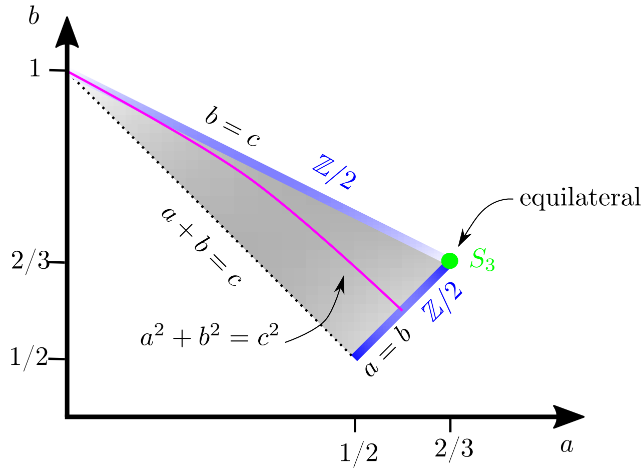



Figure 15: Picture of the moduli groupoid $\mathcal{M}^{\mathrm{unl}}$ with non-trivial automorphism groups labeled.

There is a functor

$$\mathcal{M}^{\mathrm{unl}} \to M^{\mathrm{unl}},$$

from the moduli groupoid to the moduli set, which is an equivalence on isomorphism classes of objects and collapses all morphisms to the identity. This is a first example of a *coarse moduli space*.

**Exercise 0.4.18.** Recalling the description of the moduli set $M^{\mathrm{lab}}$ of labeled triangles up to similarity from (0.2.1), show that there is a natural action of $S_3$ on the moduli set $M^{\mathrm{lab}}$ of labeled triangles up to similarity and that there is an identification $\mathcal{M}^{\mathrm{unl}} \cong [M^{\mathrm{lab}}/S_3]$.

**Exercise 0.4.19.** Define a moduli groupoid of *oriented triangles* and investigate its relation to the moduli sets/groupoids of labeled/unlabeled triangles.

For a more detailed exposition of the moduli stack of triangles, see [Beh14].

## 0.5 Motivation: why the étale topology?

Moduli stacks will be introduced in the next section by combining moduli functors with groupoids—one needs to specify families of objects over every scheme $S$ (along with identifications and pullbacks). For such data to define a *stack*, we will require that objects and their morphisms glue in the *étale topology*!

Why is the Zariski topology not sufficient for our purposes? The short answer is that there are not enough Zariski-open subsets and that étale morphisms serve as a good replacement of analytic open subsets.

### 0.5.1 What is an étale morphism anyway?

I've sometimes been baffled when a student is intimidated by étale morphisms, especially when she has already mastered the conceptually more difficult notions of say properness and flatness. One factor could be the fact that the definition is buried in [Har77, Exercises III.10.3-6] and its importance is not highlighted there.

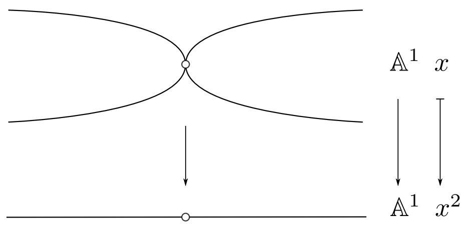



Figure 16: Picture of an étale double cover of $\mathbb{A}^1 \setminus 0$

The geometric picture to have in your mind is a covering space. There are several ways in which we can formulate an étale morphism $f\colon X \to Y$ of schemes of finite type over $\mathbb{C}$:

- $f$ is smooth of relative dimension 0 (i.e. $f$ is flat and all fibers are smooth of dimension 0);
- $f$ is flat and unramified (i.e. for all $y \in Y(\mathbb{C})$, the scheme-theoretic fiber $X_y$ is isomorphic to a disjoint union $\coprod_i \operatorname{Spec} \mathbb{C}$ of points);
- $f$ is flat and $\Omega_{X/Y} = 0$;
- for all $x \in X(\mathbb{C})$, the induced map $\widehat{\mathcal{O}}_{Y,f(x)} \to \widehat{\mathcal{O}}_{X,x}$ on completions is an isomorphism; and
- (assuming in addition that $X$ and $Y$ are smooth) for all $x \in X(\mathbb{C})$, the induced map $T_{X,x} \to T_{Y,f(x)}$ on tangent spaces is an isomorphism.

We say that $f$ is *étale at $x \in X$* if there is an open neighborhood $U$ of $x$ such that $f|_U$ is étale. See §A.3.2 for more background.

These characterizations are all equivalent, but this by no means should be clear to you—some of the proofs are quite involved. Nevertheless, if you can take the equivalences on faith, it requires very little effort to not only internalize the concept but to master its use.

**Exercise 0.5.1.** Show that $f\colon \mathbb{A}^1 \to \mathbb{A}^1, x \mapsto x^2$ is étale over $\mathbb{A}^1 \setminus 0$ but is not étale at the origin. *Try to show this for as many of the above characterizations as you can.*

### 0.5.2 What can you see in the étale topology?

Working with the étale topology is like putting on a better pair of glasses allowing you to see what you couldn't before. Or perhaps more accurately, it is like getting magnifying lenses for your algebraic geometry glasses allowing you to see what you already could with your differential geometry glasses.

**Example 0.5.2** (Reducibility of a node)**.** Consider the plane nodal cubic $C$ defined by $y^2 = x^2(x-1)$ in the plane. While there is an analytic open neighborhood of the node $p = (0,0)$ which is reducible, there is no such Zariski-open neighborhood. However, taking a 'square root' of $x-1$ yields a reducible étale neighborhood. More specifically, define $C' = \operatorname{Spec} k[x,y,t]_t/(y^2 - x^3 + x^2, t^2 - x + 1)$ and consider

$$C' \to C, \qquad (x,y,t) \mapsto (x,y)$$

Since $y^2 - x^3 + x^2 = (y - xt)(y + xt)$, we see that $C'$ is reducible.

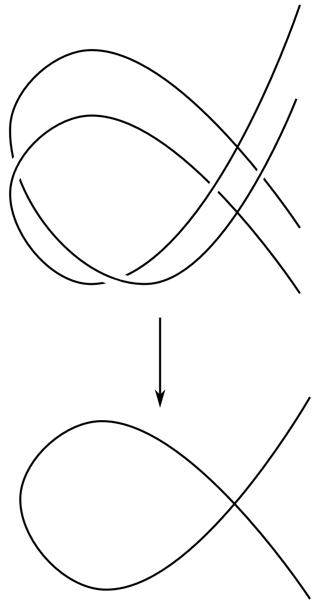


Figure 17: After an étale cover, the nodal cubic becomes reducible.

**Example 0.5.3** (Étale cohomology)**.** Sheaf cohomology for the Zariski topology can be extended to the étale topology leading to the extremely robust theory of *étale cohomology*. For example, for a smooth projective curve $C$ of genus $g$ over $\mathbb{C}$, the étale cohomology $\mathrm{H}^1(C_{\text{ét}}, \mathbb{Z}/n)$ of the finite constant sheaf $\mathbb{Z}/2$ is isomorphic to $(\mathbb{Z}/n)^{2g}$ just like the ordinary cohomology groups, while the sheaf cohomology $\mathrm{H}^1(C, \mathbb{Z}/n)$ in the Zariski topology is 0. Finally, we would be remiss without mentioning the spectacular application of étale cohomology to prove the Weil conjectures.

**Example 0.5.4** (Étale fundamental group)**.** Have you ever thought that there is a similarity between the bijection in Galois theory between intermediate field extensions and subgroups of the Galois group, and the bijection in algebraic topology between covering spaces and subgroups of the fundamental group? Well, you're in good company—Grothendieck also considered this and developed a beautiful theory of the *étale fundamental group* which packages Galois groups and fundamental groups in the same framework.

**Example 0.5.5** (Quotients by free actions of finite groups)**.** If $G$ is a finite group acting freely on a projective variety $X$, then there exists a quotient $X/G$ as a projective variety. The essential reason for this is that every $G$-orbit (or in fact every finite set of points) is contained in an affine variety $U$, which is the complement of some hypersurface. Then the intersection $V = \bigcap_g gU$ of the $G$-translates is a $G$-invariant affine open containing $Gx$ and $V/G = \operatorname{Spec} \Gamma(V, \mathcal{O}_V)^G$ (see Corollary 4.2.7). These local quotients glue to form $X/G$.

However, if $X$ is not projective, the quotient does not necessarily exist as a scheme. As with most phenomena for smooth proper varieties that are not projective, a counterexample can be constructed by using Hironaka's examples of smooth, proper 3-folds; [Har77, App. B, Ex. 3.4.1]. There is a smooth, proper 3-fold with a free action by $G = \mathbb{Z}/2$ such that there is an orbit $Gx$ not contained in any $G$-invariant affine open. This shows that $X/G$ cannot exist as a scheme; indeed, if it did, then the image of $x$ under the finite morphism $X \to X/G$ would be contained in some affine and its inverse would be an affine open containing $Gx$. See [Knu71, Ex. 1.3] or [Ols16, Ex. 5.3.2] for details.

Nevertheless, for a free action of a finite group $G$ on a scheme $X$, then every point $x \in X$ has a $G$-equivariant étale neighborhood $U_x \to X$ where $U_x$ is an affine scheme, and the quotients $U_x/G$ can be glued in the *étale topology* to construct $X/G$ as an *algebraic space* (Corollary 3.1.13). The upshot is that we can always take quotients of free actions by finite groups. This is a very desirable feature given the ubiquity of group actions in algebraic geometry but it comes at the cost of enlarging our category from schemes to algebraic spaces.

**Example 0.5.6** (Artin approximation)**.** *Artin approximation* is a powerful and extremely deep result, due to Michael Artin, which implies that most properties which hold for the completion $\widehat{\mathcal{O}}_{X,x}$ of the local ring is also true in an étale neighborhood of $x$. See Theorem A.10.9 for a precise statement. For instance, since the completion of the local ring at a nodal singularity of a curve is reducible, Artin approximation implies that there is a reducible étale neighborhood.

### 0.5.3 Working with the étale topology: descent theory

Another reason why the étale topology is so useful is that many properties of schemes and their morphisms can be checked on étale covers. In fact, almost every property that can be checked on a Zariski-open cover $\{U_i\}$ of scheme $X$ can also be checked on an étale cover $\{U_i \to X\}$; here each map $U_i \to U$ is étale and $\coprod_i U_i \to U$ is surjective. Descent theory is developed in Appendix B and is used to prove just about everything about algebraic spaces and stacks.

## 0.6 Moduli stacks

As promised, we now synthesize moduli functors with the groupoid perspective. To define a moduli stack, we need to specify

(1) families of objects;

(2) how two families of objects are isomorphic; and

(3) how families pull back under morphisms.

Notice the difference from specifying a moduli functor is that rather than specifying *when* two families are isomorphic, we specify *how*.

In other words, we need to specify an assignment

$$F\colon \mathrm{Sch} \to \mathrm{Groupoids}, \quad S \mapsto \mathrm{Fam}_S\,.$$

taking a scheme $S$ to a *groupoid* of families of objects over $S$. But what exactly do we mean by this? Groupoids form a '2-category' as they have objects (groupoids), morphisms (functors between groupoids), and 2-morphisms (natural transformations between functors). How can we precisely formulate such an assignment in down-to-earth terms? Well, we certainly need pullback functors $f^*\colon \mathrm{Fam}_T \to \mathrm{Fam}_S$ for each morphism $f\colon S \to T$. Given a composition $S \xrightarrow{f} T \xrightarrow{g} U$ of schemes, we should also have an isomorphism of functors (i.e. a 2-morphism) $\mu_{f,g}\colon (f^* \circ g^*) \xrightarrow{\sim} (g \circ f)^*$. Should the isomorphisms $\mu_{f,g}$ satisfy a compatibility condition under triples $S \xrightarrow{f} T \xrightarrow{g} U \xrightarrow{h} V$? Yes! This leads to the notion of a *pseudo-functor* but we won't spell it out here; we encourage the reader to work it out (or to look it up [Vis05, Def. 3.10], [SP, Tag 003N]). We will take a slightly different approach.

### 0.6.1 Motivating the definition of a prestack

Instead of trying to define an assignment $S \mapsto \mathrm{Fam}_S$, we will build one massive category $\mathcal{X}$ encoding all of the groupoids $\mathrm{Fam}_S$ which will live over the category Sch of schemes. Loosely speaking, the objects of $\mathcal{X}$ will be a family $a$ of objects over a scheme $S$, i.e. $a \in \mathrm{Fam}_S$, and a morphism $a \to b$ between a family $a$ over $S$ and a family $b$ over $T$ will be the data of a morphism $f: S \to T$ together with an isomorphism $a \xrightarrow{\sim} f^*b$ of $a$ and the pullback family of $b$.

A *prestack* over Sch is a category $\mathcal{X}$ together with functor $p: \mathcal{X} \to \mathrm{Sch}$, which we visualize as

$$\begin{tikzcd} \mathcal{X} \arrow[d, "p"] \\ \mathrm{Sch} \end{tikzcd} \qquad \begin{tikzcd} a \arrow[r, "\alpha"] \arrow[d, mapsto] & b \arrow[d, mapsto] \\ S \arrow[r, "f"] & T \end{tikzcd}$$

where the lower case letters $a, b$ are objects in $\mathcal{X}$ and the upper case letters $S, T$ are schemes. We say that *$a$ is over $S$* and that *$\alpha: a \to b$ is over $f: S \to T$*. Moreover, we need to require that certain natural axioms hold for $p: \mathcal{X} \to \mathrm{Sch}$. Loosely speaking, we require the existence and uniqueness of pullbacks: given a map $S \to T$ and object $b \in \mathcal{X}$ over $T$, there should exist an arrow $a \xrightarrow{\alpha} b$ over $f$ satisfying a suitable universal property; see Definition 2.3.1.

Given a scheme $S$, the *fiber category* $\mathcal{X}(S)$ is defined as the category of objects over $S$ whose morphisms are over the identity. If $\mathcal{X}$ is built from the groupoids $\mathrm{Fam}_S$ as above, then $\mathcal{X}(S) = \mathrm{Fam}_S$.

**Example 0.6.1** (Viewing a functor as a prestack)**.** A moduli functor $F: \mathrm{Sch} \to \mathrm{Sets}$ can be encoded as a moduli prestack as follows: we define the category $\mathcal{X}_F$ of pairs $(S, a)$ where $S$ is a scheme and $a \in F(S)$. A map $(S', a) \to (S, a)$ is a map $f: S' \to S$ such that $a' = f^*a$, where $f^*$ is convenient shorthand for $F(f): F(S) \to F(S')$. Observe that the fiber categories $\mathcal{X}_F(S)$ are equivalent (even equal) to the set $F(S)$.

**Example 0.6.2** (Moduli prestack of smooth curves)**.** The *moduli prestack of smooth curves* is the category $\mathcal{M}_g$ of families of smooth curves $\mathcal{C} \to S$ together with the functor $p: \mathcal{M}_g \to \mathrm{Sch}$ defined by $(\mathcal{C} \to S) \mapsto S$. A morphism $(\mathcal{C}' \to S') \to (\mathcal{C} \to S)$ is the data of maps $\alpha: \mathcal{C}' \to \mathcal{C}$ and $f: S' \to S$ such that the diagram

$$\begin{tikzcd} \mathcal{C}' \arrow[r, "\alpha"] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{C} \arrow[d] \\ S' \arrow[r, "f"] & S \end{tikzcd}$$

is cartesian. Note that in the fiber category $\mathcal{M}_g(\mathbb{C})$, an object is a smooth curve $C$ and the set of morphisms $C \to C$ is identified with the automorphism group $\mathrm{Aut}(C)$.

**Example 0.6.3** (Moduli prestack of vector bundles)**.** The *moduli prestack of vector bundles* on a smooth curve $C$ is the category $\mathrm{Bun}_{r,d}(C)$ of pairs $(E, S)$ where $S$ is a scheme and $E$ is a vector bundle on $C_S = C \times_{\mathbb{C}} S$ together with the functor $p: \mathrm{Bun}_{r,d}(C) \to \mathrm{Sch}/\mathbb{C}, (E, S) \mapsto S$. A map $(E', S') \to (E, S)$ consists of a map of schemes $f: S' \to S$ together with an isomorphism $E' \xrightarrow{\sim} (\mathrm{id} \times f)^*E$

### 0.6.2 Motivating the definition of a stack

A stack is to a prestack as a sheaf is to a presheaf. The concept could not be more intuitive: we require that objects and morphisms glue uniquely.

**Example 0.6.4** (Moduli stack of sheaves)**.** Define the category $\mathcal{X}$ over Sch of pairs $(E, S)$ where $E$ is a sheaf of abelian groups on a scheme $S$, and the functor $p\colon \mathcal{X} \to \mathrm{Sch}$ is given by $(E, S) \mapsto S$. A map $(E', S') \to (E, S)$ in $\mathcal{X}$ is a map of schemes $f\colon S' \to S$ together with a map $E \to f_* E'$ of $\mathcal{O}_{S'}$-modules whose adjoint is an isomorphism.

You already know that morphisms of sheaves glue: let $E$ and $F$ be sheaves on schemes $S$ and $T$ and let $f\colon S \to T$ be a map. If $\{S_i\}$ is a Zariski open cover of $S$, then giving a morphism $\alpha\colon (E, S) \to (F, T)$ is the same data as giving morphisms $\alpha_i\colon (E|_{S_i}, S_i) \to (F, T)$ such that $\alpha_i|_{S_{ij}} = \alpha_j|_{S_{ij}}$ [Har77, Exer. II.1.15]. You also know how sheaves glue—it is more complicated than gluing morphisms since sheaves have automorphisms and given two sheaves, we prefer to say that they are isomorphic rather than equal. If $\{S_i\}$ is a Zariski open cover of a scheme $S$, then giving a sheaf $E$ on $S$ is equivalent to giving a sheaf $E_i$ on $S_i$ and isomorphisms $\phi_{ij}\colon E_i|_{S_{ij}} \to E_j|_{S_{ij}}$ such that $\phi_{ik} = \phi_{jk} \circ \phi_{ij}$ on the triple intersection $S_{ijk}$ [Har77, Exer. II.1.22].

In a similar way, we could have considered the stack of $\mathcal{O}$-modules, quasi-coherent sheaves, or vector bundles, or we could have stacks of sheaves/$\mathcal{O}$-modules/quasi-coherent sheaves/vector bundles over a given scheme $X$ where an object over a scheme $S$ is sheaf on $X \times S$.

The definition of a stack (Definition 2.4.1) simply axiomitizes these two natural gluing concepts.

### 0.6.3 Motivating the definition of an algebraic stack

For a stack to be a geometric object, we need to specify that it is locally like a scheme in a suitable sense. Without such a condition would be like trying to study the geometry of an arbitrary ringed space $(X, \mathcal{O}_X)$ or a (possibly non-representable) functor $F\colon \mathrm{Sch} \to \mathrm{Sets}$ which is a sheaf in the big Zariski topology. If we wish to utilize our algebraic geometry toolkit (e.g. coherent sheaves, commutative algebra, cohomology, ...) to study stacks in a similar way that we study schemes, we must impose an algebraicity condition.

The conditions we impose are quite natural. Here are the definitions in increasing generality:

(1) A functor $X\colon \mathrm{Sch} \to \mathrm{Sets}$ is an *algebraic space* if objects of $X$ glue uniquely in the étale topology and there is an étale cover $\{U_i \to X\}$ where each $U_i$ is an affine scheme.

(2) A stack $\mathcal{X} \to \mathrm{Sch}$ is *Deligne–Mumford* if there is an étale cover $\{U_i \to \mathcal{X}\}$ where each $U_i$ is an affine scheme.

(3) A stack $\mathcal{X} \to \mathrm{Sch}$ is *algebraic* if there is a smooth cover $\{U_i \to \mathcal{X}\}$ where each $U_i$ is an affine scheme.

Of course, we need to make precise what étale and smooth covers are. For $\{U_i \to X\}$ to be an étale cover in (1), we require that for every map $T \to X$ of functors where $T$ is representable by a scheme, the fiber product of functors (introduced in Exercise 0.3.28) is representable by a scheme $T_i$ such that $T_i \to T$ is étale and $\coprod T_i \to T$ is surjective. Replacing 'étale cover' with 'Zariski cover' (as defined in Definition 0.3.29(3)) would be equivalent to requiring that $F$ is representable by a scheme by Exercise 0.3.30. Only minor modifications are needed to define étale/smooth covers $\{U_i \to \mathcal{X}\}$ in (2)/(3): for every morphism $T \to \mathcal{X}$ from a scheme $T$, the fiber product $U_i \times_{\mathcal{X}} T$ is representable by an algebraic space $T_i$ such that each $T_i \to T$ is étale/smooth and $\coprod_i T_i \to T$ is surjective. To make this completely rigorous will require a little more work as we need to make sense of fiber products of stacks and properties of morphisms of algebraic spaces. See Definition 3.1.6 for the precise definition of an algebraic stack.

| Algebro-geometric space | Type of object | Obtained by gluing |
|---|---|---|
| Schemes | ringed space/ sheaf | affine schemes in the Zariski topology |
| Algebraic spaces | sheaf | affine schemes in the étale topology |
| Deligne–Mumford stacks | stack | affine schemes in the étale topology |
| Algebraic stacks | stack | affine schemes in the smooth topology |

Table 2: Schemes, algebraic spaces, Deligne–Mumford stacks, and algebraic stacks are obtained by gluing affine schemes.

**Why smooth covers?** After all the fuss motivating étale morphisms above, you might be surprised to see that an algebraic stack is *smooth-locally* a scheme. For Deligne–Mumford stacks—which turn out to be precisely algebraic stacks with finite automorphism groups—étale covers are sufficient. But for algebraic stacks like $\mathrm{Bun}_{r,d}(C)$ with infinite automorphism groups, we need smooth covers. For instance, we would like to be able to form the quotient $[\operatorname{Spec}\mathbb{C}/\mathbb{G}_m]$ (which we will call the *classifying stack* $\mathbf{B}\mathbb{G}_m$) of the trivial action of $\mathbb{G}_m$ (or $\mathbb{C}^*$) on a point, and this will have no étale cover by a scheme.

### 0.6.4 Examples of moduli stacks

Constructing a smooth cover of a given moduli stack is a geometric problem inherent to the moduli problem. It can often be solved by rigidifying the moduli problem by parameterizing additional information. This concept is best absorbed in examples.

**Example 0.6.5** (Moduli stack of elliptic curves)**.** An elliptic curve $(E,p)$ is embedded into $\mathbb{P}^2$ via $\mathcal{O}_E(3p)$ such that $E$ is defined by a Weierstrass equation $y^2z = x(x-z)(x-\lambda z)$ for some $\lambda \neq 0, 1$ [Har77, Prop. IV.4.6]. Setting $U = \mathbb{A}^1 \setminus \{0,1\}$ with coordinate $\lambda$, the family $\mathcal{E} \subset U \times \mathbb{P}^2$ of elliptic curves defined by the Weierstrass equation defines map $U \to \mathcal{M}_{1,1}$ which is an étale cover.

**Example 0.6.6** (Moduli stack of smooth curves)**.** For a smooth curve $C$ of genus $g \geq 2$, the line bundle $\Omega_C^{\otimes 3}$ is very ample and defines an embedding $C \hookrightarrow \mathbb{P}(\Gamma(C, \Omega_C^{\otimes 3})) \cong \mathbb{P}^{5g-6}$. There is a Hilbert scheme $H$ (see Theorem 1.1.2) parameterizing closed subschemes of $\mathbb{P}^{5g-6}$ with the same Hilbert polynomial as $C \subset \mathbb{P}^{5g-6}$, and there is a locally closed subscheme $H' \subset H$ parameterizing *smooth* subschemes such that $\Omega_C^{\otimes 3} \cong \mathcal{O}_C(1)$. The universal subscheme over $H'$ defines a map $H' \to \mathcal{M}_g$ which is a smooth cover (see Theorem 3.1.16 for details) and thus $\mathcal{M}_g$ is an algebraic stack. We will show that it is Deligne–Mumford in Corollary 3.6.9.

**Example 0.6.7** (Moduli stack of vector bundles)**.** For every vector bundle $E$ of rank $r$ and degree $d$ on a smooth curve $C$, the twist $E(m)$ is globally generated for sufficiently large $m$. Taking $N_m = \mathrm{h}^0(C, E(m))$, we can view $E$ as a quotient $\mathcal{O}_C(-m)^{N_m} \twoheadrightarrow E$. There is a Quot scheme $Q_m$ (see Theorem 1.1.3) parameterizing such quotients that have the same Hilbert polynomial as $E$ and there is a locally closed subscheme $Q'_m \subset Q$ parameterizing vector bundle quotients $\pi\colon \mathcal{O}_C(-m)^{N_m} \twoheadrightarrow E$ such that the induced map $\Gamma(\pi \otimes \mathcal{O}_C(m))\colon \mathbb{C}^{N_m} \to \Gamma(C, E(m))$ is an isomorphism. The universal quotient over $Q'_m$ defines a map $Q'_m \to \mathrm{Bun}_{r,d}(C)$ which is smooth and the collection $\{Q'_m \to \mathrm{Bun}_{r,d}(C)\}$ for $m \gg 0$ defines a smooth cover. This shows that an $\mathrm{Bun}_{r,d}(C)$ is an algebraic stack; see Theorem 3.1.20 for details. It is not a Deligne–Mumford stack.

### 0.6.5 Quotient stacks

One of the most important examples of a stack is a quotient stack $[X/G]$ arising from an action of an algebraic group $G$ on a scheme $X$. The geometry of $[X/G]$ couldn't be simpler: it is the $G$-equivariant geometry of $X$ (see §0.6.6).

Similar to how toric varieties provide concrete examples of schemes, quotient stacks provide both concrete examples useful to gain geometric intuition of general algebraic stacks and a fertile testing ground for conjectural results. On the other hand, it turns out that many algebraic stacks are quotient stacks or are at least locally quotient stacks, and most properties that hold for quotient stacks also hold for many algebraic stacks.

**Quotient prestack:** Given an action of an algebraic group $G$ on a scheme $X$, the *quotient prestack* $[X/G]^{\mathrm{pre}}$ is the prestack whose fiber category $[X/G]^{\mathrm{pre}}(S)$ over a scheme $S$ is the quotient groupoid (or the moduli groupoid of orbits) $[X(S)/G(S)]$. This will not satisfy the gluing axioms of a stack; even when the action is free, the quotient functor $\mathrm{Sch} \to \mathrm{Sets}$ defined by $S \mapsto X(S)/G(S)$ is not a sheaf (see Exercise 0.3.25). How can we make it into a stack? Well, instead of thinking of an object of $[X/G]^{\mathrm{pre}}$ over a scheme $S$ as a morphism $f\colon S \to X$, let's think of it as a trivial $G$-bundle together with a map to $X$:

$$\begin{tikzcd} G \times S \arrow[r, "\widetilde{f}"] \arrow[d, "p_2"] & X, & (g,s) \arrow[r, mapsto] & g \cdot f(s) \\ S. \end{tikzcd}$$

Given two maps $f_1, f_2\colon S \to X$, an element of $\alpha \in G(S)$ with $f_2 = \alpha \cdot f_1$ is the same data as an isomorphism of trivial $G$-bundles $G \times S \to G \times S$, $(g,s) \mapsto (g\alpha(s), s)$; this is because any such isomorphism must be $G$-equivariant and commute with the structure maps to $S$. From this perspective, it is even more clear that $[X/G]^{\mathrm{pre}}$ is not stack even when $X$ is a point: given a Zariski cover $\{S_i\}$ of a scheme $S$, trivial $G$-bundles $G \times S_i \to S_i$ together with isomorphisms over $S_i \cap S_j$ satisfying a cocycle condition will glue to a principal $G$-bundle $P \to S$ but it will *not* necessarily be trivial. This suggests that we should define objects of a quotient stack to be principal $G$-bundles (Definition C.2.1).

**Quotient stack:** We define the *quotient stack* $[X/G]$ as the category over $\mathrm{Sch}/\mathbb{C}$

whose objects over a $\mathbb{C}$-scheme $S$ are diagrams

$$\begin{tikzcd} P \arrow[r, "f"] \arrow[d] & X \\ S & \end{tikzcd}$$

where $P \to S$ is a principal $G$-bundle and $f\colon P \to X$ is a $G$-equivariant morphism. A morphism $(P' \to S', P' \xrightarrow{f'} X) \to (P \to S, P \xrightarrow{f} X)$ is the data of a commutative diagram

$$\begin{tikzcd} P' \arrow[r, "\varphi"] \arrow[d] \arrow[rr, bend left, "f'"] \arrow[dr, phantom, "\square"] & P \arrow[r, "f"] \arrow[d] & X \\ S' \arrow[r, "g"] & S & \end{tikzcd}$$

where the left square is cartesian.

There is an object of $[X/G]$ over $X$ given by the diagram

$$\begin{tikzcd} G \times X \arrow[r, "\sigma"] \arrow[d, "p_2"] & X \\ X, & \end{tikzcd}$$

where $\sigma$ denotes the action map, and this defines a map $X \to [X/G]$. The map $X \to [X/G]$ is a principal $G$-bundle even if the action of $G$ on $X$ is not free. Let's pause to appreciate that:

> **The map $X \to [X/G]$ is a principal $G$-bundle even if the action of $G$ on $X$ is not free.**

In particular, the map $X \to [X/G]$ is smooth and thus $[X/G]$ is algebraic (see also Theorem 3.1.10). At the expense of enlarging our category from schemes to algebraic stacks, we are able to (tautologically) construct the quotient $[X/G]$ as a 'geometric space' with desirable properties.

**Example 0.6.8** (Classifying stack)**.** We define the *classifying stack* of an algebraic group $G$ as the category $\mathbf{B}G := [\operatorname{Spec}\mathbb{C}/G]$ of principal $G$-bundles $P \to S$. The projection $\operatorname{Spec}\mathbb{C} \to \mathbf{B}G$ is not only a principal $G$-bundle; it is the *universal principal $G$-bundle*. Given any other principal $G$-bundle $P \to S$, there is a unique map $S \to \mathbf{B}G$ and a cartesian diagram

$$\begin{tikzcd} P \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \operatorname{Spec}\mathbb{C} \arrow[d] \\ S \arrow[r] & \mathbf{B}G. \end{tikzcd}$$

**Example 0.6.9** (Quotients by finite groups)**.** Quotients by free actions of finite groups exist as algebraic spaces! See Corollary 3.1.13.

**Exercise 0.6.10.** What is the universal family over the quotient stack $[X/G]$?

Moduli stacks can often be described as quotient stacks, and these descriptions can be leveraged to establish properties of the moduli stack.

**Example 0.6.11** (Moduli stack of smooth curves as a quotient)**.** Reexamining Example 0.6.6, we see that the embedding of a smooth curve $C$ via $|\Omega_C^{\otimes 3}|\colon C \hookrightarrow \mathbb{P}^{5g-6}$ depends on a choice of basis $\Gamma(C, \Omega_C^{\otimes 3}) \cong \mathbb{C}^{5g-5}$ and therefore is only unique up to a projective automorphism, i.e. an element of $\mathrm{PGL}_{5g-5} = \mathrm{Aut}(\mathbb{P}^{5g-6})$. The algebraic group $\mathrm{PGL}_{5g-5}$ acts on the scheme $H'$ parameterizing smooth tricanonically embedded curves such that $\mathcal{M}_g \cong [H'/\,\mathrm{PGL}_{5g-6}]$.

**Example 0.6.12** (Moduli stack of smooth curves as a quotient)**.** For the moduli stack of vector bundles (Example 0.6.7), the presentation of a vector bundle $E$ as a quotient $\mathcal{O}_C(-m)^{N_m} \twoheadrightarrow E$ depends on a choice of basis $\Gamma(C, E(m)) \cong \mathbb{C}^{N_m}$. The algebraic group $\mathrm{PGL}_{N_m-1}$ acts on the scheme $Q'_m$ and there is an identification

$$\mathrm{Bun}_{r,d}(C) \cong \bigcup_{m \gg 0} [Q'_m/\,\mathrm{PGL}_{N_m-1}]$$

.

### 0.6.6 Geometry of a quotient stack

While the definition of the quotient stack $[X/G]$ may appear abstract, its geometry is very familiar. The table below provides a dictionary between the geometry of a quotient stack $[X/G]$ and the $G$-equivariant geometry of $X$. The stack-theoretic concepts on the left-hand side will be introduced later.

| Geometry of $[X/G]$ | $G$-equivariant geometry of $X$ |
|---|---|
| $\mathbb{C}$-point $\overline{x} \in [X/G]$ | orbit $Gx$ of $\mathbb{C}$-point $x \in X$ (with $\overline{x}$ the image of $x$ under $X \to [X/G]$) |
| automorphism group $\mathrm{Aut}(\overline{x})$ | stabilizer $G_x$ |
| function $f \in \Gamma([X/G], \mathcal{O}_{[X/G]})$ | $G$-equivariant function $f \in \Gamma(X, \mathcal{O}_X)^G$ |
| map $[X/G] \to Y$ to a scheme $Y$ | $G$-equivariant map $X \to Y$ |
| line bundle | $G$-equivariant line bundle (or $G$-linearization) |
| quasi-coherent sheaf | $G$-equivariant quasi-coherent sheaf |
| tangent space $T_{[X/G],\overline{x}}$ | normal space $T_{X,x}/T_{Gx,x}$ to the orbit |
| coarse moduli space $[X/G] \to Y$ | geometric quotient $X \to Y$ |
| good moduli space $[X/G] \to Y$ | good GIT quotient $X \to Y$ |

## 0.7 Constructing projective moduli spaces

Our motivation for algebraic stacks was to ensure that a given moduli problem $\mathcal{M}$ is representable with a universal family. While many geometric questions can be studied (and arguably should be studied) on the moduli stack $\mathcal{M}$ itself, it is often very convenient to make a trade-off: by sacrificing the existence of a universal family, we can sometimes construct a more familiar geometric space, ideally a projective variety. This allows us to utilize the much larger toolkit of projective geometry (e.g. birational geometry, intersection theory, Hodge theory, ...) to study the moduli problem.

We highlight two approaches to construct projective moduli spaces:

(1) Geometric Invariant Theory (GIT), and

(2) Intrinsic construction of coarse/good moduli spaces.

There is a beautiful interplay between the intrinsic and extrinsic approaches. Ideas from GIT have inspired techniques in each of the six steps of the intrinsic approach, and conversely the intrinsic approach sheds light back on GIT. GIT is also deeply intertwined with 19th century invariant theory, and determining the GIT semistable locus is an interesting and important problem on its own. It is valuable to keep both approaches in mind.

### 0.7.1 GIT approach

Outline of the GIT strategy

(A) Express the moduli stack $\mathcal{M}$ as a substack

$$\mathcal{M} \subset [X/G],$$

where $G$ is reductive and $X \hookrightarrow \mathbb{P}(V)$ is $G$-equivariantly embedded into the projectivization of a $G$-representation $V$.

(B) Show that a point $x \in X$ is GIT semistable if and only if $x \in \mathcal{M}$, or in other words that $\mathcal{M} = [X^{\mathrm{ss}}/G]$.

For Step A, there are often natural ways to rigidify the moduli problem by parameterizing additional data. For smooth curves, we can parameterize a basis of $\Gamma(C, \Omega_C^{\otimes 3})$ along with a curve $C$ to obtain a tricanonical embedding $C \hookrightarrow \mathbb{P}^{5g-6}$ (Example 0.6.11), or we can parameterize a basis of $\Gamma(C, \Omega_C^{\otimes k})$ for the $k$th canonical embedding $C \hookrightarrow \mathbb{P}^{(2k-1)(g-1)-1}$. For vector bundles on a smooth curve, we can parameterize a basis of $\Gamma(C, E(m))$ along with a vector bundle $E$, after making a *choice* of a sufficiently large integer $m$. The rigidified moduli problem should have a compactification which is represented by a projective variety $X$—which is Hilb and Quot in our two examples—and the choice of additional data should be governed by an action of a group $G$. For the GIT approach to succeed, we need that $G$ is reductive and that $\mathcal{M}$ is a substack of $[X/G]$. Finally, we need to choose a $G$-equivariant embedding $X \hookrightarrow \mathbb{P}(V)$ where $V$ is a finite dimensional $G$-representation, or equivalently choose a $G$-linearization of an ample line bundle on $X$.

Step B is the hardest: we must show that $\mathcal{M}$ is precisely the open substack of $[X/G]$ of GIT semistable points. Using the Hilbert–Mumford Criterion we can translate the problem to the following: a point $x \in X$ represents an object of the moduli problem $\mathcal{M}$ if and only if *Hilbert–Mumford index* $\mu(x, \lambda) \geq 0$ for every one-parameter subgroup $\lambda \colon \mathbb{G}_m \to G$. This often reduces the goal to a tractable (but often still daunting) combinatorial problem.

The GIT quotient $M := X^{\mathrm{ss}} /\!\!/ G$ is necessarily projective. One beautiful feature of GIT is that even if the moduli stack $\mathcal{M}$ is not compact, the GIT strategy provides a compactification! If $\mathcal{M}$ has only finite automorphisms or equivalently there are no strictly semistable points, then $X^{\mathrm{ss}} \to M$ is a geometric quotient and $\mathcal{M} = [X^{\mathrm{ss}}/G] \to M$ is a coarse moduli space. In the presence of infinite automorphisms, $X^{\mathrm{ss}} \to M$ is a good quotient and $\mathcal{M} \to M$ is a good moduli space.

The GIT approach is covered in detail in §6.7. We sketch the GIT construction of $\overline{M}_g$ in §5.8.4 and present a complete GIT construction of $\mathrm{Bun}_{r,d}(C)$ in Chapter 7.

### 0.7.2 Intrinsic approach

> *However I do not claim at all that [GIT] should be avoided, but only that sometimes it may be good to have an alternative.*
>
> ---
>
> FALTINGS [Fal93]

Six steps toward projective moduli

(1) **Algebraicity:** Express the moduli stack $\mathcal{M}$ as a substack

$$\mathcal{M} \subset \mathcal{X}$$

of a larger moduli stack $\mathcal{X}$. Define an object $x \in \mathcal{X}$ to be *semistable* if it is in $\mathcal{M}$; this allows us to think of $\mathcal{M}$ as the semistable locus $\mathcal{X}^{\mathrm{ss}}$. Show that $\mathcal{X}$ is an algebraic stack locally of finite type over $\mathbb{C}$.

(2) **Openness of semistability:** Show that semistability is an open condition, i.e. $\mathcal{M} = \mathcal{X}^{\mathrm{ss}} \subset \mathcal{X}$ is an open substack.

(3) **Boundedness of semistability:** Show that semistability is bounded, i.e. $\mathcal{M} = \mathcal{X}^{\mathrm{ss}}$ is of finite type over $\mathbb{C}$.

(4) **Semistable reduction:** Show that $\mathcal{M}$ satisfies the existence part of the valuative criterion for properness.

(5) **Existence of a moduli space:** Show that there is a fine/coarse/good moduli space $\mathcal{M} \to M$ where $M$ is a proper algebraic space.[5]

(6) **Projectivity:** Show that a tautological line bundle on $\mathcal{M}$ descends to an ample line bundle on $M$, i.e. $M$ is projective.

GIT magically solves all these steps at once! In Step A of the GIT approach, expressing the moduli stack $\mathcal{M}$ as a substack $[X/G]$ already implies 'boundedness.' Since GIT semistability is always an open condition, the identification in Step B of $\mathcal{M}$ with the semistable locus $[X^{\mathrm{ss}}/G]$ gives 'openness of semistability' and thus 'algebraicity' of $\mathcal{M}$. Strikingly, GIT also implies each of the other steps: 'semistable reduction,' 'existence of a moduli space', and 'projectivity.'

**Step 1 (Algebraicity).** Many moduli stacks have natural enlargements. The stack $\mathcal{M}_g$ of smooth curves and the stack $\overline{\mathcal{M}}_g$ of stable curves are both contained in the stack of all curves. The stack of semistable vector bundles on a smooth curve is contained in the stack of all vector bundles or even the stack of all coherent sheaves. It is usually easier to first show that the enlargement $\mathcal{X}$ is an algebraic stack, and then use Steps 2 and 3 to conclude that $\mathcal{M}$ itself is algebraic.

[5] The calligraphic font $\mathcal{M}$ denotes the stack while the Roman font $M$ denotes the space. This convention will be followed throughout the text.

To get started, we need to define the stacks $\mathcal{M}$ and its enlargement $\mathcal{X}$—this entails specifying families of objects along with pullbacks and identifications. To check that $\mathcal{X}$ is algebraic requires finding a smooth cover $U \to \mathcal{X}$ by a scheme. In many cases, we can even show that $\mathcal{X}$ is identified with a quotient stack $[U/G]$ in which case $U \to [U/G]$ provides a presentation. Alternatively, it is often possible to use Artin's Criteria (Theorem D.7.4) to establish algebraicity; this essentially amounts to verifying local properties of the moduli problem and in particular requires an understanding of the deformation and obstruction theory.

**Step 2 (Openness of semistability).** This translates to the following condition: for every family $\mathcal{E}$ of objects of $\mathcal{X}$ over a scheme $S$, the subset

$$\{s \in S \mid \mathcal{E}_s \text{ is semistable}\}, \tag{0.7.1}$$

where $\mathcal{E}_s$ is the pullback of $\mathcal{E}$ along $\operatorname{Spec} \kappa(s) \to S$, is an open subset of $S$. This is precisely what it means for the inclusion $\mathcal{M} = \mathcal{X}^{\mathrm{ss}} \hookrightarrow \mathcal{X}$ to be representable by open immersions: for every map $S \to \mathcal{X}$ (corresponding to the family $\mathcal{E}$), the fiber product $\mathcal{M} \times_{\mathcal{X}} S$ (which is identified set-theoretically with (0.7.1)) is an open subscheme of $S$. This step ensures that $\mathcal{M}$ is also an algebraic stack locally of finite type over $\mathbb{C}$.

**Step 3 (Boundedness of semistability).** By boundedness, we mean that the moduli stack $\mathcal{M}$ is of finite type over $\mathbb{C}$. Since algebraicity implies that $\mathcal{M}$ is *locally* of finite type over $\mathbb{C}$, boundedness translates into quasi-compactness of $\mathcal{M}$. More concretely, boundedness is equivalent to the existence of a scheme $Z$ of *finite type* over $\mathbb{C}$ and a family of objects $\mathcal{E}$ over $Z$ such that every object $E$ of $\mathcal{M}$ is isomorphic to $\mathcal{E}_z$ for some (not necessarily unique) $z \in Z$.

For example, $\mathcal{M}_g$ is bounded but the stack of all proper curves of genus $g$ and the stack $\coprod_g \mathcal{M}_g$ of all smooth curves (of any genus) are not bounded. For vector bundles, the stack $\operatorname{Bun}^{\mathrm{ss}}_{r,d}(C)$ of semistable vector bundles of fixed rank and degree is bounded. The stack of all vector bundles $\operatorname{Bun}_{r,d}(C)$ of fixed rank and degree is not bounded, nor is the stack of semistable vector bundles of arbitrary rank and degree.

**Step 4 (Semistable reduction).** The existence part of the valuative criterion for properness is the assertion that for every DVR $R$ (which you can think of as a local model of a smooth curve) with fraction field $K$ (or punctured curve) then every object $\mathcal{E}^\times$ over $K$ extends to a family of objects $\mathcal{E}$ over $R$ after possibly replacing $R$ with an extension of DVRs. In other words, every diagram

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r, "\mathcal{E}^\times"] \arrow[d] & \mathcal{M} \\ \operatorname{Spec} R, \arrow[ur, dashed, "\mathcal{E}"'] & \end{tikzcd} \tag{0.7.2}$$

has an extension after replacing $R$ with an extension. If the extension $\mathcal{E}$ over $R$ is also unique, then we say that $\mathcal{M}$ satisfies the valuative criterion for properness, and this implies properness (Theorem 3.8.5) and in particular separatedness. Arguably the usefulness of valuative criteria in algebraic geometry is best witnessed in moduli theory.

The moduli stack of smooth curves is not compact and does not satisfy the existence part of the valuative criterion.

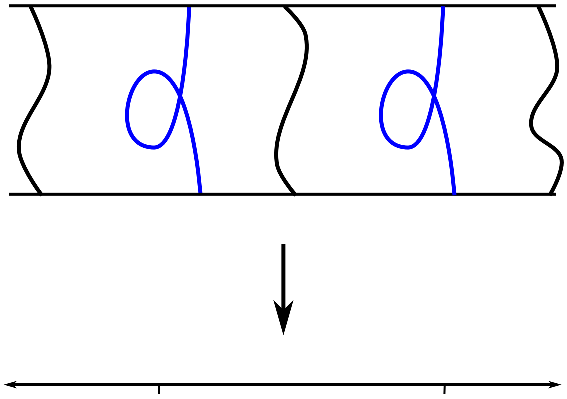



Figure 18: The family of elliptic curves $y^2z = x(x-z)(x-\lambda z)$ degenerates to the nodal cubic over $\lambda = 0, 1$.

Projective varieties are of course compact and satisfy the valuative criterion. If there's any hope to construct a projective moduli space, then the moduli stack better satisfy the existence part of the valuative criterion. Properness of $\overline{\mathcal{M}}_g$ was first proven by Deligne and Mumford in their influential paper [DM69]. We prove semistable reduction in characteristic 0 in §5.5.

For the moduli of vector bundles, semistable reduction was first proved by Mumford and Seshadri as a consequence of the GIT construction [Ses67]. An intrinsic geometric argument was later proved by Langton [Lan75]. Note that unlike stable curves, the stack $\mathrm{Bun}^{\mathrm{ss}}_{r,d}(C)$ is not separated as there may exist several non-isomorphic extensions of a vector bundle on $C_K$ to $C_R$. Nevertheless, the moduli stack $\mathrm{Bun}^{\mathrm{ss}}_{r,d}(C)$ satisfies a weaker notion of separatedness called *S-completeness*.

**Step 5 (Existence of a moduli space).** We would like to construct an algebraic space that is the best possible approximation of the moduli stack. This step depends on the automorphisms of the moduli problem:

- *No automorphisms*: in this case, the moduli stack $\mathcal{M}$ is already an algebraic space $M$, or in other words $M$ is a fine moduli space.
- *Finite automorphisms*: we must show that $\mathcal{M}$ is separated or in other words that $\mathcal{M}$ satisfies the uniqueness part (in addition to the existence part) of the valuative criterion. The Keel–Mori theorem (Theorem 4.3.11) then establishes the existence of a *coarse moduli space* $\mathcal{M} \to M$ where $M$ is a proper algebraic space. The map $\mathcal{M} \to M$ induces a bijection of $\mathbb{C}$-points and satisfies the universal property that any other map $\mathcal{M} \to Y$ to an algebraic space $Y$ factors uniquely through $M$.
- *Reductive automorphisms*: we must show that $\mathcal{M}$ is *Θ-complete* and *S-complete*—these are valuative criteria about extending $\mathbb{G}_m$-equivariant families of objects over a punctured surface which are introduced in Section 6.8.2. Given these properties, Theorem 6.8.1 yields a *good moduli space* $\mathcal{M} \to M$ where $M$ is a proper algebraic space. The map $\mathcal{M} \to M$ is no longer a bijection of $\mathbb{C}$-points as it identifies points whose closures intersect in an analogous way to the orbit closure equivalence relation in GIT. But $\mathcal{M} \to M$ does induce a bijection between *closed* $\mathbb{C}$-points of $\mathcal{M}$ (sometimes called *polystable* objects) and the $\mathbb{C}$-points of $M$, and it also satisfies the universal property for maps to algebraic spaces.

**Step 6 (Projectivity).** This is usually the hardest step. It requires a solid understanding of the geometry of the moduli problem and sometimes relies on sophisticated techniques in birational geometry. Kollár introduced a strategy in [Kol90] to verify projectivity for moduli stacks of varieties and applied it to verify the projectivity of $\overline{M}_g$. We cover Kollár's method in §5.8. Faltings constructed projective moduli spaces of vector bundles in [Fal93] without using the theory of GIT, and we borrow several of his ideas in our construction in Chapter 7.

# Chapter 1

# Hilbert and Quot schemes

We prove that the Grassmannian, Hilbert and Quot functors are representable by projective schemes. These results serve as the backbone of many results in moduli theory and more widely algebraic geometry. In particular, they are essential for establishing properties about the moduli stacks $\overline{\mathcal{M}}_g$ of stable curves and $\mathcal{V}^{\mathrm{ss}}_{r,d}$ of vector bundles over a curve. While the reader could safely treat these results as black boxes (and we encourage some readers to do this), it is also worthwhile to dive into the details. We follow Mumford's simplification [Mum66] of Grothendieck's original construction of Hilbert of Quot schemes [$\mathrm{FGA}_{\mathrm{IV}}$]. Specifically, we exploit the theory of Castelnuovo–Mumford regularity (Section 1.3) and flattening stratifications (Theorem A.2.14), which are interesting results on their own with wide-ranging applications outside moduli theory.

## 1.1 The Grassmannian, Hilbert, and Quot functors

### 1.1.1 Statements of the main theorems

The representability theorems below are formulated for a *strongly projective* morphism $X \to S$ of noetherian schemes, i.e. there exists a closed immersion $X \hookrightarrow \mathbb{P}_S(E)$ over $S$ where $E$ is a vector bundle on $S$. This is a stronger condition than the *projectivity* of $X \to S$ which only requires that $E$ is a coherent sheaf [EGA, §II.5], [SP, Tag 01W8]. On the other hand, the definition of projectivity in [Har77, II.4] requires that $X$ embeds into projective space $\mathbb{P}^n_S$ over $S$.

**Theorem 1.1.1.** *Let $S$ be a noetherian scheme and $V$ be a vector bundle on $S$ of rank $n$. For an integer $0 < k < n$, the functor*

$$\begin{aligned} \mathrm{Gr}_S(k, V) \colon \mathrm{Sch}/S &\to \mathrm{Sets} \\ (T \xrightarrow{f} S) &\mapsto \{ \text{ vector bundle quotients } V_T = f^*V \to Q \text{ of rank } k \ \} \end{aligned}$$

*is represented by a scheme strongly projective over $S$.*

If $S = \operatorname{Spec} \mathbb{Z}$ and $V = \mathcal{O}_S^n$, then $\mathrm{Gr}_S(k, V)$ is equal to the functor $\mathrm{Gr}(k, n)$ defined in Example 0.3.11. In addition, when $k = 1$, the Grassmannian $\mathrm{Gr}_S(1, V)$ is identified with the projectivization $\mathbb{P}_S(V)$ of $V$ as discussed in Exercise 0.3.12. For arbitrary $S$, we sometimes denote $\mathrm{Gr}_S(k, n) := \mathrm{Gr}_S(k, \mathcal{O}_S^n)$ and we sometimes drop the subscript $S$ when we are working over a fixed base such as $S = \operatorname{Spec} \Bbbk$ or $S = \operatorname{Spec} \mathbb{Z}$.

In the formulation of the following two theorems, we will use the convention that if $X \to S$ and $T \to S$ are morphisms of schemes, then $X_T := X \times_S T$. Similarly, if $F$ is a sheaf on $X$, then $F_T$ denotes the pullback of $F$ under $X_T \to X$. If $s \in S$ is a point, then $X_s := X \times_S \operatorname{Spec} \kappa(s)$ and $F_s := F|_{X_s} = F_{\operatorname{Spec} \kappa(s)}$. If $X \to S$ is a projective morphism, $\mathcal{O}_X(1)$ is relatively ample and $s \in S$ is a point, the *Hilbert polynomial of $F_s$* is

$$P_{F_s}(z) = \chi(X_s, F_s(z)),$$

where $F_s(z) = F_s \otimes \mathcal{O}_{X_s}(z)$. It is a fact that this defines a polynomial $P_{F_s} \in \mathbb{Q}[z]$ (c.f. [Har77, Exer III.5.2]]); for $z \gg 0$, we have $P_{F_s}(z) = \mathrm{h}^0(X_s, F_s(z))$.

**Theorem 1.1.2.** *Let $X \to S$ be a strongly projective morphism of noetherian schemes and $\mathcal{O}_X(1)$ be a relatively ample line bundle on $X$. For every polynomial $P \in \mathbb{Q}[z]$, the functor*

$$\begin{aligned} \operatorname{Hilb}^P_{X/S} \colon \mathrm{Sch}/S &\to \mathrm{Sets} \\ (T \to S) &\mapsto \left\{ \begin{array}{l} \textit{subschemes } Z \subset X_T \textit{ flat and finitely presented} \\ \textit{over } T \textit{ such that } Z_t \subset X_t \textit{ has Hilbert} \\ \textit{polynomial } P \textit{ for all } t \in T \end{array} \right\} \end{aligned}$$

*is represented by a scheme strongly projective over $S$.*

**Theorem 1.1.3.** *Let $\pi \colon X \to S$ be a strongly projective morphism of noetherian schemes, $\mathcal{O}_X(1)$ be a relatively ample line bundle on $X$, and $F$ be a coherent sheaf on $X$ which is the quotient of $\pi^*(W)(q)$ for a vector bundle $W$ on $S$ and an integer $q$. For every polynomial $P \in \mathbb{Q}[z]$, the functor*

$$\begin{aligned} \operatorname{Quot}^P_{X/S}(F) \colon \mathrm{Sch}/S &\to \mathrm{Sets} \\ (T \to S) &\mapsto \left\{ \begin{array}{l} \textit{quasi-coherent and finitely presented} \\ \textit{quotients } F_T \to Q \textit{ on } X_T \textit{ such that } Q \textit{ is} \\ \textit{flat over } T \textit{ and } Q|_{X_t} \textit{ on } X_t \textit{ has} \\ \textit{Hilbert polynomial } P \textit{ for all } t \in T \end{array} \right\} \end{aligned}$$

*is represented by a scheme strongly projective over $S$.*

The Grassmannian and the Hilbert scheme are special cases of the Quot scheme: $\mathrm{Gr}_S(k, V) \cong \operatorname{Quot}^P_{S/S}(V)$ where $P(z) = k$ is the constant polynomial and $\operatorname{Hilb}^P_{X/S} = \operatorname{Quot}^P_{X/S}(\mathcal{O}_X)$.

**Remark 1.1.4.**

(1) In the definition of the Grassmannian and Quot functor above, two quotients $V_T \xrightarrow{q} Q$ and $V_T \xrightarrow{q'} Q'$ are identified if $\ker(q) = \ker(q')$ as subsheaves of $V_T$, or equivalently there exists an isomorphism $Q \xrightarrow{\alpha} Q'$ such that the composition $V_T \xrightarrow{q} Q \xrightarrow{\alpha} Q'$ is equal to $V_T \xrightarrow{q'} Q'$. In the Hilbert functor, two subschemes of $X_T$ are identified if they are equal as subschemes (or equivalently their ideal sheaves are equal as subsheaves of $\mathcal{O}_{X_T}$).

(2) The definitions $\operatorname{Hilb}^P_{X/S}$ and $\operatorname{Quot}^P_{X/S}(F)$ depend on the relatively ample line bundle $\mathcal{O}_X(1)$ but we have suppressed this from the notation.

(3) When $T$ is noetherian, the conditions that $Z$ be finitely presented and $Q$ be of finite presentation in the definitions of $\operatorname{Hilb}^P_{X/S}$ and $\operatorname{Quot}^P_{X/S}(F)$ are superfluous.

(4) If we do not fix $P$, then $\mathrm{Hilb}(X/S)$ and $\mathrm{Quot}(F/X/S)$ are representable by schemes *locally* of finite type, and there are decompositions

$$\mathrm{Hilb}(X/S) = \coprod_P \mathrm{Hilb}^P_{X/S} \quad \text{and} \quad \mathrm{Quot}(F/X/S) = \coprod_P \mathrm{Quot}^P_{X/S}(F);$$

these functorial decompositions follows from the flatness of the quotient $Q$ and the local constancy of the Hilbert polynomial (Proposition A.2.4).

(5) Suppose that $S$ satisfies the *resolution property*, i.e. every coherent sheaf is the quotient of a vector bundle. This is satisfied if $S$ has an ample line bundle or if $S$ is regular. Then a projective morphism $X \to S$ is necessarily strongly projective. Moreover, if $F$ is a coherent sheaf on $X$, then $\pi^*\pi_*(F(q)) \to F(q)$ is surjective for $q \gg 0$ and choosing a surjection $W \twoheadrightarrow \pi_*(F(q))$ from a vector bundle $W$ on $S$, we have a surjection $\pi^*(W(-q)) \mapsto F$. Theorem 1.1.3 therefore implies that $\mathrm{Quot}^P_{X/S}(F)$ is strongly projective over $S$ if $X \to S$ is projective and $F$ is coherent.

**Caution 1.1.5.** We will abuse notation by using $\mathrm{Hilb}^P_{X/S}$, $\mathrm{Quot}^P_{X/S}(F)$ and $\mathrm{Gr}_S(k, V)$ to denote both the functor and the scheme that represents it.

### 1.1.2 Proof strategy

In §1.2, we show that $\mathrm{Gr}_S(k, V)$ is representable by a projective scheme by using the functorial Plücker embedding $\mathrm{Gr}_S(k, V) \to \mathbb{P}(\bigwedge^k V)$, which over an $S$-scheme $T$ sends a quotient $V_T \to Q$ to the line bundle quotient $\bigwedge^k V_T \to \bigwedge^k Q$.

In §1.3, we introduce Castelnuovo–Mumford regularity and exploit Mumford's result on Boundedness of Regularity (Theorem 1.3.8) to show that under the hypotheses of Theorem 1.1.3, then for $d \gg 0$, the morphism of functors

$$\begin{aligned} \mathrm{Quot}^P_{X/S}(F) &\to \mathrm{Gr}_S(P(d), \pi_* F(d)) \\ [F_T \twoheadrightarrow Q] &\mapsto [\pi_{T,*}(F_T(d)) \to \pi_{T,*}(Q(d))], \end{aligned} \tag{1.1.1}$$

defined over an $S$-scheme $T$, is well-defined. Note that for a field-valued point $s\colon \operatorname{Spec} \Bbbk \to S$ a quotient $[F_s \mapsto Q]$ is mapped to $[\mathrm{H}^0(X_s, F_s(d)) \to \mathrm{H}^0(X_s, Q(d))]$.

We show that the above functor is representable by locally closed immersions (Proposition 1.4.1). This is established by reducing to the special case where $X = \mathbb{P}_S(V)$ and $F = \pi^* W$ where $V$ and $W$ are vector bundles on $S$; this is where Boundedness of Regularity (Theorem 1.3.8) is applied.

Since $\mathrm{Gr}_S(P(d), \pi_* F(d))$ is representable by a projective scheme over $S$ (Theorem 1.1.1), this already establishes the representability and quasi-projectivity of $\mathrm{Quot}^P_{X/S}(F)$. Finally, we establish that $\mathrm{Quot}^P_{X/S}(F)$ is proper over $S$ (Proposition 1.4.2) by checking the valuative criterion which implies that $\mathrm{Quot}^P_{X/S}(F)$ is projective over $S$.

## Historical comments

Grothendieck established both the representability and projectivity of $\mathrm{Quot}^P_{\mathbb{P}^n_A/A}(F)$ where $F$ is coherent sheaf on $\mathbb{P}^n_A$ and $A$ is a noetherian ring [$\mathrm{FGA_{IV}}$, Thm. 3.2]. Our exposition largely follows Grothendieck's original strategy while incorporating Mumford's simplification to establish boundedness, i.e. finite typeness of $\mathrm{Quot}^P_{\mathbb{P}^n_A/A}(F)$. Boundedness is one of the hardest parts of the proof, and almost every boundedness argument for a moduli space in algebraic geometry ultimately relies on the boundedness of Hilb or Quot. Grothendieck's approach for boundedness of Quot was a reduction argument to Hilb (i.e. the case where $F = \mathcal{O}_X$) and relied on Chow's boundedness result for the parameter space of reduced, pure-dimensional subschemes of fixed degree. In [Mum66], Mumford introduced the concept of regularity of a coherent sheaf—now called Castelnuovo–Mumford regularity—and proved that for sufficiently large integers $m$ every subsheaf $F \subset \mathcal{O}_{\mathbb{P}^n}^r$ with fixed Hilbert polynomial is $m$-regular (Theorem 1.3.8). Mumford used this result to construct the Hilbert scheme of curves on a surface but his argument applies equally to construct $\mathrm{Quot}^P_{\mathbb{P}^n_A/A}(F)$.

Our formulation of Theorem 1.1.3 using the strong projectivity of $X \to S$ follows Altman and Kleiman [AK80, Thm. 2.6]. This chapter follows closely the excellent expositions of [Mum66, §14-15], [FGI+05, §6], [Kol96, §1], [Laz04a, §1.8], and [AK80, §2].

## 1.2 Representability and projectivity of the Grassmannian

The Grassmannian provides a warmup to the functorial approach of constructing projective moduli spaces in these notes, and it is also used in the proof of the representability of Hilb and Quot. Given its importance, we present a slow-paced expository account of the representability and projectivity of the Grassmannian. We focus first on the Grassmannian $\mathrm{Gr}(k, n) = \mathrm{Gr}_{\mathbb{Z}}(k, \mathcal{O}^n_{\mathrm{Spec}\,\mathbb{Z}})$ over $\mathbb{Z}$ parameterizing $k$-dimension quotients of a trivial vector bundle of rank $n$. The proof of the projectivity and representability of the relative Grassmannian $\mathrm{Gr}_S(k, V)$ is shown in §1.2.3.

### 1.2.1 Representability by a scheme

In this subsection, we show that $\mathrm{Gr}(k, n)$ is representable by a scheme (Proposition 1.2.3). Our strategy will be to find a Zariski open cover of $\mathrm{Gr}(k, n)$ by representable subfunctors; see Definition 0.3.29. Given a subset $I \subset \{1, \dots, n\}$ of size $k$, let $\mathrm{Gr}_I \subset \mathrm{Gr}(k, n)$ be the subfunctor where for a scheme $S$, $\mathrm{Gr}(k, n)_I(S)$ is the subset of $\mathrm{Gr}(k, n)(S)$ consisting of surjections $\mathcal{O}^n_S \overset{q}{\twoheadrightarrow} Q$ such that the composition

$$\mathcal{O}^I_S \xrightarrow{e_I} \mathcal{O}^n_S \overset{q}{\twoheadrightarrow} Q$$

is an isomorphism, where $e_I$ is the canonical inclusion.

**Lemma 1.2.1.** *For each $I \subset \{1, \dots, n\}$ of size $k$, the functor $\mathrm{Gr}_I$ is representable by affine space $\mathbb{A}^{k \times (n-k)}_{\mathbb{Z}}$*

*Proof.* We may assume that $I = \{1, \dots, k\}$. We define a map of functors $\phi \colon \mathbb{A}^{k \times (n-k)} \to \mathrm{Gr}_I$ where over a scheme $S$, a $k \times (n-k)$ matrix

$$f = \big(f_{i,j}\big)_{1 \le i \le k\,,\, 1 \le j \le n-k}$$

of global functions on $S$ is mapped to the quotient

$$\left(\begin{array}{cccc|ccc} 1 & & & & f_{1,1} & \cdots & f_{1,n-k} \\ & 1 & & & f_{2,1} & \cdots & f_{2,n-k} \\ & & \ddots & & \vdots & & \\ & & & 1 & f_{k,1} & \cdots & f_{k,n-k} \end{array}\right) : \mathcal{O}^n_S \to \mathcal{O}^k_S. \tag{1.2.1}$$

The injectivity of $\phi(S)\colon \mathbb{A}^{k\times(n-k)}(S) \to \mathrm{Gr}_I(S)$ follows from the fact that any two quotients written in the form of (1.2.1) which are equivalent in $\mathrm{Gr}_I$ are necessarily defined by the same equations. To see surjectivity, let $[\mathcal{O}_S^n \xrightarrow{q} Q] \in \mathrm{Gr}_I(S)$ where by definition $\mathcal{O}_S^I \xrightarrow{e_I} \mathcal{O}_S^n \overset{q}{\twoheadrightarrow} Q$ is an isomorphism. The tautological commutative diagram

$$\begin{tikzcd} \mathcal{O}_S^n \arrow[r, "q"] \arrow[dr, "(q\circ e_I)^{-1}\circ q"'] & Q \arrow[d, "(q\circ e_I)^{-1}"] \\ & \mathcal{O}_S^I \end{tikzcd}$$

shows that $[\mathcal{O}_S^n \overset{q}{\twoheadrightarrow} Q] = [\mathcal{O}_S^n \overset{(q\circ e_I)^{-1}\circ q}{\twoheadrightarrow} \mathcal{O}_S^I] \in \mathrm{Gr}(k,n)(S)$. Since the composition $\mathcal{O}_S^I \xrightarrow{e_I} \mathcal{O}_S^n \overset{(q\circ e_I)^{-1}}{\twoheadrightarrow} \mathcal{O}_S^I$ is the identity, the $k\times n$ matrix corresponding to $(q\circ e_I)^{-1}\circ q$ is necessarily of the same form as (1.2.1) for functions $f_{i,j} \in \Gamma(S, \mathcal{O}_S)$. Therefore $\phi(S)(\{f_{i,j}\}) = [\mathcal{O}_S^n \overset{q}{\twoheadrightarrow} Q] \in \mathrm{Gr}(k,n)(S)$. □

**Lemma 1.2.2.** *$\{\mathrm{Gr}_I\}$ is a Zariski open cover of $\mathrm{Gr}(k,n)$ where $I$ ranges over all subsets of size $k$.*

*Proof.* For a fixed subset $I$, we first show that $\mathrm{Gr}_I \subset \mathrm{Gr}(k,n)$ is an open subfunctor. To this end, we consider a scheme $S$ and a morphism $S \to \mathrm{Gr}(k,n)$ corresponding to a quotient $q\colon \mathcal{O}_S^n \to Q$. Let $C$ denote the cokernel of the composition $q\circ e_I\colon \mathcal{O}_S^I \to Q$. Notice that if $C=0$, then $q\circ e_I$ is an isomorphism. The fiber product

$$\begin{tikzcd} F_I \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & S \arrow[d, "{[\mathcal{O}_S^n \overset{q}{\twoheadrightarrow} Q]}"] \\ \mathrm{Gr}_I \arrow[r] & \mathrm{Gr}(k,n) \end{tikzcd}$$

of functors is representable by the open subscheme $U = S \setminus \mathrm{Supp}(C)$ (the reader is encouraged to verify this claim). Note that if $S$ is not noetherian, then $\mathrm{Supp}(C) \subset S$ is still closed as $C$ is finitely presented as a quasi-coherent sheaf.

To check the surjectivity of $\coprod_I F_I \to S$, let $s \in S$ be a point. Since $\kappa(s)^n \overset{q\otimes\kappa(s)}{\twoheadrightarrow} Q\otimes\kappa(s)$ is a surjection of vector spaces, there is a nonzero $k\times k$ minor, given by a subset $I$, of the $k\times n$ matrix $q\otimes\kappa(s)$. This implies that $[\kappa(s)^n \overset{q\otimes\kappa(s)}{\twoheadrightarrow} Q\otimes\kappa(s)] \in F_I(\kappa(s))$. □

Lemmas 1.2.1 and 1.2.2 together imply:

**Proposition 1.2.3.** *The functor $\mathrm{Gr}(k,n)$ is representable by a scheme.* □

**Exercise 1.2.4.** Show that $\mathrm{Gr}(k,n)$ is an integral scheme of finite type over $\mathbb{Z}$.

**Exercise 1.2.5.** Use the valuative criterion of properness to show that $\mathrm{Gr}(k,n) \to \operatorname{Spec}\mathbb{Z}$ is proper.

### 1.2.2 Projectivity of the Grassmannian

We show that the Grassmannian scheme $\mathrm{Gr}(k,n)$ is projective (Proposition 1.2.6) by explicitly providing a projective embedding. The *Plücker embedding* is the map of functors

$$\begin{aligned} P\colon \operatorname{Gr}(k,n) &\to \mathbb{P}(\textstyle\bigwedge^k \mathcal{O}^n_{\operatorname{Spec}\mathbb{Z}}) \\ [\mathcal{O}^n_S \xrightarrow{q} Q] &\mapsto [\textstyle\bigwedge^k \mathcal{O}^n_S \to \bigwedge^k Q] \end{aligned}$$

defined above over a scheme $S$. As both sides are representable by schemes, the morphism $P$ corresponds to a morphism of schemes via Yoneda's lemma.

**Proposition 1.2.6.** *The morphism* $P\colon \operatorname{Gr}(k,n) \to \mathbb{P}(\bigwedge^k \mathcal{O}^n_{\operatorname{Spec}\mathbb{Z}})$ *of schemes is a closed immersion. In particular,* $\operatorname{Gr}(k,n)$ *is a strongly projective scheme over* $\mathbb{Z}$*.*

*Proof.* A subset $I \subset \{1,\dots,n\}$ corresponds to a coordinate $x_I$ on $\mathbb{P}(\bigwedge^k \mathcal{O}^n_{\operatorname{Spec}\mathbb{Z}})$, and we set $\mathbb{P}(\bigwedge^k \mathcal{O}^n_{\operatorname{Spec}\mathbb{Z}})_I$ to be the open locus where $x_I \neq 0$. Note that $\mathbb{P}(\bigwedge^k \mathcal{O}^n_{\operatorname{Spec}\mathbb{Z}})_I \subset \mathbb{P}(\bigwedge^k \mathcal{O}^n_{\operatorname{Spec}\mathbb{Z}})$ is the subfunctor parameterizing line bundle quotients $\bigwedge^k \mathcal{O}^n_S \to L$ such that the composition $\mathcal{O}_S \xrightarrow{e_I} \bigwedge^k \mathcal{O}^n_S \to L$ (where the first map is the inclusion of the $I$th term) is an isomorphism, or in other words $\mathbb{P}(\bigwedge^k \mathcal{O}^n_{\operatorname{Spec}\mathbb{Z}})_I \cong \operatorname{Gr}(1,\binom{n}{k})_{\{I\}}$ viewing $\{I\}$ as the corresponding subset of $\{1,\dots,\binom{n}{k}\}$ of size 1. Using these functorial descriptions, one can check that there is a cartesian diagram of functors

$$\begin{tikzcd} \operatorname{Gr}(k,n)_I \arrow[r, "P_I"] \arrow[d] \arrow[dr, phantom, "\square"] & \mathbb{P}(\bigwedge^k \mathcal{O}^n_{\operatorname{Spec}\mathbb{Z}})_I \arrow[d] \\ \operatorname{Gr}(k,n) \arrow[r, "P"] & \mathbb{P}(\bigwedge^k \mathcal{O}^n_{\operatorname{Spec}\mathbb{Z}}). \end{tikzcd}$$

Since $\{\mathbb{P}(\bigwedge^k \mathcal{O}^n_{\operatorname{Spec}\mathbb{Z}})_I\}$ is a Zariski open cover, it suffices to show that each $P_I\colon \operatorname{Gr}(k,n)_I \to \mathbb{P}(\bigwedge^k \mathcal{O}^n_{\operatorname{Spec}\mathbb{Z}})_I$ is a closed immersion.

For simplicity, assume that $I = \{1,\dots,k\}$. Under the isomorphisms $\operatorname{Gr}(k,n)_I \cong \mathbb{A}^{k\times(n-k)}_{\mathbb{Z}}$ of Lemma 1.2.1 and $\mathbb{P}(\bigwedge^k \mathcal{O}^n_{\operatorname{Spec}\mathbb{Z}})_I \cong \mathbb{A}^{\binom{n}{k}-1}_{\mathbb{Z}}$, the morphism $P_I$ corresponds to the map

$$\mathbb{A}^{k\times(n-k)}_{\mathbb{Z}} \to \mathbb{A}^{\binom{n}{k}-1}_{\mathbb{Z}}$$

assigning a $k\times(n-k)$ matrix $A = \{x_{i,j}\}$ to the element of $\mathbb{A}^{\binom{n}{k}-1}_{\mathbb{Z}}$ whose $J$th coordinate, where $J \subset \{1,\dots,n\}$ is a subset of length $k$ distinct from $I$, is the $\{1,\dots,k\}\times J$ minor of the $k\times n$ block matrix

$$\left(\begin{array}{cccc|ccc} 1 & & & & x_{1,1} & \cdots & x_{1,n-k} \\ & 1 & & & x_{2,1} & \cdots & x_{2,n-k} \\ & & \ddots & & \vdots & & \\ & & & 1 & x_{k,1} & \cdots & x_{k,n-k} \end{array}\right).$$

The coordinate $x_{i,j}$ on $\mathbb{A}^{k\times(n-k)}_{\mathbb{Z}}$ is the pullback of the coordinate corresponding to the subset $\{1,\cdots,\widehat{i},\cdots,k,k+j\}$ (see Figure 1.1). This shows that the corresponding ring map is surjective thereby establishing that $P_I$ is a closed immersion. $\square$

$$x_{i,j} = \det \left( \begin{array}{cccc|ccccc} 1 & & & & x_{1,1} & \cdots & x_{1,j} & \cdots & x_{1,n-k} \\ & \ddots & & & \vdots & & \vdots & & \vdots \\ & & 1 & & x_{i,1} & \cdots & x_{i,j} & \cdots & x_{i,n-k} \\ & & & \ddots & \vdots & & \vdots & & \vdots \\ & & & 1 & x_{k,1} & \cdots & x_{k,j} & \cdots & x_{k,n-k} \end{array} \right)$$

Figure 1.1: The minor obtained by removing the $i$th column and all columns $k+1, \dots, n$ other than $k+j$ is precisely $x_{i,j}$.

**Exercise 1.2.7.** For a field $\Bbbk$, let $\mathrm{Gr}(k,n)_{\Bbbk}$ be the $\Bbbk$-scheme $\mathrm{Gr}(k,n) \times_{\mathbb{Z}} \Bbbk$, and $p \in \mathrm{Gr}(k,n)_{\Bbbk}$ be the point corresponding to a quotient $Q = \Bbbk^n/K$. Show that there is a natural bijection of the tangent space

$$T_p \,\mathrm{Gr}(k,n)_{\Bbbk} \xrightarrow{\sim} \mathrm{Hom}_{\Bbbk}(K,Q).$$

with the vector space of $\Bbbk$-linear maps $K \to Q$.

**Exercise 1.2.8.** Provide an alternative proof of the projectivity of $\mathrm{Gr}(k,n)$ as follows.

(a) Show that the functor $P\colon \mathrm{Gr}(k,n) \to \mathbb{P}(\bigwedge^k \mathcal{O}^n_{\operatorname{Spec}\mathbb{Z}})$ is injective on points and tangent spaces.

(b) Use a criterion for being a closed immersion (c.f. [Har77, Prop. II.7.3]) to show that $P\colon \mathrm{Gr}(k,n) \to \mathbb{P}(\bigwedge^k \mathcal{O}^n_{\operatorname{Spec}\mathbb{Z}})$ is a closed immersion.

(Alternatively, you could show that $P\colon \mathrm{Gr}(k,n) \to \mathbb{P}(\bigwedge^k \mathcal{O}^n_{\operatorname{Spec}\mathbb{Z}})$ is a proper monomorphism and conclude that $\mathrm{Gr}(k,n)$ is projective over $\mathbb{Z}$.)

### 1.2.3 Relative version

We now prove the relative version of the representability and strong projectivity of the Grassmannian.

*Proof of Theorem 1.1.1.* If $V$ is a vector bundle over $S$ of rank $n$, there is the relative Plücker embedding

$$\begin{aligned} P\colon \mathrm{Gr}_S(k,V) &\to \mathbb{P}_S(\textstyle\bigwedge^k V) \\ [V_T \overset{q}{\twoheadrightarrow} Q] &\mapsto [\textstyle\bigwedge^k V_T \to \bigwedge^k Q] \end{aligned} \tag{1.2.2}$$

defined above over a $S$-scheme $T$. This is a morphism of functors over $S$. Since $\mathbb{P}_S(\bigwedge^k V)$ is projective over $S$, it suffices to show that this morphism is representable by closed immersions. This property can be checked Zariski-locally: if $U \subset S$ is an open subscheme where $V$ is trivial, then the base change of $\mathrm{Gr}_S(k,V) \to \mathbb{P}_S(\bigwedge^k V)$ over $U$ is the Plücker embedding $\mathrm{Gr}_U(k, \mathcal{O}_U^n) \to \mathbb{P}_S(\bigwedge^k \mathcal{O}_U^N)$ which is a closed immersion (Proposition 1.2.6). $\square$

Since the Grassmannian functor is representable, there is a *universal quotient* $\mathcal{O}_{\mathrm{Gr}_S(k,V)} \otimes_S V \to Q_{\mathrm{univ}}$; here $\mathcal{O}_{\mathrm{Gr}_S(k,V)} \otimes_S V$ denotes the pullback of $V$ under the structure morphism $\mathrm{Gr}_S(k,V) \to S$. Under the Plücker embedding (1.2.2), the pullback of $\mathcal{O}(1)$ is identified with $\det(Q_{\mathrm{univ}})$, which we sometimes call the *Plücker line bundle*. Thus, we obtain:

**Corollary 1.2.9.** *The determinant* $\det(Q_{\mathrm{univ}})$ *of the universal quotient is a very ample line bundle on* $\mathrm{Gr}_S(k, V)$. □

**Remark 1.2.10.** For projective space $\mathbb{P}^n = \mathrm{Gr}(1, n)$, the universal quotient yields an exact sequence $0 \to \Omega_{\mathbb{P}^n}(1) \to \mathcal{O}_{\mathbb{P}^n}^{n+1} \to \mathcal{O}_{\mathbb{P}^n}(1) \to 0$, which is the dual of the Euler sequence [Har77, Ex. 8.20.1] twisted by $\mathcal{O}_{\mathbb{P}^n}(1)$.

## 1.3 Castelnuovo–Mumford regularity

The Cartan–Serre–Grothendieck theorem states that if $F$ is a coherent sheaf on a projective variety $(X, \mathcal{O}_X(1))$, then for $d \gg 0$

(1) $F(d)$ is globally generated;

(2) $\mathrm{H}^i(X, F(d)) = 0$ for $i > 0$; and

(3) the multiplication map
$$\mathrm{H}^0(X, F(d)) \otimes \mathrm{H}^0(X, \mathcal{O}(p)) \to \mathrm{H}^0(X, F(d + p))$$
is surjective for all $p \geq 0$.

Castelnuovo–Mumford regularity provides a quantitative measure of the size of $d$ necessary for the twist $F(d)$ to have the three above desired cohomological properties and in particular that the Hilbert polynomial $\chi(X, F(d))$ of $F$ evaluated at $d$ agrees with $\mathrm{h}^0(X, F(d))$.

### 1.3.1 Definition and basic properties

**Definition 1.3.1.** Let $F$ be a coherent sheaf on projective space $\mathbb{P}^n$ over a field $\Bbbk$. For an integer $m$, we say that $F$ is *$m$-regular* if
$$\mathrm{H}^i(\mathbb{P}^n, F(m - i)) = 0$$
for all $i \geq 1$.

The *regularity of $F$* is the smallest integer $m$ such that $F(m)$ is $m$-regular.

While the requirement that the $i$th cohomology of the $(m - i)$th twist vanishes may appear mysterious at first, this definition is very convenient for induction arguments on the dimension $n$ as indicated for instance by the following result.

**Lemma 1.3.2.** *Let $F$ be an $m$-regular coherent sheaf on $\mathbb{P}^n$ over a field $\Bbbk$. If $H \subset \mathbb{P}^n$ is a hyperplane avoiding the associated points of $F$, then $F|_H$ is also $m$-regular.*

*Proof.* The hypotheses imply that over an affine open subscheme $U \subset \mathbb{P}^n$, the defining equation of $H$ is a nonzerodivisor for the module $\Gamma(U, F)$. Thus $F(-1) \xrightarrow{H} F$ is injective, and for an integer $i > 0$ we have a short exact sequence
$$0 \to F(m - i - 1) \to F(m - i) \to F|_H(m - i) \to 0$$
inducing a long exact sequence on cohomology
$$\cdots \to \mathrm{H}^i(\mathbb{P}^n, F(m - i)) \to \mathrm{H}^i(H, F|_H(m - i)) \to \mathrm{H}^{i+1}(\mathbb{P}^n, F(m - i - 1)) \to \cdots$$
If $F$ is $m$-regular, then $\mathrm{H}^i(\mathbb{P}^n, F(m - i)) = \mathrm{H}^{i+1}(\mathbb{P}^n, F(m - i - 1)) = 0$. It follows that $\mathrm{H}^i(H, F|_H(m - i)) = 0$ for all $i > 0$, and thus $F|_H$ is also $m$-regular. □

**Remark 1.3.3.** It follows from the definition of regularity that if $F$ is $m$-regular, then $F(d)$ is $(m-d)$-regular. We will show in Lemma 1.3.6 that if $F$ is $m$-regular, it also $d$-regular for all $d \geq m$.

**Exercise 1.3.4.**

(a) Show that $\mathcal{O}(d)$ is $(-d)$-regular on $\mathbb{P}^n$.

(b) Show that the structure sheaf of a hypersurface $H \subset \mathbb{P}^n$ of degree $d$ is $(d-1)$-regular.

(c) Show that the structure sheaf of a smooth curve $C \subset \mathbb{P}^n$ of genus $g$ is $(2g-1)$-regular.

**Exercise 1.3.5.** Let $F$ be a coherent sheaf on $\mathbb{P}^n$ resolved by a long exact sequence of coherent sheaves. Show that if each $F_i$ is $(m+i)$-regular, then $F$ is $m$-regular.

$$\cdots \to F_2 \to F_1 \to F_0 \to F \to 0$$

Another advantage of regularity is the following lemma due to Castelnuovo.

**Lemma 1.3.6.** *Let $F$ be an $m$-regular coherent sheaf on $\mathbb{P}^n$.*

*(1) For $d \geq m$, $F$ is $d$-regular.*

*(2) The multiplication map*

$$\mathrm{H}^0(\mathbb{P}^n, F(d)) \otimes \mathrm{H}^0(\mathbb{P}^n, \mathcal{O}(k)) \to \mathrm{H}^0(\mathbb{P}^n, F(d+k))$$

*is surjective if $d \geq m$ and $k \geq 0$.*

*(3) For $d \geq m$, $F(d)$ is globally generated and $\mathrm{H}^i(\mathbb{P}^n, F(d)) = 0$ for $i \geq 1$.*

*Proof.* If $\Bbbk \to \Bbbk'$ is a field extension, then flat base change implies that $\mathrm{H}^i(\mathbb{P}^n_{\Bbbk}, F) \otimes_{\Bbbk} \Bbbk' = \mathrm{H}^i(\mathbb{P}^n_{\Bbbk'}, F \otimes_{\Bbbk} \Bbbk')$. As $\Bbbk \to \Bbbk'$ is faithfully flat, the assertions (1)–(3) can be checked after base change. We can thus assume that $\Bbbk$ is algebraically closed and in particular infinite.

For (1) and (2), we will argue by induction on $n$ with the base case of $n = 0$ being clear. If $n > 0$, since $\Bbbk$ is infinite, we may choose a hyperplane $H \subset \mathbb{P}^n$ avoiding the associated points of $F$. Since the restriction $F|_H$ is $m$-regular (Lemma 1.3.2) on $H \cong \mathbb{P}^{n-1}$, the inductive hypothesis implies that (1) and (2) hold for $F|_H$.

We prove (1) by using induction also on $d$. The base case $d = m$ holds by hypothesis. For $d > m$, the short exact sequence $0 \to F(d-i-1) \to F(d-i) \to F|_H(d-i) \to 0$ induces a long exact sequence on cohomology

$$\cdots \to \mathrm{H}^i(\mathbb{P}^n, F(d-i-1)) \to \mathrm{H}^i(\mathbb{P}^n, F(d-i)) \to \mathrm{H}^i(H, F|_H(d-i)) \to \cdots$$

For $i > 0$, the first term vanishes by the induction hypothesis on $d$ ($F$ is $(d-1)$-regular so $\mathrm{H}^i(\mathbb{P}^n, F(d-1-i)) = 0$) and the third term vanishes by the inductive hypothesis on $n$ ($F|_H$ is $m$-regular by Lemma 1.3.2 and thus $d$-regular by the inductive hypothesis on $n$ so $\mathrm{H}^i(H, F|_H(d-i)) = 0$). Thus, the second term vanishes and we have established (1).

To show (2), we use induction on $k$ in addition to $n$. We denote the multiplication map by

$$\mu_{d,k} \colon \mathrm{H}^0(\mathbb{P}^n, F(d)) \otimes \mathrm{H}^0(\mathbb{P}^n, \mathcal{O}(k)) \to \mathrm{H}^0(\mathbb{P}^n, F(d+k)).$$

While the base case $k = 0$ is clear, the inductive argument will require us to directly establish the case $k = 1$. To this end, we consider the commutative diagram

$$\begin{tikzcd} & \mathrm{H}^0(\mathbb{P}^n, F(d)) \otimes \mathrm{H}^0(\mathbb{P}^n, \mathcal{O}(1)) \arrow[r, "\nu_d \otimes \mathrm{res}"] \arrow[d, "\mu_{d,1}"] & \mathrm{H}^0(H, F|_H(d)) \otimes \mathrm{H}^0(H, \mathcal{O}_H(1)) \arrow[d] \\ \mathrm{H}^0(\mathbb{P}^n, F(d)) \arrow[ur, "\mathrm{id} \otimes H"] \arrow[r, "\alpha"] & \mathrm{H}^0(\mathbb{P}^n, F(d+1)) \arrow[r, "\nu_{d+1}"] & \mathrm{H}^0(H, F|_H(d+1)). \end{tikzcd} \tag{1.3.1}$$

As the map $\alpha$ is given by multiplication by $H \in \mathrm{H}^0(\mathbb{P}^n, \mathcal{O}(1))$, $\alpha$ factors through the the map $\mathrm{id} \otimes H$ defined by $v \mapsto v \otimes H$. It follows that $\mathrm{im}(\alpha) \subset \mathrm{im}(\mu_{d,1})$. Since $\mathrm{H}^1(\mathbb{P}^n, F(d)) = 0$ by (2), the restriction map $\nu_d \colon \mathrm{H}^0(\mathbb{P}^n, F(d)) \to \mathrm{H}^0(H, F|_H(d))$ is surjective. Likewise, since $\mathrm{H}^1(\mathbb{P}^n, \mathcal{O}) = 0$, $\mathrm{res} \colon \mathrm{H}^0(\mathbb{P}^n, \mathcal{O}(1)) \to \mathrm{H}^0(H, \mathcal{O}_H(1))$ is surjective. We conclude that the top horizontal arrow is surjective. The inductive hypothesis applied to $H = \mathbb{P}^{n-1}$ implies that the right vertical arrow is surjective. Therefore, the composition $\nu_{d+1} \circ \mu_{d,1}$ is surjective and it follows that $\mathrm{im}(\mu_{d,1})$ surjects onto $\mathrm{H}^0(H, F|_H(d+1))$. By exactness of the bottom row, we have that

$$\mathrm{H}^0(\mathbb{P}^n, F(d+1)) = \mathrm{im}(\mu_{d,1}) + \ker(\beta) = \mathrm{im}(\mu_{d,1}) + \mathrm{im}(\alpha) = \mathrm{im}(\mu_{d,1}),$$

which shows that $\mu_{d,1}$ is surjective.

If $k > 1$, we consider the commutative square

$$\begin{tikzcd} \mathrm{H}^0(\mathbb{P}^n, F(d)) \otimes \mathrm{H}^0(\mathbb{P}^n, \mathcal{O}(k-1)) \otimes \mathrm{H}^0(\mathbb{P}^n, \mathcal{O}(1)) \arrow[r] \arrow[d, "\mu_{d,k-1} \otimes \mathrm{id}"] & \mathrm{H}^0(\mathbb{P}^n, F(d)) \otimes \mathrm{H}^0(\mathbb{P}^n, \mathcal{O}(k)) \arrow[d, "\mu_{d,k}"] \\ \mathrm{H}^0(\mathbb{P}^n, F(d+k-1)) \otimes \mathrm{H}^0(\mathbb{P}^n, \mathcal{O}(1)) \arrow[r, "\mu_{d+k-1,1}"] & \mathrm{H}^0(\mathbb{P}^n, F(d+k)). \end{tikzcd}$$

The left vertical map and bottom horizontal arrow are surjective by the inductive hypothesis applied to $k-1$ and $k = 1$, respectively. It follows that $\mu_{d,k}$ is surjective.

To show (3), we know that for $k \gg 0$, $F(d+k)$ is globally generated, i.e. $\gamma_{F(d+k)} \colon \mathrm{H}^0(\mathbb{P}^n, F(d+k)) \otimes \mathcal{O}_{\mathbb{P}^n} \to F(d+k)$ is surjective. Consider the commutative square

$$\begin{tikzcd} \mathrm{H}^0(\mathbb{P}^n, F(d)) \otimes \mathrm{H}^0(\mathbb{P}^n, \mathcal{O}(k)) \otimes \mathcal{O}_{\mathbb{P}^n} \arrow[r, "\mu_{d,k} \otimes \mathrm{id}"] \arrow[d, "\gamma_{F(d)} \otimes \mathrm{id}"] & \mathrm{H}^0(\mathbb{P}^n, F(d+k)) \otimes \mathcal{O}_{\mathbb{P}^n} \arrow[d, "\gamma_{F(d+k)}"] \\ F(d) \otimes \left(\mathrm{H}^0(\mathbb{P}^n, \mathcal{O}(k)) \otimes \mathcal{O}_{\mathbb{P}^n}\right) \arrow[r, "\mathrm{id} \otimes \gamma_{\mathcal{O}(k)}"] & F(d) \otimes \mathcal{O}(k). \end{tikzcd}$$

Since the top horizontal arrow is surjective by (1), the composition from the top left to the bottom right is surjective. Given the nature of the bottom horizontal map, we see that $\gamma_{F(d)}$ must be surjective (indeed, if $V = \mathrm{im}(\gamma_{F(d)}) \subset F(d)$, then $\mathrm{im}(\mathrm{id} \otimes \gamma_{\mathcal{O}(k)} \circ \gamma_{F(d)} \otimes \mathrm{id}) = V \otimes \mathcal{O}(k)$). Finally, to see the vanishing of the higher cohomology of $F(d)$ observe that for each $i > 0$, the sheaf $F$ is $(d+i)$-regular by (2) and thus $\mathrm{H}^i(\mathbb{P}^n, F(d)) = 0$. □

One easy consequence of (1) is that if $F$ is $m$-regular, then the restriction map

$$\nu_d \colon : \mathrm{H}^0(\mathbb{P}^n, F(d)) \to \mathrm{H}^0(H, F|_H(d))$$

is surjective for all $d \geq m$. Indeed, (1) implies that $F$ is also $d$-regular and the surjectivity follows from the vanishing of $\mathrm{H}^1(\mathbb{P}^n, F(d-1))$. The following lemma—which will be used in the proof of Theorem 1.3.8—shows that we can still arrange for the surjectivity of $\nu_d$ under weaker hypotheses.

**Lemma 1.3.7.** *Let $F$ be a coherent sheaf on $\mathbb{P}^n$ and $H$ be a hyperplane avoiding the associated points of $F$. If $F|_H$ is $m$-regular and $\nu_d$ is surjective for some $d \geq m$, then $\nu_p$ is surjective for all $p \geq d$.*

*Proof.* By staring at the square in diagram (1.3.1), we see that the top arrow $\nu_d \otimes \mathrm{res}$ is surjective (as both $\nu_d$ and res are surjective) and the vertical right multiplication morphism is surjective (by applying Lemma 1.3.6(2) to the $m$-regular sheaf $F|_H$). The statement follows. □

### 1.3.2 Regularity bounds

We now turn to the following bound on the regularity of subsheaves of the trivial vector bundle established by Mumford in [Mum66, p.101].

**Theorem 1.3.8** (Boundedness of Regularity)**.** *For every pair of non-negative integers $r$ and $n$, and for every polynomial $P \in \mathbb{Q}[z]$, there exists an integer $m_0$ with the following property: for every field $\Bbbk$, every subsheaf $F \subset \mathcal{O}^r_{\mathbb{P}^n_{\Bbbk}}$ with Hilbert polynomial $P$ is $m_0$-regular.*

*Proof.* As in the proof of Lemma 1.3.6, we can assume that $\Bbbk$ is infinite. We will argue by induction on $n$. The base case of $n = 0$ holds as every sheaf $F$ on $\mathbb{P}^0$ is $m$-regular for every integer $m$.

For $n \geq 1$ and a subsheaf $F \subset \mathcal{O}^r_{\mathbb{P}^n}$ with Hilbert polynomial $P$, we can choose a hyperplane $H \subset \mathbb{P}^n$ avoiding all associated points of $\mathcal{O}^r_{\mathbb{P}^n}/F$. This ensures that $\mathrm{Tor}^1_{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{O}_H, \mathcal{O}^r_{\mathbb{P}^n}/F) = 0$ and that the short exact sequence $0 \to F \to \mathcal{O}^r_{\mathbb{P}^n} \to \mathcal{O}^r_{\mathbb{P}^n}/F \to 0$ restricts to a short exact sequence

$$0 \to F|_H \to \mathcal{O}^r_H \to \mathcal{O}^r_H/F \to 0. \tag{1.3.2}$$

As $H \cong \mathbb{P}^{n-1}$, this will allow us to apply the inductive hypothesis to $F|_H \subset \mathcal{O}^r_H$.

On the other hand, since $F \subset \mathcal{O}^r_{\mathbb{P}^n}$ is torsion free, we have a short exact sequence

$$0 \to F(-1) \xrightarrow{H} F \to F|_H \to 0, \tag{1.3.3}$$

and the Hilbert polynomial of $F|_H$ is $\chi(F|_H(d)) = \chi(F(d)) - \chi(F(d-1)) = P(d) - P(d-1)$. In particular, the Hilbert polynomial of $F|_H$ only depends on $P$ and the inductive hypothesis applied to $F|_H \subset \mathcal{O}^r_H$ gives an integer $m_1$ such that $F|_H$ is $m_1$-regular.

For $m \geq m_1 - 1$, since $\mathrm{H}^i(H, F|_H(m)) = 0$ for all $i \geq 1$, we have a long exact sequence

$$\begin{aligned} 0 \to \mathrm{H}^0(\mathbb{P}^n, F(m-1)) \to \mathrm{H}^0(\mathbb{P}^n, F(m)) \to \mathrm{H}^0(H, F|_H(m)) \to \\ \mathrm{H}^1(\mathbb{P}^n, F(m-1)) \to \mathrm{H}^1(\mathbb{P}^n, F(m)) \to 0. \end{aligned} \tag{1.3.4}$$

For $i \geq 2$, we also have isomorphisms $\mathrm{H}^i(\mathbb{P}^n, F(m-1)) \to \mathrm{H}^i(\mathbb{P}^n, F(m))$, and since $\mathrm{H}^i(\mathbb{P}^n, F(d))$ vanishes for $d \gg 0$, we can conclude that $\mathrm{H}^i(\mathbb{P}^n, F(m-1)) = 0$.

To handle $\mathrm{H}^1$, we use the inequalities $\mathrm{h}^1(\mathbb{P}^n, F(m_1)) \geq \mathrm{h}^1(\mathbb{P}^n, F(m_1+1)) \geq \cdots$, which eventually stabilize to 0. We claim that in fact that the inequalities $\mathrm{h}^1(\mathbb{P}^n, F(m_1)) > \mathrm{h}^1(\mathbb{P}^n, F(m_1+1)) > \cdots$ are strict until they become 0. To see this, we observe that there is an equality $\mathrm{h}^1(\mathbb{P}^n, F(m-1)) = \mathrm{h}^1(\mathbb{P}^n, F(m))$ for $m \geq m_1$ if and only if $\nu_m \colon \mathrm{H}^0(\mathbb{P}^n, F(m)) \to \mathrm{H}^0(H, F|_H(m))$ is surjective. If $\mathrm{h}^1(\mathbb{P}^n, F(m-1)) = \mathrm{h}^1(\mathbb{P}^n, F(m))$ for some $m \geq m_1$, then $\nu_m$ is surjective. Since $F|_H$ is $m_1$-regular, we may apply Lemma 1.3.7 to conclude that $\nu_{m'}$ is surjective for all $m' \geq m$, which in turn implies that $\mathrm{h}^1(\mathbb{P}^n, F(m'))$ is constant for $m' \geq m$, and therefore zero. This establishes the claim. Setting $m_2 = m_1 + 1 + \mathrm{h}^1(\mathbb{P}^n, F(m_1))$, we see that $\mathrm{h}^1(\mathbb{P}^n, F(m_2 - 1)) = 0$ and that $F$ is $m_2$-regular.

We now show that $m_2$ is bounded above by a constant $m_0$ independent of $F$. Since $F \subset \mathcal{O}^r_{\mathbb{P}^n}$, we have that $\mathrm{h}^0(\mathbb{P}^n, F(d)) \leq r\mathrm{h}^0(\mathbb{P}^n, \mathcal{O}(d)) = r\binom{n+d}{n}$ for any $d \geq 0$. Using the vanishing of $\mathrm{h}^i(\mathbb{P}^n, F(m_1))$ for $i \geq 2$, we have

$$\begin{aligned} \mathrm{h}^1(\mathbb{P}^n, F(m_1)) &= \mathrm{h}^0(\mathbb{P}^n, F(m_1)) - \chi(F(m_1)) \\ &\leq r\binom{n+m_1}{n} + P(m_1). \end{aligned}$$

Thus, defining $m_0 := m_1 + 1 + r\binom{n+m_1}{n} + P(m_1)$, we have that $m_2 \leq m_0$. □

**Remark 1.3.9.** The above proof establishes in fact a stronger statement. To formulate the result, we recall that every numerical polynomial $P \in \mathbb{Q}[z]$ (i.e. $P(d) \in \mathbb{Z}$ for integers $d \gg 0$) of degree $n$ can be uniquely written as

$$P(d) = \sum_{i=0}^{n} a_i \binom{d}{i}$$

for $a_i \in \mathbb{Z}$; this follows from a straightforward inductive argument (c.f. [Har77, Prop. I.7.3]). For non-negative integers $r$ and $n$, there exists a polynomial $\Lambda_{r,n} \in \mathbb{Z}[x_0, \ldots, x_n]$ with the following property: for every field $\Bbbk$, every subsheaf $F \subset \mathcal{O}^r_{\mathbb{P}^n_{\Bbbk}}$ with Hilbert polynomial $P(d) = \sum_{i=0} a_i \binom{d}{i}$ is $m_0$-regular for $m_0 = \Lambda_{r,n}(a_0, \ldots, a_n)$.

**Remark 1.3.10** (Optimal bounds)**.** Although Mumford's result on Boundedness of Regularity (Theorem 1.3.8) provides an explicit bound and is sufficient for many applications including the construction of the Quot scheme as well as for other applications, there is a more optimal bound established by Gotzmann: for a projective scheme $X \subset \mathbb{P}^N$ over a field $\Bbbk$ with Hilbert polynomial $P$, there are unique integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \geq 1$ such that $P$ can be expressed as

$$P(d) = \binom{d + \lambda_1 - 1}{\lambda_1 - 1} + \binom{d + \lambda_2 - 2}{\lambda_2 - 1} + \cdots + \binom{d + \lambda_r - r}{\lambda_r - 1},$$

and the ideal sheaf $\mathcal{I}_X$ of $X$ is $r$-regular. See [Got78], [Gre89], [Gre98, §3] and [BH93, §4.3].

**Exercise 1.3.11.** Let $C \subset \mathbb{P}^n$ be a curve of degree $d$ and genus $g$. Show that Gotzmann's bound implies that the ideal sheaf $I_C$ of $C$ is $(\binom{d}{2} + 1 - g)$-regular. Can you compare this to the bound given by the proof of Theorem 1.3.8, i.e. can you compute $\Lambda_{1,n}(1 - g, d)$ for an explicit polynomial satisfying Theorem 1.3.8?

**Remark 1.3.12.** It was shown in [GLP83] that the ideal sheaf $I_C$ of an integral, non-degenerate curve $C \subset \mathbb{P}^N$ of degree $d$ is $(d - N + 2)$-regular. It is conjectured more generally that the ideal sheaf of a smooth, non-degenerate projective variety $X \subset \mathbb{P}^N$ of dimension $n$ and degree $d$ is $(d - (N - n)) + 1)$-regular; see [GLP83] and [EG84].

**Corollary 1.3.13.** *Let $\pi \colon X \to S$ be a strongly projective morphism of noetherian schemes and $\mathcal{O}_X(1)$ be a relatively ample line bundle on $X$. Let $F$ be quotient sheaf of $\pi^*(W)(q)$ for some vector bundle $W$ on $S$ and integer $q$. Let $P \in \mathbb{Q}[z]$ be a* *polynomial. There exists an integer $m_0$ satisfying the following property for every $d \geq m_0$: for every morphism $f\colon T \to S$ inducing a cartesian square*

$$\begin{tikzcd} X_T \arrow[r, "f_T"] \arrow[d, "\pi_T"] & X \arrow[d, "\pi"] \\ T \arrow[r, "f"] & S \end{tikzcd}$$

*and every finitely presented quotient $Q = F_T/K$ flat over $S$ such that every fiber $Q_t$ on $X_t$ has Hilbert polynomial $P$, then*

*(1) $\pi_{T,*}Q(d)$ is a vector bundles of rank $P(d)$*

*(2) the comparison maps $f^*\pi_*Q(d) \to \pi_{T,*}f_T^*Q(d)$, $f^*\pi_*F(d) \to \pi_{T,*}f_T^*F(d)$ and $f^*\pi_*K(d) \to \pi_{T,*}f_T^*K(d)$ are isomorphisms;*

*(3) $R^1\pi_{T,*}K(d) = 0$ for $i > 0$; and*

*(4) the adjunction maps $\pi_T^*\pi_{T,*}Q(d) \to Q(d)$, $\pi_T^*\pi_{T,*}F_T(d) \to F_T(d)$ and $\pi_T^*\pi_{T,*}K(d) \to K(d)$ are surjective.*

*Proof.* For (2), since $\pi\colon X \to S$ is strongly projective, there is a closed immersion $i\colon X \hookrightarrow \mathbb{P}_S(V)$ where $V$ is a vector bundle on $S$. Since the statement is local on $S$ (and $S$ is quasi-compact), we may assume that $S$ is affine and that $V$ is the trivial vector bundle of rank $n+1$. We are given a surjection $\pi^*(W)(q) \twoheadrightarrow F$, and if $Q$ is a quotient of $i_*F$ with Hilbert polynomial $P$, then $Q(-q)$ is a quotient of $\pi^*W$ with Hilbert polynomial $P'$ where $P'(z) = P(z+d)$. We can therefore replace $(F, X, P)$ with $(\pi^*(W), \mathbb{P}_S(V), P')$. In particular, for every field-valued point $s\colon \operatorname{Spec} \Bbbk \to S$, $\mathbb{P}_S(V)_s \cong \mathbb{P}^n_{\Bbbk}$ and $F_s \cong \mathcal{O}^k_{\mathbb{P}^n_{\Bbbk}}$ where $\operatorname{rk}(V) = n+1$ and $\operatorname{rk}(W) = k$.

By Boundedness of Regularity (Theorem 1.3.8), there exists an integer $m_0$ depending on $n$, $r$ and $P$ such that for every field-valued point $s\colon \operatorname{Spec} \Bbbk \to S$, the kernel $K_s$ is $m_0$-regular. As $K_s$ is also $(m_0+2)$-regular (Lemma 1.3.6) and $F_s \cong \mathcal{O}^k_{\mathbb{P}^n_{\Bbbk}}$ is $(m_0+1)$-regular (in fact, it is 0-regular), it follows that $Q_s$ is $m_0$-regular (Exercise 1.3.4). By Lemma 1.3.6, for $d \geq m_0 + 2$, $K_s(d)$, $F_s(d)$ and $Q_s(d)$ are each globally generated with vanishing higher cohomology. Since $K$, $F$ and $Q$ are flat over $S$, statements (1)–(3) follow from applying Cohomology and Base Change in the form of Corollary A.7.7. For (4), to verify the surjectivity of the adjunction map $\pi_T^*\pi_{T,*}K(d) \to K(d)$ (and likewise for $F_T$ and $Q$), it suffices to check that the restriction

$$(\pi_T^*\pi_{T,*}K(d))|_{X_t} \to K_t(d) \tag{1.3.5}$$

is surjective one each fiber $X_t$ over $t \in T$. Using (2), we have identifications

$$(\pi_T^*\pi_{T,*}K(d))|_{X_t} \cong \pi_t^*(\pi_{T,*}K(d) \otimes \kappa(t)) \cong \pi_t^*\pi_{t,*}K_t(d),$$

where $\pi_t\colon X_t \to \operatorname{Spec} \kappa(t)$ and thus (1.3.5) corresponds to the adjunction map $\pi_t^*\pi_{t,*}K_t(d) \to K_t(d)$, which we know is surjective as $K_t(d)$ is globally generated. ☐

## 1.4 Representability and projectivity of Hilb and Quot

In this section, we prove the representability and projectivity of Quot (Theorem 1.1.3) and as a consequence we obtain the same for the Hilbert scheme (Theorem 1.1.2).

As before, $\pi\colon X \to S$ is a strongly projective morphism of noetherian schemes, $\mathcal{O}_X(1)$ is a relatively ample line bundle on $X$, $F$ is a quotient sheaf of $\pi^*(W)(q)$ for some vector bundle $W$ on $S$ and integer $q$, and $P \in \mathbb{Q}[z]$ is a polynomial. Our strategy is to use the morphism of functors

$$\begin{aligned} \mathrm{Quot}^P_{X/S}(F) &\to \mathrm{Gr}_S(P(d), \pi_*F(d)) \\ [F_T \twoheadrightarrow Q] &\mapsto [\pi_{T,*}F_T(d) \to \pi_{T,*}Q(d)], \end{aligned}$$

defined above over an $S$-scheme $T$. For $d \gg 0$, Corollary 1.3.13 implies that the above morphism is well-defined: indeed part (1) shows that $\pi_{T,*}Q(d)$ is a vector bundle of rank $P(d)$, part (2) shows the pullback of the coherent sheaf $\pi_*F(d)$ under $T \to S$ is identified with $\pi_{T,*}F_T(d)$, and part (3) shows that $R^1\pi_{T,*}K(d) = 0$ which implies the surjectivity of $\pi_{T,*}F_T(d) \to \pi_{T,*}Q(d)$.

### 1.4.1 Quot is locally closed in a Grassmannian

We prove that the map $\mathrm{Quot}^P_{X/S}(F) \to \mathrm{Gr}_S(P(d), \pi_*F(d))$ is representable by locally closed immersions.

**Proposition 1.4.1.** *Let $\pi\colon X \to S$ be a strongly projective morphism of noetherian schemes, $\mathcal{O}_X(1)$ be a relatively ample line bundle on $X$, and $F$ be a coherent sheaf on $X$ which is the quotient of $\pi^*(W)(q)$ for a vector bundle $W$ on $S$ and an integer $q$. For $d \gg 0$, the morphism $\mathrm{Quot}^P_{X/S}(F) \to \mathrm{Gr}_S(P(d), \pi_*F(d))$ is representable by locally closed immersions, i.e. for every morphism $T \to \mathrm{Gr}_S(P(d), \pi_*F(d))$ from a scheme, the fiber product*

$$T \times_{\mathrm{Gr}_S(P(d), \pi_*F(d))} \mathrm{Quot}^P_{X/S}(F)$$

*is representable by a locally closed subscheme of $T$.*

*Proof.* *We first reduce to the special case that $X = \mathbb{P}_S(V)$ and $F = \pi^*W$ for trivial vector bundles $V$ and $W$.* Let $i\colon X \hookrightarrow \mathbb{P}_S(V)$ be a closed immersion where $V$ is a vector bundle on $S$. The morphism of functors $\mathrm{Quot}^P_{X/S}(F) \to \mathrm{Gr}_S(P(d), \pi_*F(d))$ is defined over $S$ and its base change to an open subscheme $U \subset S$ is identified with the morphism $\mathrm{Quot}^P_{X_U/U}(F_U) \to \mathrm{Gr}_S(P(d), \pi_{U,*}F_U(d))$. Since the property of being a locally closed immersion is Zariski-local on the target, the statement is Zariski-local on $S$. We may therefore assume that $S$ is affine and that $V$ is the trivial vector bundle of rank $n+1$.

First, observe that since there is an isomorphism of functors

$$\mathrm{Quot}^P_{X/S}(F) \to \mathrm{Quot}^P_{\mathbb{P}_S(V)/S}(i_*F),$$

we may replace $(F, X)$ with $(i_*F, \mathbb{P}_S(V))$. Next using the surjection $\pi^*(W)(q) \twoheadrightarrow F$, we obtain a morphism of functors

$$\begin{aligned} \mathrm{Quot}^P_{\mathbb{P}_S(V)/S}(F) &\to \mathrm{Quot}^{P'}_{\mathbb{P}_S(V)/S}(\pi^*W) \\ [F_T \to Q] &\mapsto [(\pi^*W)_T \to F(-q)_T \to Q(-q)], \end{aligned}$$

defined over an $S$-scheme $T$, where $P'(z) = P(z-q)$. We claim that this morphism is representable by closed immersions. This claims boils down to the statement that for an $S$-scheme $T$ and quotient $\pi^*W(q)_T \twoheadrightarrow Q$, there is a closed subscheme $Z \subset T$ such that a morphism $U \to T$ factors through $Z$ if and only if the restriction

$\pi^*W(q)_U \twoheadrightarrow G_U$ factors through $F_U$. Defining $K = \ker(\pi^*W(q)_T \to F_T)$ and considering the diagram

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r] \arrow[dr] & \pi^*W(q)_T \arrow[r] \arrow[d, two heads] & F_T \arrow[r] & 0 \\
& & G, & &
\end{tikzcd}$$

we see that the claim is satisfied by taking $Z \subset T$ to be vanishing scheme of the morphism $K \to G$ (see Exercise 0.3.14(b)).

Finally, using that $\pi_*(\pi^*W(d)) = W \otimes \operatorname{Sym}^d V$, we have a commutative diagram

$$\begin{tikzcd}
\operatorname{Quot}^P_{\mathbb{P}_S(V)/S}(F) \arrow[r, hook] \arrow[d] & \operatorname{Quot}^{P'}_{\mathbb{P}_S(V)/S}(\pi^*W) \arrow[d, hook] \\
\operatorname{Gr}_S(P(d), \pi_*F(d)) \arrow[r] & \operatorname{Gr}_S(P'(d), W \otimes \operatorname{Sym}^d V).
\end{tikzcd}$$

By the above claim, the top horizontal map is a closed immersion. As $\operatorname{Gr}_S(P(d), \pi_*F(d))$ and $\operatorname{Gr}_S(P'(d), W \otimes \operatorname{Sym}^d V)$ are projective (Theorem 1.1.1), the bottom horizontal map is projective and in particular separated. If the proposition holds for $\operatorname{Quot}^{P'}_{\mathbb{P}_S(V)/S}(\pi^*W)$ and the right vertical map is a locally closed immersion, then the left vertical map is also a closed immersion by the cancellation property.

*We now handle the special case.* We first claim that $\operatorname{Quot}^P_{X/S}(F) \to \operatorname{Gr}_S(P(d), W \otimes \operatorname{Sym}^d V)$ is a monomorphism, i.e.

$$\operatorname{Quot}^P_{X/S}(F)(T) \to \operatorname{Gr}_S(P(d), \pi_*F(d))(T)$$

is injective for each scheme $T$. To see this, observe that if $F_T = Q/K$ is a quotient with Hilbert polynomial $P$, then Corollary 1.3.13 implies that there is a map of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \pi_T^*\pi_{T,*}K(d) \arrow[r] \arrow[d] & \pi_T^*\pi_{T,*}F_T(d) \arrow[r] \arrow[d] & \pi_T^*\pi_{T,*}Q(d) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K(d) \arrow[r] & F_T(d) \arrow[r] & Q(d) \arrow[r] & 0
\end{tikzcd}$$

where the vertical maps are surjections. Thus $F_T(d) \to Q(d)$ can be recovered from $\pi_{T,*}F_T(d) \to \pi_{T,*}Q(d)$ by taking the cokernel of the composition $\pi_T^*\pi_{T,*}K(d) \to \pi_T^*\pi_{T,*}F_T(d) \to F_T(d)$.

Let $T \to \operatorname{Gr}_S(P(d), W \otimes \operatorname{Sym}^d V)$ be a morphism determined by a vector bundle quotient $\gamma\colon \pi_{T,*}F_T(d) = W_T \otimes \operatorname{Sym}^d V_T \to G$ of rank $P(d)$. Define $Q$ as the quotient sheaf of $F_T$ with the property that $F_T(d) \twoheadrightarrow Q(d)$ is identified with the cokernel of $\ker(\pi_T^*\gamma) \to \pi_T^*\pi_{T,*}F_T(d) \to F_T(d)$. The fiber product

$$\begin{tikzcd}
Z \arrow[r] \arrow[d] & T \arrow[d] \\
\operatorname{Quot}^P_{X/S}(F) \arrow[r] & \operatorname{Gr}_S(P(d), W \otimes \operatorname{Sym}^d V)
\end{tikzcd}$$

is identified with the subfunctor of $T$ (or more precisely the subfunctor of $\operatorname{Mor}_S(-, T)$) consisting of morphisms $T' \to T$ such that $Q_{T'}$ is flat over $T'$ with Hilbert polynomial

$P$ (in other words, a map $T' \to T$ factors through $Z$ if and only if $Q_{T'}$ is flat over $T'$ with Hilbert polynomial $P$). By Existence of Flattening Stratifications (Theorem A.2.14), $Z$ is representable by a locally closed subscheme of $T$. $\square$

### 1.4.2 Valuative criteria for Hilb and Quot

To establish that Quot is projective, it will be sufficient to know that it is proper.

**Proposition 1.4.2.** *For every projective morphism $X \to S$ of noetherian schemes, relatively ample line bundle $\mathcal{O}_X(1)$, coherent sheaf $F$ on $X$ and polynomial $P \in \mathbb{Q}[x]$, the functor $\operatorname{Quot}^P_{X/S}(F)$ satisfies the valuative criterion for properness, i.e. for every DVR $R$ over $S$ with fraction field $K$, every flat coherent quotient $F_K \to Q^\times$ on $X_K$ with Hilbert polynomial $P$ extends uniquely to a flat coherent quotient $F_R \to Q$ on $X_R$ with Hilbert polynomial $P$.*

**Remark 1.4.3.** In other words, the proposition implies that for every commutative diagram

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & \operatorname{Quot}^P_{X/S}(F) \arrow[d] \\ \operatorname{Spec} R \arrow[r] \arrow[ur, dashed] & S, \end{tikzcd}$$

of solid arrows, there is a unique dotted arrow filling in the diagram. See §3.8 for a further discussion of the valuative criterion for functors and stacks.

*Proof.* If we write $j\colon X_K \hookrightarrow X_R$ as the open immersion, we define $Q$ as the image of the composition $F_R \to j_* F_K \to j_* Q^\times$ (where the first map is given by the adjunction $F_R \to j_* j^* F_R = j_* F_K$). Since $Q$ is a subsheaf of $j_* Q^\times$, it is torsion free over $R$ and thus flat (as $R$ is a DVR). (Locally, if $S = \operatorname{Spec} B$ is affine and $U = \operatorname{Spec} A \subset X$ is an affine open, then we can write $F|_U = \widetilde{M}$ for a finitely generated $A$-module $M$ and we have a quotient $M \otimes_B K \to N^\times$ of $A \otimes_B K$-modules where $Q^\times|_{U_K} = \widetilde{N^\times}$. Then $Q = \widetilde{N}$ where $N$ is the $A \otimes_B R$-module defined by $N := \operatorname{im}(M \otimes_B R \to M \otimes_B K \to N^\times)$. Since the $R$-module $N$ is a subsheaf of the $K$-module $N^\times$, we see that $N$ is torsion free and thus flat.) Finally, since $Q$ if flat over $R$ and $\operatorname{Spec} R$ is connected, its Hilbert polynomial is constant. $\square$

**Remark 1.4.4.** For $\operatorname{Hilb}^P_{X/S}$, the argument translates into the following: the unique extension of a closed subscheme $Z^\times \subset X_K$ is the scheme-theoretic image $Z = \operatorname{im}(Z^\times \to X_K \hookrightarrow X_R)$. The scheme $Z$ is flat over $R$ as all associated points live over the generic point of $\operatorname{Spec} R$.

### 1.4.3 Projectivity

The proof of the main theorem of this section (Theorem 1.1.3) follows from the following proposition.

**Proposition 1.4.5.** *Let $\pi\colon X \to S$ be a strongly projective morphism of noetherian schemes, $\mathcal{O}_X(1)$ be a relatively ample line bundle on $X$, and $F$ be a coherent sheaf on $X$ which is the quotient of $\pi^*(W)(q)$ for a vector bundle $W$ on $S$ and an integer $q$. For $d \gg 0$, the morphism $\operatorname{Quot}^P_{X/S}(F) \to \operatorname{Gr}_S(P(d), \pi_* F(d))$ is a closed immersions.*

*Proof.* For $d \gg 0$, the morphism

$$\mathrm{Quot}^P_{X/S}(F) \to \mathrm{Gr}_S(P(d), \pi_* F(d))$$

is a locally closed immersion of schemes defines over $S$ (Proposition 1.4.1). Since $\mathrm{Quot}^P_{X/S}(F)$ is proper over $S$ (Proposition 1.4.2), this map is a closed immersion. Since $\mathrm{Gr}_S(P(d), \pi_* F(d))$ is strongly projective over $S$ (Theorem 1.1.1), so is $\mathrm{Quot}^P_{X/S}(F)$. $\square$

Consider the diagram

$$\begin{tikzcd}
 & X \times_S \mathrm{Quot}^P_{X/S}(F) \arrow[dl, "p_1"'] \arrow[d, "p_2"] & \\
X \arrow[dr, "\pi"'] & \mathrm{Quot}^P_{X/S}(F) \arrow[r, hook] \arrow[d] & \mathrm{Gr}_S(P(d), \pi_* F(d)) \arrow[dl, "g"] \\
 & S &
\end{tikzcd}$$

As $\mathrm{Quot}^P_{X/S}(F)$ represents the Quot functor, there is a *universal quotient* $p_1^* F \to \mathcal{Q}_{\mathrm{univ}}$ on $X \times_S \mathrm{Quot}^P_{X/S}(F)$. For $d \gg 0$, we also have the universal quotient $g^* \pi_* F(d) \to Q_{\mathrm{univ}}$ on $\mathrm{Gr}_S(P(d), \pi_* F(d))$ and a composition of closed immersions

$$\begin{array}{ccccc}
\mathrm{Quot}^P_{X/S}(F) & \hookrightarrow & \mathrm{Gr}_S(P(d), \pi_* F(d)) & \hookrightarrow & \mathbb{P}_S(\bigwedge^{P(d)}(\pi_* F(d))) \\
[F_T \twoheadrightarrow Q] & \longmapsto & [\pi_{T,*} F_T(d) \to \pi_{T,*} Q(d)] & \longmapsto & \left[\bigwedge^{P(d)}(\pi_{T,*} F_T(d)) \to \bigwedge^{P(d)}(\pi_{T,*} Q(d))\right]
\end{array}$$

The pullback of $\mathcal{O}(1)$ on $\mathbb{P}_S(\bigwedge^{P(d)}(\pi_* F(d)))$ pulls back to the Plücker line bundle $\det(Q_{\mathrm{univ}})$ (Corollary 1.2.9) which in turn pulls back to $\det\left(p_{2,*}(\mathcal{Q}_{\mathrm{univ}}(d))\right)$ on $\mathrm{Quot}^P_{X/S}(F)$. We obtain:

**Corollary 1.4.6.** *For $d \gg 0$, the line bundle $\det\left(p_{2,*}(\mathcal{Q}_{\mathrm{univ}}(d))\right)$ is very ample on $\mathrm{Quot}^P_{X/S}(F)$.* $\square$

**Exercise 1.4.7.**

(a) Show that if $S$ is a noetherian scheme and $V$ is a *coherent* sheaf on $S$, then functor $\mathrm{Gr}_S(k, V)$ defined analogously to Theorem 1.1.1 is represented by a scheme projective (but not necessarily strongly projective) over $S$.

(b) Show that if $X \to S$ is a projective morphism of noetherian scheme and $F$ is a coherent sheaf on $X$ *flat* over $S$, then $\mathrm{Quot}^P_{X/S}(F) \to \mathrm{Gr}_S(P(d), \pi_* F(d))$ is well-defined for $d \gg 0$ and $\mathrm{Quot}^P_{X/S}(F)$ is projective over $S$.

### 1.4.4 Generalizations

If $\pi \colon X \to S$ is a strongly quasi-projective morphism of noetherian schemes (i.e. there is a locally closed immersion $X \hookrightarrow \mathbb{P}_S(V)$ where $V$ is a vector bundle on $S$), $\mathcal{O}_X(1)$ is a relatively ample line bundle, $F$ is a coherent sheaf on $X$ which is a quotient of $\pi^*(W)(q)$ for a vector bundle $W$ on $S$ and integer $q$, and $P \in \mathbb{Q}[z]$ is a polynomial, we can modify the functors of Hilb and Quot as follows:

$$\mathrm{Hilb}^P_{X/S}\colon \mathrm{Sch}/S \to \mathrm{Sets}$$
$$(T \to S) \mapsto \left\{ \begin{array}{l} \text{subschemes } Z \subset X_T \text{ flat, proper, and finitely} \\ \text{presented over } T \text{ such that } Z_t \subset X_t \\ \text{has Hilbert polynomial } P \text{ for all } t \in T \end{array} \right\}$$

$$\mathrm{Quot}^P_{X/S}(F)\colon \mathrm{Sch}/S \to \mathrm{Sets}$$
$$(T \to S) \mapsto \left\{ \begin{array}{l} \text{quasi-coherent quotients } F_T \to Q \text{ on } X_T \\ \text{of finite presentation which are flat and have} \\ \text{proper support over } T \text{ such that } Q|_{X_t} \text{ on } X_t \\ \text{has Hilbert polynomial } P \text{ for all } t \in T \end{array} \right\}$$

Then $\mathrm{Hilb}^P_{X/S}$ and $\mathrm{Quot}^P_{X/S}(F)$ are represented by strongly quasi-projective schemes over $S$; see [$\mathrm{FGA}_{\mathrm{IV}}$, §4], [AK80] or [FGI+05, §5.6]

If $X \to S$ is merely a separated morphism of noetherian schemes, then one can define functors $\mathrm{Hilb}(X/S)$ and $\mathrm{Quot}(F/X/S)$ as above dropping the condition on the Hilbert polynomial $P$. These functors are representable by algebraic spaces separated and locally of finite type over $S$; see [Art69b, Thm. 6.1][1] and [SP, Tag 09TQ]. Examples of Hironaka produce smooth proper (but not projective) 3-folds $X$ over $\mathbb{C}$ such that $\mathrm{Hilb}^P_{X/S}$ is not a scheme.

There are further variants and generalizations:

- Vistoli's Hilbert stack parameterizing finite and unramified morphisms to a separated scheme $X$ (or stack) [Vis91].
- Alexeev and Knutson's moduli of branch varieties parameterizing finite morphisms from a geometrically reduced proper scheme to a separated scheme $X$ [AK10].
- If $X \to S$ is not separated, then Hall and Rydh show that there is an algebraic stack locally of finite type over $S$ parameterizing quasi-finite morphism $Z \to X$ from a proper scheme [HR14].

**Exercise 1.4.8** (Schemes of morphisms)**.** For projective morphisms $X \to S$ and $Y \to S$ of noetherian schemes, consider the functor

$$\underline{\mathrm{Mor}}_S(X,Y)\colon \mathrm{Sch}/S \to \mathrm{Sets}$$
$$(T \to S) \mapsto \mathrm{Mor}_T(X_T, Y_T)$$

assigning an $S$-scheme $T$ to the set of $T$-morphisms $X_T \to Y_T$. By using a suitable Hilbert scheme $\mathrm{Hilb}^P_{X\times_S Y/X}$ parameterizing graphs $X \subset X \times_S Y$ of morphisms $X \to Y$, show that $\underline{\mathrm{Mor}}_S(X,Y)$ is representable by a projective scheme over $S$. Can we weaken the hypothesis on $X$ and $Y$?

## 1.5 An invitation to the geometry of Hilbert schemes

### 1.5.1 First examples

In this section, we work over an algebraically closed field $\Bbbk$.

[1] As pointed out in [Art74, Appendix], the representability is not true without the separated hypothesis on $X \to S$.

The Hilbert polynomial $P(z) = \sum_{i=0}^{d} a_i z^i$ of a projective scheme $X \subset \mathbb{P}^n$ encodes invariants of $X$. For instance, $\dim X$ is the degree $d$ of $P$ and $\deg X$ is the normalized leading coefficient $d!a_d$. Applying Riemann–Roch in the case of a smooth curve $C \subset \mathbb{P}^n$ gives $P(z) = \deg(C)z + (1-g)$ and for a surface $S \subset \mathbb{P}^n$ gives $P(z) = \frac{1}{2}(zH \cdot (zH - K)) + (1 - p_a)$ where $H$ is a hyperplane divisor, $K$ is the canonical divisor and $p_a = 1 - \chi(\mathcal{O}_S)$ is the arithmetic genus. In arbitrary dimension, Hirzebruch–Riemann–Roch implies that $P(z) = \int_X \mathrm{ch}(\mathcal{O}_X(z))\mathrm{td}(X)$, where $\mathrm{ch}(\mathcal{O}_X(z))$ is the Chern character and $\mathrm{td}(X)$ the Todd class.

**Example 1.5.1** (Hypersurfaces and linear subspaces)**.** A hypersurface $H \subset \mathbb{P}^n$ of degree $d$ has Hilbert polynomial

$$P(z) = \chi(\mathcal{O}_{\mathbb{P}^n}(z)) - \chi(\mathcal{O}_{\mathbb{P}^n}(z-d)) = \binom{n+z}{n} - \binom{n+z-d}{n}$$

(coming from the exact sequence $0 \to \mathcal{O}_{\mathbb{P}^n}(-d) \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_H \to 0$). We claim that $\mathrm{Hilb}^P_{\mathbb{P}^n} \cong \mathbb{P}(\Gamma(\mathbb{P}^n, \mathcal{O}(d)))$. We encourage the reader to show this and in particular establish that every subscheme $Z \subset \mathbb{P}^n$ with Hilbert polynomial $P$ is a hypersurface.

Similarly, a linear subspace $L \subset \mathbb{P}^n$ of dimension $k$ has Hilbert polynomial $P(z) = \binom{z+k}{k}$ and $\mathrm{Hilb}^P_{\mathbb{P}^n} = \mathrm{Gr}(k+1, n+1)$.

**Example 1.5.2** (Hilbert scheme of points on a curve)**.** If $C$ is a smooth projective curve, then the Hilbert scheme of $n$ points $\mathrm{Hilb}^n_C$ (viewing $n$ as the constant polynomial) is a smooth irreducible projective variety isomorphic to the symmetric product

$$\mathrm{Sym}^n C := \underbrace{C \times \cdots \times C}_{n} / S_n,$$

where $S_n$ acts by permuting the factors. The quotient exists as a projective variety since $C \times \cdots \times C$ is projective; see Exercise 4.2.8.

**Example 1.5.3** (Hilbert scheme of points on a surface)**.** T.S. Gustavsen, D. Laksov, R.M. Skjelnes

If $S$ is a smooth irreducible projective surface, then the Hilbert scheme of $n$ points $\mathrm{Hilb}^n_S$ is a smooth irreducible projective variety [Fog68]. See also [Nak99a] and [Mac07, §4]. There is a birational morphism

$$\mathrm{Hilb}^n_S \to \mathrm{Sym}^n(S) := \underbrace{S \times \cdots \times S}_{n} / S_n,$$

of projective varieties. The symmetric product $\mathrm{Sym}^n(S)$ is not smooth for $n > 1$ and this provides a resolution of singularities. For an unordered collection of (possibly non-distinct) points $(p_1, \ldots, p_n) \in \mathrm{Sym}^n(S)$, the fiber consists of all possible scheme structures on $\{p_1, \ldots, p_n\}$ of length $n$.

When $n = 1$, $\mathrm{Hilb}^1_S = S$. For $n = 2$ and the points $p_1$ and $p_2$ are equal, there is a $\mathbb{P}^1$ of scheme structures given by $k[x,y]/(x^2, xy, y^2, ay - bx)$ (with coordinates such that $p_1 = p_2 = 0$) parameterized by their "tangent direction" $[a : b] \in \mathbb{P}^1$. In this case, $\mathrm{Hilb}^2_S \to \mathrm{Sym}^2(S)$ is the blow-up of the diagonal $S \hookrightarrow \mathrm{Sym}^2(S)$ given by $p \mapsto (p,p)$. In fact, for $n > 2$, the map $\mathrm{Hilb}^n_S \to \mathrm{Sym}^n(S)$ is a blow-up along some ideal sheaf [Hai98] but the description of the ideal sheaf is more complicated.

When $X$ is of arbitrary dimension, $\mathrm{Hilb}^n_X$ is smooth at (reduced) closed subschemes $Z \subset X$ consisting of $n$ distinct smooth points of $X$. If $X$ is reduced, there is an open subscheme of $\mathrm{Hilb}^n_X$ dimension $n \dim(X)$ parameterizing $n$ distinct smooth points. Another result of Fogarty is that $\mathrm{Hilb}^n_X$ is connected as long as $X$ is connected [Fog68]. Moreover, for every projective scheme $X$, there is an irreducible component $\mathrm{Hilb}_X^n$, called the "good component," that can be identified with the blow-up of $\mathrm{Sym}^n(X)$ along some ideal sheaf [ES14].

**Example 1.5.4** (Twisted cubics)**.** The Hilbert scheme $\mathrm{Hilb}_{\mathbb{P}^3}^{3z+1}$ consists of the union of two smooth rational irreducible components $H$ and $H'$ of dimensions 12 and 15 intersecting transversely along a smooth rational subvariety of dimension 11 [PS85].

The locus $H$ is the closure of the locus $H_0$ consisting of *twisted cubics*, i.e. rational smooth curves in $\mathbb{P}^3$ of degree 3. Each twisted cubic can be represented by a map $\mathbb{P}^1 \to \mathbb{P}^3$ given by the line bundle $\mathcal{O}_{\mathbb{P}^1}(3)$ and a choice of basis of $\Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(3))$, and this representation is unique up to automorphisms of $\mathbb{P}^1$. All such curves are projectively equivalent, i.e. differ by an automorphism of $\mathbb{P}^3$, so we see that $H_0$ is identified with the homogeneous space $\mathrm{Aut}(\mathbb{P}^3)/\mathrm{Aut}(\mathbb{P}^1) = \mathrm{SL}_4/\mathrm{SL}_2$, which is smooth and irreducible of dimension 12. The locus $H_0$ is not proper as it includes families such as $\mathbb{P}^1 \hookrightarrow \mathbb{P}^3$ given by $[x,y] \mapsto [x^3, x^2y, xy^2, ty^3]$ parameterized by $t \in \mathbb{A}^1$ whose limit is a singular curve $C_0$ supported on a nodal cubic in $V(w) = \mathbb{P}^2$ (where $w$ is the 4th coordinate) but with an embedded point at the node; see [Har77, Ex. 9.8.4].

The locus $H'$ is the closure of the locus $H_0'$ consisting of subschemes $C \sqcup \{p\}$ where $C$ is a smooth cubic curve contained in a hyperplane $H$ and $p \in \mathbb{P}^3 \setminus C$. To count the dimension, observe that the choice of hyperplane $H \in \mathbb{P}(\mathrm{H}^0(\mathbb{P}^3, \mathcal{O}(1)))$ is given by 3 parameters, the choice of plane cubic $C \in \mathbb{P}(\mathrm{H}^0(H, \mathcal{O}_H(3)))$ is given by 9 parameters and the point $p \in \mathbb{P}^3 \setminus C$ is given by 3 parameters. The locus $H_0'$ is smooth and irreducible of dimension 15. Again, the locus $H_0'$ is not proper and its closure contains the limits of for instance degenerating the point $p$ to lie on the curve whose limit can be curves like $C_0$.

The intersection $H \cap H'$ consists of plane singular cubic curves with an embedded point at the singular point. This locus contains curves such as $C_0$ above but it also contains even more degenerate curves such as a triple line with an embedded point. Every curve $C \in H \cap H'$ is in fact projectively equivalent to the curve defined by $V(xz, yz, z^2, q(x,y,w))$ where $q(x,y,w)$ is a homogeneous cubic polynomial with a singular point at $(0,0,1)$. This depends on 11 parameters.

### 1.5.2 Geometric properties

**Non-emptiness.** The Hilbert scheme $\mathrm{Hilb}_{\mathbb{P}^n}^P$ is non-empty if and only if the Hilbert polynomial $P$ can be written as as

$$P(z) = \binom{z+\lambda_1-1}{\lambda_1-1} + \binom{z+\lambda_2-2}{\lambda_2-1} + \cdots + \binom{z+\lambda_r-r}{\lambda_r-1}, \tag{1.5.1}$$

integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \geq 1$. This is a result of Hartshorne [Har66b, Cor. 5.7] The necessity of this condition was already mentioned in Remark 1.3.10 in the context of Gotzmann's bounds on regularity.

**Connectedness.** Hartshorne's Connectedness Theorem asserts that the Hilbert scheme $\mathrm{Hilb}_{\mathbb{P}^n}^P$ is connected for every Hilbert polynomial $P$ [Har66b]. More generally, for every connected noetherian scheme $S$, $\mathrm{Hilb}_{\mathbb{P}^n_S/S}^P$ is connected.

The strategy of the argument is to show that every closed subscheme $Z \subset \mathbb{P}^n$ degenerates to a subscheme $V(I)$ defined by a monomial ideal. This reduces the question to the combinatorial question of connecting any two monomial ideals by a family over $\mathbb{A}^1$. This turns out to be a purely deformation and combinatorial question or as Hartshorne writes: "It also appears that the Hilbert scheme is never actually needed in the proof."

See also [Mac07, §3].

**Murphy's Law.**

> Murphy's Law for Hilbert Schemes: There is no geometric possibility so horrible that it cannot be found generically on some component of the Hilbert scheme. [HM98, p.18]

The first pathology was exhibited by Mumford: there is an irreducible component of $\mathrm{Hilb}^{14z-23}_{\mathbb{P}^3}$ which is generically non-reduced [Mum62]. Ellia—Hirschowitz—Mezzetti show that the number of irreducible components in $\mathrm{Hilb}^{az+b}_{\mathbb{P}^3}$ is not bounded by a polynomial in $a, b$ [EHM92].

Murphy's Law was made precise by Vakil [Vak06]: for every scheme $X$ finite type over $\mathbb{Z}$ and point $x \in X$, there exists a point $q = [Z \subset \mathbb{P}^n] \in \mathrm{Hilb}^P_{\mathbb{P}^n}$ of some Hilbert scheme and an isomorphism

$$\widehat{\mathcal{O}}_{X,p}[\![x_1, \ldots, x_s]\!] \cong \widehat{\mathcal{O}}_{\mathrm{Hilb}^P_{\mathbb{P}^n},q}[\![y_1, \ldots, y_t]\!]$$

for integers $s, t$. In other words, if we introduce the equivalence relation on pointed schemes $(Z, z)$ generated by $(Z, z) \sim (Z', z')$ if there exists a smooth pointed morphism $(Z', z') \to (Z, z)$, then $(X, p)$ is equivalent to $(\mathrm{Hilb}^P_{\mathbb{P}^n}, q)$.

In fact, it can be arranged that the Hilbert scheme parameterizes smooth curves in $\mathbb{P}^n$ for some $n$, or that it parameterizes smooth surfaces in $\mathbb{P}^5$ (resp. surfaces in $\mathbb{P}^4$). It turns out that various other moduli spaces also satisfy Murphy's Law: Kontsevich's moduli space of maps, moduli of canonically polarized smooth surfaces, moduli of curves with linear systems, and the moduli space of stable sheaves.

**Smoothness.** Despite Murphy's Law, many Hilbert schemes are in fact smooth. We've seen before that the Hilbert scheme of points on a smooth surface is smooth. Moreover, it is not hard to see that the Hilbert scheme $\mathrm{Hilb}^P_{\mathbb{P}^n/\Bbbk}$ of projective space over a field is smooth at every complete intersection $Z \subset \mathbb{P}^n$ (despite the obstruction space $\mathrm{H}^1(Z, N_{Z/X})$ being potentially nonzero).

A theorem of Skjelnes–Smith [SS20] states that the Hilbert scheme $\mathrm{Hilb}^P_{\mathbb{P}^n}$ is smooth if and only if $P(z)$ can be written as (1.5.1) for a partition $\lambda = (\lambda_1, \ldots, \lambda_r)$ of integers satisfying $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \geq 1$ such that one of the seven condition holds:

(1) $n = 2$;
(2) $\lambda_r \geq 2$;
(3) $\lambda = (1)$ or $\lambda = (n^{r-2}, \lambda_{r-1}, 1) = (\underbrace{n, \ldots, n}_{r-2}, \lambda_{r-1}, 1)$ where $r \geq 2$ and $n \geq \lambda_{r-1} \geq 1$;
(4) $\lambda = (n^{r-s-3}, \lambda_{r-s-2}^{s+2}, 1)$ where $r - s \geq s \geq 0$ and $n - 1 \geq \lambda_{r-s-2} \geq 3$;
(5) $\lambda = (n^{r-s-5}, 2^{s+4}, 1)$ where $r - 5 \geq s \geq 0$;
(6) $\lambda = (n^{r-3}, 1^s)$ where $r \geq 3$;
(7) $\lambda = (n + 1)$ or $r = 0$.

**Local properties.**

**Exercise 1.5.5.** Let $X$ be a projective scheme over a field $\Bbbk$ and $F$ be a coherent sheaf on $X$.

(a) Let $p \in \mathrm{Quot}^P_{X/\Bbbk}(F)$ be the point corresponding to a quotient $Q = F/K$. Show that $T_p \mathrm{Quot}^P_{X/\Bbbk}(F) \cong \mathrm{Hom}_{\mathcal{O}_X}(K, Q)$. This generalizes the exercise computing the tangent space of the Grassmannian (Exercise 1.2.7).

(b) Conclude that if $p \in \mathrm{Hilb}^P_{X/\Bbbk}$ is a point corresponding to a closed subscheme $Z \subset X$ defined by a sheaf of ideals $I$, then $T_p \mathrm{Hilb}^P_{X/\Bbbk} \cong \mathrm{H}^0(Z, N_{Z/X})$ where $N_{Z/X}$ is the normal sheaf $\mathrm{Hom}_{\mathcal{O}_Z}(I_Z/I_Z^2, \mathcal{O}_Z)$.

# Chapter 2

# Sites, sheaves, and stacks

## 2.1 Grothendieck topologies and sites

We would like to form a topology on a scheme where étale morphisms replace Zariski open subsets. This doesn't quite make sense using the conventional notion of a topological space so we instead adapt our definitions. Grothendieck topologies and stacks were introduced in [SGA4]. Our exposition closely follows [Art62], [FGI+05, Part 1], [Ols16, §2], and [SP, Tag 00UZ].

### 2.1.1 Definitions and examples

**Definition 2.1.1** (Sites)**.** A *Grothendieck topology* on a category $\mathcal{S}$ consists of the following data: for each object $X \in \mathcal{S}$, there is a set $\mathrm{Cov}(X)$ consisting of *coverings* of $X$, i.e. collections of morphisms $\{X_i \to X\}_{i \in I}$ in $\mathcal{S}$. We require that:

(1) (identity) If $X' \to X$ is an isomorphism, then $(X' \to X) \in \mathrm{Cov}(X)$.

(2) (restriction) If $\{X_i \to X\}_{i \in I} \in \mathrm{Cov}(X)$ and $Y \to X$ is a morphism, then the fiber products $X_i \times_X Y$ exist in $\mathcal{S}$ and the collection $\{X_i \times_X Y \to Y\}_{i \in I} \in \mathrm{Cov}(Y)$.

(3) (composition) If $\{X_i \to X\}_{i \in I} \in \mathrm{Cov}(X)$ and $\{X_{ij} \to X_i\}_{j \in J_i} \in \mathrm{Cov}(X_i)$ for each $i \in I$, then $\{X_{ij} \to X_i \to X\}_{i \in I, j \in J_i} \in \mathrm{Cov}(X)$.

A *site* is a category $\mathcal{S}$ with a Grothendieck topology.

**Example 2.1.2** (Topological spaces)**.** If $X$ is a topological space, let $\mathrm{Op}(X)$ denote the category of open sets $U \subset X$. There is a unique morphism $U \to V$ if and only if $U \subset V$. We say that a covering of $U$ (i.e. an element of $\mathrm{Cov}(U)$) is a collection of open immersions $\{U_i \to U\}_{i \in I}$ such that $U = \bigcup_{i \in I} U_i$. This defines a Grothendieck topology on $\mathrm{Op}(X)$.

In particular, if $X$ is a scheme, the Zariski topology on $X$ defines a site $X_{\mathrm{Zar}}$, called the *small Zariski site on $X$*

The most important sites for us will be the small and big étale sites.

**Example 2.1.3** (Small étale site)**.** If $X$ is a scheme, the *small étale site on $X$* is the category $X_{\text{ét}}$ of étale morphisms $U \to X$ such that a morphism $(U \to X) \to (V \to X)$ is simply an $X$-morphism $U \to V$ (which is necessarily étale). In other words, $X_{\text{ét}}$ is the full subcategory of $\mathrm{Sch}/X$ consisting of schemes étale over $X$. A covering of an object $(U \to X) \in X_{\text{ét}}$ is a collection of étale morphisms $\{U_i \to U\}$ such that $\coprod_i U_i \to U$ is surjective.

Later we will introduce the small étale site $\mathcal{X}_{\text{ét}}$ of an algebraic space or Deligne–Mumford stack (Definition 4.1.1) and use it to define sheaves on $\mathcal{X}$.

### Big sites

**Example 2.1.4** (Big topological site)**.** Let Top be the category of topological spaces. A covering of $U \in \text{Top}$ is a collection of open subspaces $\{U_i \hookrightarrow U\}_{i \in I}$ such that $U = \bigcup_{i \in I} U_i$.

The big étale site is the most frequently used site in these notes. It is used to define the most central notions in this book: an algebraic space is a sheaf on $\text{Sch}_{\text{ét}}$ that is étale locally a scheme (Definition 3.1.2) while an algebraic stack is a stack over $\text{Sch}_{\text{ét}}$ that is smooth-locally a scheme (Definition 3.1.6).

**Example 2.1.5** (Big étale site)**.** The *big étale site* is the category Sch where a covering of a scheme $U$ is a collection of étale morphisms $\{U_i \to U\}$ in Sch such that $\coprod_i U_i \to U$ is surjective. We denote this site as $\text{Sch}_{\text{ét}}$.

The following sites will be less important for us than the étale sites.

**Example 2.1.6** (Big Zariski site)**.** Replacing étale morphisms in Example 2.1.5 with open immersions defines the *big Zariski site* $\text{Sch}_{\text{Zar}}$.

**Example 2.1.7** (Big fppf site)**.** An fppf morphism of schemes is by definition a surjective and flat morphism locally of finite presentation; see also Definition A.2.18. The *big fppf site* $\text{Sch}_{\text{fppf}}$ is the category Sch of schemes where a covering $\{U_i \to U\}$ is a collection of morphisms such that $\coprod_i U_i \to U$ is fppf, i.e. each $U_i \to U$ is flat and locally of finite presentation, and $\coprod_i U_i \to U$ is surjective.

**Example 2.1.8** (Lisse-étale site)**.** On a scheme $X$, the *lisse-étale site* $X_{\text{lis-ét}}$ is the category of schemes smooth over $X$ where morphisms in $X_{\text{lis-ét}}$ are (not necessarily smooth) morphisms of schemes over $X$. A covering $\{U_i \to U\}$ of an $X$-scheme $U$ is a collection of $X$-morphisms such that $\coprod_i U_i \to U$ is surjective and étale.

We will later introduce the lisse-étale site of an algebraic stack (Definition 6.1.1)

Instead of defining the above sites on the category Sch of schemes, one can define the sites $\text{AffSch}_{\text{Zar}}$, $\text{AffSch}_{\text{ét}}$, $\text{AffSch}_{\text{fppf}}$ and $\text{AffSch}_{\text{lis-ét}}$ on the category of affine schemes with the same coverings. These variants sometimes appear in the literature.

**Example 2.1.9** (Localized categories and sites)**.** If $\mathcal{S}$ is a category and $S \in \mathcal{S}$, define the category $\mathcal{S}/S$ whose objects are maps $T \to S$ in $\mathcal{S}$. A morphism $(T' \to S) \to (T \to S)$ is a map $T' \to T$ over $S$. If $\mathcal{S}$ is a site, $\mathcal{S}/S$ is also a site where a covering of $T \to S$ in $\mathcal{S}/S$ is a covering $\{T_i \to T\}$ in $\mathcal{S}$.

Applying this construction to a scheme $S$ yields the big Zariski, étale, fppf and fpqc sites $(\text{Sch}/S)_{\text{Zar}}$, $(\text{Sch}/S)_{\text{ét}}$, $(\text{Sch}/S)_{\text{fppf}}$ and $(\text{Sch}/S)_{\text{fpqc}}$.

**Example 2.1.10** (Grothendieck topologies on the category of affine schemes)**.** A variant of the big sites introduced above on the category Sch of all schemes are the sites $\text{AffSch}_{\text{Zar}}$, $\text{AffSch}_{\text{ét}}$, $\text{AffSch}_{\text{fppf}}$ and $\text{AffSch}_{\text{lis-ét}}$

## 2.2 Presheaves and sheaves

Recall that if $X$ is a topological space, a presheaf of sets on $X$ is simply a contravariant functor $F\colon \text{Op}(X) \to \text{Sets}$ on the category $\text{Op}(X)$ of open sets. The sheaf axiom translates succinctly into the condition that for each covering $U = \bigcup_i U_i$, the sequence

$$F(U) \to \prod_i F(U_i) \rightrightarrows \prod_{i,j} F(U_i \cap U_j)$$

is exact (i.e. is an equalizer diagram), where the two maps $F(U_i) \rightrightarrows F(U_i \cap U_j)$ are induced by the two inclusions $U_i \cap U_j \subset U_i$ and $U_i \cap U_j \subset U_j$. Also note that the intersections $U_i \cap U_j$ can also be viewed as fiber products $U_i \times_X U_j$.

### 2.2.1 Definitions

**Definition 2.2.1** (Presheaves)**.** A *presheaf* on a category $\mathcal{S}$ is a contravariant functor $\mathcal{S} \to \mathrm{Sets}$.

**Remark 2.2.2.** If $F \colon \mathcal{S} \to \mathrm{Sets}$ is a presheaf and $S \xrightarrow{f} T$ is a map in $\mathcal{S}$, then the pullback $F(f)(b)$ of an element $b \in F(T)$ is sometimes denoted as $f^*b$ or $b|_S$.

**Definition 2.2.3** (Sheaves)**.** A *sheaf* on a site $\mathcal{S}$ is a presheaf $F \colon \mathcal{S} \to \mathrm{Sets}$ such that for every object $S$ and covering $\{S_i \to S\} \in \mathrm{Cov}(S)$, the sequence

$$F(S) \to \prod_i F(S_i) \rightrightarrows \prod_{i,j} F(S_i \times_S S_j) \tag{2.2.1}$$

is exact, where the two maps $F(S_i) \rightrightarrows F(S_i \times_S S_j)$ are induced by the two maps $S_i \times_S S_j \to S_i$ and $S_i \times_S S_j \to S_i$.

**Remark 2.2.4.** The exactness of (2.2.1) means that it is an equalizer diagram: $F(S)$ is precisely the subset of $\prod_i F(S_i)$ consisting of elements whose images under the two maps $F(S_i) \rightrightarrows F(S_i \times_S S_j)$ are equal.

**Exercise 2.2.5.** Let $F$ be a presheaf on Sch. Show that the following are equivalent:

(1) $F$ is a sheaf on $\mathrm{Sch}_{\text{ét}}$ (resp. $\mathrm{Sch}_{\mathrm{fppf}}$, $\mathrm{Sch}_{\mathrm{fpqc}}$),

(2) $F$ sends coproducts to products (i.e. $F(\coprod_i U_i) = \prod_i F(U_i)$ for schemes $U_i$) and for every surjective étale (resp. fppf, faithfully flat) morphism $S' \to S$ of schemes, the sequence $F(S) \to F(S') \rightrightarrows F(S' \times_S S')$ is exact.

(3) $F$ is a sheaf in the big Zariski topology $\mathrm{Sch}_{\mathrm{Zar}}$ and for every surjective étale (resp. fppf, faithfully flat) morphism $S' \to S$ of *affine* schemes, the sequence $F(S) \to F(S') \rightrightarrows F(S' \times_S S')$ is exact.

**Proposition 2.2.6.** *If $X \to S$ is a morphism of schemes, then* $\mathrm{Mor}_S(-, X) \colon \mathrm{Sch}/S \to \mathrm{Sets}$ *is a sheaf on* $(\mathrm{Sch}/S)_{\mathrm{fpqc}}$ *and therefore also a sheaf on* $(\mathrm{Sch}/S)_{\text{ét}}$*.*

*Proof.* As we already know that $\mathrm{Mor}_S(-, X)$ is a sheaf in the big Zariski topology, it suffices by Exercise 2.2.5 to show that if $T' \to T$ is a faithfully flat morphism of affine schemes over $S$, then the sequence

$$\mathrm{Mor}_S(T, X) \to \mathrm{Mor}_S(T', X) \rightrightarrows \mathrm{Mor}_S(T' \times_T T', X)$$

is exact. This follows from fpqc descent for morphisms of schemes (Proposition B.2.1). □

### 2.2.2 Morphisms and fiber products

A *morphism* of presheaves or sheaves is by definition a natural transformation. By Yoneda's lemma (Lemma 0.3.6), if $X$ is a scheme and $F$ is a presheaf on Sch, a morphism $\alpha\colon X \to F$ (which we interpret as a morphism of presheaves $\mathrm{Mor}(-, X) \to F$) corresponds to an element in $F(X)$, which by abuse of notation we also denote by $\alpha$.

**Exercise 2.2.7.** Recall from Proposition 2.2.6 that a scheme can be viewed as a sheaf in the big fpqc topology.

(a) Show that a surjective étale (resp. fppf, fpqc) morphism of schemes is an epimorphism of sheaves on $\mathrm{Sch}_{\text{ét}}$ (resp. $\mathrm{Sch}_{\text{fppf}}$, $\mathrm{Sch}_{\text{fpqc}}$).

(b) Show that a surjective smooth morphism of schemes is an epimorphism sheaves on $\mathrm{Sch}_{\text{ét}}$.

Given morphisms $F \xrightarrow{\alpha} G$ and $G' \xrightarrow{\beta} G$ of presheaves on a category $\mathcal{S}$, define the presheaf $F \times_G G'$ by $(F \times_G G')(S) = F(S) \times_{G(S)} G'(S)$, i.e.

$$\begin{aligned} F \times_G G' \colon \mathcal{S} &\to \mathrm{Sets} \\ S &\mapsto \{(a, b) \in F(S) \times G'(S) \mid \alpha_S(a) = \beta_S(b)\}\,. \end{aligned} \tag{2.2.2}$$

**Exercise 2.2.8.**

(a) Show that (2.2.2) is a fiber product $F \times_G G'$ in $\mathrm{Pre}(\mathcal{S})$. (This is a generalization of Exercise 0.3.28 but the same proof should work.)

(b) Show that if $F$, $G$ and $G'$ are sheaves on a site $\mathcal{S}$, then so is $F \times_G G'$. In particular, (2.2.2) is also a fiber product $F \times_G G'$ in $\mathrm{Sh}(\mathcal{S})$.

### 2.2.3 Sheafification

**Theorem 2.2.9** (Sheafification)**.** *Let $\mathcal{S}$ be a site. The forgetful functor $\mathrm{Sh}(\mathcal{S}) \to \mathrm{Pre}(\mathcal{S})$ admits a left adjoint $F \mapsto F^{\mathrm{sh}}$, called the sheafification.*

*Proof.* A presheaf $F$ on $\mathcal{S}$ is called *separated* if for every covering $\{S_i \to S\}$ of an object $S$, the map $F(S) \to \prod_i F(S_i)$ is injective (i.e. if sections glue, they glue uniquely). Let $\mathrm{Pre}(\mathcal{S})$ and $\mathrm{Sh}(\mathcal{S})$ be the categories of presheaves and sheaves, and let $\mathrm{Pre}^{\mathrm{sep}}(\mathcal{S}) \subset \mathrm{Pre}(\mathcal{S})$ be the full subcategory of separated presheaves. We will construct left adjoints

$$\begin{tikzcd} \mathrm{Sh}(\mathcal{S}) \arrow[r, hook] & \mathrm{Pre}^{\mathrm{sep}}(\mathcal{S}) \arrow[l, bend right, "\mathrm{sh}_2"'] \arrow[r, hook] & \mathrm{Pre}(\mathcal{S}). \arrow[l, bend right, "\mathrm{sh}_1"'] \end{tikzcd}$$

For $F \in \mathrm{Pre}(\mathcal{S})$, we define $\mathrm{sh}_1(F)$ by $S \mapsto F(S)/\sim$ where $a \sim b$ if there exists a covering $\{S_i \to S\}$ such that $a|_{S_i} = b|_{S_i}$ for all $i$.

For $F \in \mathrm{Pre}^{\mathrm{sep}}(\mathcal{S})$, we define $\mathrm{sh}_2(F)$ by

$$S \mapsto \left\{ (\{S_i \to S\}, \{a_i\}) \,\middle|\, \begin{array}{l} \text{where } \{S_i \to S\} \in \mathrm{Cov}(S) \text{ and } a_i \in F(S_i) \\ \text{such that } a_i|_{S_{ij}} = a_j|_{S_{ij}} \text{ for all } i, j \end{array} \right\} / \sim$$

where $(\{S_i \to S\}, \{a_i\}) \sim (\{S'_j \to S\}, \{a'_j\})$ if $a_i|_{S_i \times_S S'_j} = a'_j|_{S_i \times_S S'_j}$ for all $i, j$. The details are left to the reader. $\square$

**Remark 2.2.10** (Topos)**.** A *topos* is a category equivalent to the category of sheaves on a site. Two different sites may have equivalent categories of sheaves, and the topos can be viewed as a more fundamental invariant. While topoi are undoubtedly important in moduli theory, they will not play a role in these notes.

### 2.2.4 Effective descent for sheaves

**Proposition 2.2.11** (Effective Descent)**.** *Let $\mathcal{P}$ be one of the following properties of morphisms of schemes: open immersion, closed immersion, locally closed immersion, affine, quasi-affine, or separated and locally quasi-finite. Let $X \to Y$ be a surjective smooth (resp. fppf) morphism of schemes. Let $F$ be a sheaf on $(\mathrm{Sch}/Y)_{\text{ét}}$ (resp, $(\mathrm{Sch}/Y)_{\text{fppf}}$). Consider the fiber product*

$$\begin{tikzcd} F_X \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X \arrow[d] \\ F \arrow[r] & Y \end{tikzcd}$$

*of sheaves. If $F_X$ is a scheme and $F_X \to X$ has $\mathcal{P}$, then $F$ is a scheme and $F \to Y$ has $\mathcal{P}$.*

*Proof.* As $F_X$ is the pullback of $F$, there is a canonical isomorphism $\alpha \colon p_1^* F_X \to p_2^* F_X$ on $X \times_Y X$ satisfying the cocycle condition $p_{23}^*\alpha \circ p_{12}^*\alpha = p_{13}^*\alpha$. By Proposition B.3.1, there exists a morphism of schemes $W \to Y$ satisfying $\mathcal{P}$ that pulls back to $F_X \to X$. There is an induced injective morphism $F \to W$ of sheaves over $Y$ that pulls back to an isomorphism under $X \to Y$. Since $X \to Y$ is smooth and surjective (resp. fppf), it is an epimorphism of sheaves (Exercise 2.2.7) and it follows that $F \to W$ is an epimorphism, thus isomorphism of sheaves. $\square$

## 2.3 Prestacks

In §0.6.1, we motivated the concept of a prestack on a category $\mathcal{S}$ as a generalization of a presheaf $\mathcal{S} \to \mathrm{Sets}$. By trying to keep track of automorphisms, we were naively led to consider a 'functor' $F \colon \mathcal{S} \to \mathrm{Groupoids}$ but decided instead to package this data into one large category $\mathcal{X}$ over $\mathcal{S}$ parameterizing pairs $(a, S)$ where $S \in \mathcal{S}$ and $a \in F(S)$.

### 2.3.1 Definition of a prestack

Let $\mathcal{S}$ be a category and $p \colon \mathcal{X} \to \mathcal{S}$ be a functor of categories. We visualize this data as

$$\begin{tikzcd} \mathcal{X} \arrow[d, "p"] \\ \mathcal{S} \end{tikzcd} \qquad \begin{tikzcd} a \arrow[r, "\alpha"] \arrow[d, mapsto] & b \arrow[d, mapsto] \\ S \arrow[r, "f"] & T \end{tikzcd}$$

where the lower case letters $a, b$ are objects of $\mathcal{X}$ and the upper case letters $S, T$ are objects of $\mathcal{S}$. We say that $a$ is over $S$ and $\alpha \colon a \to b$ is over $f \colon S \to T$.

**Definition 2.3.1** (Prestacks)**.** A functor $p \colon \mathcal{X} \to \mathcal{S}$ is a *prestack over a category $\mathcal{S}$* if

(1) (pullbacks exist) for every diagram

$$\begin{tikzcd} a \arrow[r, dashed] \arrow[d, mapsto, dashed] & b \arrow[d, mapsto] \\ S \arrow[r] & T \end{tikzcd}$$

of solid arrows, there exists a morphism $a \to b$ over $S \to T$; and

(2) (universal property for pullbacks) for every diagram

$$\begin{tikzcd} a \arrow[r, dashed] \arrow[d, mapsto] \arrow[rr, bend left] & b \arrow[r] \arrow[d, mapsto] & c \arrow[d, mapsto] \\ R \arrow[r] & S \arrow[r] & T \end{tikzcd}$$

of solid arrows, there exists a unique arrow $a \to b$ over $R \to S$ filling in the diagram.

**Caution 2.3.2.** When defining and discussing prestacks, we often write $\mathcal{X}$ instead of $\mathcal{X} \to \mathcal{S}$, but when necessary, we denote the projection by $p_{\mathcal{X}} \colon \mathcal{X} \to \mathcal{S}$. We do not usually spell out the definition of the functor $\mathcal{X} \to \mathcal{S}$ as it should be clear to the reader. Moreover, when defining a prestack $\mathcal{X}$, we often only define the objects and morphisms in $\mathcal{X}$ and leave the composition law to the reader.

**Remark 2.3.3.** Axiom (2) above implies that the pullback in Axiom (1) is unique up to unique isomorphism. We often write $f^*b$ or simply $b|_S$ to indicate a *choice* of a pullback.

**Definition 2.3.4** (Fiber categories)**.** If $\mathcal{X}$ is a prestack over $\mathcal{S}$, the *fiber category* $\mathcal{X}(S)$ over $S \in \mathcal{S}$ is the category of objects in $\mathcal{X}$ over $S$ with morphisms over $\mathrm{id}_S$.

**Exercise 2.3.5.** Show that the fiber category $\mathcal{X}(S)$ is a groupoid.

**Caution 2.3.6.** Our terminology is not standard. Prestacks are usually referred to as *categories fibered in groupoids*. In the literature (c.f. [FGI⁺05, Part 1], [Ols16]) a prestack is sometimes defined as a category fibered in groupoids together with Axiom 2.4.1(1) of a stack.

It is also standard to call a morphism $b \to c$ in $\mathcal{X}$ *cartesian* if it satisfies the universal property in Axiom 2.4.1(2) and $p \colon \mathcal{X} \to \mathcal{S}$ a *fibered category* if for every diagram as in Axiom 2.4.1(1), there exists a cartesian morphism $a \to b$ over $S \to T$. With this terminology, a prestack (as we've defined it) is a fibered category where every arrow is cartesian, or equivalently where every fiber category $\mathcal{X}(S)$ is a groupoid.

### 2.3.2 Examples

**Example 2.3.7** (Presheaves are prestacks)**.** If $F \colon \mathcal{S} \to \mathrm{Sets}$ is a presheaf, we can construct a prestack $\mathcal{X}_F$ as the category of pairs $(a, S)$ where $S \in \mathcal{S}$ and $a \in F(S)$. A map $(a', S') \to (a, S)$ is a map $f \colon S' \to S$ such that $a' = f^*a$, where $f^*$ is convenient shorthand for $F(f) \colon F(S) \to F(S')$. Observe that the fiber categories $\mathcal{X}_F(S)$ are equivalent (even equal) to the set $F(S)$. We will often abuse notation by conflating $F$ and $\mathcal{X}_F$.

**Example 2.3.8** (Schemes are prestacks)**.** For a scheme $X$, applying the previous example to the functor $\mathrm{Mor}(-, X) \colon \mathrm{Sch} \to \mathrm{Sets}$ yields a prestack $\mathcal{X}_X$. This allows us to view a scheme $X$ as the prestack $\mathcal{X}_X$ where an object over a scheme $T$ is a morphism $T \to X$ of schemes. We often abuse notation by referring to $\mathcal{X}_X$ as $X$. More generally, if $\mathcal{S}$ is any category and $X \in \mathcal{S}$ is an object, we may view $X$ as a prestack over $\mathcal{S}$ where an object over $T \in \mathcal{S}$ is a morphism $T \to SX$.

**Example 2.3.9** (Prestack of smooth curves)**.** We define the prestack $\mathcal{M}$ over Sch as the category of families of smooth curves $\mathcal{C} \to S$, i.e. smooth and proper morphisms

$\mathcal{C} \to S$ (of finite presentation) of schemes such that every geometric fiber is a connected curve. A map $(\mathcal{C}' \to S') \to (\mathcal{C} \to S)$ is the data of maps $\alpha\colon \mathcal{C}' \to \mathcal{C}$ and $f\colon S' \to S$ such that the diagram

$$\begin{tikzcd} \mathcal{C}' \arrow[r, "\alpha"] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{C} \arrow[d] \\ S' \arrow[r, "f"] & S \end{tikzcd}$$

is cartesian.

The prestack $\mathcal{M}_g$ is defined as the full subcategory of $\mathcal{M}$ consisting of families of smooth curves $\mathcal{C} \to S$ where every geometric fiber has genus $g$. Note that the fiber category $\mathcal{M}_g(\Bbbk)$ over a field $\Bbbk$ is the groupoid of smooth, connected, and projective curves $C$ over $\Bbbk$ of genus $g$ such that $\mathrm{Mor}_{\mathcal{M}_g(\Bbbk)}(C, C') = \mathrm{Isom}_{\mathrm{Sch}/\Bbbk}(C, C')$.

**Exercise 2.3.10.** Verify that $\mathcal{M}$ and $\mathcal{M}_g$ are prestacks.

**Example 2.3.11** (Prestack of coherent sheaves and vector bundles)**.** Let $C$ be a fixed smooth, connected, and projective curve over an algebraically closed field $\Bbbk$. We define the prestack $\underline{\mathrm{Coh}}(C)$ over $\mathrm{Sch}/\Bbbk$ where objects are pairs $(E, S)$ where $S$ is a scheme over $\Bbbk$ and $E$ is a coherent sheaf on $C_S = C \times_\Bbbk S$ flat over $S$. A morphism $(E', S') \to (E, S)$ consists of a map of schemes $f\colon S' \to S$ together with a map $E \to (\mathrm{id} \times f)_* E'$ of $\mathcal{O}_{C_S}$-modules whose adjoint is an isomorphism (i.e. for every *choice* of pullback $(\mathrm{id} \times f)^* E$, the adjoint map $(\mathrm{id} \times f)^* E \to E'$ is an isomorphism).

The substack $\mathrm{Bun}(C) \subset \underline{\mathrm{Coh}}(C)$ is the full subcategory consisting of pairs $(E, S)$ such that $E$ is a vector bundle (i.e. locally free sheaf of finite rank). For integers $r \geq 0$ and $d$, the full subcategories $\underline{\mathrm{Coh}}_{r,d}(C) \subset \underline{\mathrm{Coh}}(C)$ (resp. $\mathrm{Bun}_{r,d}(C) \subset \mathrm{Bun}(C)$) are defined to contain only coherent sheaves (resp. vector bundles) of rank $r$ and degree $d$.

**Exercise 2.3.12.** Verify that $\underline{\mathrm{Coh}}(C)$, $\mathrm{Bun}(C)$, $\underline{\mathrm{Coh}}_{r,d}(C)$ and $\mathrm{Bun}_{r,d}(C)$ are prestacks.

Let $G \to S$ be a smooth affine group scheme.[1] A *principal $G$-bundle over an $S$-scheme $T$* is a morphism $P \to T$ with an action of $G$ on $P$ via $\sigma\colon G \times_S P \to P$ such that $P \to T$ is a $G$-invariant fppf morphism and

$$(\sigma, p_2)\colon G \times_S P \to P \times_T P, \qquad (g, p) \mapsto (gp, p)$$

is an isomorphism; in other words $G$ acts freely and transitively on $P$. Equivalently, $P \to T$ is a principal $G$-bundle if there is an étale cover $T' \to T$ such that $P \times_T T'$ is $G$-equivariant isomorphic to the trivial principal $G$-bundle $G \times_S T'$ (Proposition C.2.4). See §C.2 for background on and examples of principal bundles.

We now define classifying and quotient stacks, concepts motivated in §0.6.5.

**Definition 2.3.13** (Classifying stack)**.** We define the *classifying stack* $\mathbf{B}G$ *of* $G$ as the prestack over $\mathrm{Sch}/S$ where objects are principal $G$-bundles $P \to T$ and a morphism $(P' \to T') \to (P \to T)$ is the data of a $G$-equivariant morphism $P' \to P$ such that

$$\begin{tikzcd} P' \arrow[r] \arrow[d] & P \arrow[d] \\ T' \arrow[r] & T \end{tikzcd}$$

[1] In §6.2.2, we will define principal $G$-bundles, classifying stacks, and quotient stacks more generally for fppf group schemes.

is cartesian. We show in Example 2.4.8 that $\mathbf{B}G$ is a stack over $(\mathrm{Sch}/S)_{\text{ét}}$, which justifies called $\mathbf{B}G$ the 'classifying stack'.

**Definition 2.3.14** (Quotient prestacks and stacks)**.** Let $G \to S$ be a smooth affine group scheme acting on a scheme $U$ over $S$. We define the *quotient prestack* $[U/G]^{\mathrm{pre}}$ as the category over $\mathrm{Sch}/S$ where the fiber category over an $S$-scheme $T$ is the quotient groupoid $[U(T)/G(T)]$ whose objects are elements $u \in U(T)$. A morphism from $u' \in U(T')$ to $u \in U(T)$ is the data of a map $f\colon T' \to T$ and an element $g \in G(T')$ such that $u' = \gamma \cdot (u \circ f)$.

We define the *quotient stack* $[U/G]$ as the prestack over $\mathrm{Sch}/S$ whose objects over an $S$-scheme $T$ are diagrams

$$\begin{tikzcd} P \arrow[r] \arrow[d] & U \\ T & \end{tikzcd}$$

where $P \to T$ is a principal $G$-bundle and $P \to U$ is a $G$-equivariant morphism of schemes. A morphism $(P' \to T', P' \to U) \to (P \to T, P \to U)$ consists of a morphism $T' \to T$ and a $G$-equivariant morphism $P' \to P$ of schemes such that the diagram

$$\begin{tikzcd} P' \arrow[r] \arrow[d] \arrow[rr, bend left] \arrow[dr, phantom, "\square"] & P \arrow[r] \arrow[d] & U \\ T' \arrow[r] & T & \end{tikzcd}$$

is commutative and the left square is cartesian.

We show later that $[U/G]$ is a stack (Example 2.4.9) which is identified as the stackification of $[U/G]^{\mathrm{pre}}$ (Exercise 2.4.16).

**Exercise 2.3.15.** Verify that $[U/G]^{\mathrm{pre}}$ and $[U/G]$ are prestacks over $\mathrm{Sch}/S$.

### 2.3.3 Morphisms of prestacks

**Definition 2.3.16.**

(1) A *morphism of prestacks* $f\colon \mathcal{X} \to \mathcal{Y}$ is a functor $f\colon \mathcal{X} \to \mathcal{Y}$ such that the diagram

$$\begin{tikzcd} \mathcal{X} \arrow[rr, "f"] \arrow[dr, "p_{\mathcal{X}}"'] & & \mathcal{Y} \arrow[dl, "p_{\mathcal{Y}}"] \\ & \mathcal{S} & \end{tikzcd}$$

strictly commutes, i.e. for every object $a \in \mathrm{Ob}(\mathcal{X})$, there is an *equality* $p_{\mathcal{X}}(a) = p_{\mathcal{Y}}(f(a))$ of objects in $\mathcal{S}$.

(2) If $f, g\colon \mathcal{X} \to \mathcal{Y}$ are morphisms of prestacks, a *2-morphism* (or *2-isomorphism*) $\alpha\colon f \to g$ is a natural transformation $\alpha\colon f \to g$ such that for every object $a \in \mathcal{X}$, the morphism $\alpha_a\colon f(a) \to g(a)$ in $\mathcal{Y}$ (which is an isomorphism) is over the identity in $\mathcal{S}$. We often describe the 2-morphism $\alpha$ schematically as

$$\begin{tikzcd} \mathcal{X} \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathcal{Y}. \end{tikzcd}$$

(3) We define the category $\mathrm{MOR}(\mathcal{X}, \mathcal{Y})$ whose objects are morphisms of prestacks and whose morphisms are 2-morphisms.

(4) A *2-commutative diagram* (which we often call simply a *commutative diagram*) is a diagram

$$\begin{tikzcd} \mathcal{X}' \arrow[r, "f'"] \arrow[d, "g'"'] \arrow[dr, phantom, "\swarrow\alpha"] & \mathcal{Y}' \arrow[d, "g"] \\ \mathcal{X} \arrow[r, "f"] & \mathcal{Y} \end{tikzcd}$$

together with a 2-isomorphism $\alpha\colon g \circ f' \xrightarrow{\sim} f \circ g'$.

(5) A morphism $f\colon \mathcal{X} \to \mathcal{Y}$ of prestacks is a *monomorphism* (resp. *epimorphism*) if $f$ is fully faithful (resp. essentially surjective), and $f$ is an *isomorphism* if there exists a morphism $g\colon \mathcal{Y} \to \mathcal{X}$ of prestacks and 2-isomorphisms $g \circ f \xrightarrow{\sim} \mathrm{id}_{\mathcal{X}}$ and $f \circ g \xrightarrow{\sim} \mathrm{id}_{\mathcal{Y}}$.

**Exercise 2.3.17.** Show that every 2-morphism is an isomorphism of functors, or in other words that $\mathrm{MOR}(\mathcal{X}, \mathcal{Y})$ is a groupoid.

**Exercise 2.3.18.** Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a morphism of prestacks over a category $\mathcal{S}$.

(a) Show that $f$ is a monomorphism if and only if $f_S\colon \mathcal{X}(S) \to \mathcal{Y}(S)$ is fully faithful for every $S \in \mathcal{S}$.

(b) Show that $f$ is an isomorphism if and only if $f$ is fully faithful and essentially surjective.

A prestack $\mathcal{X}$ is *equivalent to a presheaf* if there is a presheaf $F$ and an isomorphism between $\mathcal{X}$ and the stack $\mathcal{X}_F$ corresponding to $F$ (see Example 2.3.7).

**Exercise 2.3.19.** Show that $G$ acts freely on $U$ (i.e. the action map $(\sigma, p_2)\colon G \times_S U \to U \times_S U$ is a monomorphism) if and only if $[U/G]^{\mathrm{pre}}$ (resp. $[U/G]$) is equivalent to a presheaf. We often denote these presheaves by $(U/G)^{\mathrm{pre}}$ and $U/G$.

### 2.3.4 The 2-Yoneda lemma

Yoneda's lemma (Lemma 0.3.6) states that for a presheaf $F\colon \mathcal{S} \to \mathrm{Sets}$ on a category $\mathcal{S}$ and an object $S \in \mathcal{S}$, there is a bijection $\mathrm{Mor}(S, F) \xrightarrow{\sim} F(S)$. In particular, there is a fully faithful embedding $\mathcal{S} \to \mathrm{Pre}(\mathcal{S})$, from $\mathcal{S}$ into the category of presheaves on $\mathcal{S}$, given by $S \mapsto \mathrm{Mor}(-, S)$. We will need an analogue of Yoneda's lemma for prestacks. First we recall that an object $S \in \mathcal{S}$ can be viewed as a prestack over $\mathcal{S}$, which we also denote by $S$, whose objects over $T \in \mathcal{S}$ are morphisms $T \to S$ and a morphism $(T \to S) \to (T' \to S)$ is an $S$-morphism $T \to T'$.

**Lemma 2.3.20** (The 2-Yoneda Lemma)**.** *Let $\mathcal{X}$ be a prestack over a category $\mathcal{S}$ and $S \in \mathcal{S}$. The functor*

$$\mathrm{MOR}(S, \mathcal{X}) \to \mathcal{X}(S), \qquad f \mapsto f_S(\mathrm{id}_S)$$

*is an equivalence of categories.*

*Proof.* We will construct a quasi-inverse $\Psi\colon \mathcal{X}(S) \to \mathrm{MOR}(S, \mathcal{X})$ as follows.

*On objects:* For $a \in \mathcal{X}(S)$, we define $\Psi(a)\colon S \to \mathcal{X}$ as the morphism of prestacks sending an object $(T \xrightarrow{f} S)$ (of the prestack corresponding to $S$) over $T$ to a *choice* of pullback $f^*a \in \mathcal{X}(T)$ and a morphism $(T' \xrightarrow{f'} S) \to (T \xrightarrow{f} S)$ given by an $S$-morphism $g\colon T' \to T$ to the morphism $f'^*a \to f^*a$ uniquely filling in the diagram

$$\begin{tikzcd} f'^*a \arrow[r, dashed] \arrow[d] \arrow[rr, bend left] & f^*a \arrow[r] \arrow[d, mapsto] & a \arrow[d, mapsto] \\ T' \arrow[r, "g"] & T \arrow[r, "f"] & S, \end{tikzcd}$$

using Axiom (2) of a prestack.

*On morphisms:* If $\alpha\colon a' \to a$ is a morphism in $\mathcal{X}(S)$, then $\Psi(\alpha)\colon \Psi(a') \to \Psi(a)$ is defined as the morphism of functors which maps a morphism $T \xrightarrow{f} S$ (i.e. an object in $S$ over $T$) to the unique morphism $f^*a' \to f^*a$ filling in the diagram

$$\begin{tikzcd} f^*a' \arrow[r, dashed] \arrow[d] & f^*a \arrow[d] \\ a' \arrow[r, "\alpha"] & a \end{tikzcd} \quad \text{over} \quad \begin{tikzcd} T \arrow[d, "f"] \\ S \end{tikzcd}$$

using again Axiom (2) of a prestack.

We leave the verification that $\Psi$ is a quasi-inverse to the reader. □

We will use the 2-Yoneda lemma, often without mention, throughout these notes in passing between morphisms $S \to \mathcal{X}$ and objects of $\mathcal{X}$ over $S$.

**Example 2.3.21** (Quotient stack presentations)**.** Consider the prestack $[U/G]$ in Definition 2.3.14 arising from a group action $\sigma\colon G \times_S U \to U$. The object of $[U/G]$ over $U$ given by the diagram

$$\begin{tikzcd} G \times_S U \arrow[r, "\sigma"] \arrow[d, "p_2"] & U \\ U \end{tikzcd}$$

corresponds via the 2-Yoneda Lemma (2.3.20) to a morphism $U \to [U/G]$.

**Exercise 2.3.22.**

(a) Show that there is a morphism $p\colon U \to [U/G]^{\mathrm{pre}}$ and a 2-commutative diagram

$$\begin{tikzcd} G \times_S U \arrow[r, "\sigma"] \arrow[d, "p_2"] \arrow[dr, phantom, "\not\Downarrow_\alpha"] & U \arrow[d, "p"] \\ U \arrow[r, "p"] & {[U/G]^{\mathrm{pre}}} \end{tikzcd}$$

(b) Show that $U \to [U/G]^{\mathrm{pre}}$ is a categorical quotient among prestacks, i.e. for every 2-commutative diagram

$$\begin{tikzcd} G \times_S U \arrow[r, "\sigma"] \arrow[d, "p_2"'] \arrow[dr, phantom, "\not\Downarrow_\alpha"] & U \arrow[d, "p"] \arrow[ddr, bend left, "\varphi"] & \\ U \arrow[r, "p"] \arrow[drr, bend right, "\varphi"'] & {[U/G]^{\mathrm{pre}}} \arrow[dr, phantom, "\not\Downarrow_\tau"] & \\ & & \mathcal{Z} \end{tikzcd}$$

of prestacks, there exists a morphism $\chi\colon [U/G]^{\mathrm{pre}} \to \mathcal{Z}$ and a 2-isomorphism $\beta\colon \varphi \xrightarrow{\sim} \chi \circ p$ which is compatible with $\alpha$ and $\tau$ (i.e. the two natural transformations $\varphi \circ \sigma \xrightarrow{\beta \circ \sigma} \chi \circ p \circ \sigma \xrightarrow{\chi \circ \alpha} \chi \circ p \circ p_2$ and $\varphi \circ \sigma \xrightarrow{\tau} \varphi \circ p_2 \xrightarrow{\beta \circ p_2} \chi \circ p \circ p_2$ agree.

### 2.3.5 Fiber products

We discuss fiber products for prestacks and in particular prove their existence. Recall that for morphisms $X \to Y$ and $Y' \to Y$ of presheaves on a category $\mathcal{S}$, the fiber product can be constructed as the presheaf mapping an object $S \in \mathcal{S}$ to the fiber product $X(S) \times_{Y(S)} Y'(S)$ of sets. Essentially the same construction works for morphisms $\mathcal{X} \to \mathcal{Y}$ and $\mathcal{Y}' \to \mathcal{Y}$ of prestacks but since we are dealing with groupoids rather than sets, the fiber category over an object $S \in \mathcal{S}$ should be the fiber product $\mathcal{X}(S) \times_{\mathcal{Y}(S)} \mathcal{Y}'(S)$ of groupoids. The reader may first want to work on Exercises 2.3.26 and 2.3.28 on fiber products of groupoids as a warmup to fiber products of prestacks.

**Construction 2.3.23.** Let $f\colon \mathcal{X} \to \mathcal{Y}$ and $g\colon \mathcal{Y}' \to \mathcal{Y}$ be morphisms of prestacks over a category $\mathcal{S}$. Define the prestack $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}'$ over $\mathcal{S}$ as the category of triples $(x, y', \gamma)$ where $x \in \mathcal{X}$ and $y' \in \mathcal{Y}'$ are objects over the *same* object $S := p_{\mathcal{X}}(x) = p_{\mathcal{Y}'}(y') \in \mathcal{S}$, and $\gamma\colon f(x) \xrightarrow{\sim} g(y')$ is an isomorphism in $\mathcal{Y}(S)$. A morphism $(x_1, y'_1, \gamma_1) \to (x_2, y'_2, \gamma_2)$ consists of a triple $(f, \chi, \gamma')$ where $f\colon p_{\mathcal{X}}(x_1) = p_{\mathcal{Y}'}(y'_1) \to p_{\mathcal{Y}'}(y'_2) = p_{\mathcal{X}}(x_2)$ is a morphism in $\mathcal{S}$, and $\chi\colon x_1 \xrightarrow{\sim} x_2$ and $\gamma'\colon y'_1 \xrightarrow{\sim} y'_2$ are morphisms in $\mathcal{X}$ and $\mathcal{Y}'$ over $f$ such that

$$\begin{tikzcd} f(x_1) \arrow[r, "f(\chi)"] \arrow[d, "\gamma_1"] & f(x_2) \arrow[d, "\gamma_2"] \\ g(y'_1) \arrow[r, "g(\gamma')"] & g(y'_2) \end{tikzcd}$$

commutes.

Let $p_1\colon \mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}' \to \mathcal{X}$ and $p_2\colon \mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}' \to \mathcal{X}$ denote the projections $(x, y', \gamma) \mapsto x$ and $(x, y', \gamma) \mapsto y'$. There is a 2-isomorphism $\alpha\colon f \circ p_1 \xrightarrow{\sim} g \circ p_2$ defined on an object $(x, y', \gamma) \in \mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}'$ by $\alpha_{(x,y',\gamma)}\colon f(x) \xrightarrow{\gamma} g(y')$. This yields a 2-commutative diagram

$$\begin{tikzcd} \mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}' \arrow[r, "p_2"] \arrow[d, "p_1"] \arrow[dr, phantom, "\alpha \nearrow"] & \mathcal{Y}' \arrow[d, "g"] \\ \mathcal{X} \arrow[r, "f"] & \mathcal{Y} \end{tikzcd} \tag{2.3.1}$$

**Theorem 2.3.24.** *The prestack $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}'$ together with the morphisms $p_1$ and $p_2$ and the 2-isomorphism $\alpha$ as in (2.3.1) satisfy the following universal property: for every 2-commutative diagram*

$$
\begin{tikzcd}
 & & \mathcal{Y}' \arrow[dr, "g"] & \\
\mathcal{T} \arrow[urr, bend left, "q_2"] \arrow[drr, bend right, "q_1"'] \arrow[r, phantom, "\tau\Uparrow"] & \mathcal{X}\times_{\mathcal{Y}}\mathcal{Y}' \arrow[ur, "p_2"] \arrow[dr, "p_1"'] \arrow[rr, phantom, "\alpha\Uparrow"] & & \mathcal{Y} \\
 & & \mathcal{X} \arrow[ur, "f"'] &
\end{tikzcd}
$$

*with 2-isomorphism $\tau\colon f\circ q_1 \xrightarrow{\sim} g\circ q_2$, there exists a morphism $h\colon \mathcal{T}\to\mathcal{X}\times_{\mathcal{Y}}\mathcal{Y}'$ and 2-isomorphisms $\beta\colon q_1\to p_1\circ h$ and $\gamma\colon q_2\to p_2\circ h$ yielding a 2-commutative diagram*

$$
\begin{tikzcd}
 & & \mathcal{Y}' \arrow[dr, "g"] & \\
\mathcal{T} \arrow[urr, bend left, "q_2"] \arrow[drr, bend right, "q_1"'] \arrow[r, "h"] \arrow[ur, phantom, "\Downarrow\gamma"] \arrow[dr, phantom, "\Uparrow\beta"] & \mathcal{X}\times_{\mathcal{Y}}\mathcal{Y}' \arrow[ur, "p_2"] \arrow[dr, "p_1"'] \arrow[rr, phantom, "\alpha\Uparrow"] & & \mathcal{Y} \\
 & & \mathcal{X} \arrow[ur, "f"'] &
\end{tikzcd}
$$

*such that*

$$
\begin{tikzcd}
f\circ q_1 \arrow[r, "f(\beta)"] \arrow[d, "\tau"] & f\circ p_1\circ h \arrow[d, "\alpha\circ h"] \\
g\circ q_2 \arrow[r, "g(\gamma)"] & g\circ p_2\circ h
\end{tikzcd}
$$

*commutes. The data $(h,\beta,\gamma)$ is unique up to unique isomorphism.*

*Proof.* We define $h\colon \mathcal{T}\to\mathcal{X}\times_{\mathcal{Y}}\mathcal{Y}'$ on objects by $t\mapsto \big(q_1(t), q_2(t), f(q_1(t)) \xrightarrow{\tau_t} g(q_2(t))\big)$ and on morphisms as $(t\xrightarrow{\Psi} t')\mapsto (p_{\mathcal{T}}(\Psi), q_1(\Psi), q_2(\Psi))$. There are *equalities* of functors $q_1 = p_1\circ h$ and $q_2 = p_2\circ h$ so we define $\beta$ and $\gamma$ as the identity natural transformation. The remaining details are left to the reader. $\square$

**Definition 2.3.25.** We say that a 2-commutative diagram

$$
\begin{tikzcd}
\mathcal{X}' \arrow[r] \arrow[d] \arrow[dr, phantom, "\swarrow_{\alpha}"] & \mathcal{Y}' \arrow[d] \\
\mathcal{X} \arrow[r] & \mathcal{Y}
\end{tikzcd}
$$

is *cartesian* if it satisfies the universal property of Theorem 2.3.24. We often write a cartesian diagram of stacks as

$$
\begin{tikzcd}
\mathcal{X}' \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{Y}' \arrow[d] \\
\mathcal{X} \arrow[r] & \mathcal{Y}
\end{tikzcd}
$$

where the existence of the 2-isomorphism $\alpha$ is implicit.

### 2.3.6 Examples of fiber products

This section asks the reader to verify several convenient fiber product diagrams. To get started, it is instructive to first compute fiber products of groupoids (rather than prestacks).

**Exercise 2.3.26.**

(a) If $\mathcal{C} \xrightarrow{f} \mathcal{D}$ and $\mathcal{D}' \xrightarrow{g} \mathcal{D}$ are morphisms of groupoids, define the groupoid $\mathcal{C} \times_{\mathcal{D}} \mathcal{D}'$ whose objects are triples $(c, d', \delta)$ where $c \in \mathcal{C}$ and $d' \in \mathcal{D}'$ are objects, and $\delta \colon f(c) \xrightarrow{\sim} g(d')$ is an isomorphism in $\mathcal{D}$. A morphism $(c_1, d'_1, \delta_1) \to (c_2, d'_2, \delta_2)$ is the data of morphisms $\gamma \colon c_1 \xrightarrow{\sim} c_2$ and $\delta' \colon d'_1 \xrightarrow{\sim} d'_2$ such that

$$\begin{tikzcd}
f(c_1) \arrow[r, "f(\gamma)"] \arrow[d, "\delta_1"] & f(c_2) \arrow[d, "\delta_2"] \\
g(d'_1) \arrow[r, "g(\delta')"] & g(d'_2)
\end{tikzcd}$$

commutes. Formulate a university property for fiber products of groupoids and show that $\mathcal{C} \times_{\mathcal{D}} \mathcal{D}'$ satisfies it.

(b) If $f \colon \mathcal{X} \to \mathcal{Y}$ and $g \colon \mathcal{Y}' \to \mathcal{Y}$ are morphisms of prestacks over a category $\mathcal{S}$, show that for every $S \in \mathcal{S}$, the fiber category $(\mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}')(S)$ is a fiber product $\mathcal{X}(S) \times_{\mathcal{Y}(S)} \mathcal{Y}'(S)$ of groupoids.

The following foreshadows analogous cartesian diagrams associated to quotient stacks.

**Exercise 2.3.27.**

(a) Let $G$ be a group acting on a set $U$ via $\sigma \colon G \times U \to U$. Let $[U/G]$ denote the quotient groupoid (Exercise 0.4.7) with projection $p \colon U \to [U/G]$. Show that there are cartesian diagrams

$$\begin{tikzcd}
G \times U \arrow[r, "\sigma"] \arrow[d, "p_2"] \arrow[dr, phantom, "\square"] & U \arrow[d, "p"] \\
U \arrow[r, "p"] & {[U/G]}
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
G \times U \arrow[r, "{(\sigma, p_2)}"] \arrow[d] \arrow[dr, phantom, "\square"] & U \times U \arrow[d, "p \times p"] \\
{[U/G]} \arrow[r, "\Delta"] & {[U/G] \times [U/G].}
\end{tikzcd}$$

(b) Recall from Example 0.4.3 that the classifying groupoid $BG$ of a group $G$ is the category with one object $*$ with $\mathrm{Mor}(*, *) = G$. If $x \in U$, show that there is a morphism $BG_x \to [U/G]$ of groupoids and a cartesian diagram

$$\begin{tikzcd}
Gx \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & U \arrow[d, "p"] \\
BG_x \arrow[r] & {[U/G].}
\end{tikzcd}$$

(See Proposition 3.5.16 and Remark 3.5.19 for the analogous diagrams for algebraic stacks.)

(c) Let $\phi \colon H \to G$ be a homomorphism of groups. Show that there is an induced morphism $BH \to BG$ of groupoids and that $BH \times_{BG} \mathrm{pt} \cong [G/H]$.

(d) If $K \triangleleft G$ is a normal subgroup with quotient $Q = G/K$, show that there is a cartesian diagram

$$\begin{tikzcd} Q \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & BK \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathrm{pt} \arrow[d] \\ \mathrm{pt} \arrow[r] & BG \arrow[r] & BQ. \end{tikzcd}$$

The following exercise is essential for working with quotient stacks and in particular is used to verify the algebraicity of quotient stacks (Theorem 3.1.10).

**Exercise 2.3.28.**

(a) Let $G \to S$ be a smooth affine group scheme acting on a scheme $U$ over $S$ via $\sigma\colon G \times_S U \to U$. Let $[U/G]$ be the quotient stack (Definition 2.3.14). Show that there are cartesian diagrams

$$\begin{tikzcd} G \times_S U \arrow[r, "\sigma"] \arrow[d, "p_2"] \arrow[dr, phantom, "\square"] & U \arrow[d, "p"] \\ U \arrow[r, "p"] & {[U/G]} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} G \times_S U \arrow[r, "(\sigma, p_2)"] \arrow[d] \arrow[dr, phantom, "\square"] & U \times_S U \arrow[d, "p \times p"] \\ {[U/G]} \arrow[r, "\Delta"] & {[U/G] \times_S [U/G].} \end{tikzcd}$$

(b) Show that if $P \to T$ is a principal $G$-bundle and $P \to U$ is a $G$-equivariant map, there is a morphism $T \to [U/G]$, unique up to unique isomorphism, and a cartesian diagram

$$\begin{tikzcd} P \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & U \arrow[d] \\ T \arrow[r] & {[U/G].} \end{tikzcd}$$

(We will later see that $[U/G]$ is an algebraic stack and that $U \to [U/G]$ is principal $G$-bundle (Theorem 3.1.10). In particular, the principal $G$-bundle $U \to [U/G]$ with the identity map $U \to U$ is the universal family over $[U/G]$; this corresponds via the 2-Yoneda Lemma (2.3.20) to the identity map $[U/G] \to [U/G]$.)

As with schemes, the following diagram is utilized extensively. As we will see in §3.2.2, the diagonal is used to define stabilizers and the inertia stack.

**Exercise 2.3.29** (Magic Square)**.** Let $\mathcal{X}$ be a prestack over a category $\mathcal{S}$. Let $S, T \in \mathcal{S}$ be objects which we can view as prestacks over $\mathcal{S}$ via Example 2.3.8. Show that for all morphisms $a\colon S \to \mathcal{X}$ and $b\colon T \to \mathcal{X}$, there is a cartesian diagram

$$\begin{tikzcd} S \times_{\mathcal{X}} T \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & S \times T \arrow[d, "a \times b"] \\ \mathcal{X} \arrow[r, "\Delta"] & \mathcal{X} \times \mathcal{X}. \end{tikzcd}$$

Properties of the diagonal are used to define separation conditions on algebraic stacks (see §3.3.3) and translate into properties of Isom presheaves.

**Exercise 2.3.30** (Isom presheaves)**.** Let $\mathcal{X}$ be a prestack over a category $\mathcal{S}$.

(a) For $S \in \mathcal{S}$, recall from Example 2.1.9 that the localized category $\mathcal{S}/S$ denotes the category whose objects are morphisms $T \to S$ in $\mathcal{S}$ and whose morphisms are $S$-morphisms. Show that for objects $a$ and $b$ of $\mathcal{X}$ over $S$ that the functor

$$\begin{aligned} \underline{\mathrm{Isom}}_{\mathcal{X}(S)}(a,b)\colon \mathcal{S}/S &\to \mathrm{Sets} \\ (T \xrightarrow{f} S) &\mapsto \mathrm{Mor}_{\mathcal{X}(T)}(f^*a, f^*b), \end{aligned}$$

where $f^*a$ and $f^*b$ are choices of a pullback, defines a presheaf on $\mathcal{S}/S$.

(b) Show that there is a cartesian diagram

$$\begin{tikzcd} \underline{\mathrm{Isom}}_{\mathcal{X}(S)}(a,b) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & S \arrow[d, "{(a,b)}"] \\ \mathcal{X} \arrow[r, "\Delta"] & \mathcal{X} \times \mathcal{X}. \end{tikzcd}$$

(c) Show that the presheaf $\underline{\mathrm{Aut}}_{\mathcal{X}(T)}(a) = \underline{\mathrm{Isom}}_{\mathcal{X}(T)}(a,a)$ is naturally a presheaf in groups.

(d) Show that $\mathcal{X}$ is equivalent to a sheaf if and only if the diagonal $\mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is a monomorphism.

**Exercise 2.3.31.** If $n \geq 2$, show that $[\mathbb{A}^n/\mathbb{G}_m^n] \cong \underbrace{[\mathbb{A}^1/\mathbb{G}_m] \times \cdots \times [\mathbb{A}^1/\mathbb{G}_m]}_{n \text{ times}}$.

**Exercise 2.3.32.**

(a) Show that if $H \to G$ is a morphism of smooth affine group schemes over a scheme $S$, there is an induced morphism of prestacks $\mathbf{B}H \to \mathbf{B}G$ over $\mathrm{Sch}/S$.

(b) Show that $\mathbf{B}H \times_{\mathbf{B}G} S \cong [G/H]$.

(c) If $1 \to K \to G \to Q \to 1$ is an exact sequence of smooth affine algebraic groups over a field $\Bbbk$, show that there is a cartesian diagram

$$\begin{tikzcd} Q \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathbf{B}K \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \operatorname{Spec} \Bbbk \arrow[d] \\ \operatorname{Spec} \Bbbk \arrow[r] & \mathbf{B}G \arrow[r] & \mathbf{B}Q. \end{tikzcd}$$

## 2.4 Stacks

A stack over a site $\mathcal{S}$ is a prestack $\mathcal{X}$ such that objects and morphisms glue uniquely in the Grothendieck topology of $\mathcal{S}$ (see Definition 2.4.1). Verifying a given prestack is a stack reduces to a *descent* condition on objects and morphisms with respect to the covers of $\mathcal{S}$. The theory of descent is discussed in Section B.1 and is essential for verifying the stack axioms.

### 2.4.1 Definition of a stack

**Definition 2.4.1** (Stacks)**.** A prestack $\mathcal{X}$ over a site $\mathcal{S}$ is a *stack* if the following conditions hold for all coverings $\{S_i \to S\}$ of an object $S \in \mathcal{S}$:

(1) (morphisms glue) For objects $a$ and $b$ in $\mathcal{X}$ over $S$ and morphisms $\phi_i\colon a|_{S_i} \to b$ such that $\phi_i|_{S_{ij}} = \phi_j|_{S_{ij}}$ as displayed in the diagram

$$
\begin{tikzcd}
 & a|_{S_i} \arrow[dr] \arrow[drr, bend left, "\phi_i"] & & & & & S_i \arrow[dr] & \\
a|_{S_{ij}} \arrow[ur] \arrow[dr] & & a \arrow[r, dashed] & b & \text{over} & S_{ij} \arrow[ur] \arrow[dr] & & S, \\
 & a|_{S_j} \arrow[ur] \arrow[urr, bend right, "\phi_j"'] & & & & & S_j \arrow[ur] &
\end{tikzcd}
$$

there exists a unique morphism $\phi\colon a \to b$ with $\phi|_{S_i} = \phi_i$.

(2) (objects glue) For objects $a_i$ over $S_i$ and isomorphisms $\alpha_{ij}\colon a_i|_{S_{ij}} \to a_j|_{S_{ij}}$, as displayed in the diagram

$$
\begin{tikzcd}
 & & a_i \arrow[dr, dashed] & & & & S_i \arrow[dr] & \\
a_i|_{S_{ij}} \arrow[r, "\alpha_{ij}"] & a_j|_{S_{ij}} \arrow[ur] \arrow[dr] & & a & \text{over} & S_{ij} \arrow[ur] \arrow[dr] & & S \\
 & & a_j \arrow[ur, dashed] & & & & S_j \arrow[ur] &
\end{tikzcd}
$$

satisfying the cocycle condition $\alpha_{jk}|_{S_{ijk}} \circ \alpha_{ij}|_{S_{ijk}} = \alpha_{ik}|_{S_{ijk}}$ on $S_{ijk}$, then there exists an object $a$ over $S$ and isomorphisms $\phi_i\colon a|_{S_i} \to a_i$ such that $\alpha_{ij} \circ \phi_i|_{S_{ij}} = \phi_j|_{S_{ij}}$ on $S_{ij}$.

**Remark 2.4.2.** There is an alternative description of the stack axioms analogous to the sheaf axiom of a presheaf $F\colon \mathcal{S} \to \text{Sets}$, i.e. that $F(S) \to \prod_i F(S_i) \rightrightarrows \prod_{i,j} F(S_i \times_S S_j)$ is exact for coverings $\{S_i \to S\}$. Namely, by adding an additional layer corresponding to triple intersections, the stack axiom translates to the 'exactness' of

$$\mathcal{X}(S) \longrightarrow \prod_i \mathcal{X}(S_i) \rightrightarrows \prod_{i,j} \mathcal{X}(S_i \times_S S_j) \mathrel{\substack{\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow}} \prod_{i,j,k} \mathcal{X}(S_i \times_S S_j \times_S S_k).$$

**Exercise 2.4.3.** Show that Axiom (1) is equivalent to the condition that for all objects $a$ and $b$ of $\mathcal{X}$ over $S \in \mathcal{S}$, the Isom presheaf $\underline{\text{Isom}}_{\mathcal{X}(S)}(a, b)$ (see Exercise 2.3.30) is a sheaf on $\mathcal{S}/S$.

A *morphism* of stacks is a morphism of prestacks.

**Exercise 2.4.4** (Fiber product of stacks)**.** Show that if $\mathcal{X} \to \mathcal{Y}$ and $\mathcal{Y}' \to \mathcal{Y}$ are morphisms of stacks over a site $\mathcal{S}$, then $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}'$ is also a stack over $\mathcal{S}$.

### 2.4.2 First examples of stacks

**Example 2.4.5** (Sheaves and schemes are stacks)**.** Recall that if $F$ is a presheaf on a site $\mathcal{S}$, we can construct a prestack $\mathcal{X}_F$ over $\mathcal{S}$ as the category of pairs $(a, S)$ where $S \in \mathcal{S}$ and $a \in F(S)$ (see Example 2.3.7). If $F$ is a sheaf, then $\mathcal{X}_F$ is a stack. We often abuse notation by writing $F$ also as the stack $\mathcal{X}_F$.

Since schemes are sheaves on $\text{Sch}_{\text{ét}}$ (Proposition 2.2.6), a scheme $X$ defines a stack over $\text{Sch}_{\text{ét}}$ (where objects over a scheme $S$ are morphisms $S \to X$), which we also denote as $X$.

**Example 2.4.6** (Stack of sheaves)**.** Let Sheaves be the prestack over Sch whose objects are pairs $(T, F)$ where $T$ is a scheme and $F$ is a sheaf on the Zariski topology of $T$. A morphism $(T, F) \to (T', F')$ is the data of a morphism $f: T \to T'$ of schemes and a morphism $f_*F \to F'$ of sheaves on $T'$ such that the adjoint $F \to f^{-1}F'$ is an isomorphism. Because sheaves and their morphisms glue in the Zariski topology ([Har77, Exer. II.1.15 and 22]), $\mathcal{X}$ is a stack over the big Zariski site $\mathrm{Sch}_{\mathrm{Zar}}$. The full subcategories QCoh and Bun of Sheaves parameterizing quasi-coherent sheaves and vector bundles are also stacks over $\mathrm{Sch}_{\mathrm{Zar}}$.

**Exercise 2.4.7.**

(1) Formulate and prove a more general statement for sheaves over an arbitrary site.

(2) Use fppf descent to show that the prestack QCoh (resp. Bun), parameterizing pairs $(T, F)$ where $T$ is a scheme and $F$ is a quasi-coherent sheaf on $T$ (resp. vector bundle on $T$), is a stack over $\mathrm{Sch}_{\mathrm{fppf}}$.

**Example 2.4.8** (Classifying stacks)**.** Let $G \to S$ be a smooth affine group scheme. The classifying prestack $\mathbf{B}G$ is the category over $\mathrm{Sch}/S$ classifying principal $G$-bundles $P \to T$ (see Definition 2.3.13). We claim that $\mathbf{B}G$ is a stack over $(\mathrm{Sch}/S)_{\text{ét}}$. Axiom (1) holds as morphisms to schemes glue uniquely in the étale topology (Proposition B.2.1). For Axiom (2), if $\{T_i \to T\}$ is an étale covering, $P_i \to T_i$ are objects over $T_i$, and $\alpha_{ij}: P_i \times_{T_i} T_{ij} \to P_j \times_{T_j} T_{ij}$ satisfying the cocycle condition $\alpha_{jk} \circ \alpha_{ij} = \alpha_{ik}$ on $T_{ijk}$, then the existence of a principal $G$-bundle $P \to T$ follows from Effective Descent for Principal $G$-bundles (C.2.5).

**Example 2.4.9** (Quotient stacks)**.** Let $G \to S$ be a smooth affine group scheme acting on a scheme $U$ over $S$. Let $[U/G]$ be the prestack defined in Definition 2.3.14; an object over an $S$-scheme $T$ is a diagram

$$\begin{array}{ccc} P & \longrightarrow & U \\ \downarrow & & \\ T & & \end{array}$$

where $P \to T$ is a principal $G$-bundle and $P \to U$ is a $G$-equiariant morphism of schemes. The prestack $[U/G]$ is a stack over $(\mathrm{Sch}/S)_{\text{ét}}$, which we call the *quotient stack*. Axiom (1) holds by étale descent for morphisms of schemes (B.2.1). Axiom (2) holds because in the étale topology principal $G$-bundles glue uniquely (as seen in the previous example) and morphisms of schemes do also (B.2.1).

**Example 2.4.10** (Stack of schemes over $\mathrm{Sch}_{\mathrm{Zar}}$)**.** Define Schemes as the prestack over Sch consists of morphisms $T \to S$ of schemes where a morphism $(T \to S) \to (T' \to S')$ consists of morphisms $T \to T'$ and $S \to S'$ of schemes such that the two compositions $T \to S'$ agree. The projection map takes $T \to S$ to $S$. Since schemes glue in the Zariski topology [Har77, Exer. II.2.12], Schemes is a stack over $\mathrm{Sch}_{\mathrm{Zar}}$. However, Schemes is *not* a stack over $\mathrm{Sch}_{\text{ét}}$. Schemes can be glued to algebraic spaces in the étale topology and there is a stack of algebraic spaces over $\mathrm{Sch}_{\text{ét}}$; see Exercise 4.4.15.

### 2.4.3 Moduli stack of curves

Let $\mathcal{M}_g$ denote the prestack of families of smooth curves $\mathcal{C} \to S$ of genus $g$; see Example 2.3.9.

**Proposition 2.4.11** (Moduli stack of smooth curves). *If $g \geq 2$, then $\mathcal{M}_g$ is a stack over* $\mathrm{Sch}_{\text{ét}}$.

*Proof.* Axiom (1) translates to: for families of smooth curves $\mathcal{C} \to S$ and $\mathcal{D} \to S$ of genus $g$ and commutative diagrams

$$\begin{tikzcd}
\mathcal{C}_{S_{ij}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{C}_{S_i} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] \arrow[rr, bend left, "f_i"] & \mathcal{C} \arrow[r, dashed, "f"] \arrow[d] & \mathcal{D} \arrow[dl] \\
S_{ij} \arrow[r] & S_i \arrow[r] & S &
\end{tikzcd}$$

of solid arrows for all $i, j$ (i.e. morphisms $f_i \colon \mathcal{C}_{S_i} \to \mathcal{D}$ such that $f_i|_{\mathcal{C}_{S_{ij}}} = f_j|_{\mathcal{C}_{S_{ij}}}$), there exists a unique morphism filling in the diagram (i.e. $f_i = f|_{\mathcal{C}_{S_i}}$). The existence and uniqueness of $f$ follow from étale descent for morphisms (Proposition B.2.1). The fact that $f$ is an isomorphism also follows from étale descent (Proposition B.4.1).

Axiom (2) is more difficult: we must show that given diagrams

$$\begin{tikzcd}
\mathcal{C}_i|_{S_{ij}} \arrow[r, "\alpha_{ij}"] \arrow[dr] \arrow[rr, bend left] & \mathcal{C}_j|_{S_{ij}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{C}_j \arrow[r, dashed] \arrow[d, "\pi_j"] \arrow[dr, phantom, "\square"] & \mathcal{C} \arrow[d, dashed] \\
& S_{ij} \arrow[r] & S_j \arrow[r] & S
\end{tikzcd}$$

for all $i, j$ where $\pi_i \colon \mathcal{C}_i \to S_i$ are families of smooth curves of genus $g$ and $\alpha_{ij} \colon \mathcal{C}_i|_{S_{ij}} \to \mathcal{C}_j|_{S_{ij}}$ are isomorphisms satisfying the cocycle condition $\alpha_{jk} \circ \alpha_{ij} = \alpha_{ik}$, there is family of smooth curves $\mathcal{C} \to S$ and isomorphisms $\phi_i \colon \mathcal{C}|_{S_i} \to \mathcal{C}_i$ such that $\alpha_{ij} \circ \phi_i|_{\mathcal{C}_{S_{ij}}} = \phi_j|_{\mathcal{C}_{S_{ij}}}$.

We will use the following property of families of smooth curves (see Proposition 5.1.9): for a family of smooth curves $\pi \colon \mathcal{C} \to S$, $\omega_{\mathcal{C}/S}^{\otimes 3}$ is relatively very ample on $S$ (as $g > 2$) and $F := \pi_* \omega_{\mathcal{C}/S}^{\otimes 3}$ is a vector bundle of rank $5(g-1)$. In particular, $\omega_{\mathcal{C}/S}^{\otimes 3}$ yields a closed immersion $\mathcal{C} \hookrightarrow \mathbb{P}(F)$ over $S$.

Therefore, if we set $E_i = (\pi_i)_*(\omega_{\mathcal{C}_i/S_i})$, there is a closed immersion $\mathcal{C}_i \hookrightarrow \mathbb{P}(E_i)$ over $S_i$. The isomorphisms $\alpha_{ij}$ induce isomorphisms $\beta_{ij} \colon E_i|_{S_{ij}} \to E_j|_{S_{ij}}$ satisfying the cocycle condition $\beta_{jk} \circ \beta_{ij} = \beta_{ik}$ on $S_{ijk}$. Descent for quasi-coherent sheaves (Proposition B.1.3) implies there is a quasi-coherent sheaf $E$ on $S$ and isomorphisms $\Psi_i \colon E|_{S_i} \to E_i$ such that $\beta_{ij} \circ \Psi_i|_{S_{ij}} = \Psi_j|_{S_{ij}}$. It follows again from descent that $E$ is a vector bundle (Proposition B.4.3). Pictorially, we have

$$\begin{tikzcd}
& \mathbb{P}(E_{ij}) \arrow[r] & \mathbb{P}(E_i) \arrow[r] & \mathbb{P}(E) \\
\mathcal{C}_i|_{S_{ij}} \arrow[ur, hook] \arrow[r] \arrow[d] & \mathcal{C}_i \arrow[ur, hook] \arrow[r, dashed] \arrow[d] & \mathcal{C} \arrow[ur, hook, dashed] \arrow[d, dashed] & \\
S_{ij} \arrow[r] & S_i \arrow[r] & S. &
\end{tikzcd}$$

Since the preimages of $\mathcal{C}_i \subset \mathbb{P}(E_i)$ and $\mathcal{C}_j \subset \mathbb{P}(E_j)$ in $\mathbb{P}(E_{ij})$ are equal, it follows from descent for closed subschemes (Proposition B.3.1) that there exists $\mathcal{C} \to S$ and isomorphisms $\phi_i$ such that $\alpha_{ij} \circ \phi_i|_{\mathcal{C}_{S_{ij}}} \to \phi_j|_{\mathcal{C}_{S_{ij}}}$. Since smoothness and properness are étale-local properties on the target (Proposition B.4.1), $\mathcal{C} \to S$ is smooth and proper. The geometric fibers of $\mathcal{C} \to S$ are connected genus $g$ curves since the geometric fibers of $\mathcal{C}_i \to S_i$ are. □

**Exercise 2.4.12.**

(a) Show that the prestack $\mathcal{M}_0$ is a stack on $\mathrm{Sch}_{\text{ét}}$ isomorphic to $\mathbf{B}\,\mathrm{PGL}_2$ over $\operatorname{Spec}\mathbb{Z}$.

(b) Show that the moduli stack $\mathcal{M}_{1,1}$, whose objects are families of elliptic curves (see Example 0.3.20) is a stack on $\mathrm{Sch}_{\text{ét}}$.

(c) Is the prestack $\mathcal{M}_1$, whose objects over a scheme $S$ are smooth families $\mathcal{C} \to S$ of genus 1 curves, a stack over $\mathrm{Sch}_{\text{ét}}$?

### 2.4.4 Moduli stack of coherent sheaves and vector bundles

Let $C$ be a smooth, connected, and projective curve over an algebraically closed field $\Bbbk$, and fix integers $r \geq 0$ and $d$. Recall from Example 2.3.11 that $\underline{\mathrm{Coh}}_{r,d}(C)$ (resp. $\mathrm{Bun}_{r,d}(C)$) denotes the prestack over $\mathrm{Sch}/\Bbbk$ consisting of pairs $(E, S)$ where $S$ is a $\Bbbk$-scheme and $E$ is a coherent sheaf on $C_S$ flat over $S$ (resp. vector bundle on $C_S$) of rank $r$ and degree $d$.

**Proposition 2.4.13.** *For all integers $r$ and $d$ with $r \geq 0$, $\underline{\mathrm{Coh}}_{r,d}(C)$ and $\mathrm{Bun}_{r,d}(C)$ are stacks over $(\mathrm{Sch}/\Bbbk)_{\text{ét}}$.*

*Proof.* Axioms (1) is precisely descent for morphisms of quasi-coherent sheaves (Proposition B.1.3(2)) while Axiom (2) is descent for quasi-coherent sheaves (Proposition B.1.3(1)) coupled with the fact that the property of a quasi-coherent sheaf being a coherent sheaf (resp. vector bundle) is étale-local (Proposition B.4.3). □

### 2.4.5 Stackification

Given a presheaf $F$ on a site $\mathcal{S}$, there is a sheafification $F \to F^{\mathrm{sh}}$ which is a left adjoint to the inclusion, i.e. $\mathrm{Mor}(F^{\mathrm{sh}}, G) \to \mathrm{Mor}(F, G)$ is bijective for every sheaf $G$ on $\mathcal{S}$ (Theorem 2.2.9). Similarly, there is a stackification $\mathcal{X} \to \mathcal{X}^{\mathrm{st}}$ of a prestack $\mathcal{X}$ over $\mathcal{S}$.

**Theorem 2.4.14** (Stackification)**.** *If $\mathcal{X}$ is a prestack over a site $\mathcal{S}$, there exists a stack $\mathcal{X}^{\mathrm{st}}$, which we call the* stackification*, and a morphism $\mathcal{X} \to \mathcal{X}^{\mathrm{st}}$ of prestacks such that for every stack $\mathcal{Y}$ over $\mathcal{S}$, the induced functor*

$$\mathrm{MOR}(\mathcal{X}^{\mathrm{st}}, \mathcal{Y}) \to \mathrm{MOR}(\mathcal{X}, \mathcal{Y}) \tag{2.4.1}$$

*is an equivalence of categories.*

*Proof.* As in the construction of the sheafification (see the proof of Theorem 2.2.9), we construct the stackification in stages. Most details are left to the reader.

First, given a prestack $\mathcal{X}$, we can construct a prestack $\mathcal{X}^{\mathrm{st}_1}$ satisfying Axiom (1) and a morphism $\mathcal{X} \to \mathcal{X}^{\mathrm{st}_1}$ of prestacks such that

$$\mathrm{MOR}(\mathcal{X}^{\mathrm{st}_1}, \mathcal{Y}) \to \mathrm{MOR}(\mathcal{X}, \mathcal{Y})$$

is an equivalence for all prestacks $\mathcal{Y}$ satisfying Axiom (1). Specifically, the objects of $\mathcal{X}^{\mathrm{st}_1}$ are the same as $\mathcal{X}$, and for objects $a, b \in \mathcal{X}$ over $S, T \in \mathcal{S}$, the set of morphisms $a \to b$ in $\mathcal{X}^{\mathrm{st}_1}$ over a given morphism $f \colon S \to T$ is the global sections $\Gamma(S, \underline{\mathrm{Isom}}_{\mathcal{X}(S)}(a, f^*b)^{\mathrm{sh}})$ of the sheafification of the Isom presheaf (Exercise 2.3.30).

Second, given a prestack $\mathcal{X}$ satisfying Axiom (1), we construct a stack $\mathcal{X}$ and a morphism $\mathcal{X} \to \mathcal{X}^{\mathrm{st}}$ of prestacks such that (2.4.1) is an equivalence for all stacks $\mathcal{Y}$. An object of $\mathcal{X}^{\mathrm{st}}$ over $S \in \mathcal{S}$ is given by a triple consisting of a covering $\{S_i \to S\}$, objects $a_i$ of $\mathcal{X}$ over $S_i$, and isomorphisms $\alpha_{ij} \colon a_i|_{S_{ij}} \to a_j|_{S_{ij}}$ satisfying the cocycle condition $\alpha_{jk}|_{S_{ijk}} \circ \alpha_{ij}|_{S_{ijk}} = \alpha_{ik}|_{S_{ijk}}$ on $S_{ijk}$. Morphisms

$$\big(\{S_i \to S\}, \{a_i\}, \{\alpha_{ij}\}\big) \to \big(\{T_\mu \to T\}, \{b_\mu\}, \{\beta_{\mu\nu}\}\big)$$

in $\mathcal{X}^{\mathrm{st}}$ over $S \to T$ are defined as follows: first consider the induced cover $\{S_i \times_S T_\mu \to S\}_{i,\mu}$ and choose pullbacks $a_i|_{S_i \times_S T_\mu}$ and $b_\mu|_{S_i \times_S T_\mu}$. A morphism is then the data of maps $\Psi_{i\mu} \colon a_i|_{S_i \times_S T_\mu} \to b_\mu|_{S_i \times_S T_\mu}$ for all $i, \mu$ which are compatible with $\alpha_{ij}$ and $\beta_{\mu\nu}$ (i.e. $\Psi_{j\nu} \circ \alpha_{ij} = \beta_{\mu\nu} \circ \Psi_{i\mu}$ on $S_{ij} \times_T T_{\mu\nu}$). $\square$

**Exercise 2.4.15.** Show that stackification commutes with fiber products: if $\mathcal{X} \to \mathcal{Y}$ and $\mathcal{Z} \to \mathcal{Y}$ are morphisms of prestacks, then $(\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z})^{\mathrm{st}} \cong \mathcal{X}^{\mathrm{st}} \times_{\mathcal{Y}^{\mathrm{st}}} \mathcal{Z}^{\mathrm{st}}$.

**Exercise 2.4.16.** Let $G \to S$ be a smooth affine group scheme acting on a scheme $U$ over $S$. Recall from Definition 2.3.14 that the quotient prestack $[U/G]^{\mathrm{pre}}$ and quotient stack $[U/G]$ denote the prestacks over $\mathrm{Sch}/S$ classifying trivial principal $G$-bundles (resp. principal $G$-bundles) $P \to T$ and $G$-equivariant maps $P \to U$.

(a) Show that $[U/G]^{\mathrm{pre}}$ satisfies Axiom (1) of a stack over $(\mathrm{Sch}/S)_{\text{ét}}$.

(b) Show that the $[U/G]$ is isomorphic to the stackification of $[U/G]^{\mathrm{pre}}$ over $(\mathrm{Sch}/S)_{\text{ét}}$, and that $[U/G]^{\mathrm{pre}} \to [U/G]$ is fully faithful.

**Exercise 2.4.17.** Extending Exercise 2.3.22, show that $U \to [U/G]$ is a categorical quotient among stacks.

# Chapter 3

# Algebraic spaces and stacks

## 3.1 Definitions of algebraic spaces and stacks

What are algebraic spaces, Deligne–Mumford stacks, and algebraic stacks? After giving their definitions, we will verify the algebraicity of quotient stacks $[U/G]$, the moduli stack of curves $\mathcal{M}_g$, and the moduli stack of vector bundles $\mathrm{Bun}_{r,d}(C)$.

### 3.1.1 Algebraic spaces

**Definition 3.1.1** (Morphisms representable by schemes)**.** A morphism $\mathcal{X} \to \mathcal{Y}$ of prestacks (or presheaves) over Sch is *representable by schemes* if for every morphism $T \to \mathcal{Y}$ from a scheme, the fiber product $\mathcal{X} \times_{\mathcal{Y}} T$ is a scheme.

If $\mathcal{P}$ is a property of morphisms of schemes (e.g. surjective or étale), a morphism $\mathcal{X} \to \mathcal{Y}$ of prestacks representable by schemes *has property* $\mathcal{P}$ if for every morphism $T \to \mathcal{Y}$ from a scheme, the morphism $\mathcal{X} \times_{\mathcal{Y}} T \to T$ of schemes has property $\mathcal{P}$.

**Definition 3.1.2.** An *algebraic space* is a sheaf $X$ on $\mathrm{Sch}_{\text{ét}}$ such that there exists a scheme $U$ and a surjective étale morphism $U \to X$ representable by schemes.

The map $U \to X$ is called an *étale presentation.* Morphisms of algebraic spaces are by definition morphisms of sheaves. Every scheme is an algebraic space.

### 3.1.2 Deligne–Mumford stacks

**Definition 3.1.3** (Representable morphisms)**.** A morphism $\mathcal{X} \to \mathcal{Y}$ of prestacks (or presheaves) over Sch is *representable* if for every morphism $T \to \mathcal{Y}$ from a scheme $T$, the fiber product $\mathcal{X} \times_{\mathcal{Y}} T$ is an algebraic space.

If $\mathcal{P}$ is a property of morphisms of schemes which is étale-local on the source (e.g., surjective, étale, or smooth), we say that a representable morphism $\mathcal{X} \to \mathcal{Y}$ of prestacks *has property* $\mathcal{P}$ if for every morphism $T \to \mathcal{Y}$ from a scheme and étale presentation $U \to \mathcal{X} \times_{\mathcal{Y}} T$ by a scheme, the composition $U \to \mathcal{X} \times_{\mathcal{Y}} T \to T$ has property $\mathcal{P}$.

**Definition 3.1.4.** A *Deligne–Mumford stack* is a stack $\mathcal{X}$ over $\mathrm{Sch}_{\text{ét}}$ such that there exists a scheme $U$ and a surjective, étale, and representable morphism $U \to \mathcal{X}$.

The morphism $U \to \mathcal{X}$ is called an *étale presentation.* Morphisms of Deligne–Mumford stacks are by definition morphisms of stacks. Every algebraic space is a Deligne–Mumford stack via Example 2.3.7.

**Remark 3.1.5.** If the diagonal of $\mathcal{X}$ is separated and quasi-compact, then it is in fact representable by schemes and every presentation $U \to \mathcal{X}$ is representable by schemes; see Corollary 4.4.8.

### 3.1.3 Algebraic stacks

**Definition 3.1.6.** An *algebraic stack* is a stack $\mathcal{X}$ over $\mathrm{Sch}_{\text{ét}}$ such that there exists a scheme $U$ and a surjective, smooth, and representable morphism $U \to \mathcal{X}$.

The morphism $U \to \mathcal{X}$ is called a *smooth presentation*. Morphisms of algebraic stacks are by definition morphisms of prestacks. Every scheme, algebraic space, or Deligne–Mumford stack is also an algebraic stack.

**Caution 3.1.7.** The definitions above are not standard as most authors add a representability condition on the diagonal. They are nevertheless equivalent to the standard definitions: we show in Theorem 3.2.1 that the diagonal of an algebraic space is representable by schemes and that the diagonal of an algebraic stack is representable.

**Exercise 3.1.8** (Fiber products)**.** Show that fiber products exist for algebraic spaces, Deligne–Mumford stacks, and algebraic stacks.

**Exercise 3.1.9** (Open substacks)**.** A substack $\mathcal{U} \subset \mathcal{X}$ of a stack over $\mathrm{Sch}_{\text{ét}}$ is called an *open substack* if for every map $T \to \mathcal{X}$ from a scheme, the fiber product $\mathcal{U} \times_{\mathcal{X}} T$ is an open subscheme of $T$. Show that if $\mathcal{X}$ is algebraic, then so is $\mathcal{U}$.

### 3.1.4 Algebraicity of quotient stacks

If $G \to S$ is a smooth affine group scheme acting on an algebraic space $U$ over a base scheme $S$, the quotient stack $[U/G]$ is algebraic and $U \to [U/G]$ is a principal $G$-bundle (Theorem 3.1.10).

Since we want to allow for the case that $U$ is not a scheme, we need to generalize a few definitions. An *action* of a smooth affine group scheme $G \to S$ on an algebraic space $U \to S$ is a morphism $\sigma \colon G \times_S U \to U$ satisfying the same axioms as in Definition C.1.9, and we define as in Definition 2.3.14 the quotient stack $[U/G]$ as the stackification of the prestack $[U/G]^{\mathrm{pre}}$, whose fiber category over an $S$-scheme $T$ is the quotient groupoid $[U(T)/G(T)]$. Objects of $[U/G]$ over an $S$-scheme $T$ are principal $G$-bundles $P \to T$ and $G$-equivariant morphisms $T \to U$. Since morphisms to algebraic spaces glue uniquely in the étale topology (by definition), the argument of Example 2.4.9 extends to show that $[U/G]$ is a stack. Using Definition 3.1.3, the morphism $U \to [U/G]$ is a *principal $G$-bundle* if for every morphism $T \to \mathcal{X}$ from a scheme $T$, the algebraic space $U \times_{\mathcal{X}} T$ with the induced $G$-action is a principal $G$-bundle over $T$.

**Theorem 3.1.10** (Algebraicity of Quotient Stacks)**.** *If $G \to S$ is a smooth affine group scheme acting on an algebraic space $U \to S$, the quotient stack $[U/G]$ is an algebraic stack over $S$ such that $U \to [U/G]$ is a principal $G$-bundle and in particular surjective, smooth and affine.*

*Proof.* If $T \to [U/G]$ is a morphism from an $S$-scheme corresponding to a principal $G$-bundle $P \to T$ and a $G$-equivariant map $P \to U$, there is a cartesian diagram

$$\begin{tikzcd} P \arrow[r] \arrow[d] & U \arrow[d] \\ T \arrow[r] & {[U/G]} \end{tikzcd}$$

(see Exercise 2.3.28). This shows that $U \to [U/G]$ is a principal $G$-bundle. If $U' \to U$ is an étale presentation by a scheme, then $U' \to U \to [U/G]$ provides a smooth presentation. $\square$

**Corollary 3.1.11.** *If $G \to S$ is a smooth affine group scheme, then the classifying stack $\mathbf{B}G = [S/G]$ is algebraic.* $\square$

**Example 3.1.12.** In this example, we use the alternative geometric descriptions of principal $G$-bundles from §C.2.2 to give alternative descriptions of classifying stacks. The classifying stack $\mathbf{B}\mathbb{G}_m$ (resp. $\mathbf{B}\,\mathrm{GL}_n$) is the stacks over Sch whose objects are pairs $(S, V)$ consisting of a scheme $S$ and a line bundle $V$ (resp. vector bundle of rank $n$) $V$ on $S$. Objects of the classifying stack $\mathbf{B}\,\mathrm{PGL}_n$ over a scheme $S$ can be described equivalently as either principal $\mathrm{PGL}_n$-bundles, Brauer–Severi schemes, or Azumaya algebras over $S$.

Over a field $\Bbbk$ of $\mathrm{char}(\Bbbk) \neq 2$, recall that for a non-degenerate quadratic form $q$ on an $n$-dimensional vector space $V$, the orthogonal group $O(q)$ is the subgroup of $\mathrm{GL}(V)$ containing matrices preserving $q$. For two different non-degenerate forms $q$ and $q'$, then $\mathbf{B}O(q) \cong \mathbf{B}O(q')$ as both classifying rank $n$ vector bundles equipped with non-degenerate quadratic forms even though $O(q)$ and $O(q')$ may be non-isomorphic.

**Corollary 3.1.13.** *If $G$ is a finite group acting freely on an algebraic space $U$, then the quotient sheaf $U/G$ is an algebraic space.*

*Proof.* Since the action is free, the quotient stack $[U/G]$ is equivalent to a sheaf, which we denote by $U/G$ (see Exercise 2.3.19). Theorem 3.1.10 implies that $U/G$ is an algebraic stack and that $U \to U/G$ is a principal $G$-bundle so in particular finite, étale, surjective and representable by schemes. Taking $U' \to U$ to be an étale presentation by a scheme, the composition $U' \to U \to U/G$ yields an étale presentation of $U/G$. $\square$

**Remark 3.1.14.** This resolves the troubling issue from Example 0.5.5 where we saw that the quotient of a finite group acting freely on a scheme need not exist as a scheme. In addition, it shows that the category of algebraic spaces itself is closed under taking quotients by free actions of finite groups so that we don't need to enlarge our category even more.

**Exercise 3.1.15.** Let $G \to S$ be a smooth affine group scheme acting on $S$-schemes $X$ and $Y$.

(a) Show that a $G$-equivariant morphism $X \to Y$ induces a morphism $[X/G] \to [Y/G]$ of algebraic stacks.

(b) Show that $[X/G] \to [Y/G]$ is induced by a $G$-equivariant morphism if and only if $[X/G] \to [Y/G]$ is a morphism over $\mathbf{B}G$.

### 3.1.5 Algebraicity of $\mathcal{M}_g$

The main reason that $\mathcal{M}_g$ is an algebraic stack is quite simple: every smooth, connected, and projective curve $C$ is tri-canonically embedded $C \hookrightarrow \mathbb{P}^{5g-6}$ by the very ample line bundle $\omega_C^{\otimes 3}$ and the locally closed subscheme $H' \subset \mathrm{Hilb}_P(\mathbb{P}^{5g-6})$ parameterizing smooth families of tri-canonically embedded curves provides a smooth presentation $H' \to \mathcal{M}_g$.

**Theorem 3.1.16** (Algebraicity of the stack of smooth curves)**.** *If $g \geq 2$, then $\mathcal{M}_g$ is an algebraic stack over* $\mathrm{Spec}\,\mathbb{Z}$.

*Proof.* As in the proof that $\mathcal{M}_g$ is a stack (Proposition 2.4.11), we will use Properties of Families of Smooth Curves (5.1.9): for a family of smooth curves $p\colon \mathcal{D} \to S$, $\omega_{\mathcal{D}/S}^{\otimes 3}$ is relatively very ample on $S$ and $p_*(\omega_{\mathcal{D}/S}^{\otimes 3})$ is a vector bundle of rank $5(g-1)$. It follows that $\omega_{\mathcal{D}/S}^{\otimes 3}$ defines a closed immersion $\mathcal{D} \hookrightarrow \mathbb{P}(p_*(\omega_{\mathcal{D}/S}^{\otimes 3}))$ over $S$. By Riemann–Roch, the Hilbert polynomial of a fiber $\mathcal{D}_s \hookrightarrow \mathbb{P}^{5g-6}_{\kappa(s)}$ is given by

$$P(n) := \chi(\mathcal{O}_{\mathcal{D}_s}(n)) = \deg(\omega_{\mathcal{D}_s}^{\otimes 3n}) + 1 - g = (6n-1)(g-1).$$

Let

$$H := \operatorname{Hilb}^P(\mathbb{P}^{5g-6}_{\mathbb{Z}}/\mathbb{Z})$$

be the (projective) Hilbert scheme parameterizing closed subschemes of $\mathbb{P}^{5g-6}$ with Hilbert polynomial $P$ (Theorem 1.1.2). Let $\mathcal{C} \hookrightarrow \mathbb{P}^{5g-6} \times H$ be the universal closed subscheme and let $\pi\colon \mathcal{C} \to H$ be the projection. We claim that there is a unique locally closed subscheme $H' \subset H$ consisting of points $h \in H$ satisfying

(a) $\mathcal{C}_h \to \operatorname{Spec} \kappa(h)$ is smooth and geometrically connected; and

(b) $\mathcal{C}_h \hookrightarrow \mathbb{P}^{5g-6}_{\kappa(h)}$ is embedded by the complete linear series $\omega^{\otimes 3}_{\mathcal{C}_h/\kappa(h)}$.

(c) denote $\mathcal{C}' = \mathcal{C} \times_H H'$, the line bundles $\omega^{\otimes 3}_{\mathcal{C}'/H'}$ and $\mathcal{O}_{\mathcal{C}_{H'}}(1)$ differ by a pullback of a line bundle from $H'$.

Moreover, if $T \to H$ is a morphism schemes such that (a)–(c) hold for the family $\mathcal{C}_T \to T$, then $T \to H$ factors through $H'$.

Since the condition that a fiber of a proper morphism (of finite presentation) is smooth is an open condition on the target (Corollary A.3.10), the condition on $H$ that $\mathcal{C}_h$ is smooth is open. Consider the Stein factorization [Har77, Cor. 11.5] $\mathcal{C} \to \widetilde{H} = \operatorname{Spec}_H \pi_* \mathcal{O}_{\mathcal{C}} \to H$ where $\mathcal{C} \to \widetilde{H}$ has geometrically connected fibers and $\widetilde{H} \to H$ is finite. Since the kernel and cokernel of $\mathcal{O}_H \to \pi_*\mathcal{O}_{\mathcal{C}}$ have closed support (as they are coherent), $\widetilde{H} \to H$ is an isomorphism over an open subscheme of $H$, which is precisely where the fibers of $\mathcal{C} \to H$ are geometrically connected. In summary, the set of $h \in H$ satisfying (a) is an open subscheme of $H$, which we will denote by $H_1$.

The relative canonical sheaf $\omega_{\mathcal{C}_1/H_1}$ of the family $\mathcal{C}_1 := \mathcal{C}_{H_1} \to H_1$ is a line bundle. By Proposition A.7.16, there exists a locally closed subscheme $H_2 \hookrightarrow H_1$ such that a morphism $T \to H_1$ factor through $H_2$ if and only if $\omega_{\mathcal{C}_1/H_1}|_{\mathcal{C}_T}$ and $\mathcal{O}_{\mathcal{C}}(1)|_{\mathcal{C}_T}$ differ by the pullback of a line bundle on $T$. In particular, (c) holds and for every $h \in H_2$, there is an isomorphism $\omega^{\otimes 3}_{\mathcal{C}_h/\kappa(h)} \cong \mathcal{O}_{\mathcal{C}_h}(1)$. To arrange (b), consider the restriction of the universal curve $\pi_2\colon \mathcal{C}_2 := \mathcal{C}_{H_2} \to H_2$. There is a canonical map $\alpha\colon \mathrm{H}^0(\mathbb{P}^{5g-6}_{\mathbb{Z}}, \mathcal{O}(1)) \otimes \mathcal{O}_{H_2} \to \pi_{2,*}\mathcal{O}_{\mathcal{C}_2}(1)$ of vector bundles of rank $5g-5$ on $H_2$ whose fiber over a point $h \in H_2$ is the map $\alpha_h\colon \mathrm{H}^0(\mathbb{P}^{5g-6}_{\kappa(h)}, \mathcal{O}(1)) \to \mathrm{H}^0(\mathcal{C}_h, \mathcal{O}_{\mathcal{C}_h}(1)) \cong \mathrm{H}^0(\mathcal{C}_h, \omega^{\otimes 3}_{\mathcal{C}_h/\kappa(h)})$. The closed locus defined by the support of $\operatorname{coker}(\alpha)$ is precisely the locus where $\alpha_h$ is not an isomorphism (as the vector bundles have the same rank). The subscheme $H' = H_2 \setminus \operatorname{Supp}(\operatorname{coker}(\alpha))$ satisfies (a)–(c) along with the universal property.

The group scheme $\mathrm{PGL}_{5g-5} = \underline{\mathrm{Aut}}(\mathbb{P}^{5g-6}_{\mathbb{Z}})$ over $\mathbb{Z}$ acts naturally on $H$: if $g \in \operatorname{Aut}(\mathbb{P}^{5g-6}_S)$ and $[\mathcal{D} \subset \mathbb{P}^{5g-6}_S] \in H(S)$, then $g \cdot [\mathcal{D} \subset \mathbb{P}^{5g-6}_S] = [g(\mathcal{D}) \subset \mathbb{P}^{5g-6}_S]$. The closed subscheme $H' \subset H$ is $\mathrm{PGL}_{5g-5}$-invariant and we claim that $\mathcal{M}_g \cong [H'/\mathrm{PGL}_{5g-5}]$. This establishes the theorem since $[H'/\mathrm{PGL}_{5g-5}]$ is algebraic (Theorem 3.1.10).

Consider the morphism $H' \to \mathcal{M}_g$ defined by the restriction $\mathcal{C}' \to H'$ of the universal family of the Hilbert scheme. This morphism forgets the embedding, i.e. assigns a closed subscheme $\mathcal{D} \subset \mathbb{P}^{5g-6}_S$ to the family $\mathcal{D} \to S$. This morphism is $\mathrm{PGL}_{5g-5}$-invariant and descends to a morphism $[H'/\mathrm{PGL}_{5g-5}]^{\mathrm{pre}} \to \mathcal{M}_g$ of prestacks. We claim that this map is fully faithful. To see this, observe that for a family $p\colon \mathcal{D} \to S$ in $H'$ defined by a closed subscheme $\mathcal{D} \subset \mathbb{P}^{5g-6}_S$, by (c) there is an isomorphism $\mathcal{O}_{\mathcal{D}}(1) \cong \omega_{\mathcal{D}/S}^{\otimes 3} \otimes p^*M$ for a some line bundle $M$ on $S$, and by (b), the canonical map

$$\mathrm{H}^0(\mathbb{P}^{5g-6}_{\mathbb{Z}}, \mathcal{O}(1)) \otimes \mathcal{O}_S \to p_*\mathcal{O}_{\mathcal{D}}(1) \cong p_*(\omega_{\mathcal{D}/S}^{\otimes 3} \otimes p^*M) \cong p_*\omega_{\mathcal{D}/S}^{\otimes 3} \otimes M$$

is an isomorphism. Every automorphism of $\mathcal{D} \to S$ induces an automorphism of $\omega_{\mathcal{D}/S}^{\otimes 3}$ and thus an automorphism of $p_*\omega_{\mathcal{D}/S}^{\otimes 3} \otimes M$, which in turn induces an automorphism of $\mathbb{P}^{5g-6}_S$ preserving $\mathcal{D}$. Since $\mathcal{M}_g$ is a stack (Theorem 3.1.10), the universal property of stackification yields a morphism $[H'/\mathrm{PGL}_{5g-5}] \to \mathcal{M}_g$; this map is fully faithful since $[H'/\mathrm{PGL}_{5g-5}]^{\mathrm{pre}} \to [H'/\mathrm{PGL}_{5g-5}]$ is fully faithful (Exercise 2.4.16). It remains to check that $[H'/\mathrm{PGL}_{5g-5}] \to \mathcal{M}_g$ is essentially surjective. For this, it suffices to check that if $p\colon \mathcal{D} \to S$ is a family of smooth curves, then there exists an étale cover $\{S_i \to S\}$ such that each $\mathcal{D}_{S_i}$ is in the image of $H'(S_i) \to \mathcal{M}_g(S_i)$. Since $\omega_{\mathcal{D}/S}^{\otimes 3}$ defines a closed immersion $\mathcal{D} \hookrightarrow \mathbb{P}(p_*\omega_{\mathcal{D}/S}^{\otimes 3})$ over $S$ and $p_*\omega_{\mathcal{D}/S}^{\otimes 3}$ is locally free of rank $5g-5$, we may simply take $\{S_i\}$ to be a Zariski open cover (and thus étale cover) where the restriction of $p_*\omega_{\mathcal{D}/S}^{\otimes 3}$ is free. □

**Remark 3.1.17.** The entire stack $\mathcal{M}$ of smooth curves (as defined in Example 2.3.9) is also algebraic since $\mathcal{M} = \coprod_g \mathcal{M}_g$.

**Exercise 3.1.18.** Let $\mathcal{M}_{1,1}$ be the stack over Sch where an object over a scheme $S$ is a *family of elliptic curves over* $S$, i.e. a pair $(\mathcal{E} \to S, \sigma)$ where $\mathcal{E} \to S$ is smooth proper morphism with a section $\sigma\colon S \to \mathcal{E}$ such that for every $s \in S$, the fiber $(\mathcal{E}_s, \sigma(s))$ is an elliptic curve over the residue field $\kappa(s)$.

(a) Show that $\mathcal{M}_{1,1}$ is an algebraic stack over $\mathbb{Z}$.

(b) Use the Weierstrass form $y^2 = x^3 + ax + b$ (see [Sil09, §3.1]) to show that if we invert the primes 2 and 3, there is an isomorphism

$$\mathcal{M}_{1,1} \times_{\mathbb{Z}} \mathbb{Z}[1/6] \cong [(\mathbb{A}^2 \smallsetminus V(\Delta))/\mathbb{G}_m],$$

where the action is given by $t \cdot (a, b) = (t^4 a, t^6 b)$ and $\Delta$ is the discriminant $4a^3 + 27b^2$.

(c) Define a *stable elliptic curve* over a field $\Bbbk$ as a pair $(E, p)$ where $E$ is an irreducible projective curve over $\Bbbk$ of arithmetic genus 1 with at worst nodal singularities and $p \in E(\Bbbk)$ is a smooth point. Over a scheme $S$, a *family of stable elliptic curves over* $S$ is a proper flat $\mathcal{E} \to S$ and a section $\sigma\colon S \to \mathcal{E}$ such that every fiber is a stable elliptic curve. Denoting $\overline{\mathcal{M}}_{1,1}$ as the stack over Sch classifying stable elliptic curves, show that

$$\overline{\mathcal{M}}_{1,1} \times_{\mathbb{Z}} \mathbb{Z}[1/6] \cong [(\mathbb{A}^2 \smallsetminus 0)/\mathbb{G}_m]$$

with the same action as above.

**Exercise 3.1.19.** An *$n$-pointed family of genus 0 curves* is proper, flat, and finitely presented morphism $X \to S$ of schemes with $n$ sections $\sigma_1, \ldots, \sigma_n\colon S \to X$ such that for every geometric point $s\colon \operatorname{Spec} \Bbbk \to S$, the geometric fiber $X \times_S \Bbbk$ is a smooth genus 0 curve and $\sigma_1(s), \ldots, \sigma_n(s) \in X(\Bbbk)$ are distinct.

(a) Show that the prestack $\mathcal{M}_{0,n}$ parameterizing $n$-pointed families of genus 0 curves is a stack over $\mathrm{Sch}_{\text{ét}}$.

(b) Show that $\mathcal{M}_{0,0} \cong \mathbf{B}\,\mathrm{PGL}_2$.

*Hint: Use Exercise C.2.14 to show that $X \to S$ is a Brauer–Severi scheme (i.e. there exists an étale cover $S' \to S$ such that $X \times_S S' \cong \mathbb{P}^1_{S'}$), and use the correspondence between Brauer–Severi schemes and principal* $\mathrm{PGL}_2$*-torsors (Exercise C.2.13).*

(c) Show that $\mathcal{M}_{0,1} \cong \mathbf{B}U_2$ where $U_2 \subset \mathrm{PGL}_2$ is the two-dimensional subgroup of upper triangular matrices.

(d) Show that $\mathcal{M}_{0,2} \cong \mathbf{B}\mathbb{G}_m$.

(e) Show that $\mathcal{M}_{0,3} \cong \operatorname{Spec} \mathbb{Z}$.

(f) Show that for $n > 3$, $\mathcal{M}_{0,n} \cong (\mathbb{P}^1 \setminus \{0, 1, \infty\})^{n-3} \setminus \Delta$ where $\Delta$ is the closed subscheme where at least two of the $n-3$ points are equal.

### 3.1.6 Algebraicity of $\mathrm{Bun}_{r,d}(C)$

**Theorem 3.1.20** (Algebraicity of the stack of vector bundles)**.** *Let $C$ be a smooth, connected, and projective curve over a field $\Bbbk$, and let $r$ and $d$ be integers with $r \geq 0$. The stack $\underline{\mathrm{Coh}}_{r,d}(C)$ and $\mathrm{Bun}_{r,d}(C)$ are algebraic stacks over $\operatorname{Spec} \Bbbk$. The inclusion $\mathrm{Bun}_{r,d}(C) \subset \underline{\mathrm{Coh}}_{r,d}(C)$ is an open substack.*

*Proof.* By Riemann–Roch (5.1.1),the Hilbert polynomial of a coherent sheaf $E$ of rank $r$ and degree $d$ is

$$P(n) := \chi(E(n)) = \deg(E(n)) + \mathrm{rk}(E(n))(1-g) = d + rn + r(1-g).$$

For $N \gg 0$, by Serre vanishing $E(N)$ is globally generated and $\mathrm{H}^1(C, E(N)) = 0$ for $N \gg 0$. In particular, there is a surjective map $\Gamma(C, E(N)) \otimes \mathcal{O}_C \twoheadrightarrow E(N)$ which induces an isomorphism on global sections.

For each integer $N$, we claim that there is an open substack

$$\mathcal{U}_N \subset \underline{\mathrm{Coh}}_{r,d}(C)$$

parameterizing coherent sheaves $E$ on $C$ such that $E(N)$ is globally generated and $\mathrm{H}^1(C, E(N)) = 0$. Note that since $P(N) = \mathrm{h}^0(C, E(N)) - \mathrm{h}^1(C, E(N))$, the latter condition implies that $\mathrm{h}^0(C, E(N)) = P(N)$. To verify the claim, let $S$ be a scheme and $\mathcal{E}$ a finitely presented, quasi-coherent sheaf (e.g. coherent if $S$ is noetherian) on $C \times S$ flat over $S$ with Hilbert polynomial $P$, and consider the diagram

$$\begin{tikzcd} & C \times S \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ C & & S. \end{tikzcd}$$

For a point $s \in S$, we denote by $\mathcal{E}_s$ the restriction $\mathcal{E}|_{C \times \operatorname{Spec} \kappa(s)}$ to $C \times \operatorname{Spec} \kappa(s)$. A simple application of Cohomology and Base Change (see Proposition A.7.9) implies that the locus $S' \subset S$ of points $s \in S$ such that $\mathrm{H}^1(C, \mathcal{E}_s(N)) = 0$ is open, and moreover that $\big(\mathrm{R}^1 p_{2,*}\mathcal{E}(N)\big)|_{S'} = 0$ and $\big(p_{2,*}c(N)\big)|_{S'}$ is a vector bundle of rank $P(N)$ whose construction commutes with respect to base change by maps to $S'$. The sheaf $\mathcal{F} := \operatorname{coker}\big(p_2^* p_{2,*}\mathcal{E}(N) \to \mathcal{E}(N)\big)$ has closed support and the open subset $S' \setminus p_2(\mathrm{Supp}(\mathcal{F}))$ is the locus of points $s \in S$ such that $\mathcal{E}_s(N)$ is globally generated and $\mathrm{H}^1(C, \mathcal{E}_s(N)) = 0$.

For each $N$, consider the Quot scheme

$$Q_N := \mathrm{Quot}_C^P(\mathcal{O}_C(-N)^{P(N)})$$

parameterizing quotients $\mathcal{O}_C(-N)^{P(N)} \twoheadrightarrow F$ with Hilbert polynomial $P$ (Theorem 1.1.3). A similar argument as above shows that there is an open subscheme $Q'_N \subset Q_N$ parameterizing quotients $q\colon \mathcal{O}_C(-N)^{P(N)} \twoheadrightarrow F$ such that $\mathrm{H}^0(q(N))\colon \mathrm{H}^0(C, \mathcal{O}_C)^{P(N)} \to \mathrm{H}^0(C, F(N))$ is surjective and $\mathrm{H}^1(C, F(N)) = 0$. Note that since we have specified the Hilbert polynomial, the conditions imply that $\mathrm{H}^0(q(N))$ is an isomorphism.

The Quot scheme $Q_N$ inherits a natural action from $\mathrm{GL}_{P(N)}$ such that $Q'_N$ is invariant. The morphism

$$Q'_N \to \mathcal{U}_N, \qquad [\mathcal{O}_C(-N)^{P(N)} \twoheadrightarrow F] \mapsto F,$$

is $\mathrm{GL}_{P(N)}$-equivariant and descends to a morphism $\Psi^{\mathrm{pre}}\colon [Q'_N/\,\mathrm{GL}_{P(N)}]^{\mathrm{pre}} \to \mathcal{U}_N$ of prestacks. The map $\Psi^{\mathrm{pre}}$ is fully faithful since every automorphism of a coherent sheaf $\mathcal{E}$ on $C \times S$ induces an automorphism of $p_{2,*}\mathcal{E}(N) = \mathcal{O}_S^{P(N)}$, i.e. an element of $\mathrm{GL}_{P(N)}(S)$, and this element acts on $\mathcal{O}_C(-N)^{P(N)}$ preserving the quotient $\mathcal{E}$.

Since $\underline{\mathrm{Coh}}_{r,d}(C)$ is a stack (Proposition 2.4.13), there is an induced morphism $\Psi\colon [Q'_N/\,\mathrm{GL}_{P(N)}] \to \mathcal{U}_N$ of stacks which is fully faithful (by Exercise 2.4.16) and essentially surjective (by construction). We conclude that $\mathcal{U}_N = [Q'_N/\,\mathrm{GL}_{P(m)}]$ and that

$$\underline{\mathrm{Coh}}_{r,d}(C) = \bigcup_N \,[Q'_N/\,\mathrm{GL}_{P(N)}\,].$$

The algebraicity of quotient stacks (Theorem 3.1.10) implies the algebraicity of $\underline{\mathrm{Coh}}_{r,d}(C)$.

To see that $\mathrm{Bun}_{r,d}(C) \subset \underline{\mathrm{Coh}}_{r,d}(C)$ is an open substack and in particular that $\mathrm{Bun}_{r,d}(C)$ is also an algebraic stack, let $S \to \underline{\mathrm{Coh}}_{r,d}(C)$ be a morphism corresponding to a finitely presented, quasi-coherent sheaf $\mathcal{E}$ on $C \times S$ flat over $S$. If $V \subset C \times S$ is the open locus where $\mathcal{E}$ is a vector bundle, then $S \setminus p_2(C \times S \setminus V)$ is open and identified with the fiber product $S \times_{\underline{\mathrm{Coh}}_{r,d}(C)} \mathrm{Bun}_{r,d}(C)$. □

**Remark 3.1.21.** Note that the stacks parameterizing all coherent sheaves or all vector bundles are also algebraic since

$$\underline{\mathrm{Coh}}(C) = \coprod_{r,d} \underline{\mathrm{Coh}}_{r,d}(C) \quad \text{and} \quad \mathrm{Bun}(C) = \coprod_{r,d} \mathrm{Bun}_{r,d}(C)$$

Note also that while $\mathrm{Bun}_{r,d}(C)$ itself is not quasi-compact (Definition 3.3.20), the proof establishes that every quasi-compact open substack of $\mathrm{Bun}_{r,d}(C)$ is a quotient stack.

**Exercise 3.1.22.** Modify the above argument to show that $\underline{\mathrm{Coh}}(X)$ is an algebraic stack if $X$ is a projective scheme over $\Bbbk$. More generally, show that if $X \to S$ is a strongly projective and flat morphism of noetherian schemes, then the stack $\underline{\mathrm{Coh}}(X/S)$, whose objects over an $S$-scheme $T$ are finitely presented, quasi-coherent sheaves on $X_T$ flat over $T$, is an algebraic stack.

### 3.1.7 Desideratum

We will develop the foundations of algebraic spaces and stacks in the forthcoming chapters but it is worth first highlighting some of the most important results.

### The importance of the diagonal

When overhearing others discussing algebraic stacks, you may have wondered what's all the fuss about the diagonal. Well, I'll tell you—the diagonal encodes the stackiness!

First and foremost, the diagonal $\mathcal{X} \to \mathcal{X} \times \mathcal{X}$ of an algebraic stack is representable and the diagonal $X \to X \times X$ of an algebraic space is representable by schemes (Theorem 3.2.1).

The stabilizer $G_x$ of a field-valued point $x\colon \operatorname{Spec} \Bbbk \to \mathcal{X}$ is defined as the sheaf $\underline{\mathrm{Aut}}_{\mathcal{X}(\Bbbk)}(x) = \underline{\mathrm{Isom}}_{\mathcal{X}(\Bbbk)}(x, x)$ and is identified with the fiber product $\mathcal{X} \times_{\mathcal{X} \times \mathcal{X}} \operatorname{Spec} \Bbbk$ by Exercise 2.3.30. By Representability of the Diagonal (3.2.1) the stabilizer $G_x$ is representable by a group algebraic space over $\Bbbk$. We later show that $G_x$ is in fact a group scheme of finite type over $\Bbbk$ (Corollary 4.4.29) as long as the diagonal of $\mathcal{X}$ is quasi-separated. See §3.2.2 for a further discussion of stabilizers.

For schemes (resp. separated schemes), the diagonal is an immersion (resp. closed immersion). For algebraic stacks, the diagonal is not necessarily even a monomorphism as the fiber over $(x, x)\colon \operatorname{Spec} \Bbbk \to \mathcal{X} \times \mathcal{X}$, or in other words the stabilizer $G_x$, may be non-trivial. Properties of the diagonal in fact characterize algebraic spaces and Deligne–Mumford stacks: an algebraic stack is an algebraic space (resp. Deligne–Mumford stack) if and only if $\mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is a monomorphism (resp. unramified)—see Theorems 3.6.4 and 3.6.5. An equivalent characterization is given by the properties of the stabilizer groups as in the table below:

| Type of space | Property of the diagonal | Property of stabilizers |
|---|---|---|
| algebraic space | monomorphism | trivial |
| Deligne–Mumford stack | unramified | reduced finite groups[1] |
| algebraic stack | arbitrary | arbitrary |

Table 3.1: Characterization of algebraic spaces and Deligne–Mumford stacks

As a consequence of these characterizations, we will generalize Corollary 3.1.13: the quotient of a free action of a smooth algebraic group on an algebraic space exists as an algebraic space (Corollary 3.6.7). We will also be able to establish that $\mathcal{M}_g$ is Deligne–Mumford (Corollary 3.6.9) rather than just algebraic (Theorem 3.1.16).

We now summarize additional important properties of algebraic spaces, Deligne–Mumford stacks, and algebraic stacks. The reader may consult Section 0.0.2 for a brief recap of the trichotomy of moduli spaces.

### Properties of algebraic spaces

- If $R \rightrightarrows U$ is an étale equivalence relation of schemes, the quotient sheaf $U/R$ is an algebraic space (Theorem 3.4.11).
- If $X$ is a quasi-separated algebraic space, there exists a dense open subspace $U \subset X$ which is a scheme (Theorem 4.4.1).
- If $X \to Y$ is a separated and quasi-finite morphism of noetherian algebraic spaces, then there exists a factorization $X \hookrightarrow \widetilde{X} \to Y$ where $X \hookrightarrow \widetilde{X}$ is an open immersion and $\widetilde{X} \to Y$ is finite (Zariski's Main Theorem). In particular, $X \to Y$ is quasi-affine.

[1] If the diagonal is not quasi-compact, the stabilizers will only be discrete and reduced.

**Properties of Deligne–Mumford stacks**

- If $R \rightrightarrows U$ is an étale groupoid of schemes, the quotient stack $[U/R]$ is a Deligne–Mumford stack (Theorem 3.4.11).
- If $\mathcal{X}$ is a Deligne–Mumford stack and $x \in \mathcal{X}(k)$ is a field-valued point, there exists an étale neighborhood $[\mathrm{Spec}(A)/G] \to \mathcal{X}$ of $x$ where $G$ is a finite group, which can be arranged to be the stabilizer of $x$ (Local Structure of Deligne–Mumford Stacks, Theorem 4.2.11).
- If $\mathcal{X}$ is a separated Deligne–Mumford stack, there exists a coarse moduli space $\mathcal{X} \to X$ where $X$ is a separated algebraic space (Keel–Mori Theorem, Theorem 4.3.11).
- If $\mathcal{X}$ is a Deligne–Mumford stack (e.g. algebraic space), there exists a scheme $U$ and a finite surjective morphism $U \to \mathcal{X}$ (Le Lemme de Gabber, Theorem 4.5.1).

**Properties of algebraic stacks**

- If $R \rightrightarrows U$ is a smooth groupoid of schemes, the quotient stack $[U/R]$ is an algebraic stack (Theorem 3.4.11).
- If $\mathcal{X}$ is an algebraic stack of finite type over an algebraically closed field $\Bbbk$ with affine diagonal, every point $x \in \mathcal{X}(k)$ with linearly reductive stabilizer has an affine étale neighborhood $[\mathrm{Spec}(A)/G_x] \to \mathcal{X}$ of $x$ where $G$ is a finite group (Local Structure of Algebraic Stacks).
- Let $\mathcal{X}$ be an algebraic stack of finite type over an algebraically closed field $\Bbbk$ of characteristic 0 with affine diagonal. If $\mathcal{X}$ is S-complete and $\Theta$-complete, then there exists a good moduli space $\mathcal{X} \to X$ where $X$ is a separated algebraic space of finite type over $\Bbbk$.

## Historical comments

Deligne–Mumford and algebraic stacks were first introduced in [DM69] and [Art74]—and in both cases referred to as *algebraic stacks*—with conventions slightly different than ours. Namely, [DM69, Def. 4.6] assumed in addition to the existence of an étale presentation that the diagonal is representable by schemes (which is automatic if the diagonal is separated and quasi-compact). On the other hand, [Art74, Def. 5.1] assumed in addition to the existence of a smooth presentation that the stack is locally of finite type over an excellent Dedekind domain. The term *Artin stack*—which we refrain from using—is sometimes reserved for stacks that satisfy Artin's axioms (e.g. algebraic stacks locally of finite type over an excellent scheme with quasi-compact and separated diagonal).

We follow the conventions of [Ols16] and [SP] with the exception that we work over the site $\mathrm{Sch}_{\text{ét}}$ while [SP] works over $\mathrm{Sch}_{\mathrm{fppf}}$. These two sites give equivalent notions of algebraic stacks [SP, Tag 076U].

# 3.2 Representability of the diagonal

## 3.2.1 Representability

**Theorem 3.2.1** (Representability of the Diagonal)**.**

(1) *The diagonal of an algebraic space is representable by schemes.*

(2) *The diagonal of an algebraic stack is representable.*

*Proof.* Let $X$ be an algebraic space and $U \to X$ be an étale presentation. Define the scheme $R := U \times_X U$. If $T \to X \times X$ is a morphism from a scheme, we need to show that the sheaf $Q_T = X \times_{X \times X} T$ is in fact a scheme. Since $U \to X$ is étale, surjective, and representable by schemes, so is $U \times U \to X \times X$. The base change of $T \to X \times X$ by $U \times U \to X \times X$ is a scheme $T'$ which is surjective étale over $T$. In the cartesian cube

$$\begin{tikzcd}
& Q_{T'} \arrow[rr] \arrow[dl] \arrow[dd] & & T' \arrow[dl] \arrow[dd] \\
R \arrow[rr, crossing over] \arrow[dd] & & U \times U & \\
& Q_T \arrow[rr] \arrow[dl] & & T \arrow[dl] \\
X \arrow[rr] & & X \times X, \arrow[from=uu, crossing over] &
\end{tikzcd} \tag{3.2.1}$$

$Q_T$ is a sheaf on $\mathrm{Sch}_{\acute{e}t}$ while $Q_{T'}$ is a scheme. Since $R \to U \times U$ is a separated and locally quasi-finite morphism of schemes, so is $Q_{T'} \to T'$. (If $X$ had a quasi-compact diagonal, then by Zariski's main theorem $R \to U \times U$ is quasi-affine and thus so is $Q_{T'} \to T'$.) Since $Q_T$ is a sheaf in the étale topology that pulls back to a scheme $Q_{T'}$ separated and locally quasi-finite over $T'$, we may apply Effective Descent (Proposition 2.2.11) to conclude that $Q_T$ is a scheme.

If $\mathcal{X}$ is an algebraic stack and $U \to \mathcal{X}$ is a smooth presentation, we may imitate the above argument. The fiber product $R := U \times_{\mathcal{X}} U$ is an algebraic space. If $T \to \mathcal{X} \times \mathcal{X}$ is a morphism from a scheme, its base change along $U \times U \to \mathcal{X} \times \mathcal{X}$ yields an algebraic space $T_1$ which is surjective smooth over $T$. Choose an étale presentation $T_2 \to T_1$. Then $T_2 \to T$ is a surjective smooth morphism of schemes which has a section after an étale cover $T' \to T$ (Proposition A.3.5). The composition $T' \to T_2 \to T_1 \to U \times U$ provides a lift of $T \to \mathcal{X} \times \mathcal{X}$. We obtain a diagram similar to (3.2.1) but where the left and right squares are not necessarily cartesian. The morphism $Q_{T'} \to Q_T$ is étale, surjective, and representable by schemes (as $T' \to T$ is). Choosing an étale presentation $V \to Q_{T'}$ of the algebraic space $Q_{T'}$, the composition $V \to Q_{T'} \to Q_T$ yields an étale presentation showing that $Q_T$ is an algebraic space. □

**Corollary 3.2.2.**

(1) *Every morphism from a scheme to an algebraic space is representable by schemes.*

(2) *Every morphism from a scheme to an algebraic stack is representable.*

*Proof.* This follows directly from Representability of the Diagonal (Theorem 3.2.1) and the cartesian diagram

$$\begin{tikzcd}
T_1 \times_{\mathcal{X}} T_2 \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & T_1 \times T_2 \arrow[d] \\
\mathcal{X} \arrow[r] & \mathcal{X} \times \mathcal{X}.
\end{tikzcd}$$

associated to any two maps $T_1 \to \mathcal{X}$ and $T_2 \to \mathcal{X}$ from schemes to an algebraic stack. $\square$

**Exercise 3.2.3.**

(a) If $\mathcal{X} \to \mathcal{Y}$ is a representable morphism of algebraic stacks (e.g. a morphism of algebraic spaces), then $\mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ is representable by schemes.

(b) If $\mathcal{X} \to \mathcal{Y}$ is a morphism of algebraic stacks, then $\mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ is representable.

**Exercise 3.2.4.** Show that the diagonal of a morphism $\mathcal{X} \to \mathcal{Y}$ of algebraic stacks is locally of finite type.

### 3.2.2 Stabilizer groups and the inertia stack

Now that we know the diagonal is representable, we can discuss its properties. One of the most important features of the diagonal is that it encodes the stabilizer groups.

**Definition 3.2.5** (Stabilizers)**.** If $\mathcal{X}$ is an algebraic stack and $x\colon \operatorname{Spec} \Bbbk \to \mathcal{X}$ is a field-valued point, the *stabilizer of* $x$ is defined as the group algebraic space $G_x := \underline{\mathrm{Aut}}_{\mathcal{X}(\Bbbk)}(x)$.

By Exercise 2.3.30, we can identify $G_x$ with the fiber product

$$\begin{tikzcd} G_x := \underline{\mathrm{Aut}}_{\mathcal{X}(\Bbbk)}(x) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \operatorname{Spec} \Bbbk \arrow[d, "{(x,x)}"] \\ \mathcal{X} \arrow[r, "\Delta"] & \mathcal{X} \times \mathcal{X}. \end{tikzcd}$$

The sheaf $G_x$ is representable by an algebraic space over $K$ by Representability of the Diagonal (Theorem 3.2.1). The stabilizer $G_x$ is a group algebraic space, i.e. an algebraic space $G_x$ with multiplication, inverse, and identity morphisms satisfying Definition C.1.1 (or equivalently a group object in the category of algebraic spaces). In fact, $G_x$ is actually a group scheme locally of finite type as long as the diagonal of $\mathcal{X}$ is quasi-separated (Corollary 4.4.29).

**Remark 3.2.6.** Let $G$ be a group scheme over a field $\Bbbk$ acting on a $\Bbbk$-scheme $U$ via $\sigma\colon G \times U \to U$, and let $u \in U(\Bbbk)$. The stabilizer of the image of $u$ in $[U/G]$ is the usual stabilizer group scheme, i.e. the fiber product of $(\sigma, p_2)\colon G \times U \to U \times U$ along $(u,u)\colon \operatorname{Spec} \Bbbk \to U \times U$.

**Exercise 3.2.7.**

(a) Show that the stabilizer of a field-valued point of a fiber product of algebraic stacks is the fiber product of stabilizers, i.e. for $x' \in (\mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}')(\Bbbk)$, then $G_{x'} = G_x \times_{G_y} G_{y'}$ where $x$, $y$ and $y'$ are the images of $x'$.

(b) Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks and $x \in \mathcal{X}(\Bbbk)$ be a field valued point. Show that the fiber of the diagonal $\mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ over the point $(x, x, \mathrm{id}) \in (\mathcal{X} \times_{\mathcal{Y}} \mathcal{X})(\Bbbk)$ is identified with $\ker(G_x \to G_y)$. What is the fiber of the diagonal over an arbitrary field-valued point of $\mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$?

**Exercise 3.2.8.** Let $\mathcal{X}$ be a Deligne–Mumford stack (quasi-separated Deligne–Mumford stack). An algebraic stack is *quasi-separated* if for every morphism $(a,b)\colon S \to \mathcal{X} \times \mathcal{X}$ from a scheme, the fiber product $\underline{\mathrm{Isom}}_{\mathcal{X}(S)}(a,b) \cong \mathcal{X} \times_{\Delta, \mathcal{X} \times \mathcal{X}, (a,b)} S$ is quasi-compact over $S$; see also Definition 3.3.10.

(a) For a field-valued point $x \in \mathcal{X}(\Bbbk)$, show that $G_x$ is a separated (resp. finite) étale group scheme over $\Bbbk$.

*Hint: First show that $G_x$ is an etale group algebraic space over $\Bbbk$. If $\Bbbk = \overline{\Bbbk}$, use that a section of the structure morphism $G_x \to \operatorname{Spec} \Bbbk$ is an open immersion to give an open covering of $G_x$ by schemes. Apply Proposition C.1.8 to conclude that $G_x$ is separated. For the general case, apply Effective Descent (Proposition 2.2.11).*

(b) If $\Bbbk$ is algebraically closed, show that $G_x$ is the discrete and reduced (resp. finite and reduced) group scheme corresponding to the abstract group $G_x(\Bbbk)$.

(c) Show that the diagonal of $\mathcal{X}$ is unramified.

*We will see later that these properties characterize Deligne–Mumford stacks; see Theorem 3.6.4.*

Varying the point $x$ of $\mathcal{X}$, the stabilizer group varies and naturally forms a family. We've already seen this: if $a\colon T \to \mathcal{X}$ is an object, then $\operatorname{Isom}_{\mathcal{X}(T)}(a) \to S$ is a group algebraic space such that the fiber over a point $s \in S$ is the stabilizer of the restriction $a|_{\operatorname{Spec}\kappa(s)}$ of $a$ to $\operatorname{Spec}\kappa(s)$. Applying this to the identity map $\mathrm{id}_{\mathcal{X}}\colon \mathcal{X} \to \mathcal{X}$ yields the construction of the inertia stack.

**Definition 3.2.9** (Inertia stack)**.** The *inertia stack* of an algebraic stack $\mathcal{X}$ is the fiber product

$$\begin{tikzcd} I_{\mathcal{X}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{X} \arrow[d] \\ \mathcal{X} \arrow[r] & \mathcal{X} \times \mathcal{X}. \end{tikzcd}$$

In the relative setting of a morphism $\mathcal{X} \to \mathcal{Y}$ of algebraic stacks, the *relative inertia stack* is $I_{\mathcal{X}/\mathcal{Y}} := \mathcal{X} \times_{\mathcal{X}\times_{\mathcal{Y}}\mathcal{X}} \mathcal{X}$.

The fiber of $I_{\mathcal{X}} \to \mathcal{X}$ over a field-valued point $x\colon \operatorname{Spec}\Bbbk \to \mathcal{X}$ is precisely the stabilizer $G_x$. We can therefore think of $I_{\mathcal{X}}$ as a group scheme (or really group algebraic space) over $\mathcal{X}$ incorporating all of the stabilizers of $\mathcal{X}$. If we let $(\mathrm{Sch}/\mathcal{X})_{\text{ét}}$ be the big étale site of schemes over $\mathcal{X}$, then $I_{\mathcal{X}}$ can be viewed as a sheaf of groups on $(\mathrm{Sch}/\mathcal{X})_{\text{ét}}$ where $I_{\mathcal{X}}(a) = \operatorname{Aut}_S(a)$ for $a \in \mathcal{X}(S)$. If $a' \to a$ is a morphism over $S' \to S$, there is a natural pullback functor $\alpha^*\colon \operatorname{Aut}_S(a) \to \operatorname{Aut}_{S'}(a')$ defined as follows: for $\beta \in \operatorname{Aut}_S(a)$, the image $\alpha^*(\beta)$ is the unique dotted arrow (provided by Axiom (2) of the definition of a prestack (2.3.1)) making the diagram

$$\begin{tikzcd} a' \arrow[r, dashed, "\alpha^*(\beta)"'] \arrow[rr, bend left, "\beta\circ\alpha"] & a' \arrow[r, "\alpha"'] & a \end{tikzcd} \tag{3.2.2}$$

commute. Note that if $\alpha\colon \alpha \to \alpha$ is an isomorphism over the identity, then $\alpha^*(\beta) = \alpha^{-1} \circ \beta \circ \alpha$ is conjugation by $\alpha$.

**Exercise 3.2.10.** Let $G \to S$ be a group scheme acting on a scheme $U \to S$, and let $\mathcal{X} = [U/G]$ be the quotient stack. Show that there is a cartesian diagram

$$\begin{tikzcd} S_U \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & U \arrow[d] \\ I_{\mathcal{X}} \arrow[r] & \mathcal{X} \end{tikzcd}$$

where $S_U \to U$ is the stabilizer group scheme, i.e. the fiber product of the action map $G \times U \to U \times U$ and the diagonal $U \to U \times U$.

**Example 3.2.11.** The inertia stack of the classifying stack $\mathbf{B}\mathbb{G}_m$ is $I_{\mathbf{B}\mathbb{G}_m} \cong \mathbb{G}_m \times \mathbf{B}\mathbb{G}_m$. Similarly, if we let $\mathbb{G}_m$ act on $\mathbb{G}_m \times \mathbb{A}^1$ via the product of the trivial and the scaling action and we let $V(x(t-1)) \subset \mathbb{G}_m \times \mathbb{A}^1$ be the $\mathbb{G}_m$-invariant closed subscheme, then $I_{[\mathbb{A}^1/\mathbb{G}_m]} \cong [V(x(t-1))/\mathbb{G}_m]$.

**Exercise 3.2.12.**

(a) If $G$ is a smooth affine algebraic group over a field $\Bbbk$, show that the inertia stack of $\mathbf{B}G$ is the quotient $[G/G]$ where $G$ acts on itself via conjugation. In particular, if $G$ is abelian then $I_{\mathbf{B}G} \cong G \times \mathbf{B}G$.

(b) More generally, show that if $G$ acts on a $\Bbbk$-scheme $U$, show that $I_{[U/G]} \cong [(G \times U)/G]$ where the action is given by $g \cdot (h, u) = (ghg^{-1}, gu)$.

(c) If $G$ is a finite group acting on a $\Bbbk$-scheme $U$,

$$I_{[U/G]} = \coprod_{g \in \mathrm{Conj}(G)} [U^g/C_g],$$

where $\mathrm{Conj}(G)$ is the set of conjugacy classes of elements of $G$, $C_g$ is the centralizer of $g$, and $U^g := \{x \in U \mid gx = x\}$ is the closed locus fixed by $g$.

(d) Explicitly compute the inertia stack for the quotient $[\mathbb{A}^3/S_3]$ of the permutation action.

**Exercise 3.2.13.** Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks.

(1) Show that there are morphisms $I_{\mathcal{X}/\mathcal{Y}} \to I_{\mathcal{X}} \to I_{\mathcal{Y}} \times_{\mathcal{Y}} \mathcal{X}$ of algebraic stacks over $\mathcal{X}$ such that the induced morphisms on the fibers over a field-valued point $x \in \mathcal{X}(\Bbbk)$ correspond to a left exact sequence $1 \to K_x \to G_x \to G_{f(x)}$ of algebraic groups.

(2) Show that there is a cartesian diagram

$$\begin{tikzcd} I_{\mathcal{X}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & I_{\mathcal{Y}} \times_{\mathcal{Y}} \mathcal{X} \arrow[d] \\ \mathcal{X} \arrow[r] & \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}. \end{tikzcd}$$

*Hint: An object of $I_{\mathcal{Y}} \times_{\mathcal{Y}} \mathcal{X}$ over a scheme $S$ is a quadruple $(y, \alpha, x, \beta)$ where $y \in \mathcal{X}(S)$, $\alpha\colon y \xrightarrow{\sim} y$, $x \in \mathcal{X}(S)$, and $\beta\colon y \xrightarrow{\sim} f(x)$. On the other hand, an object of $\mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ over $S$ is a triple $(x_1, x_2, \gamma)$ where $x_1, x_2 \in \mathcal{X}(S)$ and $\gamma\colon f(x_1) \xrightarrow{\sim} f(x_2)$. Define $I_{\mathcal{Y}} \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ on fiber categories by $(y, \alpha, x, \beta) \mapsto (x, x, \beta \circ \alpha \circ \beta^{-1})$. Construct a map $I_{\mathcal{X}}(S)$ to the fiber product of $\mathcal{X}(S)$ and $(I_{\mathcal{Y}} \times_{\mathcal{Y}} \mathcal{X})(S)$ over $(\mathcal{X} \times_{\mathcal{Y}} \mathcal{X})(S)$, and show that it is an equivalence.*

## 3.3 First properties

### 3.3.1 Properties of morphisms

Recall that a morphism of prestacks $\mathcal{X} \to \mathcal{Y}$ over $\mathrm{Sch}_{\text{ét}}$ is representable by schemes (resp. representable) if for every morphism $T \to \mathcal{Y}$ from a scheme, the base change $\mathcal{X} \times_{\mathcal{Y}} T$ is a scheme (resp. algebraic space); see Definitions 3.1.1 and 3.1.3. Both notions are clearly stable under base change. Morphisms representable by schemes are also clearly stable under composition, and the following lemma shows the same for representable morphisms.

**Lemma 3.3.1.**

*(1) If $X \to Y$ is a representable morphism of sheaves on* $\mathrm{Sch}_{\text{ét}}$ *and $Y$ is an algebraic space, then $X$ is an algebraic space.*

*(2) The composition of representable morphisms is representable.*

*Proof.* For the first statement, if $V \to Y$ is an étale presentation, then the base change $X_V \to X$ is a morphism of algebraic spaces which is étale, surjective and representable by schemes. Letting $U \to X_V$ be an étale presentation, then the composition $U \to X_V \to X$ is étale, surjective, and representable by schemes, and thus $X$ is an algebraic space. The second statement follows from the first. □

**Definition 3.3.2.** Let $\mathcal{P}$ be a property of morphisms of schemes.

(1) If $\mathcal{P}$ is stable under composition and base change and is étale-local (resp. smooth-local) on the source and target, a morphism $\mathcal{X} \to \mathcal{Y}$ of Deligne–Mumford stacks (resp. algebraic stacks) *has property* $\mathcal{P}$ if for all étale (resp. smooth) presentations (equivalently there exists presentations) $V \to \mathcal{Y}$ and $U \to \mathcal{X} \times_{\mathcal{Y}} V$ yielding a diagram

$$\begin{tikzcd} U \arrow[r] & \mathcal{X} \times_{\mathcal{Y}} V \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & V \arrow[d] \\ & \mathcal{X} \arrow[r] & \mathcal{Y}, \end{tikzcd}$$

the composition $U \to V$ has $\mathcal{P}$.

(2) A morphism $\mathcal{X} \to \mathcal{Y}$ of algebraic stacks representable by schemes *has property* $\mathcal{P}$ if for every morphism $T \to Y$ from a scheme, the base change $\mathcal{X} \times_{\mathcal{Y}} T \to T$ has $\mathcal{P}$.

(3) A morphism $\mathcal{X} \to \mathcal{Y}$ of algebraic stacks is an *isomorphism*, *open immersion*, *closed immersion*, *locally closed immersion*, *affine*, or *quasi-affine* if it is representable by schemes and has the corresponding property in the sense of (2).

The properties of flatness, smoothness (resp. smoothness of relative dimension $n$), surjectivity, locally of finite presentation, and locally of finite type are smooth-local on the source and target. By (1), these properties extend to morphisms of algebraic stacks. Likewise, étaleness and unramifiedness are étale-local on the source and target, and thus extend to morphisms of Deligne–Mumford stacks. These properties are stable under composition and base change.

Representable morphisms and each class of morphisms in (3) are smooth local on the target. They are even fppf local but we won't be able to show this until §6.2.

**Proposition 3.3.3.** *Let $\mathcal{P}$ be one of the following properties of morphisms of algebraic stacks: representable, isomorphism, open immersion, closed immersion, locally closed immersion, affine, or quasi-affine. Consider a cartesian diagram*

$$\begin{tikzcd} \mathcal{X}' \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{Y}' \arrow[d] \\ \mathcal{X} \arrow[r] & \mathcal{Y} \end{tikzcd}$$

*of algebraic stacks where $\mathcal{Y}' \to \mathcal{Y}$ is smooth and surjective. Then $\mathcal{X} \to \mathcal{Y}$ has $\mathcal{P}$ if and only if $\mathcal{X}' \to \mathcal{Y}'$ has $\mathcal{P}$.*

*Proof.* We will show the ($\Leftarrow$) implications as the other directions are clear. For representability, we may assume that $\mathcal{Y}$, $\mathcal{Y}'$, and $\mathcal{X}'$ are schemes and we need to show that $\mathcal{X}$ is an algebraic space. It suffices to show that the ever automorphism $\alpha\colon a \to a$ of an object $a$ over a scheme $T$ is trivial. The base change $T'$ of $a\colon T \to \mathcal{X}$ by $\mathcal{X}' \to \mathcal{X}$ is a scheme since it's also identified with $T \times_{\mathcal{Y}} \mathcal{Y}'$. Since smooth morphisms étale locally have sections (Corollary A.3.6), there is an étale cover $g\colon \widetilde{T} \to T$ that factors through $T'$. The automorphism $\alpha$ defines a section of $\underline{\mathrm{Aut}}_T(a)$ over $T$. Since $\underline{\mathrm{Aut}}_T(a)$ is a sheaf on $(\mathrm{Sch}/T)_{\text{ét}}$ and $g^*\alpha = \mathrm{id}$, we have that $\alpha = \mathrm{id}$.

For the other properties, we already know that $\mathcal{X} \to \mathcal{Y}$ is representable and it thus suffices to assume that $\mathcal{Y}$, $\mathcal{Y}'$ and $\mathcal{X}'$ are schemes and that $\mathcal{X}$ is an algebraic space. Fortunately we can apply Effective Descent (Proposition 2.2.11) to conclude that $\mathcal{X}$ is a scheme and that $\mathcal{X} \to \mathcal{Y}$ has property $\mathcal{P}$. □

**Example 3.3.4.** If $G \to S$ is a smooth affine group scheme acting on an algebraic space $U \to S$, then $[U/G] \to S$ is flat (resp. smooth, surjective, locally of finite presentation, locally of finite type) if and only if $U \to S$ is. In particular, using the quotient stack presentations in the proofs of Theorems 3.1.16 and 3.1.20, we conclude that $\mathcal{M}_g$ is locally of finite type over $\mathbb{Z}$ and that both $\underline{\mathrm{Coh}}_{r,d}(C)$ and $\mathrm{Bun}_{r,d}(C)$ are locally of finite type over $\Bbbk$.

**Exercise 3.3.5.** Assume that $\mathcal{X} \to \mathcal{Y}$ is a surjective and smooth morphism of algebraic stacks. If $T \to \mathcal{Y}$ is a morphism from a scheme, show that there exists an étale cover $T' \to T$ such that $\mathcal{X}_{T'} \to T'$ has a section.

### 3.3.2 Properties of algebraic spaces and stacks

**Definition 3.3.6** (Properties of algebraic spaces and stacks)**.** Let $\mathcal{P}$ be a property of schemes which is étale (resp. smooth) local. We say that a Deligne–Mumford stack (resp. algebraic stack) $\mathcal{X}$ *has property* $\mathcal{P}$ if for an étale (resp. smooth) presentation (equivalently for all presentations) $U \to \mathcal{X}$, the scheme $U$ has $\mathcal{P}$.

The properties of being locally noetherian, reduced, or regular are smooth-local.

**Example 3.3.7.** Let $G \to S$ be a smooth affine group scheme acting on a scheme $U$ over $S$. Then $[U/G]$ is locally noetherian, reduced, or regular if and only if $U$ is.

**Definition 3.3.8** (Substacks)**.** If $\mathcal{X}$ is an algebraic stack, a substack $\mathcal{Z} \subset \mathcal{X}$ is *closed* (resp. *open*, *locally closed*) if the induced morphism $\mathcal{Z} \to \mathcal{X}$ is a closed immersion (resp. open immersion, locally closed immersion).

**Exercise 3.3.9.** For an action of a smooth affine group scheme $G \to S$ on a scheme $U$ over $S$, show that there is an equivalence between closed (resp. open) substacks of $[U/G]$ and $G$-invariant closed (resp. open) subschemes of $U$.

### 3.3.3 Separation properties

Separation properties for algebraic stacks are defined in terms of the diagonal.

**Definition 3.3.10.**

(1) A morphism of algebraic stack $\mathcal{X} \to \mathcal{Y}$ has *affine diagonal* (resp. *quasi-affine diagonal, separated diagonal*) if the diagonal $\mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ is affine (resp. quasi-affine, separated). An algebraic stack $\mathcal{X}$ has *affine diagonal* (resp. *quasi-affine diagonal, separated diagonal*) if $\mathcal{X} \to \operatorname{Spec} \mathbb{Z}$ does.

(2) A morphism of algebraic stack $\mathcal{X} \to \mathcal{Y}$ is *quasi-separated* if the diagonal $\mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ and second diagonal $\mathcal{X} \to \mathcal{X} \times_{\mathcal{X} \times_{\mathcal{Y}} \mathcal{X}} \mathcal{X}$ are quasi-compact. An algebraic stack $\mathcal{X}$ is *quasi-separated* if it is quasi-separated over $\operatorname{Spec} \mathbb{Z}$.

(3) A representable morphism $\mathcal{X} \to \mathcal{Y}$ of algebraic stacks is *separated* if the morphism $\mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$, which is representable by schemes (Exercise 3.2.3), is proper.

Conditions on the diagonal translate to conditions on the Isom sheaves since the base change of $\mathcal{X} \to \mathcal{X} \times \mathcal{X}$ by a morphism $(a, b)\colon S \to \mathcal{X} \times \mathcal{X}$ from a scheme $S$ is identified with $\underline{\mathrm{Isom}}_{\mathcal{X}(S)}(a, b)$ (see Exercise 2.3.30), which is an algebraic space by Representability of the Diagonal (Theorem 3.2.1(2)). In particular, $\mathcal{X}$ has affine diagonal if and only if every algebraic space $\underline{\mathrm{Isom}}_{\mathcal{X}(S)}(a, b)$ is a scheme affine over $S$. Every algebraic stack with affine or quasi-affine diagonal is necessarily quasi-separated.

**Lemma 3.3.11.** *Let $S$ be an affine scheme and $G \to S$ be a smooth affine group scheme acting on an algebraic space $U$ over $S$. If $U$ has affine diagonal (resp. has quasi-affine diagonal), then so does $[U/G]$.*

*Proof.* Recall that we established that $[U/G]$ is an algebraic stack in Theorem 3.1.10. Representability of the Diagonal (Theorem 3.2.1(2)) implies that $[U/G] \to [U/G] \times_S [U/G]$ is a representable morphism. Using the cartesian diagram

$$\begin{tikzcd} G \times_S U \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & U \times_S U \arrow[d] \\ {[U/G]} \arrow[r] & {[U/G] \times_S [U/G].} \end{tikzcd}$$

Since $G$ is affine, so is the composition $G \times_S U \to U \times_S U \xrightarrow{p_1} U$. The statement follows from the cancellation law and descent. $\square$

The condition of having affine diagonal is satisfied by most moduli problems (except for example $\mathcal{M}_1$).

**Example 3.3.12.** The moduli stacks $\mathcal{M}_g$ and $\mathrm{Bun}_{r,d}(C)$ have affine diagonal and are thus quasi-separated. The statement for $\mathcal{M}_g$ follows from the above lemma and the quotient presentation $\mathcal{M}_g = [H'/\mathrm{PGL}_{5g-5}]$ in the proof of Theorem 3.1.16 where $H'$ is locally closed subscheme of a projective Hilbert scheme. We will show later that $\mathcal{M}_g$ is separated or in other words that the diagonal of $\mathcal{M}_g$ is a finite morphism.

Similarly in Theorem 3.1.20, we expressed every quasi-compact open substack of $\mathrm{Bun}_{r,d}(C)$ as a quotient stack $[Q'/\mathrm{GL}_N]$ where $Q'$ is an open subscheme of a projective Quot scheme. To see that $\mathrm{Bun}_{r,d}(C)$ has affine diagonal, it suffices to show that the base change of the along a morphism $\operatorname{Spec} A \to \mathrm{Bun}_{r,d}(C) \times \mathrm{Bun}_{r,d}(C)$ is affine. But such a morphism factors through $\mathcal{U} \times \mathcal{U}$ for some quasi-compact open substack $\mathcal{U} \subset \mathrm{Bun}_{r,d}(C)$ and we know that $\mathcal{U}$ has affine diagonal.

**Remark 3.3.13.** A quasi-separated Deligne–Mumford stack has *finite* and reduced stabilizer groups (see Exercise 3.2.8).

For morphisms of schemes, the definition of separatedness above agrees with the usual notation as the diagonal of a morphism of schemes is a closed immersion if and only if it is proper. We postpone the definition of separatedness for non-representable morphisms until Definition 3.8.1.

**Example 3.3.14.** The non-separated union $\mathbb{A}^\infty \bigcup_{\mathbb{A}^\infty \setminus 0} \mathbb{A}^\infty$ is a typical example of a non-quasi-separated scheme. For algebraic spaces and stacks, there are additional pathologies coming from actions of non-quasi-compact group schemes. For instance, $[\mathbb{A}^1/\underline{\mathbb{Z}}]$ is a non-quasi-separated algebraic space (see Example 3.9.22) while $\mathbf{B}\underline{\mathbb{Z}}$ is a non-quasi-separated algebraic stack (see Example 3.9.21).

**Exercise 3.3.15.** An action of an algebraic group $G$ over a field $\Bbbk$ on an algebraic space $U$ is called *proper* if the action map

$$\Psi\colon G \times U \to U \times U, \quad (g, u) \mapsto (gu, u)$$

is proper.

(a) Show that the action of $G$ on $U$ is proper if and only if $[U/G]$ is separated.

(b) For $u \in U(\Bbbk)$, let $\Psi_u\colon G \to U$ be the map defined by $g \mapsto gu$ (viewing $\Psi$ as a morphism over $U$ via the projections on the second component, then $\Psi_u$ is the fiber of $\Psi$ over $u$). Show that the following are equivalent:

   (i) $\Psi_u\colon G \to U$ is proper,

   (ii) $u\colon \operatorname{Spec} \Bbbk \to [U/G]$ is proper,

   (iii) $Gu \subset U$ is closed and $G_u$ is proper.

   *Hint: To show that (i) or (ii) implies (iii), replace $U$ with the reduced orbit $Gu$, use Generic Flatness (3.3.30)* to show that $\operatorname{Spec} \Bbbk \to [U/G]$ is faithfully flat, and then use fppf descent.

### 3.3.4 The topological space of a stack

We can associate a topological space $|\mathcal{X}|$ to every algebraic stack $\mathcal{X}$.

**Definition 3.3.16** (Topological space of an algebraic stack)**.** If $\mathcal{X}$ is an algebraic stack, we define *the topological space of* $\mathcal{X}$ as the set $|\mathcal{X}|$ consisting of field-valued morphisms $x\colon \operatorname{Spec} K \to \mathcal{X}$. Two morphisms $x_1\colon \operatorname{Spec} K_1 \to \mathcal{X}$ and $x_2\colon \operatorname{Spec} K_2 \to \mathcal{X}$ are identified in $|\mathcal{X}|$ if there exists field extensions $K_1 \to K_3$ and $K_2 \to K_3$ such that $x_1|_{\operatorname{Spec} K_3}$ and $x_2|_{\operatorname{Spec} K_3}$ are isomorphic in $\mathcal{X}(K_3)$. A subset $U \subset |\mathcal{X}|$ is open if there exists an open substack $\mathcal{U} \subset \mathcal{X}$ such that $U = |\mathcal{U}|$.

A morphism of stacks $\mathcal{X} \to \mathcal{Y}$ induces a continuous map $|\mathcal{X}| \to |\mathcal{Y}|$.

**Exercise 3.3.17.** Show that if $\mathcal{X}$ is an algebraic stack and $U \subset |\mathcal{X}|$ is an open subset, then there exists a reduced closed substack $\mathcal{Z} \hookrightarrow \mathcal{X}$ such that $|\mathcal{Z}| = |\mathcal{X}| \setminus U$.

**Example 3.3.18.** The topological space of the quotient stack $|[\mathbb{A}^1_\Bbbk/\mathbb{G}_m]|$ with the standard scaling action consists of two points with representatives $x_0\colon \operatorname{Spec} k \xrightarrow{0} \mathbb{A}^1 \to [\mathbb{A}^1/\mathbb{G}_m]$ and $x_1\colon \operatorname{Spec} k \xrightarrow{1} \mathbb{A}^1 \to [\mathbb{A}^1/\mathbb{G}_m]$. In particular, the inclusion of the generic point $\operatorname{Spec} \Bbbk(x) \to \mathbb{A}^1 \to [\mathbb{A}^1/\mathbb{G}_m]$ is equivalent to $x_1$.

While the stabilizer group $G_{\overline{x}}$ depends on the choice of representative $\overline{x}\colon \operatorname{Spec} \Bbbk \to \mathcal{X}$ of $x \in |\mathcal{X}|$, its dimension—which we denote by $\dim G_x$—is independent of this choice. Similarly, the properties of being smooth, unramified, affine, finite, and reduced are also independent of this choice.

**Exercise 3.3.19.** Let $x \in |\mathcal{X}|$ be a point of an algebraic stack with two representatives $x_1 \colon \operatorname{Spec} \Bbbk_1 \to \mathcal{X}$ and $x_2 \colon \operatorname{Spec} \Bbbk_2 \to \mathcal{X}$.

(1) Show that the stabilizer group $G_{x_1}$ is smooth (resp. étale, unramified, affine, finite) if and only if $G_{x_2}$ is.

(2) Show that $\dim G_{x_1} = \dim G_{x_2}$.

(3) If $\mathcal{X}$ is Deligne–Mumford and both $\Bbbk_1$ and $\Bbbk_2$ are algebraically closed, show that the abstract discrete groups corresponding to $G_{x_1}$ and $G_{x_2}$ (see Exercise 3.2.8) are isomorphic.

As a consequence of the above exercise, it makes sense to say that $x \in |\mathcal{X}|$ has *smooth* (resp. *étale*, *unramified*, *affine*, *finite*) *stabilizer*. For a Deligne–Mumford stack $\mathcal{X}$, we define the *geometric stabilizer of* $x$ as the discrete group $G = G_{\overline{x}}(\Bbbk)$ where $\overline{x} \colon \operatorname{Spec} \Bbbk \to \mathcal{X}$ is a geometric point representing $x$.

We can now define topological properties of algebraic stacks and their morphisms.

**Definition 3.3.20.** We say that an algebraic stack $\mathcal{X}$ is *quasi-compact*, *connected*, or *irreducible* if $|\mathcal{X}|$ is, and we say that $\mathcal{X}$ is *noetherian* if it is locally noetherian, quasi-compact and quasi-separated.

**Exercise 3.3.21.** Show that an algebraic stack $\mathcal{X}$ is quasi-compact if and only if there exists a smooth presentation $\operatorname{Spec} A \to \mathcal{X}$ and that a quasi-separated algebraic stack $\mathcal{X}$ is noetherian if and only if there exists a smooth presentation $\operatorname{Spec} A \to \mathcal{X}$ where $A$ is a noetherian ring.

**Example 3.3.22.** The moduli stack $\mathcal{M}_g$ is noetherian and in particular quasi-compact. This follows from the above exercise using the quotient presentation $\mathcal{M}_g = [H'/\mathrm{PGL}_{5g-5}]$ from Theorem 3.1.16. However, $\mathrm{Bun}_{r,d}(C)$ is not quasi-compact.

**Exercise 3.3.23.**

(a) Show that a morphism $\mathcal{X} \to \mathcal{Y}$ of algebraic stacks is surjective if and only if $|\mathcal{X}| \to |\mathcal{Y}|$ is surjective.

(b) Show that if $\mathcal{X} \to \mathcal{Y}$ and $\mathcal{Y}' \to \mathcal{Y}$ are morphisms of algebraic stacks, then $|\mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}'| \to |\mathcal{X}| \times_{|\mathcal{Y}|} |\mathcal{Y}'|$ is surjective.

**Exercise 3.3.24.** If $\mathcal{X}$ is a quasi-compact and locally noetherian algebraic stack, show that $|\mathcal{X}|$ is a noetherian topological space.

**Exercise 3.3.25.** Since the property of being universally open for a morphism of schemes is smooth-local on the source and target, we can define *universally open* morphisms of algebraic stacks using Definition 3.3.2(1). This property includes faithfully flat morphisms locally of finite presentation.

(a) If $f \colon \mathcal{X} \to \mathcal{Y}$ is a universally open morphism of algebraic stacks, show that $f(|\mathcal{X}|) \subset |\mathcal{Y}|$ is open and conclude that for every morphism $\mathcal{Y}' \to \mathcal{Y}$ of algebraic stacks, the map $|\mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}'| \to |\mathcal{Y}'|$ is open.

*Hint: Show that the image is identified with the open substack $\mathcal{V} \subset \mathcal{Y}$, whose objects over a scheme $T$ consist of morphisms $T \to \mathcal{Y}$ such that $\mathcal{X}_T \to T$ is surjective.*

(b) Show that if $U \to \mathcal{X}$ is a smooth presentation of an algebraic stack, then a set $\Sigma \subset |\mathcal{X}|$ is open (resp. closed) if and only if its preimage in $U$ is.

**Definition 3.3.26.** A morphism $\mathcal{X} \to \mathcal{Y}$ of algebraic stacks is *quasi-compact* if for every morphism $\operatorname{Spec} B \to \mathcal{Y}$, the fiber product $\mathcal{X} \times_{\mathcal{Y}} \operatorname{Spec} B$ is quasi-compact. We say that $\mathcal{X} \to \mathcal{Y}$ is *of finite type* if $\mathcal{X} \to \mathcal{Y}$ is locally of finite type and quasi-compact.

**Example 3.3.27.** The moduli stack $\mathcal{M}_g$ is finite type over $\mathbb{Z}$. On the other hand, $\operatorname{Bun}_{r,d}(C)$ is locally of finite type over $\Bbbk$ but not of finite type.

**Remark 3.3.28.** A quasi-compact morphism $\mathcal{X} \to \mathcal{Y}$ induces a quasi-compact morphism $|\mathcal{X}| \to |\mathcal{Y}|$ on topological spaces. The converse is true if $\mathcal{Y}$ is quasi-separated but not in general, e.g. $\operatorname{Spec} \Bbbk \to B_{\Bbbk}\mathbb{Z}$ (see Example 3.9.21).

**Exercise 3.3.29.**

(a) Let $f \colon \mathcal{X} \to \mathcal{Y}$ be a quasi-compact morphism of algebraic stacks. For a point $x \in |\mathcal{X}|$, show that $f(\overline{\{x\}}) = \overline{\{f(x)\}}$.

(b) Generalize Chevalley's criterion to algebraic stacks: if $f \colon \mathcal{X} \to \mathcal{Y}$ is a morphism of algebraic stacks locally of finite presentation, then the image $f(|\mathcal{X}|) \subset |\mathcal{Y}|$ is constructible.

(c) Show an open morphism $f \colon \mathcal{X} \to \mathcal{Y}$ of algebraic stacks (i.e. $|\mathcal{X}| \to |\mathcal{Y}|$ is open) satisfies the following lifting property: if $x \in |\mathcal{X}|$ is a point, then every specialization $y' \rightsquigarrow f(x)$ lifts to a specialization $x' \rightsquigarrow x$. Show that the converse is true for morphisms locally of finite presentation.

(d) If $\mathcal{X}$ is a quasi-separated algebraic stack, show that $|\mathcal{X}|$ is a sober topological space, i.e. every irreducible closed subset has a unique generic point.

**Exercise 3.3.30** (Generic Flatness)**.** Generalize Theorem A.2.11 to algebraic stacks: if $\mathcal{X} \to \mathcal{Y}$ is a finite type morphism of algebraic stacks with $\mathcal{Y}$ reduced, then there exists a dense open substack $\mathcal{U} \subset \mathcal{Y}$ such that the base change $X_{\mathcal{U}} \to U$ is flat and of presentation.

**Exercise 3.3.31.** Extend the characterization of locally of finite presentation morphisms given in Proposition A.1.3 to algebraic stacks: a morphism $f \colon \mathcal{X} \to \mathcal{Y}$ of algebraic stacks is locally of finite presentation if and only if for every directed system $\{\operatorname{Spec} A_\lambda\}_{\lambda \in I}$ of affine schemes over $\mathcal{Y}$, the natural map

$$\operatorname{colim}_\lambda \operatorname{MOR}_{\mathcal{Y}}(\operatorname{Spec} A_\lambda, \mathcal{X}) \to \operatorname{MOR}_{\mathcal{Y}}(\operatorname{Spec}(\operatorname{colim}_\lambda A_\lambda), \mathcal{X})$$

is bijective.

### 3.3.5 Quasi-finite and étale morphisms

A morphism of schemes is *locally quasi-finite* if it is locally of finite type and every fiber is discrete. Since this property is étale-local on the source and target, we can extend this property to morphisms of *algebraic spaces* using Definition 3.3.2.

**Definition 3.3.32.**

(1) A representable morphism $\mathcal{X} \to \mathcal{Y}$ of algebraic stacks is *locally quasi-finite* if for every morphism $T \to \mathcal{Y}$ from a scheme, the algebraic space $\mathcal{X} \times_{\mathcal{Y}} T$ is locally quasi-finite over $T$.

(2) A morphism $\mathcal{X} \to \mathcal{Y}$ of algebraic stacks is *locally quasi-finite* if is locally of finite type, the diagonal $\mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ is locally quasi-finite and for every morphism $\operatorname{Spec} \Bbbk \to \mathcal{Y}$ from a field, the topological space $|\mathcal{X} \times_{\mathcal{Y}} \operatorname{Spec} \Bbbk|$ is discrete.

(3) A morphism $\mathcal{X} \to \mathcal{Y}$ of algebraic stacks is *quasi-finite* if it is locally quasi-finite and quasi-compact.

To understand condition (2), recall that the diagonal $\mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ is always a representable morphism (Exercise 3.2.3). The diagonal is quasi-finite (resp. locally quasi-finite) if and only if for every field-valued point $x \in \mathcal{X}(\Bbbk)$ with image $y \in \mathcal{Y}(\Bbbk)$, the kernel $\ker(G_x \to G_y)$ of the induced map of stabilizer groups is finite (resp. discrete); see Exercise 3.2.7. In particular, if $\mathcal{Y}$ is a scheme, the diagonal is quasi-finite if and only if all stabilizers of $\mathcal{Y}$ are finite. For instance, if $G$ is a finite group scheme over a field $\Bbbk$ (e.g. $\mu_p$), then $\mathbf{B}G \to \operatorname{Spec} \Bbbk$ is quasi-finite. On the other hand, $\mathbf{B}\mathbb{G}_m \to \operatorname{Spec} \Bbbk$ is not quasi-finite despite that $|\mathbf{B}\mathbb{G}_m|$ is a single point.

For morphisms of schemes, the property of being étale or unramified is also étale-local on the source and target. We can therefore use Definition 3.3.2 to extend the definition of étale and unramified to morphisms of *Deligne–Mumford stacks.*

**Definition 3.3.33.** A morphism $\mathcal{X} \to \mathcal{Y}$ of algebraic stacks is *étale* (resp. *unramified*) if for every morphism $T \to \mathcal{Y}$ from a scheme, the fiber product $\mathcal{X} \times_{\mathcal{Y}} T$ is a Deligne–Mumford stack[2] such that $\mathcal{X} \times_{\mathcal{Y}} T \to T$ is étale (resp. unramified).

While étale morphisms are smooth and locally quasi-finite, the converse is not true, e.g. $B_{\Bbbk}\boldsymbol{\mu}_p \to \operatorname{Spec} \Bbbk$ over a characteristic $p$ field $\Bbbk$ (see Exercise 6.2.11). Similarly, étale morphisms are smooth of relative dimension 0, but again the converse doesn't hold, e.g. $[\mathbb{A}^1_{\Bbbk}/\mathbb{G}_m] \to \operatorname{Spec} \Bbbk$ over a field $\Bbbk$. We will later establish that every separated, quasi-finite, and representable morphism is quasi-affine (Proposition 4.4.5).

## 3.4 Equivalence relations and groupoids

**Definition 3.4.1.** An *étale (resp. smooth) groupoid of schemes* is a pair of schemes $U$ and $R$ together with étale (resp. smooth) morphisms $s\colon R \to U$ called the *source* and $t\colon R \to U$ called the *target*, and a *composition* morphism $c\colon R \times_{s,U,t} R \to R$ satisfying:

(1) (associativity) the following diagram commutes

$$\begin{tikzcd} R \times_{s,U,t} R \times_{s,U,t} R \arrow[r, "c \times \mathrm{id}"] \arrow[d, "\mathrm{id} \times c"] & R \times_{s,U,t} R \arrow[d, "c"] \\ R \times_{s,U,t} R \arrow[r, "c"] & R, \end{tikzcd}$$

(2) (identity) there exists a morphism $e\colon U \to R$ (called the *identity*) such that the following diagrams commute

$$\begin{tikzcd} & U \arrow[dl, "\mathrm{id}"'] \arrow[d, "e"] \arrow[dr, "\mathrm{id}"] & \\ U & R \arrow[l, "s"] \arrow[r, "t"'] & U \end{tikzcd} \qquad \begin{tikzcd} R \arrow[r, "e \circ s, \mathrm{id}"] \arrow[dr, "\mathrm{id}"'] & R \times_{s,U,t} R \arrow[d, "c"] & R \arrow[l, "e \circ t, \mathrm{id}"'] \arrow[dl, "\mathrm{id}"] \\ & R, & \end{tikzcd}$$

[2] A morphism $\mathcal{X} \to \mathcal{Y}$ of algebraic stacks such that each fiber product $\mathcal{X} \times_{\mathcal{Y}} T$ is Deligne–Mumford is called *relatively Deligne–Mumford* or simply *DM*. A morphism $\mathcal{X} \to \mathcal{Y}$ satisfying the weaker condition that the diagonal $\mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ is locally quasi-finite is called *quasi-DM*. See [SP, Tag 04YW].

(3) (inverse) there exists a morphism $i\colon R \to R$ (called the *inverse*) such that the following diagrams commute

$$\begin{tikzcd} R \arrow[r, "i"] \arrow[dr, "s"'] & R \arrow[r, "i"] \arrow[d, "t"] & R \arrow[dl, "s"] \\ & U & \end{tikzcd} \qquad \begin{tikzcd} R \arrow[r, "s"] \arrow[d, "{(i,\mathrm{id})}"] & U \arrow[d, "e"] \\ R \times_{s,U,t} R \arrow[r, "c"] & R \end{tikzcd} \qquad \begin{tikzcd} R \arrow[r, "t"] \arrow[d, "{(\mathrm{id},i)}"] & U \arrow[d, "e"] \\ R \times_{s,U,t} R \arrow[r, "c"] & R. \end{tikzcd}$$

We will often denote this data as $s, t\colon R \rightrightarrows U$.

If $(s,t)\colon R \to U \times U$ is a monomorphism, then we say that $s, t\colon R \rightrightarrows U$ is an *étale (resp. smooth) equivalence relation.*

If $U$ and $R$ are algebraic spaces, and the source, target, and composition are morphisms of algebraic spaces, we obtain the notion of an *étale (resp. smooth) groupoid of algebraic spaces* and similarly an *étale (resp. smooth) equivalence relation of algebraic spaces.*

We can view $R$ as a *scheme of relations on* $U$: a point $r \in R$ specifies a relation on the points $s(r), t(r) \in U$, which we sometimes write as $s(r) \xrightarrow{r} t(r)$. For every scheme $T$, the morphisms $R(T) \rightrightarrows U(T)$ define a groupoid of sets, i.e. there is composition morphism $R(T) \times_{s,U(T),t} R(T) \to R(T)$ satisfying axioms analogous to (1)–(3). We can think of an element $r \in R(T)$ as specifying a relation $u \xrightarrow{r} v$ between elements $u, v \in U(T)$. The composition morphism composes relations $u \xrightarrow{r} v$ and $v \xrightarrow{r'} w$ to the relation $u \xrightarrow{r \circ r'} w$ while the identity morphism takes $u \in U(T)$ to $u \xrightarrow{\mathrm{id}} u$ and the inverse morphism takes $u \xrightarrow{r} v$ to $v \xrightarrow{r^{-1}} u$. When $R \rightrightarrows U$ is an equivalence relation, the morphism $R(T) \to U(T) \times U(T)$ is injective, and there is thus at most one relation between any two elements of $U(T)$.

**Definition 3.4.2** (Orbits and stabilizers)**.** Let $R \rightrightarrows U$ be a smooth groupoid of algebraic spaces, and let $x\colon \operatorname{Spec} \Bbbk \to U$ be a field-valued point. The *stabilizer* $G_x$ *of* $x$ is defined as the fiber product of $(s,t)\colon R \to U \times U$ by $(x,x)\colon \operatorname{Spec} \Bbbk \to U \times U$. The *orbit* $O_R(x)$ is defined as the *set* $s(t^{-1}(u)) \subset U$.

**Remark 3.4.3.** Assuming that $U$ is defined over $\Bbbk$ and that $x \in U(\Bbbk)$, then the $\Bbbk$-points of $G_x$ are relations $\rho\colon x \xrightarrow{\sim} x$ in $R(\Bbbk)$ while the orbit $O_R(x)$ consists of points $y \in U$ such that there exists a relation $x \xrightarrow{\sim} y$ in $R$.

**Exercise 3.4.4.** Show that the identity and inverse morphism are uniquely determined.

**Example 3.4.5.** If $G \to S$ is an étale (resp. smooth) group scheme with multiplication $\mu\colon G \times_S G \to G$ acting on a scheme $U$ over $S$ via multiplication $\sigma\colon G \times U \to U$, then

$$p_2, \sigma\colon G \times_S U \rightrightarrows U$$

is an étale (resp. smooth) groupoid of schemes. The inverse $G \times_S U \to G \times_S U$ is given by $(g,u) \mapsto (g^{-1}, gu)$ and the composition is

$$(G \times_S U) \times_{\sigma,U,p_2} (G \times_S U) \to G \times_S U, \quad \big((g',u'),(g,u)\big) \mapsto (g'g, u).$$

where $u' = gu$. Here $(g,u)$ is a $T$-valued point of $G \times_S U$ and can be viewed as the relation $u \to gu$.

The following identifies projections in the groupoid with maps arising from the group action:

$$\begin{tikzcd}
U \times_{[U/G]} U \times_{[U/G]} U \arrow[r, "\sim"] \arrow[d, shift right=4, "p_{12}"'] \arrow[d, "p_{13}"] \arrow[d, shift left=4, "p_{23}"] & G \times_S G \times_S U \arrow[d, shift right=4, "\mathrm{id} \times \sigma"'] \arrow[d, "\mu \times \mathrm{id}"] \arrow[d, shift left=4, "p_{23}"] \\
U \times_{[U/G]} U \arrow[r, "\sim"] \arrow[d, shift right=2, "p_1"'] \arrow[d, shift left=2, "p_2"] & G \times_S U \arrow[d, shift right=2, "p_2"'] \arrow[d, shift left=2, "\sigma"] \\
U \arrow[r, equal] \arrow[d] & U \\
{[U/G].} &
\end{tikzcd}$$

The identification $U \times_{[U/G]} U \xrightarrow{\sim} G \times_S U$ is given by $u_2 \times_g u_1 \mapsto (g, u_1)$ where $u_2 \times_g u_1$ is shorthand notation for the triple $(u_2, u_1, g)$ (with $u_2 = gu_1$) defining an element of the fiber product. Similarly $U \times_{[U/G]} U \times_{[U/G]} U \xrightarrow{\sim} G \times_S G \times_S U$ is given by $u_3 \times_{g_2} u_2 \times_{g_1} u_1 \mapsto (g_2, g_1, u_1)$.

More generally, the $n$-fold fiber product $(U/[U/G])^n$ of $U$ over $[U/G]$ is identified with $G^{n-1} \times U$ via $u_n \times_{g_{n-1}} u_{n-1} \cdots \times_{g_1} u_1 \mapsto (g_{n-1}, \dots, g_1, u_1)$. Under these identifications, the projection $p_{\widehat{k}} \colon (U/[U/G])^{n+1} \to (U/[U/G])^n$ forgetting the $k$th term is identified with that map $G^n \times U \to G^{n-1} \times U$ taking an element $(g_n, \dots, g_1, u_1)$ to $(g_{n-1}, \dots, g_1, u_1)$ for $k = 1$, to $(g_n, \dots, g_{k+1}, g_k g_{k-1}, g_{k-2}, \dots, g_1, u_1)$ for $k = 2, \dots, n$ and to $(g_n, \dots, g_2, g_1 u_1)$ for $k = n+1$.

**Example 3.4.6.** Let $\mathcal{X}$ be a Deligne–Mumford stack (resp. algebraic stack) and $U \to \mathcal{X}$ be an étale (resp. smooth) presentation which we assume is not only representable but representable by schemes. Define the scheme $R := U \times_{\mathcal{X}} U$, the source morphism $s = p_1 \colon R \to U$, the target morphism $t = p_2 \colon R \to U$ and the composition morphism $(s \circ p_1, t \circ p_2) \colon R \times_{s,U,t} R \to R := U \times_{\mathcal{X}} U$. This gives the structure of an étale (resp. smooth) groupoid $R \rightrightarrows U$. If $X$ is an algebraic space, then $R \rightrightarrows U$ is an étale equivalence relation.

Choosing different presentations yields different groupoids which are equivalent under a notion called *Morita equivalence*; we will not use this notion in these notes.

### 3.4.1 Algebraicity of groupoid quotients

**Definition 3.4.7** (Quotient stack of a smooth groupoid)**.** Let $s, t \colon R \rightrightarrows U$ be a smooth groupoid of algebraic spaces. Define $[U/R]^{\mathrm{pre}}$ as the prestack whose objects are morphisms $T \to U$ from a scheme $T$. A morphism $(S \xrightarrow{a} U) \to (T \xrightarrow{b} U)$ is the data of a morphism of schemes $f \colon S \to T$ and an element $r \in R(S)$ such that $s(r) = a$ and $t(r) = f \circ b$.

Define $[U/R]$ to be the stackification of $[U/R]^{\mathrm{pre}}$ in the big étale topology $\mathrm{Sch}_{\text{ét}}$.

If in addition $R \rightrightarrows U$ is an equivalence relation, then $[U/R]$ is isomorphic to a sheaf (Exercise 3.4.8) and we denote it as $U/R$.

The fiber category $[U/R]^{\mathrm{pre}}(T)$ is the groupoid whose objects are $U(T)$ and morphisms are $R(T)$. The identity morphism $\mathrm{id} \colon U \to U$ defines a map $U \to [U/R]^{\mathrm{pre}}$ and therefore a map $p \colon U \to [U/R]$.

**Exercise 3.4.8.** Let $R \rightrightarrows U$ be a smooth groupoid of algebraic spaces. Show that $[U/R]$ is equivalent to a sheaf if and only if $R \rightrightarrows U$ is an equivalence relation.

**Exercise 3.4.9.** Extend Exercise 2.3.28 to show that if $s, t \colon R \rightrightarrows U$ is a smooth groupoid of algebraic spaces, the following diagrams are cartesian:

$$\begin{tikzcd} R \arrow[r, "s"] \arrow[d, "t"'] \arrow[dr, phantom, "\square"] & U \arrow[d, "p"] \\ U \arrow[r, "p"] & {[U/R]} \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} R \arrow[r, "{(s,t)}"] \arrow[d] \arrow[dr, phantom, "\square"] & U \times U \arrow[d, "p \times p"] \\ {[U/R]} \arrow[r, "\Delta"] & {[U/R] \times [U/R].} \end{tikzcd}$$

**Exercise 3.4.10.** Let $R \rightrightarrows U$ be a smooth groupoid of algebraic spaces and $x \colon \operatorname{Spec} \Bbbk \to U$ be a field-valued point. Show that the stabilizer of $x$ as defined in Definition 3.4.2 is identified with the stabilizer of $\operatorname{Spec} \Bbbk \to [U/R]$ as defined in Definition 3.2.5.

**Theorem 3.4.11.**

*(1) If $R \rightrightarrows U$ is an étale (resp. smooth) groupoid of algebraic spaces. Then $[U/R]$ is a Deligne–Mumford stack (resp. algebraic stack) and $U \to [U/R]$ is an étale (resp. smooth) presentation.*

*(2) If $R \rightrightarrows U$ be an étale equivalence relation of schemes, then $U/R$ is an algebraic space and $U \to U/R$ is an étale presentation.*

**Remark 3.4.12.** In Corollary 4.4.12, we show that in fact the quotient $U/R$ of an étale equivalence relation of *algebraic spaces* is an algebraic space, establishing that one doesn't obtain new algebro-geometric objects by considering sheaves which are étale locally algebraic spaces. This result is delayed until §4.4 as it takes more work to show that the diagonal of $U/R$ is representable by *schemes*.

More generally, if $R \rightrightarrows U$ is an fppf groupoid (resp. fppf equivalence relation) of algebraic spaces, then $[U/R]$ is an algebraic stack [SP, Tag 06FI] (resp. $U/R$ is an algebraic space [SP, Tag 04S6]). See also [Art74, Thm. 6.1] and [LMB00, Thm. 10.1].

*Proof.* For (1), we will show that $U \to \mathcal{X} := [U/R]$ is representable, surjective and smooth. Let $T \to \mathcal{X}$ be a morphism from a scheme $T$. It follows from the definition of $[U/R]$ as the stackification of $[U/R]^{\mathrm{pre}}$ that there exists an étale cover $T' \to T$ and a commutative diagram

$$\begin{tikzcd} T' \arrow[r] \arrow[d] & U \arrow[d] \\ T \arrow[r] & \mathcal{X}. \end{tikzcd}$$

In the commutative cube

$$\begin{tikzcd}[row sep=small, column sep=small] & U_{T'} \arrow[rr] \arrow[dl] \arrow[dd] & & T' \arrow[dl] \arrow[dd] \\ R \arrow[rr, crossing over] \arrow[dd] & & U & \\ & U_T \arrow[rr] \arrow[dl] & & T \arrow[dl] \\ U \arrow[rr] & & \mathcal{X} \arrow[from=uu, crossing over] & \end{tikzcd} \tag{3.4.1}$$

the front, back, top and bottom squares are cartesian where $U_T$ is a sheaf and $U_{T'}$ is a scheme. Since $T' \to T$ is a surjective étale morphism representable by schemes, so is $U_{T'} \to U_T$. This establishes that $U_T$ is an algebraic space. By descent $U_{T'}$ is surjective and étale over $T$.

For (2), it suffices to show that the diagonal of the quotient sheaf $X := U/R$ is representable by schemes. Indeed, this implies that $U \to X$ is representable by schemes via the argument of Corollary 3.2.2 and étale descent implies that $U \to X$ is étale and surjective. Let $T \to X \times X$ be a morphism from a scheme and consider the cartesian cube

$$\begin{tikzcd}
& Q_{T'} \arrow[rr] \arrow[dl] \arrow[dd] & & T' \arrow[dl] \arrow[dd] \\
R \arrow[rr, crossing over] \arrow[dd] & & U \times U & \\
& Q_T \arrow[rr] \arrow[dl] & & T \arrow[dl] \\
X \arrow[rr] & & X \times X, \arrow[from=uu, crossing over] &
\end{tikzcd} \tag{3.4.2}$$

as in (3.2.1). Since $R \to U \times U$ is separated and locally quasi-finite, so is $Q_{T'} \to T'$. Effective Descent (Proposition B.3.1) implies that sheaf $Q_T$ is a scheme. $\square$

As a consequence, we see that the affine hypothesis in Theorem 3.1.10 asserting the algebraicity of the quotient stack $[X/G]$ and classifying stack $\mathbf{B}G$ is superfluous.

**Exercise 3.4.13.** Show that if $\mathcal{X}$ is an algebraic stack (resp. algebraic space) and $U \to \mathcal{X}$ is a smooth presentation, then $\mathcal{X}$ is isomorphic to the quotient stack $[U/R]$ (resp. quotient sheaf $U/R$) of the étale groupoid (resp. equivalence relation) $R \rightrightarrows U$ where $R = U \times_{\mathcal{X}} U$.

## 3.4.2 Inducing and slicing presentations

We provide here two useful techniques to build new presentations from given ones.

First, let $\mathcal{X} = [X/H]$ be a quotient stack of a smooth algebraic group $H$ acting on a scheme $X$ over $\Bbbk$ and $H \subset G$ be an inclusion of algebraic groups. Then $H$ acts freely on $G \times X$ via $h \cdot (g, x) = (gh^{-1}, hx)$ and we let $G \times^H X$ be the algebraic space quotient $(G \times X)/H$. When $H$ is finite, this quotient exists by definition of an algebraic space and is affine (resp. quasi-projective, projective) when $X$ is by Theorem 4.3.6 (resp. Exercise 4.2.8). In the non-finite case, it follows from Corollary 3.6.7 $G \times^H X$ is an algebraic space if $X$ is noetherian. There is an action of $G$ on $G \times^H X$ via $g \cdot (g', x) = (gg', x)$.

**Exercise 3.4.14.** Show that $[X/H] \cong [(G \times^H X)/G]$.

The second method is sometimes referred to as *slicing a groupoid*. Let $U \to \mathcal{X}$ be a smooth presentation of an algebraic stack with the corresponding groupoid $s, t \colon R = U \times_{\mathcal{X}} U \rightrightarrows U$. If $g \colon U' \to U$ is a morphism, we define *the restriction of* $R \rightrightarrows U$ *along* $U' \to U$ to be the groupoid $R|_{U'} \rightrightarrows U'$ defined by the fiber product

$$\begin{tikzcd}
R|_{U'} \arrow[r, "(t', s')"] \arrow[d] \arrow[dr, phantom, "\square"] & U' \times U' \arrow[d] \\
R \arrow[r, "(t, s)"] & U \times U
\end{tikzcd}$$

**Exercise 3.4.15.**

(a) Show that $R|_{U'}$ fits into a cartesian diagram

$$\begin{tikzcd}
R|_{U'} \arrow[r] \arrow[d] & R \times_{s,U} U' \arrow[r] \arrow[d] & U' \arrow[d, "g"] \\
U' \times_{U,t} R \arrow[r] \arrow[d] & R \arrow[r, "s"] \arrow[d, "t"] & U \arrow[d] \\
U' \arrow[r, "g"] & U \arrow[r] & {[U/R]}
\end{tikzcd}$$

Assume in addition that $U' \times_{U,t} R \to R \xrightarrow{s} U$ is étale (resp. smooth).

(2) Show that $R|_{U'} \rightrightarrows U'$ is an étale (resp. smooth) groupoid.

(3) Show that there is an open immersion $[U'/R|_{U'}] \to [U/R]$.

(4) Show that $[U'/R|_{U'}] \to [U/R]$ is an isomorphism if and only if for every point $u \in U$, there exists a point $u' \in U$ and a relation $u \to g(u')$ in $R$.

## 3.5 Dimension, tangent spaces, and residual gerbes

### 3.5.1 Dimension

Recall that the dimension $\dim X$ of a scheme $X$ is the Krull dimension of the underlying topological space while the dimension $\dim_x X$ at a point $x \in X$ is the minimum dimension of open subsets containing $x$ (which is in general distinct from $\dim \mathcal{O}_{X,x}$). We now extend these definitions to algebraic spaces and stacks.

**Definition 3.5.1.**

(1) Let $X$ be a noetherian algebraic space and $x \in |X|$. We define the *dimension of $X$ at $x$* to be

$$\dim_x X = \dim_u U \in \mathbb{Z}_{\geq 0} \cup \infty$$

where $U \to X$ is an étale presentation and $u \in U$ is a preimage of $x$.

(2) Let $\mathcal{X}$ be a noetherian algebraic stack with smooth presentation $U \to \mathcal{X}$ and corresponding smooth groupoid $s, t\colon R \rightrightarrows U$, and let $u \in U$ be a preimage of $x \in |\mathcal{X}|$. We define the *dimension of $\mathcal{X}$ at $x$* to be

$$\dim_x \mathcal{X} = \dim_u U - \dim_{e(u)} R_u \in \mathbb{Z} \cup \infty$$

where $R_u$ is the fiber of $s\colon R \to U$ over $u$ and $e\colon U \to R$ denotes the identity morphism in the groupoid.

(3) If $\mathcal{X}$ is a noetherian algebraic space or stack, we define the *dimension of $\mathcal{X}$* to be

$$\dim \mathcal{X} = \sup_{x \in |\mathcal{X}|} \dim_x \mathcal{X} \in \mathbb{Z} \cup \infty.$$

**Proposition 3.5.2.** *The definition of the dimension $\dim_x \mathcal{X}$ of a noetherian algebraic stack $\mathcal{X}$ at a point $x \in |\mathcal{X}|$ is independent of the presentation $U \to \mathcal{X}$ and of the choice of preimage $u$ of $x$.*

*Proof.* The dimension of an algebraic space at a point is well defined as étale morphisms have relative dimension 0.

If $U \to \mathcal{X}$ is a smooth presentation (with $U$ a scheme) and $u \in U$ is a preimage of $x$ with residue field $\kappa(u)$, then the fiber $R_u$ is identified with the fiber product

$$\begin{tikzcd} R_u \arrow[r] \arrow[d] & R \arrow[r, "t"] \arrow[d, "s"] & U \arrow[d] \\ \operatorname{Spec} \kappa(u) \arrow[r] & U \arrow[r] & \mathcal{X}, \end{tikzcd}$$

and is a smooth algebraic space over $\kappa(u)$.

If $U' \to \mathcal{X}$ is a second presentation and $u' \in U'$ a preimage of $x$, then define the algebraic space $U'' := U \times_{\mathcal{X}} U'$. Observe that there is a cartesian diagram

$$\begin{tikzcd} U''_u \arrow[r] \arrow[d] & U'' \arrow[r] \arrow[d] & U' \arrow[d] \\ \operatorname{Spec} \kappa(u) \arrow[r] & U \arrow[r] & \mathcal{X} \end{tikzcd} \tag{3.5.1}$$

where the fiber $U''_u$ is identified with $R'_{u'}$. By Exercise 3.5.3 applied to $U'' \to U$, we have the identity

$$\dim_{u''} U'' = \dim_u U + \dim_{u''} U''_u = \dim_u U + \dim_{e'(u')} R'_{u'}. \tag{3.5.2}$$

Choose a representative $\operatorname{Spec} L \to U''$ in $|U''|$ mapping to $u$ and $u'$. Note that the compositions $\operatorname{Spec} \kappa(u) \to U \to \mathcal{X}$, $\operatorname{Spec} \kappa(u') \to U' \to \mathcal{X}$ and $\operatorname{Spec} L \to U'' \to \mathcal{X}$ all define the same point $x \in |\mathcal{X}|$. Let $R \rightrightarrows U$ and $R' \rightrightarrows U'$ be the corresponding smooth groupoids, and set $R''_{u''} = U'' \times_{\mathcal{X}} \operatorname{Spec} L$.

We need to show that

$$\dim_u U - \dim_{e(u)} R_u = \dim_{u'} U' - \dim_{e'(u')} R'_{u'}$$

and by symmetry between $U$ and $U'$, it suffices to show that

$$\dim_u U - \dim_{e(u)} R_u = \dim_{u''} U'' - \dim_{e''(u'')} R''_{u''}$$

where $e''(u'') \in |R''_{u''}|$ is the image of the map $\operatorname{Spec} L \to R''_{u''} = U'' \times_{\mathcal{X}} \operatorname{Spec} L$ defined by the identity automorphism of $u''$. By (3.5.2), this is in turn equivalent to

$$\dim_{e''(u'')} R''_{u''} = \dim_{e(u)} R_u + \dim_{e'(u')} R'_{u'}$$

This last fact follows from the cartesian cube

$$\begin{tikzcd}[row sep=small, column sep=small] & R''_{u''} \arrow[rr] \arrow[dl] \arrow[dd] & & R'_{u'} \times_{\kappa(u')} L \arrow[dl] \arrow[dd] \\ U'' \arrow[rr, crossing over] \arrow[dd] & & U' & \\ & R_u \times_{\kappa(u)} L \arrow[rr] \arrow[dl] & & \operatorname{Spec} L \arrow[dl] \\ U \arrow[rr] & & \mathcal{X}. \arrow[from=uu, crossing over] & \end{tikzcd}$$

and properties of dimension (see Exercise 3.5.3). □

**Exercise 3.5.3.**

(a) Show that the analogue of Proposition A.3.11 holds for algebraic spaces; that is, if $X \to Y$ is a smooth morphism of noetherian algebraic spaces, and if $x \in |X|$ is a point with image $y \in |Y|$, then

$$\dim_x(X) = \dim_y(Y) + \dim_x(X_y).$$

(b) If $X$ and $X'$ are noetherian algebraic spaces over a field $\Bbbk$ with $\Bbbk$-points $x$ and $x'$, show that

$$\dim_{(x,x')} X \times_{\Bbbk} X' = \dim_x X + \dim_{x'} X'.$$

(c) Let $\mathcal{X}$ be a noetherian algebraic space over a field $\Bbbk$ and $\Bbbk \to L$ be a field extension. Set $\mathcal{X}_L = \mathcal{X} \times_{\Bbbk} L$. If $x' \in |\mathcal{X}_L|$ is a point with image $x \in |\mathcal{X}|$, show that $\dim_{x'} \mathcal{X} \times_{\Bbbk} L = \dim_x \mathcal{X}$.

**Example 3.5.4.** If $U$ is a scheme of pure dimension with an action of an affine algebraic group $G$ (which is necessarily of pure dimension) over a field $\Bbbk$, then

$$\dim[U/G] = \dim U - \dim G.$$

In particular, the classifying stack has dimension $\dim \mathbf{B}G = -\dim G$ and we see that the dimension may be negative!

### 3.5.2 Tangent spaces

The *dual numbers* is the ring $\Bbbk[\epsilon] := \Bbbk[\epsilon]/\epsilon^2$ defined over a field $\Bbbk$.

**Definition 3.5.5.** If $\mathcal{X}$ is an algebraic stack and $x\colon \operatorname{Spec} \Bbbk \to \mathcal{X}$, we define the *Zariski tangent space* or simply the *tangent space* of $\mathcal{X}$ at $x$ as the set

$$T_{\mathcal{X},x} := \left\{ \text{2-commutative diagrams} \quad \begin{tikzcd} \operatorname{Spec} \Bbbk \arrow[d, hook] \arrow[dr, "x"] \arrow[d, phantom, shift left=3ex, "\alpha\ \Leftarrow"'] & \\ \operatorname{Spec} \Bbbk[\epsilon] \arrow[r, "\tau"] & \mathcal{X} \end{tikzcd} \right\} \Big/ \sim$$

or in other words the set of pairs $(\tau, \alpha)$ where $\tau\colon \operatorname{Spec} \Bbbk[\epsilon] \to \mathcal{X}$ and $\alpha\colon x \xrightarrow{\sim} \tau|_{\Bbbk}$. Two pairs are equivalent $(\tau, \alpha) \sim (\tau', \alpha')$ if there is an isomorphism $\beta\colon \tau \xrightarrow{\sim} \tau'$ in $\mathcal{X}(k[\epsilon])$ compatible with $\alpha$ and $\alpha'$, i.e. $\alpha' = \beta|_{\operatorname{Spec} \Bbbk} \circ \alpha$

**Proposition 3.5.6.** *If $\mathcal{X}$ is an algebraic stack with affine diagonal and $x \in \mathcal{X}(\Bbbk)$, then $T_{\mathcal{X},x}$ is naturally a $\Bbbk$-vector space.*

*Proof.* Scalar multiplication of $c \in \Bbbk$ on $(\tau, \alpha) \in T_{\mathcal{X},x}$ is defined as the composition $\operatorname{Spec} \Bbbk[\epsilon] \to \operatorname{Spec} \Bbbk[\epsilon] \xrightarrow{\tau} \mathcal{X}$ where the first map is defined by $\epsilon \mapsto c\epsilon$ and with the same 2-isomorphism $\alpha$.

To define addition, we will show that there is an equivalence of categories

$$\mathcal{X}(\Bbbk[\epsilon_1] \times_{\Bbbk} \Bbbk[\epsilon_2]) \to \mathcal{X}(\Bbbk[\epsilon_1]) \times_{\mathcal{X}(\Bbbk)} \mathcal{X}(\Bbbk[\epsilon_2]) \tag{3.5.3}$$

or in other words that

$$\begin{tikzcd} \operatorname{Spec} \Bbbk \arrow[r, hook] \arrow[d, hook] & \operatorname{Spec} \Bbbk[\epsilon_1] \arrow[d, hook] \\ \operatorname{Spec} \Bbbk[\epsilon_2] \arrow[r, hook] & \operatorname{Spec}(\Bbbk[\epsilon_1] \times_{\Bbbk} \Bbbk[\epsilon_2]) \end{tikzcd}$$

is a pushout among algebraic stacks with affine diagonal (see §A.8). Once this is established, we define addition of $(\tau_1, \alpha_1)$ and $(\tau_2, \alpha_2)$ by the composition $\operatorname{Spec} \Bbbk[\epsilon] \to \operatorname{Spec}(\Bbbk[\epsilon_1] \times_{\Bbbk} \Bbbk[\epsilon_2]) \to \mathcal{X}$ where the first map is defined sending both $(\epsilon_1, 0)$ and $(0, \epsilon_2)$ to $\epsilon$.

Choose a smooth morphism $(U, u) \to (\mathcal{X}, x)$ from an affine scheme $U$. Since $\mathcal{X}$ has affine diagonal $U \to \mathcal{X}$ is an affine morphism. Let $\operatorname{Spec} A_0 = \operatorname{Spec} \Bbbk \times_{\mathcal{X}} U$, $\operatorname{Spec} A_1 = \operatorname{Spec} \Bbbk[\epsilon_1] \times_{\mathcal{X}} U$ and $\operatorname{Spec} A_2 = \operatorname{Spec} \Bbbk[\epsilon_2] \times_{\mathcal{X}} U$. Since $\operatorname{Spec}(A_1 \times_A A_2)$ is clearly the pushout of $\operatorname{Spec} A_0 \hookrightarrow \operatorname{Spec} A_1$ and $\operatorname{Spec} A_0 \hookrightarrow \operatorname{Spec} A_2$ in the category of affine schemes, there are unique morphisms $\operatorname{Spec}(A_1 \times_A A_2) \to \operatorname{Spec}(\Bbbk[\epsilon_1] \times_{\Bbbk} \Bbbk[\epsilon_2])$ and $\operatorname{Spec}(A_1 \times_A A_2) \to U$ completing the diagram

$$\begin{tikzcd}
 & \operatorname{Spec} A_0 \arrow[rrr] \arrow[dl] \arrow[dd] & & & \operatorname{Spec} A_2 \arrow[dl] \arrow[dd] & \\
\operatorname{Spec} \Bbbk \arrow[rrr] \arrow[dd] & & & \operatorname{Spec} \Bbbk[\epsilon_1] \arrow[dd] \arrow[dddr, "\tau_1"] & & \\
 & \operatorname{Spec} A_1 \arrow[rrr] \arrow[dl] & & & \operatorname{Spec}(A_1 \times_A A_2) \arrow[dl] \arrow[dr] & \\
\operatorname{Spec} \Bbbk[\epsilon_2] \arrow[rrr] \arrow[drrrr, bend right, "\tau_2"'] & & & \operatorname{Spec}(\Bbbk[\epsilon_1] \times_{\Bbbk} \Bbbk[\epsilon_2]) \arrow[dr, dashed] & & U \arrow[dl] \\
 & & & & \mathcal{X} &
\end{tikzcd}$$

By the Flatness Criterion over Artinian Rings (Proposition A.2.3), we see that the map $\operatorname{Spec}(A_1 \times_A A_2) \to \operatorname{Spec}(\Bbbk[\epsilon_1] \times_{\Bbbk} \Bbbk[\epsilon_2])$ is faithfully flat. By repeating this argument on $U \times_{\mathcal{X}} U$, one argues that the $\operatorname{Spec}(A_1 \times_A A_2) \to U$ descends uniquely providing the desired dotted arrow. $\square$

**Exercise 3.5.7.** Show that $T_{\mathcal{X},x}$ is naturally a representation of $G_x$ which is given set-theoretically by: $g \cdot (\tau, \alpha) = (\tau, g \circ \alpha)$ for $g \in G_x$ and $(\tau, \alpha) \in T_{\mathcal{X},x}$.

**Example 3.5.8.** Consider a smooth, connected, and projective curve $[C] \in \mathcal{M}_g(\Bbbk)$ defined over $\Bbbk$ of genus $g \geq 2$. Deformation theory (Proposition D.1.11) implies that $T_{\mathcal{M}_g,[C]} = H^1(C, T_C)$. Since $\deg T_C < 0$, $\mathrm{H}^0(C, T_C) = 0$ and Riemann–Roch implies

$$\dim T_{\mathcal{M}_g,[C]} = \dim \mathrm{H}^1(C, T_C) = -\chi(T_C) = -(\deg T_C + (1-g)) = 3g - 3.$$

**Example 3.5.9.** Let $C$ be a smooth, connected, and projective curve over $\Bbbk$ and $E \in \mathrm{Bun}_{r,d}(C)(\Bbbk)$ be a vector bundle on $C$ of rank $r$ and degree $d$. Deformation theory (Proposition D.1.15) implies that $T_{\mathrm{Bun}_{r,d}(C),[E]} = \operatorname{Ext}^1_{\mathcal{O}_C}(E, E) = H^1(C, E \otimes E^\vee)$. By Riemann–Roch, $\chi(E \otimes E^\vee) = r^2(1-g)$. Since $\dim \operatorname{Aut}(E) = \dim_{\Bbbk} \operatorname{Hom}_{\mathcal{O}_C}(E, E) = \mathrm{H}^0(C, E \otimes E^\vee)$, we compute that

$$\dim T_{\mathrm{Bun}_{r,d}(C),[E]} = \dim \operatorname{Ext}^1_{\mathcal{O}_C}(F, F) = \dim \operatorname{Aut}(F) + r^2(g-1).$$

**Exercise 3.5.10.** Show that Proposition 3.5.6 remains true without the affine diagonal condition.

**Remark 3.5.11.** Suppose that $A \to A'$ and $A \to A''$ are homomorphisms of artinian local rings such that $A \twoheadrightarrow A'$ is surjective. If $\mathcal{X}$ is an algebraic stack, then the above argument extends to show that

$$\mathcal{X}(A_1 \times_{A_0} A_2) \to \mathcal{X}(A_1) \times_{\mathcal{X}(A_0)} \mathcal{X}(A_2) \tag{3.5.4}$$

is an equivalence of categories. This condition is usually referred to as *homogeneity*. The conditions (RS$_1$)–(RS$_2$) in Rim–Schlessinger's Criteria (Theorem D.3.11) are weaker versions of homogeneity which ensure the existence of a formal miniversal deformation space, and also appear in Artin's Axioms for Algebraicity (Theorem D.7.4).

More generally, (3.5.4) holds if $A \to A'$ and $A \to A''$ are arbitrary ring homomorphisms with $A \twoheadrightarrow A'$ surjective which shows that the Ferrand pushout $\operatorname{Spec} A' \times_A A''$ (see Section A.8) is a pushout in the category of algebraic stacks.

### 3.5.3 Residual gerbes

Attached to every point $x \in X$ of a scheme is a residue field $\kappa(x)$ and a monomorphism $\operatorname{Spec} \kappa(x) \to X$ with image $x$. The residual gerbe will provide us with an analogous property for algebraic stacks. Note that non-trivial stabilizers prevent field-valued points from being monomorphisms (e.g. $\mathbf{B}G$ for a finite group $G$).

**Definition 3.5.12.** Let $\mathcal{X}$ be an algebraic stack and $x \in |\mathcal{X}|$ be a point. We say that the *residual gerbe at $x$ exists* if there is a reduced, locally noetherian algebraic stack $\mathcal{G}_x$ and a monomorphism $\mathcal{G}_x \hookrightarrow \mathcal{X}$ such that $|\mathcal{G}_x|$ is a point mapping to $x$. The algebraic stack $\mathcal{G}_x$ is called the *residual gerbe at $x$*.

Later in Proposition 6.2.34 we will show that $\mathcal{G}_x$ is a gerbe over a field $\kappa(x)$, called the *residue field*. While not apparent from the definition, we will also show that residual gerbes are in fact unique. This will justify our terminology of calling $\mathcal{G}_x$ *the* residual *gerbe*.

In the meantime, we will be content with verifying the existence of residual gerbes at *finite type points* (see Definition 3.5.13 and Proposition 3.5.16). This statement will suffice for most of our purposes, but we will later prove that residual gerbes in fact exist for every point (as long as the stack is quasi-separated) in Proposition 6.2.34—this result is postponed as we will utilize the Fppf Criterion for Algebraicity (Theorem 6.2.1). In §6.2.6, we also provide other characterizations of residual gerbes and fields.

**Definition 3.5.13.** A point $x \in |\mathcal{X}|$ in an algebraic stack is *of finite type* if there exists a representative $\operatorname{Spec} \Bbbk \to \mathcal{X}$ of $x$ that is locally of finite type.[3]

**Remark 3.5.14.** If $X$ is a noetherian scheme, a point $x \in X$ is of finite type if and only if $x \in X$ is locally closed. Slightly more generally, a morphism $\operatorname{Spec} \Bbbk \to X$ with image $x$ is of finite type if and only if the image $x \in X$ is locally closed and $\kappa(x)/\Bbbk$ is a finite extension. Indeed, to see the nontrivial ($\Rightarrow$) implication, we replace $X$ with $\overline{\{x\}}$, and since $\operatorname{Spec} \Bbbk \to X$ is of finite type with dense image, Generic Flatness (A.2.11) implies that $\operatorname{Spec} \Bbbk \to X$ is fppf and thus its image is open.

An example of a finite type point of a scheme that is not closed is the generic point of a DVR. On the other hand, if $X$ is a scheme of finite type over a field $\Bbbk$, then every finite type point is a closed point. The analogous fact is *not* true for algebraic stacks of finite type over $\Bbbk$, e.g. $\operatorname{Spec} \Bbbk \xrightarrow{1} [\mathbb{A}^1/\mathbb{G}_m]$ is an open finite type point.

**Exercise 3.5.15.** Let $\mathcal{X}$ be an algebraic stack.

(a) Show that a point $x \in |\mathcal{X}|$ is of finite type if and only if there exists a scheme $U$, a closed point $u \in U$, and a smooth morphism $(U, u) \to (\mathcal{X}, x)$.

[3] If $\mathcal{X}$ has quasi-compact diagonal, e.g. $\mathcal{X}$ is quasi-separated (Definition 3.3.10), then every field-valued point $\operatorname{Spec} \Bbbk \to \mathcal{X}$ is automatically quasi-compact, and thus the locally of finite typeness of $\operatorname{Spec} \Bbbk \to \mathcal{X}$ is equivalent to finite typeness.

(b) Show that any algebraic stack (resp. quasi-compact algebraic stack) has a finite type point (resp. closed point).

**Proposition 3.5.16.** *If $\mathcal{X}$ is a noetherian algebraic stack and $x \in |\mathcal{X}|$ is a finite type point, then the residual gerbe $\mathcal{G}_x$ exists at $x$ and is unique. Moreover, $\mathcal{G}_x$ is a regular algebraic stack and the morphism $\mathcal{G}_x \hookrightarrow \mathcal{X}$ is a locally closed immersion. If in addition $\mathcal{X}$ is of finite type over a field $\Bbbk$ and $x \in \mathcal{X}(\Bbbk)$ has a smooth affine stabilizer $G_x$, then $\mathcal{G}_x = \mathbf{B}G_x$.*

*Proof.* We first show the existence. After replacing $\mathcal{X}$ with $\overline{\{x\}}$, we may assume that $\mathcal{X}$ is reduced and $x \in |\mathcal{X}|$ is dense. Let $\operatorname{Spec} \Bbbk \to \mathcal{X}$ be a finite type representative of $x$. By Generic Flatness (3.3.30), $\operatorname{Spec} \Bbbk \to \mathcal{X}$ is flat. Therefore the image of $\operatorname{Spec} \Bbbk \to \mathcal{X}$—which is $\{x\} \subset |\mathcal{X}|$—is open (Exercise 3.3.25). The corresponding open substack $\mathcal{G}_x \subset \mathcal{X}$ satisfies the properties of being a residual gerbe. Since $\operatorname{Spec} \Bbbk \to \mathcal{G}_x$ is fppf and the property of being regular descends under fppf morphisms (Proposition B.4.4), $\mathcal{G}_x$ is regular.

For the uniqueness, suppose that $\mathcal{G}$ and $\mathcal{G}'$ are reduced, locally noetherian algebraic stacks with monomorphisms $\mathcal{G} \hookrightarrow \mathcal{X}$ and $\mathcal{G}' \hookrightarrow \mathcal{X}$ such that $|\mathcal{G}|$ and $|\mathcal{G}'|$ are singletons mapping to $x$. Then $\mathcal{G}'' := \mathcal{G} \times_{\mathcal{X}} \mathcal{G}'$ is a nonempty algebraic stack with monomorphisms $\mathcal{G}'' \to \mathcal{G}$ and $\mathcal{G}'' \to \mathcal{G}'$. Let $\operatorname{Spec} \Bbbk \to \mathcal{G}$ be a finite type morphism from a field (which exists by Exercise 3.5.15); by Generic Flatness (3.3.30) $\operatorname{Spec} \Bbbk \to \mathcal{G}$ is fppf. The base change $\mathcal{G}'' \times_{\mathcal{G}} \operatorname{Spec} \Bbbk$ is a nonempty algebraic space equipped with a monomorphism to $\operatorname{Spec} \Bbbk$. This implies that $\mathcal{G}'' \times_{\mathcal{G}} \operatorname{Spec} \Bbbk \to \operatorname{Spec} \Bbbk$ is an isomorphism, and by fppf descent $\mathcal{G}'' \to \mathcal{G}$ is also an isomorphism. Similarly, $\mathcal{G}'' \to \mathcal{G}'$ is an isomorphism.

Suppose now that $\mathcal{X}$ is of finite type over a field $\Bbbk$ and $x \in \mathcal{X}(\Bbbk)$ has a smooth affine stabilizer $G_x$. There is a monomorphism of prestacks $\mathbf{B}G_x^{\mathrm{pre}} \to \mathcal{X}$: for a $\Bbbk$-scheme $T$, there is a unique object of $\mathbf{B}G_x^{\mathrm{pre}}$ over $T$, and this object gets mapped to the composition $T \to \operatorname{Spec} \Bbbk \xrightarrow{x} \mathcal{X}$. Similarly, a morphism over $T' \to T$ corresponds to a map $T' \to G_x$, and this gets mapped to the corresponding morphism in $\mathcal{X}$. Under stackification, this induces a monomorphism $\mathbf{B}G_x \to \mathcal{X}$, and thus $\mathbf{B}G_x$ satisfies the properties of a residual gerbe. $\square$

**Exercise 3.5.17.** Let $\mathcal{X}$ be a (possibly non-noetherian) algebraic stack and $x \in \mathcal{X}$ be a finite type point such that the stabilizer is unramified (i.e. the stabilizer group scheme of any representative is unramified). Show that the residual gerbe exists and is unique. See also [SP, Tag 06G3].

**Corollary 3.5.18.** *Let $x \in |\mathcal{X}|$ be a finite type point of a noetherian algebraic stack $\mathcal{X}$. If $(U, u) \to (\mathcal{X}, x)$ is a smooth morphism from a scheme $U$ with $u \in U$ a finite type point, then there is a cartesian diagram*

$$\begin{tikzcd} O(u) \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & U \arrow[d] \\ \mathcal{G}_x \arrow[r, hook] & \mathcal{X} \end{tikzcd} \tag{3.5.5}$$

*where $O(u)$ is identified set-theoretically with the orbit $s(t^{-1}(u))$ of the induced groupoid $s, t \colon R := U \times_{\mathcal{X}} U \rightrightarrows U$.* $\square$

**Remark 3.5.19.** If $\mathcal{X} = [U/G]$ is the quotient stack of a smooth affine algebraic group over a field $\Bbbk$ acting on a noetherian $\Bbbk$-scheme $U$ and $u \in U(\Bbbk)$, there is a cartesian diagram

$$\begin{tikzcd} Gu \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & U \arrow[d] \\ \mathbf{B}G_x \arrow[r, hook] & {[U/G].} \end{tikzcd}$$

We recover the familiar fact that orbit $Gu \hookrightarrow U$ is locally closed (C.3.3(6)).

**Corollary 3.5.20.** *A finite type point $x \in |X|$ of a noetherian algebraic space has a residue field $\kappa(x)$, i.e. there is a field $\kappa(x)$ and a locally closed immersion $\operatorname{Spec} \kappa(x) \hookrightarrow X$ with image $x$.* □

**Exercise 3.5.21.** Let $\mathcal{X}$ be a noetherian algebraic stack and $x \in |\mathcal{X}|$ be a finite type point with smooth stabilizer. Let $\overline{x}\colon \operatorname{Spec} \Bbbk \to \mathcal{X}$ be a representative of $x$. Show that $\dim \mathcal{G}_x = -\dim G_{\overline{x}}$.

## 3.6 Characterization of Deligne–Mumford stacks

### 3.6.1 Existence of minimal presentations

**Theorem 3.6.1** (Existence of Minimal Presentations)**.** *Let $\mathcal{X}$ be a noetherian algebraic stack and let $x \in |\mathcal{X}|$ be a finite type point with smooth stabilizer $G_x$. Then there exists a scheme $U$ with a closed point $u \in U$ and a smooth morphism $(U, u) \to (\mathcal{X}, x)$ of relative dimension $\dim G_x$ from a scheme $U$ such that the diagram*

$$\begin{tikzcd} \operatorname{Spec} \kappa(u) \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & U \arrow[d] \\ \mathcal{G}_x \arrow[r, hook] & \mathcal{X} \end{tikzcd}$$

*is cartesian.*

*In particular, if $G_x$ is finite and reduced, there is an étale morphism $(U, u) \to (\mathcal{X}, x)$ from a scheme.*

*Proof.* Let $(U, u) \to (\mathcal{X}, x)$ be a smooth morphism of relative dimension $n$ from a scheme $U$ such that $u \in U$ is a finite type point. From Proposition 3.5.16, the residual gerbe $\mathcal{G}_x$ at $x$ exists and is regular of dimension $-\dim G_x$ (Exercise 3.5.21). We obtain a cartesian diagram

$$\begin{tikzcd} O(u) \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & U \arrow[d] \\ \mathcal{G}_x \arrow[r, hook] & \mathcal{X}. \end{tikzcd}$$

It follows that $O(u)$ is a regular scheme of dimension $c := n - \dim G_x$. Let $f_1, \ldots, f_c \in \mathcal{O}_{O(u),u}$ be a regular sequence generating the maximal ideal at $u$. After replacing $U$ with an open affine neighborhood of $u$, we may assume that each $f_i$ is a global function on $U$. We can consider the closed subscheme $W := V(f_1, \ldots, f_c)$ which by design intersects $O(u)$ transversely at $U$, i.e. $W \cap O(u) = \operatorname{Spec} \kappa(u)$ scheme-theoretically.

By inductively applying a version of the local criterion for flatness (Corollary A.2.8) to the smooth groupoid $U \times_{\mathcal{X}} U \rightrightarrows U$ at a preimage of $u$ and the applying smooth descent, we conclude that the composition $W \hookrightarrow U \to \mathcal{X}$ is flat at $u$. Since $G_x$ is smooth, so is $\operatorname{Spec} \kappa(u) \to \mathcal{G}_x$. For flat morphisms, smoothness is a property that can be checked on fibers and thus (again arguing on $R \rightrightarrows U$ and using descent) $W \to \mathcal{X}$ is smooth at $u$. The statement follows after replacing $W$ with an open neighborhood of $u$. □

**Remark 3.6.2.** A smooth presentation $p\colon U \to \mathcal{X}$ is called a *miniversal at* $u \in U(\Bbbk)$ if $T_{U,u} \to T_{\mathcal{X},p(u)}$ is an isomorphism of $\Bbbk$-vector spaces. We will see that the above presentations are miniversal in Proposition 3.7.3.

If the stabilizer $G_x$ is not smooth, there are two candidates for 'minimal presentations.' There still exists a miniversal presentation $(U,u) \to (\mathcal{X},x)$, but its relative dimension is equal to the dimension of the Lie algebra of $G_x$ (rather than $\dim G_x$) and the fiber product $\mathcal{G}_x \times_{\mathcal{X}} U$ may be positive dimensional. For example, $\mathbf{B}\boldsymbol{\mu}_p$ is an algebraic stack in characteristic $p$ (Proposition 6.2.9) and it can be realized as the quotient of $\mathbb{G}_m$ acting on itself via $t \cdot x = t^p x$; here $\mathbb{G}_m \to \mathbf{B}\boldsymbol{\mu}_p$ is a miniversal presentation. On the other hand, there is an fppf (but not smooth) morphism $(U,u) \to (\mathcal{X},x)$ such that $\mathcal{G}_x \times_{\mathcal{X}} U \cong \operatorname{Spec} \kappa(u)$. In particular, if $\mathcal{X}$ has quasi-finite diagonal, then there is an fppf and quasi-finite morphism $(U,u) \to (\mathcal{X},x)$. In our example, the map $\operatorname{Spec} \Bbbk \to \mathbf{B}\boldsymbol{\mu}_p$ is such a presentation.

**Exercise 3.6.3.** If $\mathcal{X}$ is a (possibly non-noetherian) algebraic stack and $x \in \mathcal{X}$ is a finite type point with unramified stabilizer, show that there is an étale morphism $(U,u) \to (\mathcal{X},x)$ from a scheme $U$ where $u \in U$ is a closed point.

*Hint: Replicate the argument above using Exercise 3.5.17.*

### 3.6.2 Equivalent characterizations

**Theorem 3.6.4** (Characterization of Deligne–Mumford Stacks)**.** *Let $\mathcal{X}$ be an algebraic stack. The following are equivalent:*

*(1) the stack $\mathcal{X}$ is a Deligne–Mumford;*

*(2) the diagonal $\mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is unramified; and*

*(3) every point of $\mathcal{X}$ has a finite and reduced stabilizer group.*

*Proof.* The equivalence (2) $\iff$ (3) is essentially the definition of unramified: since the diagonal $\mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is always locally of finite type (Exercise 3.2.4), it is unramified if and only if every geometric fiber (which is either empty or isomorphic to a stabilizer) is discrete and reduced. It is not hard to see that a Deligne–Mumford stack has unramified diagonal (Exercise 3.2.8). For the converse, Existence of Minimal Presentations (Theorem 3.6.1 and Exercise 3.6.3) shows that for every finite type point $x \in \mathcal{X}$, there is an étale morphism $U \to \mathcal{X}$ from a scheme whose image contains $x$. Thus $\mathcal{X}$ is Deligne–Mumford.

See [LMB00, Thm 8.1] and [SP, Tag 06N3]. □

**Theorem 3.6.5** (Characterization of Algebraic Spaces)**.** *Let $\mathcal{X}$ be an algebraic stack whose diagonal is representable by schemes. The following are equivalent:*

*(1) the stack $\mathcal{X}$ is an algebraic space;*

*(2) the diagonal $\mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is a monomorphism; and*

*(3) every point of $\mathcal{X}$ has a trivial stabilizer.*

**Remark 3.6.6.** We will remove the pesky hypothesis that $\Delta_{\mathcal{X}}$ is representable by schemes in Theorem 4.4.10.

*Proof.* Condition (2) is equivalent to the condition that $\mathcal{X}$ is a sheaf. The implication (1) $\Rightarrow$ (2) follows from the definition of an algebraic space. For the converse, if $\mathcal{X}$ is a sheaf, then Theorem 3.6.1 implies that there exists a surjective, étale, and representable morphism $U \to \mathcal{X}$ from a scheme. Since $\Delta_{\mathcal{X}}$ is representable by schemes, so is $U \to \mathcal{X}$.

The equivalence (2) $\iff$ (3) follows from the fact that a group scheme of finite type is trivial if and only if every fiber is trivial (Proposition C.1.8). □

**Corollary 3.6.7.** *Let $G \to S$ be a smooth and affine group scheme over a scheme $S$. Let $U$ be an algebraic space over $S$ with an action of $G$. Then*

*(1) $[U/G]$ is Deligne–Mumford $\iff$ every point of $U$ has a discrete and reduced stabilizer group $\iff$ the action map $G \times U \to U \times U$ is unramified.*

*(2) $[U/G]$ is an algebraic space $\iff$ every point of $U$ has a trivial stabilizer group $\iff$ the action map $G \times U \to U \times U$ is a monomorphism.* □

**Corollary 3.6.8** (Characterization of Representable Morphisms)**.** *Let $\mathcal{X} \to \mathcal{Y}$ be a morphism of noetherian algebraic stacks whose diagonal is representable by schemes. Then $\mathcal{X} \to \mathcal{Y}$ is representable if and only if for every geometric point $x \in \mathcal{X}(\Bbbk)$, the map $G_x \to G_{f(x)}$ on automorphism groups is injective.* □

**Corollary 3.6.9.** *If $g \geq 2$, $\mathcal{M}_g$ is a Deligne–Mumford of finite type over $\mathbb{Z}$ with affine diagonal.*

*Proof.* It only remains to show that $\mathcal{M}_g$ is Deligne–Mumford and by Theorem 3.6.4 it suffices to show that for every smooth, connected, and proper curve $C$ over $\Bbbk$ that $G = \mathrm{Aut}(C)$ is discrete and reduced, or in other words that the dimension of the Lie algebra $\dim T_{G,e} = 0$. The vector space $T_{G,e}$ is identified with the automorphism group of the trivial first-order deformation of $C$. Deformation theory (Proposition D.2.6) implies that $T_{G,e} = \mathrm{H}^0(C, T_C)$, but this vector space is zero since the degree of $T_C = \Omega_C^\vee$ is $2 - 2g < 0$. □

## 3.7 Smoothness and the Infinitesimal Lifting Criteria

We state and prove the Infinitesimal Lifting Criteria (Theorem 3.7.1) which provides extremely useful functorial criteria to check that moduli stacks are smooth. We apply this criteria to establish that the moduli stacks $\mathcal{M}_g$ of smooth curves and $\mathrm{Bun}_{r,d}(C)$ of vector bundles are smooth (Propositions 3.7.4 and 3.7.5)

### 3.7.1 Infinitesimal Lifting Criteria

Since flatness and smoothness are smooth-local properties on the source and target, we have the notions of smoothness and flatness for arbitrary morphisms of algebraic stacks (Definition 3.3.2). Since étaleness and unramifiedness are étale-local on the source and smooth-local on the target, we can make sense of étale or unramified morphisms of algebraic stacks; see Definition 3.3.33.

The following criteria will be our means for establishing that moduli stacks are smooth.

**Theorem 3.7.1** (Infinitesimal Lifting Criteria for Unramifed/Étale/Smooth Morphisms)**.** *Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a finite type morphism of noetherian algebraic stacks. Consider 2-commutative diagrams*

$$\begin{tikzcd} \operatorname{Spec} A_0 \arrow[r] \arrow[d, hook] & \mathcal{X} \arrow[d, "f"] \\ \operatorname{Spec} A \arrow[r] \arrow[ur, dashed] & \mathcal{Y}, \end{tikzcd} \tag{3.7.1}$$

*of solid arrows where $A \to A_0$ is a surjection of artinian local rings with residue field $\Bbbk$ such that $\ker(A \to A_0) \cong \Bbbk$ and $\operatorname{Spec} \Bbbk \hookrightarrow \operatorname{Spec} A_0 \to \mathcal{X}$ is a finite type point.*

*Then*

*(1) $f$ is unramified if and only if for every 2-commutative diagram (3.7.1), any two liftings are isomorphic.*

*(2) $f$ is étale if and only if for every 2-commutative diagram (3.7.1), there exists a lifting which is unique up to unique isomorphism.*

*(3) $f$ is smooth if and only if for every 2-commutative diagram (3.7.1), there exists a lifting.*

**Remark 3.7.2.** To be explicit, a *lifting* of a 2-commutative diagram

$$\begin{tikzcd} S \arrow[r, "x"] \arrow[d, "g"'] \arrow[dr, phantom, "\swarrow\!\!\!\!/_{\alpha}"] & \mathcal{X} \arrow[d, "f"] \\ T \arrow[r, "y"'] & \mathcal{Y}, \end{tikzcd} \tag{3.7.2}$$

is the data of a morphism $\widetilde{x}\colon T \to \mathcal{X}$ as pictured

$$\begin{tikzcd} S \arrow[r, "x"] \arrow[d, "g"'] & \mathcal{X} \arrow[d, "f"] \\ T \arrow[r, "y"'] \arrow[ur, "\widetilde{x}"] & \mathcal{Y}, \end{tikzcd}$$

$$\beta \Uparrow \qquad \Downarrow \gamma$$

together with 2-morphisms $\beta\colon \widetilde{x} \circ g \xrightarrow{\sim} x$ and $\gamma\colon f \circ \widetilde{x} \xrightarrow{\sim} y$ such that

$$\begin{tikzcd} & f \circ x \arrow[dl, "\alpha"'] & \\ y \circ g & & f \circ \widetilde{x} \circ g \arrow[ul, "f(\beta)"'] \arrow[ll, "g^*\gamma"] \end{tikzcd}$$

commutes. A morphism $(\widetilde{x}, \beta, \gamma) \to (\widetilde{x}', \beta', \gamma')$ of liftings is a 2-morphism $\Theta\colon \widetilde{x} \to \widetilde{x}'$ such that $\beta = \beta' \circ (\Theta \circ g)$ and $\gamma = \gamma' \circ f(\Theta)$.

We can also interpret liftings using the map $\Psi\colon \mathcal{X}(T) \to \mathcal{X}(S) \times_{\mathcal{Y}(S)} \mathcal{Y}(T)$ of groupoids. The 2-commutativity of (3.7.2) defines an object $(x, y, \alpha) \in \mathcal{X}(S) \times_{\mathcal{Y}(S)} \mathcal{Y}(T)$ and the category of liftings is the fiber category over this object, e.g. a lifting is an object $\widetilde{x} \in \mathcal{X}(T)$ together with an isomorphism $\Psi(\widetilde{x}) \to (x, y, \alpha)$. For instance, the existence of a lifting translates to the essential surjectivity of $\mathcal{X}(T) \to \mathcal{X}(S) \times_{\mathcal{Y}(S)} \mathcal{Y}(T)$.

*Proof.* We handle the smooth case and leave the remaining cases to the reader. We first show that smoothness implies formal smoothness, i.e. every 2-commutative diagram (3.7.1) has a lifting. By replacing $\mathcal{X}$ with $\mathcal{X} \times_{\mathcal{Y}} \operatorname{Spec} A$ and $\mathcal{Y}$ with $\operatorname{Spec} A$, we may assume that $\mathcal{Y}$ is affine and we need to show that a section over $\operatorname{Spec} A_0$

$$\begin{tikzcd} & \mathcal{X} \arrow[d] \\ \operatorname{Spec} A_0 \arrow[ur] \arrow[r, hook] & \operatorname{Spec} A \arrow[u, dashed, bend right] \end{tikzcd}$$

extends to a section over $\operatorname{Spec} A$.

If $\mathcal{X}$ is a scheme, then the existence of a lifting is provided by the Infinitesimal Lifting Criterion for Smoothness (A.3.1) for schemes. If $\mathcal{X} = X$ is an algebraic space, we may choose a étale presentation $U \to X$ from a scheme. Since $U \to X$ is representable by schemes, it is formally smooth and we may lift $\operatorname{Spec} A_0 \to X$ to $\operatorname{Spec} A_0 \to U$. The composition $U \to X \to \operatorname{Spec} A$ is a smooth morphism of schemes, thus formally smooth, and we can lift the section over $\operatorname{Spec} A_0$ to a section over $\operatorname{Spec} A$. In general, if $\mathcal{X}$ is an algebraic stack, we can choose a smooth morphism $U \to \mathcal{X}$ from an algebraic space and a lifting $\operatorname{Spec} \Bbbk \to U$ of $\operatorname{Spec} \Bbbk \hookrightarrow \operatorname{Spec} A_0 \to \mathcal{X}$ (Proposition 4.2.14). Since we've already shown that smooth representable morphisms are formally smooth, there is a lifting $\operatorname{Spec} A_0 \to U$ of $\operatorname{Spec} A_0 \to \mathcal{X}$. Now $U \to \mathcal{X} \to \operatorname{Spec} A$ is a smooth morphism of schemes so we see that there is a section extending $\operatorname{Spec} A_0 \to U$.

Conversely, if $\mathcal{X} \to \mathcal{Y}$ is formally smooth, then choose smooth presentation $V \to \mathcal{Y}$ and $U \to \mathcal{X} \times_{\mathcal{Y}} V$. By the above argument, $U \to \mathcal{X} \times_{\mathcal{Y}} V$ is formally smooth. Since $\mathcal{X} \times_{\mathcal{Y}} V \to V$ is formally smooth, so is the composition $U \to \mathcal{X} \times_{\mathcal{Y}} V$. But as $U \to V$ is a morphism of schemes, it is formally smooth (Smooth Equivalences A.3.1). Since smoothness is a smooth-local property on the source and target, we obtain that $\mathcal{X} \to \mathcal{Y}$ is smooth.

See also [LMB00, 4.15(ii)] and [SP, Tag 0DP0]. $\square$

As a first application, we see that the presentations produced by Existence of Minimal Presentations (Theorem 3.6.1) are in fact miniversal, and that the dimension of a smooth algebraic stack can be computed in terms of its tangent space and stabilizer.

**Proposition 3.7.3.** *Let $\mathcal{X}$ be a noetherian algebraic stack and $x \in |\mathcal{X}|$ be a finite type point with smooth stabilizer. Let $f\colon (U, u) \to (\mathcal{X}, x)$ be a smooth morphism from a scheme such that $\mathcal{G}_x \times_{\mathcal{X}} U \cong \operatorname{Spec} \kappa(u)$. Then $U \to \mathcal{X}$ is miniversal at $u$, i.e. $T_{U,u} \to T_{\mathcal{X},f(u)}$ is an isomorphism of $\kappa(u)$-vector spaces.*

*In particular, if $\mathcal{X}$ is smooth over a field $\Bbbk$ and $x \in \mathcal{X}(\Bbbk)$ is a point with smooth stabilizer. Then*

$$\dim_x \mathcal{X} = \dim T_{\mathcal{X},x} - \dim G_x.$$

*Proof.* Surjectivity of $T_{U,u} \to T_{\mathcal{X},f(u)}$ follows from the Infinitesimal Lifting Criterion (Theorem 3.7.1). Let $\Bbbk = \kappa(u)$. Injectivity follows from the fact that

$$\begin{tikzcd} \operatorname{Spec} \Bbbk \arrow[r, hook] \arrow[d] & U \arrow[d] \\ \mathcal{G}_x \arrow[r, hook] & \mathcal{X} \end{tikzcd}$$

is cartesian. Indeed, if $\tau\colon \operatorname{Spec} \Bbbk[\epsilon] \to U$ is an element of $T_{U,u}$ mapping to $0 \in T_{\mathcal{X},f(u)}$, then by the definition of the residual gerbe, the composition $\operatorname{Spec} \Bbbk[\epsilon] \to U \to \mathcal{X}$

factors through $\mathcal{G}_x$ and therefore also factors through the fiber product $\operatorname{Spec} \Bbbk$. We conclude that $\tau = 0$.

For the last statement, Existence of Minimal Presentations (Theorem 3.6.1) produces a smooth morphism $(U, u) \to (\mathcal{X}, x)$ miniversal at $u$ and whose relative dimension is equal to $\dim G_x$. Therefore $\dim_x \mathcal{X} = \dim_u U - \dim G_x$ but since $U$ is smooth at $u$, we have $\dim_u U = \dim T_{U,u} = \dim T_{\mathcal{X},x}$. $\square$

### 3.7.2 Smoothness of moduli stacks

The Infinitesimal Lifting Criterion for Smoothness combined with deformation theory allows us to verify the smoothness of a moduli problem and to compute its dimension.

**Proposition 3.7.4.** *For $g \geq 2$, the Deligne–Mumford stack $\mathcal{M}_g$ is smooth over $\operatorname{Spec} \mathbb{Z}$ of relative dimension $3g - 3$.*

*Proof.* Let $\operatorname{Spec} \Bbbk \to \mathcal{M}_g$ be a morphism from a field $\Bbbk$ corresponding to smooth projective and connected curve $C \to \operatorname{Spec} \Bbbk$. Consider a diagram

$$\begin{tikzcd} \operatorname{Spec} \Bbbk \arrow[r] \arrow[rr, bend left, "{[C]}"] & \operatorname{Spec} A_0 \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\not\swarrow\alpha"] & \mathcal{M}_g \arrow[d, "f"] \\ & \operatorname{Spec} A \arrow[r] & \operatorname{Spec} \mathbb{Z}, \end{tikzcd} \tag{3.7.3}$$

where $A \to A_0$ is surjection of artinian local rings with residue field $\Bbbk$ such that $\Bbbk = \ker(A \to A_0)$. The map $\operatorname{Spec} A_0 \to \mathcal{M}_g$ corresponds to a family of curves $\mathcal{C}_0 \to \operatorname{Spec} A_0$ and a cartesian diagram

$$\begin{tikzcd} C \arrow[r, hook] \arrow[d] & \mathcal{C}_0 \arrow[r, hook, dashed] \arrow[d] & \mathcal{C} \arrow[d, dashed] \\ \operatorname{Spec} \Bbbk \arrow[r, hook] & \operatorname{Spec} A_0 \arrow[r, hook] & \operatorname{Spec} A \end{tikzcd}$$

of solid arrows: a lifting of the diagram (3.7.3) corresponds to a family $\mathcal{C} \to \operatorname{Spec} A$ extending $\mathcal{C}_0 \to \operatorname{Spec} A_0$. By Proposition D.2.6, there is cohomology class $\mathrm{ob}_C \in \mathrm{H}^2(C, T_C)$ such that $\mathrm{ob}_C = 0$ if and only if there exists a lifting. Since $C$ is a curve, $\mathrm{H}^2(C, T_C) = 0$. Finally, deformation theory gives the identification $T_{\mathcal{M}_g,[C]} = \mathrm{H}^1(C, T_C)$ which has dimension $3g-3$ by Riemann–Roch (see Example 3.5.8). Since $\dim \operatorname{Aut}(C) = 0$, we conclude that $\dim_{[C]} \mathcal{M}_g = 3g - 3$. $\square$

**Proposition 3.7.5.** *The algebraic stack $\operatorname{Bun}_{r,d}(C)$ is smooth over $\operatorname{Spec} \Bbbk$ of dimension $r^2(g-1)$.*

*Proof.* Let $[F] \in \operatorname{Bun}_{r,d}(C)(\Bbbk)$ be a vector bundle on $C$ of rank $r$ and degree $d$. Let $A \to A_0$ be a surjection of artinian local rings with residue field $\Bbbk$ such that $\Bbbk = \ker(A \to A_0)$. We need to check that every vector bundle $\mathcal{F}_0$ on $C_{A_0}$ that restricts to $F$ extends to a vector bundle $\mathcal{F}$ on $C_A$. By deformation theory (Proposition D.2.15), there is an element $\mathrm{ob}_F \in \operatorname{Ext}^2_{\mathcal{O}_C}(F, F)$ such that $\mathrm{ob}_F = 0$ if and only if there exists an extension. Since $C$ is a smooth curve, $\operatorname{Ext}^2_{\mathcal{O}_C}(F, F) = \mathrm{H}^2(C_{\Bbbk}, F \otimes F^\vee) = 0$. Deformation theory also provides an identification $T_{\operatorname{Bun}_{r,d}(C),[F]} = \operatorname{Ext}^1_{\mathcal{O}_C}(F, F)$ and a Riemann–Roch calculation yields $\dim \operatorname{Ext}^1_{\mathcal{O}_C}(F, F) = \dim \operatorname{Aut}(F) + r^2(g-1)$ (see Example 3.5.9). Therefore $\dim_{[F]} \operatorname{Bun}_{r,d}(C) = \dim \operatorname{Ext}^1_{\mathcal{O}_C}(F, F) - \dim \operatorname{Aut}(F) = r^2(g-1)$. $\square$

## 3.8 Properness and the valuative criterion

With some care, we define separatedness and properness for morphisms of algebraic stacks. Recall from Definition 3.3.10 that we say a representable morphism $\mathcal{X} \to \mathcal{Y}$ of algebraic stacks is *separated* if the diagonal $\mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ (which is representable by schemes) is proper.

**Definition 3.8.1.**

(1) A morphism $\mathcal{X} \to \mathcal{Y}$ of algebraic stacks is *universally closed* if for every morphism $\mathcal{Y}' \to \mathcal{Y}$ of algebraic stacks, the morphism $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}' \to \mathcal{Y}'$ induces a closed map $|\mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}'| \to |\mathcal{Y}'|$.

(2) A representable morphism $\mathcal{X} \to \mathcal{Y}$ of algebraic stacks is *proper* if it is universally closed, separated, and of finite type.

(3) A morphism $\mathcal{X} \to \mathcal{Y}$ of algebraic stacks is *separated* if the representable morphism $\mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ is proper.

(4) A morphism $\mathcal{X} \to \mathcal{Y}$ of algebraic stacks is *proper* if it is universally closed, separated, and of finite type.

**Remark 3.8.2.** Notice that we have not defined properness by requiring the diagonal to be a closed immersion as with schemes. Indeed, the diagonal of a morphism of algebraic stacks is not a monomorphism. For schemes or algebraic spaces, the diagonal is proper if and only if it is a closed immersion; this follows from the fact that proper monomorphisms of schemes are closed immersions.

**Remark 3.8.3.** The property of being universally closed is smooth-local on the target. We thus have equivalences: $\mathcal{X} \to \mathcal{Y}$ is universally closed $\iff$ $|\mathcal{X} \times_{\mathcal{Y}} T| \to |T|$ is closed for all maps $T \to \mathcal{Y}$ from affine schemes $\iff$ for a smooth presentation $V \to \mathcal{Y}$, the base change $\mathcal{X} \times_{\mathcal{Y}} V \to V$ is universally closed.

**Remark 3.8.4.** Recall that the stabilizer $G_x$ of a field-valued point $x \colon \operatorname{Spec} \Bbbk \to \mathcal{X}$ is given by the cartesian diagram

$$\begin{tikzcd} G_x \arrow[r] \arrow[d] & \operatorname{Spec} \Bbbk \arrow[d, "{(x,x)}"] \\ \mathcal{X} \arrow[r] & \mathcal{X} \times \mathcal{X}. \end{tikzcd}$$

If $\mathcal{X}$ is a separated algebraic stack over a scheme $S$, then $G_x$ is a proper group algebraic space over $\Bbbk$. If an addition $\mathcal{X}$ has affine diagonal, then the stabilizer group $G_x$ is proper and affine, thus finite. Since $\operatorname{Bun}_{r,d}(C)$ has affine diagonal (Example 3.3.12) and infinite automorphism groups, we see that $\operatorname{Bun}_{r,d}(C)$ is *not* separated.

We now state the valuative criteria using the notion of liftings defined formally in Remark 3.7.2. For moduli problems, the valuative criterion translates to the geometric question of whether a family of objects over a punctured curve extends to the entire curve. We will apply the valuative criterion later to verify that $\overline{\mathcal{M}}_g$ is proper (Theorem 5.5.3) and that $\operatorname{Bun}^{\mathrm{ss}}_{r,d}$ is universally closed.

**Theorem 3.8.5** (Valuative Criteria for Universally Closed/Separated/Proper Morphisms)**.** *Let $f \colon \mathcal{X} \to \mathcal{Y}$ be a finite type morphism of noetherian algebraic stacks* *with separated diagonals. Consider a 2-commutative diagram*

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] \arrow[dr, phantom, "\swarrow_{\alpha}"] & \mathcal{X} \arrow[d, "f"] \\ \operatorname{Spec} R \arrow[r] & \mathcal{Y} \end{tikzcd} \tag{3.8.1}$$

*where $R$ is a DVR with fraction field $K$. Then*

(1) *$f$ is proper if and only if for every diagram (3.8.1), there exists an extension $R \to R'$ of DVRs with the map $K \to K'$ on fraction fields having finite transcendence degree and a lifting unique up to unique isomorphism*

$$\begin{tikzcd} \operatorname{Spec} K' \arrow[r] \arrow[d] & \operatorname{Spec} K \arrow[r] \arrow[d] & \mathcal{X} \arrow[d, "f"] \\ \operatorname{Spec} R' \arrow[r] \arrow[urr, dashed] & \operatorname{Spec} R \arrow[r] & \mathcal{Y}. \end{tikzcd} \tag{3.8.2}$$

(2) *$f$ is separated if and only if every two liftings of a diagram (3.8.1) are uniquely isomorphic.*

(3) *$f$ has separated diagonal if and only if every lifting of a diagram (3.8.1) has no non-trivial automorphisms.*

(4) *$f$ is universally closed if for every diagram (3.8.1), there exists an extension $R \to R'$ of DVRs with the map $K \to K'$ on fraction fields having finite transcendence degree and a lifting as in (3.8.2).*

**Remark 3.8.6.** See also [LMB00, Thm. 7.10], [SP, Tags 0CLV and 0CLY] and [Fal03, §4].

We modify the proof of the valuative criterion for schemes (see §A.4). The starting point is the following lifting criterion for closed morphisms generalizing Lemma A.4.1.

**Lemma 3.8.7.** *Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a quasi-compact morphism of algebraic stacks. Then $f$ is closed if and only for every point $x \in |\mathcal{X}|$, every specialization $f(x) \rightsquigarrow y_0$ lifts to a specialization $x \rightsquigarrow x_0$.*

*Proof.* The statement is equivalent to the equality that $f(\overline{\{x\}}) = \overline{\{f(x)\}}$ (Exercise 3.3.29(a)). □

To compare specializations to maps from DVRs, we have the following analogue of Proposition A.4.2.

**Proposition 3.8.8.** *If $f\colon \mathcal{X} \to \mathcal{Y}$ is a finite type morphism of noetherian schemes, $x \in |\mathcal{X}|$ and $f(x) \rightsquigarrow y_0$ is a specialization, then there exists a diagram*

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & \mathcal{X} \arrow[d, "f"] & x \arrow[d, mapsto] & \\ \operatorname{Spec} R \arrow[r] & \mathcal{Y} & f(x) \arrow[r, squiggly] & y_0. \end{tikzcd}$$

*where $R$ is a DVR with fraction field $K$, the image of $\operatorname{Spec} K \to \mathcal{X}$ is $x$ and $\operatorname{Spec} R \to \mathcal{Y}$ realizes the specialization $f(x) \rightsquigarrow y_0$. In particular, every specialization $x \rightsquigarrow x_0$ in a noetherian algebraic stack is realized by a map $\operatorname{Spec} R \to \mathcal{X}$ from a DVR.*

*Proof.* Let $V \to \mathcal{Y}$ be a smooth presentation and $v_0 \in V$ be a preimage of $y_0$. Since $V \to \mathcal{Y}$ is smooth, it is an open morphism (Exercise 3.3.25) and thus there exists a specialization $v \rightsquigarrow v_0$ over $f(x) \rightsquigarrow y_0$ (Exercise 3.3.29(c)). Let $x' \in |\mathcal{X}_V|$ be a preimage of $v \in V$ and $x \in |\mathcal{X}|$. Let $U \to \mathcal{X}_V$ be a smooth presentation and $u \in U$ be a preimage of $x'$. Applying Proposition A.4.2 to the morphism $U \to V$ of schemes with $u \mapsto v$ and the specialization $v \rightsquigarrow v_0$ gives the desired diagram. $\square$

*Proof of Theorem 3.8.5.* We first show that the universally closed valuative criterion implies the others. The double diagonal $\Delta_{\Delta_f}\colon \mathcal{X} \times_{\mathcal{X}\times_{\mathcal{Y}}\mathcal{X}} \mathcal{X}$ is separated and finite type. Thus $f$ has separated diagonal if and only if $\Delta_{\Delta_f}$ universally closed, and the existence of a lift for $\Delta_{\Delta_f}$ translates to the condition that every lift for $f$ has only trivial automorphisms. Assuming $f$ has separated diagonal, then $f$ is separated if and only if the diagonal $\Delta_f\colon \mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ is universally closed (as $\Delta_f$ is finite type), and the existence of a lift for $\Delta_f$ translates to the condition that any two lifts for $f$ are isomorphic.

Suppose that the valuative criterion for universal closedness holds and that $f\colon \mathcal{X} \to \mathcal{Y}$ is not universally closed. Since the property of a morphism of algebraic stacks being closed can be checked smooth-locally on the target, we can assume that $\mathcal{Y} = Y$ is a scheme and that there exists a morphism $T \to Y$ from a scheme such that $f_T\colon \mathcal{X}_T \to T$ is not closed. We will reduce to the case that $T \to Y$ is a *finite type* morphism. By Lemma 3.8.7, there exists $z \in |\mathcal{X}_T|$ and a specialization $f_T(z) \rightsquigarrow t_0$ which doesn't lift to a specialization $z \rightsquigarrow z_0$. This implies that $\mathcal{Z} = \overline{\{z\}} \subset \mathcal{X}_T$ has trivial intersection with the fiber $(\mathcal{X}_T)_{t_0}$. If $p\colon X \to \mathcal{X}$ is a smooth presentation, then the preimage $Z$ of $\mathcal{Z}$ under $X_T \to \mathcal{X}_T$ does not meet the fiber $(X_T)_{t_0}$. Applying Lemma A.4.6 implies that after replacing $T$ with an open neighborhood of $t_0$, the morphism $T \to Y$ factors through a finite type morphism $T' \to Y$ via $g\colon T \to T'$ and that there exists a closed subscheme $Z' \subset X_{T'}$ with trivial intersection with the fiber $(X_{T'})_{g(t_0)}$ such that $\operatorname{im}(Z \hookrightarrow X_T \to X_{T'}) \subset Z'$. Letting $z' \in |\mathcal{X}_{T'}|$ be the image of $z \in |\mathcal{X}_T|$, we have that $z'$ maps to $g(f_T(z)) \in T'$ and that there is a specialization $g(f_T(z)) \rightsquigarrow g(t_0)$ which does not lift to a specialization of $z'$. By Lemma 3.8.7, this shows that $\mathcal{X}_{T'} \to T'$ is also not closed.

For a finite type morphism $T \to \mathcal{Y}$, the base change $\mathcal{X}_T \to T$ is a finite type morphism of noetherian algebraic stacks which also satisfies the valuative criterion. It therefore suffices to show that $f\colon \mathcal{X} \to \mathcal{Y}$ is closed. By Lemma 3.8.7, we need to show that given a point $x \in |\mathcal{X}|$, every specialization $f(x) \rightsquigarrow y_0$ lifts to a specialization $x \rightsquigarrow x_0$. By Proposition 3.8.8, there exists a diagram (3.8.1) such that $\operatorname{Spec} R \to \mathcal{Y}$ realizes $f(x) \rightsquigarrow y_0$ with a lift $\operatorname{Spec} K \to \mathcal{X}$ whose image is $x$. The valuative criterion implies the existence of a lift $\operatorname{Spec} R \to \mathcal{X}$, which in turn yields a specialization $x \rightsquigarrow x_0$ lifting $f(x) \rightsquigarrow y_0$.

Conversely, assume that $f\colon \mathcal{X} \to \mathcal{Y}$ is universally closed and that we are given a diagram (3.8.1). By replacing $\mathcal{Y}$ with $\operatorname{Spec} R$ and $\mathcal{X}$ with $\mathcal{X} \times_{\mathcal{Y}} \operatorname{Spec} R$, we may assume that $\mathcal{Y} = \operatorname{Spec} R$ and that we have a diagram

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r, "x"] \arrow[d] & \mathcal{X} \arrow[dl] \\ \operatorname{Spec} R & \end{tikzcd}$$

By replacing $\mathcal{X}$ with $\overline{\{x\}}$, we may assume that $\mathcal{X}$ is integral with generic point $x$. Since $\mathcal{X} \to \operatorname{Spec} R$ is closed, there exists a specialization $x \rightsquigarrow x_0$ mapping to the specialization of the generic point to the closed point in $\operatorname{Spec} R$. As $\operatorname{Spec} K \to \mathcal{X}$ is quasi-compact, Proposition 3.8.8 implies there exists a DVR $R'$ with fraction field $K'$ and a commutative diagram

$$\begin{tikzcd} \operatorname{Spec} K' \arrow[r] \arrow[d] & \operatorname{Spec} K \arrow[d] \\ \operatorname{Spec} R' \arrow[r] & \mathcal{X} \end{tikzcd}$$

such that $\operatorname{Spec} R' \to \mathcal{X}$ realizes the specialization $x \rightsquigarrow x_0$. As $\operatorname{Spec} R' \to \operatorname{Spec} R$ is surjective, we see that $R \to R'$ is an extension of DVRs and that $\operatorname{Spec} R' \to \mathcal{X}$ provides a lift of the given diagram. □

**Remark 3.8.9.** In the valuative criterion for algebraic stacks, it is necessary to allow extensions $R \to R'$. Indeed, consider $\mathcal{X} = \mathbf{B}_{\mathbb{C}}\mathbb{Z}/2$ and the DVR $R = \mathbb{C}[x]_{(x)}$ with fraction field $K = \mathbb{C}(x)$. If we let $\mathbb{Z}/2$ act on $\operatorname{Spec} \mathbb{C}(y)$ via $(-1) \cdot y = -y$, then $\operatorname{Spec} \mathbb{C}(y) \to \operatorname{Spec} \mathbb{C}(x)$ defined by $x \mapsto y^2$ is a $\mathbb{Z}/2$-torsor (corresponding to a morphism $\operatorname{Spec} K \to \mathcal{X}$) that does not extend to a $\mathbb{Z}/2$-torsor over $\operatorname{Spec} R$.

**Exercise 3.8.10.**

(a) Show that if $\mathcal{X}$ is Deligne–Mumford stack over a field $\Bbbk$ that is not an algebraic space, then there exists a map $\operatorname{Spec} K \to \mathcal{X}$ that does not extend to a map $\operatorname{Spec} R \to \mathcal{X}$, where $R$ is a DVR with fraction field $K$.

(b) Show that for a *representable* morphism $X \to Y$ of finite type of noetherian algebraic stacks, the valuative criterion for universally closed (resp. separated, proper) holds without requiring an extension of DVRs.

See Remark 5.5.5 for an explicit example illustrating the necessity of extensions in the valuative criterion for $\mathcal{M}_g$. On the other hand, for $\operatorname{Bun}_{r,d}(C)$ it is not necessary to allow for extensions of DVRs.

**Exercise 3.8.11.**

(a) If $G$ is a finite group, show that $\mathbf{B}_{\mathbb{Z}}G \to \operatorname{Spec} \mathbb{Z}$ is proper.

(b) Show that $\mathbf{B}_{\mathbb{Z}}\mathbb{G}_m \to \operatorname{Spec} \mathbb{Z}$ is universally closed but not separated.

*Try to give two arguments for each part—one using the definitions and the other using the valuative criterion.*

**Exercise 3.8.12.** Show that $\mathcal{M}_{1,1}$ is separated over $\operatorname{Spec} \mathbb{Z}$.

We later show that $\mathcal{M}_g$ (and more generally $\overline{\mathcal{M}}_{g,n}$) is proper over $\operatorname{Spec} \mathbb{Z}$ and that $\operatorname{Bun}_{r,d}(C)^{\mathrm{ss}}$ is universally closed over a field $\Bbbk$.

## 3.9 Further examples

This section provides examples of algebraic spaces, Deligne–Mumford stacks, and algebraic stacks.

### 3.9.1 Examples of algebraic spaces

**Example 3.9.1.** As discussed in Example 0.5.5, there exists a smooth proper complex 3-fold $U$ with a free action of $\mathbb{Z}/2$-action such that there is an orbit not contained in an affine open subscheme. The quotient sheaf $U/(\mathbb{Z}/2)$ is an algebraic space (Corollary 3.1.13) which is not a scheme.

**Example 3.9.2** (The bug-eyed cover)**.** Let $\Bbbk$ be field of $\operatorname{char}(k) \neq 2$. Let $\mathbb{Z}/2 = \{\pm 1\}$ act on the non-separated affine line $U = \mathbb{A}^1 \bigcup_{\mathbb{A}^1 \setminus 0} \mathbb{A}^1$ over $\Bbbk$ by swapping the origins and by $(-1) \cdot x = -x$ for $x \neq 0$. Since the orbit of an origin is not contained in an affine, the quotient sheaf $U/(\mathbb{Z}/2)$ is not representable by a scheme; it is however an algebraic space (Corollary 3.1.13).

For an alternative description, let $\mathbb{Z}/2 = \{\pm 1\}$ act on $\mathbb{A}^1$ with multiplication $\sigma \colon \mathbb{Z}/2 \times \mathbb{A}^1 \to \mathbb{A}^1$ defined by $-1 \cdot x = -x$. If we remove the non-identity element of the stabilizer of the origin, we obtain a scheme $R = (\mathbb{Z}/2 \times \mathbb{A}^1) \setminus \{(-1, 0)\}$ and an equivalence relation $\sigma, p_2 \colon R \rightrightarrows \mathbb{A}^1$. The algebraic space quotient $\mathbb{A}^1/R$ is isomorphic to $U/(\mathbb{Z}/2)$ (Exercise 3.9.3(a)) For another way to see that $X = \mathbb{A}^1/R$ is not a scheme, observe that the diagonal $X \to X \times X$ is not a locally closed immersion as there is a cartesian diagram

$$\begin{tikzcd}
(\mathbb{A}^1 \setminus 0) \sqcup \{0\} \arrow[r] \arrow[d] & \mathbb{A}^1 \arrow[d] & x \arrow[d, mapsto] \\
R \arrow[r, "(\sigma, p_2)"] \arrow[d] & \mathbb{A}^1 \times \mathbb{A}^1 \arrow[d] & (x, -x) \\
X \arrow[r] & X \times X. &
\end{tikzcd}$$

**Exercise 3.9.3.**

(a) Show that $X = \mathbb{A}^1/R$ is isomorphic to $U/(\mathbb{Z}/2)$.

(b) Show that there is a universal homeomorphism $X \to \mathbb{A}^1$ which is ramified over the origin.

(c) Show that every map to a scheme $X \to Z$ factors through $X \to \mathbb{A}^1$. (In other words, while $\mathbb{A}^1$ may be the categorical quotient of $U$ by $\mathbb{Z}/2$ (or equivalently the category quotient of $R \rightrightarrows \mathbb{A}^1$) in the category of schemes, it is distinct from the algebraic space quotient.

(d) Consider the $\mathrm{SL}_2$ action on $V_d = \operatorname{Sym}^d \Bbbk^2$, the space of homogeneous polynomials in $x$ and $y$ of degree $d$. Let $W \subset V_1 \times V_4$ be the reduced locally closed subscheme defined as the set $(L, F)$ such that $L \neq 0$ and $F$ is the square of a homogeneous quadratic with discriminant 1. Show that the induced $\mathrm{SL}_2$-action on $W$ is free (i.e. $\mathrm{SL}_2 \times W \to W \times W$ is a monomorphism) and that quotient sheaf $W/\mathrm{SL}_2$ is an algebraic space isomorphic to $\mathbb{A}^1/R$ and $U/(\mathbb{Z}/2)$.

*While the descriptions of $X$ as $\mathbb{A}^1/R$ and $U/(\mathbb{Z}/2)$ may seem pathological, this exercise shows that in fact this algebraic space also arises as a quotient of a quasi-affine variety by* $\mathrm{SL}_2$*.*

**Example 3.9.4.** Let $\mathbb{Z}/2 = \{\pm 1\}$ act on $\mathbb{A}^1_{\mathbb{C}}$ via conjugation over $\operatorname{Spec} \mathbb{R}$. Note that the action defined over $\mathbb{R}$ of $\mathbb{Z}/2$ on $\operatorname{Spec} \mathbb{C}$ is free, and therefore the product action of $\mathbb{Z}/2$ on $\mathbb{A}^1_{\mathbb{C}} = \mathbb{A}^1_{\mathbb{R}} \times_{\mathbb{R}} \mathbb{C}$ (which is trivial on the first factor) is also free. Defining $R = (\mathbb{Z}/2 \times \mathbb{A}^1_{\mathbb{C}}) \setminus \{(-1, 0)\}$, show that there is an equivalence relation $\sigma, p_2 \colon R \rightrightarrows U$ such that the algebraic space $X = \mathbb{A}^1_{\mathbb{C}}/R$ is not a scheme. (The quotient $X$ looks like $\mathbb{A}^1_{\mathbb{R}}$ except that the origin has residue field $\mathbb{C}$. )

### 3.9.2 Examples of stacks with finite stabilizers

In characteristic 0, the following examples are Deligne–Mumford stacks.

**Example 3.9.5** (Classifying stacks)**.** If $G$ is an abstract finite group scheme over a field $\Bbbk$, then the *classifying stack* $\mathbf{B}G$ *of* $G$ is the stack defined as the category of pairs $(T, P)$ where $T$ is a scheme and $P \to T$ is a $G$-torsor (Definition 2.3.14). Then $\mathbf{B}G$ is a smooth and proper algebraic stack over $\Bbbk$ of dimension 0. Properness follows from the fact the base change of $\mathbf{B}G \to \mathbf{B}G \times \mathbf{B}G$ by the smooth presentation $\operatorname{Spec} \Bbbk \to \mathbf{B}G \times \mathbf{B}G$ is the finite morphism $G \to \operatorname{Spec} \Bbbk$, and smoothness follows because smoothness is a smooth-local property on the source and $S \to \mathbf{B}G$ is a smooth presentation).

**Example 3.9.6** (Weighted projective stacks)**.** For a tuple of positive integers $(d_0, \ldots, d_n)$, let $\mathbb{G}_m$ act on $\mathbb{A}^{n+1}$ via $t \cdot (x_0, \ldots, x_n) = (t^{d_0} x_0, \ldots, t^{d_n} x_n)$. We define the *weighted projective stack* as

$$\mathcal{P}(d_0, \ldots, d_n) = [(\mathbb{A}^{n+1} \setminus 0)/\mathbb{G}_m].$$

If the $d_i$ are all 1, then we recover projective space $\mathbb{P}^n$; otherwise, $\mathcal{P}(d_0, \ldots, d_n)$ is not an algebraic space.

More generally, if $R$ is a finitely generated positively graded $\Bbbk$-algebra, we can define *stacky proj* as $\mathcal{P}\mathrm{roj}\, R = [(\operatorname{Spec}(R) \setminus 0)/\mathbb{G}_m]$, where $\mathbb{G}_m$ acts such that the weight of $x_i$ is the same as its degree.

For example, over $\mathbb{Z}[1/6]$ the stack of stable elliptic curves $\overline{\mathcal{M}}_{1,1}$ is isomorphic to $\mathcal{P}(4, 6)$ by Exercise 3.1.18(c).

**Exercise 3.9.7.**

(a) If $\Bbbk$ is a field of characteristic $p$, show that $\mathcal{P}(d_0, \ldots, d_n)$ is a Deligne–Mumford stack if and only if $p$ doesn't divide each $d_i$.

(b) Classify all the points of $\mathcal{P}(3, 3, 4, 6)$ that have non-trivial stabilizers.

(c) We say that an algebraic stack $\mathcal{X}$ has *generically trivial stabilizer* if there exists a dense open substack $U \subset \mathcal{X}$ which is an algebraic space. Provide conditions for when $\mathcal{P}(d_0, \ldots, d_n)$ has generically trivial stabilizer.

(d) Show that there is a bijective morphism $\mathcal{P}(d_0, \ldots, d_n)$ to weighted projective space $\operatorname{Proj} \Bbbk[x_0, \ldots, x_n]$, where $x_i$ has degree $d_i$. (This is an example of a coarse moduli space.)

**Example 3.9.8.** Suppose $\operatorname{char}(k) \neq 2$. Let $\mathbb{Z}/2$ act on $\mathbb{A}^2_{\Bbbk}$ via $-1 \cdot (x, y) = (-x, -y)$. Show that $[\mathbb{A}^2_{\Bbbk}/(\mathbb{Z}/2)]$ is a smooth algebraic stack over a field $\Bbbk$ and that there is a proper and bijective morphism $[\mathbb{A}^2_{\Bbbk}/(\mathbb{Z}/2)] \to Y$ where $Y$ is the singular variety $\operatorname{Spec} \Bbbk[x^2, xy, y^2]$ defined by the $\mathbb{Z}/2$-invariants of $\Gamma(\mathbb{A}^2_{\Bbbk}, \mathcal{O}_{\mathbb{A}^2_{\Bbbk}})$.

**Example 3.9.9** (Stacky curves)**.** A *stacky curve* is a one-dimensional Deligne—Mumford stack of finite type over a field $\Bbbk$.

**Exercise 3.9.10.** If $d_1$ and $d_2$ are relatively prime positive integers, show that $\mathbb{P}(d_1, d_2)$ is a smooth and proper stacky curve with generically trivial stabilizer.

**Exercise 3.9.11.** We say that a stacky curve $\mathcal{X}$ over $\Bbbk$ is *nodal* if there exists a étale presentation $U \to \mathcal{X}$ from a nodal curve (equivalently every étale presentation is a nodal curve); see Definition 5.2.1. Show that a nodal stacky curve has abelian stabilizers.

**Example 3.9.12** (Root gerbes)**.** Let $X$ be a scheme and $L$ be a line bundle. This data determines a morphism $[L] \colon X \to \mathbf{B}\mathbb{G}_m$. Let $r \colon \mathbf{B}\mathbb{G}_m \to \mathbf{B}\mathbb{G}_m$ be the morphism induced from the $r$th power map $r \colon \mathbb{G}_m \to \mathbb{G}_m$, where $t \mapsto t^r$; alternatively

$r\colon \mathbf{B}\mathbb{G}_m \to \mathbf{B}\mathbb{G}_m$ is defined functorially on objects by the assignment $L \mapsto L^{\otimes r}$. For a positive integer $r$, define the *$r$th root gerbe $X(\sqrt[r]{L})$ of $X$ and $L$* (sometimes denoted as $\sqrt[r]{L/X}$) as the fiber product

$$\begin{tikzcd} X(\sqrt[r]{L}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathbf{B}\mathbb{G}_m \arrow[d, "r"] \\ X \arrow[r, "{[L]}"] & \mathbf{B}\mathbb{G}_m. \end{tikzcd}$$

**Example 3.9.13** (Root stacks)**.** Let $X$ be a scheme, $L$ be a line bundle, and $s \in \Gamma(X, L)$ be a section. This data determines a morphism $[L, s]\colon X \to [\mathbb{A}^1/\mathbb{G}_m]$ (see Example 3.9.16). Let $r\colon [\mathbb{A}^1/\mathbb{G}_m] \to [\mathbb{A}^1/\mathbb{G}_m]$ be the morphism induced from the $r$th power map $r\colon \mathbb{A}^1 \to \mathbb{A}^1$, given by $x \mapsto x^r$, which is equivariant under the $r$th power map $r\colon \mathbb{G}_m \to \mathbb{G}_m$; alternatively $r\colon [\mathbb{A}^1/\mathbb{G}_m] \to [\mathbb{A}^1/\mathbb{G}_m]$ is defined functorially by $(L, s) \mapsto (L^{\otimes r}, s^r)$. For a positive integer $r$, define the *$r$th root stack $X(\sqrt[r]{L, s})$ of $X$ and $L$ along $s$* (sometimes denoted as $\sqrt[r]{(L, s)/X}$) as the fiber product

$$\begin{tikzcd} X(\sqrt[r]{L,s}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & {[\mathbb{A}^1/\mathbb{G}_m]} \arrow[d, "r"] \\ X \arrow[r, "{[L,s]}"] & {[\mathbb{A}^1/\mathbb{G}_m].} \end{tikzcd}$$

Observe that if $s = 0$ is the zero section, then there is an identification of $X(\sqrt[r]{L, 0})$ with the root gerbe $X(\sqrt[r]{L})$ of the previous example.

**Exercise 3.9.14.** Let $S$ be a scheme and $r$ be an integer invertible in $\Gamma(S, \mathcal{O}_S)$. (This hypothesis ensures that $\boldsymbol{\mu}_{r,S} \to S$ is an étale group scheme; it will be removed in Exercise 6.2.32.)

(a) Show that $X(\sqrt[r]{L})$ and $X(\sqrt[r]{L, s})$ are Deligne–Mumford stacks.

(b) Show that $X(\sqrt[r]{L})$ has the equivalent description as the category of tuples $(T \xrightarrow{f} X, M, \alpha)$ where $f\colon T \to X$ is a morphism from a scheme, $M$ is a line bundle on $T$ and $\alpha\colon M^{\otimes r} \xrightarrow{\sim} f^*L$ is an isomorphism. In particular, there is a line bundle $L^{1/r}$ on $X(\sqrt[r]{L})$ and an isomorphism $(L^{1/r})^{\otimes r} \xrightarrow{\sim} \pi^*L$.

(c) Show that $X(\sqrt[r]{L, s})$ has the equivalent description as the category of triples $(T \xrightarrow{f} X, M, \alpha, t)$ where $f\colon T \to X$ is a morphism from a scheme, $M$ is a line bundle on $T$, $\alpha\colon M^{\otimes r} \to f^*L$ is an isomorphism, and $t \in \Gamma(T, M)$ is a section such that $\alpha(t^{\otimes r}) = f^*s$. In particular, there is a line bundle $L^{1/r}$ on $X(\sqrt[r]{L, s})$ with a section $s^{1/r}$ together with an isomorphism $(L^{1/r})^{\otimes r} \xrightarrow{\sim} \pi^*L$ identifying $(s^{1/r})^{\otimes r}$ with $\pi^*s$.

(d) If $X = \operatorname{Spec} A$ is an affine scheme over $S$ and $L = \mathcal{O}_X$ is trivial, show that

$$X(\sqrt[r]{L}) \cong [X/\boldsymbol{\mu}_r] \quad \text{and} \quad X(\sqrt[r]{L, s}) \cong [\operatorname{Spec}\big(A[x]/(x^r - s)\big)/\boldsymbol{\mu}_r]$$

where $\boldsymbol{\mu}_r$ acts trivially on $X$ and acts on $\operatorname{Spec}\big(A[x]/(x^r - s)\big)$ via $t \cdot x = tx$.

(e) Show that the fiber of $X(\sqrt[r]{L}) \to X$ at a point $x \in X$ is isomorphic to $B\boldsymbol{\mu}_{r,\kappa(x)}$.

(f) Show that $X(\sqrt[r]{L, s}) \to X$ is an isomorphism over $X_s = \{s \neq 0\}$ and that it restricts to the root gerbe $V(s)(\sqrt[r]{(L|_{V(s)})})$ over $V(s)$. In particular, the fiber over a point $x \in X$ is either the point $x$ itself if $x \notin V(s)$ and $B\boldsymbol{\mu}_{r,\kappa(x)}$ if $x \in V(s)$.

(You will show later in Exercise 6.2.32 that $X(\sqrt[r]{L}) \to X$ and the restriction of $X(\sqrt[r]{L,s}) \to X$ along $V(s)$ are banded $\boldsymbol{\mu}_r$-gerbes.)

### 3.9.3 Examples of algebraic stacks

**Example 3.9.15.** The classifying stack $\mathbf{B}\,\mathrm{GL}_n$ over $\operatorname{Spec}\mathbb{Z}$ classifies vector bundles of rank $n$. When $n = 1$, $\mathbf{B}\mathbb{G}_m = \mathbf{B}\,\mathrm{GL}_1$ classifies line bundles. The stack $\mathbf{B}\,\mathrm{GL}_n$ is a universally closed and smooth algebraic stack over $\operatorname{Spec}\mathbb{Z}$ of relative dimension $-n^2$ with affine diagonal. However, $\mathbf{B}\,\mathrm{GL}_n$ is not separated nor Deligne–Mumford.

**Example 3.9.16.** If $\mathbb{G}_m$ acts on $\mathbb{A}^1$ over $\mathbb{Z}$ via scaling, the quotient stack $[\mathbb{A}^1/\mathbb{G}_m]$ whose objects over a scheme $T$ are pairs $(L, s)$ where $L$ is a line bundle on $T$ and $s \in \Gamma(T, L)$. The stack $[\mathbb{A}^1/\mathbb{G}_m]$ is an algebraic stack universally closed and smooth over $\operatorname{Spec}\mathbb{Z}$ of relative dimension 0 with affine diagonal. The stack $[\mathbb{A}^1/\mathbb{G}_m]$ is not separated nor Deligne–Mumford

Over a field $\Bbbk$, $[\mathbb{A}^1/\mathbb{G}_m]$ has two points—one open and one closed—corresponding to the two $\mathbb{G}_m$-orbits (see Figure 13). There is an open immersion and closed immersion

$$\operatorname{Spec}\Bbbk \hookrightarrow [\mathbb{A}^1/\mathbb{G}_m] \hookleftarrow \mathbf{B}\mathbb{G}_m.$$

The morphism $[\mathbb{A}^1/\mathbb{G}_m] \to \operatorname{Spec}\Bbbk$ identifies the two orbits and is an example of a good moduli space.

**Example 3.9.17.** Working over a field $\Bbbk$, let $\mathbb{G}_m$ act on $\mathbb{A}^2$ via $t\cdot(x, y) = (tx, t^{-1}y)$. The quotient stack $\mathcal{X} = [\mathbb{A}^2/\mathbb{G}_m]$ is a smooth algebraic stack. An object of $\mathcal{X}$ over a scheme $T$ is a triple $(L, s, t)$ where $L$ is a line bundle on $T$, $s \in \Gamma(T, L)$ and $t \in \Gamma(T, L^{-1})$. The complement $\mathcal{X} \setminus 0$ of the origin is isomorphic to the non-separated affine line. There is a morphism $\mathcal{X} \to \mathbb{A}^1$ defined by $(x, y) \mapsto xy$, which is an isomorphism over $\mathbb{A}^1 \setminus 0$ and identifies the three orbits defined by $xy = 0$.

**Example 3.9.18** (Toric stacks)**.** A fan $\Sigma$ on a lattice $L = \mathbb{Z}^n$ defines a toric variety $X(\Sigma)$, i.e. a normal separated variety with an action of $\mathbb{G}_m^n$ such that there is a dense orbit with trivial stabilizer; see [Ful93].

Meanwhile, a *stacky fan* is a pair $(\Sigma, \beta)$ where $\Sigma$ is a fan on a lattice $L$ and $\beta\colon L \to N$ is a homomorphism of lattices. As $L$ and $N$ are lattices (i.e. finitely generated free abelian groups), the $\mathbb{Z}$-linear duals define tori $T_L := D(L^\vee)$ and $T_N := D(N^\vee)$ (Example C.1.6) where $T_L$ is a torus for the toric variety $X(\Sigma)$. The map $\beta$ induces a homomorphism $T_\beta\colon T_L \to T_N$, naturally identifying $\beta$ with the induced map on lattices of 1-parameter subgroups. We can then define $G_\beta = \ker(T_\beta)$ and the *toric stack*

$$X(\Sigma, \beta) := [X(\Sigma)/G_\beta].$$

**Example 3.9.19** (Picard schemes and stacks)**.** If $X$ is a scheme over a field $\Bbbk$, the *Picard functor of* $X$ and *Picard stack of* $X$ are defined as the sheaf $\underline{\mathrm{Pic}}(X)$ and stack $\underline{\mathcal{P}\mathrm{ic}}\,(X)$ on $\mathrm{Sch}_{\text{ét}}$ by

$$\underline{\mathrm{Pic}}(X) = \text{sheafification of } T \mapsto \mathrm{Pic}(X_T)$$
$$\underline{\mathcal{P}\mathrm{ic}}\,(X)(T) = \{\text{groupoid of line bundles } L \text{ on } X_T\}$$

A morphism $(T, L) \to (T', L')$ in $\underline{\mathcal{P}\mathrm{ic}}\,(X)$ is the data of a morphism $f\colon T \to T'$ of schemes and an isomorphism $\alpha\colon L \to f^*L'$ (or more precisely a morphism $f_*L \to L'$ whose adjoint is an isomorphism).

If $X$ is proper over a field $\Bbbk$, then $\underline{\mathrm{Pic}}(X)$ is a proper scheme and the tensor product of line bundles provides it with the structure of a group scheme, hence an abelian variety. Moreover, $\underline{\mathcal{P}ic}(X)$ is a smooth algebraic stack over $\Bbbk$ and there is a morphism $\underline{\mathcal{P}ic}(X) \to \underline{\mathrm{Pic}}(X)$ such that the fiber over a line bundle $L$ is isomorphic to $\mathbf{B}\mathbb{G}_m$. The tensor product of line bundles provides $\underline{\mathcal{P}ic}(X)$ with the structure of a group stack, a notion which we will not spell out precisely.

Gerbes provide important examples of algebraic stacks, but we postpone our treatment until §6.2.5.

### 3.9.4 Pathological examples

**Exercise 3.9.20.** If $G \to S$ is a smooth and affine group scheme acting on a scheme $U$ over $S$, then the quotient stack $[U/G]$ is algebraic (Theorem 3.1.10). More generally, if $G \to S$ is only assumed smooth, show that $[U/G]$ is algebraic by identifying it with the algebraic stack quotient of the smooth groupoid $G \times U \rightrightarrows U$.

In particular, the classifying stack $\mathbf{B}G = [S/G]$ is algebraic.

**Example 3.9.21.** Consider the constant group scheme $\underline{\mathbb{Z}}$ over $\operatorname{Spec}\mathbb{Z}$ associated to the abstract discrete group $\mathbb{Z}$. Then $\mathbb{B}\underline{\mathbb{Z}}$ is a non-quasi-separated smooth algebraic stack of dimension 0.

**Example 3.9.22.** Here we provide an example of a non-quasi-separated algebraic space that is not a scheme. Let $\Bbbk$ be a characteristic 0 field. Let $\underline{\mathbb{Z}}$ act on $\mathbb{A}^1$ over $\Bbbk$ via $n \cdot x = x + n$ for $x \in \mathbb{A}^1$ and $n \in \underline{\mathbb{Z}}$. Then $X = \mathbb{A}^1/\underline{\mathbb{Z}}$ is an algebraic space that is not quasi-separated (as the action map $\underline{\mathbb{Z}} \times \mathbb{A}^1 \to \mathbb{A}^1 \times \mathbb{A}^1$ is not quasi-compact).

If $X$ were a scheme, then there would exist a non-empty open affine subscheme $U = \operatorname{Spec} A \subset X$. Since $p\colon \mathbb{A}^1 \to X$ is an étale presentation, we can compute $A$ as the subring of $\underline{\mathbb{Z}}$-invariants $\Gamma(p^{-1}(U), \mathcal{O}_{\mathbb{A}^1})^{\underline{\mathbb{Z}}}$, which the reader can check consists of only the constant functions, i.e. $A = \Bbbk$. As $X$ is obtained by gluing such affine schemes, it follows that $X = \operatorname{Spec}\Bbbk$, a contradiction.

The algebraic space $X = \mathbb{A}^1/\underline{\mathbb{Z}}$ provides a counterexample to many facts that hold for all schemes and quasi-separated algebraic spaces but fail for all algebraic spaces (e.g. see Exercise 3.9.23).

Similarly, one can consider the algebraic space quotient $\mathbb{A}^1_{\mathbb{C}}/\underline{\mathbb{Z}}^2$ where $(a,b)\cdot x = x + a + ib$. While the analytic quotient $\mathbb{C}/\underline{\mathbb{Z}}^2$ of this action is an elliptic curve over $\mathbb{C}$, the algebraic space quotient is a non-quasi-separated algebraic space that is not a scheme.

**Exercise 3.9.23.** Let $X = \mathbb{A}^1/\underline{\mathbb{Z}}$ be the algebraic space defined above.

(a) Show that $X$ is locally noetherian and quasi-compact but not noetherian.

(b) Show that the generic point $\operatorname{Spec}\Bbbk(x) \to \mathbb{A}^1 \to X$ is fixed under the $\underline{\mathbb{Z}}$-action.

(c) Show that $\operatorname{Spec}\Bbbk(x) \to X$ does not factor through a monomorphism $\operatorname{Spec} L \to X$ for a field $L$. (In other words, the generic point of $X$ does not have a residue field.)

**Example 3.9.24** (Deligne–Mumford stacks with non-separated diagonal)**.** Let $G \to S$ be a finite group scheme. If $H \subset G$ is a subgroup scheme over $S$, then $G/H$ is separated if and only if $H \subset G$ is closed. For instance, taking $G = \mathbb{Z}/2 \times \mathbb{A}^1 \to \mathbb{A}^1$ and the subgroup $H = G \setminus \{-1, 0\}$, the quotient $Q = G/H$ is the non-separated affine line and is a group scheme over $\mathbb{A}^1$ which is trivial away from the origin and where the fiber over 0 is $\mathbb{Z}/2$. In this case, $\mathbf{B}_{\mathbb{A}^1}Q$ is a Deligne–Mumford stack with non-separated diagonal; however, $\mathcal{X}$ is quasi-compact and quasi-separated (i.e. $\mathcal{X}$, the first diagonal $\Delta_{\mathcal{X}}$ and second diagonal $\Delta_{\Delta_{\mathcal{X}}}$ are quasi-compact).

# Chapter 4

# Geometry of Deligne–Mumford stacks

## 4.1 Quasi-coherent sheaves and cohomology

### 4.1.1 Sheaves

The small étale site of a Deligne–Mumford stack can be defined analogously to the small étale site of a scheme (Example 2.1.3).

**Definition 4.1.1.** If $\mathcal{X}$ is a Deligne–Mumford stack, the *small étale site of* $\mathcal{X}$ is the category $\mathcal{X}_{\text{ét}}$ of schemes étale over $\mathcal{X}$. A covering of an $\mathcal{X}$-scheme $U$ is a collection of étale morphisms $\{U_i \to U\}$ over $\mathcal{X}$ such that $\coprod_i U_i \to U$ is surjective.

We can therefore discuss sheaves of abelian groups on $\mathcal{X}_{\text{ét}}$ and their morphisms. We denote $\mathrm{Ab}(\mathcal{X}_{\text{ét}})$ as the category of abelian sheaves on $\mathcal{X}_{\text{ét}}$. For an abelian sheaf $F$ on $\mathcal{X}_{\text{ét}}$, the sections over an étale $\mathcal{X}$-scheme $U$ are denoted by $F(U)$ or $\Gamma(U, F)$; you should remember that this group depends not only on $U$ but the structure morphism $U \to \mathcal{X}$.

**Example 4.1.2** (Structure sheaf)**.** The structure sheaf $\mathcal{O}_{\mathcal{X}}$ on a Deligne–Mumford stack is defined by $\mathcal{O}_{\mathcal{X}}(U) = \Gamma(U, \mathcal{O}_U)$ on an étale $\mathcal{X}$-scheme $U$.

**Example 4.1.3** (Differentials)**.** If $\mathcal{X}$ is a Deligne–Mumford stack over a scheme $S$, the relative sheaf of differentials $\Omega_{\mathcal{X}/S}$ is defined by $\Omega_{\mathcal{X}/S}(U) = \Gamma(U, \Omega_{U/S})$.

**Example 4.1.4** (Hodge bundle)**.** Define the sheaf $\mathcal{H}$ on $\mathcal{M}_g$ (for $g \geq 2$) as follows: for every étale morphism $U \to \mathcal{M}_g$ from a scheme corresponding to a family $\mathcal{C} \to U$ of smooth curves, we set $\mathcal{H}(U) = \Gamma(\mathcal{C}, \Omega_{\mathcal{C}/U})$. We will see later that $\mathcal{H}$ is a coherent $\mathcal{O}_{\mathcal{M}_g}$-module which is locally free of rank $g$, i.e. a vector bundle.

While a sheaf $F$ on $\mathcal{X}_{\text{ét}}$ by definition only has sections defined on étale $\mathcal{X}$-schemes, one can extend the definition to a Deligne–Mumford stack $\mathcal{U}$ étale over $\mathcal{X}$. Choose étale presentations $U \to \mathcal{U}$ and $R \to U \times_{\mathcal{U}} U$ by schemes and define

$$F(\mathcal{U}) := \mathrm{Eq}(F(U) \rightrightarrows F(R)).$$

One checks that this is independent of the choice of presentation. In particular, it makes sense to discuss global sections $\Gamma(\mathcal{X}, F) := F(\mathcal{X})$ over the identity $\mathrm{id}\colon \mathcal{X} \to \mathcal{X}$.

**Exercise 4.1.5.** If $F$ is an abelian sheaf on a Deligne–Mumford stack $\mathcal{X}$, show that $\Gamma(\mathcal{X}, F) = \mathrm{Hom}_{\mathrm{Ab}(\mathcal{X}_{\text{ét}})}(\underline{\mathbb{Z}}, F)$ where $\underline{\mathbb{Z}}$ is the constant sheaf. If $F$ is an $\mathcal{O}_{\mathcal{X}}$-module, show that $\Gamma(\mathcal{X}, F) = \mathrm{Hom}_{\mathcal{O}_{\mathcal{X}}}(\mathcal{O}_{\mathcal{X}}, F)$.

Given a morphism $f\colon \mathcal{X} \to \mathcal{Y}$ of Deligne–Mumford stacks, there are functors

$$\begin{tikzcd} \mathrm{Ab}(\mathcal{X}_{\text{ét}}) \arrow[r, bend left, "f_*"] & \mathrm{Ab}(\mathcal{Y}_{\text{ét}}) \arrow[l, bend left, "f^{-1}"] \end{tikzcd}$$

where $f_*F(V) := F(V \times_{\mathcal{Y}} \mathcal{X})$ and $f^{-1}G$ is the sheafification of the presheaf

$$U \mapsto \lim_{V \to \mathcal{Y}, U \to V \times_{\mathcal{Y}} \mathcal{X}} G(V),$$

with the limit is taken over the category of pairs of étale morphisms $V \to \mathcal{Y}$ and $U \to V \times_{\mathcal{Y}} \mathcal{X}$ (i.e. étale morphisms $V \to \mathcal{Y}$ and a choice of factorization of $U \to \mathcal{X} \to \mathcal{Y}$ through $V \to \mathcal{Y}$). Note that when $f\colon \mathcal{X} \to \mathcal{Y}$ is étale, then $f^{-1}G(U) = G(U)$ for an étale $\mathcal{X}$-scheme.

**Exercise 4.1.6.** Show that $f^{-1}$ is left adjoint to $f_*$.

**Exercise 4.1.7.** If $\mathcal{X}$ is a Deligne–Mumford stack, define instead the site $\mathcal{X}_{\text{ét}'}$ as the category of *algebraic spaces* over $\mathcal{X}$ where coverings are étale coverings. Show that the categories of sheaves on $\mathcal{X}_{\text{ét}}$ and $\mathcal{X}_{\text{ét}'}$ are equivalent.

### 4.1.2 $\mathcal{O}_{\mathcal{X}}$-modules

On a Deligne–Mumford stack $\mathcal{X}$, the structure sheaf $\mathcal{O}_{\mathcal{X}}$ is a ring object in $\mathrm{Ab}(\mathcal{X}_{\text{ét}})$ and we define:

**Definition 4.1.8.** If $\mathcal{X}$ is a Deligne–Mumford stack, a *sheaf of $\mathcal{O}_{\mathcal{X}}$-modules* (or simply an *$\mathcal{O}_{\mathcal{X}}$-module*) is a sheaf $F$ on $\mathcal{X}_{\text{ét}}$ which is a module object for $\mathcal{O}_{\mathcal{X}}$ in the category of sheaves, i.e. for every étale $\mathcal{X}$-scheme $U$, $F(U)$ is an $\mathcal{O}_{\mathcal{X}}(U)$-module and the module structure is compatible with respect to restriction along étale morphisms $V \to U$ of $\mathcal{X}$-schemes.

We denote $\mathrm{Mod}(\mathcal{O}_{\mathcal{X}})$ for the category of $\mathcal{O}_{\mathcal{X}}$-modules. Given two $\mathcal{O}_{\mathcal{X}}$-modules $F$ and $G$, we can define the *tensor product* $F \otimes G := F \otimes_{\mathcal{O}_{\mathcal{X}}} G$ as the sheafification of the $\mathcal{O}_{\mathcal{X}}$-module given by $(U \to \mathcal{X}) \mapsto F(U \to \mathcal{X}) \otimes_{\mathcal{O}_{\mathcal{X}}(U \to \mathcal{X})} G(U \to \mathcal{X})$. The *Hom sheaf* $\mathscr{H}om_{\mathcal{O}_{\mathcal{X}}}(F, G)$ has sections $\mathrm{Hom}_{\mathcal{O}_U}(F|_U, G|_U)$ over an étale morphism $f\colon U \to \mathcal{X}$ from scheme, where $F|_U = f^{-1}F$ denotes the restriction of $F$ to $U_{\text{ét}}$.

Given a morphism $f\colon \mathcal{X} \to \mathcal{Y}$ of Deligne–Mumford stacks, there are functors

$$\begin{tikzcd} \mathrm{Mod}(\mathcal{O}_{\mathcal{X}}) \arrow[r, bend left, "f_*"] & \mathrm{Mod}(\mathcal{O}_{\mathcal{Y}}) \arrow[l, bend left, "f^*"] \end{tikzcd}$$

where for an $\mathcal{O}_{\mathcal{X}}$-module $F$, $f_*F$ is the pushforward as sheaves and is naturally an $\mathcal{O}_{\mathcal{Y}}$-module. For an $\mathcal{O}_{\mathcal{Y}}$-module $G$, since there is a morphism $f^{-1}\mathcal{O}_{\mathcal{Y}} \to \mathcal{O}_{\mathcal{X}}$ of sheaves of rings in $\mathcal{X}_{\text{ét}}$ and $f^{-1}G$ is a $f^{-1}\mathcal{O}_{\mathcal{Y}}$-module, it makes sense to define the pullback $\mathcal{O}_{\mathcal{X}}$-module

$$f^*G := f^{-1}G \otimes_{f^{-1}\mathcal{O}_{\mathcal{Y}}} \mathcal{O}_{\mathcal{X}}.$$

**Exercise 4.1.9.** Show that $f^*$ is left adjoint to $f_*$.

**Exercise 4.1.10.** Show that $\mathrm{Mod}(\mathcal{O}_{\mathcal{X}})$ is an abelian category.

### 4.1.3 Quasi-coherent sheaves

Let $F$ be an $\mathcal{O}_{\mathcal{X}}$-module on a Deligne–Mumford stack $\mathcal{X}$. For an étale $\mathcal{X}$-scheme $U$, we have the restriction $F|_U$ to the étale site of $U$ and the further restriction $F|_{U_{\mathrm{Zar}}}$ restricted to the Zariski topology of $U$. Note that when $X$ is a scheme, $\mathcal{O}_X$ could refer to the structure sheaf either in $X_{\text{ét}}$ or $X_{\mathrm{Zar}}$. If there is a possibility for confusion, we write either $\mathcal{O}_{X_{\mathrm{Zar}}}$ or $\mathcal{O}_{X_{\text{ét}}}$.

**Definition 4.1.11.** Let $\mathcal{X}$ be a Deligne–Mumford stack. An $\mathcal{O}_{\mathcal{X}}$-module $F$ is *quasi-coherent* if

(1) for every étale $\mathcal{X}$-scheme $U$, the restriction $F|_{U_{\mathrm{Zar}}}$ is a quasi-coherent $\mathcal{O}_{U_{\mathrm{Zar}}}$-module, and

(2) for every étale morphism $f\colon U \to V$ of étale $\mathcal{X}$-schemes, the natural morphism $f^*(F|_{V_{\mathrm{Zar}}}) \to F|_{U_{\mathrm{Zar}}}$ is an isomorphism.

A quasi-coherent $F$ on $\mathcal{X}$ is a *vector bundle* (resp. *vector bundle of rank* $r$, *line bundle*) if $F|_{U_{\mathrm{Zar}}}$ is for every morphism $U \to \mathcal{X}$ from a scheme.

If in addition $\mathcal{X}$ is locally noetherian, we say $F$ is *coherent* if $F|_{U_{\mathrm{Zar}}}$ is coherent for every morphism $U \to \mathcal{X}$ from a scheme.

We denote by $\mathrm{QCoh}(\mathcal{X})$ and $\mathrm{Coh}(\mathcal{X})$ (in the noetherian setting) the categories of quasi-coherent and coherent sheaves. The property that a quasi-coherent sheaf is a vector bundle, line bundle, or coherent (in the noetherian setting) is étale local (Proposition B.4.3), and thus it suffices to check the condition on an étale presentation.

**Examples 4.1.12.** The structure sheaf $\mathcal{O}_{\mathcal{X}}$ is a line bundle which is coherent when $\mathcal{X}$ is locally noetherian. For a Deligne–Mumford stack $\mathcal{X}$ over a scheme $S$, the relative sheaf of differentials $\Omega_{\mathcal{X}/S}$ of Example 4.1.3 is quasi-coherent since for an étale morphisms $f\colon U \to V$ of étale $\mathcal{X}$-schemes, $f^*\Omega_{V/S} \to \Omega_{U/S}$ is an isomorphism; it is a vector bundle when $\mathcal{X} \to S$ is smooth.

For $\mathcal{M}_g$ (with $g \geq 2$), the Hodge bundle $\mathcal{H}$ of Example 4.1.4 is a vector bundle of rank $g$. This follows from Proposition 5.1.9(2): for a smooth family $\pi\colon \mathcal{C} \to V$ of genus $g$ curves corresponding to a $\mathcal{M}_g$-scheme $V$, the construction of $\pi_*\Omega_{\mathcal{C}/V}$ commutes with the base change along a map $f\colon U \to V$, i.e. $f^*(\pi_*\Omega_{\mathcal{C}/V}) \xrightarrow{\sim} \pi_{U,*}\Omega_{\mathcal{C}_U/U})$, which shows quasi-coherence of $\mathcal{H}$. Moreover, $\pi_*\Omega_{\mathcal{C}/V}$ is a vector bundle on $V$ of rank $g$, which shows that $\mathcal{H}$ is also a vector bundle of rank $g$.

**Example 4.1.13.** If $G$ is a finite group viewed as a group scheme over a field $\Bbbk$, a quasi-coherent sheaf on $\mathbf{B}G$ corresponds to a representation $V$ of $G$. If $G$ acts on an affine $\Bbbk$-scheme $\operatorname{Spec} A$, a quasi-coherent sheaf on $[\operatorname{Spec} A/G]$ is the data of an $A$-module $M$ equipped with a group homomorphism $G \to \mathrm{End}_A(M)$. These descriptions follow from Exercise 4.1.16(1).

**Exercise 4.1.14** (Equivalent definition)**.** There is a general definition of a quasi-coherent module on a site $\mathcal{S}$ with a sheaf of rings $\mathcal{O}$ (see [SGA4$\frac{1}{2}$] and [SP, Tag 03DL]): an $\mathcal{O}$-module $F$ is quasi-coherent if for every object $U \in \mathcal{S}$, there is a covering $\{U_i \to U\}$ such that the restriction $F|_{U_i}$ to the localized site $\mathcal{S}/U_i$ has a free presentation

$$\mathcal{O}_{U_i}^{\oplus J} \to \mathcal{O}_{U_i}^{\oplus I} \to F|_{U_i} \to 0.$$

Show the definition of quasi-coherence above for a Deligne–Mumford stack $\mathcal{X}$ agrees with this general definition on the ringed site $(\mathcal{X}_{\text{ét}}, \mathcal{O}_{\mathcal{X}})$.

The following exercise tells us that quasi-coherence is consistent with the usual one when $\mathcal{X}$ is a scheme.

**Exercise 4.1.15.** Let $X$ be a scheme and $F$ be an $\mathcal{O}_{X_{\mathrm{Zar}}}$-module.

(a) Define a presheaf $F_{\text{ét}}$ on $X_{\text{ét}}$ as follows: for an étale map $f\colon U \to \mathcal{X}$ from a scheme, set $F_{\text{ét}}(U) = \Gamma(U, f^*F)$. Show that $F_{\text{ét}}$ is a sheaf of $\mathcal{O}_{X_{\text{ét}}}$-modules and that the assignment $F \mapsto F_{\text{ét}}$ defines an exact functor $\operatorname{Mod}(\mathcal{O}_{X_{\mathrm{Zar}}}) \to \operatorname{Mod}(\mathcal{O}_{X_{\text{ét}}})$.

(b) Show that if $F$ is a quasi-coherent $\mathcal{O}_{X_{\mathrm{Zar}}}$-module, then $F_{\text{ét}}$ is a quasi-coherent $\mathcal{O}_{X_{\text{ét}}}$-module, and that $F \mapsto F_{\text{ét}}$ is an equivalence of categories between quasi-coherent $\mathcal{O}_{X_{\mathrm{Zar}}}$-modules and quasi-coherent $\mathcal{O}_{X_{\text{ét}}}$-modules. See also [SP, Tag 03DX].

**Exercise 4.1.16** (Groupoid and functorial perspectives)**.** Let $\mathcal{X}$ be a Deligne–Mumford stack.

(1) Let $U \to \mathcal{X}$ be an étale presentation from a scheme $U$. If $G$ is a quasi-coherent sheaf on $U$ and $\alpha\colon p_1^*G \xrightarrow{\sim} p_2^*G$ is an isomorphism on $R := U \times_{\mathcal{X}} U$ satisfying the cocycle condition $p_{23}^*\alpha \circ p_{12}^*\alpha = p_{13}^*\alpha$, show that $G$ descends to a unique quasi-coherent sheaf on $\mathcal{X}$.

(2) If $F$ is a quasi-coherent sheaf on $\mathcal{X}$ and $f\colon S \to \mathcal{X}$ is a morphism from a scheme, then show that $(f^*F)|_{S_{\mathrm{Zar}}}$ is a quasi-coherent sheaf on $S$.

Given a groupoid presentation $R \rightrightarrows U$ of $\mathcal{X}$, (1) gives an equivalence between quasi-coherent sheaves on $\mathcal{X}$ and quasi-coherent sheaves on $U$ with descent datum. Meanwhile, (2) above allows us to think of a quasi-coherent sheaf $F$ on $\mathcal{X}$ as the data of a quasi-coherent sheaf $F_S$ for every map $S \to \mathcal{X}$ and compatible isomorphisms $f^*F_T \to F_S$ for every map $f\colon S \to T$ over $\mathcal{X}$. For instance, the Hodge bundle on $\mathcal{M}_g$ is the data of the sheaf $\pi_*\Omega_{\mathcal{C}/S}$ for every smooth family of curves $\pi\colon \mathcal{C} \to S$

### 4.1.4 Pushforwards and pullbacks

**Exercise 4.1.17** (Pushforward–Pullback Adjunction)**.** Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a morphism of Deligne–Mumford stacks.

(a) Show that if $G$ is a quasi-coherent $\mathcal{O}_{\mathcal{Y}}$-module, then $f^*G$ is quasi-coherent.

Assume in addition that $f$ is quasi-compact and quasi-separated.

(b) Show that if $F$ is a quasi-coherent $\mathcal{O}_{\mathcal{X}}$-module, then $f_*F$ is quasi-coherent.

(c) Show that the functors

$$\begin{tikzcd} \operatorname{QCoh}(\mathcal{X}) \arrow[r, bend left, "f_*"] & \operatorname{QCoh}(\mathcal{Y}) \arrow[l, bend left, "f^*"] \end{tikzcd}$$

are adjoints (with $f_*$ the right adjoint).

**Exercise 4.1.18.** Let $G$ be a finite group and $\Bbbk$ be a field.

(a) Under the composition $\operatorname{Spec}\Bbbk \xrightarrow{p} \mathbb{B}_{\Bbbk}G \xrightarrow{\pi} \operatorname{Spec}\Bbbk$, show that for a $G$-representation $V$, $\pi_*V = V^G$ where $V^G$ is the subspace of $G$-invariants and $p^*V = V$ forgetting the $G$-action, and that for a $\Bbbk$-vector space $W$, $\pi^*W = W$ with the trivial $G$-action and $p_*W = W \otimes p_*\Bbbk$ where $p_*\Bbbk$ is the regular representation $\Gamma(G, \mathcal{O}_G)$.

(b) Given an action of $G$ an an affine $\Bbbk$-scheme $\operatorname{Spec} A$, consider the diagram

$$\begin{tikzcd} \operatorname{Spec} A \arrow[r, "p"] & {[\operatorname{Spec} A/G]} \arrow[r, "\pi"] \arrow[d, "q"] & \operatorname{Spec} A^G \\ & \mathbf{B}G & \end{tikzcd}$$

and recall from Example 4.1.13 that a quasi-coherent sheaf on $[\operatorname{Spec} A/G]$ is an $A$-module $M$ with a group homomorphism $G \to \operatorname{End}_A(M)$. Provide explicit descriptions of the functors $p_*, p^*, \pi_*, \pi^*, q_*$ and $q^*$ on quasi-coherent sheaves.

**Exercise 4.1.19** (Flat Base Change)**.** Consider a cartesian diagram

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"] \arrow[dr, phantom, "\square"] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

of Deligne–Mumford stacks, and let $F$ be a quasi-coherent sheaf on $X$. If $g\colon Y' \to Y$ is flat and $f\colon X \to Y$ is quasi-compact and quasi-separated, the natural adjunction map

$$g^* f_* F \to f'_* g'^* F$$

is an isomorphism.

**Exercise 4.1.20.** Let $\mathcal{X}$ be a noetherian Deligne–Mumford stack. Prove the following two statements:

(a) Every quasi-coherent sheaf on $\mathcal{X}$ is a directed colimit of its coherent subsheaves.

(b) If $\mathcal{U} \subset \mathcal{X}$ is an open substack, then every coherent sheaf on $\mathcal{U}$ extends to a coherent sheaf on $\mathcal{X}$.

This exercise extends [Har77, Exer II.5.15] from schemes to Deligne–Mumford stacks; see also [LMB00, Prop. 15.4], [Ols16, Prop. 7.1.11] and [SP, Tag 01PD].

### 4.1.5 Quasi-coherent constructions

A *quasi-coherent $\mathcal{O}_{\mathcal{X}}$-algebra* on a Deligne–Mumford stack is a quasi-coherent $\mathcal{O}_{\mathcal{X}}$-module with the compatible structure of a ring object in $\operatorname{Ab}(\mathcal{X}_{\text{ét}})$. We define the *relative spectrum* $\mathcal{S}\mathrm{pec}_{\mathcal{X}}\, \mathcal{A}$ as the stack whose objects over a scheme $S$ consists of a morphism $f\colon S \to \mathcal{X}$ and a morphism $f^*\mathcal{A} \to \mathcal{O}_S$ of $\mathcal{O}_S$-algebras.

**Exercise 4.1.21.** Show that $\mathcal{S}\mathrm{pec}_{\mathcal{X}}\, \mathcal{A}$ is an algebraic stack affine over $\mathcal{X}$.

**Example 4.1.22** (Reduction)**.** Let $\mathcal{X}$ be a Deligne-Mumford stack and let $\mathcal{O}_{\mathcal{X}}^{\text{red}}$ be the sheaf of $\mathcal{O}_{\mathcal{X}}$-algebras where $\mathcal{O}_{\mathcal{X}}^{\text{red}}(U) = \Gamma(U, \mathcal{O}_U)_{\text{red}}$ for an étale $\mathcal{X}$-scheme $U$. Then $\mathcal{O}_{\mathcal{X}}^{\text{red}}$ is a quasi-coherent $\mathcal{O}_{\mathcal{X}}$-algebra and $\mathcal{X}_{\text{red}} := \mathcal{S}\mathrm{pec}_{\mathcal{X}}\, \mathcal{O}_{\mathcal{X}}^{\text{red}}$ defines the *reduction of $\mathcal{X}$*.

**Example 4.1.23** (Normalization)**.** Let $\mathcal{X}$ be an integral Deligne-Mumford stack and let $\mathcal{A}$ be the sheaf of $\mathcal{O}_{\mathcal{X}}$-algebras whose sections over an étale morphism $U \to \mathcal{X}$ from a scheme is the normalization of $\Gamma(U, \mathcal{O}_U)$. Since normalization commutes with étale extensions (Proposition A.5.4), $\mathcal{A}$ is a quasi-coherent $\mathcal{O}_{\mathcal{X}}$-algebra. The *normalization of $\mathcal{X}$* is defined as $\widetilde{\mathcal{X}} := \mathcal{S}\mathrm{pec}_{\mathcal{X}}\, \mathcal{A}$.

**Exercise 4.1.24.** Let $f: \mathcal{X} \to \mathcal{Y}$ be a quasi-compact and quasi-separated morphism of Deligne–Mumford stacks.

(a) Show that there is factorization $f: \mathcal{X} \to \mathcal{S}\mathrm{pec}\, f_* \mathcal{O}_{\mathcal{X}} \to \mathcal{Y}$.

(b) Show that $f$ is affine if and only if $\mathcal{X} \to \mathcal{S}\mathrm{pec}\, f_* \mathcal{O}_{\mathcal{X}}$ is an isomorphism.

(c) Show that $f$ is quasi-affine if and only if $\mathcal{X} \to \mathcal{S}\mathrm{pec}\, f_* \mathcal{O}_{\mathcal{X}}$ is an open immersion.

**Exercise 4.1.25.** Use Exercise 4.1.20 to show that every quasi-coherent sheaf of algebras on a noetherian Deligne–Mumford stack is a directed colimit of finite type subalgebras.

### 4.1.6 Cohomology

We develop a cohomology theory for abelian sheaves on Deligne–Mumford stacks. Despite utilizing the cohomology of quasi-coherent sheaves on schemes throughout these notes, we surprisingly have little need for cohomology on algebraic spaces and Deligne–Mumford stacks, and many of the results here are included only for completeness.

The existence of enough injective objects is shown analogously to the case of schemes [Har77, Prop. 2.2].

**Lemma 4.1.26.** *If $\mathcal{X}$ is a Deligne–Mumford stack, the categories $\mathrm{Ab}(\mathcal{X}_{\text{ét}})$ and $\mathrm{Mod}(\mathcal{O}_{\mathcal{X}})$ have enough injectives. If in addition $\mathcal{X}$ is quasi-separated, then $\mathrm{QCoh}(\mathcal{X})$ has enough injectives.*

*Proof.* Recall that a functor $R: \mathcal{A} \to \mathcal{B}$ between abelian categories with an exact left adjoint $L$ preserves injectives: for an injective $I$ in $\mathcal{A}$, we have that $\mathrm{Hom}_{\mathcal{B}}(-, R(I)) = \mathrm{Hom}_{\mathcal{A}}(L(-), I)$ is exact.

By taking $\Lambda$ to be the constant sheaf $\underline{\mathbb{Z}}$ or the structure sheaf $\mathcal{O}_{\mathcal{X}}$, the first statement will follow if we show that the category $\mathrm{Mod}(\Lambda)$ of $\Lambda$-modules has enough injectives for every sheaf of rings $\Lambda$ on $\mathcal{X}_{\text{ét}}$. Let $F$ be a $\Lambda$-module and let $U \to \mathcal{X}$ be an étale presentation. For each $u \in U$, we have a map $j_u: \{u\} \hookrightarrow U \to \mathcal{X}$ from a point and the stalk $F_u = j_u^{-1} F$ is an $\Lambda_u$-module. Choose an inclusion $F_u \hookrightarrow I_u$ into an injective $\Lambda_u$-module. Adjunction gives a map $F \to j_{u,*} I_u$, where $j_{u,*}$ is injective since $j_u^{-1}$ is exact. By taking the product, we obtain an injection $F \to \prod_{u \in U} j_{u,*} I_u$ into an injective $\Lambda$-module.

For the final statement, let $F \in \mathrm{QCoh}(\mathcal{X})$ and let $p: U = \coprod_i \operatorname{Spec} A_i \to \mathcal{X}$ be an étale presentation. Choose an injection $p^* F \hookrightarrow I$ into an injective quasi-coherent $\mathcal{O}_U$-module. The composition $F \hookrightarrow p_* p^* F \hookrightarrow p_* I$ is injective and since $p^*$ is exact, $p_* I$ is injective. □

**Remark 4.1.27.** The above argument for the existence of enough injectives in $\mathrm{Mod}(\mathcal{O}_{\mathcal{X}})$ extends to the category of $\mathcal{O}$-modules in any ringed site with enough points (see [Ols16, Thm. 2.3.2]) and is even true in any ringed site [SP, Tag 01DP]. The category of quasi-coherent sheaves on an arbitrary Deligne–Mumford stack (or even algebraic stack) is a Grothendieck abelian category [SP, Tag 0781] and any such category has enough injectives [Gro57], [SP, Tag 079H].

**Definition 4.1.28** (Cohomology)**.** Let $\mathcal{X}$ be a Deligne–Mumford stack and $F$ a sheaf of abelian groups on $\mathcal{X}_{\text{ét}}$. The *cohomology group* $\mathrm{H}^i(\mathcal{X}_{\text{ét}}, F)$ is defined as the $i$th right derived functor of the global sections functor $\Gamma\colon \mathrm{Ab}(\mathcal{X}_{\text{ét}}) \to \mathrm{Ab}$.

Given a morphism $f: \mathcal{X} \to \mathcal{Y}$ of Deligne–Mumford stacks, the *higher direct image* $R^i f_* F$ is defined as the $i$th right derived functor of $f_*\colon \mathrm{Ab}(\mathcal{X}_{\text{ét}}) \to \mathrm{Ab}(\mathcal{Y}_{\text{ét}})$.

The following is a key input to the development of quasi-coherent cohomology.

**Theorem 4.1.29.** *For a quasi-coherent $\mathcal{O}_{X_{\text{ét}}}$-module $F$ on an affine scheme $X$, $\mathrm{H}^i(X_{\text{ét}}, F) = 0$ for all $i > 0$.*

We will prove this using Čech cohomology. Čech cohomology in the étale topology is defined similarly to the case of the Zariski topology [Har77, III.4] replacing intersections $U_{i_0} \cap \cdots \cap U_{i_n}$ with fiber products $U_{i_0} \times_{\mathcal{X}} \cdots \times_{\mathcal{X}} U_{i_n}$ and considering all (possibly non-distinct) indices $i_0, \ldots, i_n$ in any order.

**Definition 4.1.30** (Čech cohomology)**.** Given an étale covering $\mathcal{U} = \{U_i \to \mathcal{X}\}_{i \in I}$ of a Deligne–Mumford stack and an abelian sheaf $F$ on $\mathcal{X}_{\text{ét}}$, the *Čech complex of $F$ with respect to $\mathcal{U}$* is $\check{\mathcal{C}}^\bullet(\mathcal{U}, F)$ where

$$\check{\mathcal{C}}^n(\mathcal{U}, F) = \prod_{(i_0, \ldots, i_n) \in I^{n+1}} F(U_{i_0} \times_{\mathcal{X}} \cdots \times_{\mathcal{X}} U_{i_n})$$

with differential

$$d^n \colon \check{\mathcal{C}}^n(\mathcal{U}, F) \to \check{\mathcal{C}}^{n+1}(\mathcal{U}, F), \qquad (s_{i_0, \ldots, i_n}) \mapsto \left( \sum_{k=0}^{n+1} (-1)^k p_{\widehat{k}}^* s_{i_0, \ldots, \widehat{i_k}, \ldots, i_n} \right)_{(i_0, \ldots, i_{n+1})}$$

where $p_{\widehat{k}} \colon U_{i_0} \times_{\mathcal{X}} \cdots \times_{\mathcal{X}} U_{i_n} \to U_{i_0} \times_{\mathcal{X}} \cdots \times_{\mathcal{X}} \widehat{U_{i_k}} \times_{\mathcal{X}} \cdots \times_{\mathcal{X}} U_{i_n}$ is the map forgetting the $k$th component (with indexing starting at 0). The *Čech cohomology of $F$ with respect to $\mathcal{U}$* is

$$\check{\mathrm{H}}^i(\mathcal{U}, F) := \mathrm{H}^i(\check{\mathcal{C}}^\bullet(\mathcal{U}, F)).$$

The following is a standard result in Čech cohomology whose proof for sites is analogous to topological spaces. It is often referred to as Cartan's criterion; see [God58, II.5.9.2], [Mil80, Prop. 2.12], [SP, Tag 03F9] or [Ols16, Prop. 2.3.15].

**Lemma 4.1.31.** *Let $\mathcal{X}$ be a Deligne–Mumford stack and let $F$ be an abelian sheaf on $\mathcal{X}_{\text{ét}}$. Suppose $\mathrm{Cov}'(\mathcal{X}) \subset \mathrm{Cov}(\mathcal{X})$ is a subset of coverings of $\mathcal{X}$ such that every covering of $\mathcal{X}$ has a refinement in $\mathrm{Cov}'(\mathcal{X})$. If for every covering $\mathcal{U} \in \mathrm{Cov}'$, $\check{\mathrm{H}}^i(\mathcal{U}, F) = 0$ for $i > 0$, then $\mathrm{H}^i(\mathcal{X}_{\text{ét}}, F) = 0$.*

With these preliminaries, we can prove Theorem 4.1.29.

*Proof of Theorem 4.1.29.* Let $X = \operatorname{Spec} A$, $F = \widetilde{M}$ be a quasi-coherent $\mathcal{O}_X$-module and $F_{\text{ét}}$ be the corresponding quasi-coherent $\mathcal{O}_{X_{\text{ét}}}$-module (Exercise 4.1.15). The set of étale coverings of the form $\mathcal{U} = \{\operatorname{Spec} B \to \operatorname{Spec} A\}$ is sufficient to refine any other covering. For the covering $\mathcal{U}$, faithful flat descent (Exercise B.1.2) implies that there is a long exact sequence

$$0 \to M \to M \otimes_A B \to M \otimes_A B \otimes_A B \to M \otimes_A B \otimes_A B \otimes_A B \to \cdots,$$

which is identified with the Čech complex $\check{\mathcal{C}}^\bullet(\mathcal{U}, F)$. This shows that $\check{\mathrm{H}}^i(\mathcal{U}, F) = 0$ for $i > 0$ and thus Lemma 4.1.31 implies that $\mathrm{H}^i(X_{\text{ét}}, F_{\text{ét}}) = 0$. $\square$

As with ordinary topological spaces [Har77, Exer. III.4.11], Čech cohomology can be computed using a covering with vanishing cohomology; see for instance [SP, Tag 03F7].

**Lemma 4.1.32.** *Let $F$ be an abelian sheaf on $\mathcal{X}_{\text{ét}}$ and $(U_i \to \mathcal{X})_{i \in I}$ an étale covering. If $\mathrm{H}^i(U_{j_0} \times_U \cdots \times_U U_{j_n}, F) = 0$ for all $i > 0$, $n \geq 0$ and $j_0, \ldots, j_n \in I$, then $\check{\mathrm{H}}^i(\mathcal{U}, F) = \mathrm{H}^i(\mathcal{X}_{\text{ét}}, F)$.*

On a scheme with affine diagonal, both the étale and Zariski cohomology of a quasi-coherent sheaf can be computed on every affine open covering. We thus obtain:

**Proposition 4.1.33.** *Let $X$ be a scheme with affine diagonal. Let $F$ be a quasi-coherent $\mathcal{O}_X$-module and let $F_{\text{ét}}$ denote the corresponding quasi-coherent $\mathcal{O}_{X_{\text{ét}}}$-module (see Exercise 4.1.15). Then $\mathrm{H}^i(X, F) = \mathrm{H}^i(X_{\text{ét}}, F_{\text{ét}})$ for all $i$.*

**Remark 4.1.34.** The same result holds in the lisse-étale or fppf topology and without the affine diagonal hypothesis; see [SP, Tag 03DW] and [Mil80, Prop. 3.7].

Of course, in addition to being convenient to develop the theory of cohomology, Čech cohomology is also an extremely effective tool to compute cohomology groups. We have the following consequence of Theorem 4.1.29 and Lemma 4.1.32.

**Proposition 4.1.35.** *Let $\mathcal{X}$ be a Deligne–Mumford stack with affine diagonal and $F$ be a quasi-coherent sheaf. If $\mathcal{U} = \{U_i \to \mathcal{X}\}$ is an étale covering with each $U_i$ affine, then $\mathrm{H}^i(\mathcal{X}_{\text{ét}}, F) = \check{\mathrm{H}}^i(\mathcal{U}, F)$.*

To compare cohomologies computed in $\mathrm{Ab}(\mathcal{X}_{\text{ét}})$, $\mathrm{Mod}(\mathcal{O}_{\mathcal{X}})$ and $\mathrm{QCoh}(\mathcal{X})$, we have.

**Proposition 4.1.36.** *Let $\mathcal{X}$ be a Deligne–Mumford stack.*

*(1) If $F$ is an $\mathcal{O}_{\mathcal{X}}$-module, then the cohomology $\mathrm{H}^i(\mathcal{X}_{\text{ét}}, F)$ of $F$ as an abelian sheaf agrees with the $i$th right derived functor of $\Gamma\colon \mathrm{Mod}(\mathcal{O}_{\mathcal{X}}) \to \mathrm{Ab}$.*

*(2) If $\mathcal{X}$ has affine diagonal and $F$ is a quasi-coherent sheaf on $\mathcal{X}$, then the cohomology $\mathrm{H}^i(\mathcal{X}_{\text{ét}}, F)$ of $F$ as an abelian sheaf agrees with the $i$th right derived functor of $\Gamma\colon \mathrm{QCoh}(\mathcal{X}) \to \mathrm{Ab}$.*

*For a morphism $f\colon \mathcal{X} \to \mathcal{Y}$ of Deligne–Mumford stacks (resp. quasi-compact morphism of Deligne–Mumford stacks with affine diagonals), then (1) (resp. (2)) holds also for the higher direct images $R^i f_* F$ of an $\mathcal{O}_{\mathcal{X}}$-module (resp. quasi-coherent sheaf): it can be computed as the $i$th right derived functor of $f_*\colon \mathrm{Mod}(\mathcal{O}_{\mathcal{X}}) \to \mathrm{Mod}(\mathcal{O}_{\mathcal{Y}})$ (resp. $f_*\colon \mathrm{QCoh}(\mathcal{X}) \to \mathrm{QCoh}(\mathcal{Y})$).*

*Proof.* For (1), we need to show that an injective object in $\mathrm{Mod}(\mathcal{O}_{\mathcal{X}})$ is acyclic in $\mathrm{Ab}(\mathcal{X}_{\text{ét}})$. This uses a standard technique in Čech cohomology. We need some notation: given an étale covering $\mathcal{U} = \{U_i \to \mathcal{X}\}_{i \in I}$, we set $U_{\underline{i}} := U_{i_0} \times_{\mathcal{X}} \cdots \times_{\mathcal{X}} U_{i_n}$ with structure morphism $j_{\underline{i}}\colon U_{\underline{i}} \to \mathcal{X}$. There is a chain complex $\underline{\mathbb{Z}}_{\mathcal{U},\bullet}$ of presheaves on $\mathcal{X}$ defined by

$$\underline{\mathbb{Z}}_{\mathcal{U},n} := \bigoplus_{\underline{i} \in I^{n+1}} j_{\underline{i},!}\underline{\mathbb{Z}}$$

where $\underline{\mathbb{Z}}$ denotes the constant presheaf and $j_{\underline{i},!}\underline{\mathbb{Z}}$ is the presheaf whose sections over an $\mathcal{X}$-scheme $V$ are $\bigoplus_{\mathrm{Mor}_{\mathcal{X}}(V, U_{\underline{i}})} \mathbb{Z}$. The differentials of $\underline{\mathbb{Z}}_{\mathcal{U},\bullet}$ are the alternating sums of the natural maps. This complex of presheaves is exact in positive degrees and has the property that for every presheaf $F$

$$\check{C}(\mathcal{U}, F) = \mathrm{Mor}_{\mathrm{PAb}(\mathcal{X}_{\text{ét}})}(\underline{\mathbb{Z}}_{\mathcal{U},\bullet}, F) = \mathrm{Mor}_{\mathrm{PMod}(\mathcal{O}_{\mathcal{X}})}(\underline{\mathbb{Z}}_{\mathcal{U},\bullet} \otimes_{\underline{\mathbb{Z}}} \mathcal{O}_{\mathcal{X}}, F),$$

where morphisms are computed in the categories $\mathrm{PAb}(\mathcal{X}_{\text{ét}})$ and $\mathrm{PMod}(\mathcal{O}_{\mathcal{X}})$ of presheaves. If $F \in \mathrm{Mod}(\mathcal{O}_{\mathcal{X}})$ is injective, then it is also injective as a presheaf of $\mathcal{O}_{\mathcal{X}}$-modules. It follows that $\check{C}(\mathcal{U}, F)$ is exact in positive degrees and thus $\check{\mathrm{H}}^i(\mathcal{U}, F) = 0$ for $i > 0$. Therefore Lemma 4.1.31 implies that $\mathrm{H}^i(\mathcal{X}_{\text{ét}}, F) = 0$. For more details, see [SP, Tag 03FD] or [Ols16, Cor. 2.3.16].

For (2), let $F \in \mathrm{QCoh}(\mathcal{X})$ be an injective object. Let $p\colon U = \coprod_i \operatorname{Spec} A_i \to \mathcal{X}$ be an étale presentation and choose an injection $p^*F \hookrightarrow G$ into an injective object $G \in \mathrm{QCoh}(U)$. Then pushforward $p_*G$ is injective (as the right adjoint $p^*$ is exact) and we have an inclusion $F \hookrightarrow p_*p^*F \hookrightarrow p_*G$ of injectives which splits. It thus suffices to show that $p_*G$ is acyclic in $\mathrm{Ab}(\mathcal{X}_{\text{ét}})$. Since $\mathcal{X}$ has affine diagonal, $p\colon U \to \mathcal{X}$ is an affine morphism. By descent and Flat Base Change (Exercise 4.1.19), $p_*$ is exact on the category of quasi-coherent sheaves. It follows that $\mathrm{H}^i(\mathcal{X}_{\text{ét}}, p_*G) = \mathrm{H}^i(U_{\text{ét}}, G) = 0$ by Theorem 4.1.29. □

It follows from (2) that for a scheme $X$ with affine diagonal and for a quasi-coherent sheaf $F$, we have that $\mathrm{H}^i(X, F) = \mathrm{H}^i(X_{\text{ét}}, F_{\text{ét}})$.

**Example 4.1.37** (Group cohomology)**.** Let $G$ be a finite group viewed as a group scheme over a field $\Bbbk$, and let $V$ be a $G$-representation. The *group cohomology* $\mathrm{H}^i(G, V)$ is defined as the $i$th right derived functor of $\mathrm{Rep}(G) \to \mathrm{Vect}_{\Bbbk}, V \mapsto V^G$. Since $\mathrm{H}^i(\mathbf{B}G_{\text{ét}}, \widetilde{V})$ can be computed in $\mathrm{QCoh}(\mathbf{B}G)$ (Proposition 4.1.36(2)) where $\widetilde{V}$ is the corresponding quasi-coherent sheaf on $\mathbf{B}G$ and there is an equivalence $\mathrm{Rep}_{\Bbbk}(G) \cong \mathrm{QCoh}(\mathbf{B}G)$, we have the identification

$$\mathrm{H}^i(G, V) \cong \mathrm{H}^i(\mathbf{B}G_{\text{ét}}, \widetilde{V}).$$

The Čech complex of $\widetilde{V}$ on $\mathbf{B}G$ corresponding to $V$ with respect to the étale cover $\mathcal{U} = \{\operatorname{Spec} \Bbbk \to \mathbf{B}G\}$ has terms

$$\check{C}^n(\mathcal{U}, V) := \widetilde{V}((\operatorname{Spec} \Bbbk/\mathbf{B}G)^{n+1}) \cong \Gamma(G, \mathcal{O}_G)^{\otimes n} \otimes V.$$

To describe the differentials, let $\mu_k\colon G^{n+1} \to G^n$ for $k = 0, \dots, n$ be defined by sending $(g_1, \dots, g_{n+1})$ to $(g_1, \dots, g_n)$ for $k = 0$ and to $(g_1, \dots, g_{k-1}, g_k g_{k+1}, g_{k+2}, \dots, g_{n+1})$ for $k = 1, \dots, n$. Let $\sigma\colon V \to \Gamma(G, \mathcal{O}_G) \otimes V$ be the coaction map. The projection $p_{\widehat{k}}\colon (\operatorname{Spec} \Bbbk/\mathbf{B}G)^{n+2} \to (\operatorname{Spec} \Bbbk/\mathbf{B}G)^{n+1}$ is identified with $\mu_{n+1-k} \otimes \mathrm{id}$ for $k = 0, \dots n$ and $\mathrm{id} \otimes \sigma$ for $k = n+1$ (see Example 3.4.5). Thus the differentials in $\check{C}^\bullet(\mathcal{U}, V)$ are described by

$$\begin{aligned} d^n\colon \Gamma(G, \mathcal{O}_G)^{\otimes n} \otimes V \to &\Gamma(G, \mathcal{O}_G)^{\otimes (n+1)} \otimes V \\ f \otimes v \mapsto &\sum_{k=0}^{n} (-1)^k \mu_{n+1-k}^*(f) \otimes v + (-1)^{n+1} f \otimes \sigma(v) \end{aligned}$$

In low degrees, we have $d^0(v) = v - \sigma(v)$ and $d^1(f_1, v) = f_1 \otimes 1 \otimes v - \mu^*(f_1) \otimes v + f_1 \otimes \sigma(v)$ where $\mu = \mu_1$ is group multiplication $G \times G \to G$.

Since $G$ is finite, there is an identification $\Gamma(G^n, \mathcal{O}_{G^n}) \otimes V \cong \mathrm{Map}(G^n, V)$ with set-theoretic maps, where a map $\phi\colon G^n \to V$ is identified with $\sum_{g \in G^n} e_g \phi(g)$ where $e_g$ denotes the function which is 1 on $g$ but otherwise 0. Thus the Čech complex $\check{C}^\bullet(\mathcal{U}, V)$ can be equivalently described as

$$0 \to V \xrightarrow{d^0} \mathrm{Map}(G, V) \xrightarrow{d^1} \mathrm{Map}(G^2, V) \xrightarrow{d^2} \cdots \tag{4.1.1}$$

where the differential $d^n$ is defined by the formula

$$\begin{aligned} (d^n\phi)(g_1, \dots, g_{n+1}) = &\phi(g_1, \dots, g_n) + \\ &\sum_{k=1}^{n} (-1)^{n+1-k} \phi(g_1, \dots, g_{k-1}, g_k g_{k+1}, \dots, g_{n+1}) + (-1)^{n+1} g_1 \phi(g_2, \dots, g_n) \end{aligned}$$

for $\phi \in \operatorname{Map}(G^n, V)$. The complex (4.1.1) is sometimes referred to as the bar resolution (except that the differential $d^n$ is usually multiplied by $(-1)^{n+1}$), and is an effective means to compute group cohomology. In low degrees, $d^0(v)(g) = v - gv$ and $d^1(\phi)(g_1, g_2) = \phi(g_1) - \phi(g_1 g_2) + g_1\phi(g_2)$.

**Exercise 4.1.38.** If $\mathcal{X}$ is a Deligne–Mumford stack and $F_i$ is a directed system of abelian sheaves in $\mathcal{X}_{\text{ét}}$, show that $\operatorname{colim}_i \mathrm{H}^i(\mathcal{X}_{\text{ét}}, F_i) \to \mathrm{H}^i(\mathcal{X}_{\text{ét}}, \operatorname{colim}_i F_i)$ is an isomorphism.

**Remark 4.1.39** (Comparison of topologies)**.** One can also define the fppf cohomology groups $\mathrm{H}^i(\mathcal{X}_{\text{fppf}}, F)$ of an abelian sheaf on the small fppf site of $\mathcal{X}$. There are some cases when this agrees with the small étale cohomology. For instance, if $G \to S$ is a *smooth, commutative*, and quasi-projective group scheme, then $\mathrm{H}^i(S_{\text{ét}}, G) = \mathrm{H}^i(S_{\text{fppf}}, G)$ [Mil80, Thm. 3.9]. For $\mathbb{G}_m$, there are identifications $\operatorname{Pic}(X) = \mathrm{H}^1(X_{\text{Zar}}, \mathcal{O}_X^*) = \mathrm{H}^1(X_{\text{ét}}, \mathbb{G}_m) = \mathrm{H}^1(X_{\text{fppf}}, \mathbb{G}_m)$ for a scheme $X$ (Hilbert's Theorem 90, [Mil80, Prop. 4.9]).

On the other hand, if $X$ is a smooth scheme over $\mathbb{C}$ and $G$ is a finite abelian group, then the classical complex cohomology $\mathrm{H}^i(X(\mathbb{C}), G)$ agrees with the étale cohomology $\mathrm{H}^i(X_{\text{ét}}, G)$ of the constant sheaf associated to $G$ [Mil80, Thm. 3.12].

**Exercise 4.1.40** (Forms of group schemes)**.** Let $G$ be an algebraic group over a field $\Bbbk$. We say that a group scheme $H \to \operatorname{Spec} \Bbbk$ is a *form of* $G$ if there is an isomorphism $G_{\overline{\Bbbk}} \cong H_{\overline{\Bbbk}}$. We call $G$ the *trivial form of* $G$.

(a) Show the algebraic group $H = \operatorname{Spec} \mathbb{R}[x, y]/(x^2 + y^2 - 1)$ over $\mathbb{R}$, with the group structure induced from the embedding $H \subset \mathrm{SL}_2$ given by

$$(x, y) \mapsto \begin{pmatrix} x & y \\ -y & x \end{pmatrix},$$

is a non-trivial form of $\mathbb{G}_{m,\mathbb{R}}$.

(b) Assume that $\operatorname{char}(\Bbbk) \neq 2$. Recall the orthogonal groups $O(q)$ defined in Exercise C.2.17 for a non-degenerate quadratic form $q$ on an $n$-dimensional vector space $V$. Show that every $O(q)$ is a form of the subgroup $O_n \subset \mathrm{GL}_n$ of orthogonal matrices.

(c) If $G$ is smooth and commutative, show that forms of $G$ are classified by $\mathrm{H}^1((\mathrm{Sch}/\Bbbk)_{\text{ét}}, \operatorname{Aut}(G))$.

**Remark 4.1.41** (Other cohomology theories)**.** See §6.1.6 for the development of sheaf cohomology on an algebraic stack. See §6.1.7 for a discussion of the Chow group of an algebraic stack, and §6.1.8 for a discussion of de Rham and singular cohomology.

## 4.2 Quotients by finite groups and the local structure of Deligne–Mumford stacks

Quotient stacks $[\operatorname{Spec} A/G]$ of an affine scheme by a finite group are a particularly nice class of Deligne–Mumford stacks. Their geometry is the $G$-equivariant geometry of $\operatorname{Spec} A$. In this section, we show that the natural map $[\operatorname{Spec} A/G] \to \operatorname{Spec} A^G$ is universal for maps to algebraic spaces (Theorem 4.3.6) and that every Deligne–Mumford stack is étale locally isomorphic to a quotient stack of the form $[\operatorname{Spec} A/G]$ (Theorem 4.2.11).

### 4.2.1 Quotients by finite groups

**Definition 4.2.1** (Geometric quotients)**.** If $G$ is a finite group acting on an algebraic space $U$, a $G$-invariant morphism $U \to X$ is a *geometric quotient* if

(1) for every algebraically closed field $\Bbbk$, the map $U \to X$ induces a bijection $U(\Bbbk)/G \xrightarrow{\sim} X(\Bbbk)$, and

(2) $U \to X$ is universal for $G$-invariant maps to algebraic spaces, i.e. , every $G$-invariant map $U \to Y$ to an algebraic space factors uniquely as

$$\begin{tikzcd} U \arrow[d] \arrow[dr] & \\ X \arrow[r, dashed] & Y. \end{tikzcd}$$

If $\pi \colon U \to X$ is a geometric quotient, we often write $X = U/G$. In the case that $G$ acts freely on $U$ (i.e. the action map $G \times U \to U \times U$ is a monomorphism), then we have already defined the algebraic space quotient $U/G$ and the map $U \to U/G$ is a geometric quotient.

If a finite group $G$ acts on an affine scheme $\operatorname{Spec} A$, then $G$ also acts on the ring $A$. We define the *invariant ring* as

$$A^G = \{f \in A \mid g \cdot f = f \text{ for all } g \in G\}.$$

We will show shortly that $\operatorname{Spec} A \to \operatorname{Spec} A^G$ is a geometric quotient (Theorem 4.2.6).

**Example 4.2.2.** Assume $\operatorname{char}(\Bbbk) \neq 2$. Let $G = \mathbb{Z}/2$ acts on $\mathbb{A}^1 = \operatorname{Spec} \Bbbk[x]$ via $-1 \cdot x = -x$, then $\Bbbk[x]^G = \Bbbk[x^2]$. The geometric quotient is the map $\mathbb{A}^1 = \operatorname{Spec} \Bbbk[x] \to \operatorname{Spec} \Bbbk[x^2] = \mathbb{A}^1$ sending $p$ to $p^2$.

Let $G = \mathbb{Z}/2$ acts on $\mathbb{A}^2 = \operatorname{Spec} \Bbbk[x, y]$ via $-1 \cdot (x, y) = (-x, -y)$. Then $\Bbbk[x, y]^G = \Bbbk[x^2, xy, y^2]$ and the geometric quotient is $\mathbb{A}^2 \to \mathbb{A}^2/G = \operatorname{Spec} \Bbbk[x^2, xy, y^2]$. By setting $A = x^2, B = xy$ and $C = y^2$, the invariant ring can be identified with $\Bbbk[A, B, C]/(B^2 - AC)$ so the quotient $\mathbb{A}^2/G$ is a cone over a conic and in particular singular.

**Lemma 4.2.3.** *If $G$ is a finite group acting on an affine scheme $\operatorname{Spec} A$, then $A^G \to A$ is integral. If $A$ is finitely generated over a noetherian ring $R$, then $A^G \to A$ is finite and $A^G$ is finitely generated over $R$.*

*Proof.* To see that $A^G \to A$ is integral, for every element $a \in A$ the product $\prod_{g \in G}(x - ga) \in A^G[x]$ is polynomial with invariant coefficients which has $a$ as a root. If $R$ is noetherian and $R \to A$ is of finite type, then $A^G \to A$ is also of finite type. As $A^G \to A$ is integral, it is finite (c.f. [AM69, Cor. 5.2]). Since $R$ is noetherian, we may conclude by the Artin–Tate Lemma (c.f. [AM69, Prop. 7.8]) that $R \to A^G$ is of finite type. $\square$

The invariant ring is compatible with flat base change.

**Lemma 4.2.4.** *Let $G$ be a finite group acting on an affine scheme $\operatorname{Spec} A$. If $A^G \to B$ is a flat ring homomorphism, then $G$ acts on the affine scheme $\operatorname{Spec}(B \otimes_{A^G} A)$ and $B = (B \otimes_{A^G} A)^G$.*

*Proof.* By definition, the invariant ring is the equalizer

$$0 \to A^G \to A \overset{p_1}{\underset{p_2}{\rightrightarrows}} \prod_{g \in G} A$$

where $p_1(f) = (f)_{g\in G}$ and $p_2(f) = (gf)_{g\in G}$. Since $A^G \to B$ is flat, we have that

$$0 \to B \to A \otimes_{A^G} B \overset{p_1}{\underset{p_2}{\rightrightarrows}} \prod_{g\in G} A \otimes_{A^G} B$$

is also exact and we conclude that $B = (B \otimes_{A^G} A)^G$. □

**Exercise 4.2.5.** Let $A^G \to B$ be a ring homomorphism and consider the commutative diagram

$$\begin{tikzcd} & \operatorname{Spec} B \otimes_{A^G} A \arrow[r] \arrow[dl] \arrow[d] \arrow[dr, phantom, "\square"] & \operatorname{Spec} A \arrow[d] \\ \operatorname{Spec}(B \otimes_{A^G} A)^G \arrow[r] & \operatorname{Spec} B \arrow[r] & \operatorname{Spec} A^G. \end{tikzcd}$$

(a) Show that $\operatorname{Spec}(B \otimes_{A^G} A)^G \to \operatorname{Spec} B$ is an integral homeomorphism.

(b) If $|G|$ is invertible in $A$, show that $B \to (B \otimes_{A^G} A)^G$ is an isomorphism.

(c) Provide an example where $B \to (B \otimes_{A^G} A)^G$ is not an isomorphism.

**Theorem 4.2.6.** *If $G$ is a finite group acting on an affine scheme* $\operatorname{Spec} A$*, then* $\operatorname{Spec} A \to \operatorname{Spec} A^G$ *is a geometric quotient. If $A$ is finitely generated over a noetherian ring $R$, then $A^G$ is also finitely generated over $R$.*

*Proof.* Consider the commutative diagram

$$\begin{tikzcd} U = \operatorname{Spec} A \arrow[d] \arrow[dr, "\widetilde{\pi}"] & \\ \mathcal{X} = [U/G] \arrow[r, "\pi"] & X = \operatorname{Spec} A^G. \end{tikzcd}$$

Since $\widetilde{\pi}$ is integral and dominant, it is surjective. To see that $\widetilde{\pi}$ is injective on $G$-orbits of geometric points, let $\Bbbk$ be an algebraically closed field and $x, x' \in U(\Bbbk)$ with $\widetilde{\pi}(x) = \widetilde{\pi}(x') \in X(\Bbbk)$. The base change $U \times_X \operatorname{Spec} \Bbbk = \operatorname{Spec}(A \otimes_{A^G} \Bbbk)$ inherits a $G$-action and the $G$-orbits $Gx, Gx' \subset U \times_X A^G$ are closed subschemes. If $Gx \neq Gx'$, then the orbits are disjoint and there exists a function $f \in A \otimes_R \Bbbk$ with $f|_{Gx} = 0$ and $f|_{Gx'} = 1$. Then $\widetilde{f} = \prod_{g\in G} gf \in (A \otimes_{A^G} \Bbbk)^G$ is a $G$-invariant function with $f'(x) = 0$ and $f'(x') = 1$. But this implies that $\widetilde{\pi}(x) \neq \widetilde{\pi}(x') \in X(\Bbbk)$, which is a contradiction.

The map $\widetilde{\pi}\colon U \to X$ is universal for $G$-invariant maps to algebraic spaces if and only if $\pi\colon \mathcal{X} = [U/G] \to X$ is universal for maps to algebraic spaces. In other words, we need to show that if $Y$ is an algebraic space, then the natural map

$$\operatorname{Map}(X, Y) \to \operatorname{Map}(\mathcal{X}, Y) \tag{4.2.1}$$

is bijective. We note that this is immediate when $Y$ is affine as $\Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) = \Gamma(X, \mathcal{O}_X)$ and the case when $Y$ is a scheme can be reduced to this case without much effort: if $g\colon \mathcal{X} \to Y$ is a map, an affine covering $Y_i$ of $Y$ induces an open covering $X_i = X \setminus \pi(\mathcal{X} \setminus g^{-1}(Y_i))$ of $X$, and $g$ restricts to a map $\pi^{-1}(X_i) \to Y_i$ which factors uniquely through $X_i$ since $\pi_*\mathcal{O}_{\mathcal{X}} = \mathcal{O}_X$; see also [GIT, §0.6]. We need to work harder to handle the case that $Y$ is an algebraic space.

For the injectivity of (4.2.1), let $h_1, h_2\colon X \to Y$ be two maps such that $h_1 \circ \pi = h_2 \circ \pi$. Let $E \to X$ be the equalizer of $h_1$ and $h_2$, i.e. the pullback of the diagonal

$Y \to Y \times Y$ along $(h_1, h_2)\colon X \to Y \times Y$. The equalizer $E \to X$ is a monomorphism and locally of finite type. By construction $\pi\colon \mathcal{X} \to X$ factors through $E \to X$ and since $\pi$ is universally closed and schematically dominant (i.e. $\mathcal{O}_X \to \pi_*\mathcal{O}_{\mathcal{X}}$ is injective), so is $E \to X$. As every universally closed and locally of finite type monomorphism is a closed immersion (see Corollary A.5.5 and Remark A.5.6), we conclude that $E \to X$ is an isomorphism.

For the surjectivity of (4.2.1), let $g\colon \mathcal{X} \to Y$ be a map. We claim that the question is étale-local on $X$. Indeed, if $V \to X$ is an étale cover and $h\colon V \to Y$ is a morphism such that the two compositions $V \times_X \mathcal{X} \to V \xrightarrow{h} Y$ and $V \times_X \mathcal{X} \to \mathcal{X} \xrightarrow{g} Y$ agree, then by the injectivity of (4.2.1) applied to the good moduli space $V \times_X V \times_X \mathcal{X} \to V \times_X V$, the two compositions $V \times_X V \rightrightarrows V \xrightarrow{h} Y$ agree and thus $h\colon V \to Y$ descends to a morphism $\overline{h}\colon X \to Y$. Étale descent also implies the commutativity of $g = \overline{h} \circ \pi$.

Since $\mathcal{X}$ is quasi-compact, we may assume that $Y$ is quasi-compact as $g\colon \mathcal{X} \to Y$ factors through a quasi-compact open algebraic subspace of $Y$. Let $Y' \to Y$ be an étale presentation from an affine scheme and let $\mathcal{X}' := \mathcal{X} \times_Y Y'$. We claim that after replacing $X$ with an étale cover $V \to X$ and $\mathcal{X}$ with the base change $\mathcal{X} \times_X V$, there is a section $s\colon \mathcal{X} \to \mathcal{X}'$ of $\mathcal{X}' \to \mathcal{X}$ in the commutative diagram

$$\begin{tikzcd}
\mathcal{X}' \arrow[r] \arrow[d, "g'"'] \arrow[dr, phantom, "\square"] & \mathcal{X} \arrow[l, bend right, "s"'] \arrow[r, "\pi"] \arrow[d, "g"] & X \arrow[dl, dashed] \\
Y' \arrow[r] & Y. &
\end{tikzcd}$$

The surjectivity of (4.2.1) follows from this claim: since $X$ and $Y'$ are affine, the equality $\Gamma(X, \mathcal{O}_X) = \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ implies that $\mathcal{X} \xrightarrow{s} \mathcal{X}' \xrightarrow{g'} Y'$ factors through $\pi\colon \mathcal{X} \to X$ via a morphism $X \to Y'$. The composition $X \to Y' \to Y$ yields the desired dotted arrow above.

We claim that limit methods allow us to reduce to the case that $X = \operatorname{Spec} A^G$ is the spectrum of a strictly henselian local ring. Indeed, for a closed point $u$ of $U := \operatorname{Spec} A$ over $x \in |\mathcal{X}|$, the strict henselization $X^{\mathrm{sh}} := \mathcal{O}^{\mathrm{sh}}_{X,\pi(x)}$ is the limit $\lim_i X_i$ over all affine étale neighborhoods $X_i \to X$ of $\pi(x)$. The base change $U^{\mathrm{sh}} := U \times_X X^{\mathrm{sh}}$ is the limit of the affine schemes $U_i := U \times_X X_i$. We also set $\mathcal{X}^{\mathrm{sh}} := \mathcal{X} \times_X X^{\mathrm{sh}} = [U^{\mathrm{sh}}/G]$ and $\mathcal{X}_i := \mathcal{X} \times_X X_i = [U_i/G]$. Since $\mathcal{X}' \to \mathcal{X}$ is locally of finite presentation, the natural map

$$\operatorname{colim}_i \operatorname{Mor}_{\mathcal{X}}(\mathcal{X}_i, \mathcal{X}') \to \operatorname{Mor}_{\mathcal{X}}(\mathcal{X}^{\mathrm{sh}}, \mathcal{X}')$$

is an equivalence; this follows from Exercise 3.3.31 using that $\operatorname{Mor}_{\mathcal{X}}(\mathcal{X}^{\mathrm{sh}}, \mathcal{X}')$ is the equalizer of $\operatorname{Mor}_{\mathcal{X}}(U^{\mathrm{sh}}, \mathcal{X}') \rightrightarrows \operatorname{Mor}_{\mathcal{X}}(G \times U^{\mathrm{sh}}, \mathcal{X}')$ and similarly for the left-hand side. A section of $\mathcal{X}' \times_{\mathcal{X}} \mathcal{X}^{\mathrm{sh}} \to \mathcal{X}^{\mathrm{sh}}$ is determined by a map $\mathcal{X}^{\mathrm{sh}} \to \mathcal{X}'$ over $\mathcal{X}$. This map extends to a morphism $\mathcal{X}_i \to \mathcal{X}'$ for some $i$, giving us the desired section.

Let $\kappa$ be the residue field of $A^G$. As $A^G \to A$ is finite, $A = A_1 \times \cdots \times A_r$ is a product of strictly henselian local rings, each finite over $A^G$ (Proposition A.9.6). If $u \in \operatorname{Spec} A_1 \subset \operatorname{Spec} A$ is a closed point, then $\operatorname{Spec} A_1$ is $G_u$-invariant and the orbit $Gu$ is in bijection with the $r$ connected components of $\operatorname{Spec} A$. There is an isomorphism $\mathcal{X} \cong [\operatorname{Spec} A_1/G_u]$; this can be verified directly by for instance slicing the groupoid $G \times \operatorname{Spec} A \rightrightarrows \operatorname{Spec} A$ by $\operatorname{Spec} A_1 \hookrightarrow \operatorname{Spec} A$ (as in Exercise 3.4.15). We may thus replace $\mathcal{X} = [\operatorname{Spec} A/G]$ with $[\operatorname{Spec} A_1/G_u]$, and we can assume that there is a unique closed point $u \in \operatorname{Spec} A$ which is set-theoretically fixed by $G$.

As $Y' \to Y$ is representable by schemes, we can write $\mathcal{X}' = [U'/G]$ for a scheme $U'$. Let $u' \in U'$ be a preimage of $u \in \operatorname{Spec} A$. As $A$ is strictly henselian and the $G$-equivariant morphism $U' \to U$ is the base change of the étale morphism $Y' \to Y$, we see that $\kappa(u') = \kappa(u)$ and $G_{u'} = G_u = G$, and moreover the stabilizers act trivially on the residue fields. Again using that $A$ is strictly henselian, there is a unique section $s\colon \operatorname{Spec} A \to U'$ with $s(u) = u'$ (Proposition A.9.3). This section is $G$-invariant because for every $g \in G$, both $s \circ g$ and $g \circ s$ are sections of $U' \to \operatorname{Spec} A \xrightarrow{g^{-1}} \operatorname{Spec} A$ with $u' \mapsto u$ and thus the sections agree. It follows that $s$ descends to a section $\mathcal{X} = [\operatorname{Spec} A/G] \to [U'/G] = \mathcal{X}'$ of $\mathcal{X}' \to \mathcal{X}$. This finishes the proof that $\operatorname{Spec} A \to \operatorname{Spec} A^G$ is a geometric quotient.

The final statement follows from Lemma 4.2.3. □

**Corollary 4.2.7.** *Let $G$ be a finite group acting freely on an affine scheme $U = \operatorname{Spec} A$, then the algebraic space quotient $U/G$ is isomorphic to $\operatorname{Spec} A^G$.* □

**Exercise 4.2.8.** Let $R$ be a noetherian ring. Let $G$ be a finite group acting on a scheme $U$ projective (resp. quasi-projective, quasi-affine) over a ring $R$. Show that there exists a geometric quotient $U \to U/G$ such that $U/G$ is a projective (resp. quasi-projective, quasi-affine) scheme over $R$.

**Exercise 4.2.9.** Suppose that $G$ is a finite group acting on an affine scheme $\operatorname{Spec} A$ of finite type over a noetherian ring $R$. If $x \in \operatorname{Spec} A$ is a closed point, show that there is an isomorphism

$$\widehat{A}^{G_x} \cong \widehat{A^G}$$

between the $G_x$-invariants of the completion at $\operatorname{Spec} A$ at $x$ and the completion of $\operatorname{Spec} A^G$ at the image of $x$.

The following exercise generalizes Theorem 4.3.6 from quotients of finite groups to quotients of finite flat groupoids.

**Exercise 4.2.10.** Let $s, t\colon R \rightrightarrows U$ be a finite flat groupoid of affine schemes, and define $A^R \subset A$ as the subring of $R$-invariants, i.e. the subring of elements $a \in A$ such that $s^*a = t^*a \in \Gamma(R, \mathcal{O}_R)$. Show that $U \to X := \operatorname{Spec} A^R$ induces a bijection $U(\Bbbk)/R(\Bbbk) \xrightarrow{\sim} X(\Bbbk)$ for every algebraically closed field $\Bbbk$ and that $U \to X$ is universal for $R$-invariant maps to algebraic spaces. Moreover, show that if $A$ is finitely generated over a noetherian ring, then so is $A^R$.

### 4.2.2 The Local Structure Theorem

We show that a Deligne–Mumford stack $\mathcal{X}$ near a point $x$ is étale locally the quotient stack $[\operatorname{Spec} A/G_x]$ of an affine scheme by the stabilizer group scheme. Conceptually, this tells us that just as schemes (resp. algebraic spaces) are obtained by gluing affine schemes in the Zariski topology (resp. étale topology), Deligne–Mumford stacks are obtained by gluing quotient stacks $[\operatorname{Spec} A/G]$ in the étale topology.[1] This has the practical application of allowing one to reduce many properties of Deligne–Mumford stacks to quotient stacks $[\operatorname{Spec} A/G]$. We will take advantage of this local structure to construct a coarse moduli space (Theorem 4.3.11).

The *geometric stabilizer* of a point $x$ of a Deligne–Mumford stack $\mathcal{X}$ is the abstract group defined as the stabilizer of any geometric point $\operatorname{Spec} \Bbbk \to \mathcal{X}$ with image $x$.

[1] Of course, Deligne–Mumford stacks are also étale locally schemes but the étale neighborhoods $([\operatorname{Spec} A/G_x], w) \to (\mathcal{X}, x)$ produced by Theorem 4.2.11 preserve the stabilizer group at $w$.

**Theorem 4.2.11** (Local Structure Theorem of Deligne–Mumford Stacks)**.** *Let $\mathcal{X}$ be a separated Deligne–Mumford stack and $x \in \mathcal{X}$ be a finite type point with geometric stabilizer $G_x$. There exists an affine étale morphism*

$$f\colon ([\operatorname{Spec} A/G_x], w) \to (\mathcal{X}, x)$$

*where $w \in [\operatorname{Spec} A/G_x]$ such that $f$ induces an isomorphism of geometric stabilizer groups at $w$.*

*Proof.* Choose a geometric point $\operatorname{Spec} \Bbbk \to \mathcal{X}$ of representing $x$. Let $(U, u) \to (\mathcal{X}, x)$ be an étale representable morphism from an affine scheme. For simplicity, we can assume that the fiber product $\mathbf{B}G_x \times_{\mathcal{X}} U \cong \operatorname{Spec} \Bbbk$ (see Theorem 3.6.1). Let $d$ be the cardinality of $G_x$. Since $\mathcal{X}$ is separated, $U \to \mathcal{X}$ is affine. Define the quasi-affine scheme

$$\mathrm{SEC}_d := \underbrace{U \times_{\mathcal{X}} \cdots \times_{\mathcal{X}} U}_{d \text{ times}} \setminus \Delta,$$

where $\Delta$ is the union of all pairwise diagonals. A map $S \to \mathrm{SEC}_d$ from a scheme is classified by a morphism $S \to \mathcal{X}$ and $d$ sections $s_1, \ldots, s_d$ of $U_S := U \times_{\mathcal{X}} S \to S$ which are disjoint (i.e. the intersection of $s_i$ and $s_j$ is empty for $i \neq j$. There is an action of $S_d$ on $\mathrm{SEC}_d$ given by permuting the sections and we define the quotient stack

$$\mathrm{ET}_d := [\mathrm{SEC}_d / S_d].$$

By the correspondence between principal $S_d$-bundles and finite étale covers of degree $d$ (Exercise C.2.8), an object of $\mathrm{ET}_d$ over a scheme $S$ corresponds to a diagram

$$\begin{tikzcd} Z \arrow[r, hook] \arrow[dr] & U_S \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & U \arrow[d] \\ & S \arrow[r] & \mathcal{X} \end{tikzcd}$$

where $Z \hookrightarrow U_S$ is a closed subscheme and $Z \to S$ is finite étale of degree $d$. Let $w \in \mathrm{ET}_d(\Bbbk)$ be the point corresponding to $Z = \operatorname{Spec} \Bbbk \times_{\mathcal{X}} U$. There is an induced morphism $\mathrm{ET}_d \to \mathcal{X}$ and a commutative diagram

$$\begin{tikzcd} \mathrm{SEC}_d \arrow[rr] \arrow[d, hook] & & \mathrm{ET}_d \arrow[dr] & \\ U \times_{\mathcal{X}} \cdots \times_{\mathcal{X}} U \arrow[rr] & & U \arrow[r] & \mathcal{X} \end{tikzcd}$$

We claim that $\mathrm{ET}_d \to \mathcal{X}$ is an étale morphism that induces an isomorphism of stabilizer groups at $w$. The étaleness follows from étale descent as $U \times_{\mathcal{X}} \cdots \times_{\mathcal{X}} U \to \mathcal{X}$ is étale. To see that the induced map on stabilizers at $w$ is an isomorphism, it suffices to assume that $\mathcal{X} = \mathbf{B}G_x$ and $U = \operatorname{Spec} \Bbbk$. In this case, there is an isomorphism $U \times_{\mathcal{X}} \cdots \times_{\mathcal{X}} U \cong G_x^{d-1}$, which we can further identify with the quotient $G_x^d / G_x$ of the diagonal action. Then $\mathrm{SEC}_d \subset U \times_{\mathcal{X}} \cdots \times_{\mathcal{X}} U$ is identified with the quotient $(G_x^d \setminus \Delta)/G_x$. The permutation $S_d$-action is transitive with stabilizer isomorphic to $G_x$, and in fact $\mathrm{SEC}_d$ is $S_d$-equivariantly isomorphic to the quotient $S_d/G$ of the regular representation $G \subset S_d$. We thus see that $\mathrm{ET}_d \cong \mathbf{B}G_x$.

Since $\mathrm{ET}_d \to \mathcal{X}$ is separated, the relative inertia stack $I_{\mathrm{ET}_d/\mathcal{X}} \to \mathrm{ET}_d$ is finite and thus an isomorphism in an open substack $[W/S_d] \subset \mathrm{ET}_d$ around $w$, where $W \subset \mathrm{SEC}_d$ is a quasi-affine scheme. It follows that $[W/S_d] \to \mathcal{X}$ is an étale representable morphism inducing an isomorphism on stabilizer groups at $w$. By quotienting out by $G_x \subset S_d$ instead, the morphism $[W/G_x] \to \mathcal{X}$ is also étale representable inducing an isomorphism of stabilizer groups at $w$. Letting $W' \subset W$ be an affine open subscheme containing $w$, we may replace $W$ with the $G_x$-invariant affine open subscheme $\bigcap_{g \in G_x} g \cdot W'$.

It remains to show that $[W/G_x] \to \mathcal{X}$ is affine. Since $\mathcal{X}$ is separated, its diagonal is affine and the morphism $W \to \mathcal{X}$ from the affine scheme $W$ is affine. The fiber product

$$\begin{tikzcd} {[W/G_x] \times_{\mathcal{X}} W} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & W \arrow[d] \\ {[W/G_x]} \arrow[r] & \mathcal{X} \end{tikzcd}$$

is affine over $[W/G_x]$ and thus isomorphic to a quotient stack $[\operatorname{Spec} B/G_x]$. On the other hand, since $[W/G_x] \to \mathcal{X}$ is representable, the quotient stack $[\operatorname{Spec} B/G_x]$ is an algebraic space and the action of $G_x$ on $\operatorname{Spec} B$ is free. By Corollary 4.2.7, $[\operatorname{Spec} B/G_x]$ is isomorphic to the affine scheme $\operatorname{Spec} B^{G_x}$. By étale descent $[W/G_x] \to \mathcal{X}$ is affine.

See also [LMB00, Thm. 6.2]. □

**Exercise 4.2.12.** Let $\mathcal{X}$ be a Deligne–Mumford stack. Show that $\mathcal{X}$ is isomorphic to a quotient stack $[U/G]$ where $U$ is an affine scheme (resp. scheme, algebraic space) and $G$ is a finite group if and only if there exists a finite étale morphism $V \to \mathcal{X}$ from an affine scheme (resp. scheme, algebraic space).

*Hint: If $V \to \mathcal{X}$ is a finite étale cover of degree $d$, consider the associated principal $S_d$-torsor $\underbrace{V \times_{\mathcal{X}} \cdots \times_{\mathcal{X}} V}_{d \text{ times}} \setminus \Delta \to \mathcal{X}$; see Exercise C.2.8.*

**Proposition 4.2.13.** *If $R \rightrightarrows U$ is a finite étale equivalence relation of affine schemes, then the algebraic space quotient $U/R$ is an affine scheme.*

*Proof.* By Exercise 4.2.12, the algebraic space $U/R$ is isomorphism to $V/G$ for the free action of a finite group $G$ on an affine scheme $V = \operatorname{Spec} B$. Theorem 4.3.6 shows that $V/G \to \operatorname{Spec} B^G$ is universal for maps to algebraic spaces and thus an isomorphism. Alternatively, this follows from Exercise 4.2.10: if $U = \operatorname{Spec} A$, then $U/R \to \operatorname{Spec} A^R$ is universal for maps to algebraic spaces and thus an isomorphism. □

With a similar technique to the proof of Theorem 4.2.11, we can prove the following useful result asserting the existence of presentations with a lift of a given field-valued point.

**Proposition 4.2.14.** *If $\mathcal{X}$ is an algebraic stack (resp. algebraic space) with separated and quasi-compact diagonal and $x \in \mathcal{X}(\Bbbk)$ is a field-valued point, then there exists a smooth (resp. étale) morphism $U \to \mathcal{X}$ from an affine scheme and a point $u \in U(\Bbbk)$ over $x$.*

*Proof.* Let $U \to \mathcal{X}$ be a smooth presentation and consider the fiber product

$$\begin{tikzcd} U_x \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & U \arrow[d] \\ \operatorname{Spec} \Bbbk \arrow[r, "x"] & \mathcal{X}. \end{tikzcd}$$

Since $\mathcal{X}$ is quasi-separated, so is $U_x$. If $u \in U_x$ is any closed point, then the inclusion $\operatorname{Spec}\kappa(u) \to U_x$ of the residue field (Proposition 3.5.16) is a closed immersion and $\Bbbk \to \kappa(u)$ is a finite separable extension of fields. Let $d = [\kappa(u) : \Bbbk]$. Following the notation of the proof of Theorem 4.2.11, we consider the open subspace $\mathrm{SEC}_d \subset U \times_{\mathcal{X}} \cdots \times_{\mathcal{X}} U$ parameterizing $d$ disjoint sections of $U \to \mathcal{X}$ and the morphism $\mathrm{ET}_d = [\mathrm{SEC}_d / S_d] \to \mathcal{X}$. As $\operatorname{Spec}\kappa(u) \to \operatorname{Spec}\Bbbk$ is finite étale of degree $d$, the closed immersion $\operatorname{Spec}\kappa(u) \hookrightarrow U_x$ defines a $\Bbbk$-point $v$ of $[V/S_d]$. This gives a commutative diagram

$$\begin{tikzcd} & & \operatorname{Spec}\Bbbk \arrow[dl, "v"'] \arrow[d, "x"] \\ \mathrm{SEC}_d \arrow[r] & \mathrm{ET}_d \arrow[r] & \mathcal{X}. \end{tikzcd}$$

The point $v \in \mathrm{ET}_d(\Bbbk)$ does not necessarily lift to $\mathrm{SEC}_d$, but we can use the following trick to choose a different quotient presentation of $\mathrm{ET}_d$ where $v$ does lift. Namely, choose a faithful representation $S_d \subset \mathrm{GL}_n$ and write $\mathrm{ET}_d \cong [V/\mathrm{GL}_n]$ where $V = \mathrm{SEC}_d \times^{S_d} \mathrm{GL}_n$. Then $v\colon \operatorname{Spec}\Bbbk \to [V/\mathrm{GL}_n]$ corresponds to a principal $\mathrm{GL}_n$-bundle $P \to \operatorname{Spec}\Bbbk$ and a $\mathrm{GL}_n$-equivariant map $P \to V$. Since principal $\mathrm{GL}_n$-bundles are in bijection to vector bundles (Exercise C.2.11), $P$ is the trivial principal $\mathrm{GL}_n$-bundle and there is a section $\operatorname{Spec}\Bbbk \to P$. The composition $V \to [V/\mathrm{GL}_n] \to \mathcal{X}$ is smooth and the composition $\operatorname{Spec}\Bbbk \to P \to V$ is a lift of $x$.

It remains to show that a $\Bbbk$-point of an algebraic space $V$ (with separated and quasi-compact diagonal) lifts to an étale presentation by a scheme. We will use the fact that a quasi-separated algebraic space has quasi-affine diagonal; this is proved in Corollary 4.4.8 using the result Proposition 4.2.13 above and the theory of quasi-coherent sheaves (4.1). We repeat the above argument by choosing an étale map $U \to V$ from an affine scheme such that the image contains $x$. Then the space of $d$ disjoint sections $\mathrm{SEC}_d$ with respect to $U \to V$ is a quasi-affine scheme with a free action of $S_d$. The quotient $\mathrm{ET}_d = \mathrm{SEC}_d / S_d$ is also quasi-affine (Exercise 4.2.8). The induced map $\mathrm{ET}_d \to \mathcal{X}$ is étale and by construction the $\Bbbk$-point $x$ lifts to a $\Bbbk$-point of $\mathrm{ET}_d$.

See also [LMB00, Thm. 6.3]. □

## 4.3 Coarse moduli spaces and the Keel–Mori Theorem

The goal of this section is to establish the Keel–Mori Theorem: every separated Deligne–Mumford stack $\mathcal{X}$ of finite type over a noetherian scheme admits a separated coarse moduli space $\pi\colon \mathcal{X} \to X$ (see Theorem 4.3.11). One can view this theorem as a way to remove the stackiness of a Deligne–Mumford stack; at the expense of sacrificing universal properties of $\mathcal{X}$ (e.g. existence of a universal family), one can replace $\mathcal{X}$ with an algebraic space without changing the underlying topological space.

We will later apply this theorem to show that the Deligne–Mumford stack $\overline{\mathcal{M}}_g$ parameterizing stable curves admits a coarse moduli space $\pi\colon \overline{\mathcal{M}}_g \to \overline{M}_g$ where $\overline{M}_g$ is a separated algebraic space, which we later show to be proper and then finally projective.

To prove Theorem 4.3.11, we will apply the Local Structure Theorem (4.2.11) to construct étale neighborhoods $[\operatorname{Spec}(A_i)/G] \to \mathcal{X}$ and show that the geometric quotients $\operatorname{Spec}(A_i^G)$ glue in the étale topology to a coarse moduli space of $\mathcal{X}$.

### 4.3.1 Coarse moduli spaces

We begin with the definition:

**Definition 4.3.1.** A morphism $\pi\colon \mathcal{X} \to X$ from an algebraic stack to an algebraic space is a *coarse moduli space* if

(1) for every algebraically closed field $\Bbbk$, the induced map $\mathcal{X}(\Bbbk)/\!\sim \to X(\Bbbk)$, from the set of isomorphism classes of objects of $\mathcal{X}$ over $\Bbbk$, is bijective, and

(2) $\pi$ is universal for maps to algebraic spaces, i.e. every map $\mathcal{X} \to Y$ to an algebraic space factors uniquely as

$$\begin{tikzcd} \mathcal{X} \arrow[d, "\pi"'] \arrow[dr] & \\ X \arrow[r, dashed] & Y. \end{tikzcd}$$

**Remark 4.3.2.** If $G$ is a finite group acting on an algebraic space $U$, then $[U/G] \to X$ is a coarse moduli space if and only if $U \to X$ is a geometric quotient (Definition 4.2.1).

**Remark 4.3.3.** In practice, we desire coarse moduli spaces with additional properties of $\pi\colon \mathcal{X} \to X$ as otherwise it is difficult to work with this notion. For instance, it is not true that this notion is stable under étale base change (or even open immersions) or that $\pi_*\mathcal{O}_{\mathcal{X}} = \mathcal{O}_X$. However, we emphasize that the Keel–Mori Theorem produces a coarse moduli space $\pi\colon \mathcal{X} \to X$ with the additional properties: (a) it is stable under flat base change, (b) $\pi_*\mathcal{O}_{\mathcal{X}} = \mathcal{O}_X$, (c) $\pi$ is proper (and in particular separated!) and (d) $\pi$ is a universal homeomorphism.

**Lemma 4.3.4.** *Let $\pi\colon \mathcal{X} \to X$ be a coarse moduli space such that for every étale morphism $X' \to X$ from an affine scheme, the base change $\mathcal{X} \times_X X' \to X'$ is a coarse moduli space. Then the natural map $\mathcal{O}_X \to \pi_*\mathcal{O}_{\mathcal{X}}$ is an isomorphism.*

*Proof.* As $\pi$ is universal for maps to algebraic spaces, we have that $\mathrm{Map}(X, \mathbb{A}^1) \to \mathrm{Map}(\mathcal{X}, \mathbb{A}^1)$ is bijective or in other words $\Gamma(X, \mathcal{O}_X) \cong \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$. For every étale map $X' \to X$, the base change $\mathcal{X}' = \mathcal{X} \times_X X' \to X'$ is also a coarse moduli space and thus $\Gamma(X', \mathcal{O}_{X'}) \cong \Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})$. This shows that $\mathcal{O}_X \to \pi_*\mathcal{O}_{\mathcal{X}}$ is isomorphism. $\square$

The property that a given map is a coarse moduli space can be checked étale locally.

**Lemma 4.3.5.** *Let $\pi\colon \mathcal{X} \to X$ be a morphism to an algebraic space. Suppose that there is an étale covering $\{X_i \to X\}$ such that $\mathcal{X} \times_X X_i \to X_i$ is a coarse moduli space for each $i$. Then $\pi\colon \mathcal{X} \to X$ is a coarse moduli space.* $\square$

*Proof.* Axiom (1) of a coarse moduli space is a condition on geometric fibers and can thus be checked étale locally, while Axiom (2) follows from the fact that algebraic spaces are sheaves in the étale topology. $\square$

**Theorem 4.3.6.** *If $G$ is a finite group acting on an affine scheme $\operatorname{Spec} A$, then $\pi\colon [\operatorname{Spec} A/G] \to \operatorname{Spec} A^G$ is a coarse moduli space. Moreover,*

*(1) the base change of $\pi$ along a flat morphism $X' \to \operatorname{Spec} A^G$ of algebraic spaces is a coarse moduli space,*

*(2) the natural map $X \to \pi_*\mathcal{O}_{\mathcal{X}}$ is an isomorphism, and*

*(3) if $A$ is finitely generated over a noetherian ring $R$, then $A^G$ is finitely generated over $R$ and $\pi$ is a proper universal homeomorphism.*

*Proof.* In Theorem 4.2.6, we showed that $\pi\colon [\operatorname{Spec} A/G] \to \operatorname{Spec} A^G$ is a coarse moduli space and that $R \to A^G$ is of finite type if $R$ is noetherian and $R \to A$ is of finite type. To see (1), it suffices by Lemma 4.3.5 to consider flat morphisms $Y' \to Y$ from an affine scheme. But in this case, the base change $\mathcal{X} \times_Y Y'$ is isomorphic to a quotient stack $[\operatorname{Spec} B/G]$ and Lemma 4.2.4 implies that $Y' \cong \operatorname{Spec} B^G$. It follows that $\mathcal{X} \times_Y Y' \to Y'$ is a coarse moduli space. Part (2) follows directly from (1) by Lemma 4.3.4. For (3), it remains to show that $\pi$ is a proper universal homeomorphism. Since $\pi$ is bijective and universally closed, its set-theoretic inverse is continuous, and thus $\pi$ is a homeomorphism. The base change of $\pi$ along a morphism $\operatorname{Spec} B \to \operatorname{Spec} A^G$ factors as $[\operatorname{Spec}(B \otimes_{A^G} A)/G] \to \operatorname{Spec}(B \otimes_{A^G} A)^G \to \operatorname{Spec} B$ where the first map is a homeomorphism by the above argument and the second is a homeomorphism by Exercise 4.2.5. We conclude that $\pi$ is a universal homeomorphism. □

### 4.3.2 Descending étale morphisms to quotients

**Proposition 4.3.7.** *Let $G$ be a finite group and $f\colon \operatorname{Spec} A \to \operatorname{Spec} B$ be a $G$-equivariant morphism of affine schemes of finite type over a noetherian ring $R$. Let $x \in \operatorname{Spec} A$ be a closed point. Assume that*

*(a) $f$ is étale at $x$ and*

*(b) the induced map $G_x \to G_{f(x)}$ of stabilizer group schemes is bijective.*

*Then there is an open affine neighborhood $W \subset \operatorname{Spec} A^G$ of the image of $x$ such that $W \to \operatorname{Spec} A^G \to \operatorname{Spec} B^G$ is étale and $\pi_A^{-1}(W) \cong W \times_{\operatorname{Spec} B^G} [\operatorname{Spec} B/G]$, where $\pi_A\colon [\operatorname{Spec} A/G] \to \operatorname{Spec} A^G$.*

**Remark 4.3.8.** In other words, after replacing $\operatorname{Spec} A^G$ with an affine neighborhood $W$ of $\pi_A(x)$ and $\operatorname{Spec} A$ with $\pi_A^{-1}(W)$, it can be arranged that the diagram

$$\begin{tikzcd} {[\operatorname{Spec} A/G]} \arrow[r, "f"] \arrow[d, "\pi_A"] & {[\operatorname{Spec} B/G]} \arrow[d, "\pi_B"] \\ \operatorname{Spec} A^G \arrow[r] & \operatorname{Spec} B^G \end{tikzcd} \tag{4.3.1}$$

is cartesian where both horizontal maps are étale.

Condition (b) can be tested on a field-valued point $\operatorname{Spec} \Bbbk \to \operatorname{Spec} A$ representing $x$ (e.g. the inclusion of the residue field).

The above proposition will be applied in the following form in the proof of the Keel–Mori Theorem (Theorem 4.3.11).

**Corollary 4.3.9.** *Let $G$ be a finite group and $f\colon \operatorname{Spec} A \to \operatorname{Spec} B$ be a $G$-equivariant morphism of affine schemes of finite type over a noetherian ring $R$. Assume that for every closed point $x \in \operatorname{Spec} A$,*

*(a) $f$ is étale at $x$ and*

*(b) the induced map $G_x \to G_{f(x)}$ of stabilizer group schemes is bijective.*

*Then $\operatorname{Spec} A^G \to \operatorname{Spec} B^G$ is étale and (4.3.1) is cartesian.* □

*Proof of Proposition 4.3.7.* Set $y = f(x)$. We first claim that the question is étale local around $\pi_B(y) \in \operatorname{Spec} B^G$. Indeed, if $Y' \to Y := \operatorname{Spec} B^G$ is an affine étale neighborhood of $\pi_B(y)$, we let $X', \mathcal{X}'$ and $\mathcal{Y}'$ denote the base changes of $X := \operatorname{Spec} A^G$, $\mathcal{X} := [\operatorname{Spec} A/G]$, and $\mathcal{Y} := [\operatorname{Spec} B/G]$. By Lemma 4.2.4, we know that $\mathcal{Y}' \cong [\operatorname{Spec} B'/G]$ with $Y' \cong \operatorname{Spec} B'^G$ and similarly for $\mathcal{X}'$ and $X'$. If the result holds after this base change, there is an open neighborhood $W' \subset X'$ containing a preimage of $\pi_A(x)$ such that $W' \hookrightarrow X' \to Y'$ is étale and such that the preimage of $W'$ in $\mathcal{X}'$ is isomorphic to $W' \times_{Y'} \mathcal{Y}'$. Taking $W$ as the image of $W'$ under $X' \to \operatorname{Spec} A^G$ and applying étale descent yields the desired claim.

We now claim that this allows us to assume that $B^G$ is strictly henselian. To see this, let $Y^{\mathrm{sh}} = \operatorname{Spec} \mathcal{O}^{\mathrm{sh}}_{Y, \pi_B(y)}$ and $X^{\mathrm{sh}}$, $\mathcal{X}^{\mathrm{sh}}$ and $\mathcal{Y}^{\mathrm{sh}}$ be the base changes of $X$, $\mathcal{X}$ and $\mathcal{Y}$ along $Y^{\mathrm{sh}} \to Y$. Suppose $U^{\mathrm{sh}} \subset X^{\mathrm{sh}}$ is an open affine subscheme of the unique point in $X^{\mathrm{sh}}$ over $x$ and the closed point of $Y^{\mathrm{sh}}$ such that $U^{\mathrm{sh}} \to Y^{\mathrm{sh}}$ is étale with $\pi^{-1}_{\mathcal{X}^{\mathrm{sh}}}(U^{\mathrm{sh}}) \cong U^{\mathrm{sh}} \times_{Y^{\mathrm{sh}}} \mathcal{Y}^{\mathrm{sh}}$. Then $Y = \lim_\lambda Y_\lambda$ is the limit of affine étale neighborhood $Y_\lambda \to Y$ of $y$ and we set $X_\lambda$, $\mathcal{X}_\lambda$ and $\mathcal{Y}_\lambda$ to be the base changes of $X$, $\mathcal{X}$ and $\mathcal{Y}$ along $Y_\lambda \to Y$. By Proposition A.6.4, the morphism $U^{\mathrm{sh}} \to X^{\mathrm{sh}}$ descends to $U_\eta \to X_\eta$ for some $\eta$. Setting $U_\lambda = U_\eta \times_{X_\eta} X_\lambda$ for $\lambda > \eta$, it follows from Proposition A.6.7 that for $\lambda \gg 0$ (a) $U_\lambda \to X_\lambda$ is an open immersion, (b) the composition $U_\lambda \to X_\lambda \to Y_\lambda$ is étale, and (c) $\pi^{-1}_{\mathcal{X}_\lambda}(U_\lambda) \cong U_\lambda \times_{Y_\lambda} \mathcal{Y}_\lambda$ (by arguing on the étale presentations of $\mathcal{X}$ and $\mathcal{Y}$).

Finally, As $B^G \to B$ is finite (Lemma 4.2.3), $B = B_1 \times \cdots \times B_r$ is a product of strictly henselian local rings (Proposition A.9.6). As in the proof of Theorem 4.3.6, we may replace $[\operatorname{Spec} B/G]$ with $[\operatorname{Spec} B_1/G_y]$ and $[\operatorname{Spec} A/G]$ with $[f^{-1}(\operatorname{Spec} B_1)/G]$ to assume that $G$ fixes $x$ and $y$ while acting trivially on the residue fields $\kappa(x) = \kappa(y)$. Thus $\operatorname{Spec} A \to \operatorname{Spec} B$ has a unique section $s\colon \operatorname{Spec} B \to \operatorname{Spec} A$ taking $y$ to $x$. The section $s$ is necessarily $G$-invariant (as in the proof Theorem 4.3.6). Thus $s$ descends to a section of $\operatorname{Spec} A^G \to \operatorname{Spec} B^G$ which gives our desired open and closed subscheme $W \subset \operatorname{Spec} A^G$. $\square$

**Remark 4.3.10.** Here's a conceptual reason why we should expect the induced map of quotients to be étale. For simplicity, assume that $R = \Bbbk$ is an algebraically closed field. Let $\widehat{A}$ and $\widehat{B}$ be the completions of the local rings at $x$ and $f(x)$. The stabilizers $G_x$ and $G_{f(x)}$ act on $\operatorname{Spec} \widehat{A}$ and $\operatorname{Spec} \widehat{B}$, respectively, and the map $\operatorname{Spec} \widehat{A} \to \operatorname{Spec} \widehat{B}$ is equivariant with respect to the map $G_x \to G_{f(x)}$. The completion $\widehat{A^G}$ of $A^G$ at the image of $x$ is isomorphic to $\widehat{A}^{G_x}$ (Exercise 4.2.9) and similarly $\widehat{B^G} = \widehat{B}^{G_{f(x)}}$. Since $f$ is étale at $x$, $\widehat{B} \to \widehat{A}$ is an isomorphism and since $G_x \to G_{f(x)}$ is bijective, the induced map $\widehat{B^G} \to \widehat{A^G}$ is an isomorphism which shows that $\operatorname{Spec} A^G \to \operatorname{Spec} B^G$ is étale at the image of $x$.

### 4.3.3 The Keel–Mori Theorem

We now state and prove the Keel–Mori Theorem.

**Theorem 4.3.11.** *Let $\mathcal{X}$ be a Deligne-Mumford stack separated and of finite type over a noetherian algebraic space $S$. Then there exists a coarse moduli space $\pi\colon \mathcal{X} \to X$ with $\mathcal{O}_X = \pi_* \mathcal{O}_{\mathcal{X}}$ such that*

*(1) $X$ is separated and of finite type over $S$,*

*(2) $\pi$ is a proper universal homeomorphism, and*

*(3) for every flat morphism $X' \to X$ of algebraic spaces, the base change $\mathcal{X} \times_X X' \to X'$ is a coarse moduli space.*

**Remark 4.3.12.** The Keel–Mori Theorem [KM97] holds more generally with the 'separated' condition on $\mathcal{X} \to S$ by the finiteness of the inertia $I_{\mathcal{X}} \to \mathcal{X}$; see Remark 4.3.13. In particular, it holds for algebraic stacks with finite but *non-reduced* automorphism groups. The theorem also holds without any noetherian or finiteness conditions; see [Con05b, Ryd13] and [SP, Tag 0DUK].

*Proof.* We first handle the case when $S = \operatorname{Spec} R$ is affine. The question is Zariski-local on $\mathcal{X}$: if $\{\mathcal{X}_i\}$ is a Zariski open covering of $\mathcal{X}$ with coarse moduli spaces $\mathcal{X}_i \to X_i$, then since coarse moduli spaces are unique (Definition 4.3.1(2)), the $X_i$'s glue to form an algebraic space $X$ and a map $\mathcal{X} \to X$, which is a coarse moduli space by Lemma 4.3.5. It thus suffices to show that every closed point $x \in |\mathcal{X}|$ has an open neighborhood that admits a coarse moduli space.

By the Local Structure Theorem of Deligne–Mumford Stacks (Theorem 4.2.11), there exists an affine étale morphism

$$f\colon \big(\mathcal{W} = [\operatorname{Spec} A/G_x], w\big) \to (\mathcal{X}, x)$$

such that $f$ induces an isomorphism of geometric stabilizer groups at $w$.

We claim that since $\mathcal{X}$ is separated, the locus $\mathcal{U}$ consisting of points $z \in |\mathcal{W}|$, such that $f$ induces an isomorphism of geometric stabilizer groups at $z$, is open. To establish this, we will analyze the natural morphism $I_{\mathcal{W}} \to I_{\mathcal{X}} \times_{\mathcal{X}} \mathcal{W}$ of relative group schemes over $\mathcal{W}$ as the fiber of this morphism over $z \in \mathcal{W}(\Bbbk)$ is precisely the morphism $G_z \to G_{f(z)}$ of stabilizers. We will exploit the cartesian diagram

$$\begin{tikzcd} I_{\mathcal{W}} \arrow[r, "\Psi"] \arrow[d] \arrow[dr, phantom, "\square"] & I_{\mathcal{X}} \times_{\mathcal{X}} \mathcal{W} \arrow[d] \\ \mathcal{W} \arrow[r] & \mathcal{W} \times_{\mathcal{X}} \mathcal{W}; \end{tikzcd}$$

see Exercise 3.2.13. Since $\mathcal{W} \to \mathcal{X}$ is representable, étale and separated, the diagonal $\mathcal{W} \to \mathcal{W} \times_{\mathcal{X}} \mathcal{W}$ is an open and closed immersion and thus so is $\Psi$. Since $I_{\mathcal{X}} \to \mathcal{X}$ is finite, so is $p_2\colon I_{\mathcal{X}} \times_{\mathcal{X}} \mathcal{W} \to \mathcal{W}$. Thus $p_2(|I_{\mathcal{X}} \times_{\mathcal{X}} \mathcal{W}| \setminus |I_{\mathcal{W}}|) \subset |\mathcal{W}|$ is closed and its complement, which is identified with the locus $\mathcal{U}$, is open.

Let $\pi_{\mathcal{W}}\colon \mathcal{W} \to W = \operatorname{Spec} A^{G_x}$ be the coarse moduli space (Theorem 4.3.6). Choose an affine open subscheme $X_1 \subset W$ containing $\pi_{\mathcal{W}}(w)$. Then $\mathcal{X}_1 = \pi_{\mathcal{W}}^{-1}(X_1)$ is isomorphic to a quotient stack $[\operatorname{Spec} A_1/G_x]$ such that $X_1 = \operatorname{Spec} A_1^{G_x}$. This provides an affine étale morphism

$$g\colon (\mathcal{X}_1 = [\operatorname{Spec} A_1/G_x], w) \to (\mathcal{X}, x)$$

which induces a bijection on all geometric stabilizer groups.

We now show that the open substack $\mathcal{X}_0 := \operatorname{im}(f)$ admits a coarse moduli space. Define $\mathcal{X}_2 := \mathcal{X}_1 \times_{\mathcal{X}} \mathcal{X}_1$ and $\mathcal{X}_3 := \mathcal{X}_1 \times_{\mathcal{X}} \mathcal{X}_1 \times_{\mathcal{X}} \mathcal{X}_1$. Since $g$ is affine, each $\mathcal{X}_i$ is of the form $[\operatorname{Spec} A_i/G_x]$ and there is a coarse moduli space $\pi_i\colon \mathcal{X}_i \to X_i = \operatorname{Spec} A_i^{G_x}$. By universality of coarse moduli spaces, there is a diagram

$$\begin{tikzcd} \mathcal{X}_3 \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] \arrow[d, "\pi_3"] & \mathcal{X}_2 \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "\pi_2"] & \mathcal{X}_1 \arrow[r, "g"] \arrow[d, "\pi_1"] & \mathcal{X}_0 = \operatorname{im}(f) \arrow[d, dashed, "\pi_0"] \\ X_3 \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & X_2 \arrow[r, shift left] \arrow[r, shift right] & X_1 \arrow[r, dashed] & X_0 \end{tikzcd} \tag{4.3.2}$$

where the natural squares commute. Since $g$ induces bijections of geometric stabilizer groups at all points, the same is true for each projection $\mathcal{X}_2 \to \mathcal{X}_1$ and $\mathcal{X}_3 \to \mathcal{X}_2$.

Corollary 4.3.9 implies that each map $X_2 \to X_1$ and $X_3 \to X_2$ is étale, and the natural squares of solid arrows in (4.3.2) are cartesian.

The universality of coarse moduli spaces induces an étale groupoid structure $X_2 \rightrightarrows X_1$. To check that this is an étale equivalence relation, it suffices to check that $X_2 \to X_1 \times X_1$ is injective on geometric points, but this follows from the observation the $|\mathcal{X}_2| \to |\mathcal{X}_1| \times |\mathcal{X}_1|$ is injective on closed points. Therefore there is an algebraic space quotient $X_0 := X_1/X_2$ and a map $X_1 \to X_0$. By étale descent along $\mathcal{X}_1 \to \mathcal{X}_0$, there is a map $\pi_0 \colon \mathcal{X}_0 \to X_0$ making the right square in (4.3.2) commute.

To argue that $\pi \colon \mathcal{X}_0 \to X_0$ is a coarse moduli space, we will use the commutative cube

$$\begin{tikzcd}
& \mathcal{X}_2 \arrow[rr] \arrow[dl] \arrow[dd] & & \mathcal{X}_1 \arrow[dl] \arrow[dd] \\
\mathcal{X}_1 \arrow[rr, crossing over] \arrow[dd] & & \mathcal{X}_0 & \\
& X_2 \arrow[rr] \arrow[dl] & & X_1 \arrow[dl] \\
X_1 \arrow[rr] & & X_0, \arrow[from=uu, crossing over] &
\end{tikzcd}$$

where the top, left, and bottom faces are cartesian. It follows from étale descent along $\mathcal{X}_1 \to \mathcal{X}_0$ that the right face is also cartesian and since being a coarse moduli space is étale local on $X_0$ (Lemma 4.3.5), we conclude that $\mathcal{X}_0 \to X_0$ is a coarse moduli space. Except for the separatedness, the additional properties in the statement are étale-local on $X_0$, so they follow from the analogous properties of the coarse moduli space $[\mathrm{Spec}(A_1)/G_x] \to \mathrm{Spec}(A_1^{G_x})$ from Theorem 4.3.6. As $\mathcal{X}_0 \to X_0$ is proper, the separatedness of $\mathcal{X}_0$ is equivalent to the separatedness of $X_0$.

Finally, the case when $S$ is a noetherian algebraic space can be reduced to the affine case by imitating the above argument to étale locally construct the coarse moduli space of $\mathcal{X}$. □

**Remark 4.3.13.** The more general case when $\mathcal{X}$ is an algebraic stack with finite inertia $I_{\mathcal{X}} \to \mathcal{X}$ (see Remark 4.3.12) is proven in an analogous but more technical manner. Namely, the use of the Local Structure Theorem for Deligne–Mumford stacks (Theorem 4.2.11) is replaced by the existence of an étale neighborhood $\mathcal{W} \to \mathcal{X}$ around every closed point such that $\mathcal{W}$ admits a finite flat presentation $V \to \mathcal{W}$ from an affine scheme and the corresponding groupoid $R := V \times_{\mathcal{W}} V \rightrightarrows V$ is a finite flat groupoid of affine schemes. This in turn is proven in an analogous way to Theorem 4.2.11 where one chooses a quasi-finite and flat surjection $U \to \mathcal{X}$ and one replaces the use of $[(U/\mathcal{X})_0^d/S_d])$ with a Hilbert stack $\mathcal{H}$ whose objects over a scheme $S$ consists of a morphism $S \to \mathcal{X}$ and a closed subscheme $Z \hookrightarrow U_S$ finite and flat (rather than finite and étale) over $S$. (Aside: it is also possible to prove this without reference to a Hilbert scheme by using étale localization of groupoidsand splitting for groupoids; see [KM97, §4] or [SP, Tags 0DU4 and 04RJ]. Finally, the existence of a coarse moduli space for quotients $[V/R]$ is proven analogously to Theorem 4.3.6 (see Exercise 4.2.10).

The Local Structure Theorem of Deligne–Mumford Stacks (Theorem 4.2.11) can also be formulated étale locally on a coarse moduli space:

**Corollary 4.3.14** (Local Structure of Coarse Moduli Spaces)**.** *Let $\mathcal{X}$ be a Deligne–Mumford stack of finite type and separated over a noetherian algebraic space $S$, and let $\pi\colon \mathcal{X} \to X$ be its coarse moduli space. For every closed point $x \in |\mathcal{X}|$ with geometric stabilizer group $G_x$, there exists a cartesian diagram*

$$\begin{tikzcd} {[\operatorname{Spec} A/G_x]} \arrow[r] \arrow[d] & \mathcal{X} \arrow[d, "\pi"] \\ \operatorname{Spec} A^{G_x} \arrow[r] & X \end{tikzcd}$$

*such that* $\operatorname{Spec} A^{G_x} \to X$ *is an étale neighborhood of* $\pi(x) \in |X|$.

*Proof.* This follows from the construction of the coarse moduli space in the proof of Theorem 4.3.11. Alternatively, it follows from the Local Structure Theorem of Deligne–Mumford stacks (Theorem 4.2.11) and Exercise 4.3.15 □

**Exercise 4.3.15.** Establish the following generalization of Proposition 4.3.7: Let $S$ be a noetherian algebraic space. Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a morphism of Deligne–Mumford stacks separated and of finite type over $S$ and

$$\begin{tikzcd} \mathcal{X} \arrow[r, "f"] \arrow[d, "\pi_{\mathcal{X}}"] & \mathcal{Y} \arrow[d, "\pi_{\mathcal{Y}}"] \\ X \arrow[r] & Y \end{tikzcd}$$

be a commutative diagram where $\pi_{\mathcal{X}}\colon \mathcal{X} \to X$ and $\pi_{\mathcal{Y}}\colon \mathcal{Y} \to Y$ are coarse moduli spaces. Let $x \in |\mathcal{X}|$ be a closed point such that

(1) $f$ is étale at $x$ and

(2) the induced map $G_x \to G_{f(x)}$ of geometric stabilizer groups is bijective.

Then there exists an open neighborhood $U \subset X$ of $\pi_{\mathcal{X}}(x)$ such that $U \to X \to Y$ is étale and $\pi_{\mathcal{X}}(U) \cong U \times_Y \mathcal{Y}$.

**Exercise 4.3.16.** Let $\mathcal{X}$ be a Deligne–Mumford stack of finite type and separated over a noetherian algebraic space $S$, and let $\pi\colon \mathcal{X} \to X$ be its coarse moduli space. Assume that the order of the stabilizer of every geometric point of $\mathcal{X}$ is invertible in $S$.

(a) Show that the functor $\pi_*$ is exact on quasi-coherent sheaves on $\mathcal{X}$.

(b) Show that for every morphism $X' \to X$ of algebraic spaces, the base change $\mathcal{X} \times_X X' \to X'$ is a coarse moduli space (see Exercise 4.2.5).

**Exercise 4.3.17.** Let $\mathcal{X}$ be a Deligne–Mumford stack of finite type and separated over a noetherian algebraic space $S$, and let $\pi\colon \mathcal{X} \to X$ be its coarse moduli space. Show that if $\mathcal{X}$ is normal, then so is $X$.

### 4.3.4 Examples

**Example 4.3.18.** Consider the moduli stack $\mathcal{M}_{1,1}$ of elliptic curves over a field $\Bbbk$ with $\operatorname{char}(\Bbbk) \neq 2, 3$. The Weierstrass form $y^2 = x(x-1)(x-\lambda)$ gives an isomorphism $\mathcal{M}_{1,1} \cong [(\mathbb{A}^1 \setminus \{0,1\})/S_3]$ (see Exercise 3.1.18) where the $S_3$-orbit of $\lambda$ is $\{\lambda, 1/\lambda, 1-\lambda, 1/(1-\lambda), \lambda/(\lambda-1), (\lambda-1)/\lambda\}$. The coarse moduli space is given by $j$-invariant

$$j\colon \mathcal{M}_{1,1} \to \mathbb{A}^1, \quad \lambda \mapsto 2^8 \frac{(\lambda^2-\lambda+1)^3}{\lambda^2(\lambda-1)^3}.$$

Indeed, one can verify $\Bbbk[\lambda]^{S_3}_{\lambda(\lambda-1)} = \Bbbk[j(\lambda)]$.

Alternatively, the Weierstrass form $y^2 = x^3 + ax + b$ gives an isomorphism $\mathcal{M}_{1,1} \cong [\mathbb{A}^2_\Delta/\mathbb{G}_m]$ (see Exercise 3.1.18(b)) where the action is given by $t \cdot (a,b) = (t^4 a, t^6 b)$ and $\Delta$ is the discriminant $4a^3 + 27b^2$. As $\Bbbk[a,b]^{\mathbb{G}_m}_\Delta = \Bbbk[a^3/\Delta]$ (noting that $\beta := b^2/\Delta$ is generated by $\alpha := a^3/\Delta$ under the relation $4\alpha + 27\beta = 1$), the coarse moduli space $\mathcal{M}_{1,1} \to \mathbb{A}^1$ is given by $(a,b) \mapsto a^3/\Delta$.

**Exercise 4.3.19.** Let $\operatorname{char}(\Bbbk) \neq 2$ and $G = \mathbb{Z}/2$.

(a) Let $G$ act on the non-separated union $X = \mathbb{A}^1 \bigcup_{x \neq 0} \mathbb{A}^1$ by exchanging the copies of $\mathbb{A}^1$. The quotient $[X/G]$ is a Deligne–Mumford stack with quasi-finite but not finite inertia, and in particular non-separated. Show nevertheless that there is a coarse moduli space $[X/G] \to \mathbb{A}^1$.

(b) Let $X$ be the non-separated union $\mathbb{A}^2 \bigcup_{x \neq 0} \mathbb{A}^2$. Let $G = \mathbb{Z}/2$ act on $X$ by simultaneously exchanging the copies of $\mathbb{A}^2$ and by acting via the involution $y \mapsto -y$ on each copy. Show that $[X/G]$ does not admit a coarse moduli space.

**Example 4.3.20.** Consider the action of $\mathrm{PGL}_2$ on the scheme $\operatorname{Sym}^4 \mathbb{P}^1 \cong (\mathbb{P}^1)^4/S_4$ (which is the coarse moduli space of $[\mathbb{P}^1)^4/S_4]$) parameterizing four unordered points in $\mathbb{P}^1$. Let $\mathcal{X} \subset [\operatorname{Sym}^4 \mathbb{P}^1/\mathrm{PGL}_2]$ be the open substack parameterizing tuples $(p_1, p_2, p_3, p_4)$ where at least three points are distinct. Consider the family $(0, 1, \lambda, \infty)$ with $\lambda \in \mathbb{P}^1$. If $\lambda \notin \{0, 1\infty\}$, then we claim that $\operatorname{Aut}(0,1,\lambda,\infty) = \mathbb{Z}/2 \times \mathbb{Z}/2$. To see this, there is a unique element $\sigma \in \mathrm{PGL}_2$ such that $\sigma(0) = \infty$, $\sigma(\infty) = 0$ and $\sigma(1) = \lambda$ which acts on $\mathbb{P}^1$ via $\sigma([x,y]) = [y, \lambda, x]$ and thus $\sigma(\lambda) = 1$. Similarly, there is an element interchanging 0 with 1 and $\lambda$ with $\infty$ and an element interchanging 0 with $\lambda$ and 1 with $\infty$. However, if $\lambda \in \{0, 1\infty\}$, then $\operatorname{Aut}(0,1,\lambda,\infty) = \mathbb{Z}/2$. We therefore see that the inertia $I_\mathcal{X} \to \mathcal{X}$ while quasi-finite is not finite and that $\mathcal{X}$ is not separated. Nevertheless, the map $\mathcal{X} \to \mathbb{P}^1$ taking $(p_1, p_2, p_3, p_4)$ to its cross-ratio is a coarse moduli space.

### 4.3.5 Descending vector bundles to the coarse moduli space

We begin with a Nakayama lemma for coherent sheaves.

**Lemma 4.3.21.** *Let $\mathcal{X}$ be a Deligne-Mumford stack separated and of finite type over a noetherian algebraic space $S$, and let $\pi \colon \mathcal{X} \to X$ be its coarse moduli space. Let $x \in |\mathcal{X}|$ be a closed point.*

*(1) If $F$ is a coherent sheaf on $\mathcal{X}$ such that $F|_{\mathcal{G}_x} = 0$, then there exists an open neighborhood $U \subset X$ of $\pi(x)$ such that $F|_{\pi^{-1}(U)} = 0$.*

*(2) If $\phi \colon F \to G$ is a morphism of coherent sheaves (resp. vector bundles of the same rank) on $\mathcal{X}$ such that $\phi|_{\mathcal{G}_x}$ is surjective, then there exists an open neighborhood $U \subset X$ of $\pi(x)$ such that $\phi|_{\pi^{-1}(U)}$ is surjective (resp. an isomorphism).*

*Proof.* For (1), the support $\operatorname{Supp}(F) \subset |\mathcal{X}|$ of $F$ is a closed subset (which follows from using descent along a presentation) and the open set $U = X \setminus \pi(\operatorname{Supp}(F))$ satisfies the conclusion. For (2), apply (1) to the coherent sheaf $\operatorname{coker}(\phi)$ noting that a surjection of vector bundles of the same rank is an isomorphism. □

**Definition 4.3.22.** A Deligne–Mumford stack $\mathcal{X}$ is *tame* if for every geometric point $x \in \mathcal{X}(\Bbbk)$, the order of $\operatorname{Aut}_{\mathcal{X}(\Bbbk)}(x)$ is invertible in $\Gamma(\mathcal{X}, \mathcal{O}_\mathcal{X})$.

**Remark 4.3.23.** If $\mathcal{X}$ is defined over a field $\Bbbk$, then this means that the order of every geometric stabilizer group is prime to the characteristic of $\Bbbk$.

**Lemma 4.3.24.** *Let $\mathcal{X}$ be a Deligne-Mumford stack separated and of finite type over a noetherian algebraic space $S$, and let $\pi\colon \mathcal{X} \to X$ be its coarse moduli space. If $\mathcal{X}$ is tame, then $\pi_*$ is exact.*

*Proof.* The question is étale-local on $X$: if $g\colon X' \to X$ is an étale cover inducing a cartesian diagram

$$\begin{tikzcd} \mathcal{X}' \arrow[r, "g'"] \arrow[d, "\pi'"] & \mathcal{X} \arrow[d, "\pi"] \\ X \arrow[r, "g"] & X \end{tikzcd}$$

then by flat base change there is an identification $g^*\pi_* = \pi'_* g'^*$ of functors on quasi-coherent sheaves. Since $g^*$ is faithfully exact, we see that $\pi_*$ is exact if and only $\pi'_*$ is. We can therefore use Corollary 4.3.14 to reduce to the case that $\mathcal{X} = [\operatorname{Spec} A/G]$ and $X = \operatorname{Spec} A^G$, and this case follows from Exercise 4.2.5. $\square$

We say that a vector bundle $F$ on $\mathcal{X}$ *descends to its coarse moduli space* $\pi\colon \mathcal{X} \to X$ if there exists a vector bundle $\overline{F}$ on $X$ and an isomorphism $F \cong \pi^*\overline{F}$. Observe that one necessary condition is that for every field-valued point $x\colon \operatorname{Spec} \Bbbk \to \mathcal{X}$, which induces a commutative diagram

$$\begin{tikzcd} \mathbf{B}G_x \arrow[r, hook, "i_x"] \arrow[d, "p"] & \mathcal{X} \arrow[d, "\pi"] \\ \operatorname{Spec} \Bbbk \arrow[r, hook] & X, \end{tikzcd}$$

the pullback $i_x^*F = p^*(\overline{F} \otimes \Bbbk)$ is trivial or in other words $G_x$ acts trivially on the fiber $F \otimes \Bbbk$.

**Proposition 4.3.25.** *Let $\mathcal{X}$ be a tame Deligne-Mumford stack separated and of finite type over a noetherian algebraic space $S$, and let $\pi\colon \mathcal{X} \to X$ be its coarse moduli space. A vector bundle $F$ on $\mathcal{X}$ descends to a vector bundle on $X$ if and only if for every field-valued point $x\colon \operatorname{Spec} \Bbbk \to \mathcal{X}$ with closed image, the action of $G_x$ on the fiber $F \otimes \Bbbk$ is trivial. In this case, $\pi_*F$ is a vector bundle and the adjunction map $\pi^*\pi_*F \to F$ is an isomorphism.*

**Remark 4.3.26.** The above condition is insensitive to field extensions and equivalent to the condition that the restriction of $F$ to the residual gerbe is trivial.

*Proof.* To see that the condition is sufficient, consider the commutative diagram

$$\begin{tikzcd} \mathcal{G}_x \arrow[r, hook] \arrow[d, "p"] & \mathcal{X} \arrow[d, "\pi"] \\ \operatorname{Spec} \kappa(x) \arrow[r, hook] & X. \end{tikzcd}$$

We break down the proof into three steps.

*Step 1: $\pi^*\pi_*F \to F$ is surjective.* It suffices by Lemma 4.3.21 to show that $(\pi^*\pi_*F)|_{\mathcal{G}_x} \to F|_{\mathcal{G}_x}$ is surjective for every closed point $x \in |\mathcal{X}|$. Since $F \to F|_{\mathcal{G}_x}$ is surjective and $\pi_*$ is exact (Lemma 4.3.24), $(\pi^*\pi_*F)|_{\mathcal{G}_x} \to \pi^*(\pi_*(F|_{\mathcal{G}_x}))|_{\mathcal{G}_x} \cong p^*p_*(F|_{\mathcal{G}_x})$ is surjective. The hypotheses imply that the adjunction $p^*p_*(F|_{\mathcal{G}_x}) \to$

$F|_{\mathcal{G}_x}$ is an isomorphism and it follows that the composition $(\pi^*\pi_*F)|_{\mathcal{G}_x} \to p^*p_*(F|_{\mathcal{G}_x}) \xrightarrow{\sim} F|_{\mathcal{G}_x}$ is surjective.

*Step 2: $\pi_*F$ is a vector bundle.* We can assume that the rank $r$ of $F$ is constant. Since being a vector bundle is an étale-local property, we can assume that $X = \operatorname{Spec} A$. The surjection $\bigoplus_{s \in \Gamma(X, \pi_*F)} A \to \pi_*F$ pulls back to a surjection $\bigoplus_{s \in \Gamma(\mathcal{X}, F)} \mathcal{O}_{\mathcal{X}} \to \pi^*\pi_*F$ and by Step 1, the composition $\bigoplus_{s \in \Gamma(\mathcal{X}, F)} \mathcal{O}_{\mathcal{X}} \to \pi^*\pi_*F \to F$ is surjective. As $F|_{\mathcal{G}_x} \cong \mathcal{O}_{\mathcal{G}_x}^r$ is trivial, for each closed point $x \in |\mathcal{X}|$, we can find $r$ sections $\phi \colon \mathcal{O}_{\mathcal{X}}^r \to F$ such that $\phi|_{\mathcal{G}_x}$ is an isomorphism. By Lemma 4.3.21, there exists an open neighborhood $U \subset X$ of $\pi(x)$ such that $\phi|_{\pi^{-1}(U)}$ is an isomorphism. Thus $\pi_*\phi \colon \mathcal{O}_X^r \to \pi_*F$ is an isomorphism over $U$ and we conclude that $\pi_*F$ is a vector bundle of the same rank as $F$.

*Step 3: $\pi^*\pi_*F \to F$ is an isomorphism.* Since $\pi^*\pi_*F \to F$ is a surjection of vector bundles of the same rank, it is an isomorphism. □

**Remark 4.3.27.** The analogous statement for coherent sheaves is not true. For example, if the characteristic is not 2, then letting $\mathbb{Z}/2$ on $\mathbb{A}^1$ via $x \mapsto -x$, we have a tame coarse moduli space $[\mathbb{A}^1/(\mathbb{Z}/2)] \to \mathbb{A}^1 = \operatorname{Spec} \operatorname{Spec} k[x^2]$. The inclusion $\mathbf{B}\mathbb{Z}/2 \hookrightarrow [\mathbb{A}^1/(\mathbb{Z}/2)]$ of 0 is a closed substack and $\mathcal{O}_{\mathbf{B}\mathbb{Z}/2}$ is a coherent sheave which does not descend. Observe that in this case, the pullback of the residue field of $0 \in \mathbb{A}^1$ is $k[x]/x^2$. This example also illustrated that the fibers of a coarse moduli space $\mathcal{X} \to X$ can be non-reduced and larger than the residual gerbe.

When $\mathcal{X}$ is not tame, we have the following variant for descending line bundles.

**Proposition 4.3.28.** *Let $\mathcal{X}$ be a Deligne-Mumford stack separated and of finite type over a noetherian algebraic space $S$, and let $\pi \colon \mathcal{X} \to X$ be its coarse moduli space. If $\mathcal{L}$ is a line bundle on $\mathcal{X}$, then for $N$ sufficiently divisible $\mathcal{L}^{\otimes N}$ descends to $X$.*

*Proof.* To be added. □

**Example 4.3.29.** Show $\operatorname{Pic}(\mathcal{M}_{1,1}) = \mathbb{Z}/12$ generated by the Hodge bundle (see Example 4.1.4).

## 4.4 When are algebraic spaces schemes?

We prove various results providing conditions for an algebraic space to be a scheme. We show:

- a quasi-separated algebraic space is a scheme on a dense open subspace (Theorem 4.4.1);
- Zariski's Main Theorem for algebraic spaces (Theorem 4.4.9);
- an algebraic space separated and locally quasi-finite over a scheme is a scheme (Corollary 4.4.7);
- if the diagonal of a Deligne–Mumford stack is separated and quasi-compact diagonal, then the diagonal is quasi-affine (and in particular representable by schemes) (Corollary 4.4.8);
- an algebraic stack with trivial stabilizers is an algebraic space (Theorem 4.4.10) generalizing Theorem 3.6.5;

- Serre's and Chevalley's Criteria for Affineness (Theorems 4.4.16 and 4.4.20) for algebraic spaces;
- if $X$ is a quasi-separated algebraic space locally of finite type over a field $\Bbbk$ such that $X_{\Bbbk}$ has the property that every finite set of points is contained in an affine (e.g. $X_{\Bbbk}$ is quasi-projective), then $X$ is a scheme (Proposition 4.4.25); and
- quasi-separated group algebraic spaces locally of finite type over a field are schemes (Theorem 4.4.26)

We also give applications to the algebraicity of quotients of étale and smooth equivalence relations (Corollary 4.4.12).

### 4.4.1 Algebraic spaces are schemes over a dense open

**Theorem 4.4.1.** *Every quasi-separated algebraic space has a dense open subspace which is a scheme.*

*Proof.* We may assume that $X$ is quasi-compact. Let $f\colon V \to X$ be an étale presentation with $V$ an affine scheme. Since $X$ is quasi-separated, $f\colon V \to X$ is quasi-compact and there exists an open algebraic subspace $U \subset X$ such that $f^{-1}(U) \to U$ is finite. By Exercise 4.2.12, $U$ is isomorphic to a quotient stack $[V/G]$ for the free action of a finite group $G$ on a scheme $V$. If $V_1 \subset V$ is a dense affine open subscheme, then $V_2 = \bigcap_{g\in G} gV_1$ is a $G$-invariant quasi-affine open subscheme of $V$ and in particular separated. Repeating this argument, we can choose a dense affine open subscheme $V_3 \subset V_2$ and now $V_4 = \bigcap_{g\in G} gV_3$ is a $G$-invariant affine open subscheme. Proposition 4.2.13 implies that $V_4/G \cong \operatorname{Spec} A^G$ is a dense affine open algebraic subspace of $U$. □

**Remark 4.4.2.** See also [Knu71, II.6.7] and [SP, Tag 06NN]. The above result is not necessarily true if $X$ is not quasi-separated, e.g. $\mathbb{A}^1/\underline{\mathbb{Z}}$ (Example 3.9.22).

**Corollary 4.4.3.** *An integral quasi-separated algebraic space has a well-defined fraction field.* □

**Exercise 4.4.4.** Let $G$ be a finite group acting on a quasi-separated algebraic space $U$. Show that there is a $G$-invariant affine open subscheme of $U$.

### 4.4.2 Zariski's Main Theorem for algebraic spaces

We now prove Zariski's Main Theorem for algebraic spaces and Deligne–Mumford stacks. Its proof relies on the theory of quasi-coherent sheaves. Specifically, we will use the fact that if $f\colon \mathcal{X} \to \mathcal{Y}$ is a quasi-compact and quasi-separated morphism of Deligne–Mumford stacks, then $f_*\mathcal{O}_X$ is a quasi-coherent sheaf of $\mathcal{O}_{\mathcal{Y}}$-algebras (Exercise 4.1.17) and there is a factorization

$$f\colon \mathcal{X} \to \mathcal{S}\mathrm{pec}_{\mathcal{Y}}\, f_*\mathcal{O}_{\mathcal{X}} \to \mathcal{Y}.$$

See §A.5 for a discussion of Zariski's Main Theorem for schemes. In this section, we follow [LMB00, Thm. A.2] (see also [SP, Tag 05W7], [Knu71, II.6.15] and [Ols16, Thm. 7.2.10]).

**Proposition 4.4.5.** *A separated, quasi-finite, and representable morphism $f\colon \mathcal{X} \to \mathcal{Y}$ of Deligne–Mumford stacks factors as the composition of an open immersion $\mathcal{X} \hookrightarrow \mathcal{S}\mathrm{pec}_{\mathcal{Y}}\, f_*\mathcal{O}_{\mathcal{X}}$ and an affine morphism $\mathcal{S}\mathrm{pec}_{\mathcal{Y}}\, f_*\mathcal{O}_{\mathcal{X}} \to \mathcal{Y}$. In particular, $f$ is quasi-affine.*

*Proof.* Since the construction of $f_*\mathcal{O}_{\mathcal{X}}$ commutes with flat base change on $\mathcal{Y}$, so does the formation of the factorization $f\colon \mathcal{X} \to \mathcal{S}\mathrm{pec}_{\mathcal{Y}}\, f_*\mathcal{O}_{\mathcal{X}} \to \mathcal{Y}$. The statement is thus étale-local on $\mathcal{Y}$. In particular, we can assume that $\mathcal{Y} = Y$ is an affine scheme and that $\mathcal{X} = X$ is an algebraic space. After replacing $Y$ with $\mathcal{S}\mathrm{pec}_Y\, f_*\mathcal{O}_X$, we can assume that $f_*\mathcal{O}_X = \mathcal{O}_Y$ and we must show that $f\colon X \to Y$ is an open immersion.

Since $X$ is quasi-compact, there is an étale presentation $\pi\colon U \to X$ from an affine scheme. Since $X$ is separated, $U \to X$ is also separated. As the composition

$$U \xrightarrow{\pi} X \xrightarrow{f} Y$$

is a quasi-finite morphism of schemes, we can apply Étale Localization of Quasi-finite Morphisms (Theorem A.5.1) around every point $y \in Y$: after replacing $Y$ with an étale neighborhood, we can assume that $U = U_1 \sqcup U_2$ with $U_1 \to Y$ finite and $(U_2)_y = \emptyset$. Then $\pi(U_1)$ is open (as $\pi$ is étale) and closed (as $U_1 \to Y$ is finite and $X \to Y$ is separated). Thus $X = X_1 \sqcup X_2$ with $X_1 = \pi(U_1)$ and $(X_2)_y = \emptyset$. This shows that $\mathcal{O}_Y = f_*\mathcal{O}_X$ is the product $\mathcal{A}_1 \times \mathcal{A}_2$ of quasi-coherent $\mathcal{O}_X$-algebras, and thus we can also decompose $Y$ as $Y_1 \sqcup Y_2$ such that $y \in Y_1$ and $f(Y_i) \subset X_i$ for $i = 1, 2$. After replacing $Y$ with $Y_1$, the composition $U \to X \to Y$ is finite and Lemma 4.4.6 implies that $X$ is affine. Thus $X = Y = \mathcal{S}\mathrm{pec}_Y\, f_*\mathcal{O}_X$. □

**Lemma 4.4.6.** *Suppose that $U \to X$ is a surjective étale morphism of algebraic spaces and $X \to Y$ is a separated morphism of algebraic spaces. If the composition $U \to X \to Y$ is finite, so is $X \to Y$.*

*Proof.* The statement is étale-local on $Y$ so we can assume that $Y$ and $U$ are affine. As $X \to Y$ is separated, $U \to X$ is also finite. Since $X$ is identified with the quotient $U/R$ of the finite étale groupoid $R := U \times_X U \rightrightarrows U$ of affine schemes, Proposition 4.2.13 implies that $X$ is affine. As $U \to Y$ is proper, so is $X \to Y$. As $X \to Y$ is a proper and quasi-finite morphism of schemes, it is finite (Corollary A.5.5).

Alternatively, the properness of $X \to Y$ follows from the properness of $U \to Y$, and we may apply Corollary 4.4.14 to conclude that the proper and quasi-finite morphism $X \to Y$ is finite. □

**Corollary 4.4.7.** *A morphism of algebraic spaces which is separated and locally quasi-finite is representable by schemes. In particular, an algebraic space separated and locally quasi-finite over a scheme is a scheme.*

*Proof.* It suffices to show that if $X \to Y = \operatorname{Spec} A$ is a separated and locally quasi-finite, then $X$ is a scheme. Since being a scheme is a Zariski-local property, we can assume that $X$ is quasi-compact. Therefore Proposition 4.4.5 applies. □

**Corollary 4.4.8.** *The diagonal of a Deligne–Mumford stack with separated and quasi-compact diagonal is quasi-affine. In particular, a quasi-separated algebraic space has quasi-affine diagonal.*

*Proof.* The diagonal is separated, quasi-finite, and representable, and we conclude by Proposition 4.4.5. □

As with the case for schemes, we can refine Proposition 4.4.5 to obtain Zariski's Main Theorem.

**Theorem 4.4.9** (Zariski's Main Theorem)**.** *A separated, quasi-finite, and representable morphism $f\colon \mathcal{X} \to \mathcal{Y}$ of noetherian Deligne–Mumford stacks factors as the composition of a dense open immersion $\mathcal{X} \hookrightarrow \widetilde{\mathcal{Y}}$ and a finite morphism $\widetilde{\mathcal{Y}} \to \mathcal{X}$.*

*Proof.* Let $\mathcal{A} \subset f_*\mathcal{O}_\mathcal{X}$ be the integral closure of $\mathcal{O}_\mathcal{Y} \to f_*\mathcal{O}_\mathcal{X}$ where the sections of $\mathcal{A}$ over an étale morphism $T \to \mathcal{Y}$ from a scheme is the integral closure of $\Gamma(T, \mathcal{O}_T) \to \Gamma(\mathcal{X} \times_\mathcal{Y} T, \mathcal{O}_{\mathcal{X}\times_\mathcal{Y} T})$. Since the integral closure is compatible under étale extensions (Proposition A.5.4), $\mathcal{A}$ is a quasi-coherent sheaf of $\mathcal{O}_\mathcal{Y}$-algebras. Using Exercise 4.1.25, write $\mathcal{A} = \operatorname{colim} \mathcal{A}_\lambda$ as the colimit of finite type $\mathcal{O}_\mathcal{Y}$-algebras. As $\mathcal{Y}$ is quasi-compact, there exists an étale presentation $p\colon U \to \mathcal{Y}$ from an affine scheme. Then the base change $\mathcal{X}_U \to U$ is a separated and quasi-finite morphism of algebraic spaces, thus a morphism of schemes by Corollary 4.4.7. We have that $p^*\mathcal{A} = \operatorname{colim} p^*\mathcal{A}_\lambda$ and by Theorem A.5.3 for $\lambda \gg 0$, the morphism $\mathcal{X}_U \to \mathcal{S}\mathrm{pec}_U\, p^*\mathcal{A}_\lambda$ is an open immersion and $\mathcal{S}\mathrm{pec}_U\, p^*\mathcal{A}_\lambda \to U$ is finite. By étale descent, $\mathcal{X} \to \mathcal{S}\mathrm{pec}_U\, \mathcal{A}_\lambda$ is an open immersion and $\mathcal{S}\mathrm{pec}_\mathcal{Y}\, \mathcal{A} \to \mathcal{Y}$ is finite. $\square$

### 4.4.3 Characterization of algebraic spaces

We now can remove the hypothesis in Theorem 3.6.5 that the diagonal is representable by schemes.

**Theorem 4.4.10** (Characterization of Algebraic Spaces II)**.** *For an algebraic stack $\mathcal{X}$, the following are equivalent:*

*(1) the stack $\mathcal{X}$ is an algebraic space,*

*(2) the diagonal $\mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is a monomorphism, and*

*(3) every point of $\mathcal{X}$ has a trivial stabilizer.*

*Proof.* We only need to show (2) $\Rightarrow$ (1). As the diagonal of $\mathcal{X}$ is a monomorphism, it is separated and locally quasi-finite. Corollary 4.4.7 implies that the diagonal $\mathcal{X}$ is representable by schemes and thus Theorem 3.6.5 applies. $\square$

This allows us to prove a more general version of Corollary 3.6.8.

**Corollary 4.4.11** (Characterization of Representable Morphisms II)**.** *A morphism $\mathcal{X} \to \mathcal{Y}$ of algebraic stacks is representable if and only if for every geometric point $x \in \mathcal{X}(\Bbbk)$, the map $G_x \to G_{f(x)}$ on automorphism groups is injective.* $\square$

**Corollary 4.4.12.**

*(1) If $X$ is a sheaf on* $\mathrm{Sch}_{\text{ét}}$ *such that there exists a surjective, étale (resp. smooth), and representable morphism $U \to X$ from an algebraic space, then $X$ is an algebraic space.*

*(2) If $R \rightrightarrows U$ is an étale (resp. smooth) equivalence relation of algebraic spaces, then the quotient $U/R$ is an algebraic space.*

**Remark 4.4.13.** The above statement holds with with 'étale' replaced with 'fppf'; see Theorem 6.2.1 and corollary 6.2.4.

*Proof.* We first handle the étale case. For (1), by taking an étale presentation of $U$ by a scheme, we may assume that $U$ is a scheme. Let $T \to X$ be a morphism from a scheme, and we must show that the algebraic space $U \times_X T$ is a scheme. Since $U \times_X T \to U \times T$ is the base change of $X \to X \times X$, it is a monomorphism, thus separated and locally quasi-finite. By Corollary 4.4.7, $U \times_X T$ is a scheme. For (2), let

$X = U/R$ be the quotient sheaf. By copying the argument of Theorem 3.4.11(1), we see that $U \to X$ is representable. The statement then follows from (1). Alternatively, Theorem 3.4.11(1) implies that $U/R$ is an algebraic stack and the statement follows from Theorem 4.4.10.

In the noetherian and smooth case, the sheaf $X$ in (1) is an algebraic stack by definiton and the quotient stack $[U/R]$ is an algebraic stack by Theorem 3.4.11. Theorem 4.4.10 implies that $X$ and $[U/R]$ are algebraic spaces. □

**Corollary 4.4.14.** *A proper and quasi-finite morphism (resp. proper monomorphism) of algebraic spaces is finite (resp. a closed immersion).*

*Proof.* Proper and quasi-finite morphisms are representable by schemes. Thus the statement follows from the corresponding result for schemes (Corollary A.5.5) and étale descent. □

**Exercise 4.4.15.** Consider the prestack $\underline{\mathrm{AlgSp}}$ over $\mathrm{Sch}_{\text{ét}}$ whose objects over a scheme $T$ are algebraic spaces over $T$ and where morphisms correspond to cartesian diagrams of algebraic spaces. Show that $\underline{\mathrm{AlgSp}}$ is a stack.

### 4.4.4 Affineness criteria

**Theorem 4.4.16** (Serre's Criterion for Affineness)**.** *Let $X$ be a quasi-compact and quasi-separated (resp. noetherian) algebraic space. If the functor $\Gamma(X, -)$ is exact on the category of quasi-coherent (resp. coherent) sheaves, then $X$ is an affine scheme.*

*Proof.* If $X$ is noetherian, then every quasi-coherent sheaf is a colimit of coherent sheaves (Exercise 4.1.20) and $\Gamma(X, -)$ commutes with colimits. Assume that $\Gamma(X, -)$ is exact on coherent sheaves. Given a surjection $p\colon F \twoheadrightarrow G$ of quasi-coherent sheaves on $X$, write $G = \mathrm{colim}_i\, G_i$ as a colimit of coherent sheaves and choose coherent subsheaves $F_i \subset p^{-1}(G_i)$ surjecting onto $G_i$. Then $\Gamma(X, F_i) \twoheadrightarrow \Gamma(X, G_i)$ and the composition $\mathrm{colim}_i\, \Gamma(X, F_i) \to \Gamma(X, F) \to \Gamma(X, G) = \mathrm{colim}_i\, \Gamma(X, G_i)$ is surjective. Thus $\Gamma(X, F) \to \Gamma(X, G)$ is surjective and we conclude that $\Gamma(X, -)$ is exact on quasi-coherent sheaves.

We show that the canonical morphism $\pi\colon X \to Y := \operatorname{Spec} \Gamma(X, \mathcal{O}_X)$ is a proper monomorphism. This gives the result as Corollary 4.4.14 implies that $X \to Y$ is a closed immersion so that $X$ is affine and $X \to Y$ is an isomorphism. As a first step, we establish:

*Claim: If $g\colon Y' \to Y$ is a morphism of algebraic spaces, then the base change $\pi'\colon X' := X \times_Y Y' \to Y'$ has the following properties:*

- *(a) $\pi'_*$ induces an equivalence of the categories of quasi-coherent sheaves on $X'$ and $Y'$.*
- *(b) $\mathcal{O}_{Y'} \to \pi'_* \mathcal{O}_{X'}$ is an isomorphism.*
- *(c) $X' \to Y'$ is a homeomorphism.*

By Flat Base Change (Exercise 4.1.19), properties (a) and (b) are étale local on $Y'$ so we may assume $Y' = \operatorname{Spec} B$. We will show that the adjunction morphisms $G \to \pi'_* \pi'^* G$ and $\pi'^* \pi'_* F \to F$ are isomorphisms for quasi-coherent sheaves $G$ and $F$ and $Y'$ and $X'$, respectively. For the first adjunction, choose a free presentation $\mathcal{O}_Y^{\oplus J} \to \mathcal{O}_Y^{\oplus J} \to g_* G \to 0$ of $G$ as an $\mathcal{O}_Y$-module. As $\pi_*$ is exact, we have a morphism of right exact sequences

$$\begin{tikzcd}
\mathcal{O}_Y^{\oplus J} \arrow[r] \arrow[d] & \mathcal{O}_Y^{\oplus I} \arrow[r] \arrow[d] & g_*G \arrow[r] \arrow[d] & 0 \\
\pi_*\pi^*(\mathcal{O}_Y^{\oplus J}) \arrow[r] & \pi_*\pi^*(\mathcal{O}_Y^{\oplus I}) \arrow[r] & \pi_*\pi^*g_*G \arrow[r] & 0
\end{tikzcd}$$

The two left vertical arrows are isomorphisms since $\pi_*\mathcal{O}_X = \mathcal{O}_Y$. Therefore $g_*G \to g_*\pi_*\pi^*G \cong g_*\pi'_*\pi'^*G$ is an isomorphism. Since $g_*$ is faithfully exact, $G \to \pi'_*\pi'^*G$ is also an isomorphism. We note that property (b) already follows from this fact by taking $G = \mathcal{O}_{Y'}$ and the fact that affine morphisms are faithfully exact on quasi-coherent sheaves.

To see the second adjunction, let $K$ and $Q$ be the kernel and cokernel of $\pi'^*\pi'_*F \to F$. As $\pi_*$ is exact and $g_*$ is faithfully exact, we see that $\pi'_*$ is exact. Since $\pi'_*\pi'^*\pi'_*F \to \pi'_*F$ is an isomorphism (using that the first adjunction is an isomorphism), we see that $\pi'_*K = \pi'_*Q = 0$. It thus suffices to show that for a quasi-coherent sheaf $F'$ on $X'$, then $F' \neq 0$ implies $\pi'_*F' \neq 0$. If $x\colon \operatorname{Spec} \Bbbk \to X'$ is a geometric point such that $x^*F \neq 0$, then by base changing by the composition $\pi' \circ x\colon \operatorname{Spec} \Bbbk \to Y'$, we may assume that $Y' = \operatorname{Spec} \Bbbk$ and that $x\colon \operatorname{Spec} \Bbbk \to X'$ is a section of $\pi'$. Since every $\Bbbk$-point of an algebraic space defined over $\Bbbk$ is a closed point, $x\colon \operatorname{Spec} \Bbbk \to X'$ is a closed immersion, and hence $F \to x_*x^*F = F \otimes \Bbbk$ is surjective. It follows from the exactness of $\pi'_*$ that $\pi'_*F \to F \otimes \Bbbk$ is surjective and hence $\pi'_*F \neq 0$. This finishes the proof of (a) and (b).

To see (c), if $y\colon \operatorname{Spec} \Bbbk \to Y$ is a geometric point, then by (b) $\Gamma(X_y, \mathcal{O}_{X_y}) = \Bbbk$ as thus the fiber $X_y$ is non-empty. On the other hand, if $x, x' \in X_y(\Bbbk)$ were distinct points each necessarily closed, then $\mathcal{O}_{X_y} \to \mathcal{O}_{\{x,x'\}}$ is surjective. Since $\pi_*$ is exact, we also get a surjection $\Bbbk = \Gamma(X_y, \mathcal{O}_{X_y}) \to \Bbbk \oplus \Bbbk$, a contradiction. To see that $\pi'$ is closed, let $Z \subset X'$ be a closed subspace and $q\colon Z \to \operatorname{im}(Z)$ denote the morphism to its scheme-theoretic image. Then $\mathcal{O}_Z \to q_*\mathcal{O}_{\operatorname{im}(Z)}$ is an isomorphism and $q_*$ is exact. Applying the surjectivity result above to $q$, we see that $q$ is surjective, and hence $\pi'(Z)$ is closed.

With the claim established, we now show that $X \to Y$ is a monomorphism and in particular separated. To see that the diagonal $\Delta\colon X \to X \times_Y X$ is an isomorphism, observe that the pushforward of $\mathcal{O}_{X \times_Y X} \to \Delta_* X$ along the first projection $p_1\colon X \times_Y X \to X$ is an isomorphism. Thus (a) applied to $p_1$ shows that $\mathcal{O}_{X \times_Y X} \to \Delta_* X$ is an isomorphism. Zariski's Main Theorem (A.5.3) implies that $\Delta$ is an open immersion. Applying (c) to $p_1$ shows that $p_1\colon |X \times_Y X| \to |X|$ is bijective. Hence $\Delta$ must be an isomorphism.

It remains to show that $X \to Y$ is of finite type. Let $U = \operatorname{Spec} A \to X$ be an étale presentation. Since $X$ is separated, $R := U \times_X U$ is a closed subscheme of $U \times_Y U = \operatorname{Spec} A \otimes_{\Gamma(X, \mathcal{O}_X)} A$. Hence $R = \operatorname{Spec} B$ is affine. Letting $s$ and $t$ denote the two maps $A \rightrightarrows B$, we have a commutative diagram

$$\begin{tikzcd}
 & A \arrow[dr] \arrow[drr, "s"] & & \\
\Gamma(X, \mathcal{O}_X) \arrow[ur] \arrow[dr] & & A \otimes_{\Gamma(X, \mathcal{O}_X)} A \arrow[r, two heads] & B. \\
 & A \arrow[ur] \arrow[urr, "t"'] & &
\end{tikzcd}$$

Since $U \to X$ is étale, $t\colon A \to B$ is of finite type and there are generators $b_1, \ldots, b_n \in B$ over $t$. For each $i$, choose a preimage $\sum_j a_{ij} \otimes a'_{ij} \in A \otimes_{\Gamma(X,\mathcal{O}_X)} A$ of $b_i$. Viewing $B$ as an $A$-algebra via $t$, then $\sum_j a'_{ij} s(a_{ij}) = b_i$ and thus we have elements $a_{ij} \in A$ such that $s(a_{ij})$ generate $B$ over $t$. Then $a_{ij} \in \Gamma(X, p_*\mathcal{O}_U) = A$ define a homomorphism $\mathcal{O}_X[z_{ij}] \to p_*\mathcal{O}_U$ of $\mathcal{O}_X$-algebras taking $z_{ij}$ to $a_{ij}$. Its pullback via $p$ is identified with $\mathcal{O}_U[z_{ij}] \to p^*p_*\mathcal{O}_U \cong t_*\mathcal{O}_R$, where the last equivalence comes from Flat Base Change (Exercise 4.1.19), and this map is surjective precisely because $s(a_{ij})$ generate $B$ over $t$. By étale descent, $\mathcal{O}_X[z_{ij}] \to p_*\mathcal{O}_U$ is surjective and therefore so is $\Gamma(X, \mathcal{O}_X)[z_{ij}] \to A$. Thus $\Gamma(X, \mathcal{O}_X) \to A$ is of finite type and by étale descent $X \to Y$ is also of finite type.

See also [Knu71, Thm. III.2.5], [Ryd15, Thm. 8.7] and [SP, Tag 07V6]. □

**Corollary 4.4.17.** *Let $X$ be a quasi-compact and quasi-separated (resp. noetherian) algebraic space. Then $X$ is an affine scheme if and only if $\mathrm{H}^i(X, F) = 0$ for every quasi-coherent (resp. coherent) sheaf $F$ and $i > 0$.*

*Proof.* If $X$ is affine, then Theorem 4.1.29 establishes the vanishing of quasi-coherent cohomology. Conversely, the vanishing of quasi-coherent (resp. coherently) cohomology implies that $\Gamma(X, -)$ is exact on the category of quasi-coherent (resp. coherent) sheaves: if $0 \to F_1 \to F_2 \to F_3 \to 0$ is exact, then $\Gamma(X, F_2) \to \Gamma(X, F_3)$ is surjective as $\mathrm{H}^1(X, F_1) = 0$. □

**Remark 4.4.18.** Given a quasi-compact and quasi-separated morphism $f\colon \mathcal{X} \to \mathcal{Y}$ of Deligne–Mumford stacks, the condition that $f_*\colon \mathrm{QCoh}(\mathcal{X}) \to \mathrm{QCoh}(\mathcal{Y})$ is exact is fppf local on $\mathcal{Y}$ (see Lemma 6.3.15). Since $R^i f_* F$ can be computed in $\mathrm{QCoh}(\mathcal{X})$, the relative versions of Theorem 4.4.16 and Corollary 4.4.17 also hold: $f$ is affine if and only if $f_*\colon \mathrm{QCoh}(\mathcal{X}) \to \mathrm{QCoh}(\mathcal{Y})$ is exact if and only if $R^i f_* F = 0$ for all $i > 0$ and $F \in \mathrm{QCoh}(\mathcal{X})$.

**Proposition 4.4.19.** *Let $X$ be a noetherian algebraic space. If $X_{\mathrm{red}}$ is a scheme (resp. quasi-affine, affine), then so is $X$.*

*Proof.* If $X_{\mathrm{red}}$ is affine, then one uses Corollary 4.4.17 to show that $X$ is affine exactly as in [Har77, Exer. III.3.1]: if $F$ is a coherent sheaf on $X$ and $I \subset \mathcal{O}_X$ denotes the nilpotent ideal defining $X_{\mathrm{red}}$, then one shows the vanishing of $\mathrm{H}^i(X, F)$ using the filtration $0 = I^N F \subset I^{N-1} F \subset \cdots \subset IF \subset F$, whose factors $I^k F / I^{k+1} F$ are supported on $X_{\mathrm{red}}$.

If $X_{\mathrm{red}}$ is quasi-affine, then $X_{\mathrm{red}} \to \operatorname{Spec} \Gamma(X, \mathcal{O}_X)_{\mathrm{red}}$ is an open immersion. Thus $X \to \operatorname{Spec} \Gamma(X, \mathcal{O}_X)$ is an open immersion and $X$ is quasi-affine. If $X_{\mathrm{red}}$ is a scheme, then every point $x \in |X|$ has an open neighborhood $U$ such that $U_{\mathrm{red}}$ is affine. Thus $U$ is affine and $X$ is a scheme. □

**Theorem 4.4.20** (Chevalley's Criterion for Affineness)**.** *Let $Y$ be a noetherian algebraic space and $X \to Y$ be a finite surjective morphism of algebraic spaces. If $X$ is affine, then so is $X$.*

*Proof.* One can argue as in [Har77, Exer. 4.1] using Corollary 4.4.17. □

There is also a cohomological criterion for ampleness generalizing [Har77, Prop. 5.3]:

**Exercise 4.4.21.** Let $X$ be a proper algebraic space over a noetherian ring. For a line bundle $L$ on $X$, show that the following are equivalent:

(1) $X$ is a scheme and $L$ is ample;

(2) for every coherent sheaf $F$ on $X$, there is an integer $n_0$ such that $\mathrm{H}^i(X, F \otimes L^n) = 0$ for $i > 0$ and $n \geq n_0$.

See also [SP, Tag 0D2W].

The following generalizes [Har77, Exer. III.5.7].

**Exercise 4.4.22.** Let $f\colon X \to Y$ be a finite surjective morphism of algebraic spaces proper over a noetherian ring. Let $L$ be a line bundle on $Y$. If $X$ is a scheme and $f^*L$ is ample, show that $Y$ is a scheme and $L$ is ample.

See also [SP, Tag 0GFB].

### 4.4.5 Effective descent along field extensions

**Lemma 4.4.23.** *Let $X$ be a quasi-separated algebraic space locally of finite type over a field $\Bbbk$. If $X_{\overline{\Bbbk}}$ is an affine scheme, then so is $X$.*

*Proof.* By Chevalley's Criterion for Affineness (Theorem 4.4.20), it suffices to show that there is a finite field extension $\Bbbk \to K$ such that $X_K$ is affine. (Note that the lemma follows directly from the strengthening of Chevelley's Criterion to integral surjective morphisms.)

The algebraic space $X$ is necessarily quasi-compact and we choose an étale presentation $U \to X$ be an affine scheme. We write $\overline{\Bbbk} = \operatorname{colim} \Bbbk_\lambda$ as the colimit of finite field extensions $\Bbbk_\lambda/\Bbbk$. Set $X_\lambda := X_{\Bbbk_\lambda}$ and $U_\lambda = U_{\Bbbk_\lambda}$. By Flat Base Change (Exercise 4.1.19), $\Gamma(X, \mathcal{O}_X) \otimes_{\Bbbk} \Bbbk_\lambda = \Gamma(X_\lambda, \mathcal{O}_{X_\lambda})$ and $\Gamma(X, \mathcal{O}_X) \otimes_{\Bbbk} \overline{\Bbbk} = \Gamma(X_{\overline{\Bbbk}}, \mathcal{O}_{X_{\overline{\Bbbk}}})$. We have a cartesian diagram

$$\begin{tikzcd}
U_{\overline{\Bbbk}} \arrow[r] \arrow[d] & U_\lambda \arrow[r] \arrow[d] & U \arrow[d] \\
X_{\overline{\Bbbk}} \arrow[r] \arrow[d, "\wr"] & X_\lambda \arrow[r] \arrow[d] & X \arrow[d] \\
\operatorname{Spec} \Gamma(X_{\overline{\Bbbk}}, \mathcal{O}_{X_{\overline{\Bbbk}}}) \arrow[r] & \operatorname{Spec} \Gamma(X_\lambda, \mathcal{O}_{X_\lambda}) \arrow[r] & \operatorname{Spec} \Gamma(X, \mathcal{O}_X)
\end{tikzcd}$$

Since $U_{\Bbbk} \to \operatorname{Spec} \Gamma(X_{\overline{\Bbbk}}, \mathcal{O}_{X_{\overline{\Bbbk}}})$ is an étale morphism of schemes, so is $U_\lambda \to \operatorname{Spec} \Gamma(X_\lambda, \mathcal{O}_{X_\lambda})$ for $\lambda \gg 0$ (Proposition A.6.7). Thus $X_\lambda \to \operatorname{Spec} \Gamma(X_\lambda, \mathcal{O}_{X_\lambda})$ is étale for $\lambda \gg 0$. Let $R = U \times_X U$ with base changes $R_\lambda := R_{\Bbbk_\lambda}$ and $R_{\overline{\Bbbk}}$. Since $R_{\overline{\Bbbk}} \to U_{\overline{\Bbbk}} \times_{\overline{\Bbbk}} U_{\overline{\Bbbk}}$ is a closed immersion, so is $R_\lambda \to U_\lambda \times_{\Bbbk_\lambda} U_\lambda$ for $\lambda \gg 0$ (Proposition A.6.7) and in particular $X_\lambda$ are separated for $\lambda \gg 0$. For $\lambda \gg 0$, since $X_\lambda$ is étale and separated over a scheme, $X_\lambda$ is a scheme (Corollary 4.4.7). We may therefore apply Proposition B.4.4 to $X$ (or Proposition A.6.7 to $X \to \operatorname{Spec} \Gamma(X, \mathcal{O}_X)$) to conclude that $X_\lambda$ is affine for $\lambda \gg 0$. $\square$

**Proposition 4.4.24.** *Let $X$ be a quasi-separated algebraic space of finite type over a field $\Bbbk$. If $X_{\overline{\Bbbk}}$ is a scheme, then there exists a finite separable field extension $\Bbbk \to K$ such that $X_K$ is a scheme.*

*Proof.* Choose an étale presentation $U \to X$ be an affine scheme and set $R = U \times_X U$. As in the proof of the previous lemma, we write $\overline{\Bbbk} = \operatorname{colim} \Bbbk_\lambda$ with $\Bbbk_\lambda/\Bbbk$ finite, and set $X_\lambda := X_{\Bbbk_\lambda}$, $U_\lambda = U_{\Bbbk_\lambda}$ and $R_\lambda = R_{\Bbbk_\lambda}$.

Let $\overline{V} \subset X_{\overline{\Bbbk}}$ be an open affine subscheme. We claim that for $\lambda \gg 0$, there exists an open subscheme $V_\lambda \subset X_\lambda$ such that $\overline{Z} = U_\lambda \times_{\Bbbk_\lambda} \overline{\Bbbk}$. Indeed, the preimage

$V' \subset U_{\overline{\Bbbk}}$ of $\overline{V}$ has the property that its two preimages in $R_{\overline{\Bbbk}}$ are equal. Using Proposition A.6.4 and Proposition A.6.7, for $\lambda \gg 0$ there is an open subscheme $V'_\lambda \subset U_\lambda$ with $V' = V'_\lambda \times_{\Bbbk_\lambda} \overline{\Bbbk}$ such that the two preimages of $V'_\lambda$ in $R_\lambda$ are equal. By étale descent, $V'_\lambda$ descends to the desired closed subscheme $V_\lambda \subset U_\lambda$.

Lemma 4.4.23 implies that $V_\lambda$ is a scheme. By covering $X_{\overline{\Bbbk}}$ with finitely many affines and choosing $\lambda$ sufficiently large, we obtain a finite field extension $K = \Bbbk_\lambda$ of $\Bbbk$ such that $X_\lambda$ is a scheme. If $\Bbbk^s \subset K$ be the separable closure of $\Bbbk$, then $X_K \to X_{\Bbbk^s}$ is a finite universal homeomorphism and by Chevalley's Theorem for Affineness (Theorem 4.4.20), the image of an affine subscheme $X_K$ in $X_{\Bbbk^s}$ is also affine. We conclude that $X_{\Bbbk^s}$ is a scheme. □

With an additional condition on $X_{\overline{\Bbbk}}$, we can conclude that $X$ is a scheme.

**Proposition 4.4.25.** *Let $X$ be a quasi-separated algebraic space locally of finite type over a field $\Bbbk$. If $X_{\overline{\Bbbk}}$ is a scheme such that every finite set of $\overline{\Bbbk}$-points is contained in an affine (e.g. $X_{\overline{\Bbbk}}$ is quasi-projective), then $X$ is a scheme.*

*Proof.* We may assume that $X$ is quasi-compact. We will show that every closed point $x \in X$ has an affine open neighborhood. Let $\operatorname{Spec} l \hookrightarrow X$ be the inclusion of the residue field of $x$ (Corollary 3.5.20) and let $\Bbbk^s$ be the separable closure of the finite field extension $\Bbbk \to l$. We have a cartesian diagram

$$\begin{tikzcd} \coprod_{i=1}^n \operatorname{Spec} A_i \arrow[r, hook] \arrow[d] & X_{\overline{\Bbbk}} \arrow[r] \arrow[d] & \operatorname{Spec} \overline{\Bbbk} \arrow[d] \\ \operatorname{Spec} l \arrow[r, hook, "x"] & X \arrow[r] & \operatorname{Spec} \Bbbk \end{tikzcd}$$

where $A_i$ is a artinian local $\Bbbk$-algebra where $n$ is the degree of the separable closure $\kappa^s \subset l$ of $\Bbbk$; here we are using that $\Bbbk^s \otimes_{\Bbbk} \overline{\Bbbk} = \prod_{i=1}^n \overline{\Bbbk}$ and that $\operatorname{Spec} l \to \operatorname{Spec} \Bbbk^s$ is a finite universal homeomorphism. The hypotheses on $X_{\overline{\Bbbk}}$ ensure that there is an affine open subscheme $\overline{U} \subset X_{\overline{\Bbbk}}$ containing the images of each $\operatorname{Spec} A_i$.

By Proposition 4.4.24, there is a finite field extension $\Bbbk \to K$ such that $X_K$ is a scheme. After enlarging $K$, we can arrange that $\overline{U}$ descends to an affine open subscheme $U' \subset U_K$ by using Proposition A.6.4 to descend the morphism $\overline{U} \to X$, Proposition A.6.7 to arrange that it is an open immersion and Proposition B.4.4 to arrange affineness. Observe that $U'$ contains all preimages of $x$ under $X_K \to X$. By taking the normal closure of $K$, we can assume $K$ is normal over $\Bbbk$. Let $G = \operatorname{Aut}(K/\Bbbk)$ so that $K^G$ is a purely inseparable field extension of $\Bbbk$. Then $G$ acts on $X_K$ freely such that $X_K/G = X_{K^G}$.

The intersection of the translates of $U'$ by elements of $G$ is a $G$-invariant quasi-affine variety $U''$. Choosing an affine in $U''$ containing all of the preimages of $x$ and intersecting again the translates of $G$, we obtain a $G$-invariant affine $V \subset X_K$ containing the preimages of $x$. Then the quotient $V/G$ is an affine subscheme of $X_{K^G}$ containing the unique preimage of $x$ (Theorem 4.3.6). Letting $W$ be the image of $V/H$ under the finite universal homeomorphism $X_{K^G} \to X$, Chevelley's Criterion for Affineness (Theorem 4.4.20) implies that $W$ is an affine neighborhood of $x$. □

### 4.4.6 Group algebraic spaces are schemes

Every quasi-separated group algebraic space over a field $\Bbbk$ is a scheme. When $\Bbbk$ is algebraically closed, this follows easily from Theorem 4.4.1 as we know there is a dense open that is a scheme, and we can translate this around by rational points. The general case relies on Proposition 4.4.25.

**Theorem 4.4.26.** *A quasi-separated group algebraic space $G$ locally of finite type over a field $\Bbbk$ is a separated scheme. The connected component of the identity $G^0$ is quasi-projective.*

**Remark 4.4.27.** If $G$ is not quasi-separated, then the above corollary does not hold, e.g. $G = \mathbb{G}_a/\underline{\mathbb{Z}}$ over $\Bbbk$ (Example 3.9.22).

Note that Proposition 4.4.19 implies that the result also holds over an Artinian base. Over a general base scheme, the statement is not true; see [Ray70, Lem. X.14].

If in addition $G$ is quasi-compact, then $G$ is quasi-projective by Proposition C.3.1(7).

*Proof.* It suffices to show that $G$ is a scheme. Indeed, a group scheme locally of finite type over a field is necessarily separated (C.3.1(1)) and the identity component $G^0$ is quasi-compact (C.3.1(4)), thus quasi-projective (C.3.1(7)).

Assume first that $\Bbbk$ is algebraically closed. There is a non-empty open subscheme $U$ of $G$ (Theorem 4.4.1) with a point $h \in U(\Bbbk)$. For every $g \in G(\Bbbk)$, left multiplication by $gh^{-1}$ defines an isomorphism $G \xrightarrow{\sim} G$ and the image $gh^{-1}U$ of $U$ is a scheme containing $g$.

The general case follows from Proposition 4.4.25 using that $G_{\overline{\Bbbk}}$ is a scheme with the property that every finite set of points is contained in an affine (Lemma 4.4.28).

See also [Art69b, Lem. 4.2] and [SP, Tag 0B8D]. □

**Lemma 4.4.28.** *Every group scheme $G$ locally of finite type over an algebraically closed field $\Bbbk$ has the property that every finite set of $\Bbbk$-points is contained in an affine open subscheme.*

*Proof.* Let $g_1, \ldots, g_n \in G(\Bbbk)$. We first use induction on $n$ to assume that all of the elements $g_i$ are in the same connected component. If not, we can write $G = W_1 \sqcup W_2$ with $r$ points in $W_1$ and $n-r$ points in $W_2$ for $0 < r < n$. By induction, there are affine opens $U_1 \subset W_1$ and $U_2 \subset W_2$ containing the $r$ and $n-r$ points, respectively. Then $U_1 \sqcup U_2$ is an affine containing each $g_i$.

By translating by $g_1^{-1}$, we may assume that $g_1, \ldots, g_n \in G^0(\Bbbk)$. Let $U \subset G^0$ be an affine open neighborhood of the identity. Since $G^0$ is irreducible (C.3.1(4)), $Ug_1^{-1} \cap \cdots \cap Ug_n^{-1}$ is non-empty and contains a closed point $h$. Since $h \in Ug_i^{-1}$, each $g_i$ is contained in the affine open $h^{-1}U$.

See also [SP, Tag 0B7S]. It is also true that every group scheme of *finite type* over a field is quasi-projective [SP, Tag 0BF7]. □

**Corollary 4.4.29.** *Let $\mathcal{X}$ be an algebraic stack with quasi-separated diagonal. Then the stabilizer of every field-valued point is a group scheme locally of finite type.*

*Proof.* By Exercise 3.2.4 the diagonal of $\mathcal{X}$ is locally of finite type. As the stabilizer is the base change of the diagonal, the statement follows from Theorem 4.4.26. □

## 4.5 Finite covers of Deligne–Mumford stacks

This section aims to prove the following theorem asserting that Deligne–Mumford stacks have finite covers by schemes.

**Theorem 4.5.1** (Le Lemme de Gabber)**.** *Let $\mathcal{X}$ be a Deligne-Mumford stack separated and of finite type over a noetherian scheme $S$. Then there exists a finite, generically étale, and surjective morphism $Z \to \mathcal{X}$ from a scheme $Z$.*

By applying Chow's Lemma (c.f [Har77, Exer. II.4.10]) to $Z$, we obtain:

**Corollary 4.5.2.** *There exists a projective, generically étale, and surjective morphism $Z \to \mathcal{X}$ from a scheme $Z$ quasi-projective over $S$.* $\square$

See also [LMB00, Thm. 16.6 and Cor. 16.6.1], [Del85], [Vis89, Prop. 2.6], [Ols16, Thm. 11.4.1] and [SP, Tag 09YC] (for the case of algebraic spaces). More generally, every separated algebraic stack of finite type over $S$ has a proper cover by a quasi-projective scheme [Ols05].

We provide two arguments, each of which uses normalization to construct a finite cover.

*Proof 1 (following [LMB00, Thm. 16.6]):*

By replacing $\mathcal{X}$ with the disjoint union of the irreducible components with their reduced stack structure, we may assume that $\mathcal{X}$ is irreducible and reduced. Every étale presentation $U \to \mathcal{X}$ is separated, quasi-finite and representable and thus factors as the composition of an open immersion $U \hookrightarrow \widetilde{\mathcal{X}}$ and a finite morphism $\widetilde{\mathcal{X}} \to \mathcal{X}$ by Zariski's Main Theorem (4.4.9). After replacing $\mathcal{X}$ with $\widetilde{\mathcal{X}}$, we may assume that $\mathcal{X}$ has a dense open subscheme. If $p\colon U \to \mathcal{X}$ is an étale presentation, there is therefore a dense open subscheme $V \subset \mathcal{X}$ such that $p^{-1}(V) \to V$ is finite étale of degree $d$. We may choose a finite étale covering $V' \to V$ such that $p^{-1}(V) \times_V V' \to V'$ is a trivial étale covering; indeed as in Proposition A.3.12, we may take $V'$ to be the complement of all pairwise diagonals in $(V'/V)^d = \underbrace{V' \times_V \cdots \times_V V'}_{d}$. Applying Zariski's Main Theorem (Theorem 4.4.9) to the composition $V' \to V \hookrightarrow \mathcal{X}$ gives a finite surjective morphism $\widetilde{\mathcal{X}} \to \mathcal{X}$ restricting to $V' \to V$. Thus after replacing $\mathcal{X}$ with $\widetilde{\mathcal{X}}$, we may assume that there is an étale presentation $U \to \mathcal{X}$ which over a dense open subscheme $j\colon V \hookrightarrow \mathcal{X}$ is a trivial étale covering, i.e. there is a cartesian diagram

$$\begin{tikzcd} \coprod_{i=1}^d V \arrow[r, hook, "j'"] \arrow[d] & U \arrow[d, "p"] \\ V \arrow[r, hook, "j"] & \mathcal{X}. \end{tikzcd}$$

We will construct a finite surjective morphism $Z \to \mathcal{X}$ from a scheme that is an isomorphism over $V$. Let $\mathcal{A} \subset j_*\mathcal{O}_{\mathcal{X}}$ be integral closure of $\mathcal{O}_{\mathcal{X}} \to j_*\mathcal{O}_{\mathcal{X}}$. Then $\pi^*\mathcal{A}$ is the integral closure of $\mathcal{O}_U$ in $j'_*\mathcal{O}_{\sqcup V} = p^*j_*\mathcal{O}_V$ (Proposition A.5.4). The idempotent $e_i \in \Gamma(U, j'_*\mathcal{O}_{\sqcup V}) = \Gamma(\sqcup V, \mathcal{O}_{\sqcup V})$, defining the $i$th copy of $V$, is integral over $\mathcal{O}_U$ and thus defines a global section $e_i \in \Gamma(U, p^*\mathcal{A})$. Now write $\mathcal{A} = \operatorname{colim}_\lambda \mathcal{C}_\lambda$ as a filtered colimit of finite type $\mathcal{O}_{\mathcal{X}}$ algebra (Exercise 4.1.25). Since $\mathcal{A}$ is integral over $\mathcal{O}_{\mathcal{X}}$, each $\mathcal{C}_\lambda$ is a finite $\mathcal{O}_{\mathcal{X}}$-algebra. For $\lambda \gg 0$, we have $e_i \in \Gamma(U, p^*\mathcal{A}_\lambda)$. The Deligne–Mumford stack $Z := \mathcal{S}\mathrm{pec}_{\mathcal{X}}\, \mathcal{A}_\lambda$ is finite over $\mathcal{X}$ and we claim that $Z$ is a scheme. To see this, consider the cartesian diagram

$$\begin{tikzcd} Z' \arrow[r] \arrow[d] & U \arrow[d, "p"] \\ Z \arrow[r] & \mathcal{X} \end{tikzcd}$$

noting that $Z'$ is a scheme since it is finite over $U$. Each idempotent $e_i$ defines a global section of $Z'$ and thus yields a decomposition $Z' = \coprod_{i=1}^d Z'_i$. Each morphism $Z'_i \to Z$ is étale, separated and birational, thus an open immersion. Since $Z' \to Z$ is surjective, the collection of $Z'_i$ defines an open covering of $Z$, and it follows that $Z$ is a scheme.

*Proof 2 (following [Vis89, Prop. 2.6]):*

We first use limit methods to reduce to the case that $S$ is of finite type over $\mathbb{Z}$ to ensure that normalizations are finite. By Noetherian Approximation (Proposition A.6.2), we may write $S = \lim_\lambda S_\lambda$ as the limit of schemes with affine transition maps where each $S_i$ is of finite type over $\mathbb{Z}$. Let $U \to \mathcal{X}$ be an étale presentation and set $R = U \times_{\mathcal{X}} U \rightrightarrows U$ be the corresponding étale groupoid equipped with source, target, identity and compositions morphisms $s$,$t$,$i$ and $c$. There exists an index 0 and schemes $U_0$ and $R_0$ of finite type over $S_0$ such that $U = U_0 \times_{S_0} S$ and $R = R_0 \times_{R_0} S$ (Proposition A.6.4(2)). For $\lambda \geq 0$, set $U_\lambda = U_0 \times_{S_0} S_\lambda$ and $R_\lambda = R_0 \times_{S_0} S_\lambda$. For $\lambda \gg 0$, there are morphisms $s_\lambda, t_\lambda \colon R_\lambda \to U_\lambda$, $i_\lambda \colon R_\lambda \to R_\lambda$ and $c_\lambda \colon R_\lambda \times_{t_\lambda, U_\lambda, s_\lambda} R_\lambda \to R_\lambda$ that base change to $s$,$t$,$i$ and $c$ (Proposition A.6.4(1)). Finally, for $\lambda \gg 0$, the morphisms $s_\lambda$ and $t_\lambda$ are étale, and $R_\lambda \to U_\lambda \times_{S_\lambda} R_\lambda$ is finite (Proposition A.6.7). It follows that $R_\lambda \rightrightarrows U_\lambda$ defines an étale groupoid of schemes and that the quotient stack $\mathcal{X}_\lambda := [U_\lambda / R_\lambda]$ is a Deligne–Mumford stack separated and of finite type over $S_\lambda$ such that $\mathcal{X} \cong \mathcal{X}_\lambda \times_{S_\lambda} S$. A finite, generically étale cover of $\mathcal{X}_\lambda$ by a scheme will pull back to a finite, generically étale cover of $\mathcal{X}$ by a scheme. This finishes the reduction.

By replacing $\mathcal{X}$ with the disjoint union of the irreducible components with their reduced stack structure, we may assume that $\mathcal{X}$ is irreducible and reduced. Let $\widetilde{\mathcal{X}}$ be the normalization of $\mathcal{X}$ (Example 4.1.23). Then $\widetilde{\mathcal{X}} \to \mathcal{X}$ is finite and so after replacing $\mathcal{X}$ with $\widetilde{\mathcal{X}}$, we may assume that $\mathcal{X}$ is also normal.

Let $\mathcal{X} \to X$ be the coarse moduli space (Theorem 4.3.11) and let $U \to \mathcal{X}$ be an étale presentation. As $\mathcal{X}$ is normal, so is $X$ (Exercise 4.3.17). We can write $U = \coprod_i U_i$ as the disjoint union of integral affine schemes $U_i$; each morphism $U_i \to \mathcal{X}$ is étale and in particular quasi-finite and dominant.

Each field extension $\operatorname{Frac}(X) \to \operatorname{Frac}(U_i)$ of fraction fields is finite, and we let $F$ be a finite normal extension of $\operatorname{Frac}(X)$ containing each $\operatorname{Frac}(U_i)$. The normalization $Y \to X$ of $X$ in $F$ is finite; here $X$ is an algebraic space and the normalization is well-defined by Proposition A.5.4. Meanwhile, by the universal property of the normalization $Y \to X$, the normalization $Y_i$ of $U_i$ in $F$ admits a morphism $Y_i \to Y$ over $X$. As $Y_i \to Y$ is separated, quasi-finite, and birational, it is an open immersion.

The automorphism group $G = \operatorname{Aut}(F/\operatorname{Frac}(X))$ acts on $Y$ over $X$ and for each pair $\alpha = (i, \sigma)$ of an integer $i$ and $\sigma \in G$, we set $Y_\alpha = \sigma(Y_i)$. We claim that $Y = \bigcup_\alpha Y_\alpha$. To see this, we first show that $G$ acts transitively on the fibers of $Y \to X$. The fixed field $F^G$ is a purely inseparable field extension of $\operatorname{Frac}(X)$ and the normalization $X' \to X$ of $X$ in $F^G$ is a universal homeomorphism. Thus to see that $G$ acts transitively on the fibers, we may assume that $\operatorname{Frac}(X) \to F$ is a Galois extension. We may also assume that $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$ with $B$ the integral closure of $A$ in $F$. Then $G$ acts on $B$ and we have inclusions $A \subset B^G \subset F^G = \operatorname{Frac}(X)$. Since $A$ is normal and $B^G$ is integral over $A$, we see that $A = B^G$. By Theorem 4.3.6, $[\operatorname{Spec} A/G] \to \operatorname{Spec} B$ is a coarse moduli space, and it follows that $G$ acts transitively on the fibers of $\operatorname{Spec} A \to \operatorname{Spec} B$. To prove the claim, observe that since $\coprod_i Y_i \to \coprod_i U_i \to X$ is surjective, each point $x \in X$ has a preimage $y \in Y_i$ for some $i$. Since $G$ acts transitively on the fibers, $\bigcup Y_\alpha$ contains the fiber of $Y \to X$ over $x$.

The claim implies that $Y$ is a scheme and that $Y \to X$ factors through $\mathcal{X}$ Zariski-locally on $Y$. Indeed, each $Y_\alpha$ is separated and quasi-finite over $U_i$ and thus a scheme by Corollary 4.4.7. Each $Y_\alpha \to X$ factors via $s_\alpha \colon Y_\alpha \to U_i \to \mathcal{X}$. After replacing $X$ with $Y$ and $\mathcal{X}$ with $\mathcal{X} \times_X Y$, we may assume that we have a coarse moduli space $\mathcal{X} \to X$ with $X$ a scheme and an open covering $X = \bigcup X_\alpha$ together with a commutative diagram

$$\begin{tikzcd} & \mathcal{X} \arrow[d] \\ X_\alpha \arrow[ur, "s_\alpha"] \arrow[r, hook] & X \end{tikzcd}$$

for each $\alpha$. We will show that after replacing $X$ with a finite cover, the sections $s_\alpha$ glue to a global section $s$. Such a section is necessarily finite since $\mathcal{X}$ is Deligne–Mumford, and this then finishes the proof as $X \to \mathcal{X}$ is a finite surjective morphism from a scheme.

To show that the sections glue, we first claim that the diagonal $\Delta_{\mathcal{X}} \colon \mathcal{X} \to \mathcal{X} \times_X \mathcal{X}$ is étale. This is a Zariski local question on $X$, so we may assume that there is a section $s \colon X \to \mathcal{X}$ of $\pi \colon \mathcal{X} \to X$. Then $s \colon X \to \mathcal{X}$ is a dominant and unramified (since $\Delta_{\mathcal{X}}$ is unramified) morphism of normal Deligne–Mumford stacks and thus étale (Proposition A.3.13). It follows that $\Delta_{\mathcal{X}} \colon \mathcal{X} \to \mathcal{X} \times_X \mathcal{X}$ is étale (and also that $\pi \colon \mathcal{X} \to X$ is étale).

Since the diagonal $\mathcal{X} \to \mathcal{X} \times_X \mathcal{X}$ is finite and étale, the scheme $J_{\alpha,\beta} := \underline{\mathrm{Isom}}_{X_{\alpha,\beta}}(s_\alpha|_{X_{\alpha,\beta}}, s_\alpha|_{X_{\alpha,\beta}})$ of isomorphisms is finite and étale over $X_{\alpha,\beta} := X_\alpha \cap X_\beta$. We may choose a finite étale cover $V_{\alpha,\beta} \to X_{\alpha,\beta}$ trivializing $J_{\alpha,\beta} \to X_{\alpha,\beta}$ (see Proposition A.3.12). By Zariski's Main Theorem (A.5.3), $V_{\alpha,\beta} \to X$ factors as an open immersion $V_{\alpha,\beta} \hookrightarrow \widetilde{X}$ and a finite morphism $\widetilde{X} \to X$. After replacing $X$ with $\widetilde{X}$, we may assume that $J_{\alpha,\beta} \to X_{\alpha,\beta}$ is trivial.

The intersection $\bigcap_\alpha X_\alpha$ is non-empty and we may choose a geometric point $x \colon \operatorname{Spec} \Bbbk \to \bigcap_\alpha X_\alpha$. All objects in the fiber $\mathcal{X}_x(\Bbbk)$ of $\mathcal{X} \to X$ over $x$ are isomorphic. We may therefore choose an object $t \in \mathcal{X}_x(\Bbbk)$ and isomorphisms $\mu_\alpha \colon t \xrightarrow{\sim} x^* s_\alpha$ for each $\alpha$. This allows us to define isomorphisms $\phi_{\alpha,\beta} \colon x^* s_\alpha \xrightarrow{\mu_\alpha^{-1}} t \xrightarrow{\mu_\beta} x^* s_\beta$. It is readily checked that the isomorphisms $\phi_{\alpha,\beta}$ satisfy the cocycle $\phi_{\alpha,\gamma} = \phi_{\beta,\gamma} \circ \phi_{\alpha,\beta}$. Each $\phi_{\alpha,\beta}$ defines a lift $\operatorname{Spec} \Bbbk \to J_{\alpha,\beta}$ of $x \colon \operatorname{Spec} \Bbbk \to X_{\alpha,\beta}$ which extends uniquely to a section $\lambda_{\alpha,\beta} \colon X_{\alpha,\beta} \to J_{\alpha,\beta}$. The triple intersections $X_\alpha \cap X_\beta \cap X_\gamma$ are connected and since the $\phi_{\alpha,\beta}$ satisfy the cocycle condition, so does $\lambda_{\alpha,\beta}$. The isomorphisms $\lambda_{\alpha,\beta}$ between $s_\alpha|_{X_{\alpha,\beta}}$ and $s_\alpha|_{X_{\alpha,\beta}}$ therefore glue to a global section of $\mathcal{X} \to X$. $\square$

**Exercise 4.5.3.** Let $X$ be a normal algebraic space of finite type over a noetherian scheme $S$. Show that there is a normal scheme $U$ with an action of a finite group $G$ such that $X$ is the quotient of $U$ by $G$, i.e. $[U/G] \to X$ is a coarse moduli space.

*Hint: After reducing to the case that $X$ is integral, choose a finite, generically étale and surjective morphism $U \to X$ from a scheme. Let $K$ be the Galois closure of the finite separable field extension $\mathrm{Frac}(U)/\mathrm{Frac}(X)$. Then take $U$ to be the integral closure of $X$ in $K$ (which is finite over $X$ as $K/\mathrm{Frac}(X)$ is separable) and take $G = \mathrm{Gal}(K/\mathrm{Frac}(X))$. See also [LMB00, Cor. 16.6.2].*

# Chapter 5

# Moduli of stable curves

## 5.1 Review of smooth curves

### 5.1.1 Curves

A *curve* over a field $\Bbbk$ is a one-dimensional scheme $C$ of finite type over $\Bbbk$. Proper curves are projective; this can be reduced to the case of smooth curves [Har77, Prop. I.6.7]. More generally, every one-dimensional separated algebraic space is a quasi-projective scheme; see [SP, Tags 0ADD and 09NZ].

If $C$ is a proper curve over a field $\Bbbk$, we define the *arithmetic genus of* $C$ or simply the *genus of* $C$ as

$$g(C) = 1 - \chi(C, \mathcal{O}_C),$$

which is equal to $\mathrm{h}^1(C, \mathcal{O}_C)$ if $C$ is geometrically connected and reduced.

For a connected, reduced, and projective curve $C$ over an algebraically closed field $\Bbbk$, the degree of a *very ample* line bundle $L$ on $C$ is defined as the number of zeros (counted with multiplicity) of any section of $L$. In other words, if $C \hookrightarrow \mathbb{P}^n$ is the projective embedding defined by $L$, then $\deg L = \dim_{\Bbbk} \Gamma(C \cap H, \mathcal{O}_{C \cap H})$, where $H$ is any hyperplane and $C \cap H$ is the scheme-theoretic intersection. Any line bundle on $C$ can be written as the difference of two very ample line bundles: if $M$ is very ample on $C$, then $M' := L \otimes M^n$ is very ample for $n \gg 0$, and $L \cong M' \otimes (M^{\otimes n})^\vee$. In this way, we also see that $L = \mathcal{O}_C(D)$ for a divisor $D = \sum_{n_i} p_i$ supported on the smooth locus of $C$, i.e. each $p_i \in C$ is a smooth point. Note that $\deg(L \otimes M) = \deg L + \deg M$, and that if $C = \bigcup_i C_i$ denotes the irreducible decomposition, then $\deg L = \sum_i \deg L|_{C_i}$.

**Theorem 5.1.1** (Riemann–Roch)**.** *Let $C$ be a connected, reduced, and projective curve of genus $g$ over an algebraically closed field $\Bbbk$. If $L$ is a line bundle on $C$, then*

$$\chi(C, L) = \deg L + 1 - g.$$

*Proof.* We can write $L = \mathcal{O}_C(D)$ for a divisor $D$ supported on the smooth locus. Since Riemann–Roch holds for $\mathcal{O}_C$, it suffices by adding and subtracting points to show that Riemann–Roch holds for $\mathcal{O}_C(D)$ if and only if it holds for $\mathcal{O}_C(D+p)$ for a smooth point $p \in C(\Bbbk)$. This follows by considering the short exact sequence

$$0 \to \mathcal{O}_C(D) \to \mathcal{O}_C(D+p) \to \kappa(p) \to 0$$

and the identity $\chi(C, \mathcal{O}_C(D+p)) = \chi(C, \mathcal{O}_C(D)) + 1$. See also [Har77, Thm IV.1.3, Exer. IV.1.9] and [Vak17, Exers. 18.4.B and S]. □

### 5.1.2 Smooth curves

We review some basic properties of smooth curves, which we will later generalize to nodal curves. If $C$ is a smooth curve, then the sheaf of differentials $\Omega_C$ is a line bundle. Serre Duality states $\Omega_C$ is a dualizing sheaf on $C$; this is a deep result that is indispensable in the study of curves.

**Theorem 5.1.2** (Serre Duality for Smooth Curves)**.** *If $C$ is a smooth projective curve over a field $\Bbbk$, then $\Omega_C$ is a dualizing sheaf, i.e. there is a linear map $\mathrm{tr}\colon H^1(C,\Omega_C) \to \Bbbk$ such that for every coherent sheaf $\mathcal{F}$, the natural pairing*

$$\mathrm{Hom}_{\mathcal{O}_C}(\mathcal{F},\Omega_C) \times H^1(C,\mathcal{F}) \to H^1(C,\Omega_C) \xrightarrow{\mathrm{tr}} \Bbbk$$

*is perfect.*

*Proof.* See [Har77, Cor. III.7.12]. □

**Remark 5.1.3.** The pairing being perfect means that the $\mathrm{Hom}_{\mathcal{O}_C}(\mathcal{F},\Omega_C)$ is identified with the dual $H^1(C,\mathcal{F})^\vee$. If $\mathcal{F}$ is a vector bundle, $\mathrm{Hom}_{\mathcal{O}_C}(\mathcal{F},\Omega_C) \cong H^0(C,\mathcal{F}^\vee \otimes \Omega_C)$ and Serre Duality gives an isomorphism

$$H^0(C,\mathcal{F}^\vee \otimes \Omega_C) \cong H^1(C,\mathcal{F})^\vee.$$

Taking $\mathcal{F} = \Omega_C$, we see that $H^1(C,\Omega_C) \cong H^0(C,\mathcal{O}_C)^\vee$ and in particular that the trace map $\mathrm{tr}\colon H^1(C,\Omega_C) \to \Bbbk$ is an isomorphism if $C$ is geometrically connected and reduced.

Combining the above version of Riemann–Roch (5.1.1) with Serre Duality leads to the more powerful version of Riemann–Roch.

**Theorem 5.1.4** (Riemann–Roch II)**.** *Let $C$ be a smooth, connected, and projective curve of genus $g$ over an algebraically closed field. If $L$ is a line bundle on $C$, then*

$$\mathrm{h}^0(C,L) - \mathrm{h}^0(C,\Omega_C \otimes L^\vee) = \deg L + 1 - g.$$

**Remark 5.1.5.** This is often written in divisor form as $\mathrm{h}^0(C,L) - \mathrm{h}^0(C,K-L) = \deg L + 1 - g$ where $K$ denotes a canonical divisor, i.e. $\Omega_C = \mathcal{O}_C(K)$.

Like Riemann–Roch, Riemann–Hurwitz (5.7.2) plays an essential role in the study of smooth curves. Riemann–Hurwitz informs us on how the sheaf of differentials behaves under finite morphisms of smooth curves; the statement is postponed until our discussion of branched covers.

### 5.1.3 Positivity of divisors on smooth curves

The following consequence of Riemann–Roch provides useful criteria to determine whether a given line bundle is base point free (equivalently globally generated), ample, or very ample.

**Corollary 5.1.6.** *Let $C$ be a connected, smooth, and projective curve over an algebraically closed field $\Bbbk$, and let $L$ be a line bundle on $C$.*

*(1) if $\deg L < 0$, then $\mathrm{h}^0(C,L) = 0$;*

*(2) if $\deg L > 0$, then $L$ is ample;*

*(3) if $\deg L \geq 2g$, then $L$ is base point free; and*

(4) *if* $\deg L \geq 2g+1$*, then* $L$ *is very ample.*

*Proof.* See [Har77, Cor. IV.3.2]. $\square$

**Remark 5.1.7.** If $g > 1$, we can use Riemann–Roch and Serre Duality to compute that: (a) $\mathrm{h}^0(C, \Omega_C) = \mathrm{h}^1(C, \mathcal{O}_C) = g$, (b) $\mathrm{h}^1(C, \Omega_C) = \mathrm{h}^0(C, \mathcal{O}_C) = 1$ and (c) $\Omega_C$ has degree $2g-2$ and is thus ample on $C$. Similarly, if $k > 1$, we have: (a) $\mathrm{h}^0(C, \Omega_C^{\otimes k}) = (2k-1)(g-1)$, (b) $\mathrm{h}^1(C, \Omega_C^{\otimes k}) = 0$ and (c) $\Omega_C^{\otimes k}$ has degree $2k(g-1)$ and is very ample if $k \geq 3$. Note that $\Omega_C$ is not very ample precisely when $C$ is *hyperelliptic*. On the other hand, if $g = 1$ then $\Omega_C \cong \mathcal{O}_C$, and if $g = 0$ then $C = \mathbb{P}^1$ and $\Omega_C = \mathcal{O}(-2)$.

### 5.1.4 Families of smooth curves

**Definition 5.1.8.** A *family of smooth curves (of genus g)* over a scheme $S$ is a smooth and proper morphism $\mathcal{C} \to S$ of schemes such that every geometric fiber is a connected curve (of genus $g$).

Recall that the relative sheaf of differentials $\Omega_{\mathcal{C}/S}$ is a line bundle on $\mathcal{C}$ such that for every geometric point $s\colon \operatorname{Spec} \Bbbk \to S$, the restriction $\Omega_{\mathcal{C}/S}|_{\mathcal{C}_s}$ is identified with $\Omega_{\mathcal{C}_s}$. More generally, for every morphism $T \to S$ of schemes, the pullback of $\Omega_{\mathcal{C}/S}$ to $\mathcal{C} \times_S T$ is canonically isomorphic to $\Omega_{\mathcal{C} \times_S T/T}$. We now show that for $k \geq 3$, the $k$th relative pluricanonical sheaf $\Omega_{\mathcal{C}/S}^{\otimes k}$ is relatively very ample, and that its pushforward is a vector bundle on $S$.

**Proposition 5.1.9** (Properties of Families of Smooth Curves)**.** *Let* $\pi\colon \mathcal{C} \to S$ *be a family of smooth curves of genus* $g \geq 2$*.*

(1) $\pi_* \mathcal{O}_{\mathcal{C}} = \mathcal{O}_S$*;*

(2) *The pushforward* $\pi_*(\Omega_{\mathcal{C}/S}^{\otimes k})$ *is a vector bundle of rank*

$$r(k) := \begin{cases} g & \text{if } k = 1 \\ (2k-1)(g-1) & \text{if } k > 1. \end{cases}$$

*whose construction commutes with base change (i.e. for a morphism* $f\colon T \to S$ *of schemes,* $f^*\pi_*(\Omega_{\mathcal{C}/S}^{\otimes k}) \cong \pi_{T,*}(\Omega_{\mathcal{C}_T/T}^{\otimes k})$*).*

(3) $R^1\pi_*\Omega_{\mathcal{C}/S}^{\otimes k}$ *is isomorphic to* $\mathcal{O}_S$ *if* $k = 1$ *and zero otherwise.*

(4) *For* $k \geq 3$*,* $\Omega_{\mathcal{C}/S}^{\otimes k}$ *is relatively very ample.*

*Proof.* Items (1)–(3) follows from Cohomology and Base Change (A.7.5) as detailed in Proposition A.7.8. For (4), observe that for every point $s \in S$, the fiber $\Omega_{\mathcal{C}/S}^{\otimes k} \otimes \kappa(s) = \Omega_{\mathcal{C}_s}^{\otimes k}$ is very ample by Corollary 5.1.6 as $\deg \Omega_{\mathcal{C}_s}^{\otimes k} = k(2g-2) > 0$. Since $\mathrm{H}^1(\mathcal{C}_s, \Omega_{\mathcal{C}_s}^{\otimes k}) = 0$, we may apply Proposition E.2.1 to conclude that $\Omega_{\mathcal{C}/S}^{\otimes k}$ is relatively very ample. $\square$

**Remark 5.1.10.** In particular, (4) above implies that every family of smooth curves is projective.

It is also true that the relative sheaf of differentials $\Omega_{\mathcal{C}/S}$ is a relative dualizing sheaf, i.e. satisfies a relative version of Serre Duality; see [Liu02, §6.4].

## 5.2 Nodal curves

### 5.2.1 Nodes

**Definition 5.2.1** (Nodes)**.** Let $C$ be a curve over a field $\Bbbk$.

- If $\Bbbk$ is algebraically closed, we say that $p \in C(\Bbbk)$ is a *node* if there is an isomorphism $\widehat{\mathcal{O}}_{C,p} \cong \Bbbk[\![x,y]\!]/(xy)$.
- If $\Bbbk$ is an arbitrary field, we say that a closed point $p \in C$ is a *node* if there exists a node $\overline{p} \in C_{\overline{\Bbbk}}$ over $p$.

We say that $C$ is a *nodal curve* (or has *at-worst nodal singularities*) if $C$ has pure dimension one and every closed point is either smooth or nodal.

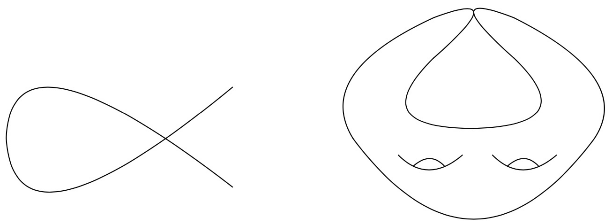


Figure 5.1: A node of a curve over $\mathbb{C}$ viewed algebraically (left-hand side) or analytically (right-hand side).

**Example 5.2.2.**

(1) The curves $\operatorname{Spec} \Bbbk[x,y]/(xy)$ and $\operatorname{Spec} \Bbbk[x,y]/(y^2 - x^2(x+1))$ have nodes at 0.

(2) The curve $C = \operatorname{Spec} \mathbb{R}[x,y]/(x^2+y^2)$ has a node at 0. Since the quadratic form $x^2+y^2$ does not split into linear factors, the completion $\widehat{\mathcal{O}}_{C,0}$ is not isomorphic to $\mathbb{R}[\![s,t]\!]/(st)$.

(3) The curve $\operatorname{Spec} \mathbb{Q}[x,y]/(x^2-2)(y^2-3)$ has a node at the point $p$ defined by the maximal ideal $(x^2-2, y^2-3)$. Note that unlike the previous example where the node 0 is a rational point, the node $p$ in this example is not a rational point and the field extension $\mathbb{Q} \to \kappa(p)$ has degree 4.

### 5.2.2 Equivalent characterizations of nodes

Recall that the singular locus $\operatorname{Sing}(C)$ of $C$ is defined scheme-theoretically as the first fitting ideal of $\Omega_C$ (see §A.3.6): locally if $C = V(f_1, \ldots, f_m) \subset \mathbb{A}^n$, then $\operatorname{Sing}(C)$ is defined by the vanishing of all $(n-1)\times(n-1)$ minors of the Jacobian matrix $J = (\frac{\partial f_j}{\partial x_i})$; note that if $C = V(f) \subset \mathbb{A}^2$ is a plane affine curve, then $\operatorname{Sing}(C) = V(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})$. We will also use properties of local complete intersections as discussed in §A.3.8.

**Proposition 5.2.3.** *Let $C$ be a pure dimension one curve over a field $\Bbbk$, and let $p \in C$ be a closed point. The following are equivalent:*

*(1) $p \in C$ is a node;*

*(2) $C$ is a local complete intersection at $p$, and $\operatorname{Sing}(C)$ is unramified over $\Bbbk$ at $p \in \operatorname{Sing}(C)$;*

*(3) $\Bbbk \to \kappa(p)$ is separable, $\mathcal{O}_{C,p}$ is reduced, $\dim \mathfrak{m}_p/\mathfrak{m}_p^2 = 2$, and there is a nondegenerate quadratic form $q \in \operatorname{Sym}^2 \mathfrak{m}_p/\mathfrak{m}_p^2$ mapping to 0 in $\mathfrak{m}_p^2/\mathfrak{m}_p^3$;*

(4) $\Bbbk \to \kappa(p)$ *is separable and* $\widehat{\mathcal{O}}_{C,p} \cong \kappa(p)[\![x,y]\!]/(q)$ *where* $q$ *is a nondegenerate quadratic form; and*

(5) *there exists a finite separable field extension* $\Bbbk \to \Bbbk'$ *and a point* $p' \in C_{\Bbbk'}$ *such that* $\widehat{\mathcal{O}}_{C_{\Bbbk'},p'} \cong \Bbbk'[\![x,y]\!]/(xy)$

*Proof.* Assuming (1), let $\overline{p} \in C_{\overline{\Bbbk}}$ be a node over $p$ and let $\mathrm{Sing}(C) \subset C$ be the scheme-theoretic singular locus. Then $\mathrm{Sing}(C) \times_{\Bbbk} \overline{\Bbbk} = \mathrm{Sing}(C_{\overline{\Bbbk}})$ and the preimage of $\mathrm{Sing}(C_{\overline{\Bbbk}})$ under $\mathrm{Spec}\, \widehat{\mathcal{O}}_{C_{\overline{\Bbbk}},\overline{p}} \to C_{\overline{\Bbbk}}$ is $\mathrm{Sing}(\mathrm{Spec}\, \widehat{\mathcal{O}}_{C_{\overline{\Bbbk}},\overline{p}})$ by properties of fitting ideals (see §A.3.6). Since $\widehat{\mathcal{O}}_{C_{\overline{\Bbbk}},\overline{p}} \cong \overline{\Bbbk}[\![x,y]\!]/(xy)$, $\mathrm{Sing}(\mathrm{Spec}\, \widehat{\mathcal{O}}_{C_{\overline{\Bbbk}},\overline{p}}) = V(x,y) = \mathrm{Spec}\, \overline{\Bbbk}$. Therefore $\mathrm{Sing}(C) \to \mathrm{Spec}\, \Bbbk$ is unramified at $p$. Since $\widehat{\mathcal{O}}_{C_{\overline{\Bbbk}},\overline{p}}$ is a complete intersection, $C$ is a local complete intersection at $p$ (Proposition A.3.15). This gives (2).

Assuming (2), since $\mathrm{Sing}(C)$ is unramified at $p$, the field extension $\Bbbk \to \kappa(p)$ is separable and there is an open neighborhood $U \subset C$ of $p$ such that $\mathrm{Sing}(U) = \mathrm{Sing}(C) \cap U = \{p\}$. In particular, $C$ and $\mathcal{O}_{C,p}$ are generically reduced. On the other hand, since $C$ is a local complete intersection, $\mathcal{O}_{C,p}$ is a dimension 1 Cohen–Macaulay local ring and thus has no embedded primes. It follows that $\mathcal{O}_{C,p}$ is reduced. Using that $C$ is a local complete intersection, we can write $\widehat{\mathcal{O}}_{C,p} = R/(f_1, \ldots, f_{n-1})$ where $R = \kappa(p)[\![x_1, \ldots, x_n]\!]$. Since $\mathrm{Sing}(C)$ is unramified at $p$, the $(n-1) \times (n-1)$ minors of the Jacobian matrix $\left(\frac{\partial f_j}{\partial x_i}\right)_{i,j}$ generate the maximal ideal $\mathfrak{m} = (x_1, \ldots, x_n) \subset \widehat{\mathcal{O}}_{C,p}$. If $\frac{\partial f_j}{\partial x_i} \in R$ is a unit for some $i$ and $j$, then the sequence $x_1, \ldots, \widehat{x_i}, x_n, f_j$ also generates $\mathfrak{m}/\mathfrak{m}^2$. We may use Lemma A.10.15 to change coordinates by replacing the generators $x_1, \ldots, x_n$ with $x_1, \ldots, \widehat{x_i}, x_n, f_j$. Eliminating $f_j$ allows us to write $R = \kappa(p)[\![x_1, \ldots, \widehat{x_i}, x_n]\!]/(f_1, \ldots, \widehat{f_j}, \ldots, f_{n-1})$. After finitely many such replacements, we can assume that $\frac{\partial f_j}{\partial x_i} \in \mathfrak{m}$ for every $i, j$. This implies that every $(n-1) \times (n-1)$ minor is in $\mathfrak{m}^{n-1}$, but since these minors generate $\mathfrak{m}$, we must have that $n = 2$. Therefore, $\widehat{\mathcal{O}}_{C,p} = \kappa(p)[\![x,y]\!]/(f)$ with $f = f_2 + f_3 + \cdots$ and each $f_i$ homogeneous of degree $i$. Since the partials $f_x$ and $f_y$ generate $(x,y)$, the quadratic form $q := f_2 \in \mathrm{Sym}^2\, \mathfrak{m}/\mathfrak{m}^2$ must be nondegenerate. This gives (3).

Assuming (3), we have that $\dim_{\kappa(p)} \mathfrak{m}^d/\mathfrak{m}^{d+1} = 2$ for every $d \geq 1$ since $q$ maps to 0 in $\mathfrak{m}^2/\mathfrak{m}^3$. A choice of elements $x_0, y_0 \in \mathfrak{m}$ mapping to a basis in $\mathfrak{m}/\mathfrak{m}^2$ induces a surjection $\kappa(p)[\![x,y]\!] \to \widehat{\mathcal{O}}_{C,p}$ (Lemma A.10.15). Since $\mathcal{O}_{C,p}$ is reduced, so is $\widehat{\mathcal{O}}_{C,p}$ (see Remark B.4.5). Therefore, we may use that $\kappa(p)[\![x,y]\!]$ is a UFD to conclude that the kernel $\kappa(p)[\![x,y]\!] \to \widehat{\mathcal{O}}_{C,p}$ is generated by an element $f$ expressed as a product of distinct irreducible elements. Thus $\widehat{\mathcal{O}}_{C,p} \cong \kappa(p)[\![x,y]\!]/(f)$ where the quadratic component $q = ax^2 + bxy + cy^2$ of $f$ is a nondegenerate quadratic form. We claim that we can modify our choice of coordinates $x_0, y_0 \in \mathfrak{m}$ so that $q(x_0, y_0) = 0 \in \widehat{\mathcal{O}}_{C,p}$, or in other words that $f = q$. We will show inductively that for each $N$, there exists elements $x_i, y_i \in \mathfrak{m}^{i+1}$ for $i = 0, \ldots, N$ such that $q(x_0 + \cdots + x_N, y_0 + \cdots + y_N) \in \mathfrak{m}^{N+3}$. Since $\widehat{\mathcal{O}}_{C,p}$ is complete, this would enable us to replace $x_0$ and $y_0$ with $\sum_i x_i$ and $\sum_i y_i$ and conclude that $\widehat{\mathcal{O}}_{C,p} \cong \kappa(p)[\![x,y]\!]/(q)$. Supposing that we've already chosen $x' = x_0 + \cdots + x_{N-1}$ and $y' = y_0 + \cdots + y_{N-1}$, then for every $x_N$ and $y_N \in \mathfrak{m}^{N+1}$, we have that

$$q(x' + x_N, y' + y_N) = q(x', y') + (2ax_0 + by_0)x_N + (bx_0 + 2cy_0)y_N \mod \mathfrak{m}^{N+3}$$

The nondegeneracy of $q = ax^2 + bxy + y^2$ implies that $2ax_0 + by_0$ and $bx_0 + 2cy_0$ are linearly independent. Since $\dim_{\kappa(p)} \mathfrak{m}^{N+2}/\mathfrak{m}^{N+3} = 2$, we may choose $x_N$ and $y_N$ such that $Q(x' + x_N, y' + y_N) \in \mathfrak{m}^{N+3}$. This completes (4).

Assuming (4) and using that $q$ is nondegenerate, we may choose a degree 2 separable field extension $\kappa(p) \to \Bbbk'$ such that $q$ splits as a product of linear forms. Thus $\widehat{\mathcal{O}}_{C,p} \otimes_{\Bbbk} \Bbbk' \cong \Bbbk'[[x,y]]/(xy)$, yielding (5). Finally, (5) implies that $p$ is a node.

See also [SP, Tags 0C49, 0C4D and 0C4E]. $\square$

**Exercise 5.2.4.** Show that $\operatorname{Spec} \Bbbk[x,y]/(f)$ has a node at 0 if and only if $f(0) = f_x(0) = f_y(0) = 0$ and the Hessian $\det \begin{pmatrix} f_{xx}(0) & f_{xy}(0) \\ f_{yx}(0) & f_{yy}(0) \end{pmatrix}$ is nonzero.

**Exercise 5.2.5.** Let $C$ be a pure dimension 1 reduced curve over a field $\Bbbk$ with normalization $\pi \colon \widetilde{C} \to C$. Show that $p \in C$ is a node if and only if $\Bbbk \to \kappa(p)$ is separable, $(\pi_* \mathcal{O}_{\widetilde{C}}/\mathcal{O}_C) \otimes \kappa(p) = 1$, and $\sum_{\pi(q)=p} [\kappa(q) \colon \kappa(p)]_{\mathrm{sep}} = 2$.

*Hint: Identify $(\pi_* \mathcal{O}_{\widetilde{C}}/\mathcal{O}_C) \otimes \kappa(p)$ with the quotient $\widetilde{A}/A$ where $A = \widehat{\mathcal{O}}_{C,p}$ and $\widetilde{A}$ is its normalization, using that normalization commutes with completion (see Remark B.4.5). To show ($\Leftarrow$), use that $\widetilde{A}$ is a product of complete DVRs to derive the structure of $A$. See also [SP, Tag 0C4A].*

**Remark 5.2.6.** The quantity $(\pi_* \mathcal{O}_{\widetilde{C}}/\mathcal{O}_C) \otimes \kappa(p)$ is referred to as the *$\delta$-invariant of $p$*, and the sum $\sum_{\pi(q)=p} [\kappa(q) \colon \kappa(p)]_{\mathrm{sep}}$ is referred to as the *number of geometric branches over $p$*. A cusp $\Bbbk[[x,y]]/(y^2 - x^3)$ has $\delta$-invariant one but has only one geometric branch.

[?]

**Proposition 5.2.7** (Local Structure of Nodes). *Let $C$ be a curve over a field $\Bbbk$. If $p \in C$ is a node, then there exists a finite separable field extension $\Bbbk \to \Bbbk'$ and étale neighborhoods*

$$\begin{tikzcd} & (U,u) \arrow[dl] \arrow[dr] & \\ (C,p) & & (\operatorname{Spec} \Bbbk'[x,y]/(xy), 0). \end{tikzcd} \tag{5.2.1}$$

*Proof.* Using the characterization of nodes from Proposition 5.2.3(5), there is a finite separable field extension $\Bbbk \to \Bbbk'$ such that $\widehat{\mathcal{O}}_{C,p} \otimes_{\Bbbk} \Bbbk' \cong \Bbbk'[[x,y]]/(xy)$. The result is now a consequence of Artin Approximation (Corollary A.10.13). $\square$

We will prove a more general statement in Theorem 5.2.18 regarding the local structure of families of nodal curves.

**Exercise 5.2.8.** Provide a proof of the Local Structure of Nodes (5.2.7) without appealing to Artin Approximation.

*Hint: Use that the normalization of a strict henselization $\mathcal{O}^{\mathrm{sh}}_{C,p}$ has two components to find an affine étale neighborhood $(\operatorname{Spec} R, u) \to (C,p)$ of $p$ with $\widetilde{R} = R_1 \times R_2$. Use the exact sequence $0 \to R \to R_1 \times R_2 \to \kappa(u) \to 0$ to construct elements $x, y \in R$ mapping to $(1,0)$, $(0,1) \in R_1 \times R_2$, and argue that $\kappa(u)[x,y]/(xy) \to R$ is étale.*

### 5.2.3 Genus formula

**Proposition 5.2.9** (Genus Formula). *Let $C$ be a connected, nodal, and projective curve over an algebraically closed field $\Bbbk$ with $\delta$ nodes and $\nu$ irreducible components.*

*Let $C_i$ denote the $i$th irreducible component with normalization $\widetilde{C}_i$ with genus $g(\widetilde{C}_i)$. The genus $g$ of $C$ satisfies*

$$g = \sum_{i=1}^{\nu} g(\widetilde{C}_i) + \delta - \nu + 1.$$

*Proof.* Let $p_1, \dots, p_\delta \in C$ denote the nodes of $C$. We claim that the normalization $\pi\colon \widetilde{C} \to C$ induces a short exact sequence

$$0 \to \mathcal{O}_C \to \pi_* \mathcal{O}_{\widetilde{C}} \to \bigoplus_i \kappa(p_i) \to 0.$$

It suffices to verify this étale-locally around a node $p_i \in C$, and so by the Local Structure of Nodes (5.2.7), we can assume that $C = \operatorname{Spec} \Bbbk[x,y]/(xy)$. In this case, $\widetilde{C} = \operatorname{Spec}(\Bbbk[x] \times \Bbbk[y])$ and the sequence above corresponds to $0 \to \Bbbk[x,y]/(xy) \to \Bbbk[x] \times \Bbbk[y] \to \Bbbk \to 0$. Alternatively, normalization commutes with completion and a direct calculation as above shows that if $A := \widehat{\mathcal{O}}_{C,p} \cong \Bbbk[\![x,y]\!]/(xy)$, then $\widetilde{A}/A \cong \Bbbk$; see also Exercise 5.2.5.

The short exact sequence induces a long exact sequence on cohomology

$$0 \to \underbrace{H^0(C, \mathcal{O}_C)}_{1} \to \underbrace{H^0(\widetilde{C}, \mathcal{O}_{\widetilde{C}})}_{\nu} \to \underbrace{\oplus_i \kappa(p_i)}_{\delta} \to \underbrace{H^1(C, \mathcal{O}_C)}_{g} \to \underbrace{H^1(\widetilde{C}, \mathcal{O}_{\widetilde{C}})}_{\sum_i g(\widetilde{C}_i)} \to 0$$

where the labels underneath indicate the dimension. The statement follows. $\square$

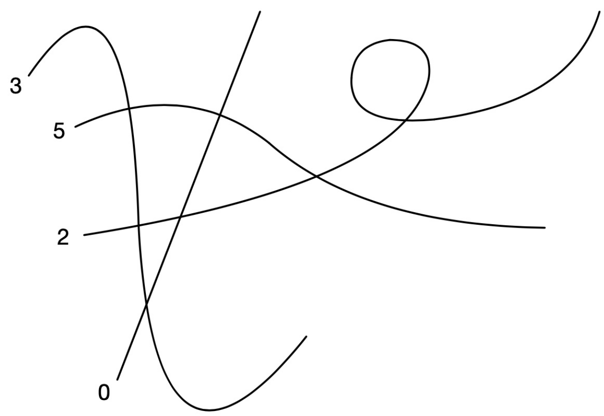



Figure 5.2: An example of a nodal curve of genus 14.

**Remark 5.2.10.** Notice that $\delta - \nu + 1$ is precisely the number of connected regions bounded by the curve $C$ as in Figure 5.2. Thus, the genus of a nodal curve can be easily computed from the picture by summing the geometric genera of the irreducible components and adding the number of bounded regions.

### 5.2.4 The dualizing sheaf

Since a nodal curve $C$ over a field $\Bbbk$ is a locally a complete intersection, $C$ is Gorenstein and there is a *dualizing line bundle* $\omega_C$ with a trace map $\operatorname{tr}_C\colon H^1(C, \omega_C) \xrightarrow{\sim} \Bbbk$; see

[Har77, III.7.11] or [Ser88, §IV]. In other words, for every coherent sheaf $F$, the natural pairing

$$\operatorname{Hom}_{\mathcal{O}_C}(F, \omega_C) \times H^1(C, F) \to H^1(C, \omega_C) \xrightarrow{\mathrm{tr}} \Bbbk$$

is perfect.

Due to its importance in the study of stable curves, we now provide an explicit description of $\omega_C$ below in the case that $\Bbbk$ is algebraically closed. Let $\Sigma := C^{\mathrm{sing}}$ be the singular locus and $U = C \setminus \Sigma$. Let $\pi\colon \widetilde{C} \to C$ be the normalization of $C$, and let $\widetilde{\Sigma}$ and $\widetilde{U}$ be the preimages of $\Sigma$ and $U$ as in the diagram

$$\begin{tikzcd} \widetilde{U} \arrow[r, hook] \arrow[d, "\wr"] & \widetilde{C} \arrow[d, "\pi"] & \widetilde{\Sigma} \arrow[l, hook'] \arrow[d] \\ U \arrow[r, hook] & C & \Sigma. \arrow[l, hook'] \end{tikzcd} \tag{5.2.2}$$

Let $\Sigma = \{z_1, \ldots, z_n\}$ be an ordering of the points and $\pi^{-1}(z_i) = \{p_i, q_i\}$. Since $\widetilde{C}$ is smooth, the sheaf of differentials $\Omega_{\widetilde{C}}$ is a dualizing sheaf and is a line bundle. There is a short exact sequence

$$0 \to \Omega_{\widetilde{C}} \to \Omega_{\widetilde{C}}(\widetilde{\Sigma}) \to \mathcal{O}_{\widetilde{\Sigma}} \to 0 \tag{5.2.3}$$

obtained by tensoring the sequence $0 \to \mathcal{O}_{\widetilde{C}}(-\widetilde{\Sigma}) \to \mathcal{O}_{\widetilde{C}} \to \mathcal{O}_{\widetilde{\Sigma}} \to 0$ with $\Omega_{\widetilde{C}}(\widetilde{\Sigma})$. As $\Omega_{\widetilde{C}}(\widetilde{\Sigma})|_{\widetilde{U}} = \Omega_{\widetilde{U}}$, we can interpret sections of $\Omega_{\widetilde{C}}(\widetilde{\Sigma})$ as rational sections of $\Omega_{\widetilde{C}}$ with at worst simple poles along $\widetilde{\Sigma}$. Evaluating (5.2.3) on an open $\widetilde{V} \subset \widetilde{C}$ yields

$$0 \longrightarrow \Gamma(\widetilde{V}, \Omega_{\widetilde{C}}) \longrightarrow \Gamma(\widetilde{V}, \Omega_{\widetilde{C}}(\widetilde{\Sigma})) \xrightarrow[s \mapsto (\mathrm{res}_y(s))]{} \bigoplus_{y \in \widetilde{V} \cap \widetilde{\Sigma}} \kappa(y), \tag{5.2.4}$$

where the last map takes a rational section $s \in \Gamma(\widetilde{V} \cap \widetilde{U}, \Omega_{\widetilde{C}})$ to the tuple whose coordinate at $y \in \widetilde{V} \cap \widetilde{\Sigma}$ is the *residue* $\mathrm{res}_y(s)$ *of* $s$ *at* $y$.

**Definition 5.2.11.** Let $C$ be a nodal curve $C$ over an algebraically closed field $\Bbbk$. Using the notation of (5.2.2) and (5.2.4), we define the subsheaf $\omega_C \subset \pi_* \Omega_{\widetilde{C}}(\widetilde{\Sigma})$ by declaring that sections along $V \subset C$ consist of rational sections $s$ of $\Omega_{\widetilde{C}}$ along $\pi^{-1}(V)$ with at worst simple poles along $\widetilde{\Sigma}$ such that for every node $z_i \in V \cap \Sigma$ with preimages $p_i, q_i \in \pi^{-1}(V)$,

$$\mathrm{res}_{p_i}(s) + \mathrm{res}_{q_i}(s) = 0.$$

The definition implies that $\omega_C$ sits in the following two exact sequences:

$$0 \longrightarrow \omega_C \longrightarrow \pi_* \Omega_{\widetilde{C}}(\widetilde{\Sigma}) \xrightarrow[s \mapsto (\mathrm{res}_{p_i}(s) + \mathrm{res}_{q_i}(s))]{} \bigoplus_{z_i \in \Sigma} \Bbbk \longrightarrow 0 \tag{5.2.5}$$

$$0 \longrightarrow \pi_* \Omega_{\widetilde{C}} \longrightarrow \omega_C \xrightarrow[s \mapsto (\mathrm{res}_{p_i}(s))]{} \bigoplus_{z_i \in \Sigma} \Bbbk \longrightarrow 0 \tag{5.2.6}$$

**Example 5.2.12** (Local calculation)**.** Let $C = \operatorname{Spec} \Bbbk[x,y]/(xy)$. Then $\widetilde{C} = \mathbb{A}^1 \sqcup \mathbb{A}^1$ with coordinates $x$ and $y$ respectively. The singular locus of $C$ is $\Sigma = \{0\}$ with preimage $\widetilde{\Sigma} = \{p, q\}$ consisting of the two origins. Then $\Gamma(\widetilde{C}, \Omega_{\widetilde{C}}) = \Gamma(\mathbb{A}^1, \Omega_{\mathbb{A}^1}) \times \Gamma(\mathbb{A}^1, \Omega_{\mathbb{A}^1})$ and $(\frac{dx}{x}, -\frac{dy}{y})$ is a rational section with opposite residues at $p$ and $q$. In fact, every section of $\Gamma(C, \omega_C)$ is of the form

$$\left(f(x)\frac{dx}{x}, g(y)\frac{-dy}{y}\right) = (f(x) + g(y) - f(0)) \cdot (\frac{dx}{x}, \frac{-dy}{y})$$

for polynomials $f(x)$ and $g(y)$ such that $f(0) = g(0)$, which is precisely the condition for $(f, g) \in \Gamma(\widetilde{C}, \mathcal{O}_{\widetilde{C}})$ to descend to a global function $f(x) + g(y) - f(0) \in \Gamma(C, \mathcal{O}_C)$. In other words, $\omega_C \cong \mathcal{O}_C$ with generator $(\frac{dx}{x}, -\frac{dy}{y})$.

**Example 5.2.13.** Let $C$ be the nodal projective plane cubic and $\mathbb{P}^1 \to C$ be the normalization with coordinates $[x : y]$ such that $0$ and $\infty$ are the fibers of the node. Observe that the rational differential $\eta := \frac{dx}{x} = -\frac{dy}{y}$ on $\mathbb{P}^1$ satisfies $\operatorname{res}_0 \eta + \operatorname{res}_\infty \eta = 0$. It is easy to see that every local section of $\omega_C$ is a multiple of $\eta$ or in other words that $\eta \colon \mathcal{O}_C \to \omega_C$ is an isomorphism.

**Exercise 5.2.14.** Let $C$ be a connected, nodal, and projective curve over an algebraically closed field $\Bbbk$.

(a) Show that if $\pi \colon C' \to C$ is an étale morphism, then $\pi^* \omega_C \cong \omega_{C'}$.

*Hint: Use the fact that normalization commutes with étale base change.*

(b) Conclude that $\omega_C$ is a line bundle.

(c) Show that $\omega_C$ is a dualizing sheaf.

*Hint: Reduce to the case of a smooth curve by considering the normalization.*

(d) If $T \subset C$ is a subcurve with complement $T^c := \overline{C \setminus T}$, show that

$$\omega_C|_T = \omega_T(T \cap T^c).$$

**Exercise 5.2.15.** Let $C$ be a connected, nodal, and projective curve over an algebraically closed field $\Bbbk$. Let $\widetilde{C} \to C$ be the normalization and $\widetilde{\Sigma} \subset \widetilde{C}$ the set of preimages of nodes. Show that there is an identification

$$\operatorname{Hom}_{\mathcal{O}_{\widetilde{C}}}(\Omega_{\widetilde{C}}(\widetilde{\Sigma}), \mathcal{O}_{\widetilde{C}}) \cong \operatorname{Hom}_{\mathcal{O}_C}(\Omega_C, \mathcal{O}_C),$$

or in other words that regular vector fields on $C$ correspond to regular vector fields on $\widetilde{C}$ vanishing at the preimages of nodes.

### 5.2.5 Nodal families

Recall that the relative singular locus $\operatorname{Sing}(\mathcal{C}/S)$ of a morphism $\mathcal{C} \to S$ with dimension one fibers is defined as the first fitting ideal of $\Omega_{\mathcal{C}/S}$; see Definition A.3.14. Syntomic morphisms are fppf morphisms whose fibers are local complete intersections; see §A.3.8.

**Proposition 5.2.16.** *Let $\mathcal{C} \to S$ be an fppf morphism of schemes and $s \in S$ a point such that the fiber $\mathcal{C}_s$ has pure dimension one. A point $p \in \mathcal{C}_s$ is a node if and only if $\mathcal{C} \to S$ is syntomic at $p$ and the relative singular locus $\operatorname{Sing}(\mathcal{C}/S) \to S$ is unramified at $p$.*

*Proof.* The conditions that $\mathcal{C} \to S$ is syntomic at $p$ and $\operatorname{Sing}(\mathcal{C}) \to S$ is unramified at $p$ are conditions on the fibers over $s$. Since $\operatorname{Sing}(\mathcal{C}/S)_s = \operatorname{Sing}(\mathcal{C}_s)$, the result follows from the equivalence of (1)-(4) of Proposition 5.2.3. $\square$

The above characterization shows that the property of being a nodal family descends under limits (Definition A.6.6).

**Lemma 5.2.17.** *The following property of morphisms of schemes descends under limits: an fppf morphism such that every fiber is a pure dimension one nodal curve.*

*Proof.* From Descending Properties of Morphisms under Limits (A.6.7), we know that the properties of being fppf, syntomic, unramified, and having connected pure one-dimensional fibers descend under limits. Since the relative singular locus commutes with base change, the result follows from Proposition 5.2.16. $\square$

A *family of nodal curves* is a proper fppf morphism $\mathcal{C} \to S$ of schemes such that every geometric fiber $\mathcal{C}_s$ is a connected nodal curve.

### 5.2.6 Local structure of nodal families

Recall that if $\mathcal{C} \to S$ is a family of smooth curves, then every point $p \in \mathcal{C}$ over $s \in S$ is étale locally isomorphic to relative affine space of dimension one. More precisely, there is a commutative diagram

$$\begin{tikzcd}
(\mathcal{C}, p) \arrow[d] & (\mathcal{C}', p') \arrow[l, hook', "\mathrm{op}"'] \arrow[r, "\text{ét}"] \arrow[d] & (S' \times_{\mathbb{Z}} \mathbb{A}^1_{\mathbb{Z}}, (s', 0)) \arrow[dl] \\
(S, s) & (S', s') \arrow[l, hook', "\mathrm{op}"'] &
\end{tikzcd}$$

where the left horizontal maps are open immersions, the right-hand map is étale map, and $S' \times_{\mathbb{Z}} \mathbb{A}^1_{\mathbb{Z}} = \mathbb{A}^1_{S'} \to S'$ is the base change of $\mathbb{A}^1_{\mathbb{Z}} \to \operatorname{Spec} \mathbb{Z}$; see Proposition A.3.5. We now give a local structure of a family of nodal curves generalizing the Local Structure of Nodes (5.2.7).

**Theorem 5.2.18** (Local Structure of Nodal Families)**.** *Let $\pi\colon \mathcal{C} \to S$ be an fppf morphism such that every geometric fiber is a curve. Let $p \in \mathcal{C}$ be a node in a fiber $\mathcal{C}_s$. There is a commutative diagram*

$$\begin{tikzcd}
(\mathcal{C}, p) \arrow[d] & (\mathcal{C}', p') \arrow[l, "\text{ét}"'] \arrow[r, "\text{ét}"] \arrow[d] & (\operatorname{Spec} A[x,y]/(xy - f), (s', 0)) \arrow[dl] \\
(S, s) & (\operatorname{Spec} A, s') \arrow[l, "\text{ét}"'] &
\end{tikzcd} \tag{5.2.7}$$

*where each horizontal map is étale and $f \in A$ is a function vanishing at $s'$.*

**Remark 5.2.19.** In other words, every family of nodal curves is étale locally on the source and target the base change of the morphism

$$\operatorname{Spec} \mathbb{Z}[x, y, t]/(xy - t) \to \operatorname{Spec} \mathbb{Z}[t]$$

by a map $\operatorname{Spec} A \to \operatorname{Spec} \mathbb{Z}[t]$ induced by a function $f \in A$.

*Proof 1 (local-to-global).*

*Step 1: Reduce to the case where $S$ is of finite type over $\mathbb{Z}$.* Use limit methods and Lemma 5.2.17.

*Step 2: Reduce to the case where $\widehat{\mathcal{O}}_{\mathcal{C}_s,p} \cong \kappa(s)[\![x,y]\!]/(xy)$.* By Proposition 5.2.3, there is a finite separable field extension $\kappa(s) \to \Bbbk'$ and a point $p' \in \mathcal{C}_s \times_{\kappa(s)} \Bbbk'$ whose completion is isomorphic to $\Bbbk'[\![x,y]\!]/(xy)$. Letting $(S', s') \to (S, s)$ be an étale morphism such that there is an isomorphism $\kappa(s') \cong \Bbbk'$ over $\kappa(s)$, we replace $S$ with $S'$.

*Step 3: Show that $\widehat{\mathcal{O}}_{\mathcal{C},p} \cong \widehat{\mathcal{O}}_{S,s}[\![x,y]\!]/(xy - \widehat{f})$ where $\widehat{f} \in \widehat{\mathfrak{m}}_s \subset \widehat{\mathcal{O}}_{S,s}$.* We claim that there exists elements $x_n, y_n \in \widehat{\mathcal{O}}_{\mathcal{C},p}$ and $f_n \in \widehat{\mathcal{O}}_{S,s}$ for $n \geq 0$ which are compatible (i.e. $x_{n+1} \equiv x_n (\text{mod } \mathfrak{m}_p^{n+1})$, $y_{n+1} \equiv y_n (\text{mod } \mathfrak{m}_p^{n+1})$, and $f_{n+1} \equiv f_n (\text{mod } \mathfrak{m}_s^{n+1})$) and such that there is an isomorphism

$$(\mathcal{O}_{S,s}/\mathfrak{m}_s^{n+1})[\![x,y]\!]/(xy - f_n) \xrightarrow{\sim} \mathcal{O}_{\mathcal{C},p}/\mathfrak{m}_s^{n+1}\mathcal{O}_{\mathcal{C},p} \tag{5.2.8}$$

induced by the map sending $x$ and $y$ to the images of $x_n$ and $y_n$. The condition that the map (5.2.8) is an isomorphism is equivalent to $x_n y_n - f_n \in \widehat{\mathfrak{m}}_s^{n+1}\widehat{\mathcal{O}}_{\mathcal{C},p}$.

We will prove this by induction. The base case $n = 0$ is handled by Step 2. Assuming the claim holds for $n$, write

$$x_n y_n - f_n = \sum_i a_i b_i \quad \text{with } a_i \in \widehat{\mathfrak{m}}_s^{n+1} \text{ and } b_i \in \widehat{\mathcal{O}}_{\mathcal{C},p}.$$

Since $x_n$ and $y_n$ generate the maximal ideal of $p$ in the fiber $\mathcal{C}_s$, and since $\kappa(s) = \kappa(p)$, we may find $a'_i \in \widehat{\mathcal{O}}_{S,s}$ and $b'_i, b''_i \in \widehat{\mathcal{O}}_{\mathcal{C},p}$ such that

$$b_i - (x_n b'_i + y_n b''_i + a'_i) \in \widehat{\mathfrak{m}}_s \widehat{\mathcal{O}}_{\mathcal{C},p}.$$

We then define

$$x_{n+1} = x_n - \sum_i a_i b''_i, \quad y_{n+1} = y_n - \sum_i a_i b'_i, \quad f_{n+1} = f_n + \sum_i a_i a'_i,$$

and check that

$$\begin{aligned} x_{n+1}y_{n+1} - f_{n+1} &= (x_n - \sum_i a_i b''_i)(y_n - \sum_i a_i b'_i) - (f_n + \sum_i a_i a'_i) \\ &= (x_n y_n - f_n) - x_n \sum_i a_i b'_i - y_n \sum_i a_i b''_i - \sum_i a_i a'_i + \sum_{i,j} a_i a_j b''_i b'_j \\ &= \sum_i a_i b_i - x_n \sum_i a_i b'_i - y_n \sum_i a_i b''_i - \sum_i a_i a'_i + \sum_{i,j} a_i a_j b''_i b'_j \\ &= \sum_i \underbrace{a_i}_{\widehat{\mathfrak{m}}_s^{n+1}} \underbrace{(b_i - x_n b'_i - y_n b''_i - a'_i)}_{\widehat{\mathfrak{m}}_s \widehat{\mathcal{O}}_{C,p}} + \sum_{i,j} \underbrace{a_i a_j}_{\widehat{\mathfrak{m}}_s^{2(n+1)}} b''_i b'_j \end{aligned}$$

is an element of $\widehat{\mathfrak{m}}_s^{n+2}\widehat{\mathcal{O}}_{C,p}$. Setting $\widehat{x} = \lim_n x_n, \widehat{y} = \lim_n y_n \in \widehat{\mathcal{O}}_{C,p}$, and $\widehat{f} = \lim_n f_n \in \widehat{\mathfrak{m}}_s$, we see that the map $\widehat{\mathcal{O}}_{S,s}[\![x,y]\!]/(xy - \widehat{f}) \to \widehat{\mathcal{O}}_{C,p}$, defined by $x \mapsto \widehat{x}$ and $y \mapsto \widehat{y}$, is an isomorphism.

*Step 4: Construct the desired étale neighborhoods.* Step 3 provides a diagram

$$\begin{tikzcd} \operatorname{Spec} \widehat{\mathcal{O}}_{S,s} \times_S \mathcal{C} \arrow[dr] & & \operatorname{Spec} \widehat{\mathcal{O}}_{S,s}[x,y]/(xy - \widehat{f}) \arrow[dl] \\ & \operatorname{Spec} \widehat{\mathcal{O}}_{S,s}. & \end{tikzcd}$$

such that the points $(s,p) \in \operatorname{Spec} \widehat{\mathcal{O}}_{S,s} \times_S \mathcal{C}$ and $(s,0) \in \operatorname{Spec} \widehat{\mathcal{O}}_{S,s}[x,y]/(xy - \widehat{f})$ have isomorphic completion, where $s$ denotes also the closed point of $\operatorname{Spec} \widehat{\mathcal{O}}_{S,s}$. A consequence of Artin Approximation (Corollary A.10.13) implies that there are étale morphisms

$$\begin{tikzcd} & (U,u) \arrow[dl] \arrow[dr] & \\ (\operatorname{Spec} \widehat{\mathcal{O}}_{S,s} \times_S \mathcal{C}, (s,p)) & & (\operatorname{Spec} \widehat{\mathcal{O}}_{S,s}[x,y]/(xy - \widehat{f}), (s,0)) \end{tikzcd} \tag{5.2.9}$$

defined over $\operatorname{Spec} \widehat{\mathcal{O}}_{S,s}$. After replacing $S$ with an open affine neighborhood of $s$, we can assume that $S = \operatorname{Spec} A$ is affine. By Neron–Popescu (A.10.4), we may write $\widehat{\mathcal{O}}_{S,s} = \operatorname{colim} B_\lambda$ as a directed colimit of smooth $A$-algebras. Set $S_\lambda = \operatorname{Spec} B_\lambda$, $\mathcal{C}_\lambda = \mathcal{C} \times_S S_\lambda$, and $U_\lambda = U \times_S S_\lambda$. For $\lambda \gg 0$, $\widehat{f} \in \widehat{\mathcal{O}}_{S,s}$ is the image of an element $f_\lambda \in B_\lambda$, and the pullbacks of $x$ and $y$ to $\Gamma(U, \mathcal{O}_U)$ are the pullbacks of elements in $\Gamma(U_\lambda, \mathcal{O}_{U_\lambda})$ under $U \to U_\lambda$. This yields a commutative diagram

$$\begin{tikzcd} & U_\lambda \arrow[dl] \arrow[dr] & \\ \mathcal{C}_\lambda \arrow[dr] & & \operatorname{Spec} B_\lambda[x,y]/(xy - f_\lambda) \arrow[dl] \\ & S_\lambda = \operatorname{Spec} B_\lambda & \end{tikzcd}$$

which base changes to (5.2.9) under $\operatorname{Spec} \widehat{\mathcal{O}}_{S,s} \to S_\lambda$. Since étaleness descends under limits (A.6.7), the maps $U_\lambda \to \mathcal{C}_\lambda$ and $U_\lambda \to \operatorname{Spec} B_\lambda[x,y]/(xy - f_\lambda)$ are étale for $\lambda \gg 0$. Letting $u_\lambda = (u, s_\lambda) \in U_\lambda$, we have a commutative diagram

$$\begin{tikzcd} (\mathcal{C}, p) \arrow[d] & (U_\lambda, u_\lambda) \arrow[l, "\text{sm}"'] \arrow[r, "\text{ét}"] \arrow[d] & (\operatorname{Spec} B_\lambda[x,y]/(xy - f_\lambda), (s_\lambda, 0)) \arrow[dl] \\ (S, s) & (\operatorname{Spec} B_\lambda, s_\lambda) \arrow[l, "\text{sm}"'] & \end{tikzcd}$$

This gives our desired diagram (5.2.7) except that the left horizontal arrows are smooth rather than étale. Since smooth maps étale locally have sections (Corollary A.3.6), there is an étale map $(\operatorname{Spec} A, s') \to (S, s)$ and a map $(\operatorname{Spec} A, s') \to (S_\lambda, s_\lambda)$ over $S$. The result follows from setting $\mathcal{C}' := U_\lambda \times_{S_\lambda} \operatorname{Spec} A$ and $p' = (u_\lambda, s')$.

See also [SP, Tag 0CBY]. □

*Proof 2 (avoiding Artin Approximation/Neron–Popescu).* We can reduce to the case where $S$ is of finite type over $\mathbb{Z}$ by Lemma 5.2.17. By Proposition 5.2.16, we may replace $\mathcal{C}$ with an open neighborhood of $p$ such that $\mathcal{C} \to S$ is syntomic and $\operatorname{Sing}(\mathcal{C}/S) \to S$ is unramified. After replacing $\mathcal{C}$ and $S$ with open neighborhoods, we may also assume that $\mathcal{C}$ and $S$ are affine and that the geometric fibers of $\mathcal{C} \to S$ are connected with at most two irreducible components. We may choose an closed immersion $S \hookrightarrow \mathbb{A}^n_{\mathbb{Z}}$ and apply Proposition A.3.16 to find a syntomic morphism $\mathcal{C}' \to \mathbb{A}^n_{\mathbb{Z}}$ extending $\mathcal{C} \to S$. The fiber $\mathcal{C}'_s$ has a node at $p$, and after replacing $\mathcal{C} \to S$ with $\mathcal{C}' \to \mathbb{A}^n_{\mathbb{Z}}$, we may assume that the base $S$ is regular. By the étale local structure of unramified morphisms (Proposition A.3.7), after replacing $\mathcal{C}$ and $S$ with étale neighborhoods, we can arrange that $\mathrm{Sing}(\mathcal{C}/S) \hookrightarrow S$ is a closed immersion.

We claim that after replacing $S$ with an open neighborhood of $s$, we can arrange that $\mathrm{Sing}(\mathcal{C}/S) = S$ or $\mathrm{Sing}(\mathcal{C}/S)$ is defined by a nonzerodivisor $f \in \Gamma(S, \mathcal{O}_S)$. This holds over the completion of $S$ at $s$ by Step 3 in the first proof above: since $\widehat{\mathcal{O}}_{\mathcal{C},p} \cong \widehat{\mathcal{O}}_{S,s}[\![x,y]\!]/(xy - \widehat{f})$ where $\widehat{f} \in \widehat{\mathfrak{m}}_s$, $\mathrm{Sing}(\mathcal{C}/S) \times_S \mathrm{Spec}\, \widehat{\mathcal{O}}_{S,s} = V(\widehat{f})$. The claim then follows from using fppf descent along $\mathrm{Spec}\, \widehat{\mathcal{O}}_{S,s} \to \mathrm{Spec}\, \mathcal{O}_{S,s}$ and properties of the ideal sheaf $\mathcal{I}$ defining $\mathrm{Sing}(\mathcal{C}/S)$. Indeed, if $\widehat{f} = 0$, then $\mathcal{I}_s = 0$ and hence $\mathcal{I}$ is zero in an open neighborhood of $s$. If $\widehat{f}$ is a nonzerodivisor, then $\mathcal{I}_s$ is a line bundle (by Proposition B.4.3) and hence $\mathcal{I}$ is defined by a nonzerodivisor in an open neighborhood of $s$.

If $\mathrm{Sing}(\mathcal{C}/S) = S$, we first claim that after replacing $\mathcal{C}$ with an étale neighborhood, we can arrange that $\mathcal{C}$ is the scheme-theoretic union $\mathcal{C}_1 \cup \mathcal{C}_1$ of closed subschemes such that $\mathrm{Sing}(\mathcal{C}/S) = \mathcal{C}_1 \cap \mathcal{C}_2$. The normalization $\widetilde{Z} \to \mathrm{Spec}\, \mathcal{O}^{\mathrm{h}}_{\mathcal{C},p}$ of the henselization is a finite morphism, and since normalization commutes with completion (see Remark B.4.5), there are two preimages in $\widetilde{Z}$ of the unique closed point. By properties of the henselization (Proposition A.9.6), $\widetilde{Z}$ is the disjoint union $\widetilde{Z} = \widetilde{Z}_1 \amalg \widetilde{Z}_2$. Therefore $\mathrm{Spec}\, \mathcal{O}^h_{\mathcal{C},p}$ is the union of the (closed) images of $\widetilde{Z}_1$ and $\widetilde{Z}_2$. This establishes the claim. After replacing $\mathcal{C}$ with an open neighborhood, we can arrange that $\mathcal{C}_1$ and $\mathcal{C}_2$ are defined by global functions $g_1, g_2 \in B := \Gamma(\mathcal{C}, \mathcal{O}_{\mathcal{C}})$ on $\mathcal{C}$ with $g_1 g_2 = 0$. Letting $S = \mathrm{Spec}\, A$, the ring map $A[x,y]/(xy) \to B$, defined by $x \mapsto g_1$ and $y \mapsto g_2$, induces a morphism $\mathcal{C} \to \mathrm{Spec}\, A[x,y]/(xy)$ over $S$. This map is étale at $p$ since it induces an isomorphism of completions at $p$.

If $\mathrm{Sing}(\mathcal{C}/S) = V(f)$ with $f \in A := \Gamma(S, \mathcal{O}_S)$, then the argument above shows that $\mathcal{C} \times_A (A/f)$ is the scheme-theoretic union $Z_1 \cup Z_1$ of effective Cartier divisors such that $\mathrm{Sing}(\mathcal{C}/S) = Z_1 \cap Z_2$. After replacing $\mathcal{C}$ with an open neighborhood, we can write each $Z_i = V(g_i)$ for global functions $g_i \in B := \Gamma(\mathcal{C}, \mathcal{O}_{\mathcal{C}})$. As the restrictions of $g_1 g_2$ and $f$ define the same closed subscheme of $\mathrm{Spec}\, \widehat{\mathcal{O}}_{\mathcal{C},p}$, we have that $f = u g_1 g_2$ for a unit $u \in B$ after replacing $\mathcal{C}$ with an open neighborhood. The ring map $A[x,y]/(xy - f) \to B$, defined by $x \mapsto u g_1$ and $y \mapsto g_2$, induces a morphism $\mathcal{C} \to \mathrm{Spec}\, A[x,y]/(xy)$ over $S$; this map is étale at $p$ since it induces an isomorphism of completions at $p$. □

One direct consequence of this local structure theorem is that if $\mathcal{C} \to S$ is an fppf morphism such every fiber is a pure dimension one curve, then the locus $\mathcal{C}^{\leq \mathrm{nod}} \subset \mathcal{C}$ of points which are smooth or nodal is open. And if we add a properness condition on $\mathcal{C} \to S$, then $\pi(\mathcal{C} \setminus \mathcal{C}^{\leq \mathrm{nod}}) \subset S$ is closed and therefore the locus of points $s \in S$ such that $\mathcal{C}_s$ is a nodal curve is the *open* subscheme $S \setminus \pi(\mathcal{C} \setminus \mathcal{C}^{\leq \mathrm{nod}})$. This will be applied later to conclude that the stack parameterizing families of nodal curves is an open substack of the stack of all curves.

**Corollary 5.2.20.** *If $\mathcal{C} \to S$ is a proper fppf morphism of schemes such that every geometric fiber is a curve, then the locus of points $s \in S$ such that $\mathcal{C}_s$ is nodal is open.* □

## 5.3 Stable curves

Stable curves were introduced in unpublished joint work by Mayer and Mumford [MM64].

### 5.3.1 Definition and equivalences

An *$n$-pointed curve* is a curve $C$ over a field $\Bbbk$ together with an ordered collection of $\Bbbk$-points $p_1, \ldots, p_n \in C$; we call the $p_i \in C$ *marked points*. A point $q \in C$ of an $n$-pointed curve is called *special* if $q$ is a node or a marked point.

**Definition 5.3.1** (Stable, semistable, and prestable curves)**.** An $n$-pointed geometrically connected, nodal, and projective curve $(C, p_1, \ldots, p_n)$ of genus $g$ over a field $\Bbbk$ is *stable* if

(1) $p_1, \ldots, p_n \in C$ are distinct smooth points,

(2) $C$ is not of genus 1 without marked points, i.e. $(g, n) \neq (1, 0)$, and

(3) every smooth rational subcurve of $C$ contains at least 3 special points.

If (1)-(2) hold, and (3) is replaced with the condition that every smooth rational subcurve contains at least 2 (rather than 3) special points, we say that $(C, p_i)$ is *semistable*. If only (1)-(2) hold, we say that $(C, p_i)$ is *prestable*.

We have the implications:

$$\boxed{\text{stable} \implies \text{semistable} \implies \text{prestable} \implies n\text{-pointed nodal.}}$$

In the unpointed case, a curve is a prestable curve if it is nodal.

**Remark 5.3.2.** Note that there are no $n$-pointed stable curves of genus $g$ if $(g, n) \in \{(0, 0), (0, 1), (0, 2), (1, 0)\}$ or equivalently $2g - 2 + n \leq 0$. We often impose the condition that $2g - 2 + n > 0$ to exclude these special cases.

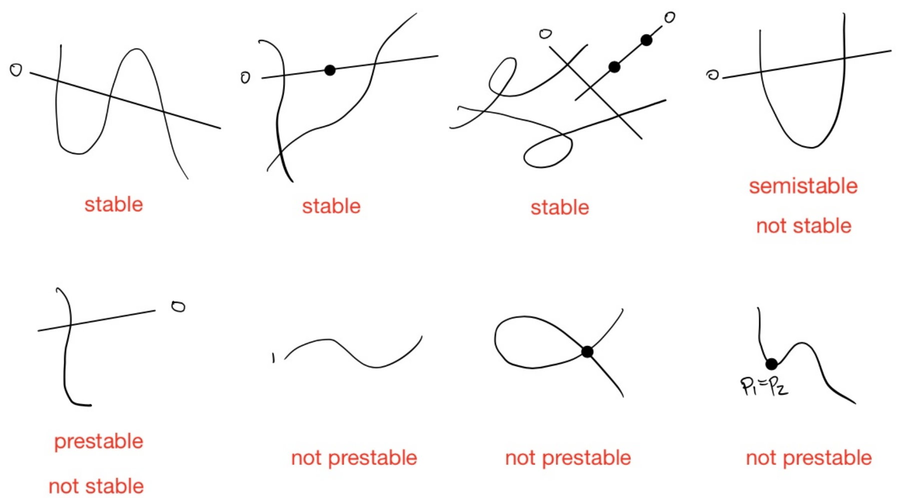



Figure 5.3: Examples of stable, semistable, and prestable curves

An automorphism of a stable curve $(C, p_1, \ldots, p_n)$ is an automorphism $\alpha \colon C \xrightarrow{\sim} C$ such that $\alpha(p_i) = p_i$. By $\mathrm{Aut}(C, p_1, \ldots, p_n)$ we denote the (abstract) group of automorphisms. Recall also that if $C$ is a geometrically connected, smooth, and projective curve of genus $g \geq 2$, then $\mathrm{Aut}(C)$ is finite [Har77, Exer. III.2.5].

**Proposition 5.3.3.** *Let $(C, p_1, \ldots, p_n)$ be an $n$-pointed prestable curve. The following are equivalent:*

*(1)* $(C, p_1, \ldots, p_n)$ *is stable,*

*(2)* $\mathrm{Aut}(C, p_1, \ldots, p_n)$ *is finite, and*

*(3)* $\omega_C(p_1 + \cdots + p_n)$ *is ample.*

*Proof.* The equivalence (1) $\Longleftrightarrow$ (2) follows from Exercise 5.3.4 and the observation that the only way a smooth prestable $n$-pointed curves $(C, p_i)$ can have a positive dimensional automorphism group is if $C = \mathbb{P}^1$ with $n \leq 2$ or if $C$ is a genus 1 curve with $n = 0$.

To see the equivalence with (3), we will use the fact that for a subcurve $T \subset C$, we have $\omega_C|_T = \omega_T(T \cap T^c)$ (Exercise 5.2.14). The line bundle $\omega_C(p_1 + \cdots + p_n)$ is ample if and only if its restriction to each irreducible component $T \subset C$

$$\omega_C(p_1 + \cdots + p_n)|_T = \omega_T\Big(\sum_{p_i \in T} p_i + (T \cap T^c)\Big) \tag{5.3.1}$$

is ample. If the genus $g(T)$ of $T$ is at least two, then $\omega_T$ is ample and thus so is (5.3.1). If $g(T) = 1$, then (5.3.1) is ample if and only if $n \geq 1$ or $T$ must meet the complement $T^C$. If $g(T) = 0$, then (5.3.1) is ample if and only if $T$ contains at least three special points. $\square$

**Exercise 5.3.4.** Let $(C, p_1, \ldots, p_n)$ be an $n$-pointed prestable curve. Let $\pi \colon \widetilde{C} \to C$ be the normalization of $C$, $\widetilde{p}_i \in \widetilde{C}$ be the unique preimage of $p_i$, and $\widetilde{q}_1, \ldots, \widetilde{q}_m \in \widetilde{C}$ be an ordering of the preimages of nodes.

(a) Show that $(C, p_i)$ is stable if and only if every connected component of $(\widetilde{C}, \widetilde{p}_i, \widetilde{q}_j)$ is stable.

(b) Show that the automorphism group scheme $\underline{\mathrm{Aut}}(C, p_i)$ is an algebraic group.

(c) Show that $\underline{\mathrm{Aut}}(C, p_i)$ is naturally a closed subgroup of $\underline{\mathrm{Aut}}(\widetilde{C}, \widetilde{p}_i, \widetilde{q}_j)$ with the same connected component of the identity, i.e. $\underline{\mathrm{Aut}}(C, p_i)^0 = \underline{\mathrm{Aut}}(\widetilde{C}, \widetilde{p}_i, \widetilde{q}_j)^0$.

(d) Provide an example where $\underline{\mathrm{Aut}}(C, p_i) \neq \underline{\mathrm{Aut}}(\widetilde{C}, \widetilde{p}_i, \widetilde{q}_j)$.

**Remark 5.3.5** (Dual graphs)**.** To add!

### 5.3.2 Positivity of $\omega_C$

**Exercise 5.3.6.** If $(C, p_1, \ldots, p_n)$ is an $n$-pointed prestable curve of genus $g$, and let $L := \omega_C(p_1 + \cdots + p_n)$.

(a) If $(C, p_i)$ is semistable, show that $L^{\otimes k}$ is base point free for $k \geq 2$,

(b) If $(C, p_i)$ is stable, show that $L^{\otimes k}$ is very ample for $k \geq 3$ and that $\mathrm{H}^1(C, (\omega_C(p_1 + \cdots + p_n))^{\otimes k}) = 0$ for $k \geq 2$.

*Hint: For (b), show that the global sections of $L^{\otimes k}$ separate points and tangent vectors. In other words, show that the maps*

$$H^0(C, L^{\otimes k}) \to \big(L^{\otimes k} \otimes \kappa(x)\big) \oplus \big(L^{\otimes k} \otimes \kappa(y)\big) \qquad H^0(C, L^{\otimes k}) \to L^{\otimes k} \otimes \mathcal{O}_{C,x}/\mathfrak{m}_x^2$$

*are surjective. Establish this using Serre Duality and a case analysis on whether $x, y$ are smooth or nodal. See also [DM69, Thm. 2], [ACG11, Lem. 10.6.1], [SP, Tag 0E8X], and [Ols16, Prop. 13.2.17].*

**Exercise 5.3.7.** If $C$ is the nodal union $C_1 \cup C_2$ of genus $i$ and $g - i$ curves along a single node $p = C_1 \cap C_2$, show that $\omega_C$ has a base point at $p$.

### 5.3.3 Families of stable curves

**Definition 5.3.8.**

(1) A *family of $n$-pointed nodal curves* is a proper, flat, and finitely presented morphism $\mathcal{C} \to S$ of schemes with $n$ sections $\sigma_1, \dots, \sigma_n \colon S \to \mathcal{C}$ such that every geometric fiber $\mathcal{C}_s$ is a (reduced) connected nodal curve.

(2) A *family of $n$-pointed stable curves* (resp. *semistable curves, prestable curves*) is a family $\mathcal{C} \to S$ of $n$-pointed nodal curves such that every geometric fiber $(\mathcal{C}_s, \sigma_1(s), \dots, \sigma_n(s))$ is stable (resp. semistable, prestable).

In a family of $n$-pointed nodal curves, marked points may lie at the nodes; this is not the case for prestable (and thus also for semistable and stable) curves.

If $\mathcal{C} \to S$ is a family of prestable curves, then $\mathcal{C} \to S$ is locally a complete intersection morphism. Thus there is a relative dualizing line bundle $\omega_{\mathcal{C}/S}$ that is compatible with base change $T \to S$ and in particular restricts to the dualizing line bundle $\omega_{\mathcal{C}_s}$ on every fiber of $\mathcal{C} \to S$; see [Har66c] or [Liu02, §6.4]. Note also that since the geometric fibers are stable curves, the image of each $\sigma_i$ is a divisor contained in the smooth locus and we can form the line bundle $\omega_{\mathcal{C}/S}(\sigma_1 + \cdots \sigma_n)$.

We have the following generalization of Proposition 5.1.9, which is proven in the same way but using the very ampleness of third tensor power of $\omega_C(p_1 + \cdots + p_n)$ in Exercise 5.3.6.

**Proposition 5.3.9** (Properties of Families of Stable Curves)**.** *Let $(\mathcal{C} \to S, \sigma_i)$ be a family of $n$-pointed stable curves of genus $g$, and set $L := \omega_{\mathcal{C}/S}(\sum_i \sigma_i)$. If $k \geq 3$, then $L^{\otimes k}$ is relatively very ample and $\pi_* L^{\otimes k}$ is a vector bundle of rank $(2k-1)(g-1) + kn$.* □

In particular, stable $n$-pointed families are projective morphisms.

**Proposition 5.3.10** (Openness of Stability)**.** *Let $(\mathcal{C} \to S, \sigma_i)$ be a family of $n$-pointed nodal curves. The locus of points $s \in S$ such that $(\mathcal{C}_s, \sigma_i(s))$ is stable is open.*

*Proof.* The locus in $S$ where $\sigma_1(s), \dots, \sigma_n(s)$ are distinct and smooth is open. We may thus assume that $(\mathcal{C} \to S, \sigma_i)$ is a family of prestable $n$-pointed curves.

Argument 1: since $\underline{\mathrm{Aut}}(\mathcal{C}/S, \sigma_1, \dots, \sigma_n) \to S$ is a finite type group scheme, upper semicontinuity of fiber dimension (C.1.8) implies that the locus of points $s \in S$ such that $\mathrm{Aut}(\mathcal{C}_s, \sigma_1(s), \dots, \sigma_n(s))$ is finite is open. By the equivalence Proposition 5.3.3(2), this open subset is identified with the stable locus.

Argument 2: the locus of points $s \in S$ such that $\omega_{\mathcal{C}/S}(\sum_i \sigma_i)|_{\mathcal{C}_s} \cong \omega_{\mathcal{C}_s}(\sum_i \sigma_i(s))$ is ample is open (Proposition E.2.1). By the equivalence Proposition 5.3.3(3), this open subset is identified with the stable locus. □

### 5.3.4 Deformation theory of stable curves

If $C$ is a smooth curve over a field $\Bbbk$, then every first-order deformation is locally trivial (Proposition D.1.8) and the set $\mathrm{Def}(C)$ of isomorphism classes of first-order deformations is naturally in bijection with $\mathrm{H}^1(C, T_C)$ (Proposition D.1.11). Moreover, automorphisms, deformations, and obstructions of higher-order deformations are classified by $\mathrm{H}^i(C, T_C)$ for $i = 0, 1, 2$ (Proposition D.2.6).

Nodal singularities on the other hand have first-order deformations that are not locally trivial, e.g. $\operatorname{Spec} \Bbbk[x, y, \epsilon]/(xy - \epsilon) \to \operatorname{Spec} \Bbbk[\epsilon]$ is non-locally trivial deformation of $C = \operatorname{Spec} \Bbbk[x,y]/(xy)$. In this section, we classify automorphisms, deformations, and obstructions of nodal curves (Proposition 5.3.11), describe first-order deformations of a nodal curve in terms of the pointed normalization and the singularities (Proposition 5.3.13), and then compute the dimensions of the groups classifying automorphisms, deformations, and obstructions of a stable curve (Proposition 5.3.14).

**Proposition 5.3.11.** *Let $A' \twoheadrightarrow A$ be a surjection of artinian local $\Bbbk$-algebras with residue field $\Bbbk$. Suppose that $J = \ker(A' \to A)$ satisfies $\mathfrak{m}_{A'} J = 0$. Let $(\mathcal{C} \to \operatorname{Spec} A, \sigma_i)$ be a family of prestable curves over $A$ and let $(C, p_i)$ be its base change to the residue field.*

*(1) The group of automorphisms of a deformation of $(\mathcal{C} \to \operatorname{Spec} A, \sigma_i)$ over $A'$ is bijective to $\operatorname{Ext}^0_{\mathcal{O}_C}(\Omega_C(\sum_i p_i), \mathcal{O}_C \otimes_{\Bbbk} J)$.*

*(2) If there exists a deformation of $(\mathcal{C} \to \operatorname{Spec} A, \sigma_i)$ over $A'$, then the set of isomorphism classes of all such deformations is a torsor under $\operatorname{Ext}^1_{\mathcal{O}_C}(\Omega_C(\sum_i p_i), \mathcal{O}_C \otimes_{\Bbbk} J)$.*

*(3) There is an element $\operatorname{ob}_{\mathcal{C}} \in \operatorname{Ext}^2_{\mathcal{O}_C}(\Omega_C(\sum_i p_i), \mathcal{O}_C \otimes_{\Bbbk} J)$ with the property that there exists a deformation of $(\mathcal{C} \to \operatorname{Spec} A, \sigma_i)$ over $A'$ if and only if $\operatorname{ob}_{\mathcal{C}} = 0$.*

*Proof.* Since nodal curves are generically smooth and local complete intersections, the unpointed case follows from Proposition D.2.11. We leave the generalization to $n$-pointed prestable curves to the reader. □

**Lemma 5.3.12.** *Let $(C, p_i)$ be a prestable curve over a field $\Bbbk$. Let $q_1, \ldots, q_s \in C$ be the nodes of $C$. Let $(\widetilde{C}, p_i, q'_j, q''_j)$ be the pointed normalization where $\pi \colon \widetilde{C} \to C$ is the normalization and $\pi^{-1}(q_j) = \{q'_j, q''_j\}$. There is a convergent spectral sequence*

$$H^p(C, \mathscr{E}xt^q_{\mathcal{O}_C}(\Omega_C(\textstyle\sum_i p_i), \mathcal{O}_C)) \Rightarrow \operatorname{Ext}^{p+q}_{\mathcal{O}_C}(\Omega_C(\textstyle\sum_i p_i), \mathcal{O}_C).$$

*such that the induced exact sequence of low-degree terms is identified with*

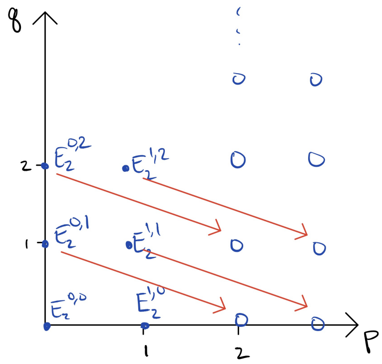
$$0 \to \mathrm{H}^1(C, \mathscr{H}om_{\mathcal{O}_C}(\Omega_C(\textstyle\sum_i p_i), \mathcal{O}_C)) \to \operatorname{Ext}^1_{\mathcal{O}_C}(\Omega_C(\textstyle\sum_i p_i), \mathcal{O}_C) \to \bigoplus_j \operatorname{Ext}^1_{\widehat{\mathcal{O}}_{C,q_j}}(\Omega_{\widehat{\mathcal{O}}_{C,q_j}}, \widehat{\mathcal{O}}_{C,q_j}) \to 0. \quad (5.3.2)$$

*Moreover, $\operatorname{Ext}^1_{\widehat{\mathcal{O}}_{C,q_j}}(\Omega_{\widehat{\mathcal{O}}_{C,q_j}}, \widehat{\mathcal{O}}_{C,q_j}) = \Bbbk$ for each $i$ and $\operatorname{Ext}^2_{\mathcal{O}_C}(\Omega_C(\sum_i p_i), \mathcal{O}_C) = 0$.*

*Proof.* For simplicity, we handle only the case without marked points, i.e. $n = 0$. As $\operatorname{Hom}_{\mathcal{O}_C}(\Omega_C, -)$ is the composition $\Gamma \circ \mathscr{H}om_{\mathcal{O}_C}(\Omega_C, -)$ of left exact functors, there is a Grothendieck spectral sequence with $E_2$-page

$$E_2^{p,q} = \mathrm{H}^p(C, \mathscr{E}xt^q_{\mathcal{O}_C}(\Omega_C, \mathcal{O}_C))$$

which converges to $\operatorname{Ext}^{p+q}_{\mathcal{O}_C}(\Omega_C, \mathcal{O}_C)$ (c.f [Wei94, Thm. 5.8.3]). Since $\dim C = 1$, we have that $E_2^{p,q} = 0$ if $p \geq 2$. We can thus draw the $E_2$ page as:




The associated exact sequence of low-degree terms is

$$0 \to E_2^{1,0} \to \operatorname{Ext}^1_{\mathcal{O}_C}(\Omega_C, \mathcal{O}_C) \to E_2^{0,1} \to E_2^{2,0} = 0.$$

As $\Omega_C$ is locally free away from the nodes, $\mathscr{E}xt^1_{\mathcal{O}_C}(\Omega_C, \mathcal{O}_C)$ is a zero-dimensional sheaf supported only at the nodes of $C$. This shows that $E_2^{1,1} = \mathrm{H}^1(C, \mathscr{E}xt^1_{\mathcal{O}_C}(\Omega_C, \mathcal{O}_C)) = 0$ and

$$E_2^{0,1} = \mathrm{H}^0(C, \mathscr{E}xt^1_{\mathcal{O}_C}(\Omega_C, \mathcal{O}_C)) = \bigoplus_j \operatorname{Ext}^1_{\mathcal{O}_{C,q_j}}(\Omega_{C,q_j}, \mathcal{O}_{C,q_j}) = \bigoplus_j \operatorname{Ext}^1_{\widehat{\mathcal{O}}_{C,q_j}}(\Omega_{\widehat{\mathcal{O}}_{C,q_j}}, \widehat{\mathcal{O}}_{C,q_j})$$

where we've used that $\widehat{\Omega}_{C,q_j} \cong \Omega_{\widehat{\mathcal{O}}_{C,q_j}}$. This gives the exact sequence (5.3.2).

Similarly, $\mathscr{E}xt^2_{\mathcal{O}_C}(\Omega_C, \mathcal{O}_C)$ is a zero-dimensional sheaf supported only at the nodes, and we have identifications

$$E_2^{0,2} = \mathrm{H}^0(C, \mathscr{E}xt^2_{\mathcal{O}_C}(\Omega_C, \mathcal{O}_C)) = \bigoplus_j \operatorname{Ext}^2_{\widehat{\mathcal{O}}_{C,q_j}}(\Omega_{\widehat{\mathcal{O}}_{C,q_j}}, \widehat{\mathcal{O}}_{C,q_j})$$

Write $\widehat{\mathcal{O}}_{C,q_j} = \Bbbk[\![x,y]\!]/(xy)$ and consider the locally free resolution

$$0 \to \widehat{\mathcal{O}}_{C,q_j} \xrightarrow{\binom{y}{x}} \widehat{\mathcal{O}}_{C,q_j}^{\oplus 2} \xrightarrow{(dx,dy)} \Omega_{\widehat{\mathcal{O}}_{C,q_j}} \to 0.$$

This allows us to compute that $\operatorname{Ext}^1_{\widehat{\mathcal{O}}_{C,q_j}}(\Omega_{\widehat{\mathcal{O}}_{C,q_j}}, \widehat{\mathcal{O}}_{C,q_j}) = \operatorname{coker}(\widehat{\mathcal{O}}_{C,q_j}^{\oplus 2} \xrightarrow{(x,y)} \widehat{\mathcal{O}}_{C,q_j}) = \Bbbk$ and $\operatorname{Ext}^2_{\widehat{\mathcal{O}}_{C,q_j}}(\Omega_{\widehat{\mathcal{O}}_{C,q_j}}, \widehat{\mathcal{O}}_{C,q_j}) = 0$. As $E_2^{0,2} = E_2^{1,1} = E_2^{2,0} = 0$, we have $\operatorname{Ext}^2_{\mathcal{O}_C}(\Omega_C, \mathcal{O}_C) = 0$. $\square$

**Proposition 5.3.13.** *Let $(C, p_i)$ be an $n$-pointed prestable curve. Let $q_1, \ldots, q_s \in C$ be the nodes of $C$. Let $(\widetilde{C}, p_i, q_j', q_j'')$ be the pointed normalization where $\pi \colon \widetilde{C} \to C$ is the normalization and $\pi^{-1}(q_j) = \{q_j', q_j''\}$. There is an exact sequence*

$$0 \to \operatorname{Def}^{\mathrm{lt}}(C, p_i) \to \operatorname{Def}(C, p_i) \to \bigoplus_j \operatorname{Def}(\widehat{\mathcal{O}}_{C,q_j}) \to 0 \tag{5.3.3}$$

*and identifications*

$$\begin{aligned} \operatorname{Def}^{\mathrm{lt}}(C, p_i) &\cong \operatorname{Def}(\widetilde{C}, p_i, q_j', q_j'') \cong \mathrm{H}^1(\widetilde{C}, T_{\widetilde{C}}(-\textstyle\sum_i p_i - \sum_j (q_j' + q_j''))) \\ \operatorname{Def}(C, p_i) &\cong \operatorname{Ext}^1_{\mathcal{O}_C}(\Omega_C(\textstyle\sum_i p_i), \mathcal{O}_C) \\ \operatorname{Def}(\widehat{\mathcal{O}}_{C,q_j}) &\cong \operatorname{Ext}^1_{\widehat{\mathcal{O}}_{C,q_j}}(\Omega^1_{\widehat{\mathcal{O}}_{C,q_j}}, \widehat{\mathcal{O}}_{C,q_j}) \cong \Bbbk. \end{aligned}$$

*Under these identifications, (5.3.3) corresponds to short exact sequence (5.3.2).*

*Proof.* For simplicity, we handle again the case without marked points. The identification $\mathrm{Def}^{\mathrm{lt}}(C) \cong \mathrm{H}^1(C, T_C)$ was given in Proposition D.1.11. If $C' \to \operatorname{Spec} \Bbbk[\epsilon]$ is a locally trivial first-order deformation of $C$, each node $q_j \colon \operatorname{Spec} \Bbbk \to C$ extends to a section $\widetilde{q}_j \colon \operatorname{Spec} \Bbbk[\epsilon] \to C$ whose image is contained in the relative singular locus of $C$ over $\Bbbk[\epsilon]$. The pointed normalization of $C$ along the sections $\widetilde{q}_j$ is a first-order deformation of the (possibly disconnected) pointed normalization $(\widetilde{C}, q'_j, q''_j)$. This gives a map $\mathrm{Def}^{\mathrm{lt}}(C) \to \mathrm{Def}(\widetilde{C}, q'_j, q''_j)$. The inverse is provided by gluing the sections of a first-order deformation $(\widetilde{C}', \widetilde{\sigma}'_j, \widetilde{\sigma}''_j)$ of $(\widetilde{C}, q'_j, q''_j)$ along nodes; more precisely, the deformation $C'$ is obtained as the pushout (see Theorem A.8.1)

$$
\begin{tikzcd}
\coprod_j (\operatorname{Spec} \Bbbk[\epsilon] \amalg \operatorname{Spec} \Bbbk[\epsilon]) \arrow[r, "q'_j \amalg q''_j"] \arrow[d] & \widetilde{C}' \arrow[d] \\
\coprod_j \operatorname{Spec} \Bbbk[\epsilon] \arrow[r] & C'.
\end{tikzcd}
$$

For the second bijection, if $C' \to \operatorname{Spec} \Bbbk[\epsilon]$ is a first-order deformation, then the ideal sheaf $I$ defining $C \hookrightarrow C'$ is the pullback of the ideal $(\epsilon) \subset \Bbbk[\epsilon]$. Since $(\epsilon) \cong \Bbbk$ as $\Bbbk[\epsilon]$-modules, we see that $I/I^2 = I \cong \mathcal{O}_C$. The right exact sequence

$$
I/I^2 \to \Omega^1_{C'} \to \Omega^1_C \to 0 \tag{5.3.4}
$$

is left exact at every smooth point of $C$. Since $C$ is generically smooth, the map $\mathcal{O}_C \cong I/I^2 \to \Omega^1_{C'}$ is generically injective, and since $\mathcal{O}_C$ is a line bundle, the map is in fact injective. The sequence (5.3.4) is therefore exact and defines an extension class in $\operatorname{Ext}^1_{\mathcal{O}_C}(\Omega_C, \mathcal{O}_C)$ (see [Har77, Exer. III.6.1]). The reader is left to verify that this defines a bijection. A similar argument gives the bijection $\mathrm{Def}(\widehat{\mathcal{O}}_{C,q_j}) \cong \operatorname{Ext}^1_{\widehat{\mathcal{O}}_{C,q_j}}(\Omega^1_{\widehat{\mathcal{O}}_{C,q_j}}, \widehat{\mathcal{O}}_{C,q_j})$, and Lemma 5.3.12 gives the identification with $\Bbbk$.

See also [DM69, Prop. 1.5] and [ACG11, §11.3]. □

Recall that automorphisms, deformations, and obstructions of a nodal curve $(C, p_i)$ are classified by $\operatorname{Ext}^i_{\mathcal{O}_C}(\Omega_C(\sum_i p_i), \mathcal{O}_C)$ for $i = 0, 1, 2$ (Proposition 5.3.11).

**Proposition 5.3.14.** *Let $(C, p_1, \ldots, p_n)$ be an $n$-pointed stable curve of genus $g$ over $\Bbbk$. Then*

$$
\dim_{\Bbbk} \operatorname{Ext}^i_{\mathcal{O}_C}(\Omega_C(\textstyle\sum_i p_i), \mathcal{O}_C) = \left\{ \begin{array}{rl} 0 & \text{if } i = 0, 2 \\ 3g - 3 + n & \text{if } i = 1 \end{array} \right.
$$

*Proof.* We may assume $\Bbbk = \overline{\Bbbk}$, and for simplicity we handle the case that there are no marked points. Let $\pi \colon \widetilde{C} \to C$ be the normalization, $\Sigma \subset C$ be the set of nodes of $C$, and $\widetilde{\Sigma} = \pi^{-1}(\Sigma)$. The vanishing of $\operatorname{Ext}^2$ was established in Lemma 5.3.12. For $\operatorname{Ext}^0$, we will use the identification $\operatorname{Hom}_{\mathcal{O}_{\widetilde{C}}}(\Omega_{\widetilde{C}}(\widetilde{\Sigma}), \mathcal{O}_{\widetilde{C}}) \cong \operatorname{Hom}_{\mathcal{O}_C}(\Omega_C, \mathcal{O}_C)$ of Exercise 5.2.15. Since the pointed normalization $(\widetilde{C}, \widetilde{\Sigma})$ is smooth and each connected component is stable (Exercise 5.3.4), the degree of the restriction of $T_{\widetilde{C}}(-\widetilde{\Sigma})$ to each connected component of $\widetilde{C}$ is strictly negative. Thus, $\operatorname{Hom}_{\mathcal{O}_{\widetilde{C}}}(\Omega_{\widetilde{C}}(\widetilde{\Sigma}), \mathcal{O}_{\widetilde{C}}) = H^0(\widetilde{C}, T_{\widetilde{C}}(-\widetilde{\Sigma})) = 0$.

To compute $\operatorname{Ext}^1$, Proposition 5.3.13 implies that there is an exact sequence

$$
0 \to \mathrm{H}^1(\widetilde{C}, T_C(-\widetilde{\Sigma}) \to \operatorname{Ext}^1_{\mathcal{O}_C}(\Omega_C, \mathcal{O}_C) \to \bigoplus_j \operatorname{Ext}^1_{\widehat{\mathcal{O}}_{C,q_j}}(\Omega_{\widehat{\mathcal{O}}_{C,q_j}}, \widehat{\mathcal{O}}_{C,q_j}) \to 0
$$

and the identification $\operatorname{Ext}^1_{\widehat{\mathcal{O}}_{C,q_j}}(\Omega_{\widehat{\mathcal{O}}_{C,q_j}}, \widehat{\mathcal{O}}_{C,q_j}) = \Bbbk$. We write $\widetilde{C} = \coprod_{i=1}^{\nu} \widetilde{C}_i$ as a union of its connected components and define $\widetilde{\Sigma}_i = \widetilde{C}_i \cap \widetilde{\Sigma}$. Using that $\Omega_{\widetilde{C}_i}$ is a line bundle, we compute using Serre Duality and Riemann–Roch that

$$\mathrm{h}^1(\widetilde{C}_i, T_{\widetilde{C}_i}(-\widetilde{\Sigma}_i)) = \mathrm{h}^0(\widetilde{C}_i, \Omega^{\otimes 2}_{\widetilde{C}_i}(\widetilde{\Sigma}_i)) = 3g(\widetilde{C}_i) - 3 + |\widetilde{\Sigma}_i|.$$

Thus

$$\begin{aligned}
\dim_{\Bbbk} \operatorname{Ext}^1_{\mathcal{O}_C}(\Omega_C, \mathcal{O}_C) &= \mathrm{h}^1(\widetilde{C}, T_{\widetilde{C}}(-\widetilde{\Sigma})) + |\Sigma| \\
&= \sum_{i=1}^{\nu} \big(3g(\widetilde{C}_i) - 3 + |\widetilde{\Sigma}_i|\big) + |\Sigma| \\
&= 3\Big(\sum_{i=1}^{\nu} g(\widetilde{C}_i) - \nu + |\Sigma|\Big) \\
&= 3g - 3
\end{aligned},$$

where we've used the Genus Formula (5.2.9) $g = \sum_{i=1}^{\nu} g(\widetilde{C}_i) - \nu + |\Sigma| + 1$. □

**Remark 5.3.15** (Consequences of deformation theory)**.** In Theorem 5.4.8, we will use deformation theory to argue that $\overline{\mathcal{M}}_{g,n}$ is a smooth Deligne–Mumford stack of dimension $3g - 3 + n$. Here's the central idea:

- $\operatorname{Ext}^0$: We've already seen that a stable curve $(C, p_i)$ has finitely many automorphism (Proposition 5.3.3). The vanishing of $\operatorname{Ext}^0$ implies that an $n$-pointed stable curve $(C, p_i)$ has no infinitesimal automorphisms, i.e. that the Lie algebra of the automorphism group scheme $\underline{\operatorname{Aut}}(C, p_i)$ is trivial. Since $\underline{\operatorname{Aut}}(C, p_i)$ is of finite type, it must be finite and discrete. Once we know that the algebraicity of the stack $\overline{\mathcal{M}}_{g,n}$, we use the Characterization of Deligne–Mumford Stacks (3.6.4) to conclude that $\overline{\mathcal{M}}_{g,n}$ is Deligne–Mumford.
- $\operatorname{Ext}^1$: Since $\operatorname{Ext}^1$ parametrizes isomorphism classes of deformations of $(C, p_i)$, it is identified with the Zariski tangent space of $\overline{\mathcal{M}}_{g,n}$ at the point corresponding to $(C, p_i)$. The computation of $\operatorname{Ext}^1$ therefore implies that $\overline{\mathcal{M}}_{g,n}$ has relative dimension $3g - 3 + n$ over $\operatorname{Spec} \mathbb{Z}$.
- $\operatorname{Ext}^2$: The vanishing of $\operatorname{Ext}^2$ implies that there are no obstructions to deforming $C$. The Infinitesimal Lifting Criterion (Theorem 3.7.1) implies that $\overline{\mathcal{M}}_{g,n}$ is smooth over $\operatorname{Spec} \mathbb{Z}$.

### 5.3.5 Stabilization of rational tails and bridges

**Definition 5.3.16** (Rational tails and bridges)**.** Let $(C, p_1, \ldots, p_n)$ be an $n$-pointed prestable curve over an algebraically closed field $\Bbbk$. We say that a smooth genus 0 subcurve $E \subset C$ is

- a *rational tail* if $E \cap E^c = 1$ (where $E^c = \overline{C \setminus E}$), and $E$ contains no marked points;
- a *rational bridge* if either $E \cap E^c = 2$ and $E$ contains no marked points, or $E \cap E^c = 1$ and $E$ contains one marked point.

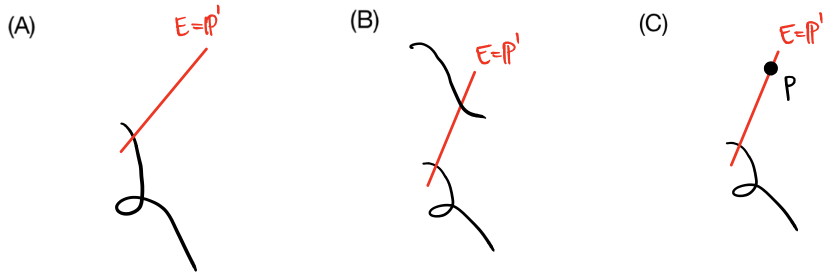



Figure 5.4: (A) features a rational tail while (B) and (C) feature rational bridges.

From the definition of stability (Definition 5.3.1), we see that if $(C, p_1, \ldots, p_n)$ is not stable and $(g, n) \neq (1, 0)$, then $C$ necessarily contains a rational tail or bridge. Note that $C$ can also contain a chain of rational tails or bridges of arbitrary length.

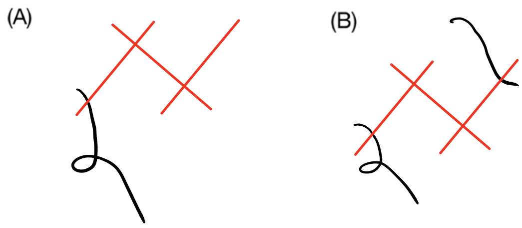



Figure 5.5: Examples of chains of rational tails and bridges

Suppose that $\mathcal{C} \to \Delta = \operatorname{Spec} R$ is a family of nodal curves over a DVR $R$ with algebraically closed residue field $\Bbbk$ such that the generic fiber $\mathcal{C}^*$ is smooth. If $E \cong \mathbb{P}^1 \subset \mathcal{C}'_0$ is a smooth rational subcurve in the central fiber, then $E^2 = -E \cdot E^c$; indeed this follows from $0 = E \cdot \mathcal{C}_0 = E \cdot E + E \cdot E^c$. Thus if $E$ is a rational tail (resp. rational bridge without a marked point), then $E^2 = -1$ (resp. $E^2 = -2$).

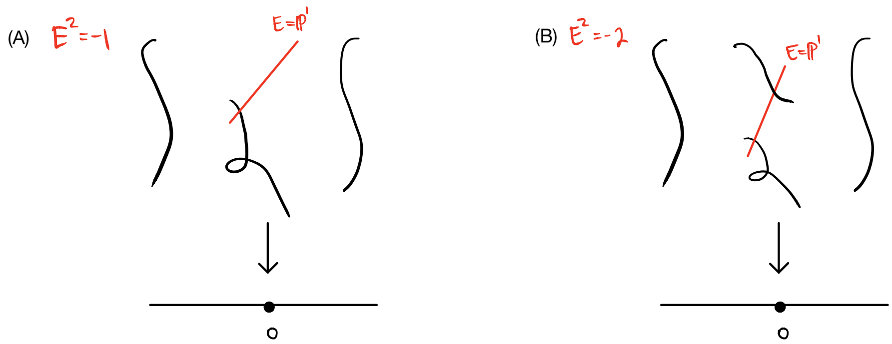



Figure 5.6: In (A) (resp. (B)), the exceptional component $E$ meets the rest of the curve at one point (resp. two points) and $E^2 = -1$ (resp. $E^2 = -2$).

**Definition 5.3.17.** Assume $2g - 2 + n > 0$. The *stabilization* of an $n$-pointed prestable curve $(C, p_1, \ldots, p_n)$ of genus $g$ over a field $\Bbbk$ is the curve $(C^{\mathrm{st}}, p'_1, \ldots, p'_n)$

where $C^{\mathrm{st}}$ is the curve obtained by contracting all rational bridges and tails $E_i$ and $p_i'$ are the images of $p_i$ under the contraction morphism $C \to C^{\mathrm{st}}$.

**Remark 5.3.18.** The contraction of a rational tail $E \subset C$ or a rational bridge with a marked point is the curve $E^c = \overline{C \setminus E}$. The contraction of a rational bridge $E \subset C$ without a marked point can be constructed as follows: since $E \cdot E^c = \dim_{\Bbbk} \Gamma(E \cap E^c, \mathcal{O}_{E \cap E^c} = 2$, the scheme-theoretic intersection $E \cdot E^c$ is isomorphic to either $\operatorname{Spec} \Bbbk \times \Bbbk$ or $\operatorname{Spec} \Bbbk'$ for a degree 2 separable field extension. The contraction $C'$ is defined as the pushout $E^c \amalg_{E \cap E^c} \operatorname{Spec} \Bbbk$ (see §A.8). A local calculation shows that the contraction $C'$ has a node at the image of $\operatorname{Spec} \Bbbk \to C'$.

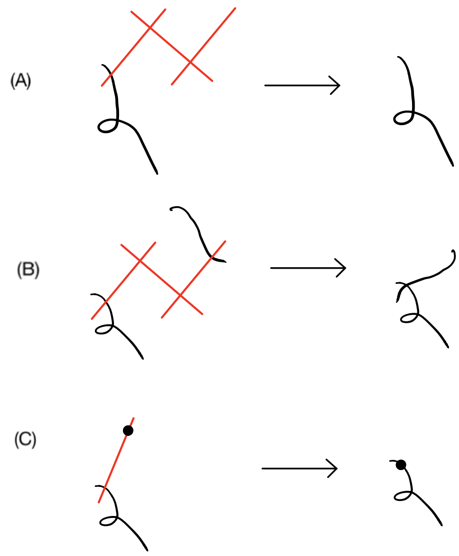



Figure 5.7: Examples of stabilizations

**Exercise 5.3.19.** Let $(C, p_1, \ldots, p_n)$ be an $n$-pointed prestable curve over a field $\Bbbk$.

(a) Show that the stabilization morphism $\pi \colon C \to C^{\mathrm{st}}$ is the unique morphism such that $(C^{\mathrm{st}}, \pi(p_1), \ldots, \pi(p_n))$ is stable, $\pi_* \mathcal{O}_C = \mathcal{O}_{C^{\mathrm{st}}}$, and $\mathrm{R}^1 \pi_* \mathcal{O}_C = 0$. See also [SP, Tag 0E7Q].

(b) If $(C, p_i)$ is semistable and $L := \omega_C(\sum_i p_i)$, show that $C^{\mathrm{st}} \cong \operatorname{Proj} \bigoplus_{d \geq 0} \mathrm{H}^0(C, L^{\otimes 4d}))$ and that $\omega_C(\sum_i p_i) \cong \pi^* \omega_{C^{\mathrm{st}}}(\sum_i p_i')$.

*Hint: Use Exercise 5.3.6 to show that $L^{\otimes 4}$ is base point free. Show that the multiplication map*

$$\mathrm{H}^0(C, L^{\otimes 2}) \otimes \mathrm{H}^0(C, L^{\otimes d}) \to \mathrm{H}^0(C, L^{\otimes (d+2)})$$

*is surjective for $d \geq 4$ to conclude that $\bigoplus_{d \geq 0} \mathrm{H}^0(C, L^{\otimes 4d})$ is finitely generated. See also [ACG11, Cor. 10.6.4].*

The construction of the stabilization extends to families of nodal curves.

**Proposition 5.3.20** (Stabilization of a Prestable Family)**.** *Let $(\mathcal{C} \to S, \sigma_1, \ldots, \sigma_n)$ be a family of $n$-pointed prestable curves of genus $g$. Assume $2g - 2 + n > 0$. Then there exists a unique morphism $\pi \colon \mathcal{C} \to \mathcal{C}^{\mathrm{st}}$ over $S$ such that*

*(1) $(\mathcal{C}^{\mathrm{st}} \to S, \sigma_i')$ is an $n$-pointed family of stable curves of genus $g$ where $\sigma_i' = \pi \circ \sigma_i$;*

(2) *for each $s \in S$, $(\mathcal{C}_s, \sigma_i(s)) \to (\mathcal{C}_s^{\mathrm{st}}, \sigma_i'(s))$ contracts all rational bridges and tails; and*

(3) *$\mathcal{O}_{\mathcal{C}^{\mathrm{st}}} = \pi_* \mathcal{O}_{\mathcal{C}}$ and $R^1\pi_*\mathcal{O}_{\mathcal{C}} = 0$ and this remains true after base change by a morphism $S' \to S$ of schemes;*

(4) *If $\mathcal{C} \to S$ is a family of semistable curves, then $\omega_{C/S}(\sum_i \sigma_i)$ is the pullback of the relatively ample line bundle $\omega_{\mathcal{C}^{\mathrm{st}}/S}(\sum_i \sigma_i')$.*

*Proof.* TO ADD. See [SP, Tag 0E7B] or [ACG11, Prop. 10.6.7]. □

## 5.4 The stack of all curves

### 5.4.1 Families of arbitrary curves

In this subsection, we redefine a *curve over a field $k$* to mean a scheme $C$ of finite type over $k$ of dimension 1 (rather than pure dimension 1). The genus of $C$ is defined as $g(C) = 1 - \chi(C, \mathcal{O}_C)$.

**Remark 5.4.1.** The reason we allow for non-pure dimensional and non-connected curves is that they may arise as deformations of connected pure one-dimensional curves; without this relaxation, the stack of all curves would fail to be algebraic. For instance, consider a rational normal curve $\mathbb{P}^1 \hookrightarrow \mathbb{P}^3$ embedded via $[x, y] \mapsto [x^3, x^2y, xy^2, ty^3]$ for every $t \neq 0$. As $t \to 0$, these curves degenerate in a flat family to a non-reduced curve $C_0$, which is supported along a plane nodal cubic and has an embedded point at the node; see [Har77, Ex. 9.8.4]. On the other hand, the curve $C_0$ deforms to the disjoint union of a plane nodal cubic and a point in $\mathbb{P}^3$.

A *family of curves* over a scheme $S$ is a proper, flat, and finitely presented morphism $\mathcal{C} \to S$ of algebraic spaces such that every fiber is a curve.

A *family of $n$-pointed curves* is a family of curves $\mathcal{C} \to S$ together with $n$ sections $\sigma_1, \dots, \sigma_n \colon S \to \mathcal{C}$ (with no condition on whether they are distinct or land in the relative smooth locus of $\mathcal{C}$ over $S$).

**Remark 5.4.2.** While every pure one-dimensional separated algebraic space over a field is in fact a scheme, in the relative setting the total family $\mathcal{C}$ may not be a scheme. There are examples of a family of prestable genus 0 curves [Ful10, Ex. 2.3] and a family of smooth genus 1 curves [Ray70, XIII 3.2] where the total family is not a scheme. Therefore, if we wish to define a *stack* of all curves, then to satisfy the decent condition, we better allow for the case that the total family is not a scheme. In the stable case however there is no difference: if $\mathcal{C} \to S$ is a family of curves (with $\mathcal{C}$ an algebraic space) such that every geometric fiber is stable, then $\omega_{\mathcal{C}/S}$ is relatively ample (Proposition 5.3.9) and $\mathcal{C} \to S$ is projective; in particular, $\mathcal{C}$ is a scheme.

**Proposition 5.4.3.** *If $\mathcal{C} \to S$ is a family of curves over a scheme $S$, there exists an étale cover $S' \to S$ such that $\mathcal{C}_{S'} \to S'$ is projective.*

*Vague sketch.* Approach 1: Local to global

For a point $s \in S$, define $S_n = \operatorname{Spec} \mathcal{O}_{S,s}/\mathfrak{m}_s^{n+1}$ and $\widehat{S} = \operatorname{Spec} \widehat{\mathcal{O}}_{S,s}$. Consider the cartesian diagram

$$\begin{tikzcd} \mathcal{C}_s = \mathcal{C}_0 \arrow[r, hook] \arrow[d] & \mathcal{C}_1 \arrow[r, hook] \arrow[d] & \cdots \\ \operatorname{Spec} \kappa(s) = S_0 \arrow[r, hook] & S_1 \arrow[r, hook] & \cdots \end{tikzcd} \qquad \begin{tikzcd} \widehat{\mathcal{C}} \arrow[r] \arrow[d] & \mathcal{C} \arrow[d] \\ \widehat{S} \arrow[r] & S \end{tikzcd}$$

*Case 1: $\mathcal{C}_s \to \operatorname{Spec} \kappa(s)$.* Since separated one-dimensional algebraic spaces are schemes and proper one-dimensional schemes are projective, there exists an ample line bundle $L_0$ on $\mathcal{C}_0$.

*Case 2: $\mathcal{C}_n \to S_n$.* The obstruction to deforming a line bundle $L_n$ on $\mathcal{C}_n$ to $L_{n+1}$ on $\mathcal{C}_{n+1}$ lives in $H^2(\mathcal{C}_0, \mathcal{O}_{\mathcal{C}_0})$ and thus vanishes as $\dim \mathcal{C}_0 = 1$. Thus there exists a compatible sequence of line bundles $L_n$ on $\mathcal{C}_n$. Since ampleness is an open condition in families and $L_0$ is ample, $L_n$ is also ample.

*Case 3: $\widehat{\mathcal{C}} \to \widehat{S}$ with $\widehat{S}$ noetherian.* Use Grothendieck's Existence Theorem: $\operatorname{Coh}(\widehat{cC}) \to \varprojlim \operatorname{Coh}(\mathcal{C}_n)$ is an equivalence of categories. The classical case is when $\widehat{\mathcal{C}} \to \widehat{S}$ is a proper morphism of schemes. Chow's Lemma for Algebraic Spaces implies that there exists a projective birational morphism $\mathcal{C}' \to \widehat{\mathcal{C}}$ of algebraic spaces such that $\mathcal{C}' \to \widehat{S}$ is projective. This allows one to reduce Grothendieck's Existence Theorem for $\widehat{\mathcal{C}} \to \widehat{S}$ to $\mathcal{C}' \to \widehat{S}$ using devissage similar to how the proper case of schemes is reduced to the projective case.

As a result, we can extend the sequence of line bundle $L_n$ to a line bundle $\widehat{L}$ on $\widehat{\mathcal{C}}$ which is ample (using again that ampleness is an open condition in families).

*Case 4: $S$ is of finite type over $\mathbb{Z}$.* For every closed point $s \in S$, apply Artin Approximation to the functor

$$\mathrm{Sch}/S \to \mathrm{Sets}, \qquad (T \to S) \mapsto \operatorname{Pic}(\mathcal{C}_T)$$

to obtain an étale neighborhood $(S', s') \to (S, s)$ of $s$ and a line bundle $L'$ on $\mathcal{C}_{S'}$ extending $L_0$. By openness of ampleness, we can replace $S'$ with an open neighborhood of $s'$ such that $L'$ is relatively ample over $S'$.

*Case 5: $S$ is an arbitrary scheme.* Apply Noetherian Approximation.

Approach 2: Explicitly extend an ample line bundle

The idea here is to use geometric methods to extend a line bundle $L_s$ on $\mathcal{C}_s$ to a line bundle on $\mathcal{C}$. If we assume in addition that every fiber of $\mathcal{C} \to S$ is generically reduced (and thus also generically smooth), then we may follow the argument of [Ols16, Cor. 13.2.5]. Choose smooth points $p_1, \ldots, p_n \in \mathcal{C}_s$ such that every irreducible one-dimensional component of $\mathcal{C}_s$ contains at least one of the $p_i$'s. Our hypothesis implies that the relative smooth locus $\mathcal{C}^0$ of $\mathcal{C} \to S$ surjects onto $S$. As smooth morphisms étale locally have sections, there is an étale neighborhood $S' \to S$ of $s$ and sections $\sigma_i \colon S' \to \mathcal{C}^0$ extending $p_i$. The line bundle $L' := \mathcal{O}_{\mathcal{C}_{S'}}(\sigma_1 + \cdots + \sigma_n)$ extends the ample line bundle $L_s := \mathcal{O}_{\mathcal{C}_s}(p_1 + \cdots + p_n)$. By openness of ampleness in families, $L'$ is relatively ample over $S'$ in an open neighborhood of $s'$.

(An alternative argument that works without restrictions is presented in [Hal13, Lem. 1.2] (based on ideas in [SGA4$\frac{1}{2}$, IV.4.1]) where one first uses Noetherian approximation and étale localization to reduce to $S = \operatorname{Spec} R$ where $R$ is an excellent strictly henselian local ring. One can then reduce to the case where $\mathcal{C}$ is a scheme by appealing to the fact that there exists a finite surjection $\mathcal{C}' \to \mathcal{C}$ from a scheme and the fact that $\mathcal{C}$ satisfies the Chevalley-Kleiman property (i.e. every finite set of points is contained in an open affine) if and only if $\mathcal{C}'$ does. Using deformation theory as above, one can further reduce to the case where $\mathcal{C}$ is reduced. Finally, one attempts to explicitly extend an ample line bundle on $\mathcal{C}_s$ by extending a function $f \in \Gamma(U, \mathcal{O}_{\mathcal{C}_s})$ to a function defined on an open neighborhood of $s \in \mathcal{C}$ so that it defines an effective Cartier divisor.) $\square$

**Remark 5.4.4.** Raynaud gives an example of a family of smooth $g = 1$ curves over an affine curve which is Zariski-locally projective but not projective [Ray70, XIII 3.1]. The examples in Remark 5.4.2 are not even Zariski-locally projective.

### 5.4.2 Algebraicity of the stack of all curves

**Definition 5.4.5.** Let $\mathcal{M}_{g,n}^{\text{all}}$ denote the category over $\text{Sch}_{\text{ét}}$ whose objects over a scheme $S$ consists of families of curves $\mathcal{C} \to S$ and $n$ sections $\sigma_1, \ldots, \sigma_n \colon S \to \mathcal{C}$. A morphism $(\mathcal{C}' \to S', \sigma_1', \ldots, \sigma_n') \to (\mathcal{C} \to S, \sigma_1, \ldots, \sigma_n)$ is the data of a cartesian diagram

$$\begin{tikzcd} \mathcal{C}' \arrow[r, "g"] \arrow[d] & \mathcal{C} \arrow[d] \\ S' \arrow[r, "f"] \arrow[u, bend left, "\sigma_i'"] & S \arrow[u, bend left, "\sigma_i"] \end{tikzcd}$$

such that $g \circ \sigma_i' = \sigma_i \circ f$.

As a stepping stone to the algebraicity of $\mathcal{M}_{g,n}^{\text{all}}$, we first show that the diagonal is representable.

**Lemma 5.4.6.** *The diagonal $\mathcal{M}_{g,n}^{\text{all}} \to \mathcal{M}_{g,n}^{\text{all}} \times \mathcal{M}_{g,n}^{\text{all}}$ is representable.*

*Proof.* For simplicity, we handle the case when $n = 0$. Let $S$ be a scheme and $S \to \mathcal{M}_g^{\text{all}} \times \mathcal{M}_g^{\text{all}}$ be a morphism corresponding to families of curves $\mathcal{C}_1 \to S$ and $\mathcal{C}_2 \to S$. Considering the cartesian diagram

$$\begin{tikzcd} \underline{\text{Isom}}_S(\mathcal{C}_1, \mathcal{C}_2) \arrow[r] \arrow[d] & S \arrow[d] \\ \mathcal{M}_g^{\text{all}} \arrow[r] & \mathcal{M}_g^{\text{all}} \times \mathcal{M}_g^{\text{all}}, \end{tikzcd}$$

we need to show that $\underline{\text{Isom}}_S(\mathcal{C}_1, \mathcal{C}_2)$ is an algebraic space. By Proposition 5.4.3, there exists an étale cover $S' \to S$ such that $\mathcal{C}_{S'} \to S'$ is projective. Since $\underline{\text{Isom}}_S(\mathcal{C}_1, \mathcal{C}_2) \times_S S' = \underline{\text{Isom}}_{S'}(\mathcal{C}_{1,S'}, \mathcal{C}_{2,S'})$, the morphism $\underline{\text{Isom}}_{S'}(\mathcal{C}_{1,S'}, \mathcal{C}_{2,S'}) \to \underline{\text{Isom}}_S(\mathcal{C}_1, \mathcal{C}_2)$ is representable, surjective and étale. We may thus assume that $\mathcal{C}_1$ and $\mathcal{C}_2$ are projective over $S$.

We will use the following fact from scheme theory: if $X \to Y$ is a morphism of schemes each proper over $S$, there exists an open subscheme $S^0 \subset S$ such that for every map $T \to S$ of schemes $X_T \xrightarrow{\sim} Y_T$ is an isomorphism if and only if $T \to S$ factors through $S_0 \subset S$.

Consider the inclusion of functors:

$$\underline{\text{Isom}}_S(\mathcal{C}_1, \mathcal{C}_2) \subset \underline{\text{Mor}}_S(\mathcal{C}_1, \mathcal{C}_2) \subset \text{Hilb}_S(\mathcal{C}_1 \times_S \mathcal{C}_2)$$

where the second inclusion assigns to a morphism $\mathcal{C}_1 \xrightarrow{\alpha} \mathcal{C}_2$ the graph $\mathcal{C}_1 \overset{\Gamma_\alpha}{\hookrightarrow} \mathcal{C}_1 \times_S \mathcal{C}_2$ (and is similarly defined on $T$-valued points). The first inclusion is a representable open immersion by the above fact. Analyzing the second inclusion, we see that a subscheme $[\mathcal{Z} \subset \mathcal{C}_1 \times_S \mathcal{C}_2] \in \text{Hilb}_S(\mathcal{C}_1 \times_S \mathcal{C}_2)(S)$ is in the image of element of $\underline{\text{Mor}}(\mathcal{C}_1, \mathcal{C}_2)(S)$ if and only if the composition $Z \hookrightarrow \mathcal{C}_1 \times_S \mathcal{C}_2 \xrightarrow{p_1} \mathcal{C}_1$ is an isomorphism (and similarly for $T$-valued points). Therefore, the above fact also establishes that the second inclusion is a representable open immersion. $\square$

**Theorem 5.4.7.** *$\mathcal{M}_{g,n}^{\text{all}}$ is an algebraic stack locally of finite type over* $\operatorname{Spec} \mathbb{Z}$.

*Sketch.*

- Suffices to show the $n=0$ case: $\mathcal{M}^{\mathrm{all}}_{g,1}$ is the universal family over $\mathcal{M}^{\mathrm{all}}_{g}$ and more generally $\mathcal{M}^{\mathrm{all}}_{g,n+1}$ is the universal family over $\mathcal{M}^{\mathrm{all}}_{g,n}$. (We will see that the same holds for $\overline{\mathcal{M}}_g$, but this is a more remarkable fact since an $n$-pointed stable curve can become unstable if a marked point is forgotten.)

- $\mathcal{M}^{\mathrm{all}}_{g}$ is a stack over $\mathrm{Sch}_{\text{ét}}$: Suppose $S' \to S$ is an étale cover of schemes, $\mathcal{C}' \to S'$ is a family of curves, and $\alpha \colon p_1^*\mathcal{C}' \to p_2^*\mathcal{C}'$ is an isomorphism over $S' \times_S S'$ satisfying the cocycle condition. The quotient of the étale equivalence relation

$$R' := p_1^*\mathcal{C}' \underset{p_2 \circ \alpha}{\overset{p_1}{\rightrightarrows}} \mathcal{C}'$$

is an algebraic space $\mathcal{C} := \mathcal{C}'/R$ and $\mathcal{C} \to S$ is a family of curves such that $\mathcal{C}_{S'} \cong \mathcal{C}'$.

- It to suffices show that for all projective curves $C_0$ over a field $\Bbbk$, there exists a representable, smooth morphism $U \to \mathcal{M}^{\mathrm{all}}_{g}$ from a scheme with $[C_0]$ in the image. Choose an embedding $C_0 \hookrightarrow \mathbb{P}^N$ such that $\mathrm{h}^1(C_0, \mathcal{O}_{C_0}(1)) = 0$, and let $P(t)$ be its Hilbert polynomial.

- Let $H := \mathrm{Hilb}^P(\mathbb{P}^N_{\mathbb{Z}}/\mathbb{Z})$ be the Hilbert scheme, which is projective over $\mathbb{Z}$ by Theorem 1.1.2. Considering the universal family

$$\begin{tikzcd} \mathcal{C} \arrow[r, hook] \arrow[d] & \mathbb{P}^N_H \arrow[dl] \\ H, & \end{tikzcd}$$

there is a point $h_0 \in H(\Bbbk)$ such that $\mathcal{C}_{h_0} = C_0$ as closed subschemes of $\mathbb{P}^N_{\Bbbk}$. Cohomology and Base Change implies that there exists an open neighborhood $H' \subset H$ of $h_0$ such that for all $s \in H'$, $\mathrm{h}^1(\mathcal{C}_s, \mathcal{O}_{\mathcal{C}_s}(1)) = 0$.

- Consider the morphism

$$H' \to \mathcal{M}^{\mathrm{all}}_{g}, \qquad [C \hookrightarrow \mathbb{P}^n] \mapsto [C],$$

which is representable by Lemma 5.4.6 and the fact that representability of the diagonal implies that every morphism from a scheme is representable (see the argument of Corollary 3.2.2).

- Claim: $H' \to \mathcal{M}^{\mathrm{all}}_{g}$ is smooth.

We will use the Infinitesimal Lifting Criterion (Theorem 3.7.1)—even though we don't yet know $\mathcal{M}^{\mathrm{all}}_{g}$ is algebraic, we may still use this criterion as we know that $H' \to \mathcal{M}^{\mathrm{all}}_{g}$ is representable. It thus suffices to show for all maps $S \to \mathcal{M}^{\mathrm{all}}_{g}$ from a scheme the induced morphism $H'_S \to S$ is a smooth morphism of algebraic spaces. We need to check that for all surjections $A \to A_0$ of artinian local rings with residue field $\Bbbk$ such that $\Bbbk = \ker(A \to A_0)$ and for all diagrams

$$
\begin{tikzcd}
\operatorname{Spec} \Bbbk \arrow[rrd, bend left, "{[C \subset \mathbb{P}^N_{\Bbbk}]}"] \arrow[rd] & & \\
& \operatorname{Spec} A_0 \arrow[r, "{[\mathcal{C}_0 \subset \mathbb{P}^N_{A_0}]}"] \arrow[d, hook] & H' \arrow[d, "f"] \\
& \operatorname{Spec} A \arrow[r, "{[\mathcal{C}]}"] \arrow[ru, dashed, "{[\mathcal{C} \subset \mathbb{P}^N_A]}"] & \mathcal{M}_g^{\mathrm{all}}
\end{tikzcd}
\tag{5.4.1}
$$

of solid arrows, there exists a dotted arrow. The existence of a dotted arrow in the above diagram is equivalent to the existence of a dotted arrow in the below diagram

$$
\begin{tikzcd}
& \mathbb{P}^N_{\Bbbk} & & \mathbb{P}^N_{A_0} & & \mathbb{P}^N_A \arrow[ld] \\
C \arrow[ru, hook] \arrow[rr] \arrow[d] & & \mathcal{C}_0 \arrow[ru, hook] \arrow[rr] \arrow[d] & & \mathcal{C} \arrow[ru, hook, dashed] \arrow[d] & \\
\operatorname{Spec} \Bbbk \arrow[rr, hook] & & \operatorname{Spec} A_0 \arrow[rr, hook] & & \operatorname{Spec} A &
\end{tikzcd}
$$

of solid arrows: a lifting of the diagram (3.7.3) corresponds to a family $\mathcal{C} \to \operatorname{Spec} A$ extending $\mathcal{C}_0 \to \operatorname{Spec} A_0$. By Proposition D.2.6, there is a cohomology class $\mathrm{ob} \in H^2(C, T_C)$ such that $\mathrm{ob} = 0$ if and only if there exists a lifting. Since $C$ is a curve, $H^2(C, T_C) = 0$.

- Use deformation theory to extend $\mathcal{C}_0 \hookrightarrow \mathbb{P}^N_{A_0}$ to $\mathcal{C} \hookrightarrow \mathbb{P}^N_A$. We will use the simplifying assumption that $C$ is a complete local intersection; the general case is handled by more advanced deformation theory (see [Hal13, Prop. 4.2]). This implies that the ideal sheaf $\mathcal{I}$ defining $C \hookrightarrow \mathbb{P}^N_{\Bbbk}$ is cut out locally by a regular sequence and that $\mathcal{I}/\mathcal{I}^2$ is a vector bundle on $C$ fitting into an exact sequence
$$0 \to \mathcal{I}/\mathcal{I}^2 \to \Omega_{\mathbb{P}^N_{\Bbbk}}|_C \to \Omega_C \to 0.$$
Applying $\operatorname{Hom}_{\mathcal{O}_C}(-, \mathcal{O}_C)$ gives a long exact sequence where the relevant terms for us are
$$\operatorname{Hom}_{\mathcal{O}_C}(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_C) \to \operatorname{Ext}^1_{\mathcal{O}_C}(\Omega_C, \mathcal{O}_C) \to \operatorname{Ext}^1_{\mathcal{O}_C}(\Omega_{\mathbb{P}^N_{\Bbbk}}|_C, \mathcal{O}_C) = H^1(C, T_{\mathbb{P}^N_{\Bbbk}}|_C).$$
The first term classifies embedded deformations of $\mathcal{C}_0 \hookrightarrow \mathbb{P}^N_{A_0}$ over $A_0$ to $\mathcal{C}' \hookrightarrow \mathbb{P}^N_A$ over $A$ while the second term classifies deformations of $\mathcal{C}_0$ over $A_0$ to $\mathcal{C}'$ over $A$. The boundary map $\operatorname{Hom}_{\mathcal{O}_C}(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_C) \to \operatorname{Ext}^1_{\mathcal{O}_C}(\Omega_C, \mathcal{O}_C)$ assigns an embedded deformation $[\mathcal{C}' \hookrightarrow \mathbb{P}^N_A]$ to $[\mathcal{C}']$.
Finally, we have the restriction of the Euler sequence to $C$
$$0 \to \mathcal{O}_C \to \mathcal{O}_C(1)^{\oplus(N+1)} \to T_{\mathbb{P}^N}|C \to 0.$$
Since $H^2(C, \mathcal{O}_C) = 0$ (as $\dim C = 1$) and $H^1(C, \mathcal{O}_C(1)) = 0$ (as $[C] \in H'$), we conclude that $H^1(C, T_{\mathbb{P}^N}|C) = 0$. Thus, our given deformation $[\mathcal{C}] \in \operatorname{Ext}^1_{\mathcal{O}_C}(\Omega_C, \mathcal{O}_C)$ maps to 0 in $H^1(C, \mathcal{O}_C(1))$, and thus is the image of an embedded deformation $[\mathcal{C} \hookrightarrow \mathbb{P}^N_A] \in \operatorname{Hom}_{\mathcal{O}_C}(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_C)$.

□

### 5.4.3 Algebraicity of $\overline{\mathcal{M}}_{g,n}$: openness and boundedness of stable curves

Consider the inclusions of prestacks

$$\mathcal{M}_{g,n} \subset \overline{\mathcal{M}}_{g,n} \subset \mathcal{M}_{g,n}^{\mathrm{pre}} \subset \mathcal{M}_{g,n}^{\leq \mathrm{nodal}} \subset \mathcal{M}_{g,n}^{\mathrm{all}} \tag{5.4.2}$$

$\mathcal{M}_{g,n}^{\leq \mathrm{nodal}}$) denotes the full subcategory of $\mathcal{M}_{g,n}^{\mathrm{all}}$ consisting of $n$-pointed families $(\mathcal{C} \to S, \sigma_1, \ldots, \sigma_n)$ of stable curves (resp. semistable, prestable, and nodal curves).

- By Theorem 5.4.7, $\mathcal{M}_{g,n}^{\mathrm{all}}$ is an algebraic stack locally of finite type over $\operatorname{Spec} \mathbb{Z}$.
- $\mathcal{M}_{g,n}^{\leq \mathrm{nodal}} \subset \mathcal{M}_{g,n}^{\mathrm{all}}$ is an open substack: this is equivalent to showing that $\mathcal{C} \xrightarrow{\pi} S$ is a family of curves (with $\mathcal{C}$ possibly an algebraic space) then the locus $\{s \in S \mid \mathcal{C}_s \text{ is nodal}\} \subset S$ is open. This is established in Corollary 5.2.20 when $\mathcal{C}$ is a scheme by relying on the Local Structure of Nodes (Theorem 5.2.18). In general, we can choose an étale cover $g \colon \mathcal{C}' \to \mathcal{C}$ by a scheme and use the fact that a point $p \in \mathcal{C}$ is a node in its fiber $\mathcal{C}_{\pi(p)}$ if and only if a preimage $p' \in \mathcal{C}'$ of $p$ is a node in its fiber $\mathcal{C}'_{\pi(p)}$.
- $\mathcal{M}_{g,n}^{\mathrm{pre}} \subset \mathcal{M}_{g,n}^{\leq \mathrm{nodal}}$ is an open substack: for a family $(\mathcal{C} \to S, \{\sigma_i\})$ of nodal curves, the locus $\{s \in S \mid \sigma_i(s) \text{ are disjont and smooth}\}$ is open.
- $\overline{\mathcal{M}}_{g,n} \subset \mathcal{M}_{g,n}^{\mathrm{pre}}$ is an open substack: this was shown in Proposition 5.3.10. The condition that a prestable curve $(\mathcal{C} \to S, \{\sigma_i\})$ is stable is equivalent to the ampleness of $\omega_{\mathcal{C}/S}(\sigma_1 + \cdots \sigma_n)$, and ampleness is an open condition on $S$.
- $\mathcal{M}_{g,n} \subset \overline{\mathcal{M}}_{g,n}$ is an open substack: the condition that a stable curve $(\mathcal{C} \to S, \{\sigma_i\})$ is smooth is an open condition on $S$.

It follows that each prestack featured in (5.4.2) is an algebraic stack locally of finite type over $\operatorname{Spec} \mathbb{Z}$.

**Theorem 5.4.8.** *If $2g-2+n > 0$, then $\overline{\mathcal{M}}_{g,n}$ is a quasi-compact Deligne–Mumford stack smooth over $\operatorname{Spec} \mathbb{Z}$ of relative dimension $3g-3+n$.* □

*Proof.* To show the boundedness of $\overline{\mathcal{M}}_{g,n}$ (i.e. finite typeness or equivalently quasi-compactness), we will appeal to the fact that if $(C, p_1, \ldots, p_n)$ is an $n$-pointed stable curve over a field $\Bbbk$, then the third power of the twist of the dualizing sheaf $(\omega_{C/\Bbbk}(p_1 + \cdots p_n))^{\otimes 3}$ is very ample (Exercise 5.3.6). Let $P(t)$ be the Hilbert polynomial of $C \hookrightarrow \mathbb{P}^N_{base}$ embedded via $(\omega_{C/\Bbbk}(p_1 + \cdots p_n))^{\otimes 3}$; this is independent of $[C, \{p_i\}] \in \overline{\mathcal{M}}_{g,n}$. Consider the closed subscheme

$$H \subset \operatorname{Hilb}^P(\mathbb{P}^N_{\mathbb{Z}}/\mathbb{Z}) \times (\mathbb{P}^N)^n$$

of an embedded curve and $n$ points $(C \hookrightarrow \mathbb{P}^N, p_1, \ldots, p_n)$ such that $p_i \in C$. There is a forgetful functor

$$H \to \mathcal{M}_{g,n}^{\mathrm{all}} \qquad [C \hookrightarrow \mathbb{P}^N, p_1, \ldots, p_n] \mapsto (C, p_1, \ldots, p_n).$$

Since $\operatorname{Hilb}^P(\mathbb{P}^N_{\mathbb{Z}}/\mathbb{Z})$ is a projective scheme (Theorem 1.1.2) and in particular quasi-compact and the image of $|H| \to |\mathcal{M}_{g,n}^{\mathrm{all}}|$ contains $\overline{\mathcal{M}}_{g,n}$, we conclude that $\overline{\mathcal{M}}_{g,n}$ is quasi-compact.

At this point, we've shown that $\overline{\mathcal{M}}_{g,n}$ is an algebraic stack of finite type over $\operatorname{Spec} \mathbb{Z}$. We now invoke each part of Proposition 5.3.14 characterizing automorphisms, deformations, and obstructions of stable curve exactly as in the proof of the analogous fact for $\overline{\mathcal{M}}_g$ (Proposition 3.7.4). Indeed, $\mathrm{Ext}^0_{\mathcal{O}_C}(\Omega_C(\sum_i p_i), \mathcal{O}_C) = 0$ implies that the Lie algebra of $\mathrm{Aut}(C, \{p_i\})$ is trivial and thus that $\mathrm{Aut}(C, \{p_i\})$ is a finite and reduced group scheme. By the Characterization of Deligne–Mumford stacks (Theorem 3.6.4), we conclude that $\overline{\mathcal{M}}_{g,n}$ is Deligne–Mumford. Since $\mathrm{Ext}^2_{\mathcal{O}_C}(\Omega_C(\sum_i p_i), \mathcal{O}_C) = 0$, there are no obstructions to deforming stable curves and the Infinitesimal Lifting Criterion (Theorem 3.7.1) implies that $\overline{\mathcal{M}}_{g,n}$ is smooth over $\operatorname{Spec}\mathbb{Z}$. Finally, since $\dim_{\Bbbk} \mathrm{Ext}^1_{\mathcal{O}_C}(\Omega_C(\sum_i p_i), \mathcal{O}_C) = 3g-3+n$ and there is a bijection of this Ext group with the Zariski tangent space of $[C, \{p_i\}] \in \overline{\mathcal{M}}_{g,n} \times_{\mathbb{Z}} \Bbbk$, we see that $\overline{\mathcal{M}}_{g,n} \to \operatorname{Spec}\mathbb{Z}$ has relative dimension $3g-3+n$. $\square$

**Exercise 5.4.9.** Show that $\overline{\mathcal{M}}_{g,n}$ is algebraic by explicitly presenting it as a quotient stack of a locally closed subscheme of the Hilbert scheme.

*Hint: Follow the proof of Theorem 3.1.16.*

## 5.5 Stable reduction: properness of $\overline{\mathcal{M}}_{g,n}$

This section discusses stable reduction of curves. Following the exposition of [HM98, §3.C], we give a complete proof in characteristic 0 relying on the birational geometry of surfaces and specifically the existence of embedded resolutions for curves in surfaces (Theorem E.1.2).

**Theorem 5.5.1** (Stable Reduction)**.** *Let $R$ be a DVR with fraction field $K$, and set $\Delta = \operatorname{Spec} R$ and $\Delta^* = \operatorname{Spec} K$. If $(\mathcal{C}^* \to \Delta^*, s_1^*, \ldots, s_n^*)$ is a family of $n$-pointed stable curves of genus $g$, then there exists a finite cover $\Delta' \to \Delta$ of spectrums of DVRs and a family $(\mathcal{C}' \to \Delta', s_1', \ldots, s_n')$ of stable curves extending $\mathcal{C}^* \times_{\Delta^*} \Delta'^* \to \Delta'^*$.*

**Remark 5.5.2.** This theorem was first established in [DM69] by embedding the generic fiber into its Jacobian and reducing the statement to semistable reduction for abelian varieties, which had been established in [SGA7-I, SGA7-II]. Interestingly, Gieseker also established this theorem by using GIT rather than the geometry of families of curves over a DVR [Gie82]. Later arguments due to Artin–Winters [AW71] and Saito [Sai87] follow essentially the strategy outlined below. See [SP, Tag 0C2Q] or Remark 5.5.8 for more background.

After introducing the basic strategy to establish Stable Reduction in Section 5.5.1, we prove Stable Reduction (Theorem 5.5.1) in characteristic 0 in Section 5.5.3. We also illustrate in Sections 5.5.4 and 5.5.5 how one can explicitly compute the stable limit of a given family $\mathcal{C}^* \to \Delta^*$ of stable curves: while the proof of Stable Reduction offers a strategy, additional care and techniques are needed to get an explicit handle on the stable limit. Finally, in Section 5.5.6, we prove the uniqueness of the stable limit (Proposition 5.5.15) in arbitrary (possibly mixed) characteristic.

Stable Reduction (5.5.1)implies the properness of $\overline{\mathcal{M}}_{g,n}$ via the Valuative Criterion for Properness (Theorem 3.8.5).

**Theorem 5.5.3.** *If $2g-2+n > 0$, the Deligne–Mumford stack $\overline{\mathcal{M}}_{g,n}$ is proper over* $\operatorname{Spec}\mathbb{Z}$. $\square$

By applying the Keel–Mori Theorem (4.3.11), we obtain:

**Corollary 5.5.4.** *If $2g-2+n > 0$, there exists a coarse moduli space $\overline{\mathcal{M}}_{g,n} \to \overline{M}_{g,n}$ where $\overline{M}_{g,n}$ is a proper algebraic space over* $\operatorname{Spec}\mathbb{Z}$. $\square$

### 5.5.1 Basic strategy

We provide the basic strategy to exhibit the existence of stable reduction for a given family $\mathcal{C}^* \to \Delta^*$ of stable curves. For simplicity of notation, we assume that there are no marked points, i.e. $n = 0$.

Throughout, we use the notation: $\Delta = \operatorname{Spec} R$ for a DVR $R$, $\Delta^* = \operatorname{Spec} K$ with $K$ the fraction field of $R$, $t \in R$ uniformizer, and $0 = (t) \in \operatorname{Spec} R$ the unique closed point.

*Step 0: Reduce to the case where $\mathcal{C}^* \to \Delta^*$ is smooth.* If $\mathcal{C}^*$ has $k$ nodes, then possibly after a finite extension of $K$ we can arrange that each node is given by a $K$-point $p_i \in \mathcal{C}^*(K)$. Let $(\widetilde{\mathcal{C}}^*, \widetilde{p}_1, \ldots, \widetilde{p}_{2k})$ be the pointed normalization. By induction on the genus $g$ (relying on stable reduction for $2k$*-pointed curves* of genus $< g$), we perform stable reduction on each connected component and then take the nodal union along sections. After possibly an extension of $K$ (and $R$), this produces a family of curves $\mathcal{C} \to \Delta$ extending $\mathcal{C}^* \to \Delta^*$.

*Step 1: Find some flat extension $\mathcal{C} \to \Delta$.*

Using that $\omega_{\mathcal{C}^*/\Delta^*}^{\otimes 3}$ is very ample (Proposition 5.3.9), we may embed $\mathcal{C}^*$ as a closed subscheme of $\mathbb{P}^{5g-6} \times \Delta^*$. The scheme-theoretic image $\mathcal{C}$ of $\mathcal{C}^* \hookrightarrow \mathbb{P}^{5g-6} \times \Delta$ is flat over $\Delta$ using the Flatness Criterion over Smooth Curves (Proposition A.2.2) and the fact the closure doesn't introduce any embedded points in the central fiber. Thus we have a proper flat family of curves $\mathcal{C} \to \Delta$ extending $\mathcal{C}^* \to \Delta^*$. (This is the same argument that establishes the valuative criterion for the properness of the Hilbert scheme.)

*Step 2: Use Embedded Resolutions to find a resolution of singularities $\widetilde{\mathcal{C}} \to \mathcal{C}$ so that the reduced central fiber $(\widetilde{\mathcal{C}}_0)_{\mathrm{red}}$ is nodal.*

Applying Embedded Resolutions (Theorem E.1.2), there is a finite sequence of blow ups at closed points of $\mathcal{C}_0$ yielding a projective birational morphism

$$\begin{tikzcd} \widetilde{\mathcal{C}} \arrow[r] \arrow[dr] & \mathcal{C} \arrow[d] \\ & \Delta \end{tikzcd}$$

such that $\widetilde{\mathcal{C}}$ is regular, $\widetilde{\mathcal{C}} \to \Delta$ is a (flat) family of curves and such that the preimage $\widetilde{\mathcal{C}}_0$ of $\mathcal{C}_0$ has set-theoretic normal crossings, i.e. $(\widetilde{\mathcal{C}}_0)_{\mathrm{red}}$ is nodal. Replace $\mathcal{C}$ with $\widetilde{\mathcal{C}}$.

*Step 3: Take a ramified base extension $\Delta' = \operatorname{Spec} R \to \operatorname{Spec} R = \Delta$ by $t \mapsto t^m$ such that the central fiber of the normalization of $\mathcal{C} \times_\Delta \Delta'$ becomes reduced and nodal.*

We will explain the details of this step in Section 5.5.3. This is where we use the characteristic 0 assumption. Replacing $\mathcal{C}$ with the normalization $\widetilde{\mathcal{C}'}$ of $\widetilde{\mathcal{C}'} = \mathcal{C} \times_\Delta \Delta'$, we may assume that $\mathcal{C} \to \Delta$ is a prestable family (i.e. nodal family) of curves with $\mathcal{C}$ regular.

*Step 4: After taking the minimal model $\widetilde{\mathcal{C}}_{\min} \to \mathcal{C}$, contract all rational tails and bridges in the central fiber.*

In other words, we take the stable model of the family $\widetilde{\mathcal{C}}_{\min} \to \Delta$ as in Proposition 5.3.20. Alternatively as we argue in Section 5.5.3, one can explicitly contract the rational tails (smooth rational $-1$ curves) and rational bridges (smooth rational $-2$) curves.

**Remark 5.5.5.** Add example showing why we must allow for extensions of DVRs.

### 5.5.2 Semistable reduction

In Step 4 above, if we stop after contracting only rational tails (and not the rational bridges), i.e. the smooth rational $-1$ curves, then we obtain a family $\mathcal{C} \to \Delta$ of semistable curves such that $\mathcal{C}$ is regular (by Theorem E.1.5). This is called Semistable Reduction, an important variant of Stable Reduction.

**Theorem 5.5.6** (Semistable Reduction)**.** *Let $R$ be a DVR with fraction field $K$, and set $\Delta = \operatorname{Spec} R$ and $\Delta^* = \operatorname{Spec} K$. If $\mathcal{C}^*$ is a smooth projective curve over $\Delta^*$, there exists a cover $\Delta' \to \Delta$ of spectrums of DVRs and a family $\mathcal{C}' \to \Delta'$ of semistable curves extending $\mathcal{C}^* \times_{\Delta^*} \Delta'^* \to \Delta'^*$ such that $\mathcal{C}'$ is regular.*

### 5.5.3 Proof of stable reduction in characteristic 0

*Proof of Theorem 5.5.1 in characteristic 0.* Following Steps 0-2 in the basic strategy discussed in Section 5.5.1, we may assume that $\mathcal{C} \to \Delta$ is a generically smooth family of stable curves such that the reduced central fiber $(\mathcal{C}_0)_{\mathrm{red}}$ is nodal and $\mathcal{C}$ is regular.

*Step 3: Perform a base change $\Delta' \to \Delta$ such that the normalization of the total family $\mathcal{C} \times_\Delta \Delta'$ has a reduced and nodal central fiber.* Around every point $p \in \mathcal{C}_0$, we can choose local coordinates $x, y$ (either étale locally or formally locally at $p$) such that the morphism $\mathcal{C} \to \Delta$ can be described explicitly as follows:

- If $p \in (\mathcal{C}_0)_{\mathrm{red}}$ is a smooth point, then $(x, y) \mapsto x^a$ and the multiplicity of the irreducible component of $\mathcal{C}_0$ containing $p$ is $a$.
- If $p \in (\mathcal{C}_0)_{\mathrm{red}}$ is a separating node (i.e. $\mathcal{C}_0 \setminus p$ is disconnected), then $(x, y) \mapsto x^a y^b$ and the two components of $\mathcal{C}_0$ containing $p$ have multiplicities $a$ and $b$.
- If $p \in (\mathcal{C}_0)_{\mathrm{red}}$ is a non-separating node, then $(x, y) \mapsto x^a y^a$ and the components of $\mathcal{C}_0$ containing $p$ has multiplicity $a$.

Let $m$ be the least common multiple of the multiplicities of the irreducible components of $\mathcal{C}_0$. Let $\Delta' = \operatorname{Spec} R \to \operatorname{Spec} R = \Delta$ be defined by $t \mapsto t^m$ where $t$ denotes a uniformizing parameter. Let $\mathcal{C}' := \mathcal{C} \times_\Delta \Delta'$ and $\widetilde{\mathcal{C}}'$ be its normalization Let $\rho$ be a primitive $m$th root of unity. If $p \in (\mathcal{C}_0)_{\mathrm{red}}$ is a smooth point, then $\mathcal{C}'$ locally around the unique preimage of $p$ is defined by $x^a = t^m$ which factors as $\prod_{i=0}^{a-1}(x - \rho^i t^{m/a})$. Thus $p \in \mathcal{C}$ has $a$ preimages in $\widetilde{\mathcal{C}}'$ and each preimage is locally defined by $x = \rho^i t^{m/a}$ and is thus a smooth point in the central fiber $\widetilde{\mathcal{C}}'_0$. If $p \in (\mathcal{C}_0)_{\mathrm{red}}$ is a node defined by $x^a y^b$, then one computes that each preimage of $p$ is locally defined by $t^k = xy$ (see Exercise 5.5.7) and thus is a reduced and nodal point in $\widetilde{\mathcal{C}}'_0$. Note that if $k > 1$, then $\widetilde{\mathcal{C}}'$ has an $A_{k-1}$-singularity at the preimage.

We now replace $\mathcal{C}$ with $\widetilde{\mathcal{C}}'$. At the expense of introducing singularities into the total family, we have arranged the central fiber to be reduced and nodal.

*Step 4: Take a minimal resolution of $\mathcal{C}$ and contract curves with negative self-intersection.* Let $\mathcal{C}' \to \mathcal{C}$ be a Minimal Resolution (Theorem E.1.1) which replaces each $A_k$-singular with a chain of $\lfloor \frac{k}{2} \rfloor$ rational curves. At this stage $\mathcal{C}' \to \Delta$ is a prestable family of curves, i.e. a proper flat family of reduced nodal curves, such that the total family $\mathcal{C}'$ is regular. The central fiber $\mathcal{C}'_0$ however may not be stable.

If $\mathcal{C}'_0$ is not stable, it contains either a rational tail or bridge as in Figure 5.6. Each rational tail $E$ that has self-intersection $-1$ can be blown down by Castelnuovo's Contraction Theorem (E.1.5). Contracting all rational tails yields a projective birational morphism $\mathcal{C}' \to \mathcal{C}'_{\min}$, which is the Minimal Model (Corollary E.1.6).

Replacing $\mathcal{C}$ with $\mathcal{C}'_{\min}$, we obtain a semistable family $\mathcal{C} \to \Delta$ of curves such that the total family $\mathcal{C}$ is regular.

Finally, we apply the stabilization construction (Proposition 5.3.20) to obtain a morphism $\mathcal{C} \to \mathcal{C}^{\mathrm{st}}$ contracting each rational bridge and where $\mathcal{C}^{\mathrm{st}} \to \Delta$ is a stable family of curves. We note that $\mathcal{C}^{\mathrm{st}}$ is precisely the relative canonical model of $\mathcal{C}$ (Proposition 5.3.20(4)). Alternatively, one can realize this final step by iteratively contracting each rational bridge $E$ since each such subcurve satisfies $E^2 = -2$. Indeed, a version of Castelnuovo's Contraction Theorem is valid even if $E^2 < -1$ (the only difference is that the contracted surface may not be regular) and the contraction yields a family of stable curves. □

**Exercise 5.5.7.** Let $a, b, m$ be positive integers such that both $a$ and $b$ divide $m$. Let $X = \operatorname{Spec} \Bbbk[x,y,t]/(t^m - x^a y^b)$ and $\widetilde{X} \to X$ be its normalization. Show that each preimage of the origin is locally defined by $t^k = xy$ and in particular is a reduced and nodal point in the fiber over $t = 0$.

**Remark 5.5.8.** The above argument fails if the residue field of $R$ has positive characteristic $p$. Indeed, in Step 3, if any of the multiplicities of the components of the central fiber are divisible by $p$, then the extension $\operatorname{Spec} R \to \operatorname{Spec} R$ given by $t \mapsto t^m$ is not tamely ramified and the base change $\mathcal{C} \times_\Delta \Delta'$ may remain non-reduced.

A different approach is therefore needed in positive characteristic. The approach of [AW71] starts as above by taking a resolution of singularities $\mathcal{C}$ of some family of curves over $R$ extending $\mathcal{C}^*$. One then chooses an extension $K \to K'$ (and a corresponding extension $R \to R'$ of DVRs) such that $\mathcal{C}^*$ has a $K'$-point and such that the $l$-torsion $\operatorname{Pic}(\mathcal{C}^*_{K'})[l] \cong (\mathbb{Z}/l\mathbb{Z})^{2g}$ for a sufficiently large prime $l \neq p$. This magically forces the central fiber of $\mathcal{C} \times_R R'$ to be reduced and nodal! See [AW71] or [SP, Tag 0E8C].

### 5.5.4 First examples

In these examples, $\Delta = \operatorname{Spec} R$ where $R$ is a DVR with uniformizing parameter $t$.

**Example 5.5.9** (Nodal elliptic curves)**.** Consider the family of elliptic curves $(\mathcal{C}^* \to \Delta^*, \sigma)$ defined by the equation $y^2 z = x(x-z)(x-tz)$ in $\mathbb{P}^2 \times \Delta$ and the section $\sigma(t) = [0,1,0]$. The stable limit in $\overline{\mathcal{M}}_{1,1}$ as $t \to 0$ is the nodal cubic $y^2 z = x^2(x-z)$; see Figure 18.

**Example 5.5.10** (Colliding marked points)**.** Let $C$ be a smooth curve and consider the constant family $\mathcal{C} = C \times \Delta$. Let $p \in C$ be a $\Bbbk$-point and $\sigma_1 \colon \Delta \to \mathcal{C}$ be the constant section $t \mapsto p$. Suppose that $\sigma_2 \colon \Delta \to \mathcal{C}$ is another section meeting $\sigma_1$ transversely at $(p, 0) \in \mathcal{C}$ as shown below:

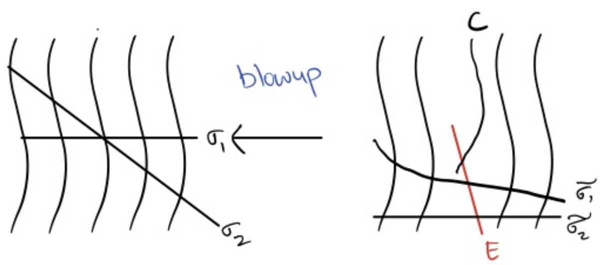



Figure 5.8:

To obtain the stable limit, we simply blow up the surface at $(p, 0)$. The stable limit is the nodal union of $C$ and $\mathbb{P}^1$ at $p$.

For a more involved example of colliding points, consider again the constant family $C \times \Delta$ with sections locally defined by $(\sigma_1, \sigma_2, \sigma_3) = (t^2, -t^2, 4t)$.

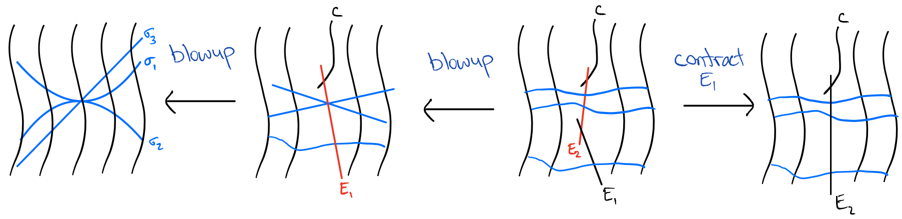



Figure 5.9:

After blowing up twice, the sections become disjoint but the central fiber is unstable as the exceptional component $E_1 \cong \mathbb{P}^1$ only has one node and one marked point. The stable limit is obtained by contracting $E_1$.

**Example 5.5.11** (A node degenerating to a cusp)**.** Consider a smooth curve $C$ with two points $p, q \in C$. Gluing $p$ and $q$ yields a nodal curve. Now if we fix $p$ and slide $q$ toward $p$, we have a family of nodal curves $\mathcal{C}^* \to C \setminus p$ as in Figure 5.10. For instance, this family could be defined locally by $y^2 = x^3 + tx^2$ in which case we have an extension $\mathcal{C} \to C$ where the central fiber $\mathcal{C}_p$ (given by $t = 0$) has a cusp. We want to compute the stable limit.

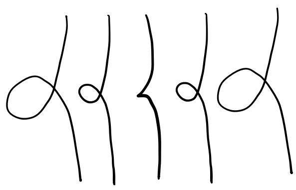


Figure 5.10: What is the stable limit of the above nodal degeneration?

In this case, the base curve is $C$ itself but it would be no different to work over $\operatorname{Spec} \mathcal{O}_{C,p}$. The pointed normalization of the family $\mathcal{C}^*$ extends to a family $C \times C \to C$ with the diagonal section $\Delta$ and the constant section $\Gamma_p = \{p\} \times C$.

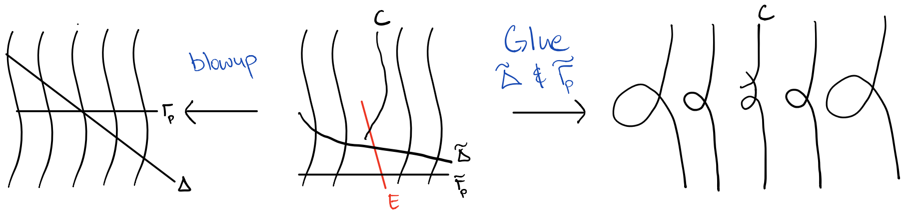



Figure 5.11: Recipe for computing the stable reduction

We first find the stable limit of the pointed normalization exactly as in Example 5.5.10: we blowup so that the strict transforms $\widetilde{\Delta}$ and $\widetilde{\Gamma}_p$ become disjoint. We then glue the sections $\widetilde{\Delta}$ and $\widetilde{\Gamma}_p$ to obtain a family $\mathcal{C} \to C$ of nodal curves where the central fiber is the nodal union of $C$ and a rational nodal curve at the point $p \in C$.

The above examples are too simple to reveal the general stable reduction procedure as no base changes were needed.

### 5.5.5 Explicit stable reduction

The biggest challenge in explicitly computing the stable limit of a family $\mathcal{C}^* \to \Delta^*$ following the basic strategy of Section 5.5.1 is in Step 3: computing the normalization $\widetilde{\mathcal{C}'}$ of the family $\mathcal{C}' = \mathcal{C} \times_\Delta \Delta'$ obtained by base changing $\mathcal{C} \to \Delta$ along a ramified cover $\Delta' \to \Delta$ defined by $t \mapsto t^m$. It is often simpler to factor $\Delta' \to \Delta$ as a composition of prime order base changes and use the following observation.

**Proposition 5.5.12.** *Let $\mathcal{C} \to \Delta$ be a generically smooth, proper, and flat family such that $(\mathcal{C}_0)_{\mathrm{red}}$ is nodal. As a divisor on $\mathcal{C}$, we may write $\mathcal{C}_0 = \sum a_i D_i$ where $a_i$ is the multiplicity of the irreducible component $D_i$. Let $\Delta' \to \Delta$ be defined by $t \mapsto t^p$ where $p$ is prime, and set $\mathcal{C}' := \mathcal{C} \times_\Delta \Delta'$ with normalization $\widetilde{\mathcal{C}'}$. Then $\widetilde{\mathcal{C}'} \to \mathcal{C}$ is a branched cover ramified over $\sum (a_i \mod p) D_i$.*

**Example 5.5.13** (Stable Reduction of an $A_{2k+1}$-singularity)**.** Suppose $\mathcal{C} \to \Delta$ is a generically smooth family degenerating to a $A_{2k+1}$-singularity in the central fiber such that the local equation around the singular point is $y^2 = x^{2k+1} + t$. In particular, the total family $\mathcal{C}$ is smooth. Figure 5.12 provides a pictorial representation of the stable reduction procedure.

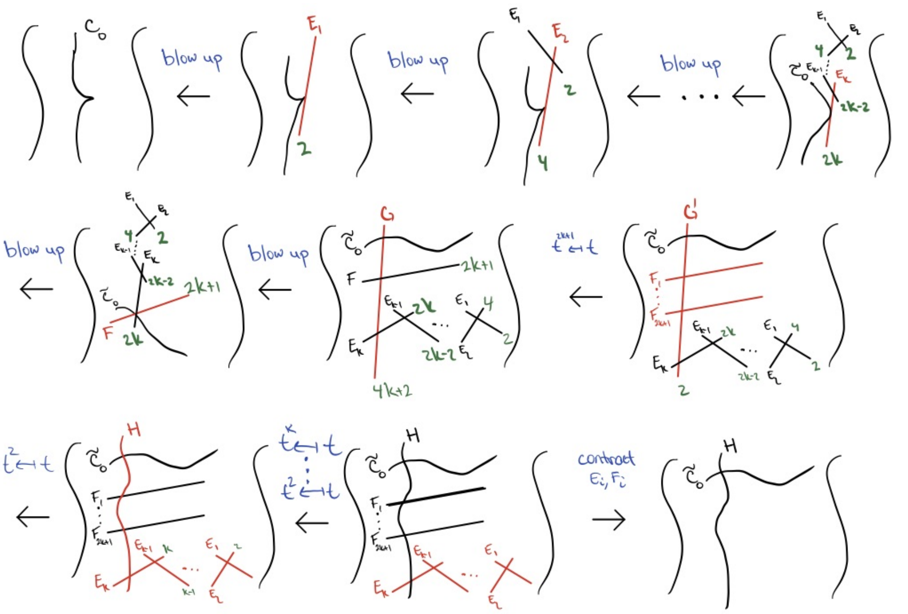



Figure 5.12: Recipe for computing the stable limit of a $A_{2k+1}$-singularity. blow upThe altered components in each step are colored in red, while the green numbers indicate the multiplicity of the component.

We are already given a flat limit $\mathcal{C} \to \Delta$ so we may begin with Step 2.

*Step 2: Repeatedly blow up to find a resolution of singularities $\widetilde{\mathcal{C}} \to \mathcal{C}$ so that the reduced central fiber $(\widetilde{\mathcal{C}}_0)_{\mathrm{red}}$ is nodal.*

We repeatedly blow up the (reduced) singular point in the central fiber. To keep track of the local equations, we will always use local coordinates $x, y$ on the original surface and $\widetilde{x}, \widetilde{y}$ on the new surface. In one chart of the blowup, $\widetilde{x} = x, \widetilde{y} = y/x$ with exceptional divisor $\widetilde{x} = 0$ while in the other chart, $\widetilde{x} = x/y, \widetilde{y} = y$ with exceptional divisor $\widetilde{y} = 0$.

For the first blow up, the preimage of $y^2 - x^{2k+1}$ in the chart $\widetilde{x} = x, \widetilde{y} = y/x$ is given by $\widetilde{x}^2(\widetilde{y}^2 - \widetilde{x}^{2k-1})$ and in the other chart by $\widetilde{y}^2(1 - \widetilde{x}^{2k+1}\widetilde{y}^{2k-1})$. The exceptional divisor $E_1$ has multiplicity 2.

For the second blow up, the preimage of $x^2(y^2 - x^{2k-1})$ in the chart $\widetilde{x} = x, \widetilde{y} = y/x$ is given by $\widetilde{x}^4(\widetilde{y}^2 - \widetilde{x}^{2k-3})$ and in the other chart by $\widetilde{x}^2\widetilde{y}^4(1 - \widetilde{x}^{2k-1}\widetilde{y}^{2k-3})$ (where $\widetilde{x}$ defines $E_1$ and $\widetilde{y}$ defines $E_2$). The new exceptional divisor $E_2$ has multiplicity 4.

After $k$ blow ups, one obtains a surface with local equation $x^{2k}(y^2 - x)$ at the singular point in the central fiber. The equation $y^2 - x$ defines the normalization $\widetilde{\mathcal{C}}_0$ of the original central fiber, and $x$ defines the exceptional divisor $E_k$ which has multiplicity $2k$. There is a chain of nodally attached exceptional divisors $E_k, \dots, E_1$ such that the multiplicity of $E_i$ is $2i$.

Blowing up again, the strict transform of $x^{2k}(y^2 - x)$ in the chart $\widetilde{x} = x/y, \widetilde{y} = y$ becomes $\widetilde{x}^{2k}\widetilde{y}^{2k+1}(\widetilde{y} - \widetilde{x})$ where $\widetilde{x}$ defines $E_k$, $\widetilde{y}$ defines the new exceptional divisor $F$ which has multiplicity $2k+1$, and $\widetilde{y} - \widetilde{x}$ defines $\widetilde{\mathcal{C}}_0$.

Blowing up one final time, the strict transform of $x^{2k}y^{2k+1}(y - x)$ in the chart $\widetilde{x} = x, \widetilde{y} = y/x$ becomes $\widetilde{x}^{4k+2}\widetilde{y}^{2k+1}(\widetilde{y} - 1)$ where $\widetilde{x}$ defines the new exceptional divisor $G$ which has multiplicity $4k+2$, $\widetilde{y}$ defines $F$ and $\widetilde{y} - 1$ defines $\widetilde{\mathcal{C}}_0$. In particular, the (non-reduced) central fiber is set-theoretically nodal.

*Step 3: Perform a base change $\Delta' \to \Delta$ such that the normalization of the total family $\mathcal{C} \times_\Delta \Delta'$ has a reduced and nodal central fiber.*

We begin by base changing by $\Delta' \to \Delta, t \mapsto t^{2k+1}$ and normalizing. For this analysis, we assume that $2k+1$ is prime, but one can inductively apply the same process to a prime factorization of $2k+1$ and obtain the same result in the end; the only difference is in the numerics of the multiplicities of the exceptional components $E_i$, but these can be resolved in the last step in the same way.

By applying Proposition 5.5.12, the new surface is a degree $2k+1$ cover ramified over $\widetilde{\mathcal{C}}_0 + \sum_i E_i$ as the other components of the central fiber have multiplicities divisible by $2k+1$. The preimage $G'$ of $G$ is $2k+1$ degree cover of $\mathbb{P}^1$ ramified over two points, each with ramification index $2k$. By Riemann–Hurwitz, the genus $g(G')$ of $G'$ satisfies $2g(G') - 2 = (2k+1)(g(\mathbb{P}^1) - 2) + R$ and since the ramification divisor $R$ has degree $2(2k)$, we see that $g(G') = 0$. Meanwhile, the preimage of $F$ is the disjoint union of $2k+1$ smooth rational curves $F_1, \dots, F_{2k+1}$. Over $\Delta$, the new special fiber is

$$(2k+1)\widetilde{\mathcal{C}}_0 + (4k+2)G' + (2k+1)\sum_i F_i + (2k+1)\sum_i 2iE_i$$

which over $\Delta'$ becomes $\widetilde{\mathcal{C}}_0 + 2G' + \sum_i F_i + \sum_i 2iE_i$

We now base change by $\Delta' \to \Delta, t \mapsto t^2$ and normalize. By Proposition 5.5.12, the new surface is a $2:1$ cover ramified over $\widetilde{\mathcal{C}}_0 + \sum_i F_i$. The preimage $H$ of $G' \cong \mathbb{P}^1$ is a $2:1$ cover ramified over $2k+2$ points, with one of those points being the node $H \cap \widetilde{\mathcal{C}}_0$. Thus $G'$ is a hyperelliptic curve of genus $g$ attached to $\widetilde{\mathcal{C}}_0$ at a ramification point (otherwise known as a Weierstrass point). The new central fiber over $\Delta'$ becomes reduced except for the components $E_i$ with multiplicity $i$.

Finally, we inductively base change and normalize by the ramified covers defined by $t \mapsto t^k, \dots, t \mapsto t^2$ so that the central fiber becomes reduced and nodal.

*Step 4: Contract rational tails in the central fiber.*

The exceptional components $F_i$ are smooth rational $-1$ curves that we can contract. We then inductive contract $E_1, E_2, \ldots, E_k$ (note that while $E_1$ is a $-1$ curve, $E_2$ is a $-2$ curve but becomes a $-1$ curve once $E_1$ is contracted). In the end, we obtain a reduced central fiber, the nodal union of the normalization $\widetilde{\mathcal{C}}_0$ of the original central fiber and a hyperelliptic genus $k$ curve $H$. The node in $H$ is a ramification point of the $2:1$ cover $H \to \mathbb{P}^1$ while the node in $\widetilde{\mathcal{C}}_0$ is the preimage of the singular point of $\mathcal{C}_0$.

The above example begs the following questions:

- Precisely which hyperelliptic curve $H$ appears in the stable limit?
- How does the stable limit depend on the choice of degeneration? By calculating the deformation space of a $A_{2k+1}$-singularity, one sees that every degeneration can be written locally as $y^2 = x^{2k+1} + a_{2k-1}(t)x^{2k-1} + \cdots + a_0(t)$ for polynomials $a_{2k-1}, \ldots, a_0$. In other words, we are asking how the stable limit depends on $a_i(t)$. In particular, what happens when the total family of the surface is singular (e.g. $y^2 = x^{2k+1} + t^2$)?

These questions are addressed in detail in [HM98, §3.C] in the case of a cusp $y^2 = x^3$ (i.e. $k = 1$). The reader is also encouraged to refer to *loc. cit.* for additional examples of stable reduction and other aspects of this story.

**Exercise 5.5.14.** Work out the stable reduction of a smooth family of curves degenerating to an $A_{2k+2}$-singularity with local equation $y^2 = x^{2k+2} + t$.

### 5.5.6 Separatedness of $\overline{\mathcal{M}}_{g,n}$

We now show that the stable limit is unique. The following proposition establishes via the Valuative Criterion for Separatedness (Theorem 3.8.5) that $\overline{\mathcal{M}}_{g,n}$ is separated.

**Proposition 5.5.15.** *Let $R$ be a DVR with fraction field $K$, and set $\Delta = \operatorname{Spec} R$ and $\Delta^* = \operatorname{Spec} K$. If $(\mathcal{C} \to \Delta, \sigma_1^*, \ldots, \sigma_n^*)$ and $(\mathcal{D} \to \Delta, \tau_1^*, \ldots, \tau_n^*)$ are families of $n$-pointed stable curves, then every isomorphism $\alpha^* \colon \mathcal{C}^* \to \mathcal{D}^*$ over $\Delta^*$ with $\tau_i^* = \alpha^* \circ \sigma_i^*$ of the generic fibers as pictured*

$$\begin{tikzcd}
\mathcal{C}^* \arrow[r, "\alpha^*"] \arrow[dr] & \mathcal{D}^* \arrow[d] & \mathcal{C} \arrow[r, dashed, "\alpha"] \arrow[dr] & \mathcal{D} \arrow[d] \\
& \Delta^* \arrow[rr, hook] & & \Delta
\end{tikzcd}$$

*extends to a unique isomorphism $\alpha \colon \mathcal{C} \to \mathcal{D}$ over $\Delta$ with $\tau_i = \alpha \circ \sigma_i$.*

*Proof.* We will prove the case when there are no marked points ($n = 0$) and the generic fiber $\mathcal{C}^* \cong \mathcal{D}^*$ is smooth over $\Delta^*$. We leave the general case to the reader.

Let $\widetilde{\mathcal{C}} \to \mathcal{C}$ and $\widetilde{\mathcal{D}} \to \mathcal{D}$ be the minimal resolutions (Theorem E.1.1). Let $\Gamma \subset \widetilde{\mathcal{C}} \times_\Delta \widetilde{\mathcal{D}}$ be the closure of the graph $\mathcal{C}^* \xrightarrow{(\mathrm{id}, \alpha^*)} \mathcal{C}^* \times_{\Delta^*} \mathcal{D}^*$ of $\alpha^*$ and let $\widetilde{\Gamma} \to \Gamma$

be the minimal resolution. We have a commutative diagram

$$\begin{tikzcd} & \widetilde{\Gamma} \arrow[dl] \arrow[dr] & \\ \widetilde{\mathcal{C}} \arrow[d] & & \widetilde{\mathcal{D}} \arrow[d] \\ \mathcal{C} \arrow[dr] & & \mathcal{D} \arrow[dl] \\ & \Delta. & \end{tikzcd} \tag{5.5.1}$$

Since $\widetilde{\Gamma} \to \widetilde{\mathcal{C}}$ and $\widetilde{\Gamma} \to \widetilde{\mathcal{C}}$ are birational morphisms of smooth projective surfaces over $\Delta$ and the relative dualizing sheaves are line bundles, we have identifications of the pluricanonical sections

$$\Gamma(\widetilde{\mathcal{C}}, \omega_{\widetilde{\mathcal{C}}/\Delta}^{\otimes k}) \cong \Gamma(\widetilde{\Gamma}, \omega_{\widetilde{\Gamma}/\Delta}^{\otimes k}) \cong \Gamma(\widetilde{\mathcal{D}}, \omega_{\widetilde{\mathcal{D}}/\Delta}^{\otimes k})$$

for each non-negative integer $k$; see [Har77, Thm. II.8.19]. Furthermore, we know that $\mathcal{C}$ and $\mathcal{D}$ are the stable models of $\widetilde{\mathcal{C}}$ and $\widetilde{\mathcal{C}}$ obtained by contracting rational tails and bridges (Proposition 5.3.20). Thus we have an isomorphism

$$\mathcal{C} \cong \operatorname{Proj} \bigoplus_k \Gamma(\widetilde{\mathcal{C}}, \omega_{\widetilde{\mathcal{C}}/\Delta}^{\otimes k}) \cong \operatorname{Proj} \bigoplus_k \Gamma(\widetilde{\mathcal{D}}, \omega_{\widetilde{\mathcal{D}}/\Delta}^{\otimes k}) \cong \mathcal{D}$$

extending $\alpha^* \colon \mathcal{C}^* \to \mathcal{D}^*$. □

**Remark 5.5.16.** We can also argue more explicitly using our understanding of the birational geometry of surfaces. First, notice that the local structure of the surface $\mathcal{C}$ or $\mathcal{D}$ around a node $z$ in the central fiber is of the form $xy = t^{n+1}$, where $t \in R$ is a uniformizer (Theorem 5.2.18). This is an $A_n$-surface singularity and in particular normal, and its preimage under the resolution $\widetilde{\mathcal{C}} \to \mathcal{C}$ is a chain $E_1 \cup \cdots \cup E_n$ of rational bridges with $E_i^2 = -2$. By construction, there are no smooth rational $-1$ curves in the fibers of $\widetilde{\mathcal{C}} \to \mathcal{C}$ and $\widetilde{\mathcal{D}} \to \mathcal{D}$, and since $\mathcal{C}$ and $\mathcal{D}$ are families of stable curves, they have no rational tails and thus no smooth rational $-1$ curves. We conclude that $\widetilde{\mathcal{C}}$ and $\widetilde{\mathcal{D}}$ are birational smooth surfaces over $\Delta$ with no smooth rational $-1$ curves whose generic fibers $\mathcal{C}^*$ and $\mathcal{D}^*$ are isomorphic.

By the Structure Theorem of Birational Morphisms of Surfaces (Theorem E.1.3), both $\widetilde{\Gamma} \to \widetilde{\mathcal{C}}$ and $\widetilde{\Gamma} \to \widetilde{\mathcal{D}}$ are the compositions of finite sequences of blow ups at closed points. Since $\widetilde{\Gamma}$ is minimal over $\Gamma$, there are no smooth rational $-1$ curves in $\widetilde{\Gamma}$ that get contracted under both $\widetilde{\Gamma} \to \widetilde{\mathcal{C}}$ and $\widetilde{\Gamma} \to \widetilde{\mathcal{D}}$.

We now claim that $\widetilde{\Gamma} \to \widetilde{\mathcal{C}}$ and $\widetilde{\Gamma} \to \widetilde{\mathcal{D}}$ are isomorphism. Suppose for instance that $\widetilde{\Gamma} \to \widetilde{\mathcal{C}}$ is not an isomorphism. Then there is a smooth rational $-1$ curve $E \subset \widetilde{\Gamma}$ not contracted under $\widetilde{\Gamma} \to \widetilde{\mathcal{D}}$ and let $E_{\widetilde{\mathcal{D}}} \subset \widetilde{\mathcal{D}}$ be its image. On the one hand, since blowing up only decreases the self-intersection number (indeed, if we write the pre-image of $E_{\widetilde{\mathcal{D}}}$ in $\widetilde{\Gamma}$ as $E + F$, then the projection formula implies that $E_{\widetilde{\mathcal{D}}}^2 = E \cdot (E+F) = E^2 + E \cdot F$), we have that $E_{\widetilde{\mathcal{D}}}^2 \geq E^2 = -1$. The Hodge Index Theorem for Exceptional Curves (Theorem E.1.4) implies however that the self-intersection of $E_{\widetilde{\mathcal{D}}}$ must be negative, and we conclude that $E_{\widetilde{\mathcal{D}}}^2 = -1$. On the other hand, since $E_{\widetilde{\mathcal{D}}}$ is not a smooth rational $-1$ curve, $E_{\widetilde{\mathcal{D}}}$ must be a singular curve and one of the blow ups in the composition $\widetilde{\Gamma} \to \widetilde{\mathcal{D}}$ must be along a singular point of $E_{\widetilde{\mathcal{D}}}$. But this implies that exceptional locus $F$ of $\widetilde{\Gamma} \to \widetilde{\mathcal{D}}$ intersects $E$ with multiplicity at least 2 so that $E^2_{\widetilde{\mathcal{D}}} \geq E^2 + 2$, a contradiction.

We finish the proof as before by observing that both $\mathcal{C}$ and $\mathcal{D}$ are the stable models of $\widetilde{\mathcal{C}} \cong \widetilde{\mathcal{D}}$. Since the stable model is unique (Proposition 5.3.20), there is an isomorphism $\mathcal{C} \xrightarrow{\sim} \mathcal{D}$ extending $\mathcal{C}^* \xrightarrow{\sim} \mathcal{D}^*$.

## 5.6 Gluing and forgetful morphisms

### 5.6.1 Gluing morphisms

**Proposition 5.6.1.** *There are finite morphisms of algebraic stacks*

$$\begin{gathered} \overline{\mathcal{M}}_{i,n} \times \overline{\mathcal{M}}_{g-i,m} \to \overline{\mathcal{M}}_{g,n+m-2} \\ \big((C, p_1, \ldots, p_n), (C', p'_1, \ldots, p'_m)\big) \mapsto (C \cup C', p_1, \ldots, p_{n-1}, p'_1, \ldots, p'_m). \end{gathered} \tag{5.6.1}$$

*and*

$$\begin{gathered} \overline{\mathcal{M}}_{g-1,n} \to \overline{\mathcal{M}}_{g,n-2} \\ (C, p_1, \ldots, p_n) \mapsto (C/_{p_{n-1} \sim p_n}, p_1, \ldots, p_{n-2}). \end{gathered} \tag{5.6.2}$$

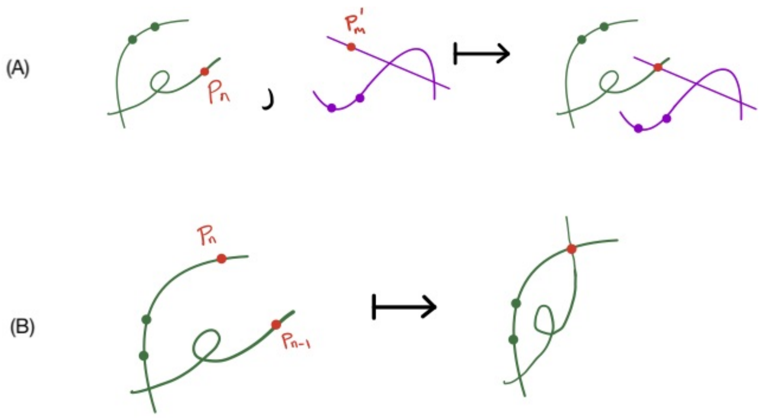



Figure 5.13: (A) is an example of (5.6.1) while (B) is an example of (5.6.2)

**Remark 5.6.2.** To simplify the notation, we chose to write only the case of gluing the $n$th marked point $p_n$ and the $m$th marked point $p'_m$ curve in (5.6.1), and likewise only case of gluing the $p_{n-1}$ and $p_n$ in (5.6.2). Clearly the same holds for the gluing of any two marked points.

*Sketch.* To simplify the notation, we will establish the proposition in the following two cases:

(a) In (5.6.1), we assume $n = m = 1$.

(b) In (5.6.2), we assume $n = 2$.

Note that once we establish the existence of the morphisms of algebraic stacks, it follows from Stable Reduction (Theorem 5.5.1) that the morphisms are proper. By inspection, they are clearly representable and have finite fibers; thus the morphisms are finite.

Case (a): Let $(\mathcal{C} \xrightarrow{\pi} S, \sigma)$ and $(\mathcal{C}' \xrightarrow{\pi'} S, \sigma')$ be two families of 1-pointed stable curves over a scheme $S$.

*Argument 1 (pushout construction):* Consider the pushout diagram

$$\begin{tikzcd}
S \arrow[r, "\sigma"] \arrow[d, "\sigma'"] & \mathcal{C} \arrow[d] \\
\mathcal{C}' \arrow[r] & \widetilde{\mathcal{C}}
\end{tikzcd}$$

which exists by Ferrand's Theorem on the Existence of Pushouts (Theorem A.8.1). We claim that $\widetilde{\mathcal{C}} \to S$ is a family of stable curves. First, note that $\widetilde{\mathcal{C}} \to S$ is proper as there is a finite cover $\mathcal{C} \sqcup \mathcal{C}' \to \mathcal{C}$ with $\mathcal{C} \sqcup \mathcal{C}'$ proper over $S$. One can use properties of pushouts to show that $\widetilde{\mathcal{C}} \to S$ is flat (missing details). It remains to show that the geometric fibers of $\widetilde{\mathcal{C}} \to S$ are stable curves and in particular nodal.

For every point $s \in S$, since $\sigma(s)$ is a smooth point of $\mathcal{C}$, there is an étale neighborhood $\operatorname{Spec} A[x] \to \mathcal{C}$ of $\sigma(s)$ which pulls back to an étale neighborhood $\operatorname{Spec} A \to S$ of $s$. Since an étale morphism from an affine scheme extend over closed immersions (missing reference), there is an étale neighborhood $\operatorname{Spec} A[y] \to \mathcal{C}'$ of $\sigma'(s)$ which also pulls back to $\operatorname{Spec} A \to S$. The geometric pushout of $[\operatorname{Spec} A[x] \leftarrow \operatorname{Spec} A \to \operatorname{Spec} A[y]]$ is $\operatorname{Spec} A[x,y]/(xy)$, and we have a commutative cube

$$\begin{tikzcd}
& \operatorname{Spec} A \arrow[rr, hook] \arrow[dl] \arrow[dd] & & \operatorname{Spec} A[y] \arrow[dl] \arrow[dd] \\
\operatorname{Spec} A[x] \arrow[rr, hook] \arrow[dd] & & \operatorname{Spec} A[x,y]/(xy) \arrow[dd] & \\
& S \arrow[rr, hook] \arrow[dl] & & \mathcal{C} \arrow[dl] \\
\mathcal{C}' \arrow[rr, hook] & & \widetilde{\mathcal{C}} &
\end{tikzcd}$$

We see by Proposition A.8.5 that $\operatorname{Spec} A[x,y]/(xy) \to \widetilde{\mathcal{C}}$ is an étale neighborhood of the image of $s$. This shows that $\widetilde{\mathcal{C}} \to S$ is nodal along $S$, and since $\widetilde{\mathcal{C}}$ is either isomorphic to $\mathcal{C}$ or $\mathcal{C}'$ outside $S$, we see that $\widetilde{\mathcal{C}} \to S$ is a nodal family of curves. Finally one checks (missing details) that $\widetilde{\mathcal{C}}_s$ is identified with the nodal union $\mathcal{C}_s$ and $\mathcal{C}_{s'}$, which is stable.

*Argument 2 (*Proj *construction):* We know that $\omega_{\mathcal{C}}(\sigma)$ is ample. There is a surjection $\omega_{\mathcal{C}}(\sigma) \to \mathcal{O}_{\sigma_1}$ and for each $k \geq 0$, the pushforward of the surjection $(\omega_{\mathcal{C}}(\sigma))^{\otimes k} \to \mathcal{O}_{\sigma_1}$ under $\pi \colon \mathcal{C} \to S$ is $\pi_*(\omega_{\mathcal{C}}(\sigma)^{\otimes k}) \to \mathcal{O}_S$. We have a similar construction for $\pi' \colon \mathcal{C}' \to S$, and we can consider the fiber product of quasi-coherent $\mathcal{O}_S$-modules

$$\begin{tikzcd}
\mathcal{A}_k \arrow[r] \arrow[d] & \pi_*(\omega_{\mathcal{C}}(\sigma)^{\otimes k}) \arrow[d] \\
\pi_*(\omega_{\mathcal{C}'}(\sigma')^{\otimes k}) \arrow[r] & \mathcal{O}_S
\end{tikzcd}$$

One checks that $\mathcal{A} := \bigoplus_{k \geq 0} \mathcal{A}_k$ is a finitely generated quasi-coherent $\mathcal{O}_S$-algebra and that $\widetilde{\mathcal{C}} := \mathcal{P}\mathrm{roj}_S\, \mathcal{A}$ is a family of stable curves over $S$ such that $\widetilde{\mathcal{C}}_s$ is the nodal union $\mathcal{C}_s$ of $\mathcal{C}_{s'}$.

Case (b): Let $(\mathcal{C} \to S, \sigma_1, \sigma_2)$ be a 2-pointed family of stable curves over a scheme $S$.

*Argument 1 (pushout construction):* We use the pushout diagram

$$\begin{tikzcd} S \sqcup S \arrow[r, "\sigma_1 \sqcup \sigma_2"] \arrow[d] & \mathcal{C} \arrow[d] \\ S \arrow[r] & \widetilde{\mathcal{C}} \end{tikzcd}$$

By the étale local properties of pushouts (Proposition A.8.5), the local structure of $\widetilde{\mathcal{C}}$ is determined by the pushout diagram

$$\begin{tikzcd} \operatorname{Spec} A \times A \arrow[r, "(0,1)"] \arrow[d] & \operatorname{Spec} A[t] \arrow[d] \\ \operatorname{Spec} A \arrow[r] & \operatorname{Spec} A \times_{A \times A} A[t]. \end{tikzcd}$$

The subalgebra $A \times_{A \times A} A[t] \subset A[t]$ consists of functions $f \in A[t]$ such that $f(0) = f(1) \in A$. The elements $x := t^2 - 1$ and $y := t^3 - t$ generate $A \times_{A \times A} A[t]$ as an $A$-algebra and since $x$ and $y$ satisfy $y^2 = x^2(x+1)$, we see that $A \times_{A \times A} A[t] \cong A[x,y]/(y^2 - x^2(x+1))$.

*Argument 2 (Proj construction):* One defines $\widetilde{\mathcal{C}} := \mathcal{P}\mathrm{roj}_S \bigoplus_{k \geq 0} \mathcal{A}_k$ where $\mathcal{A}_k$ is defined as the fiber product

$$\begin{tikzcd} \mathcal{A}_k \arrow[r] \arrow[d] & \mathcal{O}_S \arrow[d, "\Delta"] \\ \pi_*(\omega_{\mathcal{C}}(\sigma_1)^{\otimes k}) \sqcup \pi_*(\omega_{\mathcal{C}}(\sigma_2)^{\otimes k}) \arrow[r] & \mathcal{O}_S \sqcup \mathcal{O}_S. \end{tikzcd}$$

$\square$

### 5.6.2 Boundary divisors of $\overline{\mathcal{M}}_g$

Define the closed substacks

$$\begin{aligned} \delta_0 &= \operatorname{im}(\overline{\mathcal{M}}_{g-1,2} \to \overline{\mathcal{M}}_g) \\ \delta_i &= \operatorname{im}(\overline{\mathcal{M}}_{i,1} \times \overline{\mathcal{M}}_{g-i,1} \to \overline{\mathcal{M}}_g) \end{aligned}$$

where $i = 1, \ldots, \lfloor g/2 \rfloor$.

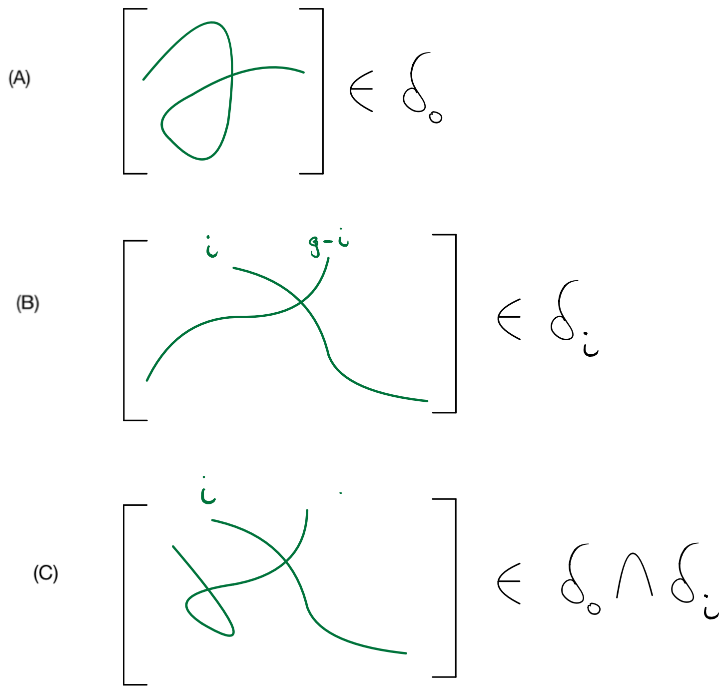



Figure 5.14: Examples of stable curves in the boundary.

Once we show that $\mathcal{M}_g$ is dense in $\overline{\mathcal{M}}_g$, it will follow that $\delta_0$ and $\delta_i$ are the closure of the locus of curves with a single node as featured in (A) and (B) of Figure 5.14.

To see that $\delta_0$ and $\delta_i$ are divisors in $\overline{\mathcal{M}}_g$, we can do a simple dimension count. As $\overline{\mathcal{M}}_{g-1,2} \to \overline{\mathcal{M}}_g$ and $\overline{\mathcal{M}}_{i,1} \times \overline{\mathcal{M}}_{g-i,1} \to \overline{\mathcal{M}}_g$ are finite morphisms, we compute that $\dim \delta_0 = \dim \overline{\mathcal{M}}_{g-1,2} = 3(g-1) - 3 + 2 = 3g - 4$ and that $\dim \delta_i = \dim \overline{\mathcal{M}}_{i,1} + \dim \overline{\mathcal{M}}_{g-i,1} = (3i - 3 + 1) + (3(g-i) - 3 + 1) = 3g - 4$.

By analyzing the formal deformation space of a stable curve, one can show that more is true: $\delta = \delta_0 \cup \cdots \cup \delta_{\lfloor \frac{g}{2} \rfloor}$ is a normal crossings divisor.

### 5.6.3 The forgetful morphism

**Proposition 5.6.3.** *There is a morphism of algebraic stacks*

$$\begin{aligned} \overline{\mathcal{M}}_{g,n} &\to \overline{\mathcal{M}}_{g,n-1} \\ (C, p_1, \ldots, p_n)) &\mapsto (C^{\mathrm{st}}, p_1, \ldots, p_{n-1}). \end{aligned}$$

*where* $(C^{\mathrm{st}}, p_1, \ldots, p_{n-1})$ *is the stable model of* $(C, p_1, \ldots, p_{n-1})$.

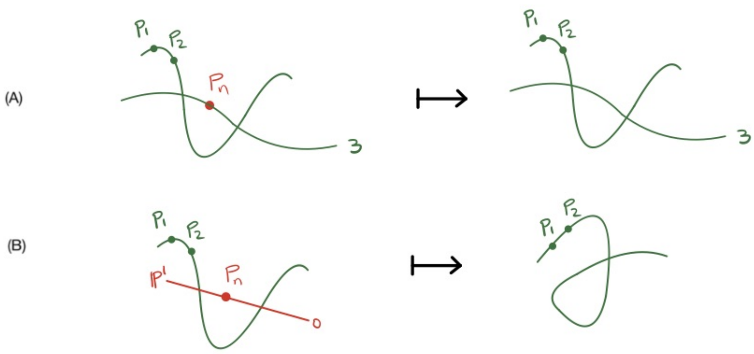


Figure 5.15: In (A), the $n$th point is simply forgotten. In (B), if $p_n$ is forgotten, the curve is no longer stable and we must contract the rational bridge.

*Proof.* If $(\mathcal{C} \to S, \sigma_1, \ldots, \sigma_n)$ is an $n$-pointed family of stable curves, then if we forget the $n$th section, the $(n-1)$-pointed family $(\mathcal{C} \to S, \sigma_1, \ldots, \sigma_{n-1})$ may not be stable. However, we have already constructed the stable model $(\mathcal{C}^{\mathrm{st}} \to S, \sigma_1, \ldots, \sigma_{n-1})$ in Proposition 5.3.20. □

### 5.6.4 The universal family $\overline{\mathcal{M}}_{g,1} \to \overline{\mathcal{M}}_g$

Let $\mathcal{U}_g \to \overline{\mathcal{M}}_g$ be the universal family: this is a proper and flat morphism of algebraic stacks whose geometric fibers are genus $g$ curves. (The existence of the universal family follows from applying descent and the 2-Yoneda Lemma (2.3.20) to the identity morphism $\mathrm{id}\colon \overline{\mathcal{M}}_g \to \overline{\mathcal{M}}_g$.) Objects of $\mathcal{U}_g$ over a scheme $S$ correspond to a family of stable curves $\mathcal{C} \to S$ and a section $\sigma\colon S \to \mathcal{C}$ (that may land in the relative singular locus).

There is a morphism of algebraic stacks

$$\overline{\mathcal{M}}_{g,1} \to \mathcal{U}_g$$

sending $(\mathcal{C} \to S, \sigma)$ to $(\mathcal{C}^{\mathrm{st}} \to S, \sigma^{\mathrm{st}})$ where $\pi\colon \mathcal{C} \to \mathcal{C}^{\mathrm{st}}$ is the stabilization of $\mathcal{C} \to S$ (see Proposition 5.3.20) and $\sigma^{\mathrm{st}} = \pi \circ \sigma$. This yields a commutative diagram

$$\begin{tikzcd} \overline{\mathcal{M}}_{g,1} \arrow[r] \arrow[dr] & \mathcal{U}_g \arrow[d] \\ & \overline{\mathcal{M}}_g, \end{tikzcd}$$

where $\overline{\mathcal{M}}_{g,1} \to \overline{\mathcal{M}}_g$ is the forgetful morphism of Proposition 5.6.3.

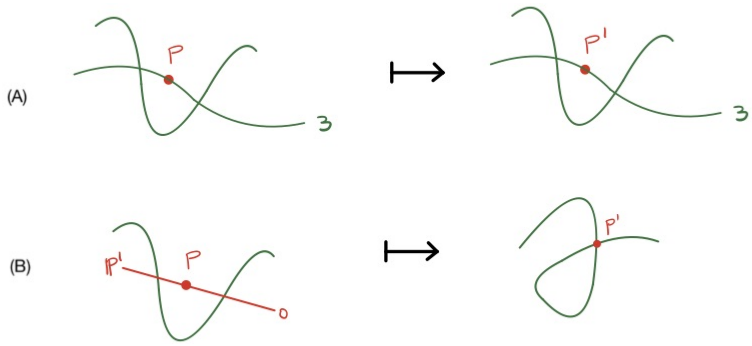



Figure 5.16: In Example (A), $\overline{\mathcal{M}}_{g,1} \to \mathcal{U}_g$ sends $(C,p)$ to itself while in Example (B), the morphism sends $(C,p)$ to the curve $(C',p')$ obtained by contracting the rational bridge.

**Proposition 5.6.4.** *The morphism $\overline{\mathcal{M}}_{g,1} \to \mathcal{U}_g$ is an isomorphism over $\overline{\mathcal{M}}_g$. In other words, the morphism $\overline{\mathcal{M}}_{g,1} \to \overline{\mathcal{M}}_g$, which forgets the marked point and stabilizes the curve, is the universal family.*

*Proof.* TO ADD $\square$

**Exercise 5.6.5.** Show that the above arguments can be modified to show that $\overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ is a universal family.

## 5.7 Irreducibility

In this section, we show that the algebraic stack $\overline{\mathcal{M}}_{g,n}$ is irreducible over an algebraically closed field $\Bbbk$. After reviewing properties of branched coverings in §5.7.1, we provide the classical topological argument due to Clebsch and Hurwitz in the late 19th century establishing irreducibility of $M_g$ in characteristic 0 (Theorem 5.7.12). We then provide a purely algebraic argument for the irreducibility of $\overline{\mathcal{M}}_{g,n}$ (Theorem 5.7.15) by using admissible covers to show that every smooth curve degenerates to a singular stable curve and induction on the genus to show that the boundary $\delta = \overline{\mathcal{M}}_g \setminus \mathcal{M}_g$ is connected. Finally, in §5.7.4, we provide the arguments from the seminal papers from 1969 of Deligne and Mumford [DM69] and Fulton [Ful69] which establish the irreducibility of $\overline{M}_{g,n}$ in positive characteristic (where Fulton's argument has the restriction $p > g+1$) by reduction to characteristic 0.

We begin with a few remarks regarding the relations between the connectedness/irreducibility of $\overline{\mathcal{M}}_{g,n}$, $\mathcal{M}_g$, and their coarse moduli spaces. Since $\overline{\mathcal{M}}_{g,n}$ is a smooth algebraic stack, its irreducibility is equivalent to its connectedness. Moreover, since $\overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ is the universal family (Proposition 5.6.4) and in particular has connected fibers, it suffices to verify the connectedness of $\overline{\mathcal{M}}_g$. We thus have equivalences

$$\begin{aligned} \overline{\mathcal{M}}_{g,n} \text{ irreducible} &\iff \overline{\mathcal{M}}_{g,n} \text{ connected} \\ &\iff \overline{\mathcal{M}}_g \text{ connected} \\ &\iff \mathcal{M}_g \text{ connected and dense in } \overline{\mathcal{M}}_g \end{aligned}$$

Finally, we note that since the coarse moduli space $\overline{\mathcal{M}}_{g,n} \to \overline{M}_{g,n}$ induces a homeomorphism $|\overline{\mathcal{M}}_{g,n}| \xrightarrow{\sim} |\overline{M}_{g,n}|$ on topological spaces, each statement above can be equivalently stated in terms of the coarse moduli space.

### 5.7.1 Branched coverings

If $f\colon C \to D$ is a finite separable morphism of smooth connected curves and $P \in C(\Bbbk)$ with image $Q$, then the *ramification index* $e_P$ is the integer $e$ such that under the map $\mathcal{O}_{D,Q} \to \mathcal{O}_{C,P}$ a uniformizer $t \mapsto us^e$ maps to a unit times the $e$th power of a uniformizer. We say that $f$ is *ramified at* $P$ if $e_P > 1$, *tamely ramified at* $P$ if $\operatorname{char} k = 0$ or $\operatorname{char}(k) \not| e_p$, and *unramified at* $P$ if $e_P = 1$.

There is a short exact sequence of differentials

$$0 \to f^*\Omega_D \to \Omega_C \to \Omega_{C/D} \to 0. \tag{5.7.1}$$

Indeed, the sequence above is always right exact. Since $f^*\Omega_D$ and $\Omega_C$ are line bundles, injectivity for the left map is equivalent to the map being nonzero. However, $K(D) \to K(C)$ is separable so $\Omega_{C/D} \otimes K(C) = \Omega_{K(C)/K(D)} = 0$, and thus $f^*\Omega_D \to \Omega_C$ is nonzero at the generic point. Examining the sequence above at the stalks at a point $P \in C(\Bbbk)$, the differential $dt$ maps to $d(us^{e_P}) = eus^{e_P-1}ds + s^{e_P}du$. If $f$ is tamely ramified at $p$, then $(\Omega_{C/D})_P \cong \mathcal{O}_{C,p}\langle ds\rangle/(s^{e_P-1}ds)$ and $\operatorname{length}(\Omega_{C/D})_P = \dim \Omega_{C/D} \otimes \kappa(p) = e_P - 1$.

If $f$ is ramified at $P$, then the scheme-theoretic fiber over $f(P)$ at $P$ is isomorphic to $\operatorname{Spec} \kappa(P)$, and thus this agrees with the definition of unramified in Unramified Equivalences A.3.4. Moreover, since $f$ is flat, $f$ is unramified at $P$ if and only if $f$ is étale at $P$.

**Definition 5.7.1.** Let $\Bbbk$ be an algebraically closed field.

(1) A *branched covering* is a finite separable morphism $f\colon C \to D$ of smooth connected curves over $\Bbbk$.

(2) A *simply branched covering* is a branched covering such that

- there is at most one ramification point in every fiber, and
- every ramification point $P \in C(\Bbbk)$ is tamely ramified with ramification index $e_P = 2$.

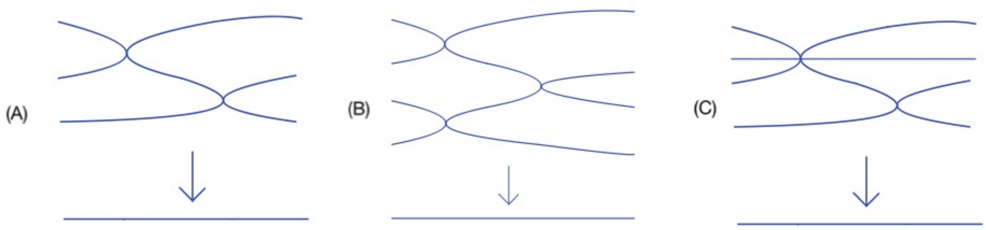



Figure 5.17: Examples of branched coverings over $\mathbb{P}^1$: (A) is simply branched while (B) and (C) are not. While the picture may suggest that the source curve $C$ is not smooth, $C$ is in fact smooth over the base field $\Bbbk$. However, the map $C \to \mathbb{P}^1$ is not smooth, and the pictures above are designed to reflect the singularities of $C$ over $\mathbb{P}^1$.

**Theorem 5.7.2** (Riemann–Hurwtiz)**.** *If $f\colon C \to D$ is a branched covering and $R = \sum_{P \in C(\Bbbk)} \operatorname{length}(\Omega_{C/D})_P \cdot P$ is the ramification divisor on $C$, then $\Omega_C \cong f^*\Omega_D \otimes \mathcal{O}_C(R)$ and*

$$2g(C) - 2 = \deg(f)(2g(D) - 2) + \deg R.$$

*In particular, $f\colon C \to \mathbb{P}^1$ is simply branched, then it is ramified over $2g + 2d - 2$ distinct points.*

*Proof.* This follows directly from the exact sequence (5.7.1). See also [Har77, Prop. IV.2.3] $\square$

**Example 5.7.3.** For a local model of a branched cover, consider the map $f\colon \mathbb{A}^1 \to \mathbb{A}^1$ defined by $x \mapsto x^n$. The relative sheaf of differentials is $\Omega_{\mathbb{A}^1/\mathbb{A}^1} = k[x]\langle dx\rangle/(nx^{n-1}dx)$ and thus if $\operatorname{char}(\Bbbk)$ does not divide $n$, then $f\colon \mathbb{A}^1 \to \mathbb{A}^1$ is étale over $\mathbb{A}^1$ and ramified at 0 with index $n-1$.

**Exercise 5.7.4.** Show that every branched covering is étale locally isomorphic to $\mathbb{A}^1 \to \mathbb{A}^1, x \mapsto x^n$ around a branched point of index $n-1$.

**Lemma 5.7.5.** *Let $C$ be a smooth, connected, and projective curve of genus $g$ over an algebraically closed field $\Bbbk$ of characteristic 0. If $L$ is a line bundle of degree $d \geq g+1$, then for a general linear series $V \subset \mathrm{H}^0(C, L)$ of dimension 2, $C \xrightarrow{V} \mathbb{P}^1$ is simply branched.*

*Proof.* We proceed with a dimension count. Since $\mathrm{h}^0(C, L) = d+1-g$, the dimension of the Grassmannian $\mathrm{Gr}(2, \mathrm{H}^0(L))$ of 2-dimensional subspaces is $2(d-g-1)$. Since $\operatorname{char}(\Bbbk) = 0$, every finite morphism $C \to \mathbb{P}^1$ is automatically separable. Thus, if $C \xrightarrow{V} \mathbb{P}^1$ is not simply branched, then one of the following three conditions must hold:

(a) $V$ has a base point;

(b) there exists a ramification point with index $> 2$; or

(c) there exists two ramification points in the same fiber.

We handle only case (b) and leave the other cases to the reader. There must exist a section $s \in V$ vanishing to order 3 at a point $p \in C$, i.e. $s \in \mathrm{H}^0(C, L(-3p))$. The dimension of $V \in \mathrm{Gr}(2, \mathrm{H}^0(L))$ having a branched point at $p \in C$ with index at least 3 can be calculated as

$$\dim \mathbb{P}\mathrm{H}^0(L(-3p)) + \dim \mathbb{P}(\mathrm{H}^0(L)/\langle s\rangle) = 2d - 2g - 4.$$

Varying $p \in C$, the locus of all $V \in \mathrm{Gr}(2, \mathrm{H}^0(L))$ failing condition (b) is thus $2d - 2g - 3 = \dim \mathrm{Gr}(2, \mathrm{H}^0(L)) - 1$. $\square$

For a branched cover $C \to \mathbb{P}^1$, we denote by $\mathrm{Aut}(C/\mathbb{P}^1) = 1$ the group of automorphisms $C \to C$ over $\mathbb{P}^1$.

**Lemma 5.7.6.** *If $C \to \mathbb{P}^1$ is a simply branched cover of degree $d > 2$ in characteristic 0, then $\mathrm{Aut}(C/\mathbb{P}^1)$ is trivial.*

*Proof.* Every automorphism $C \to C$ over $\mathbb{P}^1$ must fix the $2g+2d-2$ branched points, but this contradicts the classical result of Mayer, which asserts that there are no non-trivial automorphisms of a smooth curve fixing more than $2g+2$ points. $\square$

The above lemma shows that no stacky issues arise when defining moduli spaces of simply branched covering. We define

$$H_{d,b} := \{C \to \mathbb{P}^1 \text{ simply branched covering of degree } d \text{ over } b \text{ points}\}$$

as the moduli space of simply branched coverings where

$$b = 2g + 2d - 2.$$

The moduli space $H_{d,b}$ can be defined either as a topological space (if $\Bbbk = \mathbb{C}$) or as an algebraic space; we leave the details to the reader. Denoting $\mathrm{Sym}^b \mathbb{P}^1 \setminus \Delta$ as the variety of $b$ unordered distinct points in $\mathbb{P}^1$ (which can also be written as the complement $\mathbb{P}^b \setminus \Delta$ of the discriminant hypersurface), we have a diagram

$$\begin{tikzcd} & H_{d,b} \arrow[dl] \arrow[dr] & \\ \mathcal{M}_g & & \operatorname{Sym}^b \mathbb{P}^1 \setminus \Delta \end{tikzcd} \tag{5.7.2}$$

where a simply branched covering $[C \to \mathbb{P}^1]$ gets mapped to $[C]$ under $H_{d,b} \to \mathcal{M}_g$ and the $b$ branched points under $H_{d,b} \to \operatorname{Sym}^b \mathbb{P}^1 \setminus \Delta$.

**Lemma 5.7.7.** *In characteristic* 0*, the morphism* $H_{d,b} \to \operatorname{Sym}^b \mathbb{P}^1 \setminus \Delta$ *is finite and étale.*

*Proof.* We only establish étaleness. It is straightforward to see that $H_{d,b} \to \operatorname{Sym}^b \mathbb{P}^1 \setminus \Delta$ is a topological covering space. Consulting Figure 5.18, given a branched covering $f \colon C \to \mathbb{P}^1$ and a branched point $p \in C$, we can choose an analytic open neighborhood $U \subset \mathbb{P}^1$ around $f(p)$ such that $f^{-1}(U) \to U$ is isomorphic to an open neighborhood of $\mathbb{C} \to \mathbb{C}, x \mapsto x^n$. For every other point $q' \in U$, we can construct a branched cover $C' \to \mathbb{P}^1$ which outside $U$ is the same as $C \to \mathbb{P}^1$ and over $U$ is locally isomorphic to $x \mapsto x^n$ but centered over $q'$ (rather than $f(p)$).

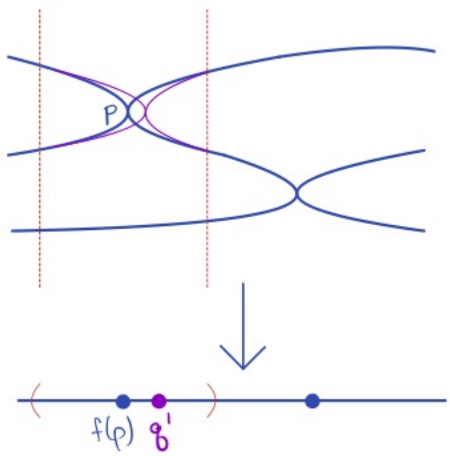



Figure 5.18:

For an algebraic argument, it suffices to show that for a covering $f \colon C \to \mathbb{P}^1$ simply branched over $p_1, \ldots, p_b$, the map

$$\operatorname{Def}(C \xrightarrow{f} \mathbb{P}^1) \to \operatorname{Def}(\{p_i\}_{i=1}^b \subset \mathbb{P}^1)$$

on first-order deformation spaces is bijective. There is an identification $\operatorname{Def}(C \xrightarrow{f} \mathbb{P}^1) = \mathrm{H}^0(C, N_f)$ where $N_f$ sits in a short exact sequence

$$0 \to T_C \to f^* T_{\mathbb{P}^1} \to N_f \to 0.$$

On cohomology, this induces a short exact sequence

$$0 \to \mathrm{H}^0(C, f^* T_{\mathbb{P}^1}) \to \mathrm{H}^0(C, N_f) \to \mathrm{H}^1(C, T_C) \to 0.$$

Riemann–Roch allows us to compute $h^0(C, f^*T_{\mathbb{P}^1}) = 2d+1-g$ and $h^1(T_C) = 3g-3$, and thus $\dim \mathrm{Def}(C \xrightarrow{f} \mathbb{P}^1) = h^0(C, N_f) = 2d + 2g - 2 = b$ is the same as the dimension of $\mathrm{Def}(\{p_i\}_{i=1}^b \subset \mathbb{P}^1)$. We leave the remaining details to the reader. □

#### 5.7.1.1 Relation between algebraic and topological branched coverings

The Clebsch–Hurwitz argument below relies on the following correspondence between topological, analytic, and algebraic branched coverings. (Topological and analytic coverings can be defined analogously to algebraic coverings—to be added.) This can be viewed as a version of the Riemann Existence Theorem.

**Proposition 5.7.8.** *Over $\mathbb{C}$, there are natural bijections*

$$\{C \to \mathbb{P}^1 \text{ algebraic branched coverings}\} \longleftrightarrow \{C \to \mathbb{P}^1 \text{ topological branched coverings}\}$$
$$\longleftrightarrow \{C \to \mathbb{P}^1 \text{ analytic branched coverings}\}$$

*Proof.* An algebraic branched covering is clearly topological, and if $C \to \mathbb{P}^1$ is a topological covering, then the holomorphic structure on $\mathbb{P}^1$ induces naturally a holomorphic structure on $C$ such that $C \to \mathbb{P}^1$ is analytic. The Riemann Existence Theorem implies that every holomorphic branched covering is in fact algebraic. □

#### 5.7.1.2 Monodromy actions

Let $C \to \mathbb{P}^1$ be a (topological) branched covering over $\mathbb{C}$ and $B \subset \mathbb{P}^1$ its ramification locus. Choose a base point $p \in \mathbb{P}^1 \setminus B$. The *monodromy action of* $\pi_1(\mathbb{P}^1 \setminus B, p)$ *on the fiber* $\pi^{-1}(p)$ is defined as follows: for $\gamma \in \pi_1(\mathbb{P}^1 \setminus B, p)$ and $q \in \pi^{-1}(p)$, then the path $\gamma\colon [0,1] \to \mathbb{P}^1$ lifts uniquely to a path $\widetilde{\gamma}\colon [0,1] \to C$ such that $\widetilde{\gamma}(0) = q$ and the action is defined by $\gamma \cdot q = \widetilde{\gamma}(1)$.

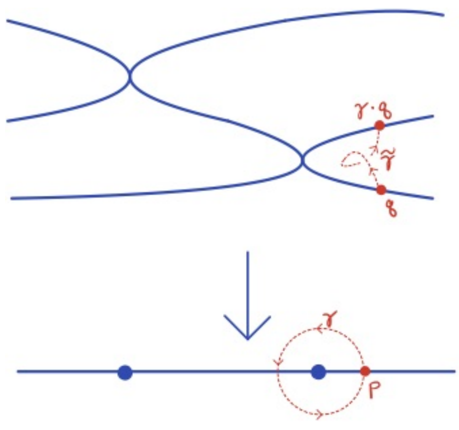



Figure 5.19:

We now summarize some of the key properties of the monodromy action.

**Proposition 5.7.9.** *Let $B \subset \mathbb{P}^1$ be a finite subset, $p \in \mathbb{P}^1 \setminus B$ be a point, and $d > 0$ a positive integer. There is a natural bijection between topological branched coverings $C \to \mathbb{P}^1$ of degree $d$ and group homomorphisms $\rho\colon \pi_1(X \setminus B, x) \to S_d$ such that $\mathrm{im}(\rho) \subset S_d$ is a transitive subgroup. Here two branched covers $C \to \mathbb{P}^1$ and $C' \to \mathbb{P}^1$ are equivalent if there is an isomorphism $C \to C'$ over $\mathbb{P}^1$, and two homomorphisms*

*$\rho, \rho' \colon \pi_1(X \setminus B, x) \to S_d$ are equivalent if they differ by an inner automorphism of $S_d$, i.e. $\exists h \in S_d$ such that $\rho' = \mathrm{h}^{-1}\rho h$.*

*Moreover, if we let $\sigma_1, \dots, \sigma_b$ be simple loops around the $b$ distinct points of $B$, then $\pi_1(\mathbb{P}^1 \setminus B, x) = \langle \sigma_i | \sigma_1 \cdots \sigma_b = 1 \rangle$, and under this correspondence a simply branched cover corresponds to a homomorphism $\pi_1(X \setminus B, p) \to S_d$ such that each $\sigma_i$ maps to a transposition.*

**Remark 5.7.10.** Recall that by definition of a branched covering $C \to \mathbb{P}^1$, the curve $C$ is necessarily connected. This is the reason for the condition above that $\mathrm{im}(\rho) \subset S_d$ is transitive: every group homomorphism $\rho \colon \pi_1(X \setminus B, x) \to S_d$ corresponds to a possibly non-connected branched covering $C \to \mathbb{P}^1$, and $C$ is connected if and only if $\mathrm{im}(\rho) \subset S_d$ is transitive.

**Remark 5.7.11.** Like with Riemann–Hurwitz, the fact that the base is $\mathbb{P}^1$ plays no role: the above proposition holds for arbitrary branched covers of smooth curves (except for the explicit description of $\pi_1$).

### 5.7.2 The Clebsch–Hurwitz argument

We now provide the classical argument due to Clebsch [Cle73] and Hurwitz [Hur1891] that $\mathcal{M}_g$ is connected over $\mathbb{C}$. For a modern treatment, see [Ful69, §1]. This argument uses a single non-algebraic input, namely Riemann's Existence Theorem in the form of Proposition 5.7.8. There are of course other non-algebraic approaches, e.g. using Teichmüller theory.

By taking $d \geq g+1$, we know that every smooth, connected, and projective complex curve $C$ of genus $g$ admits a map $C \to \mathbb{P}^1$, which is a covering simply branched over $b = 2d + 2g - 2$ points (Lemma 5.7.5). This shows that the map

$$H_{d,b} \to \mathcal{M}_g, \qquad [C \to \mathbb{P}^1] \mapsto [C]$$

is surjective, where $H_{d,b}$ is the moduli space of coverings $C \to \mathbb{P}^1$ simply branched over $b$ points. The connectedness of $H_{d,b}$ thus implies the connectedness of $\mathcal{M}_g$.

**Theorem 5.7.12** (Clebsch, Hurwitz)**.** *$H_{d,b}$ is connected.*

*Proof.* We will use the diagram

$$\begin{tikzcd} & H_{d,b} \arrow[dl] \arrow[dr, "\beta"] & \\ \mathcal{M}_g & & \operatorname{Sym}^b \mathbb{P}^1 \setminus \Delta \end{tikzcd}$$

where $H_{d,b} \to \mathcal{M}_g$ is surjective (Lemma 5.7.5) and $\beta \colon H_{d,b} \to \operatorname{Sym}^b \mathbb{P}^1 \setminus \Delta$ is finite and étale (Lemma 5.7.7).

For every finite set $B = \{p_1, \dots, p_b\} \subset \mathbb{P}^1$ of $b = 2d+2g-2$ points and $p \in \mathbb{P}^1 \setminus B$, the fundamental group $\pi_1(\mathbb{P}^1 \setminus B, p) = \langle \sigma_i | \sigma_1 \cdots \sigma_b = 1 \rangle$ acts on the fiber $\pi^{-1}(p)$ of a simply branched covering $\pi \colon C \to \mathbb{P}^1$. Similarly, $\pi_1(\operatorname{Sym}^b \mathbb{P}^1 \setminus \Delta, B)$ acts on the fiber $H_{d,B} := \beta^{-1}(B)$ of $\beta \colon H_{d,b} \to \operatorname{Sym}^b \mathbb{P}^1 \setminus \Delta$. Using Proposition 5.7.9, we have bijections

$$\begin{aligned} H_{d,b} = \beta^{-1}(B) &= \{\text{coverings } C \to \mathbb{P}^1 \text{ simply branched over } B\} \\ &= \{\text{group homomorphisms } \pi_1(\mathbb{P}^1 \setminus B, p) \xrightarrow{\rho} S_d \text{ such that} \\ &\qquad \mathrm{im}(\rho) \subset S_d \text{ is transitive and each } \rho(\sigma_i) \text{ is a transposition}\} \\ &= \{(\tau_1, \dots, \tau_b) \in (S_d)^b \mid \text{each } \tau_i \text{ is a transposition and } \tau_1 \cdots \tau_b = 1\}. \end{aligned}$$

The connectedness of $H_{d,b}$ is equivalent to the transitivity of the action of $\pi_1(\mathrm{Sym}^b \mathbb{P}^1 \setminus \Delta, B)$ on the fiber $H_{d,B}$. The strategy of proof is to find loops in $\mathrm{Sym}^b \mathbb{P}^1 \setminus \Delta$ that act on $(\tau_1, \ldots, \tau_b) \in H_{d,B}$ in a prescribed way and to find enough loops so that we can show that each orbit contains the element

$$\boldsymbol{\tau}^* := \big(\underbrace{(12),(12),(13),(13),\ldots,(1\,d-1),(1\,d-1)}_{2(d-2)},\underbrace{(1d),(1d),\cdots,(1d)}_{2g+2}\big).$$

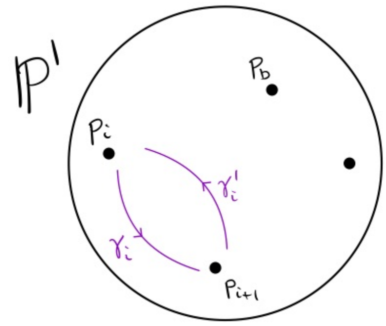



Figure 5.20:

Referring to Figure 5.20, we define the loop

$$\begin{aligned}\Gamma_i \colon [0,1] &\to \mathrm{Sym}^b \mathbb{P}^1 \setminus \Delta \\ t &\mapsto (p_1, \ldots, p_{i-1}, \gamma_i(t), \gamma_i'(t), p_{i+2}, \ldots, p_b).\end{aligned}$$

One checks that

$$\Gamma_i \cdot (\tau_1, \ldots, \tau_b) = (\tau_1, \ldots, \tau_{i-1}, \tau_i^{-1}\tau_{i+1}\tau_i, \tau_i, \tau_{i+2}, \ldots, \tau_b)$$

and that for every element $(\tau_1, \ldots, \tau_b) \in H_{d,B}$, there exists a sequence $\Gamma_{i_1}, \ldots, \Gamma_{i_k}$ of loops such that $\boldsymbol{\tau}^* = \Gamma_{i_1}\Gamma_{i_2}\cdots\Gamma_{i_k} \cdot (\tau_1, \ldots, \tau_b)$. We leave the details of this combinatorial problem to the reader.

□

### 5.7.3 Irreducibility using admissible covers

We now give a completely algebraic argument of the irreducibility of $\overline{\mathcal{M}}_g$ in characteristic 0. The main idea is to show that every smooth curve of genus $g$ degenerates in a one-dimensional family to a singular stable curve (Proposition 5.7.13) and to show the connectedness of $\delta = \overline{\mathcal{M}}_g \setminus \mathcal{M}_g$ using the inductive structure of the boundary and explicitly the gluing maps of Proposition 5.6.1. The most challenging aspect of this argument is in degenerating a smooth curve to a singular stable curve. To achieve this, we will use the theory of admissible covers. We follow the treatment in Fulton's appendix of the paper [HM82] by Harris and Mumford that introduced admissible covers as a means to compute the Kodaira dimension of $\overline{M}_g$.

**Proposition 5.7.13.** *Let $C$ be a smooth, connected, and projective curve of genus $g$ over an algebraically closed field $\Bbbk$ of characteristic 0. There exists a connected curve $T$ with points $t_1, t_2 \in T$ and a family $\mathcal{C} \to T$ of stable curves such that $\mathcal{C}_{t_1} \cong C$ and $\mathcal{C}_{t_2}$ is a singular stable curve.*

*Proof.* By Lemma 5.7.5, for $d \gg 0$ there exists a finite covering $C \to \mathbb{P}^1$ of degree $d$ simply branched over $b = 2g + 2d - 2$ distinct points $p_1, \dots, p_b \in \mathbb{P}^1$. This defines a $b$-pointed stable curve $G = [\mathbb{P}^1, \{p_i\}] \in M_{0,n}$. By Lemma 5.7.7, we may assume that $G \in M_{0,n}$ is general. Since $\overline{M}_{0,n}$ is connected, $G$ degenerates to the $b$-pointed rational curve $(D_0, q_1, \dots, q_b)$ which is the nodal union of a chain of $b - 2$ $\mathbb{P}^1$'s where $q_1, q_2$ lie on the first $\mathbb{P}^1$, $q_3$ on the second $\mathbb{P}^1$, and so on with $q_{b-1}, q_b$ lying on the last $\mathbb{P}^1$; see Figure 5.21.

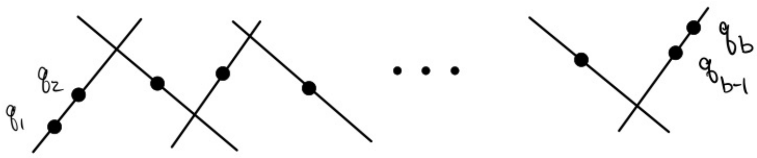



Figure 5.21:

In other words, there is a DVR $R$ with fraction field $K$ and a map $\Delta = \operatorname{Spec} R \to \overline{M}_{0,n}$ corresponding to a $b$-pointed stable family $(\mathcal{D} \to \Delta, \sigma_i)$ such that the generic fiber $(\mathcal{D}^*, \sigma_i^*)$ is isomorphic to $G = (\mathbb{P}^1, \{g_i\})$ and the special fiber to $(D_0, \{q_i\})$. We have a simply branched covering $\mathcal{C}^* \to \Delta^*$ which fits into a diagram

$$\begin{tikzcd}
\mathcal{C}^* \arrow[r, hook, dashed] \arrow[d] & \mathcal{C} \arrow[d, dashed] \\
\mathcal{D}^* \arrow[r, hook] \arrow[d] & \mathcal{D} \arrow[d] \\
\Delta^* \arrow[r, hook] & \Delta
\end{tikzcd}$$

and extends to a finite morphism $\mathcal{C} \to \mathcal{D}$ by taking $\mathcal{C}$ as the integral closure of $\mathcal{O}_{\mathcal{D}}$ in $K(\mathcal{C}^*)$.

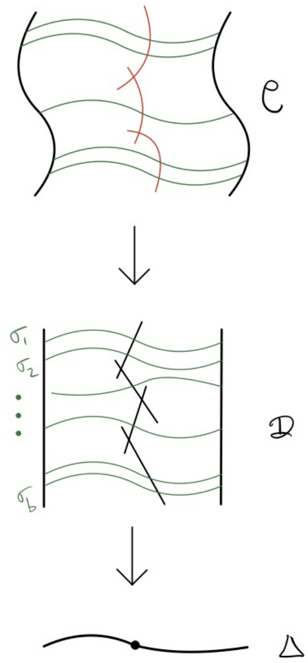



Figure 5.22:

Purity of the branch locus implies that the ramification of $\mathcal{C} \to \mathcal{D}$ is a divisor when restricted to the relative smooth locus of $\mathcal{C} \to \mathcal{D}$. Therefore, the central fiber $\mathcal{C}_0 \to \mathcal{D}_0$ is ramified over $\sigma_1(0), \dots, \sigma_b(0)$ and possibly over irreducible components of $\mathcal{D}_0$ (where $\mathcal{C}_0$ may be non-reduced). As in the proof of stable reduction, after a suitable base change $\Delta \to \Delta, t \mapsto t^m$ and replacing $\mathcal{C}$ with the normalization $\mathcal{C} \times_\Delta \Delta$, we can arrange that $\mathcal{C}_0 \to \mathcal{D}_0$ is ramified only over $\sigma_i(0)$ and possibly over nodes of $\mathcal{D}_0$. By an analysis of possible extensions $\mathcal{C} \to \mathcal{D}$, one can show that $\mathcal{C}_0$ is a nodal curve (missing details). Therefore $\mathcal{C} \to \Delta$ is a family of nodal curves.

Since $\mathcal{C}_0$ necessarily has nodes, we are done if $\mathcal{C}_0$ is a stable curve! Otherwise, we can contract rational tails and bridges to obtain the stable model $\mathcal{C}^{\mathrm{st}} \to \Delta$ (Proposition 5.3.20). We must check that $\mathcal{C}_0^{\mathrm{st}}$ is not smooth. Let $T \subset \mathcal{C}_0^{\mathrm{st}}$ be a smooth irreducible component. Applying Riemann–Hurwitz to the induced morphism $T \to \mathbb{P}^1 \subset \mathcal{D}_0$ shows that $2g(T) - 2 = -2d + R$ where $R$ is the degree of the ramification divisor on $T$. If the component $\mathbb{P}^1 \subset \mathcal{D}_0$ is a rational tail (i.e. is either the first or last $\mathbb{P}^1$ in the chain), then $R \leq 2 + (d-1)$ as $T \to \mathbb{P}^1$ is simply ramified over the two marked points and has index at worst $d-1$ over the node. On the other hand, if $\mathbb{P}^1 \subset \mathcal{D}_0$ is a rational bridge, then $R \leq 1 + 2(d-1)$. In either case, we have $R \leq 2d - 1$ and $2g(T) - 2 \leq -2 + (2d-1) = 1$ which establishes that $g(T) = 0$. We've shown every smooth irreducible component of $\mathcal{C}_0^{\mathrm{st}}$ is rational, which immediately implies that $\mathcal{C}_0^{\mathrm{st}}$ is singular. □

**Proposition 5.7.14.** *If we assume that $\overline{\mathcal{M}}_{g',n'}$ is irreducible for all $g' < g$, then the boundary $\delta = \overline{\mathcal{M}}_g \setminus \mathcal{M}_g$ is connected.*

*Proof.* We write $\delta = \delta_0 \cup \cdots \cup \delta_{\lfloor g/2 \rfloor}$ where $\delta_0 = \mathrm{im}(\overline{\mathcal{M}}_{g-1,2} \to \overline{\mathcal{M}}_g)$ and $\delta_i = \mathrm{im}(\overline{\mathcal{M}}_{i,1} \times \overline{\mathcal{M}}_{g-i,1} \to \overline{\mathcal{M}}_g)$ as defined in §5.6.2 using the gluing maps from Proposition 5.6.1. The hypotheses imply that $\delta_0$ and $\delta_i$ are connected (and even irreducible). But on the other hand, the boundary divisors $\delta_i$ intersect! Namely, for every $i, j = 0, \dots, \lfloor g/2 \rfloor$, the intersection $\delta_i \cap \delta_j$ contains curves as in Figure 5.23.

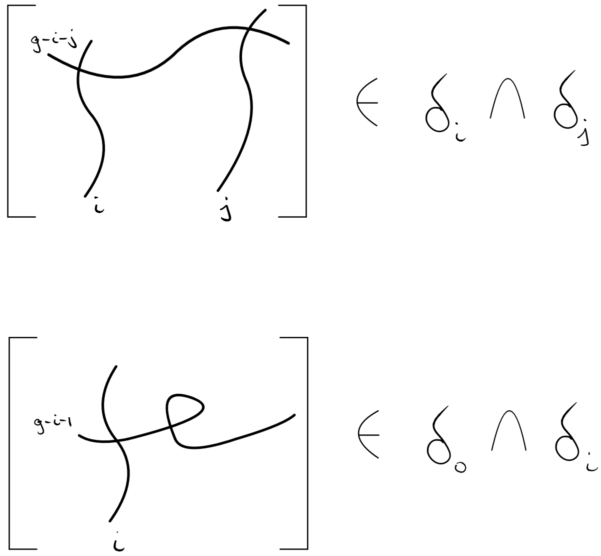



Figure 5.23:

$\square$

**Theorem 5.7.15.** *$\overline{\mathcal{M}}_{g,n}$ is irreducible.*

*Proof.* Since $\overline{\mathcal{M}}_{g,n}$ is smooth (Theorem 5.4.8), the irreducibility of $\overline{\mathcal{M}}_{g,n}$ is equivalent to its connectedness. Since $\overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ is the universal family (Proposition 5.6.4) and in particular has connected fibers, it suffices to verify the connectedness of $\overline{\mathcal{M}}_g$. Since every smooth curve degenerates to a stable singular curve in the boundary $\delta = \overline{\mathcal{M}}_g \setminus \mathcal{M}_g$ (Proposition 5.7.13) and the boundary $\delta$ itself is connected (Proposition 5.7.14) by induction on $g$, we obtain that $\overline{\mathcal{M}}_g$ is connected. $\square$

**Remark 5.7.16** (Admissible Covers)**.** The above argument was motivated by the theory of admissible covers introduced by Harris and Mumford [HM82]. Admissible covers are a generalization of simply branched covers $C \to \mathbb{P}^1$ where the source *and* target curves are allowed to have nodal singularities. The main goal is to extend the map $\mathcal{H}_{d,b} \to \mathcal{M}_g$ taking $[C \to \mathbb{P}^1] \to [C]$ to a map $\overline{\mathcal{H}}_{d,b} \to \overline{\mathcal{M}}_g$ over the boundary where $\overline{\mathcal{H}}_{d,b}$ also has a moduli interpretation.

An *admissible cover of degree $d$* over a stable $b$-pointed genus 0 curve $(B, p_1, \dots, p_b)$ is a morphism $f \colon C \to B$ such that

(a) $f^{-1}(B^{\mathrm{sm}}) = C^{\mathrm{sm}}$ and $C^{\mathrm{sm}} \to B^{\mathrm{sm}}$ is simply branched of degree $d$ over the points $p_i$, i.e. each ramification index is 2 and there is at most one ramification point in every fiber; and

(b) for every node $q \in B$ and every node $r \in C$ over $q$, the local structure (either formally or étale) of $C \to B$ at $r$ is of the form $\Bbbk[x,y]/(xy) \to \Bbbk[x,y]/(xy)$ defined by $(x,y) \mapsto (x^m, y^m)$ for some $m$.

This definition extends to families of admissible covers, and the stack $\overline{\mathcal{H}}_{d,b}$ parameterizing admissible covers of degree $d$ branched over $b$ points is a proper Deligne–Mumford stack.

The total space $C$ of an admissible cover need not be stable. Nevertheless, using the contraction morphism (Proposition 5.3.20), there is a morphism $\overline{\mathcal{H}}_{d,b} \to \overline{\mathcal{M}}_g$ sending an admissible cover $[C \to B]$ to the stable model $C^{\mathrm{st}}$ of $C$. There is also a *finite* morphism $\overline{\mathcal{H}}_{d,b} \to \overline{M}_{0,n}$ sending $[C \to B]$ to $(B, \{p_i\})$ where $p_i \in B$ are the branched points. To summarize, there is a diagram

$$\begin{tikzcd} & \overline{\mathcal{H}}_{d,b} \arrow[dl] \arrow[dr] & \\ \overline{\mathcal{M}}_g & & \overline{M}_{0,n} \end{tikzcd}$$

extending the uncompactified diagram (5.7.2).

The argument of Proposition 5.7.13 can be rewritten in this language. For $d \gg 0$, given a smooth curve $[C] \in \mathcal{M}_g$, we choose a preimage $[C \to \mathbb{P}^1] \in \overline{\mathcal{H}}_{d,b}$ (Lemma 5.7.5). By Lemma 5.7.7, we can assume that the branched points $g_1, \ldots, g_b \in \mathbb{P}^1$ are general. Since $\overline{M}_{0,n}$ is connected, there is a map $\Delta = \operatorname{Spec} R \to \overline{M}_{0,n}$ (where $R$ is a DVR) such that the generic point maps to $(\mathbb{P}^1, \{g_i\})$ and the closed points maps to the $b$-pointed stable curve $(D_0, q_1, \ldots, q_b)$ of Figure 5.21. Since $\overline{\mathcal{H}}_{d,b} \to \overline{M}_{0,n}$ is finite, we may use the valuative criterion to lift $\Delta \to \overline{M}_{0,n}$ to $\Delta \to \overline{\mathcal{H}}_{d,b}$ such that the image of the generic point is $[C \to \mathbb{P}^1]$. The composition $\Delta \to \overline{\mathcal{H}}_{d,b} \to \overline{\mathcal{M}}_g$ gives the desired degeneration.

### 5.7.4 Irreducibility in positive characteristic: Deligne–Mumford and Fulton's arguments

The year 1969 was a remarkable year for mathematics in part due to the seminal contributions of Deligne and Mumford's paper [DM69] and Fulton's paper [Ful69]. The papers provided independent arguments for the irreducibility of $\overline{M}_g$ in positive characteristic (where Fulton's argument has the restriction that $p > g+1$). Both papers relied on the connectedness of $M_g$ over $\mathbb{C}$ and the time, there was no purely algebraic argument; the algebraic argument establishing Theorem 5.7.15 used admissible covers and became available only in 1982. The connectedness of $M_g$ over $\mathbb{C}$ is a classical result. Clebsch and Hurwitz's arguments in the 19th century (featured in Theorem 5.7.12) used the Hurwitz space of branched covers and used on a single non-algebraic input, namely the Riemann's Existence Theorem. There are of course other non-algebraic arguments, e.g. using the Teichmüller space.

#### 5.7.4.1 Deligne–Mumford's first argument

The first argument appearing [DM69] is very similar in spirit to the argument in §5.7.3. As with most results, there are many approaches to construct a proof, and the first approach in [DM69, §3] reflects the state of technology at the time.

For a field $\Bbbk$ of characteristic $p$, the argument for irreducibility of $M_g \times_{\mathbb{Z}} \Bbbk$ proceeds along three steps:

*Step 1: There is no proper connected component of $M_g \times_{\mathbb{Z}} \Bbbk$.*

Let $W(\Bbbk)$ be the Witt vectors for $\Bbbk$; $W(\Bbbk)$ is a noetherian complete local ring whose generic point $\eta$ has characteristic 0 and whose closed point 0 has residue field is $\Bbbk$. (For example, $W(\mathbb{F}_p) = \mathbb{Z}_p$ is the ring of $p$-adics.) We now use the existence of a quasi-projective coarse moduli space $\mathcal{M}_g \to M_g$ over $W(\Bbbk)$ as established in [GIT]. (Although appearing in the definitive book on GIT, this would not be viewed as a "GIT construction" today as it relies on some ad hoc techniques and doesn't use the Hilbert–Mumford criterion. Indeed, the standard GIT toolkit only became available in positive characteristic in 1975 after Haboush resolved Mumford's conjecture [Hab75] and in the relative setting in 1977 after Seshadri's paper [Ses77].)

Choosing a projective compactification $M_g \subset X$ over $W(\Bbbk)$, the connectedness of the generic fiber of $M_g \to \operatorname{Spec} W(\Bbbk)$ ensures that the generic fiber of $X_\eta$ is also connected. The scheme $M_g$ is normal as GIT quotients (or alternatively coarse moduli spaces) preserve normality. By taking the normalization of $X$, we can assume that $X$ is also normal. Zariski's connectedness theorem implies that the number of connected components in a fiber $X_w$ is independent of $w \in W(\Bbbk)$. Thus, $X_0$ is also connected.

Suppose $Y \subset M_g \times_{W(\Bbbk)} \Bbbk$ is a proper connected component. Then $Y \subset M_g \times_{W(\Bbbk)} \Bbbk \subset X_0$ is an open subscheme but also a closed subscheme since $Y$ is proper. Since $X_0$ is connected, we conclude that $Y = M_g \times_{W(\Bbbk)} \Bbbk$ is proper and irreducible. To obtain a contradiction, denote by $A_{g,\Bbbk}$ the moduli of principally polarized $g$-dimensional abelian varieties over $\Bbbk$ and consider the morphism

$$\Theta\colon M_g \times_{W(\Bbbk)} \Bbbk \to A_{g,\Bbbk}, \qquad C \mapsto \operatorname{Jac}(C)$$

assigning to a smooth curve $C$ its Jacobian $\operatorname{Jac}(C)$. The properness of $M_g \times_{W(\Bbbk)} \Bbbk$ implies that the image would be closed, but there are explicit examples where the closure of the image of $\Theta$ contains products of lower dimensional Jacobians.

*Step 2: There is no connected component of $\overline{M}_g \times_{\mathbb{Z}} \Bbbk$ consisting entirely of smooth curves.*

Let $\overline{M}_{g,1}, \ldots, \overline{M}_{g,r}$ be the connected components of $\overline{M}_g$. For each $i$, Step 1 implies that $\overline{M}_{g,i}$ is not proper. Let $\Delta = \operatorname{Spec} \Bbbk[\![t]\!]$ and $\Delta^* = \operatorname{Spec} \Bbbk(\!(t)\!) \to M_{g,i} := M_g \cap \overline{M}_{g,i}$ be a morphism that does not extend to $\Delta$. By Stable reduction, after possibly replacing $\Delta$ with a finite extension, $\Delta^* \to M_{g,i}$ extends to a morphism $\Delta \to \overline{M}_g$. This shows that $\overline{M}_{g,i} \setminus M_{g,i}$ is non-empty.

*Step 3: The boundary $\delta = \overline{M}_g \setminus M_g$ is connected.*

Steps 1 and 2 show that every smooth curve degenerates to a singular stable curve (Proposition 5.7.13). This step proceeds precisely as in Proposition 5.7.14 but without using the formalism of the moduli $\overline{\mathcal{M}}_{g,n}$ of $n$-pointed stable curves and the gluing morphisms.

#### 5.7.4.2 Deligne–Mumford's second argument

The stack $\overline{\mathcal{M}}_g$ of stable curves is smooth and proper over $\operatorname{Spec} \mathbb{Z}$. Zariski's connectedness theorem implies that for every smooth and proper morphism $X \to Y$ of schemes, the number of connected components of a geometric fiber is a locally constant function on $Y$. (In fact, for a flat and proper morphism $X \to Y$, this function is lower semi-continuous, and it is enough for the fibers of $X \to Y$ to be geometrically normal in order to show constancy.) This fact extends to morphisms of algebraic stacks. Applying this fact to the morphism $\overline{\mathcal{M}}_g \to \operatorname{Spec} \mathbb{Z}$, we see that the connectedness $\overline{\mathcal{M}}_g \times_{\mathbb{Z}} \mathbb{C}$ implies the connectedness of every geometric fiber. In [DM69, §5], the connectedness of $\overline{\mathcal{M}}_g \times_{\mathbb{Z}} \mathbb{C}$ is argued by relating it to the moduli of Teichmüller structures of level $n$ and the connectedness of the Teichmüller space [Man39].

#### 5.7.4.3 Fulton's argument

In [Ful69], Fulton defines the Hurwitz scheme $H_{d,b}$ of simply branched covers over $\mathbb{Z}$ and shows that there is a diagram

$$\begin{tikzcd}
& H_{d,b} \arrow[dl] \arrow[dr] & \\
\mathcal{M}_g & & \operatorname{Sym}^b \mathbb{P}^1 \setminus \Delta
\end{tikzcd}$$

defined over $\mathbb{Z}$. He shows that the map $H_{d,b} \to \operatorname{Sym}^d \mathbb{P}^1 \setminus \Delta$, taking a simply branched cover to its branch locus, is étale. Moreover, if all primes $p \leq g+1$ are inverted, then $H_{d,b} \to \operatorname{Sym}^d \mathbb{P}^1 \setminus \mathbb{P}^1$ is finite; examples are given where is not finite over primes $p \leq g+1$. Fulton then establishes a "reduction theorem" allowing him to deduce the connectedness of $H_{d,b} \times_{\mathbb{Z}} \overline{\mathbb{F}}_p$ from $H_{d,b} \times_{\mathbb{Z}} \mathbb{C}$ for primes $p > g+1$.

## 5.8 Projectivity

In this section, we prove that the coarse moduli space $\overline{M}_{g,n}$ is projective (Theorem 5.8.14). We follow the approach introduced by Kollár in [Kol90] partially building on ideas of Viehweg (see [Vie95]). We will primarily focus on the unpointed coarse moduli space $\overline{M}_g$ as this will be enough to deduce the projectivity of $\overline{M}_{g,n}$.

To introduce the general strategy to establish projectivity, we need to introduce some terminology. Let $\pi \colon \mathcal{U}_g \to \overline{\mathcal{M}}_g$ be the universal family and for each integer $k \geq 1$ define the *kth pluricanonical bundle* as the vector bundle

$$\pi_*(\omega_{\mathcal{U}_g/\overline{\mathcal{M}}_g}^{\otimes k}) \tag{5.8.1}$$

on $\overline{\mathcal{M}}_g$. Its rank $r(k)$ can be computed via Riemann–Roch:

$$r(k) := \begin{cases} g & \text{if } k = 1 \\ (2k-1)(g-1) & \text{if } k > 1. \end{cases} \tag{5.8.2}$$

We obtain line bundles on $\overline{\mathcal{M}}_g$ by taking the determinant

$$\lambda_k := \det \pi_*(\omega_{\mathcal{U}_g/\overline{\mathcal{M}}_g}^{\otimes k}).$$

These provide natural candidates of line bundles on $\overline{\mathcal{M}}_g$ that descend to ample line bundles on $\overline{M}_g$.

**Strategy for projectivity:** Show that for $k \gg 0$, a positive power of $\lambda_k$ descends to an *ample* line bundle on the coarse moduli space $\overline{M}_g$.

*Outline of this section:* In §5.8.1, we prove Kollar's Criterion for ampleness (Theorem 5.8.5). In §5.8.2, we setup the application of Kollár's Criterion to $\overline{M}_g$ by establishing Proposition 5.8.13: projectivity of $\overline{M}_g$ follows from (a) Stable Reduction (Theorem 5.5.1) and (b) the nefness of $\pi_*(\omega_{\mathcal{C}/T}^{\otimes k})$ for a family of stable curves $\mathcal{C} \to T$ over a smooth projective curve and for $k \gg 0$ (Theorem 5.8.17). In §5.8.3, we prove this nefness statement which finishes the proof of projectivity. Finally, in §5.8.4, we compare this argument to the GIT construction of $\overline{M}_g$.

### 5.8.1 Kollár's criteria

In this section, we prove Kollár's Criterion for projectivity (Theorem 5.8.5), which we will apply to show that $\lambda_k$ is ample on $\overline{M}_g$ for $k \gg 0$. We first extend ampleness criteria of §E.2.5 to proper algebraic spaces and in particular establish that the Nakai–Moishezon criterion still holds (Theorem 5.8.4).

**Lemma 5.8.1.** *Let $\mathcal{X}$ be a proper Deligne-Mumford stack with coarse moduli space $\mathcal{X} \to X$. Suppose that $L$ is a line bundle on $\mathcal{X}$ satisfying*

*(a) $L$ is semiample (i.e. $L^N$ is base point free for some $N > 0$); and*

*(b) for every map $f : T \to \mathcal{X}$ from a proper integral curve such that $f(T) \subset |\mathcal{X}|$ is not a single point, $\deg L|_T > 0$.*

*Then for some $N > 0$, $L^{\otimes N}$ descends to an ample line bundle. In particular, $X$ is projective.*

**Remark 5.8.2.** Lemma E.2.16 handles the case when $\mathcal{X}$ is a scheme. Even though we won't actually quote this lemma, it provides a basic technique underlying many ampleness arguments, e.g. the Nakai-Moishezon criterion.

*Proof.* For $N$ sufficiently divisible, consider the diagram

$$\begin{tikzcd} \mathcal{X} \arrow[d] \arrow[dr] & \\ X \arrow[r] & \mathbb{P}(H^0(\mathcal{X}, L^N)). \end{tikzcd}$$

Property (a) implies that $\mathcal{X} \to \mathbb{P}(H^0(\mathcal{X}, L^N))$ is well-defined and (b) implies that it doesn't contract curves. The universal property for coarse moduli spaces gives the existence of the factorization $X \to \mathbb{P}(H^0(\mathcal{X}, L^N))$, which also doesn't contract curves. Thus $X \to \mathbb{P}(H^0(\mathcal{X}, L^N))$ is quasi-finite and proper (as both $X$ and projective space are proper), and thus finite by Zariski's Main Theorem. It follows that the pullback $M$ of $\mathcal{O}(1)$ under $X \to \mathbb{P}(H^0(\mathcal{X}, L^N))$ is ample; moreover, the pullback of $M$ under $\mathcal{X} \to X$ is $L^{\otimes N}$. $\square$

**Remark 5.8.3.** The semiampleness condition in (a) can be very challenging to verify in practice. Remember that in the GIT approach, semiampleness is hard-coded into the definition of semistability (see Remark 5.8.22). If $G$ is a reductive group acting linearly on projective space $\mathbb{P}(V)$ and $L = \mathcal{O}(1)$ is the corresponding $G$-line bundle on $[\mathbb{P}(V)/G]$, then a nonzero vector $v \in V$ is in the stable base locus of $L$ (i.e. $s(v) = 0$ for all $s \in \Gamma([\mathbb{P}(V)/G], L^{\otimes d})$ and $d > 0$) if and only if $0 \in \overline{Gv} \subset V$ if and only if there exists a one-parameter subgroup $\lambda \colon \mathbb{G}_m \to G$ such that $\lim_{t \to 0} \lambda(t) \cdot v = 0$. This latter equivalence is the Hilbert–Mumford criterion and can sometimes be verified combinatorially.

On the other hand, in Kollár's Criterion, the existence of sufficient sections of the line bundle follows from the bigness of suitable vector bundles.

**Theorem 5.8.4** (Nakai–Moishezon Criterion)**.** *If $X$ is a proper algebraic space, a line bundle $L$ is ample if and only if for all irreducible closed subvarieties $Z \subset X$, $c_1(L)^{\dim Z} \cdot Z > 0$*

*Proof.* By Le Lemme de Gabber (Theorem 4.5.1), there exists a finite surjection $f \colon X' \to X$ from a scheme $X$, and $L$ is ample if and only if $f^*L$ is ample (Exercise 4.4.22). The statement then follows for the Nakai–Moishezon Criterion for schemes (Theorem E.2.18). $\square$

Let $X$ be a proper algebraic space over $\Bbbk$. Let $W \to Q$ be a surjection of vector bundles of rank $w$ and $q$. Suppose that $W$ has structure group $G \to \mathrm{GL}_w$. There is a *classifying map*

$$\begin{array}{ccc} X & \to & [\mathrm{Gr}(q, \Bbbk^w)/G] \\ x & \mapsto & [W \otimes \kappa(x) \twoheadrightarrow Q \otimes \kappa(x)] \end{array}$$

which is well-defined because a choice of isomorphism $W \otimes \kappa(x) \cong \kappa(x)^w$ of the fiber of $W$ over $x$ is well-defined up to the structure group $G$. Thus, the image of $x$ is identified with the quotient $[\kappa(x)^w \cong W \otimes \kappa(x) \twoheadrightarrow Q \otimes \kappa(x)] \in \mathrm{Gr}(q, \Bbbk^w)$.

For simplicity, we state the following criterion in characteristic 0. The criterion first appears in [Kol90, Lem. 3.9] with improvements from [KP17, Thm. 4.1].

**Theorem 5.8.5** (Kollár's Criterion)**.** *Let $X$ be a proper algebraic space over a field $\Bbbk$ of characteristic* 0*. Let $W \twoheadrightarrow Q$ be a surjection of vector bundles of rank $w$ and $q$, where $W$ has structure group $G \to \mathrm{GL}_w$. Suppose that*

*(a) The classifying map $X(\Bbbk) \to \mathrm{Gr}(q, \Bbbk^w)(\Bbbk)/G(\Bbbk)$ has finite fibers; and*

*(b) $W$ is nef.*

*Then* $\det Q$ *is ample.*

**Remark 5.8.6.** Condition (a) is equivalent to the map $|X| \to |[\mathrm{Gr}(q, \Bbbk^w)/G]|$ on topological spaces having finite fibers. This set-theoretic condition is weaker than the quasi-finiteness[1] of $X \to [\mathrm{Gr}(q, \Bbbk^w)/G]$, as the latter condition also requires that for every $x \in X(\Bbbk)$ only finitely many elements of $G(\Bbbk)$ leave $\ker(W \otimes \kappa(x) \to Q \otimes \kappa(x))$ invariant (or equivalently that the image of $x$ in $[\mathrm{Gr}(q, \Bbbk^w)/G]$ has finite stabilizer.) In fact, Condition (a) is equivalent to quasi-finiteness of the projection morphism $\mathrm{im}(X \to X \times [\mathrm{Gr}(q, \Bbbk^w)/G]) \to [\mathrm{Gr}(q, \Bbbk^w)/G]$ from the scheme-theoretic image of the graph of the classifying map; it is this property that we will use in the proof.

**Remark 5.8.7.** An easy case of this theorem is when $W$ is the trivial vector bundle so that there is a reduction of structure group to the trivial group $G = \{1\}$. In this case, the classifying map $X \to \mathrm{Gr}(q, \Bbbk^w)$ is quasi-finite by condition (a) and proper since both $X$ and $\mathrm{Gr}(q, \Bbbk^w)$ are proper. Thus $X \to \mathrm{Gr}(q, \Bbbk^w)$ is finite and $\det(Q)$ is ample as its the pullback of the ample line bundle on $\mathrm{Gr}(q, \Bbbk^w)$ defining the Plücker embedding.

Note that the above theorem does not require that the image of $X$ lands in the $G$-stable locus of $\mathrm{Gr}(q, \Bbbk^w)$. However, if this is true, then we have a commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[dr] & {[\mathrm{Gr}(q, \Bbbk^w)^{\mathrm{ss}}/G]} \arrow[d] \\ & \mathrm{Gr}(q, \Bbbk^w)/\!\!/G \end{tikzcd}$$

where $\mathrm{Gr}(q, \Bbbk^w)/\!\!/G$ denotes the projective GIT quotient. Since the image of $X$ lands in the *stable* locus, $X \to \mathrm{Gr}(q, \Bbbk^w)/\!\!/G$ is quasi-finite; as it's also proper, we conclude that it's finite. Moreover, we obtain ampleness of $(\det Q)^w \otimes (\det W)^{-q}$, the pullback of the ample line bundle $\mathrm{Gr}(q, \Bbbk^w)/\!\!/G$ coming from GIT. This is a stronger ampleness statement than merely the ampleness of $\det Q$.

[1] Recall that a morphism $f\colon \mathcal{X} \to \mathcal{Y}$ of algebraic stacks is *quasi-finite* if $|\mathcal{X}| \to |\mathcal{Y}|$ has finite fibers and the relative inertia $I_{\mathcal{X}/\mathcal{Y}}$ is quasi-finite (or equivalently for every field-valued point $x \in \mathcal{X}(\Bbbk)$ the morphism $\mathrm{Aut}_{\mathcal{X}(\Bbbk)}(x) \to \mathrm{Aut}_{\mathcal{Y}(\Bbbk)}(f(x))$ has finite cokernel).

**Remark 5.8.8.** The nefness of $\det(Q)$ is an immediate consequence of the nefness of $W$ as $\det(Q) = \bigwedge^q Q$ is a quotient of $\bigwedge^q W$, which is nef by Proposition E.2.27. The proof will proceed by reducing the ampleness of $\det Q$ to its bigness, which in turn is established by using the quasi-finiteness and nefness to express $\det Q$ as the sum of an effective line bundle and a big, globally generated line bundle.

*Proof of Theorem 5.8.5.* We will verify the Nakai–Moishezon criterion: for each irreducible subvariety $Z \subset X$, we verify that $\det(Q)|_Z$ is big. Since both conditions (a) and (b) also hold for $Z$ and the restrictions $W|_Z \twoheadrightarrow Q|_Z$, it suffices to verify that if $X$ is an integral scheme with $W \twoheadrightarrow Q$ satisfying (a) and (b), then $\det(Q)$ is big.

The property of bigness (unlike ampleness) is conveniently invariant under birational maps (and we desire this flexibility because, in the proof of Proposition 5.8.9 below, we will make a series of reductions where we perform blowups to resolve the indeterminacy locus of certain rational maps). In fact, for a generically quasi-finite and proper morphism $f: Y \to X$ of integral schemes, the projection formula implies that $\det(f^*Q)^{\dim Y} = \deg(f)\det(Q)^{\dim X} > 0$ and thus $\det(Q)$ is big if and only if $f^*(\det Q)$ is big. By Le Lemme de Gabber (Corollary 4.5.2), there exists a projective, generically quasi-finite and surjective morphism $f: Y \to X$ from a projective integral scheme. By taking the normalization, we can assume that $Y$ is normal. The theorem therefore follows from the bigness of $f^*(\det Q)$, which is the conclusion of the following proposition. □

**Proposition 5.8.9.** *Let $Y$ be a normal projective integral scheme over a field $\Bbbk$ of characteristic 0. Let $W \twoheadrightarrow Q$ be a surjection of vector bundles of rank $w$ and $q$, where $W$ has structure group $G \to \mathrm{GL}_w$. Suppose that*

*(a′) The classifying map $Y(\Bbbk) \to \mathrm{Gr}(q, \Bbbk^w)(\Bbbk)/G(\Bbbk)$ generically has finite fibers;*

*(b) $W$ is nef.*

*Then $\det Q$ is big.*

**Remark 5.8.10.** Condition (a′) means that there is a non-empty open subscheme $U \subset Y$ such that $U(\Bbbk) \to \mathrm{Gr}(q, \Bbbk^w)(\Bbbk)/G(\Bbbk)$ has finite fibers.

Note that the difference in the hypotheses between Theorem 5.8.5 and Proposition 5.8.9 is that we relaxed the condition on the classifying map from having finite fibers to generically having finite fibers, but now we assume that $Y$ is already projective (in addition to being normal and integral). Also the conclusion is weaker in that it asserts the bigness of $\det(Q)$ rather than the ampleness.

*Proof.*

*Step 1: Use the universal basis map to lift the classifying map to a morphism $\mathbb{P} \setminus \Delta \to \mathrm{Gr}(q, \Bbbk^w)$ where $\mathbb{P} \subset \mathbb{P}_Y((W^\vee)^{\oplus w})$ is a closed subscheme and $\Delta \subset \mathbb{P}$ is a divisor.*

Define $\widetilde{\mathbb{P}} := \mathbb{P}_Y((W^\vee)^{\oplus w})$ as the projective space of matrices whose columns belong to $W$, and let $\widetilde{\pi}: \widetilde{\mathbb{P}} \to Y$ denote the projection. There is a *universal basis map*

$$\mathcal{O}_{\widetilde{\mathbb{P}}}^{\oplus w} \to \widetilde{\pi}^* W \otimes \mathcal{O}_{\widetilde{\mathbb{P}}}(1) \tag{5.8.3}$$

defined by the isomorphisms

$$H^0(\widetilde{\mathbb{P}}, \widetilde{\pi}^* W \otimes \mathcal{O}_{\widetilde{\mathbb{P}}}(1)) \cong H^0(Y, \widetilde{\pi}_*(\widetilde{\pi}^* W \otimes \mathcal{O}_{\widetilde{\mathbb{P}}}(1))) \cong H^0(Y, W \otimes (W^\vee)^{\oplus w}).$$

The universal basis map (5.8.3) restricts to an isomorphism on the complement $\widetilde{\mathbb{P}} \setminus \Delta$ where $\Delta \subset \widetilde{\mathbb{P}}$ is the divisor of matrices with determinant 0, and thus provides a trivialization of $(\widetilde{\pi}^* W \otimes \mathcal{O}_{\widetilde{\mathbb{P}}}(1))|_{\widetilde{\mathbb{P}}\setminus\Delta}$. Note also that there is a natural $\mathrm{PGL}_w$ action on $\widetilde{\mathbb{P}}$ which is free on $\widetilde{\mathbb{P}}\setminus\Delta$ and such that $\widetilde{\pi}\colon \widetilde{\mathbb{P}}\setminus\Delta \to Y$ is a $\mathrm{PGL}_w$-torsor and fits into the cartesian diagram

$$\begin{tikzcd} \widetilde{\mathbb{P}}\setminus\Delta \arrow[r] \arrow[d] & \mathrm{Gr}(q,\Bbbk^w) \arrow[r] \arrow[d] & \operatorname{Spec}\Bbbk \arrow[d] \\ Y \arrow[r] & {[\mathrm{Gr}(q,\Bbbk^w)/\mathrm{PGL}_w]} \arrow[r] & B\,\mathrm{PGL}_w\,. \end{tikzcd}$$

We can also consider the fiber product with respect to the $G$-action

$$\mathbb{P}\setminus\Delta := Y \times_{[\mathrm{Gr}(q,\Bbbk^w)/G]} \mathrm{Gr}(q,\Bbbk^w).$$

The inclusion $\mathbb{P}\setminus\Delta \hookrightarrow \widetilde{\mathbb{P}}\setminus\Delta$ is a closed immersion, and we define $\mathbb{P}\subset\widetilde{\mathbb{P}}$ to be the closure of $\mathbb{P}\setminus\Delta$, where we abuse notation by using the same symbol $\Delta$ for the divisor in $\widetilde{\mathbb{P}}$ and its intersection in $\mathbb{P}$. One way to see that $\mathbb{P}\setminus\Delta = Y\times_{BG}\operatorname{Spec}\Bbbk \hookrightarrow Y\times_{B\,\mathrm{PGL}_w}\operatorname{Spec}\Bbbk = \widetilde{\mathbb{P}}\setminus\Delta$ is a closed immersion is to realize it as the base change of the diagonal $BG\to BG\times_{B\,\mathrm{PGL}_w} BG$; here we use that $BG\to B\,\mathrm{PGL}_w$ is separated (it is in fact even affine since $G$ is reductive). Alternatively, one can view $\mathbb{P}\subset\widetilde{\mathbb{P}}$ as the closure of a generic $G$-orbit in $\widetilde{\mathbb{P}}$.

In summary, we have a cartesian diagram

$$\begin{tikzcd} \mathbb{P}\setminus\Delta \arrow[r] \arrow[d, "\pi"] & \mathrm{Gr}(q,\Bbbk^w) \arrow[r, hook] \arrow[d] & \mathbb{P}(\bigwedge^q \Bbbk^w) \arrow[d] \\ Y \arrow[r] & {[\mathrm{Gr}(q,\Bbbk^w)/G]} \arrow[r, hook] & {[\mathbb{P}(\bigwedge^q \Bbbk^w)/G]} \end{tikzcd}$$

where the right-hand square is given by the Plücker embedding. The map $\mathbb{P}\setminus\Delta\to Y$ extends to a map $\pi\colon\mathbb{P}\to Y$ (i.e. the composition $\mathbb{P}\hookrightarrow\widetilde{\mathbb{P}}\xrightarrow{\widetilde{\pi}} Y$). The map $\mathbb{P}\setminus\Delta\to\mathrm{Gr}(q,\Bbbk^w)$ is defined by the restriction of the composition

$$\mathcal{O}_{\mathbb{P}}^{\oplus w} \to \pi^* W\otimes\mathcal{O}_{\mathbb{P}}(1) \to \pi^* Q\otimes\mathcal{O}_{\mathbb{P}}(1) \tag{5.8.4}$$

of the universal basis map (5.8.3) with the quotient $\pi^*W\to\pi^*Q$. The image of (5.8.4) may not be locally free and thus the rational map $\mathbb{P}\dashrightarrow\mathrm{Gr}(q,\Bbbk^w)$ may not be defined everywhere.

*Step 2: Blowup $\mathbb{P}$ in order to extend the map $\mathbb{P}\setminus\Delta\to\mathrm{Gr}(q,\Bbbk^w)$.*

(Note that if (5.8.4) is surjective, then $\mathbb{P}\setminus\Delta\to\mathrm{Gr}(q,\Bbbk^w)$ extends to a morphism $\mathbb{P}\to\mathrm{Gr}(q,\Bbbk^w)$ such that the pullback of the Plücker line bundle is $\pi^*(\det Q)\otimes\mathcal{O}_{\mathbb{P}}(q)$.)

We blowup the image ideal sheaf of (5.8.4) (more precisely, if $I\subset\pi^*(\det Q)\otimes\mathcal{O}_{\mathbb{P}}(q)$ denotes the image subsheaf of (5.8.4), we blowup $I\otimes(\pi^*(\det Q)\otimes\mathcal{O}_{\mathbb{P}}(q))^\vee\subset\mathcal{O}_{\mathbb{P}}$). This yields a map $g\colon\mathbb{P}'\to\mathbb{P}$ which is an isomorphism over $\mathbb{P}\setminus\Delta$ and such that $\mathbb{P}\setminus\Delta\to\mathrm{Gr}(q,\Bbbk^w)$ extends to a morphism $\gamma\colon\mathbb{P}'\to\mathrm{Gr}(q,\Bbbk^w)$. This yields a commutative diagram

$$\begin{tikzcd} \mathbb{P}' \arrow[d, "g"] \arrow[dr, "\gamma"] \arrow[dd, bend right, "\pi'"'] & \\ \mathbb{P} \arrow[r, dashed] \arrow[d, "\pi"] & \mathrm{Gr}(q,\Bbbk^w) \arrow[d] \\ Y \arrow[r] & {[\mathrm{Gr}(q,\Bbbk^w)/G]}. \end{tikzcd}$$

The effective divisor $E \subset \mathbb{P}'$ satisfies

$$g^*(\pi^*(\det Q) \otimes \mathcal{O}_{\mathbb{P}}(q)) \cong \gamma^* \mathcal{O}_{\mathrm{Gr}(q,\Bbbk^w)}(1) \otimes \mathcal{O}_{\mathbb{P}'}(E). \tag{5.8.5}$$

where $\mathcal{O}_{\mathrm{Gr}(q,\Bbbk^w)}(1)$ denotes the Plücker line bundle.

*Step 3: Use the generic quasi-finiteness to show that $\gamma^*(\mathcal{O}_{\mathrm{Gr}(q,\Bbbk^w)}(m)) \otimes \pi'^* H^\vee$ is effective for some $m > 0$, where $H$ is a ample line bundle on $Y$.*

(Note that under the stronger assumption that the classifying map $Y \to [\mathrm{Gr}(q, \Bbbk^w)/G]$ is generically quasi-finite, then $\gamma \colon \mathbb{P}' \to \mathrm{Gr}(q, \Bbbk^w)$ is also generically quasi-finite. Thus $\gamma^* \mathcal{O}_{\mathrm{Gr}(q,\Bbbk^w)}(1)$ is big and Kodaira's Lemma (Proposition E.2.9) immediately gives the desired statement.)

Let $Z$ be the scheme-theoretic image of the graph $Y \to Y \times [\mathrm{Gr}(q, \Bbbk^w)/G]$ of the classifying map. The hypothesis that $Y(\Bbbk) \to \mathrm{Gr}(q, \Bbbk^w)(\Bbbk)/G(\Bbbk)$ is generically quasi-finite implies that $Z \to [\mathrm{Gr}(q, \Bbbk^w)/G]$ is generically quasi-finite. Consider the commutative diagram

$$\begin{tikzcd}
\mathbb{P}' \arrow[dr, "p"] \arrow[rrrd, bend left, "\gamma"] & & & \\
\mathbb{P} \setminus \Delta \arrow[u, hook] \arrow[r] \arrow[d, "\pi"] & Z' \arrow[r, hook] \arrow[d] \arrow[rr, bend left, "\gamma'"] & Y \times \mathrm{Gr}(q, \Bbbk^w) \arrow[r] \arrow[d] & \mathrm{Gr}(q, \Bbbk^w) \arrow[d] \\
Y \arrow[r] & Z \arrow[r, hook] & Y \times [\mathrm{Gr}(q, \Bbbk^w)/G] \arrow[r] & {[\mathrm{Gr}(q, \Bbbk^w)/G]}
\end{tikzcd}$$

where the squares are cartesian and where $Z'$ is the scheme-theoretic image of $\mathbb{P} \setminus \Delta \to Y \times \mathrm{Gr}(q, \Bbbk^w)$ (and also of $\mathbb{P}' \to Y \times \mathrm{Gr}(q, \Bbbk^w)$). We see that $\gamma' \colon Z' \to \mathrm{Gr}(q, \Bbbk^w)$ is also generically quasi-finite and it follows that $\gamma'^*(\mathcal{O}_{\mathrm{Gr}(q,\Bbbk^w)}(1))$ is big. If we denote by $H'$ the pullback of $H$ to $Z'$, then by Kodaira's Lemma (Proposition E.2.9), $\gamma'^*(\mathcal{O}_{\mathrm{Gr}(q,\Bbbk^w)}(m)) \otimes H'^\vee$ is effective on $Z$ for some $m > 0$. Its pullback $p^*(\gamma'^*(\mathcal{O}_{\mathrm{Gr}(q,\Bbbk^w)}(m)) \otimes H'^\vee) \cong \gamma^*(\mathcal{O}_{\mathrm{Gr}(q,\Bbbk^w)}(m)) \otimes \pi'^* H^\vee$ is also effective.

*Step 4: Pushforward a section to construct a map $\pi_* \mathcal{O}_{\mathbb{P}}(mq)^\vee \to (\det Q)^{\otimes m} \otimes H^\vee$.*

Using (5.8.5), we see that

$$\begin{aligned}
\gamma^*(\mathcal{O}_{\mathrm{Gr}(q,\Bbbk^w)}(m)) \otimes \pi'^* H^\vee &\cong \pi'^*((\det Q)^{\otimes m} \otimes H^\vee) \otimes g^* \mathcal{O}_{\mathbb{P}}(mq) \otimes \mathcal{O}_{\mathbb{P}'}(-mE) \\
&\subset \pi'^*((\det Q)^{\otimes m} \otimes H^\vee) \otimes g^* \mathcal{O}_{\mathbb{P}}(mq)) \\
&\cong g^*\big(\pi^*((\det Q)^{\otimes m} \otimes H^\vee) \otimes \mathcal{O}_{\mathbb{P}}(mq)\big)
\end{aligned}$$

and therefore we may choose a nonzero section

$$\mathcal{O}_{\mathbb{P}'} \to g^*\big(\pi^*((\det Q)^{\otimes m} \otimes H^\vee) \otimes \mathcal{O}_{\mathbb{P}}(mq)\big).$$

Pushing forward under $g \colon \mathbb{P}' \to \mathbb{P}$ and using the projection formula gives a nonzero section

$$\mathcal{O}_{\mathbb{P}} \to \pi^*((\det Q)^{\otimes m} \otimes H^\vee) \otimes \mathcal{O}_{\mathbb{P}}(mq)$$

and pushing forward again under $\pi \colon \mathbb{P} \to Y$ gives a nonzero section

$$\mathcal{O}_Y \to (\det Q)^{\otimes m} \otimes H^\vee \otimes \pi_* \mathcal{O}_{\mathbb{P}}(mq)$$

which we rearrange as

$$\pi_* \mathcal{O}_{\mathbb{P}}(mq)^\vee \to (\det Q)^{\otimes m} \otimes H^\vee. \tag{5.8.6}$$

*Step 5: Show that the nefness of $W$ implies the nefness of $\pi_*\mathcal{O}_{\mathbb{P}}(mq)^\vee$.*

We compare $\pi_*\mathcal{O}_{\mathbb{P}}(mq)$ to $\pi_*\mathcal{O}_{\widetilde{\mathbb{P}}}(mq) \cong \mathrm{Sym}^{mq}((W^\vee)^{\oplus w})$ (and their duals) under the closed immersion $\mathbb{P} \hookrightarrow \widetilde{\mathbb{P}}$ (where we are using $\pi$ to denote both projections $\mathbb{P} \to Y$ and $\widetilde{\mathbb{P}} \to Y$). For $m \gg 0$, the map $\pi_*\mathcal{O}_{\widetilde{\mathbb{P}}}(mq) \to \pi_*\mathcal{O}_{\mathbb{P}}(mq)$ is surjective and dualizes to an inclusion

$$(\pi_*\mathcal{O}_{\mathbb{P}}(mq))^\vee \hookrightarrow (\pi_*\mathcal{O}_{\widetilde{\mathbb{P}}}(mq))^\vee \cong \mathrm{Sym}^{mq}((W)^{\oplus w})$$

of vector bundles on $Y$. Since $W$ is nef, so is $\mathrm{Sym}^{mq}((W)^{\oplus w})$ (Proposition E.2.27) and therefore so is $(\pi_*\mathcal{O}_{\mathbb{P}}(mq))^\vee$ (Proposition 5.8.11).

*Step 6: Conclude that $\det Q$ is big.*

(Note that if (5.8.6) is surjective, the line bundle quotient $N := (\det Q)^{\otimes m} \otimes H^\vee$ is nef. Thus $(\det Q)^{\otimes m} \cong H \otimes N$ is written as the sum of an ample and nef divisor, which is necessarily big.)

Blowing up the image ideal sheaf of (5.8.6), we obtain a birational morphism $s\colon Y' \to Y$ and a quotient line bundle $s^*\pi_*\mathcal{O}_{\mathbb{P}}(mq)^\vee \twoheadrightarrow N \subset s^*(\det Q)^{\otimes m} \otimes s^*H^\vee$ which is nef. As $N$ is nef and $s^*H$ is big and globally generated, the sub-line bundle $s^*H \otimes N \subset s^*(\det Q)^{\otimes m}$ is big. The difference of $s^*(\det Q)^{\otimes m}$ and $s^*H \otimes N$ is effective. Since the sum of a big and globally generated line bundle is big, we can conclude that $s^*(\det Q)^{\otimes m}$ is big, which in turn implies that $\det Q$ is big. $\square$

The proof above used the following property of nefness of vector bundles complementing the basic results from Section E.2.7.

**Proposition 5.8.11.** *Let $X$ be a scheme of finite type over an algebraically closed field $\Bbbk$ of characteristic $0$ and $W$ be a vector bundle of rank $w$. Let $G$ be a reductive group and suppose that $W$ admits a reduction of the structure group $G \to \mathrm{GL}_w$. Let $V \subset W$ be a $G$-subbundle corresponding a $G$-invariant subspace $\Bbbk^v \subset \Bbbk^w$. If $W$ is nef, then so is $V$.*

*Proof.* In characteristic 0, representations of reductive groups are completely reducible. Therefore $\Bbbk^v \subset \Bbbk^w$ has a $G$-invariant complement $\Bbbk^{w-v} \subset \Bbbk^w$. Since this expresses $V$ as a quotient of $W$, we see that $V$ is nef. $\square$

### 5.8.2 Application to $\overline{M}_g$

To apply Kollár's Criterion to $\overline{M}_g$, we will make use of multiplication maps between pluricanonical bundles and their symmetric products. Given a morphism $S \to \overline{\mathcal{M}}_g$ corresponding to a family of stable curves $\pi\colon \mathcal{C} \to S$ and an integer $d \geq 0$, we will consider the *multiplication map*

$$\mathrm{Sym}^d \pi_*(\omega_{\mathcal{C}/S}^{\otimes k}) \to \pi_*(\omega_{\mathcal{C}/S}^{\otimes dk}). \tag{5.8.7}$$

For a stable curve $C$ defined over a field $\Bbbk$, this multiplication map is

$$\mathrm{Sym}^d H^0(C, \omega_C^{\otimes k}) \to H^0(C, \omega_C^{\otimes dk})$$

and its kernel consists of degree $d$ equations cutting out the image of $C \xrightarrow{|\omega_C^{\otimes k}|} \mathbb{P}^{r(k)-1}$. If $k \geq 3$, then $\omega_{\mathcal{C}/S}^{\otimes k}$ is relatively very ample and thus $\mathcal{C} \to S$ can be recovered from the kernel of the multiplication map.

**Remark 5.8.12.** We emphasize here that this construction depends on $k$ and $d$, the same two integers on which the GIT construction depends (see Section 5.8.4).

**Proposition 5.8.13.** *Let $g \geq 2$. Assume that*

*(a) $\overline{\mathcal{M}}_g$ is a proper Deligne–Mumford stack; and*

*(b) There exists a $k_0 > 0$ such that for every family of stable curves $\mathcal{C} \to T$ over a smooth projective curve $T$, $\pi_*(\omega_{\mathcal{C}/T}^{\otimes k})$ is nef for $k \geq k_0$.*

*Then for $k \gg 0$ and $N$ sufficiently divisible, the line bundle $\lambda_k^{\otimes N}$ on $\overline{\mathcal{M}}_g$ descends to an ample line bundle on the coarse moduli space $\overline{M}_g$. In particular, $\overline{M}_g$ is projective.*

*Proof.* Consider the universal curve $\mathcal{C} = \mathcal{U}_g$ over $S = \overline{\mathcal{M}}_g$. Choose integers $k$ and $d$ such that

- $\omega_{\mathcal{C}/S}^{\otimes k}$ is relatively very ample and $R^1\pi_*\omega_{\mathcal{C}/S}^{\otimes k} = 0$;
- Every stable curve $C \overset{|\omega_C^{\otimes k}|}{\hookrightarrow} \mathbb{P}^{r(k)-1}$ is cut out by equations of degree $d$; and
- $\pi_*(\omega_{\mathcal{C}/S}^{\otimes k})$ is nef.

The conditions imply that the multiplication map

$$W := \operatorname{Sym}^d \pi_*(\omega_{\mathcal{C}/S}^{\otimes k}) \twoheadrightarrow \pi_*(\omega_{\mathcal{C}/S}^{\otimes dk}) =: Q$$

is surjective. Let $w = \binom{r(k)+d-1}{d}$ and $q = r(dk)$ be the ranks of $W$ and $Q$, respectively. Note that $W$ has a reduction of the structure group to $G := \mathrm{SL}_{r(k)}$. The classifying map

$$\begin{aligned} \overline{\mathcal{M}}_g &\to [\mathrm{Gr}(q, \Bbbk^w)/G] \\ [C] &\mapsto \big[\underbrace{\operatorname{Sym}^d H^0(C, \omega_C^{\otimes k})}_{\Gamma(\mathbb{P}^{r(k)-1}, \mathcal{O}(d))} \twoheadrightarrow \underbrace{H^0(C, \omega_C^{\otimes dk})}_{\Gamma(C, \mathcal{O}(d))}\big] \end{aligned}$$

is injective as the conditions on $d$ and $k$ imply that the kernel of the multiplication map uniquely determines $C$.

Let $X \to \overline{\mathcal{M}}_g$ be a finite cover where $X$ is a proper algebraic space (Theorem 4.5.1). By Kollár's Criterion (Theorem 5.8.5), the pullback of $\lambda_k$ to $X$ is ample for $k \gg 0$. By Proposition 4.3.28, for $N$ sufficiently divisible, $\lambda_k^{\otimes N}$ descends to a line bundle $L$ on $\overline{M}_g$. Since the pullback of $L$ under the finite morphism $X \to \overline{\mathcal{M}}_g \to \overline{M}_g$ is ample, we conclude by Exercise 4.4.22 that $L$ is ample. □

In the next section, we will establish condition (b), the nefness of the pluricanonical bundles. This will allow us to conclude:

**Theorem 5.8.14.** *If $2g - 2 + n > 0$, then $\overline{M}_{g,n}$ is projective.*

*Proof.* It suffices to handle the $n = 0$ case as $\overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ is the universal family (Proposition 5.6.4) and is a projective morphism (Proposition 5.3.9). The fact that $\lambda_k$ descends to an ample line bundle on $\overline{M}_g$ follows from Proposition 5.8.13 as Condition (a) is a consequence of Stable Reduction (see Theorem 5.5.3) while (b) is Theorem 5.8.17. □

**Remark 5.8.15.** It also possible to show projectivity of $\overline{M}_{g,n}$ directly using Kollár's Criterion applied to the determinant of $\pi_*(L^k)$ where $L := \omega_{\mathcal{U}_{g,n}/\overline{\mathcal{M}}_{g,n}}(\sigma_1 + \cdots + \sigma_n)$ and $\mathcal{U}_{g,n} \to \overline{\mathcal{M}}_{g,n}$ is the universal family with sections $\sigma_1, \ldots, \sigma_n$.

**Remark 5.8.16.** The criteria of Proposition 5.8.13 for ampleness generalizes to every moduli of polarized varieties (see [Kol90, Thm. 2.6]); this was one of the original motivations of Kollár's paper. In recent years, Kollár's Criterion has been applied in more and more general settings to establish projectivity, e.g. Hassett's moduli space of weighted pointed curves [Has03], the moduli of stable varieties of any dimension [KP17], and the moduli of K-polystable Fano varieties [CP21, XZ20, LXZ21].

### 5.8.3 Nefness of pluricanonical bundles

In this section, we establish that $\pi_*(\omega_{\mathcal{C}/T}^{\otimes k})$ is nef for every $k \geq 2$.

**Theorem 5.8.17.** *For family of stable curves $\mathcal{C} \to T$ over a smooth projective curve $T$, $\pi_*(\omega_{\mathcal{C}/T}^{\otimes k})$ is nef for $k \geq 2$.*

*Proof sketch:* Let $\Bbbk$ be the base field.

*Step 1: Reduction to characteristic $p$.* Assume that $\operatorname{char}(\Bbbk) = 0$. Since $\mathcal{C}$ and $T$ are finite type over $\Bbbk$, their defining equations only involve finitely many coefficients of $\Bbbk$. Thus there exists a finitely generated $\mathbb{Z}$-subalgebra $A \subset \Bbbk$ and a cartesian diagram

$$\begin{tikzcd} \mathcal{C} \arrow[r] \arrow[d] & \widetilde{\mathcal{C}} \arrow[d] \\ T \arrow[r] \arrow[d] & \widetilde{T} \arrow[d] \\ \operatorname{Spec} \Bbbk \arrow[r] & \operatorname{Spec} A \end{tikzcd}$$

where $\widetilde{C}$ and $\widetilde{T}$ are schemes of finite type over $A$. By possibly enlarging $A$, we can arrange that $\widetilde{T} \to \operatorname{Spec} A$ is a smooth and projective family of curves and that $\widetilde{C} \to \widetilde{T}$ is a family of stable curves. Finally, by restricting along a morphism $\operatorname{Spec} R \to \operatorname{Spec} A$ from a DVR such that the images of the closed and generic points have characteristic $p$ and 0, respectively, we may assume that $A$ is a DVR. Since nefness is an open condition for such proper flat families (Proposition E.2.28), it suffices to prove the theorem when $\operatorname{char}(\Bbbk) = p > 0$.

*Step 2: Second reductions.* We reduce to the case where

(a) $\mathcal{C}$ is a smooth and minimal surface;

(b) $\mathcal{C} \to T$ is generically smooth; and

(c) the genus of $T$ is at least 2

(details to be added). These conditions imply that $\mathcal{C}$ is of general type.

*Step 3: Positive characteristic case.* Let $p = \operatorname{char}(\Bbbk)$. If $\pi_*(\omega_{\mathcal{C}/T}^{\otimes k})$ is not nef, then there exists a quotient line bundle $\pi_*(\omega_{\mathcal{C}/T}^{\otimes k}) \twoheadrightarrow M^\vee$ where $d = \deg M > 0$. Consider the absolute Frobenius morphisms $F\colon \mathcal{C} \to \mathcal{C}$ and $F\colon T \to T$ which fit into a commutative diagram

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d] & \mathcal{C} \arrow[d] \\ T \arrow[r, "F"] & T. \end{tikzcd}$$

By properties of the dualizing sheaf, we have $F^*\pi_*(\omega_{\mathcal{C}/T}^{\otimes k}) = \pi_*(\omega_{\mathcal{C}/T}^{\otimes k})$. Since $\deg F^* M = pd$, we can apply the Frobenius repeatedly to arrange that $d$, the degree of $M$, is as large as we want. Specifically, we can arrange that $M \cong \omega_T^{\otimes k} \otimes L$ where $L$ is a very ample line bundle on $T$. (This was the entire point of reducing to characteristic $p$: to repeatedly apply the Frobenius to jack-up the degree.)

The surjection $\pi_*(\omega_{\mathcal{C}/T}^{\otimes k}) \twoheadrightarrow M^\vee \cong (\omega_T^{\otimes k} \otimes L)^\vee$ yields a surjection

$$\pi_*(\omega_{\mathcal{C}/T}^{\otimes k}) \otimes \omega_T^{\otimes k} \otimes L \twoheadrightarrow \mathcal{O}_T$$

Since $\mathrm{h}^1(T, \mathcal{O}_T) \geq 2$, we have $\mathrm{h}^1(T, \pi_*(\omega_{\mathcal{C}/T}^{\otimes k}) \otimes \omega_T^{\otimes k} \otimes L) \geq 2$. Using the Leray spectral sequence to relate $H^1(\pi_*(\omega_{\mathcal{C}/T}^{\otimes k}) \otimes \omega_T^{\otimes k} \otimes L)$ to $H^1(\mathcal{C}, \omega_{\mathcal{C}}^{\otimes k} \otimes \pi^* L)$, one can show that $\mathrm{h}^1(\mathcal{C}, \omega_{\mathcal{C}}^{\otimes k} \otimes \pi^* L) \geq 2$ (details omitted). This however contradicts Bombieri–Ekedahl vanishing in the form of Lemma 5.8.19 with $D = \pi^* L$. $\square$

**Remark 5.8.18.** For families of *smooth* curves, $\pi_*(\omega_{\mathcal{C}/T})$ is nef; this fact is somewhat easier and was known earlier. If $\mathcal{C} \to S$ has no hyperelliptic fibers, then Max Noether's theorem on projective normality implies that $\mathrm{Sym}^d \pi_*(\omega_{\mathcal{C}/T}) \to \pi_*(\omega_{\mathcal{C}/T}^{\otimes d})$ is surjective. Therefore, the nefness of $\pi_*(\omega_{\mathcal{C}/T})$ implies the nefness of both $\mathrm{Sym}^d \pi_*(\omega_{\mathcal{C}/T})$ and the quotient $\pi_*(\omega_{\mathcal{C}/T}^{\otimes d})$ (Proposition E.2.27).

**Lemma 5.8.19.** *Let $S$ be a smooth projective surface over an algebraically closed field $\Bbbk$ which is minimal and of general type. Let $D$ be an effective divisor with $D^2 = 0$. If $\mathrm{char}(\Bbbk) \neq 2$, then $H^1(S, \omega_S^{\otimes n}(D)) = 0$ for all $n \geq 2$. If $\mathrm{char}(\Bbbk) = 2$, then $\mathrm{h}^1(S, \omega_S^{\otimes n}(D)) \leq 1$ for all $n \geq 2$.*

*Proof.* Bombieri–Ekedahl vanishing (Theorem E.3.1) implies that $H^1(S, K_S^{\otimes -n}) = 0$ for all $n \geq 1$. The Serre dual of this statement is that $H^1(S, K_S^{\otimes n}) = 0$ for all $n \geq 2$. The statement follows from using the short exact sequence

$$0 \to \omega_S^{\otimes n} \to \omega_S^{\otimes n}(D) \to \omega_S^{\otimes n}|_D \to 0$$

and adjunction (details omitted). $\square$

### 5.8.4 Projectivity via Geometric Invariant Theory

The Geometric Invariant Theory (GIT) construction depends on two integers:

- $k$, the multiple of the dualizing sheaf used to obtain an embedding $C \xrightarrow{|\omega_C^{\otimes k}|} \mathbb{P}^{r(k)-1}$. We need $k \geq 3$ for $\omega_C^{\otimes k}$ to be very ample for a stable curve $C$ but we need $k \geq 5$ for the GIT construction to yield $\overline{M}_g$.
- $d$, the degree of the equations that we use to embed the Hilbert scheme of $k$-canonically embedded curves into a Grassmannian. We need $d \gg 0$ to obtain an embedding of the Hilbert scheme.

Assuming that $k \geq 3$, a stable curve $C$ of genus $g$ is pluricanonically embedded via

$$C \xrightarrow{|\omega_C^{\otimes k}|} \mathbb{P}^{r(k)-1}$$

where $r(k) = (2k-1)(g-1)$. Let $P(t) = \chi(C, \omega^{\otimes kt}) = (2kt-1)(g-1)$ be the Hilbert polynomial of $C$ in $\mathbb{P}^{r(k)-1}$. Let $H' \subset \mathrm{Hilb}^P(\mathbb{P}^{r(k)-1})$ be the locally closed subscheme of the Hilbert scheme parameterizing stable curves $[C \hookrightarrow \mathbb{P}^{r(k)-1}]$ embedded via $\omega_C^{\otimes k}$. Note that $\mathrm{PGL}_{r(k)}$ acts naturally on $\mathrm{Hilb}^P(\mathbb{P}^{r(k)-1})$ and that the subscheme $H'$ is $\mathrm{PGL}_{r(k)}$-invariant.

**Exercise 5.8.20.** Extend Theorem 3.1.16 by establishing that:

(a) $H' \subset \mathrm{Hilb}^P(\mathbb{P}^{r(k)-1}$ is a locally closed $\mathrm{PGL}_{r(k)}$-invariant subscheme, and

(b) $\overline{\mathcal{M}}_g \cong [H'/\mathrm{PGL}_{r(k)}]$.

Let $H = \overline{H'} \subset \mathrm{Hilb}^P(\mathbb{P}^{r(k)-1}$ be the closure of $H'$. For $d \gg 0$, we have an embedding into the Grassmannian of $P(d)$-dimensional quotients of $\Gamma(\mathbb{P}^{r(k)-1}, \mathcal{O}(d))$

$$\begin{aligned} H &\hookrightarrow \mathrm{Gr}(P(d), \Gamma(\mathbb{P}^{r(k)-1}, \mathcal{O}(d)) \\ [C \hookrightarrow \mathbb{P}^{r(k-1)}] &\mapsto \left[\Gamma(\mathbb{P}^{r(k)-1}, \mathcal{O}(d)) \twoheadrightarrow \Gamma(C, \mathcal{O}(d))\right] \end{aligned}$$

Note that there is a natural identification of this quotient with the multiplication map

$$\begin{tikzcd} \Gamma(\mathbb{P}^{r(k)-1}, \mathcal{O}(d)) \arrow[r, two heads] \arrow[d, equal] & \Gamma(C, \mathcal{O}(d)) \arrow[d, equal] \\ \mathrm{Sym}^d H^0(C, \omega_C^{\otimes k}) \arrow[r, two heads] & H^0(C, \omega_C^{\otimes dk}). \end{tikzcd}$$

Let $\mathcal{O}_{\mathrm{Gr}}(1)$ be the very ample line bundle on $\mathrm{Gr}(P(d), \Gamma(\mathbb{P}^{r(k)-1}, \mathcal{O}(d))$ obtained via the Plücker embedding

$$\begin{aligned} \mathrm{Gr}(P(d), \Gamma(\mathbb{P}^{r(k)-1}, \mathcal{O}(d)) &\hookrightarrow \mathbb{P}(\wedge^{P(d)}\Gamma(\mathbb{P}^{r(k)-1}, \mathcal{O}(d)) \\ [\Gamma(\mathbb{P}^{r(k)-1}, \mathcal{O}(d)) \twoheadrightarrow \Gamma(C, \mathcal{O}(d))] &\mapsto [\wedge^{P(d)}\Gamma(\mathbb{P}^{r(k)-1}, \mathcal{O}(d)) \twoheadrightarrow \wedge^{P(d)}\Gamma(C, \mathcal{O}(d))]; \end{aligned}$$

see Section 1.2. Finally, let $L_d = \mathcal{O}_{\mathrm{Gr}}(1)|_H$ be the very ample line bundle on $H$ obtained by restricting $\mathcal{O}(1)$ under the composition

$$H \hookrightarrow \mathrm{Gr}(P(d), \Gamma(\mathbb{P}^{r(k)-1}, \mathcal{O}(d)) \hookrightarrow \mathbb{P}(\wedge^{P(d)}\Gamma(\mathbb{P}^{r(k)-1}, \mathcal{O}(d)) \tag{5.8.8}$$

As each morphism in (5.8.8) is $\mathrm{PGL}_{r(k)}$-equivariant, the line bundle $L_d$ inherits a $\mathrm{PGL}_{r(k)}$-linearization.

**Definition 5.8.21.** A point $h \in H$ is said to be *GIT semistable with respect to $L_d$* if there exists an equivariant section $s \in \Gamma(H, L_d^{\otimes N})^{\mathrm{PGL}_{r(k)}}$ with $N > 0$ such that $s(h) \neq 0$. The *semistable locus* $H^{\mathrm{ss}}$ consisting of GIT semistable points is an open $\mathrm{PGL}_{r(k)}$-invariant subscheme.

**Remark 5.8.22.** Stack-theoretically, the $\mathrm{PGL}_{r(k)}$-linearization $L_d$ defines a line bundle, which we will also denote by $L_d$, on the quotient stack $[H/\mathrm{PGL}_{r(k)}]$ and the open substack $[H^{\mathrm{ss}}/\mathrm{PGL}_{r(k)}]$ is the largest open substack such the restriction of $L_d$ is semiample. In other words, $h \in H$ is GIT semistable if and only $h$ does not lie in the stable base locus of $L_d$ on $[H/\mathrm{PGL}_{r(k)}]$.

**Remark 5.8.23.** These definitions clearly extend to the action of every algebraic group $G$ on a projective scheme $X$ embedded $G$-equivariantly $X \hookrightarrow \mathbb{P}^N$ by a $G$-linearization $L$. One of the main results of GIT is that if $G$ is reductive, then the graded ring $\bigoplus_{N \geq 0} \Gamma(H, L_d^{\otimes N})^{\mathrm{PGL}_{r(k)}}$ is finitely generated and that the morphism

$$X^{\mathrm{ss}} \to X^{\mathrm{ss}}/\!\!/G := \mathrm{Proj} \bigoplus_{N \geq 0} \Gamma(H, L_d^{\otimes N})^{\mathrm{PGL}_{r(k)}}$$

is a good quotient. Note that $X^{\mathrm{ss}}$ is precisely the maximal locus where the rational map $X \dashrightarrow X^{\mathrm{ss}}/\!\!/G$ is defined.

The GIT construction of $\overline{M}_g$ rests on the following difficult theorem:

**Theorem 5.8.24.** *Let $k \geq 5$ and $d \gg 0$. For $h = [C \hookrightarrow \mathbb{P}^{r(k)-1}] \in H$, the curve $C$ is stable if and only if $h \in H$ is GIT semistable with respect to $L_d$.*

**Remark 5.8.25.** This theorem can be established using the Hilbert–Mumford criteria. It is rather difficult to explicitly exhibit sections of $\Gamma(H, L_d^{\otimes N})^{\mathrm{PGL}_{r(k)}}$ and the Hilbert–Mumford criteria allows us to verify that a given point $h \in H$ is semistable by checking that for each one-parameter subgroup $\lambda \colon \mathbb{G}_m \to \mathrm{PGL}_{r(k)}$, the Hilbert–Mumford index $\mu(h, L_d)$, defined as the weight of $\mathbb{G}_m$ on the line in the affine cone $\mathbb{A}^{r(k)}$ over $\lim_{t\to 0} \lambda(t) \cdot h \in H \subset \mathbb{P}^{r(k)-1}$, is negative. The beauty of the Hilbert–Mumford criterion is that it magically guarantees the existence of sections for you! Nevertheless, verifying the Hilbert–Mumford criterion, even for a smooth pluricanonical embedded curve, is no easy task.

Given Theorem 5.8.24, we obtain $\overline{M}_g$ as the projective variety

$$\overline{M}_g = \operatorname{Proj} \Gamma(H, L_d^{\otimes N})^{\mathrm{PGL}_{r(k)}}.$$

**Remark 5.8.26.** As a spectacular corollary of Theorem 5.8.24, one obtains an alternative proof of Stable Reduction (Theorem 5.5.1) in arbitrary characteristic. This is perhaps surprising as the GIT argument uses rather little about the geometry of stable curves and their families.

**Remark 5.8.27** (The ample cone)**.** For each $k \geq 5$ and $d \gg 0$, GIT constructs a line bundle on $\overline{\mathcal{M}}_g$ which descends to an ample line bundle on $\overline{M}_g$. This class can be expressed as

$$r(k)\lambda_{dk} - r(dk)\lambda_k.$$

Grothendieck–Riemann–Roch can be used to express each of the line bundles $\lambda_k$ as a linear combination of $\lambda_1$ and $\delta$, the boundary divisor. The asymptotic limit of this class as $d$ goes to infinity is proportional to

$$(12 - \frac{4}{k})\lambda_1 - \delta.$$

Taking $k = 5$, shows that $11.2\lambda - \delta$ is ample.

However, even more is true! By bootstrapping the positivity deduced from GIT, Cornalba and Harris showed that $a\lambda - \delta$ is ample if and only if $a > 11$, thus determining the ample cone of $\overline{M}_g$ in the $\lambda_1\delta$-plane of $\mathrm{NS}^1(\overline{\mathrm{M}}_\mathrm{g})$ [CH88].

# Chapter 6

# Geometry of algebraic stacks

## 6.1 Quasi-coherent sheaves and quotient stacks

We will define quasi-coherent sheaves on an algebraic stack in the same way that we did for Deligne–Mumford stacks in §4.1 but using the lisse-étale site on $\mathcal{X}$ instead of the small étale site. The entirety of §4.1 on sheaves, $\mathcal{O}_{\mathcal{X}}$-modules, and quasi-coherent sheaves remains valid for algebraic stacks (with the same affine diagonal hypotheses).

### 6.1.1 Sheaves and $\mathcal{O}_{\mathcal{X}}$-modules

To develop abelian sheaf theory on an algebraic stack, we use the lisse-étale site.

**Definition 6.1.1** (Lisse-étale site)**.** The *lisse-étale site* $\mathcal{X}_{\text{lis-ét}}$ on an algebraic stack $\mathcal{X}$ is the category of schemes smooth over $\mathcal{X}$ where morphisms are arbitrary maps of schemes smooth over $\mathcal{X}$. A covering $\{U_i \to U\}$ is a collection of morphisms such that $\coprod_i U_i \to U$ is surjective and étale.

This allows us to discuss sheaves of abelian groups on $\mathcal{X}_{\text{lis-ét}}$ and their morphisms. Extending §4.1.1, we can define sections $\Gamma(\mathcal{U}, F)$ or $F(\mathcal{U})$ of an abelian sheaf on an algebraic stack $\mathcal{U}$ smooth over $\mathcal{X}$. The structure sheaf $\mathcal{O}_{\mathcal{X}}$, defined as $\mathcal{O}_{\mathcal{X}}(U) = \Gamma(U, \mathcal{O}_U)$, is a ring object in the abelian category $\mathrm{Ab}(\mathcal{X}_{\text{lis-ét}})$. We can therefore define $\mathcal{O}_{\mathcal{X}}$-modules as in Definition 4.1.8 and the abelian category $\mathrm{Mod}(\mathcal{O}_{\mathcal{X}})$ of $\mathcal{O}_{\mathcal{X}}$-modules. Given a morphism $f\colon \mathcal{X} \to \mathcal{Y}$ of algebraic stacks, there are adjoint functors

$$\mathrm{Ab}(\mathcal{X}_{\text{lis-ét}}) \underset{f^{-1}}{\overset{f_*}{\rightleftarrows}} \mathrm{Ab}(\mathcal{Y}_{\text{lis-ét}}) \qquad \mathrm{Mod}(\mathcal{O}_{\mathcal{X}}) \underset{f^*}{\overset{f_*}{\rightleftarrows}} \mathrm{Mod}(\mathcal{O}_{\mathcal{Y}}).$$

Given two $\mathcal{O}_{\mathcal{X}}$-modules $F$ and $G$, the *tensor product* $F \otimes G := F \otimes_{\mathcal{O}_{\mathcal{X}}} G$ is the sheafification of $U \mapsto F(U) \otimes_{\mathcal{O}_{\mathcal{X}}(U)} G(U)$, and the *Hom sheaf* $\mathscr{H}om_{\mathcal{O}_{\mathcal{X}}}(F, G)$ is the sheaf given by $U \mapsto \mathrm{Hom}_{\mathcal{O}_U}(F|_U, G|_U)$, where $F|_U$ denotes the restriction of $F$ to $U_{\text{lis-ét}}$.

### 6.1.2 Quasi-coherent sheaves

Following §4.1.3, given an $\mathcal{O}_{\mathcal{X}}$-module $F$ on an algebraic stack $\mathcal{X}$ and a smooth $\mathcal{X}$-scheme $U$, we let $F|_U$ be the restriction of $F$ to the lisse-étale site of $U$ and $F|_{U_{\mathrm{Zar}}}$ the further restriction to the small Zariski site.

**Definition 6.1.2.** Let $\mathcal{X}$ be an algebraic stack. An $\mathcal{O}_{\mathcal{X}}$-module $F$ is *quasi-coherent* if

(1) for every smooth $\mathcal{X}$-scheme $U$, the restriction $F|_{U_{\mathrm{Zar}}}$ is a quasi-coherent $\mathcal{O}_{U_{\mathrm{Zar}}}$-module, and

(2) for every morphism $f\colon U \to V$ of smooth $\mathcal{X}$-schemes, the natural morphism $f^*(F|_{V_{\mathrm{Zar}}}) \to F|_{U_{\mathrm{Zar}}}$ is an isomorphism.

A quasi-coherent sheaf $F$ on $\mathcal{X}$ is a *vector bundle* (resp. *vector bundle of rank* $r$, *line bundle*) if $F|_{U_{\mathrm{Zar}}}$ is for every smooth $\mathcal{X}$-scheme $U$.

If in addition $\mathcal{X}$ is locally noetherian, we say $F$ is *coherent* if $F|_{U_{\mathrm{Zar}}}$ is coherent for every smooth $\mathcal{X}$-scheme $U$

We denote by $\mathrm{QCoh}(\mathcal{X})$ and $\mathrm{Coh}(\mathcal{X})$ (in the noetherian setting) the categories of quasi-coherent and coherent sheaves. We encourage the reader to check that the equivalent formulations of quasi-coherent given in Exercises 4.1.14 to 4.1.16 still hold, and that the above definition of quasi-coherence is consistent with the definition of quasi-coherence on a Deligne–Mumford stack (Definition 4.1.11) and with the usual definition on a scheme. For a quasi-compact and quasi-separated morphism $f\colon \mathcal{X} \to \mathcal{Y}$ of algebraic stacks, $f_*$ and $f^*$ preserve quasi-coherence (by the same argument as for Exercise 4.1.17).

**Exercise 6.1.3.** Let $G$ be an affine algebraic group over a field $\Bbbk$. Recall that a $G$-representation is a $\Bbbk$-vector space with a dual action $\sigma\colon V \to \Gamma(G, \mathcal{O}_G) \otimes_{\Bbbk} V$ satisfying two natural compatibility conditions (see §C.1.3).

(a) Show that $\mathrm{QCoh}(\mathbf{B}G)$ is equivalent to the category $\mathrm{Rep}(G)$ of $G$-representations.

(b) If $\operatorname{Spec} A$ is an affine $\Bbbk$-scheme with a $G$-action, show that a quasi-coherent sheaf on $[\operatorname{Spec} A/G]$ is the data of an $A$-module $M$ together with a coaction $\sigma\colon M \to \Gamma(G, \mathcal{O}_G) \otimes_{\Bbbk} M$ over $\Bbbk$ (i.e. a map of $\Bbbk$-vector spaces giving $M$ the structure of a $G$-representation) such that multiplication $A \otimes_{\Bbbk} M \to M$ is a map of $G$-representations. This extends Example 4.1.13 where $G$ is finite.

(c) Considering the diagram

$$\begin{tikzcd} \operatorname{Spec} A \arrow[r, "p"] & {[\operatorname{Spec} A/G]} \arrow[r, "\pi"] \arrow[d, "q"] & \operatorname{Spec} A^G \\ & \mathbf{B}G, & \end{tikzcd}$$

extend Exercise 4.1.18 by providing descriptions of the functors $p_*, p^*, \pi_*, \pi^*, q_*$ and $q^*$ on quasi-coherent sheaves.

(d) If $U$ is a $\Bbbk$-scheme with an action of $G$, then a *line bundle with a $G$-action* is a line bundle $L$ on $U$ together with an isomorphism $\alpha\colon \sigma^* L \xrightarrow{\sim} p_2^* L$ satisfying a cocycle condition $p_{23}^*\alpha \circ (\mathrm{id}_G \times \sigma)^*\alpha = (\mu \times \mathrm{id}_U)^*\alpha$; see C.3.4. Show that a line bundle with a $G$-action is the same as a line bundle on the quotient stack $[U/G]$.

**Example 6.1.4.** If $G$ and $H$ are affine algebraic groups over a field $\Bbbk$ such that $\mathbf{B}G \cong \mathbf{B}H$, then $G$ and $H$ have equivalent categories of representations. For example, if $O(q)$ and $O(q')$ are orthogonal groups with respect to non-degenerate quadratic forms $q$ and $q'$ on an $n$-dimensional $\Bbbk$-vector space $V$, then $\mathbf{B}O(q) \cong \mathbf{B}O(q')$ (see Example 3.1.12), and thus $O(q)$ and $O(Q')$ have equivalent categories of representations.

Recall that one of the first examples we gave of a quasi-coherent sheaf on a Deligne–Mumford stack was the Hodge line bundle on $\mathfrak{M}_g$ (Examples 4.1.12), which we later generalized the pluricanonical line bundles $\lambda_k$ on $\overline{\mathfrak{M}}_g$ (see §5.8). These line bundles played an important role in our argument for the projectivity of $\overline{M}_g$ and are equally essential in the study of its geometry. Determinantal line bundles play a similar role in the study of the moduli stack of vector bundles.

**Example 6.1.5** (Determinantal line bundles)**.** Consider the stack $\mathcal{M} := \mathrm{Bun}_{C,r,d}$ of vector bundles on a smooth, connected, and projective curve $C$ over $\Bbbk$. Consider the diagram

$$\begin{tikzcd} & C \times \mathcal{M} \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ C & & \mathcal{M}. \end{tikzcd}$$

The projection $p_2\colon C \times \mathcal{M} \to \mathcal{M}$ is representable, projective, and smooth of relative dimension 1. For every vector bundle $F$ on $C \times \mathcal{M}$, applying Proposition A.7.10 and smooth descent shows that the cohomology $\mathrm{R}^i p_{2,*}F$ (as defined below) is computed as a 2-term complex $[K^0 \to K^1]$ of vector bundles and that the line bundle

$$\det \mathrm{R}p_{2,*}F := \det(K_0) \otimes \det(K_1)^\vee$$

is well-defined on $\mathcal{M}$. Note that if $\operatorname{rk} K_0 = \operatorname{rk} K_1$, i.e. $\operatorname{rk} \mathrm{R}p_{2,*}F = 0$, then we have a map $\det K^0 \to \det K^1$ of line bundles and the corresponding map $\mathcal{O}_{\mathcal{M}} \to \det(K^0)^\vee \otimes \det(K_1)$ defines a section of the dual $(\det \mathrm{R}p_{2,*}F)^\vee$.

Let $\mathcal{E}_{\mathrm{univ}}$ be the universal vector bundle on $C \times \mathcal{M}$. For every vector bundle $V$ on $C$, we define the *determinantal line bundle*

$$\mathcal{L}_V := \big(\det \mathrm{R}p_{2,*}(\mathcal{E}_{\mathrm{univ}} \otimes p_1^* V)\big)^\vee.$$

associated to $V$.

**Example 6.1.6.** If $\mathcal{X}$ is an algebraic stack of finite presentation over a scheme $S$, then the relative sheaf of differentials $\Omega_{\mathcal{X}/S}$ on $\mathcal{X}_{\text{lis-ét}}$, defined on a smooth $\mathcal{X}$-scheme $U$ by $\Omega_{\mathcal{X}/S}(U) = \Omega_{U/S}$, is not quasi-coherent. This is because for a non-étale map $f\colon U \to V$ of smooth $\mathcal{X}$-schemes, $f^*\Omega_{V/S} \to \Omega_{U/S}$ is not an isomorphism. This differs from the Deligne–Mumford case where the sheaf $\Omega_{\mathcal{X}/S}$ on $\mathcal{X}_{\text{ét}}$ is quasi-coherent (Examples 4.1.12). When $\mathcal{X}$ is Deligne–Mumford, $\Omega_{\mathcal{X}/S}$ extends to a quasi-coherent sheaf on $\mathcal{X}_{\text{lis-ét}}$ by defining $\Omega_{\mathcal{X}/S}(U)$, for a smooth map $f\colon U \to \mathcal{X}$ from a scheme, to be the global sections of the sheaf $f^*\Omega_{\mathcal{X}/S}$ on $U_{\text{lis-ét}}$.

Exercises 4.1.19 and 4.1.20 generalize to algebraic stacks.

**Proposition 6.1.7** (Flat Base Change)**.** *Consider a cartesian diagram*

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] \arrow[dr, phantom, "\square"] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

*of algebraic stacks, and let $F$ be a quasi-coherent sheaf on $X$. If $g\colon Y' \to Y$ is flat and $f\colon X \to Y$ is quasi-compact and quasi-separated, the natural adjunction map*

$$g^* f_* F \to f'_* g'^* F$$

*is an isomorphism.* $\square$

**Proposition 6.1.8.** *Let $\mathcal{X}$ be a noetherian algebraic stack. Every quasi-coherent sheaf on $\mathcal{X}$ is a directed colimit of its coherent subsheaves. If $\mathcal{U} \subset \mathcal{X}$ is an open substack, then every coherent sheaf on $\mathcal{U}$ extends to a coherent sheaf on $\mathcal{X}$.*

**Exercise 6.1.9.** Let $\mathcal{X} \to \mathcal{Y}$ be a smooth affine morphism of noetherian algebraic stacks with affine diagonal.

(1) Show that there is a vector bundle $\Omega_{\mathcal{X}/\mathcal{Y}}$ on $\mathcal{X}$ with the property that if $V \to \mathcal{Y}$ is a morphism from a scheme, the pullback of $\Omega_{\mathcal{X}/\mathcal{Y}}$ to $X_V := \mathcal{X} \times_{\mathcal{Y}} V$ is $\Omega_{\mathcal{X}_V/V}$.

(2) Given a commutative diagram

$$\begin{tikzcd} \operatorname{Spec} A_0 \arrow[r, "f_0"] \arrow[d, hook] & \mathcal{X} \arrow[d] \\ \operatorname{Spec} A \arrow[r] \arrow[ur, dashed] & \mathcal{Y} \end{tikzcd}$$

where $A \twoheadrightarrow A_0$ is a surjection of noetherian rings with square-zero kernel $J$, show that the set of liftings is a torsor under $\operatorname{Hom}_{A_0}(f_0^*\Omega_{\mathcal{X}/\mathcal{Y}}, J)$ and in particular is non-empty.

(3) Can you weaken the hypotheses?

### 6.1.3 Quasi-coherent constructions

Extending the constructions of §4.1.5 on a Deligne–Mumford stack to an algebraic stack $\mathcal{X}$, a *quasi-coherent $\mathcal{O}_{\mathcal{X}}$-algebra* is a quasi-coherent $\mathcal{O}_{\mathcal{X}}$-module $\mathcal{A}$ with a compatible structure as a ring object. The *relative spectrum* $\mathcal{S}\mathrm{pec}_{\mathcal{X}}\,\mathcal{A}$, defined as the stack of pairs $(f, \alpha)$ where $f\colon S \to \mathcal{X}$ is a morphism from a scheme and $\alpha\colon f^*\mathcal{A} \to \mathcal{O}_S$ is a map of $\mathcal{O}_S$-algebras, is an algebraic stack affine over $\mathcal{X}$. On a noetherian algebraic stack, every quasi-coherent $\mathcal{O}_{\mathcal{X}}$-algebra is a directed colimit of finite type subalgebras.

The *reduction of* $\mathcal{X}$ is $\mathcal{X}_{\mathrm{red}} := \mathcal{S}\mathrm{pec}_{\mathcal{X}}\,\mathcal{O}_{\mathcal{X}}^{\mathrm{red}}$ where $\mathcal{O}_{\mathcal{X}}^{\mathrm{red}}$ is the sheaf of $\mathcal{O}_{\mathcal{X}}$-algebras defined by $\mathcal{O}_{\mathcal{X}}^{\mathrm{red}}(U) = \Gamma(U, \mathcal{O}_U)_{\mathrm{red}}$ for a smooth $\mathcal{X}$-scheme $U$. If $\mathcal{X}$ is integral, the *normalization of* $\mathcal{X}$ is defined as $\widetilde{\mathcal{X}} := \mathcal{S}\mathrm{pec}_{\mathcal{X}}\,\mathcal{A}$, where $\mathcal{A}$ is the $\mathcal{O}_{\mathcal{X}}$-algebra whose ring of sections over a smooth $\mathcal{X}$-scheme $U$ is the normalization of $\Gamma(U, \mathcal{O}_U)$; this is well-defined since normalization commutes with smooth base change (Proposition A.5.4). For a quasi-compact and quasi-separated morphism $f\colon \mathcal{X} \to \mathcal{Y}$ of algebraic stacks, there is a factorization $f\colon \mathcal{X} \to \mathcal{S}\mathrm{pec}\, f_*\mathcal{O}_{\mathcal{X}} \to \mathcal{Y}$. The morphism $f$ is affine if and only if $\mathcal{X} \to \mathcal{S}\mathrm{pec}\, f_*\mathcal{O}_{\mathcal{X}}$ is an isomorphism, and quasi-affine if and only if $\mathcal{X} \to \mathcal{S}\mathrm{pec}\, f_*\mathcal{O}_{\mathcal{X}}$ is an open immersion.

The proof of Zariski's Main Theorem (4.4.9) in the case of Deligne–Mumford stacks extends to algebraic stacks.

**Theorem 6.1.10** (Zariski's Main Theorem)**.** *A separated, quasi-finite, and representable morphism $f\colon \mathcal{X} \to \mathcal{Y}$ of noetherian algebraic stacks factors as the composition of a dense open immersion $\mathcal{X} \hookrightarrow \widetilde{\mathcal{Y}}$ and a finite morphism $\widetilde{\mathcal{Y}} \to \mathcal{X}$.* □

### 6.1.4 Picard groups

If $\mathcal{X}$ is an algebraic stack, we let $\operatorname{Pic}(\mathcal{X})$ denote the set of isomorphism classes of line bundles on $\mathcal{X}$. It is an abelian group under tensor product.

**Example 6.1.11.** If $G$ is an affine algebraic group over a field $\Bbbk$, then $\mathrm{Pic}(\mathbf{B}G)$ is equivalent to the group of characters $G \to \mathbb{G}_m$. For example, $\mathrm{Pic}(\mathbf{B}\mathbb{G}_m) = \mathbb{Z}$, $\mathrm{Pic}(\mathbf{B}\,\mathrm{GL}_n) = \mathbb{Z}$, and $\mathrm{Pic}(\mathrm{PGL}_n) = \{0\}$.

**Exercise 6.1.12.** Let $\mathcal{X}$ be a smooth and irreducible algebraic stack over a field $\Bbbk$.

(a) If $\mathcal{D} \subset \mathcal{X}$ is a reduced substack with complement $\mathcal{U}$, show that there is a naturally defined line bundle $\mathcal{O}(\mathcal{D})$ (generalizing the usual construction for schemes) such that $\mathcal{O}(\mathcal{D})|_{\mathcal{U}} \cong \mathcal{O}_{\mathcal{U}}$.

(b) If $V$ is a vector bundle on $\mathcal{X}$, show that

$$\mathrm{Pic}(\mathbb{A}(V)) = \mathrm{Pic}(\mathcal{X}) \quad \text{and} \quad \mathrm{Pic}(\mathbb{P}(V)) = \mathrm{Pic}(\mathcal{X}) \times \mathbb{Z}.$$

**Exercise 6.1.13.** Let $\mathbb{G}_m$ acts on $\mathbb{A}^n$ over a field $\Bbbk$ with weights $d_1, \ldots, d_n$. Let $\mathcal{O}(1)$ be the line bundle on $[\mathbb{A}^n/\mathbb{G}_m]$ corresponding to the projection $[\mathbb{A}^n/\mathbb{G}_m] \to \mathbf{B}\mathbb{G}_m$.

(a) Show that $\mathrm{Pic}([\mathbb{A}^n/\mathbb{G}_m]) \cong \mathbb{Z}$ generated by $\mathcal{O}(1)$.

(b) Show that the restriction $\mathrm{Pic}([\mathbb{A}^n/\mathbb{G}_m]) \to \mathrm{Pic}(\mathcal{P}(d_1, \ldots, d_n))$ is an isomorphism, where $\mathcal{P}(d_1, \ldots, d_n)$ is the weighted projective stack (see Example 3.9.6).

(c) If $f \in \Gamma(\mathbb{A}^n, \mathcal{O}_{\mathbb{A}^n})$ is a homogenous polynomial of degree $d$ such that $V(f) \subset \mathbb{A}^n$ is reduced, show that $\mathcal{O}(V(f)) \cong \mathcal{O}(d)$.

**Exercise 6.1.14.** Let $\Bbbk$ be a field with $\mathrm{char}(\Bbbk) \neq 2, 3$.

(a) Show that $\mathrm{Pic}(\overline{\mathcal{M}}_{1,1}) = \mathbb{Z}$.

*Hint: Use the description* $\overline{\mathcal{M}}_{1,1} = [(\mathbb{A}^2 \smallsetminus 0)/\mathbb{G}_m]$ *of Exercise 3.1.18(c) where* $\mathbb{G}_m$ *acts with weights 4 and 6. Show that the restriction* $\mathrm{Pic}(\mathbb{A}^2/\mathbb{G}_m]) \to \mathrm{Pic}(\overline{\mathcal{M}}_{1,1})$ *is an equivalence.*

(b) Show that $\mathrm{Pic}(\mathcal{M}_{1,1}) = \mathbb{Z}/12$.

*Hint: Show that the restriction* $\mathrm{Pic}(\overline{\mathcal{M}}_{1,1}) \to \mathrm{Pic}(\mathcal{M}_{1,1})$ *is surjective and that the image of* $\mathcal{O}(\Delta) = \mathcal{O}(12)$ *is trivial. Show that the images of* $\mathcal{O}(4)$ *and* $\mathcal{O}(6)$ *are non-trivial by considering their restrictions to the residual gerbes of the unique elliptic curves with* $\mathbb{Z}/4$ *and* $\mathbb{Z}/6$ *automorphism groups. See also [Mum65].*

### 6.1.5 Global quotient stacks and the resolution property

**Definition 6.1.15.** An algebraic stack $\mathcal{X}$ is a *global quotient stack* if there exists an isomorphism $\mathcal{X} \cong [U/\,\mathrm{GL}_n]$ where $U$ is an algebraic space.

In other words, $\mathcal{X}$ is a global quotient stack if and only if there is a principal $\mathrm{GL}_n$-bundle $U \to \mathcal{X}$ from an algebraic space, or equivalently a representable morphism $\mathcal{X} \to \mathbf{B}\,\mathrm{GL}_n$.

**Exercise 6.1.16.** Show that a noetherian algebraic stack $\mathcal{X}$ is a global quotient stack if and only if there exists a vector bundle $E$ on $\mathcal{X}$ such that for every geometric point $x\colon \mathrm{Spec}\,\Bbbk \to \mathcal{X}$ with closed image, the stabilizer $G_x$ acts faithfully on the fiber $E \otimes \Bbbk$.

*Hint: Use the correspondence between principal* $\mathrm{GL}_n$*-bundles and vector bundles from Exercise C.2.11.*

**Exercise 6.1.17.** Let $\mathcal{X} \to \mathcal{Y}$ be a surjective, flat, and projective morphism of noetherian algebraic stacks. If $\mathcal{X}$ is a quotient stack, show that $\mathcal{Y}$ is a quotient stack.

Being a quotient stack is also related to the following notion:

**Definition 6.1.18.** A noetherian algebraic stack has the *resolution property* if every coherent sheaf is the quotient of a vector bundle.

A smooth or quasi-projective scheme over a field has the resolution property. More generally, a scheme admitting an "ample family" of line bundles has the resolution property, and this implies that every noetherian normal $\mathbb{Q}$-factorial scheme with affine diagonal has the resolution property [BS03].

**Proposition 6.1.19.** *Let $G$ be an affine algebraic group over a field $\Bbbk$ acting on a quasi-projective $\Bbbk$-scheme $U$. Assume that there is an ample line bundle $L$ with an action of $G$ (e.g. $U$ is quasi-affine and $L = \mathcal{O}_U$). Then $[U/G]$ has the resolution property.*

**Remark 6.1.20.** It is a general fact that every line bundle on a normal scheme over $\Bbbk$ has a positive power that has a $G$-action.

*Proof.* The line bundle $L$ corresponds to a line bundle $\mathcal{L}$ on $[U/G]$ which is relatively ample with respect to the morphism $p\colon [U/G] \to \mathbf{B}G$. For a coherent sheaf $F$ on $[U/G]$, the natural map

$$\mathcal{L}^{-N} \otimes p^* p_*(\mathcal{L}^N \otimes F) \twoheadrightarrow F$$

is surjective for $N \gg 0$. The pushforward $p_*(\mathcal{L}^N \otimes F)$ is a quasi-coherent sheaf on $\mathbf{B}G$, i.e. a $G$-representation, which we can write as a union of finite dimensional $G$-representations $V_i$ (C.3.3(1)). We therefore obtain a surjection $\mathrm{colim}_i(\mathcal{L}^{-N} \otimes p^* V_i) \twoheadrightarrow F$. Since $F$ is coherent, $\mathcal{L}^{-N} \otimes p^* V_i \twoheadrightarrow F$ is surjective for $i \gg 0$. □

Totaro established an interesting converse [Tot04], which was later generalized by Gross [Gro17].

**Theorem 6.1.21.** *Let $\mathcal{X}$ be a quasi-separated normal algebraic stack of finite type over a field $\Bbbk$. Assume that the stabilizer group at every closed point is smooth and affine. Then the following are equivalent:*

*(1) $\mathcal{X}$ has the resolution property,*

*(2) $\mathcal{X} \cong [U/\operatorname{GL}_n]$ with $U$ quasi-affine, and*

*(3) $\mathcal{X} \cong [\operatorname{Spec} A/G]$ with $G$ an affine algebraic group.*

*In particular, $\mathcal{X}$ has affine diagonal.*

**Remark 6.1.22.** While the normal hypothesis on $\mathcal{X}$ and smoothness hypothesis on the stabilizers are unnecessary, the affineness hypothesis on the stabilizers is necessary, e.g. the classifying stack $\mathbf{B}E$ of an elliptic curve has the resolution property.

*Proof.* The implications that (2) and (3) imply (1) were established in Proposition 6.1.19.

To see (3) ⇒ (2), it suffices to find a faithful representation $G \hookrightarrow \operatorname{GL}_N$ such that $\operatorname{GL}_N/G$ is quasi-affine. Indeed, in this case, $[\operatorname{Spec} A/G] \cong [(\operatorname{Spec} A \times^G \operatorname{GL}_N)/\operatorname{GL}_N]$ (Exercise 3.4.14) and $\operatorname{Spec} A \times^G \operatorname{GL}_N$ is affine over $\operatorname{GL}_N/G$. We begin by choosing a faithful representation $G \subset \operatorname{GL}_n$. By C.3.3C.3.1, there is a $\operatorname{GL}_n$-representation $V$ and a $\Bbbk$-point $x \in \mathbb{P}(V)$ with stabilizer $G$. Under the action of $\operatorname{GL}_n \times \mathbb{G}_m$ on $\mathbb{A}(V)$ (where $\mathbb{G}_m$ acts via scaling), the stabilizer of a lift $\widetilde{x} \in \mathbb{A}(V)$ of $x$ is $G$. The map $(\operatorname{GL}_n \times \mathbb{G}_m)/G \hookrightarrow \mathbb{A}(V)$, defined by $g \mapsto g\widetilde{x}$, is a locally closed immersion and thus

$(\mathrm{GL}_n \times \mathbb{G}_m)/G$ is quasi-affine. Under the natural inclusion $\mathrm{GL}_n \times \mathbb{G}_m \hookrightarrow \mathrm{GL}_{n+1}$, the quotient $\mathrm{GL}_{n+1}/(\mathrm{GL}_n \times \mathbb{G}_m)$ is affine (and is sometimes called the "Steifel manifold"). The composition $\mathbf{B}G \to \mathbf{B}(\mathrm{GL}_n \times \mathbb{G}_m) \to \mathbf{B}\,\mathrm{GL}_{n+1}$ is quasi-affine and therefore so is $\mathrm{GL}_{n+1}/G$.

Conversely for (2) $\Rightarrow$ (3), we may choose a $\mathrm{GL}_n$-equivariant open immersion $U \hookrightarrow \operatorname{Spec} A$ into an affine scheme of finite type over $\Bbbk$. Indeed, the morphism $p\colon [U/\mathrm{GL}_n] \to \mathbf{B}\,\mathrm{GL}_n$ is quasi-affine and $[U/\mathrm{GL}_n] \to \operatorname{Spec}_{\mathbf{B}\,\mathrm{GL}_n} p_*\mathcal{O}_{[U/\mathrm{GL}_n]}$ is an open immersion. By writing $p_*\mathcal{O}_{[U/\mathrm{GL}_n]} = \operatorname{colim}_\lambda \mathcal{A}_\lambda$ as a colimit of finite type $\mathcal{O}_{\mathbf{B}\,\mathrm{GL}_n}$-algebras, then limit methods imply that $[U/\mathrm{GL}_n] \to \operatorname{Spec}_{\mathbf{B}\,\mathrm{GL}_n} \mathcal{A}_\lambda$ is an open immersion for $\lambda \gg 0$. Let $Z \subset \operatorname{Spec} A$ be the reduced complement of $U$. By Proposition C.3.4(2), there is a $\mathrm{GL}_n$-equivariant morphism $f\colon \operatorname{Spec} A \to \mathbb{A}^r$ such that $f^{-1}(0) = Z$. This induces an affine morphism $U \to \mathbb{A}^r \smallsetminus 0$. The complement $\mathbb{A}^r \setminus 0$ can be realized as the quotient $\mathrm{GL}_r/H$ where $H \subset \mathrm{GL}_r$ is the subgroup consisting of matrices whose last row is $(0, \ldots, 0, 1)$; $H$ is identified with the semi-direct product $\mathbb{G}_a^{r-1} \rtimes \mathrm{GL}_{r-1}$. In the $\mathrm{GL}_n$-equivariant cartesian diagram

$$\begin{tikzcd} P \arrow[r] \arrow[d] & \mathrm{GL}_r \arrow[d] \\ U \arrow[r] & \mathbb{A}^r \smallsetminus 0, \end{tikzcd}$$

$P$ is affine over $\mathrm{GL}_r$, thus affine. We conclude using the equivalent $[U/\mathrm{GL}_n] \cong [P/(\mathrm{GL}_n \times H)]$.

It remains to show (1) $\Rightarrow$ (2). We first show that $\mathcal{X} \cong [U/\mathrm{GL}_n]$ with $U$ an algebraic space. Given a vector bundle $E$ on $\mathcal{X}$ of rank $n$, the frame bundle $\mathrm{Frame}(E)$ is a principal $\mathrm{GL}_n$-torsor and $\mathcal{X} \cong [\mathrm{Frame}(E)/\mathrm{GL}_n]$ (Exercise 6.1.16).

For every closed point $x \in \mathcal{X}$, let $i_x\colon \mathcal{G}_x \hookrightarrow \mathcal{X}$ be the inclusion of the residual gerbe (Proposition 3.5.16). Let $\kappa(x) \to \Bbbk$ be a finite field extension trivializing $\mathcal{G}_x$, i.e. there is a map $\widetilde{x}\colon \operatorname{Spec}\Bbbk \to \mathcal{X}$ representing $x$ inducing a finite cover $p\colon \mathbf{B}G_{\widetilde{x}} \to \mathcal{G}_x$. Since $G_{\widetilde{x}}$ is affine, we can choose a faithful representation $W$. Using the resolution property, there is a vector bundle $E$ and a surjection $E \twoheadrightarrow (i_x \circ p)_* W$. The associated frame bundle $\mathrm{Frame}(E) \to \mathcal{X}$ has trivial stabilizers over $x$. In other words, the kernel subgroup $S_E \subset I_{\mathcal{X}}$ of $E$ (i.e. the subgroup stack of the inertia stack parameterizing elements acting trivially on $E$) is trivial over $x$. If $F$ is another vector bundle, then $S_{E \oplus F} \subset S_E$ is a closed subgroup. Since $I_{\mathcal{X}}$ is noetherian, we can inductively enlarge the vector bundle $E$ so that $U := \mathrm{Frame}(E)$ is an algebraic space and $\mathcal{X} \cong [U/\mathrm{GL}_n]$.

Since $\mathcal{X}$ is normal, $U$ is also normal and we may apply Exercise 4.5.3 to conclude that $U$ is the coarse moduli space of the action of a finite group $H$ acting on a normal scheme $U'$. Let $p\colon U' \to U$ be the quotient morphism, and let $U'_1, \ldots, U'_r$ be an affine covering of $U'$ with reduced complements $Z'_1, \ldots, Z'_r$. Then $F := p_*(\bigoplus_i I_{Z'_i})$ is a coherent sheaf on $U$. Moreover, since $q\colon U \to [U/\mathrm{GL}_n]$ is affine, $q^* q_* F \twoheadrightarrow F$ is surjective, and by writing $q_* F$ as a colimit of coherent sheaves, we may find a coherent sheaf $G$ on $\mathcal{X} \cong [U/\mathrm{GL}_n]$ and a surjection $q^* G \to F$. Since $\mathcal{X}$ has the resolution property, we see that there is even a vector bundle $G$ and a surjection $q^* G \to F$. Since $p\colon U' \to U$ is affine, we have a surjection $p^* q^* G \twoheadrightarrow p^* F \twoheadrightarrow \bigoplus_i I_{Z'_i}$. Let $V = \mathrm{Frame}(G)$ and consider the cartesian diagram

$$\begin{tikzcd} U'_V \arrow[r] \arrow[d, "\beta"] & U_V \arrow[r] \arrow[d] & V \arrow[d] \\ U' \arrow[r, "p"] & U \arrow[r, "q"] & \mathcal{X} \end{tikzcd}$$

where the horizontal arrows are principal $\mathrm{GL}_n$-bundles and the vertical arrows are $\mathrm{GL}_m$-bundles where $m = \mathrm{rk}(G)$. Since the pullback of $G$ to $V$ is trivial, the pullback $\beta^*(\bigoplus_i I_{Z'_i})$ is globally generated. This implies that $\beta^{-1}(Z'_i)$ is defined by global functions on $U'_V$ and that the complement $\beta^{-1}(U'_i)$ is covered by affine opens of the form $\{f \neq 0\}$ for $f \in \Gamma(U'_V, \mathcal{O}_{U'_V})$. This implies that $\mathcal{O}_{U'_V}$ is ample and that $U'_V$ is a quasi-affine scheme. Since $\beta\colon U'_V \to U_V$ is the quotient by a finite group, $U_V$ is also quasi-affine (Exercise 4.2.8). We have thus shown that $\mathcal{X} \cong [U_V/(\mathrm{GL}_n \times \mathrm{GL}_m)]$. Under the embedding $\mathrm{GL}_n \times \mathrm{GL}_m \hookrightarrow \mathrm{GL}_{n+m}$, the quotient $\mathrm{GL}_{n+m}/(\mathrm{GL}_n \times \mathrm{GL}_m)$ is quasi-affine. Setting $W = U_V \times^{(\mathrm{GL}_n \times \mathrm{GL}_m)} \mathrm{GL}_{n+m}$, we conclude that $\mathcal{X} \cong [W/\mathrm{GL}_{n+m}]$. □

### 6.1.6 Sheaf cohomology

Abelian sheaf cohomology for algebraic stacks can be developed using essentially the same approach as we used in §4.1.6 for Deligne–Mumford stacks.

**Lemma 6.1.23.** *If $\mathcal{X}$ is an algebraic stack, the categories* $\mathrm{Ab}(\mathcal{X}_{\text{lis-ét}})$ *and* $\mathrm{Mod}(\mathcal{O}_{\mathcal{X}})$ *have enough injectives. If in addition $\mathcal{X}$ is quasi-separated, then* $\mathrm{QCoh}(\mathcal{X})$ *has enough injectives.*

*Proof.* The argument of Lemma 4.1.26 generalizes. □

**Definition 6.1.24** (Cohomology)**.** Let $\mathcal{X}$ be an algebraic stack and $F$ a sheaf of abelian groups on $\mathcal{X}_{\text{lis-ét}}$. The *cohomology group* $\mathrm{H}^i(\mathcal{X}_{\text{lis-ét}}, F)$ is defined as the $i$th right derived functor of the global sections functor $\Gamma\colon \mathrm{Ab}(\mathcal{X}_{\text{lis-ét}}) \to \mathrm{Ab}$.

Given a morphism $f\colon \mathcal{X} \to \mathcal{Y}$ of algebraic stacks, the *higher direct image* $R^i f_* F$ is defined as the $i$th right derived functor of $f_*\colon \mathrm{Ab}(\mathcal{X}_{\text{lis-ét}}) \to \mathrm{Ab}(\mathcal{Y}_{\text{lis-ét}})$.

**Definition 6.1.25** (Čech cohomology)**.** Given a smooth covering $\mathcal{U} = \{U_i \to \mathcal{X}\}_{i \in I}$ of algebraic stacks and an abelian sheaf $F$ on $\mathcal{X}_{\text{lis-ét}}$, the *Čech complex of $F$ with respect to $\mathcal{U}$* is $\check{C}^\bullet(\mathcal{U}, F)$ where

$$\check{C}^n(\mathcal{U}, F) = \prod_{(i_0,\ldots,i_n) \in I^{n+1}} F(U_{i_0} \times_{\mathcal{X}} \cdots \times_{\mathcal{X}} U_{i_n})$$

with differential

$$d^n\colon \check{C}^n(\mathcal{U}, F) \to \check{C}^{n+1}(\mathcal{U}, F), \qquad (s_{i_0,\ldots,i_n}) \mapsto \left( \sum_{k=0}^{n+1} (-1)^k p_{\widehat{k}}^* s_{i_0,\ldots,\widehat{i_k},\ldots,i_n} \right)_{(i_0,\ldots,i_{n+1})}$$

where $p_{\widehat{k}}\colon U_{i_0} \times_{\mathcal{X}} \cdots \times_{\mathcal{X}} U_{i_n} \to U_{i_0} \times_{\mathcal{X}} \cdots \times_{\mathcal{X}} \widehat{U_{i_k}} \times_{\mathcal{X}} \cdots \times_{\mathcal{X}} U_{i_n}$ is the map forgetting the $k$th component (with indexing starting at 0). The *Čech cohomology of $F$ with respect to $\mathcal{U}$* is

$$\check{\mathrm{H}}^i(\mathcal{U}, F) := \mathrm{H}^i(\check{C}^\bullet(\mathcal{U}, F)).$$

The arguments of Theorem 4.1.29 and Propositions 4.1.33, 4.1.35 and 4.1.36 as well as Exercise 4.1.38 extend.

**Theorem 6.1.26.** *For a quasi-coherent $\mathcal{O}_{X_{\text{lis-ét}}}$-module $F$ on an affine scheme $X$,* $\mathrm{H}^i(X_{\text{lis-ét}}, F) = 0$ *for all $i > 0$.* □

**Proposition 6.1.27.** *Let $\mathcal{X}$ be an algebraic stack with affine diagonal and $F$ be a quasi-coherent sheaf. If $\mathcal{U} = \{U_i \to \mathcal{X}\}$ is an étale covering with each $U_i$ affine, then* $\mathrm{H}^i(\mathcal{X}_{\text{lis-ét}}, F) = \check{\mathrm{H}}^i(\mathcal{U}, F)$. □

**Proposition 6.1.28.** *If $X$ is a scheme with affine diagonal and $F$ be a quasi-coherent sheaf, then $\mathrm{H}^i(X, F) = \mathrm{H}^i(X_{\text{lis-ét}}, F_{\text{lis-ét}})$ for all $i$, where $F_{\text{lis-ét}}$ is the sheaf of $\mathcal{O}_{X_{\text{lis-ét}}}$-module defined by $F_{\text{ét}}(U) = \Gamma(U, f^*F)$ for a smooth map $f\colon U \to \mathcal{X}$ from a scheme.*

*Similarly, if $\mathcal{X}$ is a Deligne–Mumford stack with affine diagonal and $F$ is a quasi-coherent sheaf, then $\mathrm{H}^i(\mathcal{X}, F) = \mathrm{H}^i(\mathcal{X}_{\text{lis-ét}}, F_{\text{lis-ét}})$ for all $i$.* □

**Proposition 6.1.29.** *Let $\mathcal{X}$ be an algebraic stack.*

*(1) If $F$ is an $\mathcal{O}_{\mathcal{X}}$-module, then the cohomology $\mathrm{H}^i(\mathcal{X}_{\text{lis-ét}}, F)$ of $F$ as an abelian sheaf agrees with the $i$th right derived functor of $\Gamma\colon \mathrm{Mod}(\mathcal{O}_{\mathcal{X}}) \to \mathrm{Ab}$.*

*(2) If $\mathcal{X}$ has affine diagonal and $F$ is a quasi-coherent sheaf on $\mathcal{X}$, then the cohomology $\mathrm{H}^i(\mathcal{X}_{\text{lis-ét}}, F)$ of $F$ as an abelian sheaf agrees with the $i$th right derived functor of $\Gamma\colon \mathrm{QCoh}(\mathcal{X}) \to \mathrm{Ab}$.*

*For a morphism $f\colon \mathcal{X} \to \mathcal{Y}$ of algebraic stacks (resp. quasi-compact morphism of algebraic stacks with affine diagonals), then (1) (resp. (2)) holds also for the higher direct images $R^if_*F$ of an $\mathcal{O}_{\mathcal{X}}$-module (resp. quasi-coherent sheaf): it can be computed as the $i$th right derived functor of $f_*\colon \mathrm{Mod}(\mathcal{O}_{\mathcal{X}}) \to \mathrm{Mod}(\mathcal{O}_{\mathcal{Y}})$ (resp. $f_*\colon \mathrm{QCoh}(\mathcal{X}) \to \mathrm{QCoh}(\mathcal{Y})$).* □

**Remark 6.1.30.** If $\mathcal{X}$ does not have affine diagonal, then the sheaf cohomology $\mathrm{H}^i(\mathcal{X}_{\text{lis-ét}}, F)$ of a quasi-coherent sheaf may differ from the $i$th right derived functor of $\Gamma(\mathcal{X}, -)\colon \mathrm{QCoh}(\mathcal{X}) \to \mathrm{Ab}$.

**Proposition 6.1.31.** *If $\mathcal{X}$ is an algebraic stack and $F_i$ is a directed system of abelian sheaves in $\mathcal{X}_{\text{lis-ét}}$, then $\mathrm{colim}_i\, \mathrm{H}^i(\mathcal{X}, F_i) \to \mathrm{H}^i(\mathcal{X}, \mathrm{colim}_i\, F_i)$ is an isomorphism.* □

### 6.1.7 Chow groups

Following [Tot99] and [EG98], we introduce the Chow groups of a quotient stack. Let $G$ be a smooth affine algebraic group over an algebraically closed field $\Bbbk$ of dimension $g$, and let $X$ be an $n$-dimensional scheme of finite type over $\Bbbk$. For each $i$, choose an $r$-dimensional $G$-representation $V$ such that there is a nonempty open subscheme $U \subset \mathbb{A}(V)$ such that (a) $G$ acts freely on $U$, (b) the quotient $U/G$ is a scheme, and (c) $\operatorname{codim} \mathbb{A}(V) \setminus U > n - i - g$. Such representations exist. We define the $(i-g)$*th equivariant Chow group of* $X$ or equivariantly the $i$*th Chow group of* $[X/G]$ as

$$\mathrm{CH}^G_{i-g}(X) = \mathrm{CH}_i([X/G]) := \mathrm{CH}_{i+r}(X \times^G U).$$

This definition is independent of the choice of representation. The definition is forced upon us if we desire invariance of Chow groups under vector bundles and open immersions of high codimension:

$$\begin{tikzcd} X \times^G U \arrow[r, hook, "\text{open}"] & {[(X \times \mathbb{A}(V))/G]} \arrow[d, "\text{vect bdl}"] \\ & {[X/G]} \end{tikzcd} \qquad \begin{tikzcd} \mathrm{CH}_{i+r}(X \times^G U) & \mathrm{CH}_{i+r}([(X \times \mathbb{A}(V))/G]) \arrow[l, "\sim"'] \\ & \mathrm{CH}_i([X/G]) \arrow[u, "\wr"'] \end{tikzcd}$$

If $[X/G]$ is smooth of pure dimension $d = n - g$, then we define

$$\mathrm{CH}^i_G(X) = \mathrm{CH}^i([X/G]) := \mathrm{CH}_{d-i}([X/G])$$
$$\mathrm{CH}^*_G(X) = \mathrm{CH}^*([X/G]) := \bigoplus_i \mathrm{CH}^i((X/G]).$$

The intersection product gives a ring structure, and we call $\mathrm{CH}^*_G(X)$ the *equivariant Chow ring of $X$* and $\mathrm{CH}^*([X/G])$ the *Chow ring of $[X/G]$*.

**Example 6.1.32** ($\mathrm{CH}^*(\mathbf{B}\mathbb{G}_m)$)**.** Let $V$ be the $r$-dimensional $\mathbb{G}_m$-representation with equal weights 1. Then $\mathbb{G}_m$ acts freely on $\mathbb{A}^n \setminus 0$, and for $-1 \geq i > -r-1$, we have that $\mathrm{CH}_i(\mathbf{B}\mathbb{G}_m) = \mathrm{CH}_{i+r}(\mathbb{P}^{r-1}) = \mathbb{Z}$. It follows that $\mathrm{CH}_i(\mathbf{B}\mathbb{G}_m) = \mathbb{Z}$ for $i \leq -1$ and is 0 otherwise. Therefore $\mathrm{CH}^j(\mathbf{B}\mathbb{G}_m) = \mathbb{Z}$ for $j \geq 0$, and $\mathrm{CH}^*(\mathbf{B}\mathbb{G}_m) = \mathbb{Z}[x]$. More generally, if $T \cong \mathbb{G}_m^r$ is a rank $r$ torus, then $\mathrm{CH}^*(\mathbf{B}T)$ is isomorphic to the character ring $\mathbb{Z}[x_1, \ldots, x_r]$ of $T$.

We summarize some of the important properties of equivariant Chow groups.

**Properties 6.1.33.**

(1) *(Independent of quotient presentation) If $[X/G] \cong [X'/G']$, then $\mathrm{CH}_i^G(X) \cong \mathrm{CH}_i^{G'}(X')$, and in particular the definition of $\mathrm{CH}_i([X/G])$ is independent of the quotient presentation. The definition of Chow groups can be extended to finite type algebraic stacks over $\Bbbk$; see [Kre99].*

(2) *(Vector bundle invariance) If $Y \to X$ is $G$-equivariant and a Zariski-local affine fibration of relative dimension $r$ (e.g. the total space of a rank $r$ vector bundle), then $\mathrm{CH}_*^G(X) \cong \mathrm{CH}_{*+r}^G(Y)$.*

(3) *(Excision sequence) If $\mathcal{Z} \subset \mathcal{X} = [X/G]$ is a closed substack with complement $\mathcal{U}$, then there is a right exact sequence*
$$\mathrm{CH}_*(\mathcal{Z}) \to \mathrm{CH}_*(\mathcal{X}) \to \mathrm{CH}_*(\mathcal{U}) \to 0.$$

(4) *(Comparison with coarse moduli space) If $\mathcal{X} \cong [U/G]$ is a separated Deligne–Mumford stack with coarse moduli space $X$, then $\mathrm{CH}_*(\mathcal{X}) \otimes \mathbb{Q} \cong \mathrm{CH}_*(X) \otimes \mathbb{Q}$.*

(5) *(Functoriality and self-intersection) Flat morphisms induce pullback maps on Chow groups while proper morphisms induce pushforward maps. If $\mathcal{X} = [X/G]$ is smooth and $i\colon \mathcal{Z} \hookrightarrow \mathcal{X}$ is a smooth substack of pure codimension $d$, then there is pullback $i^*\colon \mathrm{CH}^*(\mathcal{X}) \to \mathrm{CH}^*(\mathcal{Z})$ given by intersection with $\mathcal{Z}$ such that $i^* i_* \alpha = c_d(\mathcal{N}_{\mathcal{Z}/\mathcal{X}}) \cap \alpha$ for $\alpha \in \mathrm{CH}^*(\mathcal{Z})$, where $c_d$ is the top Chern class of the normal bundle.*

(6) *Let $T$ be a torus acting on a smooth scheme $X$ such that $T = T_1 \times T_2$ is a product of two tori with $T_2$ acting trivially. Then $\mathrm{CH}^*_T(X) \cong \mathrm{CH}^*_{T_1}(X) \otimes \mathrm{CH}^*(\mathbf{B}T_2)$.*

(7) *If $G$ is a connected reductive group with maximal torus $T$, and $X$ is a smooth scheme with a $G$-action, then the Weyl group $W = N_G(T)/T$ acts on $\mathrm{CH}^*_T(X)$ and $\mathrm{CH}^*_G(X)_{\mathbb{Q}} = \mathrm{CH}^*_T(X)^W_{\mathbb{Q}}$.*

**Exercise 6.1.34.**

(a) Let $\mathcal{P}(d_0, \ldots, d_n)$ be the weighted projective stack of Example 3.9.6. Show that $\mathrm{CH}^*(\mathcal{P}(d_0, \ldots, d_n)) \cong \mathbb{Z}[x]/(d_1 \cdots d_n x^{n+1})$.

(b) If $\mathrm{char}(\Bbbk) \neq 2, 3$, show that $\mathrm{CH}^*(\mathcal{M}_{1,1}) \cong \mathbb{Z}[x]/(12x)$ and $\mathrm{CH}^*(\overline{\mathcal{M}}_{1,1}) \cong \mathbb{Z}[x]/(24x^2)$. (Compare with Exercise 6.1.14).

(c) Let $\mathbb{G}_m$ act on $\mathbb{P}^n$ with weights $d_0, \ldots, d_n$. Show that $A^*([\mathbb{P}^n/\mathbb{G}_m]) = \mathbb{Z}[h,t]/p(h,t)$ where $p(h,t) = \sum_{i=0}^n h^i e_i(a_0 t, \ldots, a_n t)$ and $e_i$ is the $i$th symmetric polynomial.

### 6.1.8 de Rham and singular cohomology

We quickly discuss the de Rham and singular cohomology of an algebraic stack following [Beh04].

**Analyticification.** If $\mathcal{X}$ is a smooth algebraic stack over $\mathbb{C}$ with affine diagonal, there is an analyticification $\mathcal{X}^{\mathrm{an}}$, analogous to the analyticification of a finite type $\mathbb{C}$-scheme, such that $\mathcal{X}^{\mathrm{an}}$ is a *differentiable stack.* If $U_0 \to \mathcal{X}$ is a smooth presentation by a scheme so that $\mathcal{X}$ is the quotient of the smooth groupoid $U_1 := U_0 \times_{\mathcal{X}} U_0 \rightrightarrows U_0$, then $U_0^{\mathrm{an}} \to \mathcal{X}^{\mathrm{an}}$ is a smooth presentation and $\mathcal{X}^{\mathrm{an}}$ is the quotient of the *Lie groupoid* $U_1^{\mathrm{an}} \rightrightarrows U_0^{\mathrm{an}}$.

**De Rham cohomology of a differential stack.** Given a differentiable stack $\mathcal{X}$ with a smooth presentation $U_0 \to \mathcal{X}$, we can define a *simplicial manifold* $U_\bullet$

$$\cdots U_3 \overset{\textstyle\rightarrow}{\underset{\textstyle\rightarrow}{\rightrightarrows}} U_2 \overset{\textstyle\rightarrow}{\rightrightarrows} U_1 \longrightarrow U_0, \quad \text{where } U_p := \underbrace{U_0 \times_{\mathcal{X}} \cdots \times_{\mathcal{X}} U_0}_{p \text{ times}} \tag{6.1.1}$$

with maps $\partial_i \colon U_p \to U_{p-1}$ forgetting the $i$th term along with degeneracy maps $s_i \colon U_{p-1} \to U_p$ inserting an identity morphism in the $i$th term. This defines a double complex $\Omega^q(U_p)$ with differentials given by exterior differentiation $d \colon \Omega^{q-1}(U_p) \to \Omega^q(U_p)$ and $\partial := \sum_{i=0}^p (-1)^i \partial_i^* \colon \Omega^q(U_{p-1}) \to \Omega^q(U_p)$. We define the *de Rham complex* $C^\bullet_{\mathrm{dR}}(\mathcal{X})$ as the total complex

$$C^k_{\mathrm{dR}}(\mathcal{X}) := \bigoplus_{p+q=k} \Omega^q(U_p),$$

with differential $\delta \colon C^k_{\mathrm{dR}}(\mathcal{X}) \to C^{k+1}_{\mathrm{dR}}(\mathcal{X})$ defined by $\delta(\omega) = \partial(\omega) + (-1)^p d(\omega)$ for $\omega \in \Omega^p(U_q)$. The *de Rham cohomology* is

$$\mathrm{H}^n_{\mathrm{dR}}(\mathcal{X}) := \mathrm{H}^n(C^\bullet_{\mathrm{dR}}(\mathcal{X})),$$

and is independent of the choice of presentation. As with the case of smooth manifolds, there is an identification of $\mathrm{H}^n_{\mathrm{dR}}(\mathcal{X})$ with the sheaf cohomology of the constant sheaf $\mathbb{R}$ on the big smooth site of smooth manifolds over $\mathcal{X}$.

**Singular homology/cohomology of a topological stack.** For a *topological stack* $\mathcal{X}$, one can replicate the constructions of singular homology and cohomology. Let $U_0 \to \mathcal{X}$ be a presentation and $U_\bullet$ be the simplicial topological space as in (6.1.1). For each $p$, we have the singular chain complex $C_\bullet(U_p)$ with differentials $d : C_q(U_p) \to C_{q-1}(U_p)$. This defines a double complex $C_q(U_p)$ using the differential $\partial = \sum_{i=0}^p (-1)^i \partial_j \colon C_q(U_p) \to C_q(U_{p-1})$ induced by the maps $\partial_i \colon U_p \to U_{p-1}$. We define the *singular chain complex* $C_\bullet(\mathcal{X})$ *of* $\mathcal{X}$ as the total complex

$$C_k(\mathcal{X}) := \bigoplus_{p+q=k} C_q(U_p)$$

with the differential $\delta \colon C_k(\mathcal{X}) \to C_{k-1}(\mathcal{X})$ given by $\delta(\gamma) = (-1)^{p+q}\partial(\gamma) + (-1)^q d(\gamma)$ for $\gamma \in C_q(U_p)$. For an abelian group $A$, we can therefore define the *singular homology groups of* $\mathcal{X}$ *with coefficients in* $A$ as

$$\mathrm{H}_n(\mathcal{X}, A) := \mathrm{H}_n(C_\bullet(\mathcal{X}) \otimes_{\mathbb{Z}} A).$$

Dualizing, we define the *singular cochain complex* $C^\bullet(\mathcal{X})$ by $C^n(\mathcal{X}) := \mathrm{Hom}(C_n(\mathcal{X}), \mathbb{Z})$ and the *singular cohomology groups of* $\mathcal{X}$ *with coefficients in* $A$ as

$$\mathrm{H}^n(\mathcal{X}, A) := \mathrm{H}^n(C^\bullet(\mathcal{X}) \otimes_{\mathbb{Z}} A).$$

**Comparisons.**

- There are pairings $\mathrm{H}_k(\mathcal{X},\mathbb{Z}) \otimes \mathrm{H}^k(\mathcal{X},\mathbb{Z}) \to \mathbb{Z}$ which after tensoring with $\mathbb{Q}$ gives identifications $\mathrm{H}^k(\mathcal{X},\mathbb{Q}) \cong \mathrm{H}_k(\mathcal{X},\mathbb{Q})^\vee$.
- If $G$ is a topological group acting on a space $U$, then the *equivariant cohomology* is defined as $\mathrm{H}^*_G(U,A) := \mathrm{H}^*(EG \times^G U, A)$, where $EG$ is a contractible space with a free action of $G$, and there is an identification $\mathrm{H}^*([U/G],A) = \mathrm{H}^*_G(U,A)$.
- For a differential stack $\mathcal{X}$, there is an identification $\mathrm{H}^*_{\mathrm{dR}}(\mathcal{X}) = \mathrm{H}^*(\mathcal{X},\mathbb{R})$.
- If $\mathcal{X}$ is a *topological Deligne–Mumford stack* (e.g. the topological stack associated to a separated Deligne–Mumford stack of finite type over $\mathbb{C}$) with coarse moduli space $\mathcal{X} \to X$, then $\mathrm{H}^*(\mathcal{X},\mathbb{Q}) = \mathrm{H}^*(X,\mathbb{Q})$.

## 6.2 The fppf topology and gerbes

This section is not essential for the proofs of the two main theorems of this book and is included for completeness. We prove that algebraic spaces/stacks are sheaves/stacks in the fppf topology and that quotients by fppf groupoids/equivalence relations are algebraic. One upshot is that $\mathbf{B}G$ is an algebraic stack for any (non-necessarily smooth) algebraic group, e.g. $\boldsymbol{\mu}_p$ in characteristic $p$.

We also introduce gerbes, a central topic in the theory of stacks. For us, we want to know that residual gerbes are gerbes (justifying the terminology) and later that the moduli stack $\mathrm{Bun}^{\mathrm{s}}(C)_{r,d}$ of stable vector bundles is a $\mathbb{G}_m$-gerbe over its coarse moduli space.

### 6.2.1 Fppf criterion for algebraicity

**Theorem 6.2.1** (Fppf Criterion for Algebraicity)**.**

*(1) If $X$ is a sheaf on $\mathrm{Sch}_{\mathrm{fppf}}$ such that there exists an fppf representable morphism $U \to X$ from a scheme, then $X$ is an algebraic space.*

*(2) If $\mathcal{X}$ is a stack over $\mathrm{Sch}_{\mathrm{fppf}}$ such that there exists an fppf representable morphism $U \to \mathcal{X}$ from a scheme, then $\mathcal{X}$ is an algebraic stack.*

*Proof.* To add. □

Algebraic spaces are by definition sheaves in the big étale topology but it turns out they are also sheaves in the big fppf topology.

**Proposition 6.2.2.**

*(1) An algebraic space $X$ over a scheme $S$ is a sheaf on $(\mathrm{Sch}/S)_{\mathrm{fppf}}$.*

*(2) An algebraic stack $\mathcal{X}$ over a scheme $S$ is a stack over $(\mathrm{Sch}/S)_{\mathrm{fppf}}$.*

*Proof.* To add.

□

This allows us to finally prove that many properties of representable morphisms of algebraic stacks descend in the fppf topology. Smooth descent was established in Proposition 3.3.3

**Proposition 6.2.3.** *Let $\mathcal{P}$ be one of the following properties of morphisms of algebraic stacks: representable, isomorphism, open immersion, closed immersion,* *locally closed immersion, affine, or quasi-affine. Consider a cartesian diagram*

$$\begin{tikzcd} \mathcal{X}' \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{Y}' \arrow[d] \\ \mathcal{X} \arrow[r] & \mathcal{Y} \end{tikzcd}$$

*of algebraic stacks where $\mathcal{Y}' \to \mathcal{Y}$ is fppf. Then $\mathcal{X} \to \mathcal{Y}$ has $\mathcal{P}$ if and only if $\mathcal{X}' \to \mathcal{Y}'$ has $\mathcal{P}$.*

### 6.2.2 Fppf groupoids and quotient stacks

If $R \rightrightarrows U$ is an fppf equivalence relation of algebraic spaces, we define $U/R$ as the sheafification in big fppf topology $\mathrm{Sch}_{\mathrm{fppf}}$ of the presheaf $T \mapsto U(T)/R(T)$. Likewise, if $s, t\colon R \rightrightarrows U$ is an fppf groupoid of algebraic spaces, we define $[U/R]$ as the stackification in $\mathrm{Sch}_{\mathrm{fppf}}$ of the prestack $[U/R]^{\mathrm{pre}}$, whose fiber category over a scheme $T$ is the category of $T$-points of $U$ where a morphism from $a \in U(T)$ to $b \in U(T)$ is an element $r \in R(R)$ such that $s(r) = a$ and $t(r) = b$.

The definitions of $U/R$ and $[U/R]$ are consistent with the quotient of a smooth equivalence relation or groupoid as defined in Definition 3.4.7 using in the big étale topology $\mathrm{Sch}_{\text{ét}}$. This is because the sheafification $U/R$ in $\mathrm{Sch}_{\text{ét}}$ is an an algebraic space by Corollary 4.4.12 and thus a sheaf in the fppf topology by Proposition 6.2.2. Similarly, the stackification $[U/R]$ over $\mathrm{Sch}_{\text{ét}}$ is an algebraic stack by Theorem 3.4.11 and thus a stack in the fppf topology by Proposition 6.2.2.

**Corollary 6.2.4.**

*(1) If $R \rightrightarrows U$ is an fppf equivalence relation of algebraic spaces, then the quotient $U/R$ is an algebraic space.*

*(2) If $R \rightrightarrows U$ is an fppf groupoid of algebraic spaces, then the quotient $[U/R]$ is an algebraic stack.*

*Proof.* To add. □

We will now show that a quotient stack arising from the action of an fppf group algebraic space is an algebraic stack; this was shown for the action of a smooth affine group scheme in Corollary 3.1.11. We first need to generalize the definition of a principal $G$-bundle given in Definition C.2.1 for an action by an fppf group algebraic space.

**Definition 6.2.5** (Principal $G$-bundles)**.** If $G \to S$ is an fppf group algebraic space, then a *principal $G$-bundle over an $S$-scheme $T$* is an algebraic space $P$ with an action of $G$ via $\sigma\colon G \times_S P \to P$ such that $P \to X$ is a $G$-invariant fppf morphism and $(\sigma, p_2)\colon G \times_S P \to P \times_T P$ is an isomorphism. *Morphisms of principal $G$-bundles* are $G$-equivariant morphisms of schemes. We say that a principal $G$-bundle $P \to T$ is *trivial* if there is a $G$-equivariant isomorphism $P \cong G \times T$.

When $G \to S$ is smooth, then every principal $G$-bundle $P \to T$ is trivialized by the smooth cover $P \to T$ and since smooth morphisms étale locally have sections, there is an étale cover $T' \to T$ such that $P_{T'}$ is trivial.

**Remark 6.2.6.** It is important to require that $P$ is an algebraic space and not a scheme since we want principal $G$-bundles to satisfy descent and to be equivalent to the notion of a $G$-torsor (Definition 6.2.12). If $G \to S$ is affine, then $P$ is automatically a scheme and the above definition thus agrees with Definition C.2.1. Indeed, $P$ is a sheaf in the fppf topology (Proposition 6.2.2) and if $U \to P$ is an étale presentation, then $P \to T$ pulls back under the fppf composition $U \to P \to T$ to the affine morphism $G \times_S U \to U$. By Effective Descent (Proposition 2.2.11), $P \to T$ is affine and in particular that $P$ is a scheme.

Raynaud provides an example of an abelian variety $G$ and a principal $G$-bundle that is not scheme [Ray70, XIII 3.2].

**Definition 6.2.7** (Quotient stacks)**.** Let $G \to S$ be an fppf group algebraic space acting on an algebraic space $U$ over $S$. We define the *quotient stack* $[U/G]$ as the category over $\mathrm{Sch}/S$ whose objects over an $S$-scheme $T$ are diagrams

$$\begin{tikzcd} P \arrow[r] \arrow[d] & U \\ T & \end{tikzcd} \tag{6.2.1}$$

where $P \to T$ is a principal $G$-bundle and $P \to U$ is a $G$-equivariant morphism of schemes. A morphism $(P' \to T', P' \to U) \to (P \to T, P \to U)$ consists of a morphism $T' \to T$ and a $G$-equivariant morphism $P' \to P$ of schemes such that the diagram

$$\begin{tikzcd} P' \arrow[r] \arrow[d] \arrow[rr, bend left] \arrow[dr, phantom, "\square"] & P \arrow[r] \arrow[d] & U \\ T' \arrow[r] & T & \end{tikzcd}$$

is commutative and the left square is cartesian.

**Definition 6.2.8** (Classifying stacks)**.** Let $G \to S$ be an fppf group algebraic space. The *classifying stack* $\mathbf{B}G$ of $G$ is defined as the quotient stack $[S/G]$. It classifies principal $G$-bundles $P \to T$.

**Proposition 6.2.9.** *If $G \to S$ is an fppf group algebraic space acting on an algebraic space $U$ over $S$, then the quotient stack $[U/G]$ is an algebraic stack. In particular, the classifying stack $\mathbf{B}G$ is algebraic.*

*Proof.* Given a map $T \to [U/G]$ corresponding to an object (6.2.1), there is a cartesian diagram

$$\begin{tikzcd} P \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & U \arrow[d] \\ T \arrow[r] & {[U/G]} \end{tikzcd}$$

of stacks over $\mathrm{Sch}_{\mathrm{fppf}}$; this extends Exercise 2.3.28. As $P \to T$ is an fppf morphism of algebraic spaces, $U \to [U/G]$ is an fppf representable morphism. It follows from Theorem 6.2.1 that $[U/G]$ is an algebraic stack. □

**Exercise 6.2.10.** Let $G \to S$ be an fppf group algebraic space acting on an algebraic space $U$ over $S$.

(a) Generalize Exercise 2.4.16 by showing that the stackification of the prestack $[U/G]^{\mathrm{pre}}$ in the fppf topology is $[U/G]$.

(b) Provide an example where the stackification of $[U/G]^{\mathrm{pre}}$ in the étale topology is not isomorphic to $[U/G]$.

Recalling that $\boldsymbol{\mu}_{n,\mathbb{Z}}$ is the subgroup of $\mathbb{G}_{m,\mathbb{Z}}$ defined by $\operatorname{Spec} \mathbb{Z}[x]/(x^n - 1)$, we can now deduce that $\mathbf{B}\boldsymbol{\mu}_{n,\mathbb{Z}}$ is an algebraic stack. If $\Bbbk$ is a field of characteristic $p$, then $\boldsymbol{\mu}_n := \boldsymbol{\mu}_{n,\Bbbk}$ is smooth if and only if $p$ doesn't divide $n$.

**Exercise 6.2.11.** Let $\Bbbk$ be a field.

(a) Exhibit an explicit smooth presentation of $\mathbf{B}\boldsymbol{\mu}_n$.

(b) Show that $\mathbf{B}\boldsymbol{\mu}_n$ is equivalent to the stack over $(\mathrm{Sch}/\Bbbk)_{\text{ét}}$ whose objects over a scheme $T$ are pairs $(L, \alpha)$ consisting of a line bundle $L$ on $T$ and a trivialization $\alpha\colon \mathcal{O}_T \xrightarrow{\sim} L^{\otimes n}$.

(c) Show that $\mathbf{B}\boldsymbol{\mu}_n$ is a smooth and proper algebraic stack of dimension 0.

(d) Show that $\mathbf{B}\boldsymbol{\mu}_n$ is a Deligne–Mumford stack if and only if $n$ is prime to the characteristic.

(e) If $x\colon \operatorname{Spec} \Bbbk \to \mathbf{B}\boldsymbol{\mu}_n$ denotes the canonical presentation, compute the tangent space $T_{\mathbf{B}\boldsymbol{\mu}_n, x}$.

### 6.2.3 Torsors

If $G$ is a sheaf of groups, then a $G$-torsor is a sheaf of sets locally isomorphic to $G$.

**Definition 6.2.12** (Torsors)**.** Let $\mathcal{S}$ be a site and $G$ a sheaf of (not necessarily abelian) groups on $\mathcal{S}$. A *$G$-torsor on $\mathcal{S}$* is a sheaf $P$ of sets on $\mathcal{S}$ with a left action $\sigma\colon G \times P \to P$ of $G$ such that

(1) for every object $T \in \mathcal{S}$, there exists a covering $\{T_i \to T\}$ such that $P(T_i) \neq 0$ for each $i$, and

(2) the action map $(\sigma, p_2)\colon G \times P \to P \times P$ is an isomorphism.

If $T \in \mathcal{S}$ is an object and $G$ is a sheaf of groups on the localized site $\mathcal{S}/T$ (Example 2.1.9), then a *$G$-torsor over $T$* is by definition a $G$-torsor on the $\mathcal{S}/T$.

*Morphisms of $G$-torsors* are $G$-equivariant morphisms of sheaves. We say that a $G$-torsor $P$ is *trivial* if $P$ is $G$-equivalently isomorphic to $G$.

**Exercise 6.2.13.** Show that Any morphism of $G$-torsors is an isomorphism.

**Example 6.2.14.** Let $\mathcal{X}$ be a stack over a site $\mathcal{S}$, and let $a, b \in \mathcal{X}$ be objects over $S \in \mathcal{S}$. The sheaf $\underline{\mathrm{Isom}}_S(a, b)$ of isomorphisms is a torsor for $\underline{\mathrm{Aut}}(a)$ under the action given by precomposition.

Given a morphism $f\colon T' \to T$ and a $G$-torsor $P$ over $T$, the restriction $P|_{T'}$ is the sheaf on $\mathcal{S}/T'$ whose whose sections over a $T'$-scheme $S$ are $P(S)$; the restriction $P|_{T'}$ is naturally a $G$-torsor over $T'$.

**Exercise 6.2.15.** Let $\mathcal{S}$ be a site with a final object $S$ and $G$ be a sheaf of groups on $\mathcal{S}$.

(a) Show that Axiom (1) is equivalent to $P \to S$ being an epimorphism of sheaves.

(b) If $P$ is a $G$-torsor, show that $S$ is isomorphic to the quotient sheaf $P/G$.

(c) Show that a $G$-torsor $P$ is trivial if and only if there exists a section $s\colon S \to P$ of the structure morphism $P \to S$.

(d) Show that a sheaf $P$ of sets on $\mathcal{S}$ with a left action by $G$ is a $G$-torsor if and only if there exists a covering $\{S_i \to S\}$ and isomorphisms $P|_{S_i} \cong G|_{S_i}$ of $G|_{S_i}$-torsors.

**Example 6.2.16** (Principal $G$-bundles)**.** If $G \to S$ is an fppf group scheme, then there is an equivalence of categories between $G$-torsors in the fppf topology and principal $G$-bundles (as defined in Definition 6.2.5). To see this, first suppose that $P \to T$ is a principal $G$-bundle over an $S$-scheme $T$, i.e. $P \to T$ is an fppf morphism of algebraic spaces where $G$ is equipped with a free and transitive action of $G \times_S T$. Since algebraic spaces are sheaves in the fppf topology (Proposition 6.2.2), we may view $G \times_S T$ as a sheaf of groups on $(\mathrm{Sch}/T)_{\mathrm{fppf}}$ and $P$ as a sheaf of sets on $(\mathrm{Sch}/T)_{\text{ét}}$. Since every principal $G$-bundle is locally trivial in the fppf topology (Proposition C.2.4), $P$ is a $G \times_S T$-torsor on $(\mathrm{Sch}/T)_{\mathrm{fppf}}$. Conversely, given a $G \times_S T$-torsor $P$ on $(\mathrm{Sch}/T)_{\mathrm{fppf}}$, then by Exercise 6.2.15 there is an fppf cover $T' \to T$ such that $P \times_T T' \cong G \times_T T'$. Therefore, $P \times_T T' \to P$ is an fppf morphism from an algebraic space, and Corollary 6.2.4 implies that $P$ is an algebraic space. It follows that $P \to T$ is a principal $G$-bundle.

If in addition $G \to S$ is smooth, then there is an equivalence of categories between $G$-torsors in the étale topology and principal $G$-bundles. This holds because every principal $G$-bundle $P \to T$ is étale locally trivial and therefore $P$ is a $G \times_S T$-torsor on $(\mathrm{Sch}/T)_{\text{ét}}$.

### 6.2.4 Gerbes

Gerbes are a 2-categorical generalization of torsors. While torsors are locally isomorphic to a sheaf of groups $G$, gerbes are locally isomorphic to classifying stacks $\mathbf{B}G$. Gerbes are central figures in moduli theory; for the purposes of this book, our main examples are residual gerbes (Proposition 6.2.34) and banded $\mathbb{G}_m$-gerbes such as the map $\underline{\mathcal{P}ic}_X \to \underline{\mathrm{Pic}}_X$ from the Picard stack to Picard scheme Theorem 6.2.55) and the map $\mathrm{Bun}^{\mathrm{s}}_{r,d}(C) \to M^{\mathrm{s}}_{r,d}(C)$ from the stack of stable vector bundles to its coarse moduli space.

**Definition 6.2.17** (Gerbes)**.** A stack $\mathcal{X}$ over a site $\mathcal{S}$ is called a *gerbe* if

(1) for every object $T \in \mathcal{S}$, there exists a covering $\{T_i \to T\}$ in $\mathcal{S}$ such that each fiber category $\mathcal{X}(T_i)$ is non-empty; and

(2) for objects $x, y \in \mathcal{X}$ over $T \in \mathcal{S}$, there exists a covering $\{T_i \to T\}$ and isomorphisms $x|_{T_i} \xrightarrow{\sim} y|_{T_i}$ for each $i$.

We say that a gerbe $\mathcal{X}$ is *trivial* if there is a section $\mathcal{S} \to \mathcal{X}$ of $\mathcal{X} \to \mathcal{S}$. When $\mathcal{S}$ has a final object $S$, then the triviality of a gerbe $\mathcal{X}$ is equivalent to the existence of an element of $\mathcal{X}(S)$.

**Example 6.2.18.** If $G$ is a sheaf of groups on a site $\mathcal{S}$, then we extend Definition 6.2.8 by defining the *classifying prestack of* $G$ as the category $\mathbf{B}G$ over $\mathcal{S}$ consisting of pairs $(P, T)$ where $T \in \mathcal{S}$ and $P$ is $G$-torsor over $\mathcal{S}/T$ (Definition 6.2.12). A morphism $(P', T') \to (P, T)$ is the data of a morphism $T' \to T$ in $\mathcal{S}$ and an isomorphism $P' \to P|_{T'}$ of $G$-torsors, where $P|_{T'}$ denotes the restriction of $P$ along $T' \to T$.

The classifying stack $\mathbf{B}G$ is a gerbe over $\mathcal{S}$ because every $G$-torsor over $T$ is locally isomorphic to the trivial $G$-torsor $G \times T$.

**Exercise 6.2.19** (Gerbes are locally classifying stacks)**.** Let $\mathcal{S}$ be a site with a final object $S \in \mathcal{S}$, and let $\mathcal{X}$ be a stack over $\mathcal{S}$. Show that $\mathcal{X}$ is a gerbe if and only if there exists a covering $\{S_i \to S\}$ and sheaves of groups $G_i$ on the localized site $\mathcal{S}/S_i$ (Example 2.1.9) such that there is an isomorphism $\mathcal{X} \times_S \mathcal{S}/S_i \cong \mathbf{B}G_i$ over $\mathcal{S}/S_i$.

**Exercise 6.2.20.** Let $S$ be a scheme and let $\mathcal{X}$ be a gerbe over $(\mathrm{Sch}/S)_{\mathrm{fppf}}$. If the diagonal $\mathcal{X} \to \mathcal{X} \times_S \mathcal{X}$ is representable, show that $\mathcal{X}$ is an algebraic stack.

Important examples of gerbes $\mathcal{X}$ are those *banded* by a sheaf of groups. This means that $\mathcal{X}$ is equipped with the additional data of a natural isomorphism $G(T) \to \mathrm{Aut}_T(x)$ for every object $x \in \mathcal{X}(t)$.

**Definition 6.2.21** (Banded $G$-gerbes)**.** Let $G$ be an *abelian* sheaf on a site $\mathcal{S}$. A stack $\mathcal{X}$ over $\mathcal{S}$ is a *gerbe banded by* $G$ (or a *banded* $G$*-gerbe* or simply a $G$*-gerbe*) is a gerbe together with the data of isomorphisms $\psi_x \colon G|_T \to \underline{\mathrm{Aut}}_T(x)$ of sheaves for each object $x \in \mathcal{X}(T)$. We require that for each isomorphism $\alpha \colon x \xrightarrow{\sim} y$ over $T$, the diagram

$$\begin{tikzcd} & G|_T \arrow[dl, "\psi_x"'] \arrow[dr, "\psi_y"] & \\ \underline{\mathrm{Aut}}_T(x) \arrow[rr, "\mathrm{Inn}_\alpha"] & & \underline{\mathrm{Aut}}_T(y). \end{tikzcd} \tag{6.2.2}$$

commutes, where $\mathrm{Inn}_\alpha(\tau) = \alpha\tau\alpha^{-1}$. The data of the isomorphisms $\psi_x$ is called the *band* of $\mathcal{X}$.

A *morphism of banded* $G$*-gerbes* is a morphism of stacks compatible with the bands.

**Remark 6.2.22.** Here's another way to think about a band of a gerbe. Let $\mathcal{X}_{\mathcal{S}}$ be the *restricted site* whose underlying category is $\mathcal{X}$ and where a covering of $a \in \mathcal{X}(S)$ is a covering of $S$. Then the inertia stack $I_{\mathcal{X}} = \mathcal{X} \times_{\mathcal{X} \times \mathcal{X}} \mathcal{X}$ is a sheaf of groups on $\mathcal{X}_{\mathcal{S}}$: for $a \in \mathcal{X}(S)$, we have $I_{\mathcal{X}}(a) = \mathrm{Isom}_S(a)$. The compatibility condition (6.2.2) ensures that there is an isomorphism $\psi \colon G|_{\mathcal{X}} \to I_{\mathcal{X}}$ of sheaves on $\mathcal{X}_{\mathcal{S}}$.

**Example 6.2.23** (The trivial banded gerbe)**.** If $G$ is an abelian sheaf on a site $\mathcal{S}$ with a final object $S$, then the classifying stack $\mathbf{B}G$ of Example 6.2.18 is a banded $G$-gerbe and is trivial (i.e. $\mathbf{B}G(S) \neq \emptyset$). A banded $G$-gerbe $\mathcal{X}$ over $\mathcal{S}$ is trivial if and only if $\mathcal{X} \cong \mathbf{B}G$. We refer to $\mathbf{B}G$ as the *trivial banded* $G$*-gerbe*.

**Exercise 6.2.24** (Band associated to a gerbe)**.** Let $\mathcal{S}$ be a site with a final object $S$. Let $\mathcal{X}$ be an *abelian gerbe* over $\mathcal{S}$, i.e. a gerbe $\mathcal{X}$ such that $\mathrm{Aut}_T(a)$ is abelian for every object $a \in \mathcal{X}(T)$. Show that there is a sheaf of groups $G$ on $S$ such that $\mathcal{X}$ is banded by $G$.

*Hint: Use Axiom (1) of a gerbe to find a covering* $\{X_i \to X\}$ *and elements* $a_i \in \mathcal{X}(X_i)$*. Use Axiom (2) to glue the sheaves* $G_i := \underline{\mathrm{Aut}}_{X_i}(a_i)$ *to a sheaf* $G$*.*

### 6.2.5 Algebraic gerbes

Attached to any algebraic stack $\mathcal{Y}$ is the big fppf site $(\mathrm{Sch}/\mathcal{Y})_{\mathrm{fppf}}$ of schemes over $\mathcal{Y}$: the underlying category of $(\mathrm{Sch}/\mathcal{Y})_{\mathrm{fppf}}$ is $\mathcal{Y}$ and a covering of an object $y \in \mathcal{Y}(T)$ is a covering of $T$. Moreover, if $\mathcal{X} \to \mathcal{Y}$ is a morphism of algebraic stacks, then $\mathcal{X}$ is a stack over $(\mathrm{Sch}/\mathcal{Y})_{\mathrm{fppf}}$ thanks to Proposition 6.2.2.

**Definition 6.2.25** (Gerbes and banded $G$-gerbes)**.** A morphism $\mathcal{X} \to \mathcal{Y}$ of algebraic stacks is a *gerbe* if $\mathcal{X}$ is a gerbe over the big fppf site $(\mathrm{Sch}/\mathcal{Y})_{\mathrm{fppf}}$.

If $G \to S$ is a commutative fppf group scheme, a morphism $\mathcal{X} \to \mathcal{Y}$ of algebraic stacks over $S$ is called a *banded* $G$*-gerbe* (or simply $G$*-gerbe*) if $\mathcal{X}$ is a gerbe over $(\mathrm{Sch}/\mathcal{Y})_{\mathrm{fppf}}$ banded by the sheaf of groups $G \times_S \mathcal{Y}$.

We say that an algebraic stack $\mathcal{X}$ is a *gerbe* (resp. *banded* $G$*-gerbe*) if there exists a morphism $\mathcal{X} \to X$ to an algebraic space which is a gerbe (resp. banded $G$-gerbe).

If $G \to S$ is a commutative fppf group scheme, the classifying stack $\mathbf{B}G \to S$ is a banded $G$-gerbe. Note that a banded $G$-gerbe $\mathcal{X} \to X$ over an algebraic space $X$ is trivial if and only if $\mathcal{X} \cong \mathbf{B}G \times_S X$.

**Proposition 6.2.26.** *Let $\mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks.*

*(1) The morphism $\mathcal{X} \to \mathcal{Y}$ is a gerbe if and only if there exists an fppf morphism $V \to \mathcal{Y}$ from a scheme and an fppf group algebraic space $G \to V$ such that $\mathcal{X} \times_{\mathcal{Y}} V \cong \mathbf{B}G$.*

*(2) If $\mathcal{X} \to \mathcal{Y}$ is a gerbe, then $\mathcal{X} \to \mathcal{Y}$ is a smooth morphism.*

*(3) If $G \to S$ is an fppf group scheme and $\mathcal{X} \to X$ is a banded $G$-gerbe over an algebraic space $X$, then there exists an étale cover $X' \to X$ such that $\mathcal{X} \times_X X' \cong X' \times_S \mathbf{B}G$.*

*Proof.* We first prove (1). For ($\Rightarrow$), Exercise 6.2.19 implies that there is an fppf morphism $V \to \mathcal{Y}$ and a sheaf of groups $G$ on $V$ such that $\mathcal{X} \times_{\mathcal{Y}} V \cong \mathbf{B}G$. Since $\mathcal{X} \times_{\mathcal{Y}} V$ is an algebraic stack, its diagonal is representable, and thus $G$ is an algebraic space. Conversely, suppose that $\mathcal{X} \times_{\mathcal{Y}} V \cong \mathbf{B}G$ for an fppf morphism $V \to \mathcal{Y}$ and an fppf group algebraic space $G \to V$. To see Axiom (1) of a gerbe, if $a \in (\mathrm{Sch}/\mathcal{Y})$ is an object over a scheme $T$, then $V_T := V \times_{\mathcal{Y}} T \to T$ is an fppf covering and since $\mathcal{X} \times_{\mathcal{Y}} V_T \cong \mathbf{B}G_{V_T}$, there is an object of $\mathcal{X}$ over $V_T$. Similarly for Axiom (2), if $x_1, x_2 \in \mathcal{X}$ are objects over $y \in \mathcal{Y}(T)$, then pull backs of $x_1$ and $x_2$ become isomorphic under the fppf covering $V_T \to T$.

Part (2) follows from (1) since $\mathbf{B}G \to V$ is a smooth morphism; indeed smoothness is an fppf local property on the source (Proposition B.4.2). For part (3), $\mathcal{X} \to X$ has a section after base changing by the *smooth* and surjective morphism $\mathcal{X} \to X$. Choosing a smooth presentation $U \to \mathcal{X}$, then the composition $U \to \mathcal{X} \to X$ is smooth and the base change $\mathcal{X} \times_X U \to U$ has a section. Since smooth morphisms étale locally have sections (Corollary A.3.6), there is an étale cover $X' \to X$ factoring through $U \to X$. It follows that $\mathcal{X} \times_X X' \to X'$ has a section or in other words that $\mathcal{X} \times_X X' \cong X' \times_S \mathbf{B}G$. $\square$

**Exercise 6.2.27.** Show that a morphism $\mathcal{X} \to \mathcal{Y}$ of algebraic stacks is a gerbe if and only if $\mathcal{X} \to \mathcal{Y}$ and $\mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ are fppf.

We will later show that an algebraic stack $\mathcal{X}$ is a gerbe if and only if $I_{\mathcal{X}} \to \mathcal{X}$ is fppf (Proposition 6.2.45).

**Exercise 6.2.28.** Show that there is a non-trivial isomorphism

$$\alpha\colon \mathbf{B}(\mathbb{Z}/2) \times (\mathbb{A}^1 \smallsetminus 0) \to \mathbf{B}(\mathbb{Z}/2) \times (\mathbb{A}^1 \smallsetminus 0)$$

of trivial banded $\mathbb{Z}/2$-gerbes over $\mathbb{A}^1$ which glues to a non-trivial banded $\mathbb{Z}/2$-gerbe over $\mathbb{P}^1$.

**Exercise 6.2.29.** Let $1 \to K \to G \to Q \to 1$ be a short exact sequence of affine algebraic groups over $\Bbbk$ such that $K$ is commutative. Show that $\mathbf{B}G \to \mathbf{B}Q$ is a banded $K$-gerbe which is trivial if and only if the sequence splits.

**Exercise 6.2.30.** Assume that $\mathrm{char}(\Bbbk) \neq 2, 3$. Recall from Exercise 3.1.18(c) that the moduli stack of stable elliptic curves has a quotient description $\overline{\mathcal{M}}_{1,1} = \mathcal{P}(4,6) := [(\mathbb{A}^2 \smallsetminus 0)/\mathbb{G}_m]$ where $\mathbb{G}_m$ acts with weights 4 and 6.

(a) Show that the $j$-line $\pi\colon \mathcal{M}_{1,1} \to \mathbb{A}^1$ is a trival banded $\mathbb{Z}/2$-gerbe over $\mathbb{A}^1 \setminus \{0, 1728\}$.

*Hint: Construct a family of elliptic curves over $\mathbb{A}^1_{\Bbbk}\setminus\{0,1728\}$ via the Weierstrass equation*

$$y^2z + xyz = x^3 - \frac{36}{t-1728}xz^2 - \frac{1}{t-1728}z^3,$$

*where $t$ is the coordinate on $\mathbb{A}^1$, where the discriminant $\Delta = t^2/(t-1728)^3$. See [Sil09, Prop. III.1.4(c)].*

(b) Consider the map $\overline{\mathcal{M}}_{1,1} = \mathcal{P}(4,6) \to \mathcal{P}(2,3)$ induced the homomorphism $\mathbb{G}_m \to \mathbb{G}_m, t \mapsto t^2$; note that $\mathcal{P}(2,3)$ is the banded $\mathbb{Z}/2$-gerbe obtained by rigidifying along the hyperelliptic involution (see Proposition 6.2.45), and the restriction along $\mathcal{P}(2,3) \setminus \{0,1728,\infty\}$ is tje gerbe from (a). Show that $\overline{\mathcal{M}}_{1,1} \to \mathcal{Y}$ is non-trivial.

*Hint: If it's trivial, show that there are torsion line bundles contradicting that* $\operatorname{Pic}(\overline{\mathcal{M}}_{1,1}) = \mathbb{Z}$ *from Exercise 6.1.14(a).*

(c) Show that the rigidification $\mathcal{M}_{1,1} \to \mathcal{P}(2,3) \setminus \infty$ is also non-trivial.

*Hint: If it's trivial, show that $\mathcal{M}_{1,1}$ has three 2-torsion line bundles contradicting that* $\operatorname{Pic}(\mathcal{M}_{1,1}) = \mathbb{Z}/12$ *from Exercise 6.1.14(b).*

(d) For $g \geq 2$, let $\mathcal{H}_g \subset \mathcal{M}_g$ be the closed substack classifying hyperelliptic curves. Show that the rigidification $\mathcal{H}_g \to \mathcal{Y}$ along the hyperelliptic involution is a non-trivial banded $\mathbb{Z}/2$-gerbe.

**Exercise 6.2.31.**

(a) Show that every gerbe $\mathcal{X}$ over an algebraic space that is étale locally isomorphic to $\mathbf{B}\mathbb{Z}/2$ is in fact banded by $\mathbb{Z}/2$.

(b) Give an example of a gerbe over an algebraic space that is étale locally isomorphic to $\mathbf{B}\mathbb{G}_m$ but not banded by $\mathbb{G}_m$.

*Hint: Consider the classifying stack of a form of $\mathbb{G}_m$ (see Exercise 4.1.40).*

**Exercise 6.2.32** (Root gerbes and root stacks revisited)**.** Recall that root gerbes and stacks were introduced in Examples 3.9.12 and 3.9.13.

(a) Since we now know how to construct quotient stacks by actions of $\boldsymbol{\mu}_r$ over any base scheme $S$, show that Exercise 3.9.14 still holds without the condition that $r$ is invertible in $\Gamma(S, \mathcal{O}_S)$.

(b) Given a scheme $X$, a line bundle $L$, and a section $s \in \Gamma(X, L)$, show that $X(\sqrt[r]{L}) \to X$ and the restriction of $X(\sqrt[r]{L,s}) \to X$ along $V(s)$ are banded $\boldsymbol{\mu}_r$-gerbes.

(c) Show that $X(\sqrt[r]{L}) \to X$ is the trivial $\boldsymbol{\mu}_r$-gerbe if and only if $L$ has an $r$th root.

(d) Consider an exact sequence $1 \to \mu_r \to \mathbb{G}_m \to \mathbb{G}_m \to 1$ and a $\mathbb{G}_m$-torsor $P''$ corresponding to a line bundle $L''$. Show that $X(\sqrt[r]{L})$ is isomorphic to the gerbe of trivializations $\mathcal{G}_{P''}$ defined in Exercise 6.2.38(b).

### 6.2.6 Residual gerbes revisited

Given an algebraic stack $\mathcal{X}$ and $x \in |\mathcal{X}|$, recall from Definition 3.5.12 that *the residual gerbe at $x$* (if it exists) is a reduced, locally noetherian algebraic stack $\mathcal{G}_x$ with a monomorphism $\mathcal{G}_x \hookrightarrow \mathcal{X}$ such that $|\mathcal{G}_x|$ is a point mapping to $x$. We've already shown that the residual gerbe at a finite type point exists (Proposition 3.5.16).

Residual gerbes are unique.

**Lemma 6.2.33.** *Let $\mathcal{X}$ be a noetherian algebraic stack and $x \in |\mathcal{X}|$ be a point. If the residual gerbe at $x$ exists, it is unique.*

*Proof.* To add. $\square$

We now establish the existence of residual gerbes at all points and moreover show that they are in fact gerbes.

**Proposition 6.2.34.** *If $\mathcal{X}$ is a noetherian algebraic stack and $x \in |\mathcal{X}|$ is a point, then the residual gerbe $\mathcal{G}_x$ exists and is a gerbe over a field $\kappa(x)$, called the* residue field of $x$.

*Proof.* To add. $\square$

If $\mathcal{X}$ is a quasi-separated algebraic stack of finite type over a field $\Bbbk$ and $x \in \mathcal{X}(\Bbbk)$, then $\mathcal{G}_x = \mathbf{B}G_x$ (Proposition 3.5.16). More generally, we have:

**Exercise 6.2.35.** Let $\mathcal{X}$ be a noetherian algebraic stack and $x \in |\mathcal{X}|$ be a finite type point.

(1) For any representative $\overline{x}\colon \operatorname{Spec} \Bbbk \to \mathcal{X}$ of $x$, there is a cartesian diagram

$$\begin{tikzcd} B_{\Bbbk} G_{\overline{x}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{G}_x \arrow[r, hook] \arrow[d] & \mathcal{X} \\ \operatorname{Spec} \Bbbk \arrow[r] & \operatorname{Spec} \kappa(x). \end{tikzcd}$$

(2) If the stabilizer of $x$ is smooth, show that there is a finite separable extension $\kappa(x) \to \Bbbk$ and a representative of $x$ over $\Bbbk$.

**Exercise 6.2.36.** Let $C \subset \mathbb{P}^2_{\Bbbk}$ be a non-split quadric over a field $\Bbbk$, and let $\Bbbk \to \Bbbk'$ be a quadratic extension such that $C \times_{\Bbbk} \Bbbk' \cong \mathbb{P}^1_{\Bbbk'}$. Let $D \subset C$ be a divisor of degree 6 and let $X \to \mathbb{P}^1_{\Bbbk'}$ be the double cover ramified over $D \times_{\Bbbk} \Bbbk'$. Show that the residual gerbe of $[X] \in \mathcal{M}_2$ is non-trivial and has residue field $\Bbbk$.

To give some context for the above exercise, consider the rigidification $\mathcal{M}_2 \to \mathcal{Y}$ of the hyperelliptic involution is a non-trivial banded $\mathbb{Z}/2$-gerbe (see Exercise 6.2.30(d)). The restriction to the locus $\mathcal{M}_2^\circ \subset \mathcal{M}_2$ of curves whose only non-trivial automorphism is the hyperelliptic involution is the coarse moduli space $\mathcal{M}_2^\circ \to M_2^\circ$. This is a non-trivial banded $\mathbb{Z}/2$-gerbe and the generic fiber (over the residue field of the generic point of $M_2$) is also a non-trivial gerbe. Exercise 6.2.36 on the other hand shows that the reduced fibers of $\mathcal{M}_2^\circ \to M_2^\circ$ over closed points can be non-trivial gerbes.

### 6.2.7 Cohomological characterization

The following exercises provide cohomological characterizations of torsors and gerbes for an abelian sheaf $G$ on the small fppf site $S_{\mathrm{fppf}}$ of a scheme $S$. If $G$ is represented by a smooth, commutative, and quasi-projective group scheme, then it turns out that $\mathrm{H}^i((\mathrm{Sch}/S)_{\mathrm{fppf}}, G) = \mathrm{H}^i((\mathrm{Sch}/S)_{\text{ét}}, G)$ (see Remark 4.1.39) and thus in this case we can use étale cohomology. For an extra challenge, try to prove these statements for abelian sheaves over any site. The reader may consult [Gir71] and [Ols16, §12] for detailed proofs.

**Exercise 6.2.37** (Torsors)**.** Let $S$ be a scheme.

(a) If $G$ is an abelian sheaf on $S_{\mathrm{fppf}}$, show that $\mathrm{H}^1(S_{\mathrm{fppf}}, G)$ is in bijective correspondence with isomorphism classes of $G$-torsors.

*Hint: Imitate the proof using Čech cohomology that* $\mathrm{H}^1(X, \mathcal{O}_X^*) = \mathrm{Pic}(X)$ *for a scheme* $X$*.*

(b) Let $0 \to G' \to G \to G'' \to 0$ be an exact sequence of abelian sheaves on $S_{\mathrm{fppf}}$, and let

$$0 \to \mathrm{H}^0(S_{\mathrm{fppf}}, G') \to \mathrm{H}^0(S_{\mathrm{fppf}}, G) \to \mathrm{H}^0(S_{\mathrm{fppf}}, G'') \xrightarrow{\delta}$$
$$\to \mathrm{H}^1(S_{\mathrm{fppf}}, G') \xrightarrow{\alpha} \mathrm{H}^1(S_{\mathrm{fppf}}, G) \xrightarrow{\beta} \mathrm{H}^1(S_{\mathrm{fppf}}, G'') \to \cdots$$

be the corresponding long exact sequence. Show that under the bijection in (a), the boundary map $\delta$ assigns a section $S \to G''$ to the $G'$-torsor defined by the fiber product $G \times_{G''} S$. Show also that $\alpha$ assigns a $G'$-torsor $P'$ to the quotient $P' \times^{G'} G := (P' \times G)/G'$ while $\beta$ assigns a $G$-torsor $P$ to $P \times^G G''$.

**Exercise 6.2.38** (Gerbes)**.** Let $S$ be a scheme.

(a) If $G$ is an abelian sheaf on $S_{\mathrm{fppf}}$, show that $\mathrm{H}^2(\mathcal{S}, G)$ is in bijective correspondence with isomorphism classes of $G$-banded gerbes.

*Hint: Let* $0 \to G \to I^0 \xrightarrow{d^0} I^1 \xrightarrow{d^1} I^2 \xrightarrow{d^2} \cdots$ *be an injective resolution. For a cohomology class* $\alpha \in \mathrm{H}^2(\mathcal{S}, G)$*, define a stack* $\mathcal{G}_\alpha$ *over* $\mathcal{S}$ *as follows. Choose* $\tau \in \Gamma(\mathcal{S}, I^2)$ *with* $d^2(\tau) = 0$ *such that the image of* $\tau$ *in* $\mathrm{H}^2(\mathcal{S}, G)$ *is* $\alpha$*. Define* $\mathcal{G}_\alpha$ *as the category of pairs* $(S, \sigma)$ *consisting of an object* $S \in \mathcal{S}$ *and a section* $\sigma \in \Gamma(S, I^1)$ *with* $d^1(\sigma) = \tau|_S$*. A morphism* $(S', \sigma') \to (S, \sigma)$ *is the data of a morphism* $f\colon S' \to S$ *and an element* $\rho \in \Gamma(S', I^0)$ *with boundary* $d^0(\rho) = \sigma' - f^*(\sigma)$*. Show that* $\mathcal{G}_\alpha$ *is a* $G$*-banded gerbe and that the assignment* $\alpha \mapsto \mathcal{G}_\alpha$ *gives the stated bijection.*

Let $0 \to G' \to G \to G'' \to 0$ be an exact sequence of abelian sheaves on $S_{\mathrm{fppf}}$, and let

$$\cdots \to \mathrm{H}^1(S_{\mathrm{fppf}}, G'') \xrightarrow{\delta} \mathrm{H}^2(S_{\mathrm{fppf}}, G') \xrightarrow{\alpha} \mathrm{H}^2(S_{\mathrm{fppf}}, G) \xrightarrow{\beta} \mathrm{H}^2(S_{\mathrm{fppf}}, G'') \to \cdots$$

be the corresponding long exact sequence.

(b) Show that under the bijection in (a), the boundary map $\delta$ assigns a $G''$-torsor $P'' \to S$ to the *gerbe of trivializations* $\mathcal{G}_{P''}$. The objects of the prestack $\mathcal{G}_{P''}$ over an $S$-scheme $T$ is a pair $(P, \alpha)$ consisting of a $G$-torsor $P \to T$ and a trivialization $\alpha\colon P \times^G G'' \cong P'' \times_S T$ of $G''$-torsors. Morphisms in $\mathcal{G}_{P''}$ are morphisms of $G$-torsors compatible with the trivializations.

(c) Suppose that $G'$, $G$, and $G''$ are represented by commutative and affine algebraic groups over a field $\Bbbk$. Show that if $P'' \to S$ is a $G''$-torsor, then $\mathcal{G}_{P''}$ is identified with both the quotient stack $[P''/G]$ and the fiber product of $\mathbf{B}G \to \mathbf{B}G''$ and the map $S \to \mathbf{B}G''$ corresponding the $G''$-torsor $P''$.

**Exercise 6.2.39** (Group structure)**.** If $G$ is an abelian sheaf on the small fppf site $S_{\mathrm{fppf}}$ of a scheme $S$, show that the group laws of $\mathrm{H}^1(S_{\mathrm{fppf}}, G)$ and $\mathrm{H}^2(S_{\mathrm{fppf}}, G)$ can be described geometrically as follows:

(a) The product of two $G$-torsors $P_1$ and $P_2$ is the *contracted product* $P_1 \wedge^G P_2$ defined as the sheaf quotient $(P_1 \times P_2)/G$ where $h \cdot (p_1, p_2) = (h^{-1}p_1, hp_2)$ with the $G$-action specified by $g \cdot (p_1, p_2) = (gp_1, p_2) = (p_1, gp_2)$. The inverse of a $G$-torsor $P$ is the sheaf $P$ with the inverted $G$-action: $g \cdot p = g^{-1}p$.

(b) The product of two banded $G$-gerbes $(\mathcal{X}_1, \psi_{1,x})$ and $(\mathcal{X}_2, \psi_{2,x})$ is the *contracted product* $\mathcal{X}_1 \wedge^G P_2$, which is defined as the rigidification $(\mathcal{X}_1 \times \mathcal{X}_2) /\!\!/ G$ (see Proposition 6.2.45) of the product $\mathcal{X}_1 \times \mathcal{X}_2$ along the subgroup $(\psi_1, \psi_2)\colon G|_{\mathcal{X}_1 \times \mathcal{X}_2} \to I_{\mathcal{X}_1 \times \mathcal{X}_2}$ defined by the bands $\psi_1$ and $\psi_2$. The inverse $(\mathcal{X}, \psi_x)^{-1} = (\mathcal{X}, \psi_x^{-1})$ inverts the band.

**Remark 6.2.40** (Banded $\mu_n$-gerbes over $\mathbb{P}^1$)**.** Over an algebraically closed field $\Bbbk$, isomorphism classes of banded $\mu_n$-gerbes over $\mathbb{P}^1$ are in bijection with $\mathbb{Z}/n\mathbb{Z}$. To see this, observe that the exact sequence $1 \to \mu_n \to \mathbb{G}_m \xrightarrow{n} \mathbb{G}_m \to 1$ induces an exact sequence on cohomology

$$\mathrm{H}^1(\mathbb{P}^1_{\text{ét}}, \mathbb{G}_m) \xrightarrow{n} \mathrm{H}^1(\mathbb{P}^1_{\text{ét}}, \mathbb{G}_m) \to \mathrm{H}^2(\mathbb{P}^1_{\text{ét}}, \mu_n) \to \mathrm{H}^2(\mathbb{P}^1_{\text{ét}}, \mathbb{G}_m).$$

Since $\mathrm{H}^1(\mathbb{P}^1_{\text{ét}}, \mathbb{G}_m) = \mathrm{Pic}(\mathbb{P}^1_{\text{ét}}) = \mathbb{Z}$, we can use the fact that $\mathrm{H}^2(\mathbb{P}^1_{\text{ét}}, \mathbb{G}_m) = 0$ to conclude that $\mathrm{H}^2(\mathbb{P}^1_{\text{ét}}, \mu_n) = \mathbb{Z}/n\mathbb{Z}$. The image of a line bundle $\mathcal{O}(d)$ is equivalent to the root stack $\mathbb{P}^1(\sqrt[n]{\mathcal{O}(d)})$, and this gerbe is trivial if and only if $n$ divides $d$. The gerbe $\mathbb{P}^1(\sqrt[n]{\mathcal{O}(1)})$ is isomorphic to the quotient stack $[(\mathbb{A}^2 \smallsetminus 0)/\mathbb{G}_m]$ where $t \cdot (x, y) = (t^n x, t^n y)$.

**Exercise 6.2.41** ($\mathbb{G}_m$-gerbes and twisted sheaves)**.** Let $\mathcal{X} \to X$ be a $\mathbb{G}_m$-gerbe over an algebraic space $X$. We say that a coherent sheaf $F$ on $\mathcal{X}$ is *1-twisted* if for every field-valued point $\operatorname{Spec} \Bbbk \to \mathcal{X}$, the $\mathbb{G}_m$-representation corresponding to the pullback of $E$ under $\mathbf{B}\mathbb{G}_m = \mathbf{B}G_x \to \mathcal{X}$ decomposes as a direct sum of one-dimensional representations of *weight one*. Show that the $\mathbb{G}_m$-gerbe $\mathcal{X} \to X$ is trivial if and only if there exists a 1-twisted line bundle on $\mathcal{X}$.

**Exercise 6.2.42** (Azumaya algebras)**.** An *Azumaya algebra of rank $r^2$* over a noetherian scheme $X$ is a (possibly non-commutative) associative $\mathcal{O}_X$-algebra $A$ which is coherent as an $\mathcal{O}_X$-module and such that there is an étale covering $X' \to X$ where $A \otimes_{\mathcal{O}_X} \mathcal{O}_{X'}$ is isomorphic to the matrix algebra $M_r(\mathcal{O}_X)$. We say that $A$ is trivial if it is isomorphic to $M_r(\mathcal{O}_X)$. By Exercise C.2.15, Azumaya algebras are in bijection with principal $\mathrm{PGL}_r$-bundles (which are also in bijection with Brauer–Severi schemes).

Let $A$ be an Azumaya algebra over a noetherian scheme $X$ of rank $r^2$.

(a) Define the *gerbe of trivializations of $A$* as the stack $\mathcal{G}_A$ over $(\mathrm{Sch}/X)_{\text{ét}}$ where an object over a $X$-scheme $T$ is a pair $(E, \alpha)$ consisting of a vector bundle $E$ on $T$ of rank $r$ and a trivialization $\alpha\colon \mathrm{End}_{\mathcal{O}_X}(E) \xrightarrow{\sim} A \otimes_{\mathcal{O}_X} \mathcal{O}_T$. Morphisms in $\mathcal{G}_A(T)$ are isomorphisms of vector bundles compatible with the trivializations. Show that $\mathcal{G}_A \to X$ is a banded $\mathbb{G}_m$-gerbe.

(b) Let $P_A$ be the principal $\mathrm{PGL}_r$-bundle corresponding to $A$. Identify $\mathcal{G}_A$ with the gerbe of trivializations $\mathcal{G}_{P_A}$ defined in Exercise 6.2.38(b) with respect to the $\mathrm{PGL}_r$-torsor $P_A$ and the surjection $\mathrm{GL}_r \to \mathrm{PGL}_r$.

(c) The exact sequence $1 \to \mathbb{G}_{m,X} \to \mathrm{GL}_{r,X} \to \mathrm{PGL}_{r,X} \to 1$ of sheaves on $X_{\text{ét}}$ induces a boundary map

$$\delta\colon \mathrm{H}^1(X_{\text{ét}}, \mathrm{PGL}_r) \to \mathrm{H}^2(X_{\text{ét}}, \mathbb{G}_m).$$

Show that $\delta(P_A) = [\mathcal{G}_A] \in \mathrm{H}^2(X_{\text{ét}}, \mathbb{G}_m)$ and that this is an $r$-torsion element.

(d) Show that the Azumaya algebra $A$ is trivial if and only if $\mathcal{G}_A$ is trivial.

(e) Use the quarternions to construct a non-trivial $\mathbb{G}_m$-gerbe over $\operatorname{Spec} \mathbb{R}$.

**Remark 6.2.43** (Brauer groups)**.** Two Azumaya algebras $A$ and $A'$ on a noetherian scheme $X$ are *similar* if there exists vector bundles $E$ and $E'$ on $X$ such that $A \otimes_{\mathcal{O}_X} \mathrm{End}_{\mathcal{O}_X}(E) \cong A \otimes_{\mathcal{O}_X} \mathrm{End}_{\mathcal{O}_X}(A')$. This defines an equivalence relation, and the *Brauer group of* $X$ is the set $\mathrm{Br}(X)$ of equivalence classes of Azumaya algebras. The set $\mathrm{Br}(X)$ becomes a group under the operators $[A] \cdot [A'] = [A \otimes A']$ and $[A]^{-1} = [A^{\mathrm{op}}]$ (where $A^{\mathrm{op}}$ is the opposite algebra with same elements and addition as $A$ but with multiplication reversed: $a \cdot^{A^{\mathrm{op}}} b = b \cdot^{A} a$).

The exact sequence $1 \to \mathbb{G}_{m,X} \to \mathrm{GL}_{r,X} \to \mathrm{PGL}_{r,X} \to 1$ induces a boundary map $\mathrm{H}^1(X_{\text{ét}}, \mathrm{PGL}_r) \to \mathrm{H}^2(X_{\text{ét}}, \mathbb{G}_m)$. Viewing $\mathrm{H}^1(X_{\text{ét}}, \mathrm{PGL}_r)$ as the set of Azumaya algebras of rank $r^2$ and $\mathrm{H}^2(X_{\text{ét}}, \mathbb{G}_m)$ as the set of banded $\mathbb{G}_m$-gerbes, the boundary map assigns an Azumaya algebra $A$ to the gerbe of trivializations $\mathcal{G}_A$, which is an $r$-torsion element (see Exercise 6.2.42). Two Azumaya algebras $A$ and $A'$ (of possibly different rank) are similar if and only if $\mathcal{G}_A \cong \mathcal{G}_{A'}$, and thus there is an injective map

$$\mathrm{Br}(X) \hookrightarrow \mathrm{Br}'(X) := \mathrm{H}^2(X_{\text{ét}}, \mathbb{G}_m)_{\mathrm{tors}}, \qquad A \mapsto \mathcal{G}_A,$$

into the *cohomological Brauer group* $\mathrm{Br}'(X)$. See [Mil80, §IV.2] and [Gro68] for additional background.

Grothendieck asked whether $\mathrm{Br}(X) \hookrightarrow \mathrm{Br}'(X)$ is surjective. This is known in some cases. The strongest result is due to Gabber: if $X$ admits an ample line bundle [dJ03]. It is however open in general, even for smooth separated schemes over a field.

**Exercise 6.2.44.** Let $X$ be a noetherian scheme and $\mathcal{X} \to X$ be a banded $\mathbb{G}_m$-gerbe corresponding to a cohomology class $[\mathcal{X}] \in \mathrm{H}^2(X_{\text{ét}}, \mathbb{G}_m)$.

(a) Show that the following are equivalent:

  (i) There exists an Azumaya algebra $A$ on $X$ such that $\mathcal{X} \cong \mathcal{G}_A$, i.e. $[\mathcal{X}]$ is in the image of $\mathrm{Br}'(X) \to \mathrm{Br}(X)$,

  (ii) $\mathcal{X}$ is a global quotient stack, and

  (iii) there exists a 1-twisted vector bundle $E$ on $\mathcal{X}$ (see Exercise 6.2.41).

(b) Let $X$ be a normal separated surface over $\mathbb{C}$ such that $\mathrm{H}^2(X, \mathbb{G}_m)$ contains a non-torsion element $\alpha$; for an example, see [Gro68, II.1.11.b]. Conclude that the banded $\mathbb{G}_m$-gerbe corresponding to $\alpha$ is not a global quotient stack.

(c) Let $Y = \operatorname{Spec} \mathbb{C}[x, y, z]/(xy - z^2)$. Show that there is a non-trivial involution $\alpha$ of $(Y \smallsetminus 0) \times \mathbf{B}(\mathbb{Z}/2)$ such that the stack $\mathcal{X}$, obtained by gluing the trivial banded $\mathbb{Z}/2$-gerbes over $Y$ along $\alpha$, is a banded $\mathbb{Z}/2$-gerbe over the non-separated union $Y \bigcup_{Y \smallsetminus 0} Y$ which is not a global quotient stack.

See also [EHKV01].

### 6.2.8 Rigidification

**Proposition 6.2.45.** *Let $\mathcal{X}$ be an algebraic stack such that $I_{\mathcal{X}} \to \mathcal{X}$ is fppf. Let $X$ be the sheaf on $\mathrm{Sch}_{\mathrm{fppf}}$ defined by the sheafification of the functor assigning a scheme $S$ to the set of isomorphism classes $\mathcal{X}(S)/\sim$ of objects. Then $X$ is an algebraic space and $\mathcal{X} \to X$ is a gerbe.*

*Proof.* To show that $X$ is an algebraic space, it suffices to show that $\mathcal{X} \to X$ is a smooth representable morphism. In this case, a smooth presentation $U \to \mathcal{X}$ induces a smooth presentation $U \to X$, and it follows from Corollary 4.4.12 (or Theorem 6.2.1) that $X$ is an algebraic space. As gerbes are smooth morphisms, it suffices to show that for every morphism $S \to X$ from a scheme, the fiber product $\mathcal{X} \times_X S \to S$ is a gerbe. By construction, there is an fppf cover $S' \to S$ and a morphism $a' \colon S' \to \mathcal{X}$ lifting the composition $S' \to S \to X$. Since the property of being a gerbe is fppf local, after replacing $S$ with $S'$, we may assume that $S \to X$ lifts to a map $a \colon S \to \mathcal{X}$. We claim that there is an isomorphism

$$\Psi \colon \mathcal{X} \times_X S \to \mathbf{B}\underline{\mathrm{Aut}}_S(a).$$

An object of the fiber product $\mathcal{X} \times_X S$ consists of a pair $(f, a')$ where $f \colon T \to S$ is a map of schemes and $b \in \mathcal{X}(T)$ such that $T \to S \to X$ and $T \xrightarrow{b} \mathcal{X} \to X$ agree. Define $\Psi(f, b)$ as the principal $\underline{\mathrm{Aut}}_S(a)$-bundle $\underline{\mathrm{Isom}}_T(f^*a, b)$. Observe that $\Psi(f, f^*a)$ maps to the trivial bundle.

Since $\mathcal{X} \times_X S$ and $\mathbf{B}\underline{\mathrm{Aut}}_S(a)$ are both stacks in the fppf topology, we may verify that $\Psi$ is essentially surjective fppf locally: if $P \to T$ is a principal $\underline{\mathrm{Aut}}_S(a)$-bundle, then there is an fppf cover $T' \to T$ such that $P \times_T T'$ is the trivial bundle, which we've seen is in the essential image. Similarly, we may verify that $\Psi$ is fully faithful fppf locally. Let $(f, b), (f', b') \in (\mathcal{X} \times_X S)(T)$. Since the objects $f^*a, b, b' \in \mathcal{X}(T)$ map to the same $T$-valued point of $X$, by the construction of $X$, there is an fppf cover $T' \to T$ such that their pullbacks become isomorphic. By replacing $T$ with $T'$, we may assume that $f^*a \simeq b \simeq b'$ are isomorphic. In this case, the full faithfulness claim is clear as both $\Psi(f, b)$ and $\Psi(f, b')$ are trivial bundles.

Alternatively, we may construct $X$ directly. Let $U \to \mathcal{X}$ be a smooth presentation and $R \rightrightarrows U$ the corresponding smooth groupoid. The stabilizer groupoid scheme $S_U = R \times_{U \times U} U = I_{\mathcal{X}} \times_{\mathcal{X}} U$ is fppf over $U$. There is an fppf equivalence relation $S_U \times_U R \rightrightarrows R$ where one arrow is given by composition and the other is projection. By Corollary 6.2.4, the fppf quotient $R' := R/(S_U \times_U R)$ is an algebraic space. There is an induced fppf equivalence relation $R' \rightrightarrows U$ and $X$ is isomorphic to the fppf quotient $U/R'$.

See also [LMB00, Cor. 10.8] and [SP, Tags 06QD and 06QJ]. □

We now consider a more general situation. If $\mathcal{X}$ is an algebraic stack, then the inertia stack $I_{\mathcal{X}}$ can be viewed as a group scheme over the big étale site $(\mathrm{Sch}/\mathcal{X})_{\text{ét}}$ of $\mathcal{X}$. As a group functor, $I_{\mathcal{X}}$ assigns an object $a \in \mathcal{X}(S)$ to the group $\mathrm{Aut}_S(a)$, and a morphism $\alpha \colon a' \to a$ over $S' \to S$ to the natural pullback map $\alpha^* \colon \mathrm{Aut}_S(a) \to \mathrm{Aut}_{S'}(a')$ (see (3.2.2)). Given $a \colon S \to \mathcal{X}$, there is a canonical isomorphism $I_{\mathcal{X}} \times_{\mathcal{X}} S \cong \underline{\mathrm{Aut}}_S(a)$ of group schemes over $S$.

Suppose that $\mathcal{H} \subset I_{\mathcal{X}}$ is a closed subgroup scheme over $\mathcal{X}$ such that $\mathcal{H} \to \mathcal{X}$ is fppf. This is equivalent to requiring that for every $a \in \mathcal{X}(S)$, there is a closed fppf subgroup scheme $\mathcal{H}_a \subset \underline{\mathrm{Aut}}_S(a)$ over $S$ such that if $a' \to a$ is a morphism over $S' \to S$, the canonical isomorphism $\underline{\mathrm{Aut}}_{S'}(a') \cong \underline{\mathrm{Aut}}_S(a) \times_S S'$ restricts to an isomorphism $\mathcal{H}_{a'} \cong \mathcal{H}_a \times_S S'$. If $\alpha \colon a \xrightarrow{\sim} a$ is an automorphism over the identity, then the canonical isomorphism $\alpha^* \colon \mathrm{Aut}_S(a) \to \mathrm{Aut}_S(a)$ is conjugation by $\alpha$. In particular, $\mathcal{H}_a \subset \underline{\mathrm{Aut}}_S(a)$ is a *normal* group scheme.

Frequently in applications when $\mathcal{X}$ is defined over scheme $S$, the closed subgroup $\mathcal{H} \subset I_{\mathcal{X}}$ is obtained by the pullback of a fppf group scheme $H \to S$, i.e. $\mathcal{H} = H \times_S \mathcal{X}$.

**Definition 6.2.46** (Rigidification)**.** Let $\mathcal{X}$ be an algebraic stack and $\mathcal{H} \subset I_{\mathcal{X}}$ be an fppf closed subgroup scheme over $\mathcal{X}$. The *rigidification* $\mathcal{X} \mathbin{/\!\!/} \mathcal{H}$ is defined as the stackification in $\mathrm{Sch}_{\mathrm{fppf}}$ of the prestack with the same objects as $\mathcal{X}$ and where the set of morphisms beteen $b \in \mathcal{X}(T)$ and $a \in \mathcal{X}(S)$ over $f \colon T \to S$ is defined as $\mathrm{Mor}(b, a) = \mathrm{Mor}_{\mathcal{X}(T)}(b, f^*a)/\mathcal{H}(T)$.

If $\mathcal{X}$ is defined over $S$ and $\mathcal{H} = H \times_S \mathcal{X}$ is the base change of an fppf group scheme $H \to S$, then we write $\mathcal{X} \mathbin{/\!\!/} H := \mathcal{X} \mathbin{/\!\!/} \mathcal{H}$.

One can think of the subgroup $\mathcal{H}$ as giving an action of $\mathbf{B}\mathcal{H}$ on $\mathcal{X}$ and the rigidification $\mathcal{X} /\!\!/ \mathcal{H}$ as the quotient $\mathcal{X}/\mathbf{B}\mathcal{H}$.

**Example 6.2.47.** If $I_{\mathcal{X}} \to \mathcal{X}$ is fppf, then we can take $\mathcal{H} = I_{\mathcal{X}}$ and the rigidification $\mathcal{X} /\!\!/ I_{\mathcal{X}}$ is the algebraic space $X$ constructed in Proposition 6.2.45.

**Proposition 6.2.48.** *Let $\mathcal{X}$ be an algebraic stack and $\mathcal{H} \subset I_{\mathcal{X}}$ be an fppf closed subgroup scheme over $\mathcal{X}$. The rigidification $\mathcal{X} /\!\!/ \mathcal{H}$ is an algebraic stack such that*

*(1) the natural morphism $\pi \colon \mathcal{X} \to \mathcal{X} /\!\!/ \mathcal{H}$ is a gerbe;*

*(2) for every object $a \in \mathcal{X}(S)$, the natural map $\operatorname{Aut}_S(a) \to \operatorname{Aut}_S(\pi(a))$ is surjective with kernel $\mathcal{H}(S)$;*

*(3) a morphism $f \colon \mathcal{X} \to \mathcal{Y}$ factors uniquely through $\mathcal{X} /\!\!/ \mathcal{H}$ if and only if for every object $a \in \mathcal{X}(S)$, the composition $\mathcal{H}(S) \subset \ker(\operatorname{Aut}_{\mathcal{X}(S)}(a) \to \operatorname{Aut}_{\mathcal{Y}(S)}(f(a)))$; and*

*(4) if $\mathcal{H}$ is a commutative group scheme, then $\mathcal{H}$ descends to an fppf group scheme $H \to X$ such that $\mathcal{X} \to X$ is banded $H$-gerbe. If in addition $\mathcal{X}$ is defined over a scheme $S$ and $\mathcal{H} = H \times_S \mathcal{X}$ is the pullback of a commutative fppf group scheme $H \to S$, then $\mathcal{X} \to X$ is a banded $H$-gerbe.*

*Proof.* To show that $\mathcal{X}$ is algebraic, it suffices to show that $\pi \colon \mathcal{X} \to \mathcal{X} /\!\!/ \mathcal{H}$ is a smooth representable morphism: if $U \to \mathcal{X}$ is a smooth presentation, then so is the composition $U \to \mathcal{X} \to \mathcal{X} /\!\!/ \mathcal{H}$. If $g \colon S \to \mathcal{X} /\!\!/ \mathcal{H}$, then by the definition of $\mathcal{X} /\!\!/ \mathcal{H}$ as the stackification, there is an fppf cover $S' \to S$ such that $S' \to S \to \mathcal{X} /\!\!/ \mathcal{H}$ lifts to a map $a' \colon S' \to \mathcal{X}$. By replacing $S$ with $S'$, we may assume that $g \colon S \to \mathcal{X} /\!\!/ \mathcal{H}$ lifts to a morphism $a \colon S \to \mathcal{X}$.

We claim that there is an isomorphism

$$\Psi \colon \mathcal{X} \times_{\mathcal{X} /\!\!/ \mathcal{H}} S \to \mathbf{B}\mathcal{H}_a.$$

Since $\mathcal{H}_a \to S$ is fppf, the classifying stack $\mathbf{B}\mathcal{H}_a$ is algebraic (Proposition 6.2.9) and smooth over $S'$ (Proposition B.4.2), and the isomorphism $\Psi$ would imply that $\mathcal{X} \to \mathcal{X} /\!\!/ \mathcal{H}$ is smooth and representable. An object of $\mathcal{X} \times_{\mathcal{X} /\!\!/ \mathcal{H}} S$ consists of a triple $(f, b, \alpha)$ where $f \colon T \to S$, $b \in \mathbb{X}(T)$, and $\alpha \colon g \circ f \xrightarrow{\sim} \pi \circ b$. Define $\Psi(f, b, \alpha)$ as the principal $\mathcal{H}_a$-bundle $T \times_{\underline{\operatorname{Isom}}_T(f^* a, b)/\mathcal{H}_a} \underline{\operatorname{Isom}}_T(f^a, b)$. Noting that $\Psi(f, f^*a, \mathrm{id})$ is the trivial bundle, the proof that $\Psi$ is an isomorphism follows exactly as in Proposition 6.2.45. The remaining statements are left to the reader.

See also [ACV03, Thm. 5.1.5], [AGV08, §C], [Rom05, §5], and [AOV08, §A]. $\square$

**Example 6.2.49** (Ridigification of $\operatorname{Bun}_{r,d}(C)$)**.** Moduli stacks of sheaves provide interesting examples of rigidification since there is a canonical scaling $\mathbb{G}_m$-action on sheaves. Recall that $\operatorname{Bun}_{r,d}(C)$ is the moduli stack of vector bundles of rank $r$ and degree $d$ on a fixed smooth, connected, and projective curve $C$ over an algebraically closed field $\Bbbk$. For any vector bundle $\mathcal{E}$ on $C \times S$ where $S$ is a $\Bbbk$-scheme, there is a canonical closed immersion $i_{\mathcal{E}} \colon \mathbb{G}_{m,S} \to \underline{\operatorname{Aut}}(\mathcal{E})$ of group schemes over $S$. Thus, $\mathbb{G}_m := \mathbb{G}_{m,\operatorname{Bun}_{r,d}} \subset I_{\operatorname{Bun}_{r,d}}$ is a closed fppf group scheme of the inertia stack, and we can construct the rigidification

$$\operatorname{Bun}_{r,d}(C) /\!\!/ \mathbb{G}_m.$$

Over the substack $\operatorname{Bun}_{r,d}^{\text{simple}}(C)$ of simple bundles (i.e. vector bundles $E$ with $\operatorname{Aut}(E) = \Bbbk^*$), the rigidification $\operatorname{Bun}_{r,d}^{\text{simple}}(C) /\!\!/$ is an algebraic space and $\operatorname{Bun}_{r,d}^{\text{simple}}(C) \to \operatorname{Bun}_{r,d}^{\text{simple}}(C) /\!\!/$ is a banded $\mathbb{G}_m$-gerbe.

**Exercise 6.2.50.**

(a) If $H$ is a commutative fppf group scheme over $S$, show that $\mathbf{B}H \mathbin{/\!\!/} H \cong S$. More generally, show that if $\mathcal{X} \to X$ is a banded $H$-gerbe, then $X \cong \mathcal{X} \mathbin{/\!\!/} H$.

(b) Let $G \to S$ be an fppf group scheme acting on a $S$-scheme $U$. Suppose that $H \subset G$ is a central commutative fppf subgroup scheme acting trivially on $U$. Show that $[U/G] \mathbin{/\!\!/} H \cong [U/(G/H)]$.

**Exercise 6.2.51.** Let $\mathcal{X} \to S$ be a smooth, integral, and separated Deligne–Mumford stack over a scheme $S$. Let $\operatorname{Spec} K \to \mathcal{X}$ be a representative of the generic point. Show that the closure $\mathcal{H} \subset I_{\mathcal{X}}$ of generic fiber $I_{\mathcal{X}} \times_{\mathcal{X}} K$ of the inertia is a closed étale subgroup scheme and that the rigidification $\mathcal{X} \mathbin{/\!\!/} \mathcal{H}$ is a smooth, integral, and separated Deligne–Mumford stack over $S$ with generically trivial inertia.

**Exercise 6.2.52.** Let $\mathcal{X}$ be an algebraic stack over a scheme $S$, $H \to S$ be an fppf group scheme, and $H \times_S \mathcal{X} \subset I_{\mathcal{X}}$ a closed subgroup scheme. Show that the rigidification $\mathcal{X} \mathbin{/\!\!/} H$ can be given the moduli interpretation where an object over a scheme $S$ is a pair $(\mathcal{G}, f)$ where $\mathcal{G} \to S$ is a banded $H$-gerbe and $f \colon \mathcal{G} \to \mathcal{X}$ is an $H$-equivariant morphism (i.e. for every object $a \in \mathcal{G}(T)$ over an $S$-scheme $T$, the composition $H(T) \xrightarrow{\sim} \operatorname{Aut}_T(a) \to \operatorname{Aut}_T(f(a))$ agrees with the inclusion $H(T) \hookrightarrow \operatorname{Aut}_T(f(a))$ given by the subgroup $H \times_S \mathcal{X} \subset I_{\mathcal{X}}$.

### 6.2.9 Picard stacks and spaces

If $X \to S$ is a proper flat morphism of noetherian schemes, define the *Picard stack*

$$\underline{\mathcal{P}\mathrm{ic}}_{X/S}$$

as the stack over $(\mathrm{Sch}/S)_{\text{ét}}$ whose objects over an $S$-scheme $T$ are line bundles on $X_T = X \times_S T$ and whose morphisms are isomorphisms of line bundles. This is an open substack of $\underline{\mathrm{Coh}}(X/S)$ whose objects over an $S$-scheme $T$ are coherent sheaves on $X_T$ flat over $T$. The stack $\underline{\mathrm{Coh}}(X/S)$ is an algebraic stack locally of finite type over $S$; see Exercise 3.1.22 for the case when $X \to S$ is strongly projective and Exercise D.7.8 in general. Therefore $\underline{\mathcal{P}\mathrm{ic}}_{X/S}$ is also algebraic and locally of finite type over $S$.

On the other hand, there are several candidates for Picard functors:

(1) The *naive Picard functor* (or *absolute Picard functor*) is

$$\underline{\mathrm{Pic}}^{\mathrm{naive}}_{X/S} \colon \mathrm{Sch}/S \to \mathrm{Gps}, \qquad (T \to S) \mapsto \mathrm{Pic}(X_T).$$

(2) The *Picard functor*

$$\underline{\mathrm{Pic}}_{X/S} \colon \mathrm{Sch}/S \to \mathrm{Gps},$$

is the fppf sheafification of $\underline{\mathrm{Pic}}^{\mathrm{naive}}_{X/S}$.

(3) The *relative Picard functor* is

$$\underline{\mathrm{Pic}}^{\mathrm{rel}}_{X/S} \colon \mathrm{Sch}/S \to \mathrm{Gps}, \qquad (T \to S) \mapsto \mathrm{Pic}(X_T)/\operatorname{Pic}(T),$$

where an object over an $S$-scheme $T$ is a line bundle $L$ on $X_T$, and two line bundles $L$ and $L'$ on $X_T$ are identified if there exists a line bundle $M \in \mathrm{Pic}(T)$ such that $L \cong L' \otimes f_T^* M$.

(4) We can define the *rigidification of the Picard stack*

$$\underline{\mathcal{P}\mathrm{ic}}_{X/S} /\!\!/ \mathbb{G}_m$$

under the hypothesis that $\mathcal{O}_T \xrightarrow{\sim} f_{T,*}\mathcal{O}_{X_T}$ is an isomorphism for any map $T \to S$. This hypothesis implies that for a line bundle $L$ on $X_T$, there is a canonical isomorphism

$$\mathbb{G}_m(T) = \Gamma(T, \mathcal{O}_T)^* \xrightarrow{\sim} \Gamma(X_T, \mathcal{O}_{X_T})^* = \mathrm{Aut}(L),$$

which further implies that the inertia stack $I_{\underline{\mathcal{P}\mathrm{ic}}_{X/S}}$ is isomorphic to fppf group scheme $\mathbb{G}_m := \mathbb{G}_{m,\underline{\mathcal{P}\mathrm{ic}}_{X/S}}$ over $\underline{\mathcal{P}\mathrm{ic}}_{X/S}$.

While the Picard functor $\underline{\mathrm{Pic}}_{X/S}$ and the rigidification $\underline{\mathcal{P}\mathrm{ic}}_{X/S} /\!\!/ \mathbb{G}_m$ are sheaves in the big fppf topology by definition, it may seem surprising that $\underline{\mathrm{Pic}}^{\mathrm{rel}}_{X/S}$ is also a sheaf under relatively mild hypotheses.

**Proposition 6.2.53.** *Let $f\colon X \to S$ be a proper flat morphism of noetherian schemes such that $\mathcal{O}_S \xrightarrow{\sim} f_*\mathcal{O}_X$ is an isomorphism and this holds after base change, i.e. for every map $g\colon T \to S$, the map $\mathcal{O}_T \xrightarrow{\sim} f_{T,*}\mathcal{O}_{X_T}$ is an isomorphism. Then*

*(1) $\underline{\mathrm{Pic}}_{X/S}$ is representable by an algebraic space locally of finite type over $S$,*

*(2) the map $\underline{\mathcal{P}\mathrm{ic}}_{X/S} \to \underline{\mathrm{Pic}}_{X/S}$ from the Picard stack is a banded $\mathbb{G}_m$-gerbe and $\underline{\mathrm{Pic}}_{X/S} \cong \underline{\mathcal{P}\mathrm{ic}}_{X/S} /\!\!/ \mathbb{G}_m$, and*

*(3) if in addition there is a section $s\colon S \to X$, then $\underline{\mathrm{Pic}}_{X/S} \cong \underline{\mathrm{Pic}}^{\mathrm{rel}}_{X/S}$.*

**Remark 6.2.54.** The condition that $\mathcal{O}_S \xrightarrow{\sim} f_*\mathcal{O}_X$ is an isomorphism universally after base change holds if $f\colon X \to S$ is a flat proper morphism with geometrically connected and reduced fibers (Lemma A.7.11).

In [$\mathrm{FGA_V}$, Thm. 3.1], Grothendieck proved that $\underline{\mathrm{Pic}}_{X/S}$ is a scheme if $X \to S$ is a projective flat morphism of noetherian schemes with geometrically integral fibers. See [Mum66, §20-21] and [Kle05, §9.4] for an exposition and various generalizations. The representability as an algebraic space above was first established by Artin [Art69b, Thm. 7.3], and this holds with the slighly weaker hypothesis that $f$ is *cohomologically flat in dimension* 0, i.e. the formation of $f_*\mathcal{O}_X$ commutes with base change.

*Proof.* As pointed out above, the condition that $\mathcal{O}_S \xrightarrow{\sim} f_*\mathcal{O}_X$ holds after base change implies that the inertia stack of $\underline{\mathcal{P}\mathrm{ic}}_{X/S}$ is isomorphic to the fppf group scheme $\mathbb{G}_m := \mathbb{G}_{m,\underline{\mathcal{P}\mathrm{ic}}_{X/S}}$. Therefore Proposition 6.2.45 (or alternatively Proposition 6.2.48) implies that the rigidification $\underline{\mathcal{P}\mathrm{ic}}_{X/S} /\!\!/ \mathbb{G}_m$ is an algebraic space locally of finite type over $S$. Moreover, $\underline{\mathcal{P}\mathrm{ic}}_{X/S} /\!\!/ \mathbb{G}_m$ is identified with the Picard functor $\underline{\mathrm{Pic}}_{X/S}$ by the definition of rigidification. This gives both (1) and (2).

To identify $\underline{\mathrm{Pic}}^{\mathrm{rel}}_{X/S}$ with $\underline{\mathrm{Pic}}_{X/S}$, it suffices to prove that $\underline{\mathrm{Pic}}^{\mathrm{rel}}_{X/S}$ is a sheaf in the big fppf topology. To this end, it will be convenient to identify $\underline{\mathrm{Pic}}^{\mathrm{rel}}_{X/S}$ with the prestack $\mathcal{P}^{s-\mathrm{rig}}$, called the *s-rigidification*, whose fiber category over an $S$-scheme $T$ is

$$\mathcal{P}^{s-\mathrm{rig}}(T) = \{(L, \alpha) \mid L \in \mathrm{Pic}(X_T) \text{ and } \alpha\colon \mathcal{O}_T \xrightarrow{\sim} s_T^* L\},$$

where a morphism $(L, \alpha) \sim (L', \alpha')$ is the data of an isomorphism $\beta\colon L \to L'$ of line bundles such that $\alpha' = s_T^*\beta \circ \alpha$.

The advantage of considering $\mathcal{P}^{s-\mathrm{rig}}$ is that it is a straightforward application of fppf descent of quasi-coherent sheaves to check that $\mathcal{P}^{s-\mathrm{rig}}$ is a stack over the big fppf topology $(\mathrm{Sch}/S)_{\mathrm{fppf}}$. To show that $\underline{\mathrm{Pic}}^{\mathrm{rel}}_{X/S}$ is a sheaf, we can therefore verify that the natural map

$$\mathcal{P}^{s-\mathrm{rig}} \to \underline{\mathrm{Pic}}^{\mathrm{rel}}_{X/S}, \qquad (L,\alpha) \mapsto L \tag{6.2.3}$$

is an equivalence.

We first check that $\mathcal{P}^{s-\mathrm{rig}}$ is equivalent to a functor, i.e. the functor (6.2.3) is faithful. We must show that if $\beta\colon L \xrightarrow{\sim} L$ is an automorphism with $s_T^*\beta = \mathrm{id}_{s_T^* L}$, then $\beta = \mathrm{id}_L$. Since $\mathcal{O}_T \to f_{T,*}\mathcal{O}_{X_T}$ is an isomorphism, the pullback map $f_T^*\colon \mathrm{H}^0(T,\mathcal{O}_T) \to \mathrm{H}^0(X_T,\mathcal{O}_{X_T})$ is an isomorphism; as $s_T$ is a section, its inverse is given by $s_T^*$. The composition

$$s_T^*\colon \mathrm{Hom}_{\mathcal{O}_{X_T}}(L,L) \cong \mathrm{H}^0(X_T,\mathcal{O}_{X_T}) \xrightarrow{s_T^*} \mathrm{H}^0(T,\mathcal{O}_T) \cong \mathrm{Hom}_{\mathcal{O}_T}(s_T^*L, s_T^*L)$$

is an isomorphism of groups, and thus $\beta = \mathrm{id}_L$.

To see that (6.2.3) is full (i.e. the induced map on the functors of isomorphism classes is injective), let $(L,\alpha) \in \mathcal{P}^{s-\mathrm{rig}}(T)$ be an element such that there is a line bundle $M$ on $T$ and an isomorphism $\beta\colon L \xrightarrow{\sim} f_T^*M$. The isomorphism

$$\gamma\colon \mathcal{O}_{X_T} = f_T^*\mathcal{O}_T \xrightarrow{f_T^*\alpha} f_T^*s_T^*L \xrightarrow{f_T^*s_T^*\beta} f_T^*s_T^*f_T^*M = f_T^*M$$

satisfies $s_T^*\gamma = s_T^*\beta\circ\alpha$ and shows that $(L,\alpha)$ is isomorphic to $(\mathcal{O}_{X_T},\mathrm{id}) \in \mathcal{P}^{s-\mathrm{rig}}(T)$.

Finally, to see that the functor (6.2.3) is essentially surjective, let $L \in \mathrm{Pic}(X_T)$ and define

$$L' = L \otimes (f_T^*s_T^*L)^\vee.$$

The images of $L$ and $L'$ are equal in $\underline{\mathrm{Pic}}^{\mathrm{rel}}_{X/S}(T)$, and $s_T^*L' \cong s_T^*L \otimes (s_T^*f_T^*s_T^*L)^\vee \cong \mathcal{O}_T$ defines an isomorphism $\alpha'\colon \mathcal{O}_T \xrightarrow{\sim} s_T^*L'$ such that $L \in \underline{\mathrm{Pic}}^{\mathrm{rel}}_{X/S}(T)$ is the image of $(L',\alpha') \in \mathcal{P}^{s-\mathrm{rig}}(T)$. $\square$

Over an algebraically closed field $\Bbbk$, it is remarkably easy to verify that the Picard functor $\underline{\mathrm{Pic}}_X := \underline{\mathrm{Pic}}_{X/\Bbbk}$ is a scheme.

**Theorem 6.2.55.** *Let $X$ be a proper integral scheme over an algebraically closed field $\Bbbk$.*

*(1) $\underline{\mathrm{Pic}}_X$ is an algebraic group over $\Bbbk$, and in particular a disjoint union of quasi-projective schemes.*

*(2) $\underline{\mathrm{Pic}}_X \cong \underline{\mathrm{Pic}}^{\mathrm{rel}}_X$ and $\underline{\mathcal{P}\mathrm{ic}}_X \to \underline{\mathrm{Pic}}_X$ is a banded $\mathbb{G}_m$-gerbe,*

*(3) If $X$ is smooth, the connected component of the identity $\underline{\mathrm{Pic}}^0_X$ is projective.*

*(4) If $\mathrm{char}(\Bbbk) = 0$, then $\underline{\mathrm{Pic}}_X$ is smooth of dimension $\mathrm{h}^0(X,\mathcal{O}_X)$. In particular, $\underline{\mathrm{Pic}}^0_X$ is an abelian variety.*

*Proof.* As $\Bbbk$ is algebraically closed and $X$ is integral, the structure map $f\colon X \to \mathrm{Spec}\,\Bbbk$ has a section and $\mathcal{O}_T \xrightarrow{\sim} f_{T,*}\mathcal{O}_{X_T}$ is an isomorphism for any $\Bbbk$-scheme $T$. Proposition 6.2.53 implies that $\underline{\mathrm{Pic}}_X$ is an algebraic space locally of finite type over $\Bbbk$, and that (2) holds. The Picard stack $\underline{\mathcal{P}\mathrm{ic}}_X$ is quasi-separated (it even has affine diagonal) and it follows that $\underline{\mathrm{Pic}}_X$ is also quasi-separated. This is enough to show that $\underline{\mathrm{Pic}}_X$ is a separated scheme and that $\underline{\mathrm{Pic}}^0_X$ is quasi-projective. It is a direct consequence of Theorem 4.4.26, but it's worth recalling the argument: $\underline{\mathrm{Pic}}_X$ has a dense open subspace which is a scheme (Theorem 4.4.1), the group structure $\underline{\mathrm{Pic}}_X$ allows us to translate this open to show that $\underline{\mathrm{Pic}}_X$ is an algebraic group, which is automatically separated with quasi-projective connected components (Proposition C.3.1). This gives (1).

For (3), it suffices to show that $\underline{\mathrm{Pic}}^0_X$ is proper. As we already know it is separated, we only need to verify the existence part of the valuative criterion for properness: let $R$ be a DVR over $\Bbbk$ with fraction field $K$ and $L$ be a line bundle on $X_K$. As $X_R$ is regular, the line bundle $L$ extends to a line bundle $\widetilde{L}$ on $X_R$ (for instance, if $L = \mathcal{O}(D)$ for a divisor $D \subset X_K$, then take $\widetilde{L} = \mathcal{O}(\overline{D})$).

The smoothness in (4) follows from the fact that algebraic groups are smooth in characteristic 0 (Proposition C.3.1). If $L$ is a line bundle on $X$, then the Zariski tangent spaces of the Picard stack and Picard scheme agree, and deformation theory (Proposition D.1.15) implies that $T_{\underline{\mathrm{Pic}}_X, L} \cong \mathrm{H}^1(X, \mathcal{O}_X)$. □

**Remark 6.2.56.** As a consequence of the representability of $\underline{\mathrm{Pic}}_X \cong \underline{\mathrm{Pic}}^{\mathrm{rel}}_X$, there is a *universal family* (or *Poincaré family*) $\mathcal{P}$ on $X \times \underline{\mathrm{Pic}}_X$ that satisfies the following: for any $\Bbbk$-scheme $T$ and any line bundle $L$ on $X_T$, there is a unique morphism $T \to \underline{\mathrm{Pic}}_X$ such that

$$L \cong \mathcal{P}|_{X \times T} \otimes p_2^* M$$

for some line bundle $M$ on $T$.

The connected component of the identity $\underline{\mathrm{Pic}}^0_X$ has the functorial description of parameterizing line bundles $L$ *algebraically equivalent* to $\mathcal{O}_X$ (i.e. there is a connected $\Bbbk$-scheme $T$ with points $t_0, t_1 \in T(\Bbbk)$ and a family of line bundles $\mathcal{L}$ on $X_T$ such that $L_{t_0} \cong L$ and $L_{t_1} \cong \mathcal{O}_X$). When $X$ is a smooth curve, $\underline{\mathrm{Pic}}^0_X$ parameterizes degree 0 line bundles.

**Remark 6.2.57.** The theorem's conclusion holds if $X$ is an integral proper *algebraic space*. The only missing ingredient is the algebraicity of the Picard stack, and this can be shown using Artin's Axioms similar to Exercise D.7.8.

In characteristic $p$, Igusa showed that $\mathrm{Pic}_X$ may fail to be reduced [Igu55]. We also note that when $X$ is not normal (e.g. a nodal or cuspidal curve), then $\underline{\mathrm{Pic}}^0_X$ is not projective. Altman and Kleiman [AK80] provide a compactification of $\underline{\mathrm{Pic}}^0_X$ by classifying rank 1 torsion free sheaves.

Picard functors and schemes have a fascinating history as they were one of the first examples of moduli spaces constructed in algebraic geometry. See Kleiman's article [Kle05] for a beautiful account of the history and a broader discussion of the properties of Picard schemes.

## 6.3 Affine Geometric Invariant Theory and good moduli spaces

Good moduli spaces capture the stack-intrinsic properties of quotients that appear in Geometric Invariant Theory (GIT). In the affine case, GIT concerns the action of a linearly reductive group on an affine scheme. Recall that an affine algebraic group $G$ over a field $\Bbbk$ is *linearly reductive* if the functor $\mathrm{Rep}(G) \to \mathrm{Vect}_{\Bbbk}$, taking a $G$-representation $V$ to its $G$-invariants $V^G$, is exact. Examples include:

- finite discrete groups $G$ whose order is not divisible by $\mathrm{char}(\Bbbk)$ (Maschke's Theorem (C.4.4));
- tori $\mathbb{G}_m^n$ and diagonalizable group schemes (Proposition C.1.13); and
- reductive algebraic groups (e.g. $\mathrm{GL}_n$, $\mathrm{SL}_n$ and $\mathrm{PGL}_n$) in $\mathrm{char}(\Bbbk) = 0$ (Theorem C.4.7).

See §C.4 for further equivalences, properties, and a discussion of linearly reductive groups.

Given an action of $G$ on an affine $\Bbbk$-scheme $\operatorname{Spec} A$, the inclusion $A^G \hookrightarrow A$ induces a commutative diagram

$$\begin{tikzcd} \operatorname{Spec} A \arrow[d] \arrow[dr, "\widetilde{\pi}"] & \\ {[\operatorname{Spec} A/G]} \arrow[r, "\pi"] & \operatorname{Spec} A^G. \end{tikzcd}$$

Let's observe the following two properties of $\pi\colon [\operatorname{Spec} A/G] \to \operatorname{Spec} A^G$:

(1) $\Gamma([\operatorname{Spec} A/G], \mathcal{O}_{[\operatorname{Spec} A/G]}) = A^G$; this follows from the definition of global sections.

(2) The functor $\pi_*\colon \operatorname{QCoh}([\operatorname{Spec} A/G]) \to \operatorname{QCoh}(\operatorname{Spec} A^G)$ is exact. This holds because functor $\pi_*$ takes a quasi-coherent $\mathcal{O}_{\mathcal{X}}$-module $\widetilde{M}$, corresponding to an $A$-module $M$ with a $G$-action, to $\widetilde{M^G}$ (Exercise 6.1.3) and is therefore exact by the defining property of linear reductivity.

In this case, following the terminology of Mumford and Seshadri, we say that $\operatorname{Spec} A \to \operatorname{Spec} A^G$ is a *good quotient* or *GIT quotient*, and $\operatorname{Spec} A^G$ is sometimes denoted as $(\operatorname{Spec} A)/\!\!/G$. See §6.7 for a more general discussion of good quotients and the projective case of GIT.

### 6.3.1 Good moduli spaces

The definition of a good moduli space is inspired by properties of GIT quotients and specifically properties of the morphisms $\pi\colon [\operatorname{Spec} A/G] \to \operatorname{Spec} A^G$ and $\pi\colon [X^{\mathrm{ss}}/G] \to X^{\mathrm{ss}}/\!\!/G := \operatorname{Proj} \bigoplus_{d\geq 0} \Gamma(X, \mathcal{O}_X(d))^G$, where $G$ is linearly reductive and $X \subset \mathbb{P}(V)$ is a $G$-invariant closed subscheme of a projectivized $G$-representation.

**Definition 6.3.1** (Good moduli spaces)**.** A quasi-compact and quasi-separated morphism $\pi\colon \mathcal{X} \to X$ from an algebraic stack $\mathcal{X}$ to an algebraic space $X$ is a *good moduli space* if

(1) $\mathcal{O}_X \to \pi_*\mathcal{O}_{\mathcal{X}}$ is an isomorphism, and

(2) $\pi_*\colon \operatorname{QCoh}(\mathcal{X}) \to \operatorname{QCoh}(X)$ is exact.

**Example 6.3.2** (Basic example: affine GIT)**.** If $G$ is a linearly reductive group over a field $\Bbbk$ acting on an affine $\Bbbk$-scheme $\operatorname{Spec} A$, then $[\operatorname{Spec} A/G] \to \operatorname{Spec} A^G$ is a good moduli space.

**Example 6.3.3** (Concrete examples)**.** If $\mathbb{G}_m$ acts on $\mathbb{A}^n$ over a field $\Bbbk$ via $t \cdot (x_1, \ldots, x_n) = (tx_1, \ldots, tx_n)$, then $[\mathbb{A}^n/\mathbb{G}_m] \to \operatorname{Spec} \Bbbk$ is a good moduli space. Observe that a nonzero $\Bbbk$-point $[\mathbb{A}^n/\mathbb{G}_m]$ is not closed and contains 0 in its closures, or in other words every $\mathbb{G}_m$-orbit contains 0 in its closure. Note that $[\mathbb{A}^n/\mathbb{G}_m]\backslash 0 = \mathbb{P}^{n-1}$.

If $\mathbb{G}_m$ acts on $\mathbb{A}^2$ via $t \cdot (x, y) = (tx, t^{-1}y)$, then $[\mathbb{A}^2/\mathbb{G}_m] \to \operatorname{Spec} \Bbbk[xy] = \mathbb{A}^1$ is a good moduli space. The fiber over $a \neq 0 \in \mathbb{A}^1$ under the good quotient $\mathbb{A}^2 \to \mathbb{A}^1$ is the hyperbola $xy = a$ in $\mathbb{A}^2$ and the fiber under the good moduli space $[\mathbb{A}^2/\mathbb{G}_m] \to \mathbb{A}^1$ is the point $\operatorname{Spec} \Bbbk \cong [V(xy - a)/\mathbb{G}_m]$. The fiber over the origin is the union of the three orbits $\{(x, 0) | x \neq 0\} \cup \{(0, y) | y \neq 0\} \cup \{0, 0\}$ in $\mathbb{A}^2$. Note that $[\mathbb{A}^2/\mathbb{G}_m] \setminus 0 = \mathbb{A}^1 \bigcup_{\mathbb{A}^1\setminus 0} \mathbb{A}^1$ is the non-separated affine line

**Example 6.3.4** (Tame coarse moduli spaces)**.** If $\mathcal{X}$ is a separated Deligne–Mumford stack of finite type over a noetherian scheme $S$, then the Keel–Mori Theorem (4.3.11) implies that there exists a coarse moduli space $\pi\colon \mathcal{X} \to X$. We say that the coarse moduli space $\mathcal{X} \to X$ is *tame* if every automorphism group has order prime to the characteristic, i.e. invertible in $\Gamma(S, \mathcal{O}_S)$. A tame coarse moduli space is a good moduli space. Indeed, this will follow from the fact that the property of being a good moduli space is local on the base in the étale topology (Lemma 6.3.20) and the Local Structure of Coarse Moduli Spaces (4.3.14). If $\mathcal{X}$ has quasi-finite stabilizers, then in fact every good moduli space $\pi\colon \mathcal{X} \to X$ is a coarse moduli space and $\pi$ is separated; see Proposition 6.3.28.

The goal of this section is to establish the following theorem.

**Theorem 6.3.5.** *Let $\pi\colon \mathcal{X} \to X$ be a good moduli space where $\mathcal{X}$ is a quasi-separated algebraic stack defined over an algebraic space $S$. Then*

*(1) $\pi$ is surjective and universally closed;*

*(2) For closed substacks $\mathcal{Z}_1, \mathcal{Z}_2 \subset \mathcal{X}$, $\mathrm{im}(\mathcal{Z}_1 \cap \mathcal{Z}_2) = \mathrm{im}(\mathcal{Z}_1) \cap \mathrm{im}(\mathcal{Z}_2)$. For geometric points $x_1, x_2 \in \mathcal{X}(\Bbbk)$, $\pi(x_1) = \pi(x_2) \in X(\Bbbk)$ if and only if $\overline{\{x_1\}} \cap \overline{\{x_2\}} \neq \emptyset$ in $|\mathcal{X} \times_S \Bbbk|$. In particular, $\pi$ induces a bijection between closed points in $\mathcal{X}$ and closed points in $X$;*

*(3) If $\mathcal{X}$ is noetherian, so is $X$. If $\mathcal{X}$ is of finite type over $S$ and $S$ is noetherian, then $X$ is of finite type over $S$ and $\pi_*$ preserves coherence, i.e. for $F \in \mathrm{Coh}(\mathcal{X})$, $\pi_* F \in \mathrm{Coh}(X)$; and*

*(4) If $\mathcal{X}$ is noetherian, then $\pi$ is universal for maps to algebraic spaces.*

**Remark 6.3.6.** In (2), the images and intersections are taken scheme-theoretically. Note that since $\pi$ is closed, the set-theoretic image of a closed substack $\mathcal{Z}$ is identified with the topological space of its scheme-theoretic image $\mathrm{im}(\mathcal{Z})$. If $I \subset \mathcal{O}_{\mathcal{X}}$ is the sheaf of ideals defining $\mathcal{Z}$, the image $\mathrm{im}(\mathcal{Z})$ is defined by $\pi_* I \subset \pi_* \mathcal{O}_{\mathcal{X}} = \mathcal{O}_X$.

In the case of affine GIT where we have a good moduli space $\pi\colon [\operatorname{Spec} A/G] \to \operatorname{Spec} A$ and a good quotient $\widetilde{\pi}\colon \operatorname{Spec} A \to \operatorname{Spec} A^G$, this theorem translates to:

**Corollary 6.3.7** (Affine GIT)**.** *Let $G$ be a linearly reductive algebraic group over an algebraically closed field $\Bbbk$. Then $\widetilde{\pi}\colon U = \operatorname{Spec} A \to U/\!\!/G := \operatorname{Spec} A^G$ satisfies: Then*

*(1) $\widetilde{\pi}$ is surjective and for every $G$-invariant closed subscheme $Z \subset U$, $\mathrm{im}(Z) \subset U/\!\!/G$ is closed. The same holds for the base change $T \to U/\!\!/G$ by a morphism from a scheme;*

*(2) For closed $G$-invariant closed subschemes $Z_1, Z_2 \subset U$, $\mathrm{im}(Z_1 \cap Z_2) = \mathrm{im}(Z_1) \cap \mathrm{im}(Z_2)$. In particular, for $x_1, x_2 \in X(\Bbbk)$, $\widetilde{\pi}(x_1) = \widetilde{\pi}(x_2)$ if and only if $\overline{Gx_1} \cap \overline{Gx_2} \neq \emptyset$ and $\widetilde{\pi}$ induces a bijection between closed $G$-orbits of $\Bbbk$-points in $U$ and $\Bbbk$-points of $U/\!\!/G$.*

*(3) If $A$ is noetherian, so is $A^G$. If $A$ is finite generated over $\Bbbk$, then $A^G$ is also finitely generated over $\Bbbk$ and for every finitely generated $A$-module $M$ with a $G$-action, $M^G$ is a finitely generated $A^G$-module; and*

*(4) If $A$ is noetherian, then $\widetilde{\pi}$ is universal for $G$-invariant maps to algebraic spaces.* □

**Remark 6.3.8.** If $Z \subset U = \operatorname{Spec} A$ is defined by a $G$-invariant ideal $I$, then (1) implies that $\pi(Z)$ is defined by $I^G \subset A^G$. If $Z_1, Z_2$ are defined by $G$-invariant ideals

$I_1, I_2 \subset A$, then (2) implies that $(I_1 + I_2)^G = I_1^G + I_2^G$. In particular, if $Z_1$ and $Z_2$ are disjoint, then so are $\operatorname{im}(Z_1)$ and $\operatorname{im}(Z_2)$ and we can write $1 = f_1 + f_2$ with $f_1 \in I_1^G$ and $f_2 \in I_2^G$; the function $f_1$ restricts to 0 on $Z_1$ and 1 on $Z_2$. We see that $G$-invariant functions separate disjoint $G$-invariant closed subschemes.

**Remark 6.3.9** (Hilbert's 14th problem)**.** Hilbert's 14th problem asks when the invariant ring $A^G$ is finitely generated. While it is not true for every group $G$, Hilbert showed it is true when $G$ is linearly reductive—this is what (3) above asserts. Hilbert's original argument in [Hil1890] is so elegant and played such an important role in the development of modern algebra that we reproduce it here. Our proof of Theorem 6.3.5(3)—while similar in spirit—will not be as explicit.

Let $f_1, \ldots, f_n$ be $\Bbbk$-algebra generators of $A$ and let $V \subset A$ be a finite dimensional $G$-invariant subspace containing each $f_i$ (Proposition C.3.3(1)). Then we have a surjection $\operatorname{Sym}^* V = \Bbbk[x_1, \ldots, x_m] \twoheadrightarrow A$ of $\Bbbk$-algebras with $G$-actions and we set $I = \ker(\Bbbk[x_1, \ldots, x_m] \to A)$. Since $G$ is linearly reductive, $A^G = (\Bbbk[x_1, \ldots, x_m]/I)^G = \Bbbk[x_1, \ldots, x_m]^G/I^G$ and we can assume that $A = \Bbbk[x_1, \ldots, x_m]$ is the polynomial ring so that $A^G$ is a graded $\Bbbk$-algebra whose degree 0 component is $\Bbbk$. It therefore suffices to show that the ideal $J_+ := \sum_{d>0} A_d^G \subset A^G$ is finitely generated since its generators would then generate $A^G$ as a $\Bbbk$-algebra.

Hilbert first showed that every ideal in $A = \Bbbk[x_1, \ldots, x_n]$ is finitely generated—this is what is referred to today as Hilbert's Basis Theorem and was developed by Hilbert precisely to make this argument. It follows that $J_+A \subset A$ is finitely generated by homogenous invariants $f_1, \ldots, f_n \in A^G$. We will show that they also generate $J_+$ as an ideal in $A^G$. For $f \in A_d^G$, we can write

$$f = \sum_{i=1}^{n} f_i g_i \tag{6.3.1}$$

with $g_i \in A$ a homogeneous (not necessarily invariant) function of degree $d - \deg f_i$ (with $g_i = 0$ if $\deg f_i > d$). Since $G$ is linearly reductive, there is a $\Bbbk$-linear map $R\colon A \to A^G$ called the *Reynolds operator* (see Remark C.4.6), which is the identity on $A^G$, respects the grading, and satisfies $R(xy) = xR(y)$ for $x \in A^G$ and $y \in A$. Applying $R$ to (6.3.1) shows that $f = R(f) = \sum_i f_i R(g_i)$ with $R(g_i) \in A^G$ and thus $f$ lies in the ideal in $A^G$ generated by the $f_i$.[1]

Hilbert gave a constructive proof of this theorem in [Hil1893], which required the development of the Syzygy Theorem, the Nullstellensatz, a version of Noether normalization, and a version of the Hilbert–Mumford criterion. We strongly encourage you to read [Hil1890] and [Hil1893] (or Hilbert's translated lecture notes [Hil93]).

**Remark 6.3.10** (Reductivity in positive characteristic)**.** In characteristic $p$, every smooth linear reductive group is an extension of a torus by a finite group prime to the characteristic. In particular, $\mathrm{GL}_n$ is not linearly reductive (see Example C.4.8). In characteristic $p$, there are the following variant notions for an affine algebraic group $G$ over an algebraically closed field $\Bbbk$:

(1) $G$ *is reductive* if $G$ is smooth and every smooth, connected, unipotent, and normal subgroup of $G$ is trivial, and

(2) $G$ *is geometrically reductive* if for every surjection $V \to W$ of $G$-representations and $w \in W^G$, there exists $n > 0$ such that $w^{p^n}$ is in the image of $\operatorname{Sym}^{p^n} V \to \operatorname{Sym}^{p^n} W$.

[1] For an alternative argument that $A^G$ is noetherian, linear reductivity can be used to show that $JA \cap A^G = J$ for every ideal $J \subset A^G$ (see Lemma 6.3.22(5)). If $J_1 \subset J_2 \subset \cdots \subset A^G$ is an ascending chain of ideals, then the ascending chain $J_1 A \subset J_2 A \subset \cdots \subset A$ terminates, which implies that the original sequence $J_1 = J_1 A \cap A^G \subset J_2 = J_2 A \cap A^G \subset \cdots \subset A^G$ also terminates.

It is a deep theorem due to Haboush [Hab75] that these notions are equivalent when $G$ is smooth. See also §C.4.2-C.4.3 for further properties, equivalences, and discussion.

Geometric reductivity (sometimes called semi-reductivity) was introduced by Mumford in [GIT, preface] in an effort to extend GIT—originally developed for linearly reductive groups—to reductive groups in positive characteristic. Indeed, it is precisely the geometric reductivity property that yield the same geometric properties that we saw for affine GIT quotients by linearly reductive groups: if $G$ is geometrically reductive acting on an affine $\Bbbk$-scheme $\operatorname{Spec} A$, then $\widetilde{\pi}\colon \operatorname{Spec} A \to \operatorname{Spec} A^G$ satisfies Corollary 6.3.7(1)-(4) (with the exception that the noetherianess of $A$ does not necessarily imply the noetherianess of $A^G$). The arguments are not substantially more complicated than the linearly reductive case. See [Nag64], [MFK94, App. 1.C], [New78, §3], [Dol03, §3.4], [Spr77, §2] and [DC71, §2].

Likewise, the notion of a good moduli space can be extended to characterize quotients by geometrically reductive groups: in [Alp14], a quasi-compact and quasi-separated morphism $\pi\colon \mathcal{X} \to X$, from an algebraic stack to an algebraic space, is called an *adequate moduli space* if (1) $\mathcal{O}_X \to \pi_*\mathcal{O}_{\mathcal{X}}$ is an isomorphism and (2) for every surjection $\mathcal{A} \to \mathcal{B}$ of quasi-coherent $\mathcal{O}_{\mathcal{X}}$-algebras, then every section $s$ of $\pi_*(\mathcal{B})$ over a smooth morphism $\operatorname{Spec} A \to \mathcal{Y}$ has a positive power that lifts to a section of $\pi_*(\mathcal{A})$). An adequate moduli space satisfies Theorem 6.3.5(1)-(4) (except again for the noetherian implication). If $G$ is geometrically reductive, then $\pi\colon [\operatorname{Spec} A/G] \to \operatorname{Spec} A^G$ is an adequate moduli space. In characteristic 0, an adequate moduli space is necessarily good.

In this book, we restrict to linearly reductive groups and good moduli spaces since the proofs of the basic properties are more elementary in this case and probably best seen first. In addition, there is currently no analogue of the Local Structure Theorem for Algebraic Stacks (6.5.1) around points with reductive stabilizers.

### 6.3.2 Cohomologically affine morphisms

The exactness condition on the pushforward $\pi_*$ in the definition of a good moduli space (Definition 6.3.1(2)) is a non-representable analogue of affineness.

**Definition 6.3.11** (Cohomologically affine)**.** A quasi-compact and quasi-separated morphism $f\colon \mathcal{X} \to \mathcal{Y}$ of algebraic stacks is *cohomologically affine* if

$$f_*\colon \operatorname{QCoh}(\mathcal{X}) \to \operatorname{QCoh}(\mathcal{Y})$$

is exact. A quasi-compact and quasi-separated algebraic stack $\mathcal{X}$ is *cohomologically affine* if $\mathcal{X} \to \operatorname{Spec} \mathbb{Z}$ is.

**Example 6.3.12.** An affine algebraic group $G$ over a field $\Bbbk$ is linearly reductive (Definition C.4.1) if and only if $\mathbf{B}G$ is cohomologically affine.

**Remark 6.3.13.** By Serre's Criterion for Affineness (4.4.16), an algebraic space is cohomologically affine if and only if it is an affine scheme. An algebraic stack $\mathcal{X}$ with affine diagonal is cohomologically affine if and only if $\mathrm{H}^i(\mathcal{X}, F) = 0$ for all $i > 0$ and every quasi-coherent sheaf $F$; this follows because the cohomology $\mathrm{H}^i(\mathcal{X}, F)$ can be computed in $\operatorname{QCoh}(\mathcal{X})$ for such stacks $\mathcal{X}$ by Proposition 6.1.29(2). This is not true for algebraic stacks with non-affine diagonal, e.g. $\mathbf{B}E$ for an elliptic curve $E$.

Likewise, a morphism $f\colon \mathcal{X} \to \mathcal{Y}$ of algebraic stacks, with both $\mathcal{X}$ and $\mathcal{Y}$ having affine diagonal, is cohomologically affine if and only if $\mathrm{R}^i f_*(F) = 0$ for all $i > 0$ and every quasi-coherent sheaf $F$. If in addition $f$ is representable, then $f$ is cohomologically affine if and only if it is affine (see Corollary 6.3.16 below).

**Remark 6.3.14** (Noetherian case)**.** If $\mathcal{X}$ is noetherian, then a quasi-compact, quasi-separated morphism $f\colon \mathcal{X} \to \mathcal{Y}$ is cohomologically affine if and only if $f_*\colon \operatorname{Coh}(\mathcal{X}) \to \operatorname{QCoh}(\mathcal{Y})$ is exact. This holds because every quasi-coherent sheaf is a colimit of coherent sheaves (Proposition 6.1.8) and $f_*$ commutes with colimits. Since cohomology also commutes with colimits (Proposition 6.1.31), a morphism $f\colon \mathcal{X} \to \mathcal{Y}$ of noetherian algebraic stacks, both with affine diagonal, is cohomologically affine if and only if $\mathrm{R}^i f_*(F) = 0$ for all $i > 0$ and every *coherent* sheaf $F$.

**Lemma 6.3.15.** *Consider a cartesian diagram*

$$\begin{tikzcd} \mathcal{X}' \arrow[r, "g'"] \arrow[d, "\pi'"] & \mathcal{X} \arrow[d, "\pi"] \\ \mathcal{Y}' \arrow[r, "g"] & \mathcal{Y}. \end{tikzcd}$$

*of algebraic stacks.*

*(1) If $g$ is faithfully flat and $\pi'$ is cohomologically affine, then $\pi$ is cohomologically affine.*

*(2) If $\mathcal{Y}$ has quasi-affine diagonal (e.g. a quasi-separated algebraic space) and $\pi$ is cohomologically affine, then $\pi'$ is cohomologically affine.*

*Proof.* For (1), by Flat Base Change (6.1.7) there is an equivalence $g^*\pi_* \simeq \pi'_* g'^*$ of functors defined on categories of quasi-coherent sheaves. Since $\pi'_*$ and $g'^*$ are exact and $g^*$ is faithfully exact, $\pi_*$ is exact.

For (2), we first show that if $g$ is quasi-affine and $\pi$ is cohomologically affine, then $\pi'$ is also cohomologically affine. It suffices to handle the cases that $g$ is an open immersion and $g$ is affine. If $g$ is an open immersion and $F' \twoheadrightarrow G'$ is a surjection in $\operatorname{QCoh}(\mathcal{X}')$, we define $G = \operatorname{im}(g'_* F' \to g'_* G')$. Note that $g'^* G \cong G'$. Since $\pi_*$ is exact, $\pi_* g'_* F \twoheadrightarrow \pi_* G$. If we apply $g^*$ and use the identifies $g^*\pi_* \simeq \pi'_* g'^*$ and $g'^* g'_* \simeq \mathrm{id}$, we obtain a surjection $\pi'_* F' \twoheadrightarrow \pi'_* g'^* G \cong \pi'_* G'$. On the other hand, if $g$ is affine then $g_*$ is faithfully exact. Since $\pi_*$ and $g'_*$ are are exact, the identity $g_* \pi'_* \simeq \pi_* g'_*$ implies that $\pi'_*$ is also exact. To show (2), we may assume that $\mathcal{Y}$ and $\mathcal{Y}'$ are quasi-compact and we can choose a smooth presentation $Y = \operatorname{Spec} A \to \mathcal{Y}$, which will be quasi-affine (since $\mathcal{Y}$ has quasi-affine diagonal). Then the base change $\mathcal{X}_Y \to Y$ of $\pi$ along $Y \to \mathcal{Y}$ is cohomologically affine. To check that the base change $\mathcal{X}'_Y \to \mathcal{Y}'_Y$ is cohomologically affine, it suffices by (1) to check this after base changing by a smooth presentation $Y' = \operatorname{Spec} A' \to \mathcal{Y}' \times_{\mathcal{Y}} Y$ but this holds as $Y' \to Y$ is affine. Since $\mathcal{X}'_Y \to \mathcal{Y}'_Y$ is cohomologically affine so is $\pi'\colon \mathcal{X}' \to \mathcal{Y}'$ by invoking (1) again. □

**Corollary 6.3.16.** *Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a representable and cohomologically affine morphism of algebraic stacks where $\mathcal{Y}$ has quasi-affine diagonal, then $f$ is affine.*

*Proof.* Under the hypotheses, both affine and cohomologically affine morphisms descend under faithfully flat morphisms, and we can reduce to the case where $\mathcal{X}$ is an algebraic space and $\mathcal{Y}$ is an affine scheme which is Serre's Criterion for Affineness (4.4.16). □

### 6.3.3 Properties of linearly reductive groups

Recall that an affine algebraic group $G$ over a field $\Bbbk$ is *linearly reductive* if the functor $\operatorname{Rep}(G) \to \operatorname{Vect}_{\Bbbk}$, defined by $V \mapsto V^G$, is exact (Definition C.4.1). This is equivalent to the map $\mathbf{B}G \to \operatorname{Spec} \Bbbk$ being cohomologically affine.

**Proposition 6.3.17.** *Let $1 \to K \to G \to Q \to 1$ be an exact sequence of affine algebraic groups over a field $\Bbbk$. Then $G$ is linearly reductive if and only if both $K$ and $Q$ are.*

*Proof.* We will use the cartesian diagram

$$\begin{tikzcd} Q \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathbf{B}K \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \operatorname{Spec} \Bbbk \arrow[d] \\ \operatorname{Spec} \Bbbk \arrow[r] & \mathbf{B}G \arrow[r] & \mathbf{B}Q \end{tikzcd}$$

of Exercise 2.3.32(c). To see ($\Rightarrow$), note that $\mathbf{B}K \to \mathbf{B}G$ is affine by descent since $Q$ is affine. Therefore the composition $\mathbf{B}K \to \mathbf{B}G \to \operatorname{Spec} \Bbbk$ is cohomologically affine and $K$ is linearly reductive. If $V$ is a $Q$-representation, then its pullback under $q\colon \mathbf{B}G \to \mathbf{B}Q$ is the $G$-representation induced by the projection $G \to Q$ and in particular $K$ acts trivially. On the other hand, the pushforward of a $G$-representation $W$ under $q\colon \mathbf{B}G \to \mathbf{B}Q$ is the $Q$-representation $W^K$. Thus, the adjunction $V \to q_* q^* V$ is an isomorphism and $\Gamma(\mathbf{B}Q, -) = \Gamma(\mathbf{B}G, q^* -)$ is exact.

For the converse, descent (Lemma 6.3.15(2)) implies that $\mathbf{B}G \to \mathbf{B}Q$ is cohomologically affine and thus so is the composition $\mathbf{B}G \to \mathbf{B}Q \to \operatorname{Spec} \Bbbk$. $\square$

**Proposition 6.3.18.** *Let $H$ be a linearly reductive algebraic group over an algebraically closed field $\Bbbk$. If $H$ acts freely on an affine scheme $U$ over $\Bbbk$, then the algebraic space quotient $U/H$ is affine.*

*Proof.* The algebraic space $U/H$ and the good quotient $\operatorname{Spec} A^H$ are both universal for maps to algebraic spaces Theorem 6.3.5(4). Alternatively, the composition $U/H \to \mathbf{B}H \to \operatorname{Spec} \Bbbk$ is an affine morphism followed by a cohomologically affine morphism. It follows from Serre's Criterion for Affineness (4.4.16) that $U/H$ is affine. $\square$

In particular, if $H$ is a linearly reductive subgroup of an affine algebraic group $G$, then the quotient $G/H$ is affine. Matsushima's Theorem provides a converse.

**Proposition 6.3.19** (Matsushima's Theorem)**.** *Let $G$ be a linearly reductive group over an algebraically closed field $\Bbbk$.*

*(1) A subgroup $H$ of $G$ is linearly reductive if and only if $G/H$ is affine.*

*(2) Given an action of $G$ on an algebraic space $U$ of finite type over $\Bbbk$ and a $\Bbbk$-point $u \in U$ with stabilizer $G_u$, then $G_u$ is linearly reductive if and only if the orbit $Gu$ is affine.*

*Proof.* Part (2) follows from (1) since $Gu = G/G_u$. For (1), the ($\Rightarrow$) implication follows from Proposition 6.3.18. For the converse, consider the cartesian diagram

$$\begin{tikzcd} G/H \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \operatorname{Spec} \Bbbk \arrow[d] \\ \mathbf{B}H \arrow[r] & \mathbf{B}G. \end{tikzcd}$$

If $G/H$ is affine, then by smooth descent $\mathbf{B}H \to \mathbf{B}G$ is affine and therefore $\mathbf{B}H \to \mathbf{B}G \to \operatorname{Spec} k$ is cohomologically affine, i.e. $H$ is linearly reductive. $\square$

### 6.3.4 First properties of good moduli spaces

**Lemma 6.3.20.** *Consider a cartesian diagram*

$$\begin{tikzcd} \mathcal{X}' \arrow[r, "g'"] \arrow[d, "\pi'"'] \arrow[dr, phantom, "\square"] & \mathcal{X} \arrow[d, "\pi"] \\ X' \arrow[r, "g"] & X \end{tikzcd}$$

*of algebraic stacks where $X$ and $X'$ are quasi-separated algebraic spaces.*

(1) *If $g$ is faithfully flat and $\pi'$ is a good moduli space, then $\pi$ is a good moduli space.*

(2) *If $\pi$ is a good moduli space, so is $\pi'$.*

(3) *For $F \in \mathrm{QCoh}(\mathcal{X})$ and $G \in \mathrm{QCoh}(X)$, the adjunction map $\pi_* F \otimes G \to \pi_*(F \otimes \pi^* G)$ is an isomorphism. In particular, the adjunction map $G \xrightarrow{\sim} \pi_* \pi^* G$ is an isomorphism.*

(4) *For $F \in \mathrm{QCoh}(\mathcal{X})$, then the adjunction map $g^* \pi_* F \xrightarrow{\sim} \pi'_* g'^* F$ is an isomorphism.*

(5) *For a quasi-coherent sheaf of ideals $J \subset \mathcal{O}_X$, the natural map $J \to \pi_*(\pi^{-1} J \cdot \mathcal{O}_{\mathcal{X}})$ is an isomorphism.*

*Proof.* If $g \colon X' \to X$ is flat, then the pullback of the natural map $\mathcal{O}_X \to \pi_* \mathcal{O}_{\mathcal{X}}$ under $g$ is the map $\mathcal{O}_{X'} \to \pi'_* \mathcal{O}_{\mathcal{X}'}$. Thus (1) and the case of (2) when $g$ is flat follows from Lemma 6.3.15 and descent. Note that since $X'$ is quasi-separated, it has quasi-affine diagonal (Corollary 4.4.8).

Before proving the general case of (2), we first prove (3). Choose an étale presentation $U \to X$ with $U$ the disjoint union of affine schemes. Since the base change $\pi_U \colon \mathcal{X}_U \to U$ is a good moduli space (by the flat case of (2)) and the adjunction map $\mathrm{id} \to \pi_* \pi^*$ pulls back to the adjunction map $\mathrm{id} \to \pi_{U,*} \pi_U^*$, we may assume that $X = \operatorname{Spec} A$ is affine. If $G_2 \to G_1 \to G \to 0$ is a free presentation, then the projection maps $\pi_* F \otimes G_i \to \pi_*(F \otimes \pi^* G_i)$ are isomorphisms. Since $\pi_* F \otimes -$ and $\pi_*(F \otimes \pi^* -)$ are right exact, we have a commutative diagram

$$\begin{tikzcd} \pi_* F \otimes G_2 \arrow[r] \arrow[d] & \pi_* F \otimes G_1 \arrow[r] \arrow[d] & \pi_* F \otimes G \arrow[r] \arrow[d] & 0 \\ \pi_*(F \otimes \pi^* G_2) \arrow[r] & \pi_*(F \otimes \pi^* G_1) \arrow[r] & \pi_*(F \otimes \pi^* G) \arrow[r] & 0 \end{tikzcd}$$

Since the left two vertical maps are isomorphisms, so is the right one.

For (2), we must show that $\mathcal{O}_{X'} \to \pi'_* \mathcal{O}_{\mathcal{X}'}$ is an isomorphism as Lemma 6.3.15(2) already established that $\pi'_*$ is exact. We can assume that $X$ and $X'$ are affine. In this case, $g_*$ is faithfully exact so it suffices to show that

$$g_* \mathcal{O}_{X'} \to g_* \pi'_* \mathcal{O}_{\mathcal{X}'} \cong \pi_* g'_* \mathcal{O}_{\mathcal{X}'} \cong \pi_* \pi^* g_* \mathcal{O}_{X'} \tag{6.3.2}$$

is an isomorphism, where the last equivalence uses the identity $g'_* \pi'^* \mathcal{O}_{X'} \cong \pi^* g_* \mathcal{O}_{X'}$ following from the affineness of $g$. Thus the composition (6.3.2) is the adjunction isomorphism of (3) applied to $F = g_* \mathcal{O}_{X'}$.

For (4), we know by Flat Base Change (6.1.7) that (4) is fppf local on $X$ and $X'$ and that it holds when $g$ is flat. We may therefore reduce to when $X' \to X$ is a morphism of affine schemes. By factoring $X' \to X$ as a closed immersion followed by a flat morphism, we can further reduce to the case that $X' \hookrightarrow X$ is a closed immersion defined by a quasi-coherent sheaf of ideals $J \subset \mathcal{O}_X$. We aim to show that $\pi_* F / J\pi_* F \cong \pi_*(F/(\pi^{-1}J \cdot \mathcal{O}_{\mathcal{X}})F)$. Using the exactness of $\pi_*$, this is equivalent to the inclusion $J\pi_* F \hookrightarrow \pi_*((\pi^{-1}J \cdot \mathcal{O}_{\mathcal{X}})F)$ being surjective. The sheaf $(\pi^{-1}J \cdot \mathcal{O}_{\mathcal{X}})F$ is the image of $\pi^* J \otimes F \to F$. By the exactness of $\pi_*$, the pushforward $\pi_*((\pi^{-1}J \cdot \mathcal{O}_{\mathcal{X}})F)$ is the image of $\pi_*(\pi^* J \otimes F) \to \pi_* F$, but by (3) this is identified with the image of $J \otimes \pi_* F \to \pi_* F$.

For (5), if $Z \subset X$ is the closed subspace defined by $J$, then the preimage ideal sheaf $\pi^{-1}J \cdot \mathcal{O}_{\mathcal{X}}$ defines the preimage $\pi^{-1}(Z)$. The exactness of $\pi_*$ implies that there is a commutative diagram of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & J \arrow[r] \arrow[d] & \mathcal{O}_X \arrow[r] \arrow[d] & \mathcal{O}_Z \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \pi_*(\pi^{-1}J \cdot \mathcal{O}_{\mathcal{X}}) \arrow[r] & \pi_* \mathcal{O}_{\mathcal{X}} \arrow[r] & \pi_* \mathcal{O}_{\pi^{-1}(Z)} \arrow[r] & 0.
\end{tikzcd}$$

As $\mathcal{X} \to X$ and $\pi^{-1}(Z) \to Z$ are good moduli spaces, the right two vertical arrows are isomorphisms and so is the left arrow. ☐

**Remark 6.3.21.** The isomorphism $\pi_* F \otimes G \to \pi_*(F \otimes \pi^* G)$ in (3) is similar to the projection formula but holds even if $G$ is not locally free. It holds as long as $\pi$ is cohomologically affine.

**Lemma 6.3.22.** *Let $\pi \colon \mathcal{X} \to X$ be a good moduli space with $X$ quasi-separated.*

*(1) If $\mathcal{A}$ is a quasi-coherent sheaf of $\mathcal{O}_{\mathcal{X}}$-algebras, then $\mathcal{S}\mathrm{pec}_{\mathcal{X}}\, \mathcal{A} \to \mathcal{S}\mathrm{pec}_X\, \pi_* \mathcal{A}$ is a good moduli space.*

*(2) If $\mathcal{Z} \subset \mathcal{X}$ is a closed substack defined by a sheaf of ideals $I$ and $\operatorname{im} \mathcal{Z} \subset X$ is the scheme-theoretic image, i.e. the closed subspace defined by $\pi_* I \subset \mathcal{O}_X$, then $\mathcal{Z} \to \operatorname{im} \mathcal{Z}$ is a good moduli space.*

*Proof.* For (1), since $\mathcal{X} \times_X \mathcal{S}\mathrm{pec}_X\, \pi_* \mathcal{A} \to \mathcal{S}\mathrm{pec}_X\, \pi_* \mathcal{A}$ is cohomologically affine by Lemma 6.3.15 and $\mathrm{Spec}_{\mathcal{X}}\, \mathcal{A} \to \mathcal{X} \times_X \mathcal{S}\mathrm{pec}_X\, \pi_* \mathcal{A}$ is affine, it follows that $\mathcal{S}\mathrm{pec}_{\mathcal{X}}\, \mathcal{A} \to \mathcal{S}\mathrm{pec}_X\, \pi_* \mathcal{A}$ is cohomologically affine and therefore a good moduli space as the push forward of $\mathcal{O}_{\mathcal{S}\mathrm{pec}_{\mathcal{X}} \mathcal{A}}$ is $\mathcal{O}_{\mathcal{S}\mathrm{pec}_X \pi_* \mathcal{A}}$ by construction. Applying (1) to $\mathcal{Z} = \mathcal{S}\mathrm{pec}_{\mathcal{X}}(\mathcal{O}_{\mathcal{X}}/I)$ recovers (2) using that $\pi_*(\mathcal{O}_{\mathcal{X}}/I) = \mathcal{O}_X / \pi_* I$. ☐

The above lemmas allow us to give quick proofs of the first two parts of Theorem 6.3.5.

*Proof of Theorem 6.3.5(1).* As $\mathcal{X}$ is quasi-separated, so is $X$. For every field-valued point $x \in X(\Bbbk)$, consider the base change $\mathcal{X} \times_X \operatorname{Spec} \Bbbk$. By Lemma 6.3.20(2), $\mathcal{X}_x \to \operatorname{Spec} \Bbbk$ is a good moduli space and in particular $\Gamma(\mathcal{X}_x, \mathcal{O}_{\mathcal{X}_x}) = \Bbbk$. It follows that $\mathcal{X}_x$ is non-empty and that $\pi \colon \mathcal{X} \to X$ is surjective. For a closed substack $\mathcal{Z} \subset \mathcal{X}$, Lemma 6.3.22(2) implies that $\mathcal{Z} \to \operatorname{im} \mathcal{Z}$ is a good moduli space and therefore also surjective. Thus, the set-theoretic image $\pi(\mathcal{Z})$ is identified with the scheme-theoretic image $\operatorname{im} \mathcal{Z}$ and is therefore closed. Since good moduli spaces are stable under base change, they are universally closed. ☐

*Proof of Theorem 6.3.5(2).* For two substacks $\mathcal{Z}_1, \mathcal{Z}_2 \subset X$ defined by ideal sheaves $I_1, I_2 \subset \mathcal{O}_X$, we apply the exact functor $\pi_*$ to the short exact sequence $0 \to I_1 \to$

$I_1 + I_2 \to I_2/I_1 \cap I_2 \to 0$ and surjection $I_2 \to I_2/I_1 \cap I_2$ to obtain a commutative diagram

$$\begin{tikzcd} & & \pi_* I_2 \arrow[d] \arrow[dr, two heads] & & \\ 0 \arrow[r] & \pi_* I_1 \arrow[r] & \pi_*(I_1 + I_2) \arrow[r, two heads] & \pi_* I_2/\pi_*(I_1 \cap I_2) \arrow[r] & 0. \end{tikzcd}$$

It follows that the natural inclusion $\pi_* I_1 + \pi_* I_2 \to \pi_*(I_1 + I_2)$ is surjective. $\square$

### 6.3.5 Finite typeness of good moduli spaces

We will use the fact that a good moduli space $\mathcal{X} \to X$ is universally submersive and show that finite typeness descends under universally submersive morphisms. Recall from §A.4.3 that a morphism $f\colon X \to Y$ of schemes is *universally submersive* if $f$ is surjective and $Y$ has the quotient topology, and these properties are stable under base change. This notion extends to morphisms of algebraic stacks. Examples of universally submersive morphisms include fppf morphisms and universally closed morphisms.

**Proposition 6.3.23.** *Consider a commutative diagram*

$$\begin{tikzcd} X \arrow[d] \arrow[dr] & \\ Y \arrow[r] & S \end{tikzcd}$$

*of noetherian schemes where $X \to Y$ is universally submersive. If $X \to S$ is of finite type, then so is $Y \to S$.*

*Proof.* We can assume that $S = \operatorname{Spec} R$ and $Y = \operatorname{Spec} B$ are affine. Since a noetherian ring $B$ is of finite type over $R$ if and only if the reductions of the irreducible components of $\operatorname{Spec} B$ are of finite type over $R$, we can assume that $B$ is an integral domain.

By Generic Flatness (A.2.11) and Raynaud-Gruson Flatification (A.2.16), there is a commutative diagram

$$\begin{tikzcd} \widetilde{X} \arrow[r] \arrow[d] & X \arrow[d, "f"] \arrow[dr] & \\ Y' \arrow[r, "g"] & Y \arrow[r] & S \end{tikzcd}$$

where $\widetilde{X} \to Y'$ is flat, $Y' = \operatorname{Bl}_I Y \to Y$ is the blow-up along an ideal $I \subset B$ and $\widetilde{X}$ is the strict transform of $X$, i.e. the closure of $(Y' \setminus g^{-1}(V(I))) \times_Y X$ in the base change $Y' \times_Y X$. We claim that $\widetilde{X} \to Y'$ is surjective. As $g\colon Y' \to Y$ is an isomorphism over $U = Y \setminus V(I)$ and $f\colon X \to Y$ is surjective, we know that $g^{-1}(U) \subset Y'$ is contained in the image. If $y' \in Y'$ is a point, we can choose a map $\operatorname{Spec} R \to Y'$ from a DVR whose generic point maps to $g^{-1}(U)$ and whose special point maps to $y'$. Since $X \to Y$ is universally submersive, there exists an extension of DVRs $R \to R'$ and a lift $\operatorname{Spec} R' \to X$ (see Exercise A.4.7). The induced map $\operatorname{Spec} R' \to X \times_Y Y'$ factors through $\widetilde{X}$, and we see that $y'$ is thus in the image of $\widetilde{X}$.

Since $X$ is of finite type over $S$, so is $\widetilde{X}$. Using faithfully flat descent, $Y' \to S$ is also of finite type. To show that $Y \to S$ is of finite type, we may choose generators

$a_1, \ldots, a_n \in I$ so that $Y' = \bigcup_i \operatorname{Spec} B_i$ where $B_i = B\langle f_j/f_i \rangle \subset K = \operatorname{Frac}(B)$ is the subalgebra generated by $B$ and the elements $f_j/f_i$ for $j \neq i$. Write $B = \bigcup_\lambda B_\lambda$ as a union of its finitely generated $R$-subalgebras. For $\lambda \gg 0$, each $f_i \in B_\lambda$ and we set $I_\lambda = (f_1, \ldots, f_n) \subset B_\lambda$. Since $Y'$ is finite type over $B$, each $B_i$ is finitely generated over $B$ and thus for $\lambda \gg 0$, we see that in the diagram

$$\begin{tikzcd} B_{\lambda,i} \arrow[r, equal] \arrow[d, hook] & B_\lambda\langle f_j/f_i \rangle \arrow[r, hook] & \operatorname{Frac}(B_\lambda) \arrow[d, hook] \\ B_i \arrow[r, equal] & B\langle f_j/f_i \rangle \arrow[r, hook] & \operatorname{Frac}(B) \end{tikzcd}$$

the inclusion $B_{\lambda,i} \hookrightarrow B_i$ is surjective. It follows that $Y' = \operatorname{Bl}_I \operatorname{Spec} B = \operatorname{Bl}_{I_\lambda} \operatorname{Spec} B_\lambda$ for $\lambda \gg 0$. Considering the composition

$$g_\lambda \colon Y' \xrightarrow{g} Y = \operatorname{Spec} B \xrightarrow{p_\lambda} \operatorname{Spec} B_\lambda,$$

the push forward of the injection $\mathcal{O}_Y \hookrightarrow g_*\mathcal{O}_{Y'}$ along $p_\lambda$ yields an inclusion $p_{\lambda,*}\mathcal{O}_Y \hookrightarrow g_{\lambda,*}\mathcal{O}_{Y'}$. But $g_{\lambda,*}\mathcal{O}_{Y'}$ is a coherent module on $\operatorname{Spec} B_\lambda$ and thus so is $p_{\lambda,*}\mathcal{O}_Y$. This shows that $B$ is a finite $B_\lambda$-module and thus finitely generated as an $R$-algebra. □

We apply this proposition to show that good moduli spaces are of finite type.

*Proof of Theorem 6.3.5(3).* If $\mathcal{X}$ is noetherian, if $I_1 \subset I_2 \subset \cdots$ is an ascending chain of ideal sheaves of $\mathcal{O}_X$, then $\pi^{-1}I_1 \cdot \mathcal{O}_\mathcal{X} \subset \pi^{-1}I_2 \cdot \mathcal{O}_\mathcal{X} \subset$ is an ascending chain of ideal sheaves of $\mathcal{O}_\mathcal{X}$ which terminates. By Lemma 6.3.22(5), $I_n = \pi_*(\pi^{-1}I_n \cdot \mathcal{O}_\mathcal{X})$ and therefore the chain $I_1 \subset I_2 \subset \cdots$ terminates and $X$ is noetherian.

Assume now that $S$ is noetherian and $\mathcal{X}$ is of finite type over $S$. As $\mathcal{X} \to X$ is universally closed (Theorem 6.3.5(1)), it is also universally submersive. Choose a smooth presentation $U \to \mathcal{X}$ from a scheme. Since $U \to \mathcal{X}$ is universally submersive, so is the composition $U \to \mathcal{X} \to X$. Since $U \to S$ is of finite type and $X$ is noetherian, Proposition 6.3.23 implies that $X \to S$ is also of finite type.

Given a coherent sheaf $F$ on $\mathcal{X}$, to show that the pushforward $\pi_*F$ is coherent, we may assume that $X = \operatorname{Spec} A$ is affine and that $\mathcal{X}$ is irreducible. We first handle the case when $\mathcal{X}$ is reduced. By noetherian induction, we can assume that $\pi_*F$ is coherent if $\operatorname{Supp}(F) \subsetneq \mathcal{X}$. The maximal torsion subsheaf $F_{\mathrm{tors}} \subset F$ has support strictly contained in $\mathcal{X}$. Using the exact sequence $0 \to F_{\mathrm{tors}} \to F \to F/F_{\mathrm{tors}} \to 0$ and the exactness of $\pi_*$, we see the coherence of $\pi_*(F/F_{\mathrm{tors}})$ implies the coherence of $\pi_*F$. In other words, we can assume that $F$ is torsion free. In this case, every section $s \colon \mathcal{O}_\mathcal{X} \to F$ is injective. We now argue by induction on the dimension of the vector space $\xi^*F$ where $\xi \colon \operatorname{Spec} K \to \mathcal{X}$ is a field-valued point whose image is the generic point. If $F$ has no sections, then $\pi_*F = 0$ is coherent. Otherwise, a section induces a short exact sequence $0 \to \mathcal{O}_\mathcal{X} \to F \to F/\mathcal{O}_\mathcal{X} \to 0$ and $\xi^*(F/\mathcal{O}_\mathcal{X})$ has strictly smaller dimension. By again appealing to the exactness of $\pi_*$, we see that the coherence of $\pi_*(F/\mathcal{O}_\mathcal{X})$ implies the coherence of $\pi_*F$. Finally, to reduce to the reduced case, let $I \subset \mathcal{O}_\mathcal{X}$ be the ideal sheaf defining $\mathcal{X}_{\mathrm{red}} \hookrightarrow \mathcal{X}$. Then for some $N > 0$, we have that $I^N = 0$. By examining the exact sequences $0 \to \pi_*(I^{k+1}F) \to \pi_*(I^kF) \to \pi_*(I^kF/I^{k+1}F) \to 0$ and using that $\pi_*(I^kF/I^{k+1}F)$ is coherent (since $I^kF/I^{k+1}F$ is supported on $\mathcal{X}_{\mathrm{red}}$)), we conclude by induction that $\pi_*F$ is coherent. □

### 6.3.6 Universality of good moduli spaces

We now complete the proof of Theorem 6.3.5 by showing that $\pi\colon \mathcal{X} \to X$ is universal for maps to algebraic spaces. Our argument follows the same logic for coarse moduli spaces in Theorem 4.3.6.

*Proof of Theorem 6.3.5(4).* We need to show that every diagram

$$\begin{tikzcd} \mathcal{X} \arrow[d, "\pi"'] \arrow[dr, "f"] & \\ X \arrow[r, dashed] & Y \end{tikzcd} \tag{6.3.3}$$

has a unique filling, or in other words that the natural map $\mathrm{Mor}(X,Y) \to \mathrm{Mor}(\mathcal{X},Y)$ is bijective.

The uniqueness follows as in the proof of Theorem 4.3.6 and uses only that $\pi\colon \mathcal{X} \to X$ is universally closed, schematically dominant and surjective: if $h_1, h_2\colon X \to Y$ are two fillings of (6.3.3), then $\pi\colon \mathcal{X} \to X$ factors through the equalizer $E \to X$ of $h_1$ and $h_2$. Since $E \to X$ is universally closed, locally of finite type, surjective, and a monomorphism, it is an isomorphism.

For existence, the case when $Y$ is affine is easy:

$$\begin{aligned} \mathrm{Mor}(X,Y) = \mathrm{Hom}(\Gamma(Y,\mathcal{O}_Y), \Gamma(X,\mathcal{O}_X)) &= \\ \mathrm{Hom}(\Gamma(Y,\mathcal{O}_Y), \Gamma(\mathcal{X},\mathcal{O}_{\mathcal{X}})) &= \mathrm{Mor}(\mathcal{X},Y). \end{aligned}$$

(Although unnecessary for the argument below, the case when $Y$ is a scheme is also straightforward: if $\{Y_i\}$ is an affine cover of $Y$ and we set $\mathcal{X}_i := f^{-1}(Y_i) \subset \mathcal{X}$ with complement $\mathcal{Z}_i$, then $\mathcal{X} \setminus \pi^{-1}(\pi(\mathcal{Z}_i)) \to X \setminus \pi(\mathcal{Z}_i)$ is a good moduli space and $\mathcal{X} \setminus \pi^{-1}(\pi(\mathcal{Z}_i)) \subset \mathcal{X}_i$. By the affine case, we have unique factorizations $X \setminus \pi(\mathcal{Z}_i) \to Y_i$ and since $\bigcap_i \pi(\mathcal{Z}_i) = \emptyset$, these maps glue to the desired map $X \to Y$; see also [GIT, §0.6].)

For the general case, since $\mathcal{X}$ is quasi-compact, the map $\mathcal{X} \to Y$ factors through a quasi-compact subspace, so we can further assume that $Y$ is quasi-compact. We can also use étale descent and limit methods to reduce to the case that $X = \operatorname{Spec} A$ where $A$ is a strictly henselian local ring. This reduction works just as in the case of coarse moduli spaces (Theorem 4.3.6). Since $A$ is local, there is a unique closed point $x \in \mathcal{X}$; let $\mathcal{G}_x \hookrightarrow \mathcal{X}$ be the closed immersion of the residual gerbe (Proposition 3.5.16).

Let $(Y' = \operatorname{Spec} B, y') \to (Y, f(x))$ be an étale presentation. The base change $\mathcal{X}' := \mathcal{X} \times_Y Y' \to \mathcal{X}$ is an étale, separated, surjective and representable morphism. Let $x' \in \mathcal{X}'$ be a preimage of $x \in \mathcal{X}$ and $\mathcal{U}' \subset \mathcal{X}'$ be a quasi-compact open substack containing $x'$.

$$\begin{tikzcd} \mathcal{U}' \arrow[r, hook] & \mathcal{X}' \arrow[r] \arrow[dd] & \mathcal{X} \arrow[d, "\pi"] \arrow[dd, bend left, "f"] \\ & & X \\ & Y' \arrow[r] & Y \end{tikzcd}$$

Then $\mathcal{U}' \to \mathcal{X}$ is a quasi-finite, separated, and representable morphism, and Zariski's Main Theorem (6.1.10) implies that there is a factorization $\mathcal{U}' \to \widetilde{\mathcal{X}} \to \mathcal{X}$ with $\mathcal{U}' \hookrightarrow \widetilde{\mathcal{X}}$ an open immersion and $\widetilde{\mathcal{X}} \to \mathcal{X}$ a finite morphism. Writing $\widetilde{\mathcal{X}} = \mathcal{S}\mathrm{pec}_{\mathcal{X}}\, \mathcal{A}$ for a coherent sheaf of algebras $\mathcal{A}$, Lemma 6.3.22(1) implies that $\widetilde{\pi}\colon \widetilde{\mathcal{X}} \to \widetilde{X} := \operatorname{Spec}_X \pi_*\mathcal{A}$ is a good moduli space and we know from Theorem 6.3.5(3) that $\pi_*\mathcal{A}$ is coherent. As $\widetilde{X} \to X = \operatorname{Spec} A$ is finite with $A$ henselian, we can write $\widetilde{X} = \coprod_i \operatorname{Spec} A_i$ with each $A_i$ a henselian local ring (Proposition A.9.3). Replace $\widetilde{\mathcal{X}}$ with the copy of $\widetilde{\mathcal{X}}_i := \widetilde{\pi}^{-1}(\operatorname{Spec} A_i)$ containing $x'$ and replace $\mathcal{U}'$ with $\widetilde{\mathcal{X}}_i \cap \mathcal{U}'$. Then $\widetilde{\mathcal{X}}$ has a unique closed point which is the point $x' \in \mathcal{U}'$ and thus the complement $\widetilde{\mathcal{X}} \setminus \mathcal{U}'$ is empty, i.e. $\mathcal{U}' = \widetilde{\mathcal{X}}$. We conclude that $\mathcal{U}' \to \mathcal{X}$ is a finite étale morphism, and since it induces an isomorphism of residual gerbes at $x'$, the map has degree one; it follows that $\mathcal{U}' \to \mathcal{X}$ is an isomorphism. Since $Y'$ is an affine, the morphism $\mathcal{X} \cong \mathcal{U}' \to Y'$ factors through a map $X \to Y'$, and thus $f\colon \mathcal{X} \to Y$ factors through the composition $X \to Y' \to Y$. $\square$

### 6.3.7 Luna's Fundamental Lemma

We will apply the following result in our construction of good moduli spaces (Theorem 6.8.1), in the refinements of the Local Structure Theorem for Algebraic Stacks (6.5.1), and in the proof of Luna's Étale Slice Theorem (6.5.4), but it appears in many other arguments as well.

**Theorem 6.3.24** (Luna's Fundamental Lemma)**.** *Consider a commutative diagram*

$$\begin{tikzcd} \mathcal{X}' \arrow[r, "f"] \arrow[d, "\pi'"] & \mathcal{X} \arrow[d, "\pi"] \\ X' \arrow[r, "g"] & X \end{tikzcd} \tag{6.3.4}$$

*where $f\colon \mathcal{X}' \to \mathcal{X}$ is a separated and representable morphism of noetherian algebraic stacks, each with affine diagonal, and where $\pi$ and $\pi'$ are good moduli spaces. Let $x' \in \mathcal{X}'$ be a point such that*

*(a) $f$ is étale at $x'$,*

*(b) $f$ induces an isomorphism of stabilizer groups at $x'$, and*

*(c) $x' \in \mathcal{X}'$ and $x = f(x') \in \mathcal{X}$ are closed points.*

*Then there is an open neighborhood $U' \subset X'$ of $\pi'(x')$ such that $U' \to X$ is étale and such that $U' \times_X \mathcal{X} \cong \pi'^{-1}(U')$.*

**Remark 6.3.25.** This result is really saying two things: (1) $g$ is étale at $\pi'(x')$ and (2) after replacing $X'$ with an open neighborhood of $\pi'(x')$ the diagram (6.3.4) is cartesian. In the case of quotients by finite groups, this was established in Proposition 4.3.7. Luna's original formulation [Lun73, p. 94] was the case when $\mathcal{X}' \cong [\operatorname{Spec} A'/G]$ and $\mathcal{X} \cong [\operatorname{Spec} A/G]$ with $G$ linearly reductive and where $\mathcal{X}' \to \mathcal{X}$ is induced by a $G$-equivariant map $\operatorname{Spec} A' \to \operatorname{Spec} A$.

*Proof.* We will adapt the argument of Theorem 6.3.5(4). Since the question is étale local on $X$, limit methods (see the proof of Proposition 4.3.7) allow us to assume that $X = \operatorname{Spec} A$ with $A$ a strictly henselian local ring. If $\mathcal{U}' \subset \mathcal{X}'$ is the étale locus of $f$, then $\mathcal{X}' \setminus \pi'^{-1}(\pi'(\mathcal{X}' \setminus \mathcal{U}'))$ contains $x'$ since $\pi'(x')$ and $\pi'(\mathcal{X}' \setminus \mathcal{U}')$ are disjoint by Theorem 6.3.5(2). We can therefore replace $\mathcal{X}'$ with $\mathcal{X}' \setminus \pi'^{-1}(\pi'(\mathcal{X}' \setminus \mathcal{U}'))$ and assume that $f$ is étale.

By Zariski's Main Theorem (6.1.10), we may choose a factorization $\mathcal{X}' \to \widetilde{\mathcal{X}} = \mathcal{S}\mathrm{pec}_{\mathcal{X}}\, \mathcal{A} \to \mathcal{X}$ with $\mathcal{X}' \hookrightarrow \widetilde{\mathcal{X}}$ an open immersion and $\widetilde{\mathcal{X}} \to \mathcal{X}$ a finite morphism.

Then $\widetilde{\mathcal{X}} \to \widetilde{X} := \operatorname{Spec}_X \pi_* \mathcal{A}$ is a good moduli space and $\widetilde{X} \to X$ is finite. As $A$ is henselian, we can write $\widetilde{X} = \coprod_i \operatorname{Spec} A_i$ with each $A_i$ a henselian local ring. If $U' = \operatorname{Spec} A_i$ denotes the connected component containing the image of $x'$, then $\widetilde{\pi}^{-1}(U') \subset \widetilde{\mathcal{X}}$ is an open substack containing a unique closed point, which is necessarily $x'$; it follows that $\mathcal{X}' = \pi^{-1}(U')$. Since $\mathcal{X}' \to \mathcal{X}$ is a finite étale morphism of degree one (as it preserves residual gerbes at $x'$), we see that $f\colon \mathcal{X}' \to \mathcal{X}$ is an isomorphism and thus so is $g\colon X' \to X$. □

**Corollary 6.3.26.** *With the same hypotheses as Theorem 6.3.24, suppose that $f$ is étale and that for all closed points $x' \in \mathcal{X}'$*

*(a) $f(x') \in \mathcal{X}$ is closed, and*

*(b) $f$ induces an isomorphism of stabilizer groups at $x'$.*

*Then $g\colon X' \to X$ is étale and (6.3.4) is cartesian.* □

### 6.3.8 Finite covers of good moduli spaces

**Proposition 6.3.27.** *Consider a commutative diagram*

$$\begin{tikzcd} \mathcal{X}' \arrow[r, "f"] \arrow[d, "\pi'"] & \mathcal{X} \arrow[d, "\pi"] \\ X' \arrow[r, "g"] & X \end{tikzcd}$$

*where $\mathcal{X}$ and $\mathcal{X}'$ are noetherian algebraic stacks with affine diagonal, and $\pi$ and $\pi'$ are good moduli spaces. Assume that*

*(a) $f\colon \mathcal{X}' \to \mathcal{X}$ is quasi-finite, separated and representable,*

*(b) $f$ maps closed points to closed points, and*

*(c) $g$ is finite.*

*Then $f$ is finite.*

*Proof.* By Zariski's Main Theorem (6.1.10), there is a factorization $\mathcal{X}' \to \widetilde{\mathcal{X}} = \mathcal{S}\mathrm{pec}_{\mathcal{X}} \mathcal{A} \to \mathcal{X}$ with $\mathcal{X}' \hookrightarrow \widetilde{\mathcal{X}}$ an open immersion and $\widetilde{\mathcal{X}} \to \mathcal{X}$ a finite morphism. Then $\widetilde{X} = \operatorname{Spec}_X \pi_* \mathcal{A}$ is a finite over $X$ and $\widetilde{\mathcal{X}} \to \widetilde{X}$ is a good moduli space. By replacing $\mathcal{X} \to X$ with $\widetilde{\mathcal{X}} \to \widetilde{X}$, we can assume that $f$ is an open immersion. By replacing $\mathcal{X}$ with the fiber product $X' \times_X \mathcal{X}$, we can further reduce to the case that $X' = X$. For every closed point $x \in X$, let $x' \in \mathcal{X}'$ be the unique closed point over $x$. By (b), $f(x') \in \mathcal{X}$ is the unique closed point over $x$. Since $\mathcal{X}'$ contains all the closed points of $\mathcal{X}$, $f\colon \mathcal{X}' \to cX$ is an isomorphism. □

**Proposition 6.3.28.** *Suppose $\mathcal{X}$ is a noetherian algebraic stack with affine diagonal and a good moduli space $\pi\colon \mathcal{X} \to X$. If the diagonal $\mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is quasi-finite, then it is finite (i.e. $\pi\colon \mathcal{X} \to X$ is separated).*

*Proof.* We claim that $\mathcal{X} \times_X \mathcal{X} \to X$ is a good moduli space. By Lemma 6.3.15, the projection $p_1\colon \mathcal{X} \times_X \mathcal{X} \to \mathcal{X}$ is cohomologically affine and therefore so is the composition $\mathcal{X} \times_X \mathcal{X} \xrightarrow{p_1} \mathcal{X} \xrightarrow{\pi} X$. On the other hand, if $U \to \mathcal{X}$ is a smooth presentation, then $p_1\colon U \times_X \mathcal{X} \to U$ is a good moduli space (Lemma 6.3.20) and in particular $\mathcal{O}_U \xrightarrow{\sim} p_{1,*}\mathcal{O}_{U \times_X \mathcal{X}}$. It follows from descent that $\mathcal{O}_{\mathcal{X}} \xrightarrow{\sim} p_{1,*}\mathcal{O}_{\mathcal{X} \times_X \mathcal{X}}$ and thus $\mathcal{O}_X \xrightarrow{\sim} (\pi \circ p_1)_* \mathcal{O}_{\mathcal{X} \times_X \mathcal{X}}$; the claim follows.

The diagonal $\mathcal{X} \to \mathcal{X} \times_X \mathcal{X}$ is a quasi-finite, separated, and representable morphism that sends closed points to closed points and induces an isomorphism on good moduli spaces. Proposition 6.3.27 implies that $\mathcal{X} \to \mathcal{X} \times_X \mathcal{X}$ is finite. Note that since $\mathcal{X}$ has affine diagonal, the finiteness of the diagonal is equivalent to its properness. $\square$

### 6.3.9 Descending vector bundles

**Proposition 6.3.29.** *Let $\mathcal{X}$ be a noetherian algebraic stack and $\pi\colon \mathcal{X} \to X$ be a good moduli space. A vector bundle $F$ on $\mathcal{X}$ descends to a vector bundle on $X$ if and only if for every field-valued point $x\colon \operatorname{Spec} \Bbbk \to \mathcal{X}$ with closed image, the action of $G_x$ on the fiber $F \otimes \Bbbk$ is trivial. In this case, $\pi_* F$ is a vector bundle and the adjunction map $\pi^*\pi_* F \to F$ is an isomorphism.*

*Proof.* We follow the argument in the case of a tame coarse moduli space (Proposition 4.3.25). The condition is clearly necessary. To see that the condition is sufficient, consider the commutative diagram

$$\begin{tikzcd} \mathcal{G}_x \arrow[r, hook] \arrow[d, "p"'] & \mathcal{X} \arrow[d, "\pi"] \\ \operatorname{Spec} \kappa(x) \arrow[r, hook] & X. \end{tikzcd}$$

We first claim that $\pi^*\pi_* F \to F$ is surjective. For every closed point $x \in \mathcal{X}$, the hypotheses imply that $p^*p_*(F|_{\mathcal{G}_x}) \cong F|_{\mathcal{G}_x}$. Applying $\pi^*\pi_*(-)|_{\mathcal{G}_x}$ to the surjection $F \to F|_{\mathcal{G}_x}$ and using the exactness of $\pi_*$, we obtain that $(\pi^*\pi_*F)|_{\mathcal{G}_x} \to \pi^*(\pi_*(F|_{\mathcal{G}_x}))|_{\mathcal{G}_x} \cong p^*p_*(F|_{\mathcal{G}_x}) \cong F|_{\mathcal{G}_x}$ is surjective. The claim now follows from Lemma 6.3.30.

To show that $\pi_* F$ is a vector bundle, we may assume that $X = \operatorname{Spec} A$ is affine and that the rank $r$ of $F$ is constant. The surjection $\bigoplus_{s \in \Gamma(X, \pi_* F)} A \to \pi_* F$ pulls back to a surjection $\bigoplus_{s \in \Gamma(\mathcal{X}, F)} \mathcal{O}_{\mathcal{X}} \to \pi^*\pi_* F$ and by the above claim, the composition $\bigoplus_{s \in \Gamma(\mathcal{X}, F)} \mathcal{O}_{\mathcal{X}} \to \pi^*\pi_* F \to F$ is surjective. As $F|_{\mathcal{G}_x} \cong \mathcal{O}^r_{\mathcal{G}_x}$ is trivial, for each closed point $x \in |\mathcal{X}|$, we can find $r$ sections $\phi\colon \mathcal{O}^r_{\mathcal{X}} \to F$ such that $\phi|_{\mathcal{G}_x}$ is an isomorphism. By Lemma 6.3.30, there exists an open neighborhood $U \subset X$ of $\pi(x)$ such that $\phi|_{\pi^{-1}(U)}$ is an isomorphism. Thus $\pi_*\phi\colon \mathcal{O}^r_X \to \pi_* F$ is an isomorphism over $U$ and we conclude that $\pi_* F$ is a vector bundle of the same rank as $F$. Finally, since $\pi^*\pi_* F \to F$ is a surjection of vector bundles of the same rank, it is an isomorphism.

The case of a good quotient is due to Kempf. See also [KKV89, Prop. 4.2], [Alp13, Thm. 10.3] and [Ryd20, Thm. B]. $\square$

**Lemma 6.3.30.** *Let $\mathcal{X}$ be a noetherian algebraic stack and $\pi\colon \mathcal{X} \to X$ be a good moduli space. Let $x \in |\mathcal{X}|$ be a closed point.*

*(1) If $F$ is a coherent sheaf on $\mathcal{X}$ such that $F|_{\mathcal{G}_x} = 0$, then there exists an open neighborhood $U \subset X$ of $\pi(x)$ such that $F|_{\pi^{-1}(U)} = 0$.*

*(2) If $\phi\colon F \to G$ is a morphism of coherent sheaves (resp. vector bundles of the same rank) on $\mathcal{X}$ such that $\phi|_{\mathcal{G}_x}$ is surjective, then there exists an open neighborhood $U \subset X$ of $\pi(x)$ such that $\phi|_{\pi^{-1}(U)}$ is surjective (resp. an isomorphism).*

*Proof.* The argument of Lemma 4.3.21 applies. $\square$

## 6.4 Coherent Tannaka duality and coherent completeness

We prove a version of Tannaka duality for noetherian algebraic stacks with affine diagonal (Theorem 6.4.1). We also introduce the notion of an algebraic stack $\mathcal{X}$ being coherently complete along a closed substack $\mathcal{X}_0$ (Definition 6.4.4) and show that certain quotient stacks with a unique closed point are coherently complete (Theorem 6.4.11). This includes the important examples of $[\mathbb{A}^1/\mathbb{G}_m]_R$ and $\Phi_R$ defined in §6.8.2 where $R$ is a complete DVR.

The combined power of Tannaka duality and coherent completeness allows us to extend compatible maps $\mathcal{X}_n \to \mathcal{Y}$ from the $n$th nilpotent thickenings of $\mathcal{X}_0$ to a morphism $\mathcal{X} \to \mathcal{Y}$ (Corollary 6.4.8). This technique is used in an essential way in the proof of the Local Structure Theorem for Algebraic Stacks (6.5.1) and also appears in many other arguments—it becomes a powerful new addition in our algebraic stack toolkit.

### 6.4.1 Coherent Tannaka Duality

A classical theorem of Gabriel [Gab62] states that two noetherian schemes $X$ and $Y$ are isomorphic if and only if their abstract categories $\mathrm{Coh}(X)$ and $\mathrm{Coh}(Y)$ of coherent sheaves are equivalent, or in other words that a scheme $X$ can be recovered from the category $\mathrm{Coh}(X)$. In representation theory, classical Tannaka duality by Saavedra Rivano [SR72] (see also Deligne and Milne's article [DMOS82, Ch. II]) asserts that an affine group scheme $G$ over a field $\Bbbk$ can be recovered from the tensor category $\mathrm{Rep}^{\mathrm{fd}}(G)$ of finite dimensional representations and its forgetful functor $\mathrm{Rep}^{\mathrm{fd}}(G) \to \mathrm{Vect}_{\Bbbk}$.

Combining these two facts, one might hope that an algebraic stack $\mathcal{X}$ is recovered by the tensor category $\mathrm{Coh}(\mathcal{X})$.[2] Following a brilliant observation of Lurie [Lur04], we will not only confirm this expectation, but we will show that in fact a tensor functor $\mathrm{Coh}(\mathcal{Y}) \to \mathrm{Coh}(\mathcal{X})$ is enough to recover a morphism $\mathcal{X} \to \mathcal{Y}$ of algebraic stacks.

**Theorem 6.4.1** (Coherent Tannaka Duality)**.** *For noetherian algebraic stacks $\mathcal{X}$ and $\mathcal{Y}$ with affine diagonal, the functor*

$$\mathrm{MOR}(\mathcal{X}, \mathcal{Y}) \to \mathrm{MOR}^{\otimes}(\mathrm{Coh}(\mathcal{Y}), \mathrm{Coh}(\mathcal{X})), \qquad f \mapsto f^* \tag{6.4.1}$$

*is an equivalence of categories, where $\mathrm{MOR}^{\otimes}(\mathrm{Coh}(\mathcal{Y}), \mathrm{Coh}(\mathcal{X}))$ denotes the category of right exact additive tensor functors $\mathrm{Coh}(\mathcal{Y}) \to \mathrm{Coh}(\mathcal{X})$ of symmetric monoidal abelian categories where morphisms are tensor natural transformations.*

**Remark 6.4.2.** A *symmetric monoidal category* is a category $\mathcal{A}$ endowed with a bifunctor $\otimes \colon \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ and a unit $1 \in \mathcal{A}$ together with associativity isomorphisms $\alpha_{A,B,C} \colon A \otimes (B \otimes C) \xrightarrow{\sim} (A \otimes B) \otimes C$, left and right unit isomorphisms $l_A \colon 1 \otimes A \xrightarrow{\sim} A \xrightarrow{\sim} A$ and $r_A \colon A \otimes 1 \xrightarrow{\sim} A$, and commutativity isomorphisms $s_{A,B} \colon A \otimes B \cong B \otimes A$ (with $s_{A,B} \circ s_{B,A} = \mathrm{id}$) satisfying certain coherence conditions [Mac71, §XI.1]. A *tensor functor* $F \colon \mathcal{A} \to \mathcal{B}$ between symmetric monoidal abelian categories is a functor equipped with isomorphisms $\Phi_{A,B} \colon F(A) \otimes F(B) \xrightarrow{\sim} F(A \otimes B)$ and $\varphi \colon 1_{\mathcal{B}} \xrightarrow{\sim} F(1_{\mathcal{A}})$ compatible with the isomorphisms $\alpha_{A,B,C}$, $l_A$, $r_A$ and $s_{A,B}$ [Mac71, §XI.2]. A

[2] The structure as an abelian category is not enough, e.g. $\mathrm{Coh}(B\mathbb{Z}/2) \cong \mathrm{Coh}(\operatorname{Spec} \Bbbk \coprod \operatorname{Spec} \Bbbk)$ in $\mathrm{char}(\Bbbk) \neq 2$.

*tensor natural transformation* between tensor functors is a natural transformation of functors compatible with the isomorphisms $\Phi_{A,B}$ and $\varphi$ [Mac71, §XI.2].

A *symmetric monoidal abelian category* (resp. *symmetric monoidal $R$-linear abelian category* for a ring $R$) is a symmetric monoidal (resp. $R$-linear) abelian category $\mathcal{A}$ such that $\otimes : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ is additive (resp. $R$-linear) in each variable. A tensor functor is *additive* or *$R$-linear* if the underlying functor is. When $\mathcal{X}$ and $\mathcal{Y}$ are defined over a noetherian ring $R$, then Theorem 6.4.1 induces an equivalence

$$\mathrm{MOR}_R(\mathcal{X}, \mathcal{Y}) \xrightarrow{\sim} \mathrm{MOR}_R^{\otimes}(\mathrm{Coh}(\mathcal{Y}), \mathrm{Coh}(\mathcal{X}))$$

between morphisms over $R$ and right exact $R$-linear tensor functors.

*Proof.* Since every quasi-coherent sheaf on a noetherian algebraic stack is a colimit of its coherent subsheaves (Proposition 6.1.8), every right exact tensor functor $F\colon \mathrm{Coh}(\mathcal{Y}) \to \mathrm{Coh}(\mathcal{X})$ extends to a tensor functor $F\colon \mathrm{QCoh}(\mathcal{Y}) \to \mathrm{QCoh}(\mathcal{X})$ preserving colimits. Likewise every tensor natural transformation between functors of coherent sheaves extends uniquely to one defined on quasi-coherent sheaves.

*Fully faithfulness:* Let $f, g\colon \mathcal{X} \to \mathcal{Y}$. Choose a smooth presentation $p\colon U \to \mathcal{Y}$ where $U$ is an affine scheme. Since the question is smooth-local on $\mathcal{X}$, after replacing $\mathcal{X}$ with $\mathcal{X} \times_{f,\mathcal{Y},p} U$, we may assume there is a factorization $f\colon \mathcal{X} \xrightarrow{\widetilde{f}} U \xrightarrow{p} \mathcal{Y}$. Likewise, we may assume there is a factorization $g\colon \mathcal{X} \xrightarrow{\widetilde{g}} V \xrightarrow{q} \mathcal{Y}$ where $V$ is an affine scheme. Since $\mathcal{Y}$ has affine diagonal, $p\colon U \to \mathcal{Y}$ is affine and we have identifications

$$\mathrm{Mor}_{\mathcal{Y}}(\mathcal{X}, U) \cong \mathrm{Hom}_{\mathcal{O}_{\mathcal{Y}}-\mathrm{alg}}(p_*\mathcal{O}_U, f_*\mathcal{O}_{\mathcal{X}}) \cong \mathrm{Hom}_{\mathcal{O}_{\mathcal{X}}-\mathrm{alg}}(f^*p_*\mathcal{O}_U, \mathcal{O}_{\mathcal{X}})$$

Therefore $\widetilde{f}$ and $\widetilde{g}$ correspond to sections $s_{\widetilde{f}}\colon f^*p_*\mathcal{O}_U \to \mathcal{O}_{\mathcal{X}}$ and $s_{\widetilde{g}}\colon g^*q_*\mathcal{O}_V \to \mathcal{O}_{\mathcal{X}}$. A 2-isomorphism $\alpha\colon f \to g$ (i.e. a morphism in $\mathrm{MOR}(\mathcal{X}, \mathcal{Y})$) is identified with a factorization

$$\begin{tikzcd} \mathcal{X} \arrow[r, "(\widetilde{f},\widetilde{g},\alpha)"] \arrow[dr, "\mathrm{id}"'] & U \times_{\mathcal{X}} V \arrow[d, "\pi"] \\ & \mathcal{X} \end{tikzcd}$$

which is the same data as a section $s_\alpha$ of $\mathcal{O}_{\mathcal{X}} \to f^*\pi_*\mathcal{O}_{U \times_{\mathcal{X}} V}$. Letting $\alpha^*\colon f^* \to g^*$ be the image of $\alpha$ under (6.4.1), i.e. the pullback tensor natural transformation, the section $s_\alpha$ can be written as

$$f^*\pi_*\mathcal{O}_{U\times_{\mathcal{X}} V} \cong f^*(p_*\mathcal{O}_U) \otimes f^*(q_*\mathcal{O}_V) \xrightarrow{\mathrm{id}\otimes\alpha^*_{q_*\mathcal{O}_V}} f^*(p_*\mathcal{O}_U) \otimes g^*(q_*\mathcal{O}_V) \xrightarrow{s_{\widetilde{f}}\otimes s_{\widetilde{g}}} \mathcal{O}_S.$$

To see the faithfulness of (6.4.1), if $\alpha, \alpha'\colon f \to g$ are 2-isomorphisms with $\alpha^* = \alpha'^*$, then $\alpha^*_{q_*\mathcal{O}_V} = \alpha'^*_{q_*\mathcal{O}_V}$ and therefore the two sections $s_\alpha$ and $s_{\alpha'}$ are equal and $\alpha = \alpha'$. For the fullness of (6.4.1), let $\beta\colon f^* \to g^*$ be a tensor natural transformation. Then $\mathrm{id}\otimes\beta_{q_*\mathcal{O}_V}$ defines a section $f^*\pi_*\mathcal{O}_{U\times_{\mathcal{X}} V} \to \mathcal{O}_S$ and thus a 2-isomorphism $\alpha\colon f \to g$ such that $\beta_{q_*\mathcal{O}_V} = \alpha^*_{q_*\mathcal{O}_V}$. To see that $\beta_E = \alpha^*_E$ for every $E \in \mathrm{QCoh}(\mathcal{Y})$, note that the factorization $g = q \circ \widetilde{g}$ yields a splitting of $g^*E \to g^*(q_*q^*E)$. Since $f^*$ and $g^*$ commute with direct sums, it suffices to assume that $E = q_*G$ for $G \in \mathrm{QCoh}(V)$. Writing $G = \mathrm{colim}(\mathcal{O}_V^{\oplus I} \to \mathcal{O}_V^{\oplus J})$ as a colimit of free $\mathcal{O}_V$-modules, we can conclude that $\beta_{q_*G} = \alpha^*_{q_*G}$ since $f^*$ and $g^*$ commute with colimits and $q_*$ is exact.

*Essential surjectivity:* Let $F\colon \mathrm{QCoh}(\mathcal{Y}) \to \mathrm{QCoh}(\mathcal{X})$ be a tensor functor preserving colimits.

*The affine case:* If $\mathcal{X} = \operatorname{Spec} A$ and $\mathcal{Y} = \operatorname{Spec} B$ are noetherian affine schemes, then we have a map

$$\phi\colon B \cong \operatorname{End}(\mathcal{O}_{\mathcal{Y}}) \xrightarrow{F} \operatorname{End}(\mathcal{O}_{\mathcal{X}}) = A.$$

We claim that $\phi$ is a ring homomorphism and that there is a functorial isomorphism $F(N) = N \otimes_B A$ for $N \in \operatorname{Mod}_B$. For $b, b' \in B$, consider the commutative diagrams

$$\begin{tikzcd} \mathcal{O}_{\mathcal{Y}} \otimes \mathcal{O}_{\mathcal{Y}} \arrow[r] \arrow[d, "b\otimes b'"] & \mathcal{O}_{\mathcal{Y}} \arrow[d, "bb'"] & & A \otimes A \arrow[r] \arrow[d, "\phi(b)\otimes\phi(b')"] & A \arrow[d, "\phi(bb')"] \\ \mathcal{O}_{\mathcal{Y}} \otimes \mathcal{O}_{\mathcal{Y}} \arrow[r] & \mathcal{O}_{\mathcal{Y}} \arrow[r, phantom, "\overset{F}{\longmapsto}"] & & A \otimes A \arrow[r] & A \end{tikzcd}$$

where the horizontal maps correspond to multiplication. The commutativity of the right square is implied by the fact that $F$ preserves tensor products. This shows that $\phi(b)\phi(b') = \phi(bb')$. For a $B$-module $N$, choose a free presentation $B^{\oplus J} \to B^{\oplus I} \to N \to 0$. Since both $F$ and $- \otimes_B A$ are right exact and preserve direct sums, applying them to the free presentation yields an identification $F(N) \cong N \otimes_B A$ as both are cokernels of $A^{\oplus J} \to A^{\oplus I}$. One checks similarly that this identification is functorial.

*Reduction to the case that $\mathcal{X}$ is affine:* Choose a smooth presentation $g\colon U \to \mathcal{X}$ from an affine scheme and consider the diagram

$$\begin{tikzcd} U \times_X U \arrow[d, shift right, "p_2"'] \arrow[d, shift left, "p_1"] & \\ U \arrow[d, "g"] \arrow[dr, dotted, "g^* \circ F"] & \\ \mathcal{X} \arrow[r, dotted, "F"] & \mathcal{Y} \end{tikzcd}$$

where the dashed arrow $\mathcal{X} \dashrightarrow \mathcal{Y}$ is denoting that we have a tensor functor $\operatorname{QCoh}(\mathcal{Y}) \to \operatorname{QCoh}(\mathcal{X})$ in the other direction. Assuming that the result holds when $U$ is affine, there is a morphism $h\colon U \to \mathcal{Y}$ and an isomorphism $h \xrightarrow{\sim} g^* \circ F$ of functors. By full faithfulness, there is an isomorphism $p_1 \circ h \xrightarrow{\sim} p_2 \circ h$ satisfying the cocycle condition, and thus smooth descent implies that there is a unique morphism $f\colon \mathcal{X} \to \mathcal{Y}$ with $F \simeq f^*$.

*Reduction to the case that $\mathcal{Y}$ is affine:* Let $\mathcal{X} = \operatorname{Spec} A$ and choose a smooth presentation $q\colon V = \operatorname{Spec} C \to \mathcal{Y}$. Since $\mathcal{Y}$ has affine diagonal, $q$ is an affine morphism. Define $B := F(q_*\mathcal{O}_V)$ which is an $A$-algebra since $q_*\mathcal{O}_V$ is an $\mathcal{O}_{\mathcal{Y}}$-algebra. Consider the diagram

$$\begin{tikzcd} \operatorname{Spec} B \arrow[r, dotted, "F'"] \arrow[d] & V = \operatorname{Spec} C \arrow[d, "q"] \\ \operatorname{Spec} A = \mathcal{X} \arrow[r, dotted, "F"] & \mathcal{Y} \end{tikzcd}$$

where $F'\colon \operatorname{Mod}_C \to \operatorname{Mod}_B$ is the right exact tensor functor sending $M$ to $F(q_*\widetilde{M})$ (which is a module over $B = F(q_*\mathcal{O}_V)$ because $q_*\widetilde{M}$ is a $q_*\mathcal{O}_V$-module). By the affine case, $F'$ is induced by a morphism $f' \colon \operatorname{Spec} B \to \operatorname{Spec} C$. We can extend the above diagram into

$$\begin{tikzcd}
\operatorname{Spec} B \otimes_A B \arrow[r, "f''"] \arrow[d, shift right] \arrow[d, shift left] & V \times_{\mathcal{Y}} V \arrow[d, shift right] \arrow[d, shift left] \\
\operatorname{Spec} B \arrow[r, "f'"] \arrow[d] & V = \operatorname{Spec} C \arrow[d, "q"] \\
\operatorname{Spec} A = \mathcal{X} \arrow[r, dotted, "F"] & \mathcal{Y}.
\end{tikzcd}$$

Since $q$ is affine, $V \times_{\mathcal{Y}} V$ is affine and the top square (under either set of projections) is cartesian.

If we could show that $A \to B$ is faithfully flat, we would be done as the full faithfulness in the affine case would imply that $f'$ descends to our desired morphism $f \colon \mathcal{X} \to \mathcal{Y}$. This seems hard to directly check, but we do know already that the maps $B \rightrightarrows B \otimes_A B$ are faithfully flat as they correspond to base changes of the smooth maps $V \times_{\mathcal{Y}} V \rightrightarrows V$. We will show instead that $A \to B$ is universally injective. Since faithful flatness descends under universal injectivity maps (Proposition A.2.21(4)), the faithful flatness of $A \to B$ follows from the universal injectivity.

*Universal injectivity of $A \to B$:* Recall from Definition A.2.20 that an injective map of $A$-modules is called universally injective if it remains injective after tensoring by every $A$-module. By Proposition A.2.21(3), this notion is local under faithfully flat morphisms and thus extends to morphisms $F \to G$ of quasi-coherent sheaves on an algebraic stack.

Since $q \colon V \to \mathcal{Y}$ is faithfully flat, $\mathcal{O}_{\mathcal{Y}} \to q_* \mathcal{O}_V$ is universally injective (Proposition A.2.21(1)). We write $q_* \mathcal{O}_V = \operatorname{colim} Q_i$ as a colimit of coherent subsheaves (Proposition 6.1.8) and we may assume that each $Q_i$ contains the image of $\mathcal{O}_{\mathcal{Y}} \to q_* \mathcal{O}_V$. Then $\mathcal{O}_{\mathcal{Y}} \to Q_i$ is also universally injective and since $Q_i$ is coherent, $\mathcal{O}_{\mathcal{Y}} \to Q_i$ is a split injection smooth-locally on $\mathcal{Y}$ (Proposition A.2.21(2)). Applying $F$ to $\mathcal{O}_{\mathcal{Y}} \to q_* \mathcal{O}_V = \operatorname{colim} Q_i$ and using that it preserves colimits, we have a factorization

$$\begin{tikzcd}
 & F(Q_i) \arrow[d] \\
A = F(\mathcal{O}_{\mathcal{Y}}) \arrow[r] \arrow[ur] & B = F(q_* \mathcal{O}_V) = \operatorname{colim} F(Q_i).
\end{tikzcd}$$

It suffices to show that $A \to F(Q_i)$ is universally injective. We will show in fact that it is a split injection. As $\mathcal{O}_{\mathcal{Y}} \to Q_i$ is smooth-locally split, the map on duals $Q_i^\vee \to \mathcal{O}_{\mathcal{Y}}^\vee = \mathcal{O}_{\mathcal{Y}}$ is surjective. Applying $F$, we have a surjection $F(Q_i^\vee) \to F(\mathcal{O}_{\mathcal{Y}}) = A$ (using right exactness) and we can choose an element $\lambda \in F(Q_i^\vee)$ mapping to 1. Under the natural map $F(Q_i^\vee) \to F(Q_i)^\vee$, the element $\lambda$ is sent to a map $F(Q_i) \to A$, which one checks to be a section of the given map $A \to F(Q_i)$.

See also [Lur04], [HR19b], [BHL17] and [SP, Tag 0GRR]. □

**Remark 6.4.3** (Relation to classical Tannaka duality)**.** If $G$ is an affine group scheme over a field $\Bbbk$, then the category $\mathcal{C} = \operatorname{Rep}^{\mathrm{fd}}(G)$ of finite dimensional representations is a symmetric monoidal $\Bbbk$-linear category and there is a tensor functor $\omega \colon \operatorname{Rep}^{\mathrm{fd}}(G) \to \operatorname{Vect}_{\Bbbk}$. For $\Bbbk$-algebra $R$, let $\omega_R$ denote the composition $\operatorname{Rep}^{\mathrm{fd}}(G) \to \operatorname{Vect}_{\Bbbk} \xrightarrow{- \otimes_{\Bbbk} R} \operatorname{Mod}_R$ and let $\operatorname{Aut}^{\otimes}(\omega_R)$ denote the group of tensor natural isomorphisms of $\omega_R$.

Then $G$ is recovered as the functor $\underline{\mathrm{Aut}}^{\otimes}(\omega)$ on affine $\Bbbk$-schemes assigning $R$ to $\mathrm{Aut}^{\otimes}(\omega_R)$ [DMOS82, II.2.8].

On the other hand, Coherent Tannaka Duality for Algebraic Stacks (Theorem 6.4.1) implies that for every noetherian $\Bbbk$-algebra $R$, there is an equivalence of categories

$$\mathrm{MOR}_{\Bbbk}(\operatorname{Spec} R, \mathbf{B}G) \xrightarrow{\sim} \mathrm{MOR}^{\otimes}(\mathrm{Rep}(G)^{\mathrm{fd}}, \mathrm{Mod}_R).$$

In this way, we see that $\mathrm{Rep}(G)^{\mathrm{fd}}$ determines $\mathbf{B}G$. To recover $G$, the fiber functor $\omega\colon \mathrm{Rep}^{\mathrm{fd}}(G) \to \mathrm{Vect}_{\Bbbk}$ corresponds to a morphism $p\colon \operatorname{Spec}\Bbbk \to \mathbf{B}G$ and $G = \mathrm{Aut}_{\Bbbk}(p)$. For example, if $O(q)$ and $O(q')$ are orthogonal groups with respect to non-degenerate quadratic forms $q$ and $q'$ of the same dimension, then $\mathrm{Rep}(O(q)) \cong \mathrm{Rep}(O(q'))$ even though $O(q)$ and $O(q')$ may not be isomorphic; in this case the two maps $\operatorname{Spec}\Bbbk \to \mathbf{B}O(q)$ and $\operatorname{Spec}\Bbbk \to \mathbf{B}O(q')$ define two different fiber functors on the same category.

The classical version also provides conditions when the data of $(\mathcal{C}, \omega)$ is isomorphic to the category of representations of a group scheme. Namely, we say $\mathcal{C}$ is *rigid* if for every object of $X \in \mathcal{C}$, there is a 'dual' $X^{\vee} \in \mathcal{C}$, i.e. an object $X^{\vee}$ such that $X^{\vee} \otimes - \colon \mathcal{C} \to \mathcal{C}$ is right adjoint to $X \otimes - \colon \mathcal{C} \to \mathcal{C}$. If $\mathcal{C}$ is a rigid symmetric monoidal $\Bbbk$-linear abelian category with $\mathrm{End}(1) = \Bbbk$ and $\omega\colon \mathcal{C} \to \mathrm{Vect}_{\Bbbk}$ is an exact faithful $\Bbbk$-linear tensor functor, then $\underline{\mathrm{Aut}}^{\otimes}(\omega)$ is represented by an affine group scheme $G$ over $\Bbbk$ and there is a tensor equivalence $\mathcal{C} \cong \mathrm{Rep}^{\mathrm{fd}}(G)$ under which $\omega$ corresponds to the forgetful functor [DMOS82, II.2.11]. Moreover, $G$ is of finite type over $\Bbbk$ if and only if $\mathcal{C}$ has a tensor generator.

### 6.4.2 Coherent completeness

Coherent Tannaka Duality becomes especially powerful when combined with coherent completeness.

**Definition 6.4.4.** A noetherian algebraic stack $\mathcal{X}$ is *coherently complete along a closed substack* $\mathcal{X}_0$ if the natural functor

$$\mathrm{Coh}(\mathcal{X}) \to \varprojlim \mathrm{Coh}(\mathcal{X}_n), \quad F \mapsto (F_n)$$

is an equivalence of categories, where $\mathcal{X}_n$ denotes the $n$th nilpotent thickening of $\mathcal{X}_0$ and $F_n$ is the pullback of $F$ to $\mathcal{X}_n$.

**Remark 6.4.5.** If $\mathcal{I} \subset \mathcal{O}_{\mathcal{X}}$ is the coherent sheaf of ideals defining $\mathcal{X}_0$, then $\mathcal{X}_n$ is defined by $\mathcal{I}^{n+1}$. Letting $i_n\colon \mathcal{X}_n \hookrightarrow \mathcal{X}_{n+1}$ denote the natural inclusion, an object in $\varprojlim \mathrm{Coh}(\mathcal{X}_n)$ corresponds to a sequence $F_n \in \mathrm{Coh}(\mathcal{X}_n)$ of coherent sheaves together with maps $\alpha_n\colon i_{n,*}F_n \to F_{n+1}$ inducing isomorphism $F_n \to i_n^* F_{n+1}$. A morphism $(F_n, \alpha_n) \to (F'_n, \alpha'_n)$ is a sequence of maps $\phi_n\colon F_n \to F'_n$ such that $\phi_{n+1} \circ \alpha_n = \alpha_{n+1} \circ i_{n,*}\phi_n$.

**Example 6.4.6.** If $(R, \mathfrak{m})$ is a complete noetherian local ring, then the Artin–Rees Lemma [AM69, Prop. 10.9] implies that $\operatorname{Spec} R$ is coherently complete along $\operatorname{Spec} R/\mathfrak{m}$. The same is true if $R = \lim R/I^n$ is a noetherian $I$-adically complete ring.

**Example 6.4.7.** Grothendieck's Existence Theorem (D.4.4) asserts that if $X$ is a proper scheme over a complete local ring $(R, \mathfrak{m})$ and $X_0 = X \times_R R/\mathfrak{m}$, then $X$ is coherently complete along $X_0$. If $\mathcal{X}$ is a proper Deligne–Mumford stack or more generally a proper algebraic stack over $\operatorname{Spec} R$, the same is true. The result also holds if $R$ is an $I$-adically complete noetherian ring. See [Ols05, Thm. 1.4] or [Con05a, Thm. 4.1].

**Corollary 6.4.8** (Coherent Tannaka Duality)**.** *Let $\mathcal{X}$ and $\mathcal{Y}$ be noetherian algebraic stacks with affine diagonal. Suppose that $\mathcal{X}$ is coherently complete along $\mathcal{X}_0$. Then there is an equivalence of categories*

$$\mathrm{MOR}(\mathcal{X},\mathcal{Y}) \to \varprojlim \mathrm{MOR}(\mathcal{X}_n,\mathcal{Y}), \quad f \mapsto (f_n),$$

*where $f_n \colon \mathcal{X}_n \to Y$ denotes the restriction of $f$ to the $n$th nilpotent thickening $\mathcal{X}_n$ of $\mathcal{X}_0$.*

*Proof.* This follows from the equivalences

$$\begin{aligned} \mathrm{MOR}(\mathcal{X},\mathcal{Y}) &\simeq \mathrm{MOR}^{\otimes}\big(\mathrm{Coh}(\mathcal{Y}),\mathrm{Coh}(\mathcal{X})\big) && \text{(Coherent Tannaka Duality)}\\ &\simeq \mathrm{MOR}^{\otimes}\big(\mathrm{Coh}(\mathcal{Y}),\varprojlim \mathrm{Coh}(\mathcal{X}_n)\big) && \text{(coherent completeness)}\\ &\simeq \varprojlim \mathrm{MOR}^{\otimes}\big(\mathrm{Coh}(\mathcal{Y}),\mathrm{Coh}(\mathcal{X}_n)\big) && \\ &\simeq \varprojlim \mathrm{MOR}(\mathcal{X}_n,\mathcal{Y}) && \text{(Coherent Tannaka Duality).} \end{aligned}$$

□

**Remark 6.4.9.** If $\mathcal{X}$ and $\mathcal{Y}$ are defined over a noetherian ring $R$, then there is an equivalence $\mathrm{MOR}_R(\mathcal{X},\mathcal{Y}) \to \varprojlim \mathrm{MOR}_R(\mathcal{X}_n\mathcal{Y})$. This follows in the same way using the Tannaka duality equivalence between the category of morphisms $\mathcal{X} \to \mathcal{Y}$ over $R$ and the category of right exact $R$-linear tensor functors (Remark 6.4.2).

For example, to show that there is map $\operatorname{Spec} A \to \mathcal{Y}$ from the spectrum of a noetherian $I$-adically complete ring $A$, it suffices to construct compatible maps $\operatorname{Spec} A/I^n \to \mathcal{Y}$. This is only easy to see directly if $A$ is local.

**Exercise 6.4.10.** Let $G$ be an affine algebraic group acting on a separated noetherian algebraic space $W$ over $\Bbbk$. Let $W_0 \subset W$ be a $G$-invariant closed subspace and let $W_n$ be its $n$th nilpotent thickenings. Suppose that $[W/G]$ is coherently complete along a closed substack $[W_0/G]$. For every noetherian algebraic space $X$ over $\Bbbk$ with affine diagonal equipped with an action of $G$, the natural map on equivariant maps

$$\mathrm{Mor}^G(W,X) \to \varprojlim_n \mathrm{Mor}^G\big(W_n,X\big)$$

is bijective.

*Hint: reduce to Corollary 6.4.8 by using that a $G$-equivariant map $W \to X$ corresponds to a morphism $[W/G] \to [X/G]$ over $\mathbf{B}G$, i.e $\mathrm{Mor}^G(W,X) = \{*\} \times_{\mathrm{Mor}([W/G],\mathbf{B}G)} \mathrm{Mor}([W/G],[X/G])$.*

### 6.4.3 Coherent completeness of quotient stacks

The coherent completeness result that we will exploit through the rest of the book and in particular in the proof of the Local Structure Theorem for Algebraic Stacks (6.5.1) is the following:

**Theorem 6.4.11.** *Let $\Bbbk$ be an algebraically closed field and $R$ be a complete noetherian local $\Bbbk$-algebra with residue field $\Bbbk$. Let $G$ be a linearly reductive group over $\Bbbk$ acting on an affine scheme $\operatorname{Spec} A$ of finite type over $R$. Suppose that $A^G = R$ and that there is a $G$-fixed $\Bbbk$-point $x \in \operatorname{Spec} A$. Then $[\operatorname{Spec} A/G]$ is coherently complete along the closed substack $\mathbf{B}G$ defined by $x$.*

**Example 6.4.12.** If $\mathbb{G}_m$ acts diagonally on $\mathbb{A}^r$, then $[\mathbb{A}^r/\mathbb{G}_m]$ is coherently complete along the origin $B\mathbb{G}_m$. In other words a $\mathbb{G}_m$-equivariant coherent sheaf on $\mathbb{A}^r$ is equivalent to a compatible family of $\mathbb{G}_m$-equivariant modules over $\Bbbk[x_1,\ldots,x_r]/(x_1,\ldots,x_r)^{n+1}$.

**Remark 6.4.13.** We have a commutative diagram

$$\begin{tikzcd} \mathbf{B}G \arrow[r, hook] & {[\operatorname{Spec} A/G] \times_{A^G} \Bbbk} \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & {[\operatorname{Spec} A/G]} \arrow[d] \\ & \operatorname{Spec} \Bbbk \arrow[r, hook] & \operatorname{Spec} A^G. \end{tikzcd}$$

A formal consequence of the above theorem is that $[\operatorname{Spec} A/G]$ is also coherently complete with respect to the fiber $[\operatorname{Spec} A/G] \times_{A^G} \Bbbk$. This version is analogous to Grothendieck's Existence Theorem (6.4.7), but the coherent completeness along $\mathbf{B}G$ is a substantially stronger statement, e.g. for $[\mathbb{A}^n/\mathbb{G}_m]$ where the fiber of $[\mathbb{A}^n/\mathbb{G}_m] \to \operatorname{Spec}\Bbbk$ is everything.

*Proof of Theorem 6.4.11.* We need to show that $\operatorname{Coh}(\mathcal{X}) \to \varprojlim \operatorname{Coh}(\mathcal{X}_n)$ is an equivalence of categories, where $\mathcal{X} = [\operatorname{Spec} A/G]$ and $\mathcal{X}_n$ is the $n$th nilpotent thickening of $\mathbf{B}G \hookrightarrow \mathcal{X}$ of the inclusion of the residual gerbe at $x$.

*Full faithfulness:* Suppose that $F$ and $F'$ are coherent $\mathcal{O}_{\mathcal{X}}$-modules, and let $F_n$ and $F'_n$ denote the restrictions to $\mathcal{X}_n$, respectively. We need to show that

$$\operatorname{Hom}_{\mathcal{O}_{\mathcal{X}}}(F,F') \to \varprojlim \operatorname{Hom}_{\mathcal{O}_{\mathcal{X}}}(F_n,F'_n)$$

is bijective. Since $\mathcal{X}$ has the resolution property (Proposition 6.1.19), we can find a resolution $F_2 \to F_1 \to F \to 0$. by vector bundles. This induces a diagram

$$\begin{tikzcd} 0 \arrow[r] & \operatorname{Hom}_{\mathcal{O}_{\mathcal{X}}}(F,F') \arrow[r] \arrow[d] & \operatorname{Hom}_{\mathcal{O}_{\mathcal{X}}}(F_1,F') \arrow[r] \arrow[d] & \operatorname{Hom}_{\mathcal{O}_{\mathcal{X}}}(F_2,F') \arrow[d] \\ 0 \arrow[r] & \varprojlim \operatorname{Hom}_{\mathcal{O}_{\mathcal{X}}}(F_n,F'_n) \arrow[r] & \varprojlim \operatorname{Hom}_{\mathcal{O}_{\mathcal{X}}}(F_{1,n},F'_n) \arrow[r] & \varprojlim \operatorname{Hom}_{\mathcal{O}_{\mathcal{X}}}(F_{2,n},F'_n) \end{tikzcd}$$

with exact rows. We may therefore assume that $F$ is a vector bundle. In this case,

$$\operatorname{Hom}_{\mathcal{O}_{\mathcal{X}}}(F,F') = \operatorname{Hom}_{\mathcal{O}_{\mathcal{X}}}(\mathcal{O}_{\mathcal{X}}, F^\vee \otimes F')$$
$$\operatorname{Hom}_{\mathcal{O}_{\mathcal{X}}}(F_n,F'_n) = \operatorname{Hom}_{\mathcal{O}_{\mathcal{X}}}\big(\mathcal{O}_{\mathcal{X}_n}, (F_n^\vee \otimes F'_n)\big).$$

Therefore, we can also assume that $F = \mathcal{O}_{\mathcal{X}}$ and we are reduced to showing that

$$\Gamma(\mathcal{X},F') \to \varprojlim \Gamma\big(\mathcal{X}_n, F'_n\big) \tag{6.4.2}$$

is an isomorphism. Writing $F' = \widetilde{M}$ where $M$ is a finitely generated $A$-module with an action of $G$ and letting $\mathfrak{m} \subset A$ be the maximal ideal for $x$, then $\Gamma(\mathcal{X}_n, F'_n) = M^G/(\mathfrak{m}^n M)^G$ since $G$ is linearly reductive. We must therefore verify that

$$M^G \to \varprojlim M^G/(\mathfrak{m}^n M)^G \tag{6.4.3}$$

is an isomorphism. To this end, we first show that

$$\bigcap_{n \geq 0} \big(\mathfrak{m}^n M\big)^G = 0. \tag{6.4.4}$$

or in other words that (6.4.3) is injective. Let $N := \bigcap_{n\geq 0} \mathfrak{m}^n M$. The Artin–Rees lemma [AM69, Prop. 10.9] applied to $N \subset M$ implies that there exists an integer $c$ such that $\mathfrak{m}^n M \cap N = \mathfrak{m}^{n-c}(\mathfrak{m}^c M \cap N)$ for all $n \geq c$. Taking $n = c+1$, we see that $N = \mathfrak{m}N$ so $N \otimes_A A/\mathfrak{m} = 0$. Since the support of $N$ is a closed $G$-invariant subscheme of $\operatorname{Spec} A$ which does not contain $x$, it follows that $N = 0$.

Note also that since $G$ is linearly reductive, $M^G$ is a finitely generated $A^G$-module (Corollary 6.3.7(3)). We next establish that (6.4.3) is an isomorphism if $A^G$ is artinian. In this case, $\{(\mathfrak{m}^n M)^G\}$ automatically satisfies the Mittag–Leffler condition (it is a sequence of artinian $A^G$-modules). Therefore, taking the inverse limit of the exact sequences $0 \to (\mathfrak{m}^n M)^G \to M^G \to M^G/(\mathfrak{m}^n M)^G \to 0$ and applying (6.4.4) yields an exact sequence

$$0 \to 0 \to M^G \to \varprojlim M^G/(\mathfrak{m}^n M)^G \to 0$$

and shows that (6.4.3) is an isomorphism. To establish (6.4.3) in the general case, let $J = (\mathfrak{m}^G)A \subset A$ and observe that

$$M^G \cong \varprojlim M^G/\big(\mathfrak{m}^G\big)^n M^G \cong \varprojlim \big(M/J^n M\big)^G, \tag{6.4.5}$$

since $G$ is linearly reductive. For each $n$, we know that

$$\big(M/J^n M\big)^G \cong \varprojlim_l M^G/\big((J^n + \mathfrak{m}^l)M\big)^G \tag{6.4.6}$$

using the artinian case proved above. Finally, combining (6.4.5) and (6.4.6) together with the observation that $J^n \subset \mathfrak{m}^l$ for $n \geq l$, we conclude that

$$\begin{aligned} M^G &\cong \varprojlim_n \big(M/J^n M\big)^G \\ &\cong \varprojlim_n \varprojlim_l M^G/\big((J^n + \mathfrak{m}^l)M\big)^G \\ &\cong \varprojlim_l M^G/\big(\mathfrak{m}^l M\big)^G. \end{aligned}$$

*Essential surjectivity:* The linear reductivity of $G$ implies that every coherent sheaf $F = \widetilde{M}$ on $[\operatorname{Spec} A/G]$ decomposes as a direct sum

$$M = \bigoplus_{\rho \in \Gamma} M^{(\rho)}, \tag{6.4.7}$$

where $\Gamma$ denotes the set of isomorphism classes of irreducible representations of $G$ and $M^{(\rho)}$ is the isotypic component corresponding to $\rho$; explicitly if $W_\rho$ denotes the irreducible representation corresponding to $\rho$, then $M^{(\rho)} = \operatorname{Hom}^G_{\Bbbk}(W_\rho, M) \otimes W_\rho$. Moreover, the decomposition (6.4.7) is compatible with the $A$-module structure of $M$ and the decomposition $A = \bigoplus_{\rho \in \Gamma} A^{(\rho)}$.

Let us also note that if $F = \widetilde{M} \in \operatorname{Coh}(\mathcal{X})$ with restrictions $M_n = M/\mathfrak{m}^{n+1}M$, then applying (6.4.3) to $M \otimes W_\rho^\vee$ shows that $M^{(\rho)} = \varprojlim M_n^{(\rho)}$. By Theorem 6.3.5(3), we also know that $M^{(\rho)}$ is a finitely generated $A^G$-module. In particular, $A^{(\rho)} = \varprojlim (A/\mathfrak{m}^{n+1})^{(\rho)}$ is a finitely generated $A^G$-module.

This suggests that if $F_n = \widetilde{M_n}$ is a compatible system of coherent $\mathcal{O}_{\mathcal{X}_n}$-modules with $M_n = \bigoplus_\rho M_n^{(\rho)}$, we define

$$M^{(\rho)} := \varprojlim M_n^{(\rho)} \qquad \text{and} \qquad M := \bigoplus_{\rho \in \Gamma} M^{(\rho)}. \tag{6.4.8}$$

To see that $M$ is an $A$-module with a $G$-action, let $\rho, \gamma \in \Gamma$ be irreducible representations and let $\Lambda \subset \Gamma$ denote the finite set of nonzero irreducible representations appearing $W_\rho \otimes W_\gamma$. Taking limits of the maps $A_n^{(\rho)} \otimes_{(A/\mathfrak{m}^{n+1})^G} M_n^{(\gamma)} \to \bigoplus_{\lambda \in \Lambda} M_n^{(\lambda)}$, defines multiplication

$$A^{(\rho)} \otimes_A M^{(\gamma)} \to \varprojlim \left(A_n^{(\rho)} \otimes_{(A/\mathfrak{m}^{n+1})^G} M_n^{(\gamma)}\right) \to \varprojlim \Big(\bigoplus_{\lambda \in \Lambda} M_n^{(\lambda)}\Big) \cong \bigoplus_{\lambda \in \Lambda} M^{(\lambda)}.$$

Note that we also have $M/\mathfrak{m}^{n+1}M \cong M_n$ by construction.

We need to show that the $A$-module $M$ of (6.4.8) is finitely generated. The coherent sheaf $F_0 = \widetilde{M_0}$ on $\mathcal{X}_0 = \mathbf{B}G$ is a finite dimensional $G$-representation and we can consider the coherent $\mathcal{O}_{\mathcal{X}}$-module $F_0 \otimes_{\mathcal{O}_{\mathcal{X}_0}} \mathcal{O}_{\mathcal{X}}$ or equivalently the $A$-module $M_0 \otimes_{\Bbbk} A$ with its natural $G$-action. Since $\mathcal{X}$ is cohomologically affine, the functor

$$\mathrm{Hom}_{\mathcal{O}_{\mathcal{X}}}(F_0 \otimes_{\mathcal{O}_{\mathcal{X}_0}} \mathcal{O}_{\mathcal{X}}, -) = \Gamma(\mathcal{X}, (F_0^\vee \otimes_{\mathcal{O}_{\mathcal{X}_0}} \mathcal{O}_{\mathcal{X}}) \otimes_{\mathcal{O}_{\mathcal{X}}} -)$$

is exact. Apply the functor to the surjection $M \twoheadrightarrow M_0$ induces a map

$$M_0 \otimes_{\Bbbk} A \to M \tag{6.4.9}$$

which we would like to show is surjective. We do know that the restrictions $M_0 \otimes_{\Bbbk} (A/\mathfrak{m}^{n+1}) \to M_n$ are surjective as its cokernel is a coherent module on $\mathcal{X}_n$ not supported at the unique closed point.

As above, we first handle the case that $A^G$ is artinian. Since $A^{(\rho)} \xrightarrow{\sim} \varprojlim (A/\mathfrak{m}^n)^{(\rho)}$ is a finitely generated $A^G$-module, it follows that $(A/\mathfrak{m}^n)^{(\rho)}$ stabilizes to $A^{(\rho)}$ for $n \gg 0$. Since (6.4.9) induces surjections $M_0 \otimes_{\Bbbk} (A/\mathfrak{m}^{n+1}) \to M_n$, it follows that the modules $M_n^{(\rho)}$ stabilize to $M_\infty^{(\rho)}$ for $n \gg 0$ and that $M = \bigoplus_\rho M_\infty^{(\rho)}$ is finitely generated. In the general case, let $X_m = \operatorname{Spec} A^G/(\mathfrak{m} \cap A^G)^{m+1}$ and consider the cartesian diagram

$$\begin{tikzcd}
\mathcal{X} \times_X X_m \arrow[r, hook, "i_m"] \arrow[d, "\pi_m"] & \mathcal{X} \times_X X_{m+1} \arrow[r, hook] \arrow[d, "\pi_{m+1}"] & \cdots \arrow[r, hook] & \mathcal{X} \arrow[d, "\pi"] \\
X_m \arrow[r, hook, "j_m"] & X_{m+1} \arrow[r, hook] & \cdots \arrow[r, hook] & X.
\end{tikzcd}$$

For each $m$, we may consider the $n$th nilpotent thickenings $\mathcal{Z}_{m,n}$ of $\mathcal{X}_0 \hookrightarrow \mathcal{X} \times_X X_m$ which are closed substacks $\mathcal{X}_n$. Since $X_m$ is the spectrum of artinian ring, the restrictions $F_n|_{\mathcal{Z}_{m,n}}$ extend to a coherent sheaf $H_m = \widetilde{N}_m$ on $\mathcal{X} \times_X X_m$. Moreover, there is a canonical isomorphism between $H_m$ and the restriction of $H_{m+1}$ to $\mathcal{X} \times_X X_m$. By Lemma 6.3.20(4), the adjunction morphism $j_m^* \pi_{m+1,*} \xrightarrow{\sim} \pi_{m,*} i_m^*$ is an isomorphism on quasi-coherent sheaves. This implies that $N_{m+1}^{(\rho)} = \Gamma(\mathcal{X} \times_X X_{m+1}, H_{m+1} \otimes W_\rho^\vee)$ restricts to $N_m^{(\rho)}$ and that $M^{(\rho)} = \varprojlim N_m^{(\rho)}$ is a finitely generated $A^G$-module. By Nakayama's lemma, the map (6.4.9) is surjective on each $\rho$-isotypical component. Thus (6.4.9) is surjective and $M$ is finitely generated.

For an alternative (but similar) argument for essential surjectivity, we first choose a surjection $E \twoheadrightarrow F_0$ from a vector bundle $E$ on $\mathcal{X}$. For this we can either apply the resolution property of $\mathcal{X}$ (Proposition 6.1.19) or take $E = F_0 \otimes_{\mathcal{O}_{\mathcal{X}_0}} \mathcal{O}_{\mathcal{X}}$ as above. Since each $F_{n+1} \to F_n$ is surjective and $\mathrm{Hom}_{\mathcal{O}_{\mathcal{X}}}(E, -) = \Gamma(\mathcal{X}, E^\vee \otimes_{\mathcal{O}_{\mathcal{X}}} -)$ is exact, we can lift $E \twoheadrightarrow F_0$ to compatible maps $E \to F_n$, each which is surjective (Nakayama's lemma). The sequence $(\ker(E_n \to F_n))$ not necessarily an adic sytem of coherent sheaves on $\mathcal{X}_n$ as the restriction $\ker(E_{n+1} \to F_{n+1})$ to $\mathcal{X}_n$ may not be

$\ker(E_n \to F_n)$. But we can modify it as follows: For each $l \geq m \geq n$, the images of $\ker(E_l \to F_l)$ in $E_m$ stabilize to $K'_m$ for $l \gg m$ and $K'_m/\mathfrak{m}^{n+1}K'_m$ stabilize to $K_n$ for $m \gg n$ (see also [SP, Tag 087X]). Then $(K_n) \in \varprojlim \mathrm{Coh}(\mathcal{X}_n)$ is an adic sequence. Repeating the construction, we can find a vector bundle $E'$ on $\mathcal{X}$ and compatible surjections $E' \to K_n$. By full faithfulness, there is a morphism $E' \to E$ extending the maps $E'_n \to E_n$. Then $\mathrm{coker}(E' \to E)$ is a coherent $\mathcal{O}_{\mathcal{X}}$ extending $(F_n)$.

See also [AHR20, Thm. 1.3] and [AHR19, Thm. 1.6]. □

**Exercise 6.4.14.** If $S$ is a noetherian affine scheme, show that $[\mathbb{A}^1/\mathbb{G}_m]_S$ is coherently complete along $\mathbf{B}G_{m,S}$.

## 6.5 Local structure of algebraic stacks

We establish a local structure theorem for algebraic stack around points with linearly reductive stabilizer. The main theorem (Theorem 6.5.1) implies that quotient stacks of the form $[\operatorname{Spec} A/G]$, where $G$ is linearly reductive, are the building blocks of algebraic stacks near points with linearly reductive stabilizers in a similar way to how affine schemes are the building blocks of schemes and algebraic spaces. When $\mathcal{X}$ is Deligne–Mumford, we've already seen an analogous Local Structure Theorem for Deligne–Mumford Stacks (4.2.11). The local structure theorem will be applied to construct good moduli spaces in a similar way to how the result for Deligne–Mumford stacks was used to prove the Keel–Mori Theorem (4.3.11) on the existence of coarse moduli spaces.

**Theorem 6.5.1** (Local Structure Theorem for Algebraic Stacks)**.** *Let $\mathcal{X}$ be an algebraic stack of finite type over an algebraically closed field $\Bbbk$ with affine diagonal. For every point $x \in \mathcal{X}(\Bbbk)$ with linearly reductive stabilizer $G_x$, there exists an affine étale morphism*

$$f\colon ([\operatorname{Spec} A/G_x], w) \to (\mathcal{X}, x)$$

*which induces an isomorphism of stabilizer groups at $w$.*

**Remark 6.5.2.** In the case that $x \in \mathcal{X}$ is a smooth point, then one can say more: there is also an étale morphism

$$[\operatorname{Spec} A/G_x], w) \to ([T_{\mathcal{X},x}/G_x], 0)$$

where $T_{\mathcal{X},x}$ is the Zariski tangent space equipped as a $G_x$-representation. This addendum follows from the proof but also follows from applying Luna's Étale Slice Theorem (6.5.4) to $[\operatorname{Spec} A/G_x]$. The upshot is that we can reduce étale local properties of $\mathcal{X}$ to $G_x$-equivariant properties of $T_{\mathcal{X},x}$; for moduli problems, this translates into studying the first-order deformation space as a representation under the automorphism group.

By combining this theorem with Luna's Fundamental Lemma (6.3.24), we obtain the following result.

**Corollary 6.5.3** (Local Structure for Good Moduli Spaces)**.** *Let $\mathcal{X}$ be an algebraic stack of finite type over an algebraically closed field $\Bbbk$ with affine diagonal. Suppose that there exists a good moduli space $\pi\colon \mathcal{X} \to X$. Then for every closed point $x \in \mathcal{X}$,* *there exists an étale neighborhood $W \to X$ of $\pi(x)$ and a cartesian diagram*

$$\begin{tikzcd}
{[\operatorname{Spec} A/G_x]} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{X} \arrow[d, "\pi"] \\
W = \operatorname{Spec} A^{G_x} \arrow[r] & X.
\end{tikzcd}$$

□

*Section outline:* We first discuss Luna's Étale Slice Theorem (6.5.4), a beautiful argument providing an explicit construction of an étale neighborhood in the case that $\mathcal{X}$ is already known to have the form $[\operatorname{Spec} B/G]$ with $G$ reductive. The proof of the Local Structure Theorem (6.5.1) is far less explicit requiring: (1) deformation theory, (2) coherent completeness, (3) Coherent Tannaka Duality and (4) Artin Approximation or Equivariant Artin Algebraization.

Letting $\mathcal{T} = [T_{\mathcal{X},x}/G_x]$, deformation theory produces an embedding $\mathcal{X}_n \hookrightarrow \mathcal{T}_n$ of the $n$th nilpotent thickenings of $x$ and 0. The key step in the proof is to show that the system of closed morphisms $\{\mathcal{X}_n \to \mathcal{X}\}$ algebraizes. The first step is effectivization: the fiber product $\widehat{\mathcal{T}} := \mathcal{T} \times_T \operatorname{Spec} \widehat{\mathcal{O}}_{T,\pi(0)}$, where $\pi \colon \mathcal{T} \to T := T_{\mathcal{X},x}/\!\!/G_x$, is coherently complete (Theorem 6.4.11). We can thus construct a closed substack $\widehat{\mathcal{X}} \hookrightarrow \widehat{\mathcal{T}}$ extending $\mathcal{X}_n \hookrightarrow \mathcal{T}$ and then apply Coherent Tannaka Duality (6.4.8) to construct a morphism $\widehat{\mathcal{X}} \to \mathcal{X}$ extending $\mathcal{X}_n \to \mathcal{X}$.

If $x \in \mathcal{X}$ is smooth, Artin Approximation over the GIT quotient $T_{\mathcal{X},x}/\!\!/G_x$ produces an étale neighborhood $U \to T_{\mathcal{X},x}/\!\!/G_x$ such that $\pi^{-1}(U) \to \mathcal{X}$ algebraizes $\widehat{\mathcal{T}} \to \mathcal{X}$. In the general case, Artin Approximation cannot handle this final step and we need to establish an equivariant version of Artin Algebraization (Theorem 6.5.14).

### 6.5.1 Luna's Étale Slice Theorem

The local structure theorem was inspired by Luna's étale slice theorem in equivariant geometry.

**Theorem 6.5.4** (Luna's Étale Slice Theorem)**.** *Let $G$ be a linearly reductive group over an algebraically closed field $\Bbbk$ and let $X$ be an affine scheme of finite type over $\Bbbk$ with an action of $G$. If $x \in X(\Bbbk)$ has linearly reductive stabilizer, then there exists a $G_x$-invariant, locally closed, and affine subscheme $W \subset X$ such that the induced map*

$$[W/G_x] \to [X/G] \tag{6.5.1}$$

*is affine étale. If in addition the orbit $Gx \subset X$ is closed, then there is a cartesian diagram*

$$\begin{tikzcd}
{[W/G_x]} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & {[X/G]} \arrow[d] \\
W/\!\!/G_x \arrow[r] & X/\!\!/G
\end{tikzcd}$$

*where $W/\!\!/G_x \to X/\!\!/G$ is also étale.*

*Moreover, if $x \in X$ is a smooth point and if we denote by $N_x = T_{X,x}/T_{Gx,x}$ the normal space to the orbit, then it can be arranged that there is an $G_x$-invariant étale morphism $W \to N_x$ which is the pullback of an étale map $W/\!\!/G_x \to N_x/\!\!/G_x$ of GIT quotients.*

**Remark 6.5.5.** One can also formulate the statement $G$-equivariantly: $G$ acts naturally on the quotient $G \times^{G_x} W := (G \times W)/G_x$ and there is an identification $[W/G_x] \cong [(G \times^{G_x} W)/G]$ and likewise $W/\!\!/G_x \cong (G \times^{G_x} W)/\!\!/G$ (see Exercise 3.4.14). The morphism (6.5.1) corresponds to an étale $G$-equivariant morphism $G \times^{G_x} W \to X$.

If the orbit $Gx$ is closed, then Matsushima's Theorem (6.3.19) implies that the stabilizer $G_x$ is linearly reductive.

The proof will rely on the existence of a $G_x$-invariant morphism $X \to T_{X,x}$, which we refer to as the *Luna map*.

**Lemma 6.5.6** (Luna map)**.** *Let $G$ be a linearly reductive group over an algebraically closed field $\Bbbk$ and let $X$ be an affine scheme of finite type over $\Bbbk$ with an action of $G$. If $x \in X(\Bbbk)$ has linearly reductive stabilizer, there exists a $G_x$-equivariant morphism*

$$f\colon X \to T_{X,x} \tag{6.5.2}$$

*sending $x$ to the origin. If $X$ is smooth at $x$, then $f$ is étale at $x$.*

*Proof.* Letting $X = \operatorname{Spec} A$ and $\mathfrak{m} \subset A$ be the maximal ideal of $x$, then $\mathfrak{m}$ and $\mathfrak{m}/\mathfrak{m}^2$ are $G_x$-representations and we see that $G_x$ acts naturally on the tangent space $T_{X,x} := \operatorname{Spec} \operatorname{Sym}^* \mathfrak{m}/\mathfrak{m}^2$. Since $G_x$ is linearly reductive, the surjection $\mathfrak{m} \to \mathfrak{m}/\mathfrak{m}^2$ of $G_x$-representations has a section $\mathfrak{m}/\mathfrak{m}^2 \hookrightarrow \mathfrak{m}$. This induces a $G_x$-equivariant ring map $\operatorname{Sym}^* \mathfrak{m}/\mathfrak{m}^2 \to A$ and thus a $G_x$-equivariant morphism $f\colon \operatorname{Spec} A \to T_{X,x}$ sending $x$ to the origin. If $x \in X$ is smooth, then since $f$ induces an isomorphism of tangent spaces at $x$, we conclude that $f$ is étale at $x$ (Étale Equivalences A.3.2). $\square$

*Proof of Theorem 6.5.4.* Since $X$ is affine and of finite type, we can choose a finite dimensional $G$-representation $V$ and a $G$-equivariant closed immersion $X \hookrightarrow \mathbb{A}(V)$ (Proposition C.3.4). If $W \subset \mathbb{A}(V)$ is an affine $G_x$-invariant locally closed subscheme such that $[W/G_x] \to [\mathbb{A}(V)/G]$ is étale, then the same is true for $W' := W \cap X \subset X$ and $[W'/G_x] \to [X/G]$. We can therefore immediately reduce to the case that $x \in X$ is smooth. In this case, there is a Luna map $f\colon X \to T_{X,x}$ (see (6.5.6)) which is $G_x$-invariant, étale at $x$, and with $f(x) = 0$. The subspace $T_{Gx,x} \subset T_{X,x}$ is $G_x$-invariant and again since $G_x$ is linearly reductive, the surjection $T_{X,x} \to N_x = T_{X,x}/T_{Gx,x}$ has a section $N_x \hookrightarrow T_{X,x}$. We define $W$ as the preimage of $N_x$ under $f$:

$$\begin{tikzcd} W \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\square"] & N_x \arrow[d, hook] \\ X \arrow[r, "f"] & T_{X,x}. \end{tikzcd}$$

Since the maps $f\colon [W/G_x] \to [X/G]$ and $g\colon [W/G_x] \to [N_x/G_x]$ induce an isomorphism of tangent spaces and stabilizer groups at $w$, they are both étale at $x \in W$ (or equivalently the $G$-equivariant maps $G \times^{G_x} W \to X$ and $G \times^{G_x} W \to G \times^{G_x} N_x$ are étale at $(\mathrm{id}, x)$). We have a commutative diagram

$$\begin{tikzcd} {[N_x/G_x]} \arrow[d] & {[W/G_x]} \arrow[l, "g"'] \arrow[r, "f"] \arrow[d] & {[X/G]} \arrow[d] \\ N_x/\!\!/G_x & W/\!\!/G_x \arrow[l] \arrow[r] & X/\!\!/G_x \end{tikzcd}$$

where both $f$ and $g$ are étale at $x$, preserve stabilizer groups at $x$ and map $x$ to closed points. We can therefore apply Luna's Fundamental Lemma (6.3.24) to replace $W$ with a $G_x$-equivariant, open, and affine neighborhood of $x$ so that the above squares are cartesian. $\square$

When $\mathcal{X}$ is already known to be a quotient stack of a normal quasi-projective scheme, the Local Structure Theorem follows from a direct argument. This case is sufficient to handle many moduli problems, e.g. $\mathrm{Bun}^{\mathrm{ss}}_{r,d}(C)$ in characteristic 0.

**Exercise 6.5.7.** If $G$ is a connected affine algebraic group over an algebraically closed field $\Bbbk$ acting on a normal finite type $\Bbbk$-scheme $X$, and $x \in X(\Bbbk)$ has linearly reductive stabilizer, show that there is a $G_x$-invariant, locally closed, and affine subscheme $W \hookrightarrow X$ such that $[W/G_x] \to [X/G]$ is étale.

*Hint: Sumihiro's Theorem on Linearizations (C.3.12) to reduce to the case that $X = \mathbb{P}(V)$. Choose a homogenous polynomial $f$ not vanishing at $x$ such that $\mathbb{P}(V)_f$ is $G_x$-invariant and then argue as in the proof of Luna's Étale Slice Theorem by considering the $G_x$-equivariant étale map $\mathbb{P}(V)_f \to T_x\mathbb{P}(V)$.*

### 6.5.2 Deformation theory

In our proof of the Local Structure Theorem (6.5.1), we will need some deformation theory of algebraic stacks in the form of the following two propositions.

**Proposition 6.5.8.** *Consider a commutative diagram*

$$\begin{tikzcd} \mathcal{W} \arrow[r, "f"] \arrow[d, hook] & \mathcal{X} \arrow[d] \\ \mathcal{W}' \arrow[r] \arrow[ur, dashed] & \mathcal{Y} \end{tikzcd}$$

*of noetherian algebraic stacks with affine diagonal where $\mathcal{X} \to \mathcal{Y}$ is smooth and affine and $\mathcal{W}' \hookrightarrow \mathcal{W}'$ is a closed immersion defined by a square-zero sheaf of ideals $J$. If $\mathcal{W}$ is cohomologically affine, there exists a lift in the above diagram.*

*Proof.* When $\mathcal{W}$ is affine, the statement follows from the Infinitesimal Lifting Criterion (A.3.1). To reduce to this case, let $U' \to \mathcal{W}'$ be a smooth presentation with $U'$ an affine scheme and set $U = U' \times_{\mathcal{W}'} \mathcal{W}$. Since $\mathcal{W}$ has affine diagonal, each $n$-fold fiber product $(U/\mathcal{W})^n := U \times_{\mathcal{W}} \cdots \times_{\mathcal{W}} U$ is affine. We have a commutative diagram

$$\begin{tikzcd} (U/\mathcal{W})^2 \arrow[r, shift left] \arrow[r, shift right] \arrow[d, hook] & U \arrow[r, "q_1"] \arrow[d, hook] & \mathcal{W} \arrow[r, "f"] \arrow[d, hook] & \mathcal{X} \arrow[d] \\ (U'/\mathcal{W}')^2 \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & U' \arrow[r] \arrow[urr, "f'_U"] & \mathcal{W}' \arrow[r] \arrow[ur, dashed, "f'"] & \mathcal{Y}. \end{tikzcd}$$

where we have chosen a lift $f'_U \colon U' \to \mathcal{X}$. Defining the coherent sheaf $F = f^*(\Omega^\vee_{\mathcal{X}/\mathcal{Y}}) \otimes J$ on $\mathcal{W}$, we know by Exercise 6.1.9 that the set of lifts $U' \to \mathcal{X}$ is a torsor under $\Gamma(U, q_1^*F)$ so that any other lift differs from $f'_U$ by an element of $\Gamma(U, q_1^*F)$. Because $\mathcal{X} \to \mathcal{Y}$ is representable, to check that $f'_U$ descends to a morphism $f' \colon \mathcal{W}' \to \mathcal{X}$, we need to arrange that $f'_U \circ p_1 = f'_U \circ p_2$. Let $q_n \colon (U/\mathcal{W})^n \to \mathcal{W}$. The difference $f'_U \circ p_1 - f'_U \circ p_2$ can be viewed an element of $\Gamma((U/\mathcal{W})^2, q_2^*F)$.

Since $q_1 \colon U \to \mathcal{W}$ is a surjective, smooth, and affine morphism, there is an exact sequence of quasi-coherent sheaves

$$0 \to F \to q_{1,*}q_1^*F \to q_{2,*}q_2^*F \to q_{3,*}q_3^*F \to \cdots ;$$

see Exercise B.1.2. Since $\mathcal{W}$ is cohomologically affine, taking global sections yields an exact sequence

$$\begin{array}{ccccc}
\Gamma(U, q_1^*F) & \xrightarrow{d_0} & \Gamma((U/\mathcal{W})^2, q_2^*F) & \xrightarrow{d_1} & \Gamma((U/\mathcal{W})^3, q_3^*F) \\
s & \longmapsto & p_1^*s - p_2^*s & & \\
 & & s & \longmapsto & p_{12}^*s - p_{13}^*s + p_{23}^*s.
\end{array}$$

One checks that $d_1(f'_U \circ p_1 - f'_U \circ p_2) = 0$ so there exists an element $s \in \Gamma(U, q_1^*F)$ with $d_0(s) = f'_U \circ p_1 - f'_U \circ p_2$. After modifying the lift $f'_U$ by $s$, we see that $f'_U \circ p_1 - f'_U \circ p_2 = 0$ so that $f'_U$ descends to $f' \colon \mathcal{W}' \to \mathcal{X}$. $\square$

**Remark 6.5.9.** Alternatively, one can show that the obstruction to this deformation problem lies in $\mathrm{Ext}^1_{\mathcal{O}_\mathcal{W}}(f^*\Omega_{\mathcal{X}/\mathcal{Y}}, J) = \mathrm{H}^1(\mathcal{W}, f^*(\Omega^\vee_{\mathcal{X}/\mathcal{Y}}) \otimes J)$, which vanishes since $\mathcal{W}$ is cohomologically affine. The above result holds more generally [Ols06, Thm. 1.5].

**Proposition 6.5.10.** *Let $\mathcal{W} \hookrightarrow \mathcal{W}'$ be a closed immersion of algebraic stacks of finite type over $\Bbbk$ with affine diagonal defined by a square-zero sheaf of ideals $J$. Let $G$ be an affine algebraic group over $\Bbbk$. If $\mathcal{W}$ is cohomologically affine, then every principal $G$-bundle $\mathcal{P} \to \mathcal{W}$ extends to a principal $G$-bundle $\mathcal{P}' \to \mathcal{W}'$.*

*Proof.* Our proof will use smooth descent and the deformation theory of principal $G$-bundles over schemes (Exercise D.2.9). Let $U' \to \mathcal{W}'$ be a smooth presentation from an affine scheme and let $U := \mathcal{W} \times_{\mathcal{W}'} U'$. Since $\mathcal{W}$ has affine diagonal, each $n$-fold fiber products $(U/\mathcal{W})^n = U \times_\mathcal{W} \cdots \times_\mathcal{W} U$ is affine and we denote the projection by $q_n \colon (U/\mathcal{W})^n \to \mathcal{W}$. By descent theory, the principal $G$-bundle $\mathcal{P} \to \mathcal{W}$ corresponds to a principal $G$-bundle $P \to U$ together with an isomorphism $\alpha \colon p_1^*P \xrightarrow{\sim} p_2^*P$ on $(U/\mathcal{W})^2$ satisfying the cocycle condition $p_{23}^*\alpha \circ p_{12}^*\alpha = p_{13}^*\alpha$ on $(U/\mathcal{W})^3$. Letting $F = \mathfrak{g} \otimes J$ be the coherent sheaf on $\mathcal{W}$ where $\mathfrak{g}$ denotes the Lie algebra of $G$, we know by Exercise D.2.9 that the deformation theory of $q_n^*\mathcal{P} \to (U/\mathcal{W})^n$ with respect to the closed immersion $(U/\mathcal{W})^n \hookrightarrow (U'/\mathcal{W}')^n$ is controlled by $q_n^*F$.

Since $U$ is affine, we can choose a deformation $P' \to U'$ of $P \to U$. We can also choose an isomorphism $\alpha' \colon p_1^*P' \xrightarrow{\sim} p_2^*P'$ on $(U'/\mathcal{W}')^2$ lifting $\alpha$ where any other choice of an isomorphism differs by an element of $\Gamma((U/\mathcal{W})^2, q_2^*F)$. The isomorphism $(p_{13}^*\alpha')^{-1} \circ p_{23}^*\alpha' \circ p_{12}^*\alpha'$ restricts to the identity on $(U/\mathcal{W})^3$ and thus corresponds to an element $\Psi \in \Gamma((U/\mathcal{W})^3, q_3^*F)$. If $\Psi = 0$, then descent theory implies that $P' \to U'$ descends to the desired principal $G$-bundle $\mathcal{P}' \to \mathcal{W}'$. Since $0 \to F \to q_{1,*}q_1^*F \to q_{2,*}q_2^*F \to \cdots$ is an exact sequence and $\mathcal{W}$ is cohomologically affine, taking global sections gives an exact sequence

$$\begin{array}{ccccc}
\Gamma((U/\mathcal{W})^2, q_2^*F) & \xrightarrow{d_2} & \Gamma((U/\mathcal{W})^3, q_3^*F) & \xrightarrow{d_3} & \Gamma((U/\mathcal{W})^4, q_4^*F) \\
s & \longmapsto & p_{12}^*s - p_{13}^*s + p_{23}^*s & & \\
 & & s & \longmapsto & p_{123}^*s - p_{134}^*s + p_{124}^*s - p_{234}^*s.
\end{array} \tag{6.5.3}$$

While $\Psi$ may be nonzero, one can check that $d_3(\Psi) = 0$ and thus there exists an element $s \in \Gamma((U/\mathcal{W})^2, q_2^* F)$ such that $d_2(s) = \Psi$. Thus modifying the isomorphism $\alpha'$ by $s$, we see that we can arrange the cocycle condition to hold. $\square$

**Remark 6.5.11.** The deformation question is equivalent to deforming the morphism $f\colon \mathcal{W} \to \mathbf{B}G$ classified by $P \to \mathcal{W}$ to a morphism $\mathcal{W}' \to \mathbf{B}G$, which is analogous to Proposition 6.5.8 except that $\mathcal{X} = \mathbf{B}G \to \mathcal{Y} = \operatorname{Spec} \Bbbk$ is not affine. The obstruction to deforming a principal $G$-bundle lies in the group $\mathrm{H}^2(\mathcal{W}, \mathfrak{g} \otimes J)$. When $\mathcal{W} \to \mathbf{B}G$ is representable, one can see this as a consequence of [Ols06, Thm. 1.5] (see Remarks D.5.12 and D.7.5): the obstruction lies in $\operatorname{Ext}^1_{\mathcal{O}_{\mathcal{W}}}(Lf^* L_{\mathbf{B}G/\Bbbk}, J)$. Under the composition $\operatorname{Spec} \Bbbk \xrightarrow{p} \mathbf{B}G \to \operatorname{Spec} \Bbbk$, we have an exact triangle $p^* L_{\mathbf{B}G/\Bbbk} \to L_{\Bbbk/\Bbbk} \to L_{\Bbbk/\mathbf{B}G}$. Since $L_{\Bbbk/\Bbbk} = 0$, we obtain that $p^* L_{\mathbf{B}G/\Bbbk} = L_{\Bbbk/\mathbf{B}G}[-1] \cong \mathfrak{g}^\vee[-1]$ and $L_{\mathbf{B}G/\Bbbk} \cong \mathfrak{g}^\vee[-1]$, where the Lie algebra $\mathfrak{g}$ is equipped with the adjoint representation. Thus $\operatorname{Ext}^1_{\mathcal{O}_{\mathcal{W}}}(Lf^* L_{\mathbf{B}G/\Bbbk}, J) = \mathrm{H}^1(\mathcal{W}, f^*\mathfrak{g}[1] \otimes J) = \mathrm{H}^2(\mathcal{W}, \mathfrak{g} \otimes J)$. Since $\mathcal{W}$ is cohomologically affine with affine diagonal, this cohomology group is 0 and the obstruction vanishes.

Here's a third approach in the case that $\mathcal{W} = [\operatorname{Spec} A/G]$ where $G$ is linearly reductive and $A^G$ is an artinian $\Bbbk$-algebra. Since $\mathcal{W}$ is a global quotient stack, there exists a vector bundle $E$ on $\mathcal{W}$ such that the stabilizer groups act faithfully on the fibers (Exercise 6.1.16). Generalizing the deformation theory of vector bundles on schemes (Proposition D.2.15), the obstruction to deforming $E$ to a vector bundle $E'$ lies in $\mathrm{H}^2(\mathcal{W}, \mathscr{E}nd_{\mathcal{O}_{\mathcal{W}}}(E) \otimes J)$ which vanishes as $\mathcal{W}$ is cohomologically affine. Since the stabilizer groups also act faithfully on the fibers of $E'$, we have that $\mathcal{W}' \cong [V'/\operatorname{GL}_n]$ where $V'$ is an algebraic space. Then $\mathcal{W} \cong [V/\operatorname{GL}_n]$ with $V_{\mathrm{red}} = V'_{\mathrm{red}}$. Since $\mathcal{W}$ is cohomologically affine and $V \to \mathcal{W}$ is affine, $V$ is cohomologically affine and thus affine by Serre's Criterion for Affineness (4.4.16). It follows that $V'$ is also affine (Proposition 4.4.19). Since $\Gamma(\mathcal{W}', \mathcal{O}_{\mathcal{W}'})$ is an artinian $\Bbbk$-algebra and has no non-trivial affine étale covers, Luna's Étale Slice Theorem (6.5.4) implies that we can arrange that $\mathcal{W}' \cong [\operatorname{Spec} A'/G]$.

We will also need the following criteria for morphisms to be closed immersions or isomorphisms.

**Lemma 6.5.12.** *Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a representable morphism of algebraic stacks of finite type over an algebraically closed field $\Bbbk$ with affine diagonal. Assume that $|\mathcal{X}| = \{x\}$ and $|\mathcal{Y}| = \{y\}$ consist of a single point and that $f$ induces an isomorphism $\mathcal{X}_0 := \mathbf{B}G_x$ with $\mathcal{Y}_0 := \mathbf{B}G_y$. Let $\mathfrak{m}_x \subset \mathcal{O}_{\mathcal{X}}$ and $\mathfrak{m}_y \subset \mathcal{O}_{\mathcal{Y}}$ be the ideal sheaves defining $\mathcal{X}_0$ and $\mathcal{Y}_0$, and let $f_1\colon \mathcal{X}_1 \to \mathcal{Y}_1$ be the induced morphism between the first nilpotent thickenings of $\mathcal{X}_0$ and $\mathcal{Y}_0$.*

*(1) If $f_1$ is a closed immersion, then so is $f$.*

*(2) If $f_1$ is a closed immersion and there is an isomorphism $\bigoplus_{n \geq 0} \mathfrak{m}_y^n/\mathfrak{m}_y^{n+1} \cong \bigoplus_{n \geq 0} \mathfrak{m}_x^n/\mathfrak{m}_x^{n+1}$ of graded $\mathcal{O}_{\mathcal{X}_0}$-modules, then $f$ is an isomorphism.*

*Proof.* Choose a smooth presentation $V = \operatorname{Spec} B \to \mathcal{Y}$ from an affine scheme such that $V \times_{\mathcal{Y}} \mathcal{Y}_0 \cong \operatorname{Spec} \Bbbk$ (Theorem 3.6.1). Then $B$ is a local artinian $\Bbbk$-algebra as $\mathcal{Y}$ consists of only one point. The base change $U = V \times_{\mathcal{Y}} \mathcal{X}$ is an algebraic space and since $U_{\mathrm{red}} = V_{\mathrm{red}}$ is a point, it follows from Proposition 4.4.19 that $U = \operatorname{Spec} A$ with $A$ a local artinian $\Bbbk$-algebra. We can therefore assume that $f\colon \operatorname{Spec} A \to \operatorname{Spec} B$ is a morphism of local artinian schemes.

For (1), we need to show that if $B/\mathfrak{m}_B^2 \to A/\mathfrak{m}_A^2$ is surjective, so is $B \to A$. We first claim that the inclusion $\mathfrak{m}_B A \hookrightarrow \mathfrak{m}_A$ is surjective. By Nakayama's Lemma, it suffices to show that $\mathfrak{m}_B A/\mathfrak{m}_A \mathfrak{m}_B A \to \mathfrak{m}_A/\mathfrak{m}_A^2$ is surjective, but this follows from the hypothesis that the composition $\mathfrak{m}_A/\mathfrak{m}_A^2 \to \mathfrak{m}_B A/\mathfrak{m}_A \mathfrak{m}_B A \to \mathfrak{m}_A/\mathfrak{m}_A^2$ is surjective. Since $B/\mathfrak{m}_B \to A/\mathfrak{m}_B A = A/\mathfrak{m}_A$ is surjective, another application of Nakayama's Lemma shows that $B \to A$ is surjective. See also [Har77, Lem. II.7.4] for a related criterion.

For (2), since $\dim_{\Bbbk} \mathfrak{m}_B^n/\mathfrak{m}_B^{n+1} = \dim_{\Bbbk} \mathfrak{m}_A^n/\mathfrak{m}_A^{n+1}$, the surjections $\mathfrak{m}_B^n/\mathfrak{m}_B^{n+1} \to \mathfrak{m}_A^n/\mathfrak{m}_A^{n+1}$ are isomorphisms and it follows that $f$ is an isomorphism. □

### 6.5.3 Proof of the Local Structure Theorem—smooth case

*Proof of Theorem 6.5.1—smooth case.* Since the $\Bbbk$-point $x \in \mathcal{X}$ is locally closed (Proposition 3.5.16), by replacing $\mathcal{X}$ by an open substack we may assume that $x \in \mathcal{X}$ is a closed point. Let $\mathcal{I}$ be the coherent sheaf of ideals defining $\mathcal{X}_0 := \mathbf{B}G_x \hookrightarrow \mathcal{X}$ and set $\mathcal{X}_n$ to be the $n$th nilpotent thickening defined by $\mathcal{I}^{n+1}$. The Zariski tangent space $T_{\mathcal{X},x}$ can be identified with the normal space $(\mathcal{I}/\mathcal{I}^2)^\vee$ to the orbit, viewed as a $G_x$-representation. (Note that when $\mathcal{X} = [X/G]$ with $G$ a smooth affine algebraic group, then $T_{\mathcal{X},x}$ is identified with the normal space to the orbit $T_{X,\widetilde{x}}/T_{Gx,\widetilde{x}}$ for a point $\widetilde{x} \in X(\Bbbk)$ over $x$.)

Define the quotient stack $\mathcal{T} = [T_{\mathcal{X},x}/G_x]$ and let $\mathcal{T}_0 = \mathbf{B}G_x$ be the closed substack supported at the origin and $\mathcal{T}_n$ its $n$th nilpotent thickenings. We claim that there are compatible isomorphisms $\mathcal{X}_n \cong \mathcal{T}_n$. Since $G_x$ is linearly reductive, $\mathcal{X}_0 = \mathbf{B}G_x$ is cohomologically affine. By the deformation theory of principal $G_x$-bundles (Proposition 6.5.10), we can inductively extending the principal $G_x$-bundle $\operatorname{Spec} \Bbbk \to \mathcal{X}_0$ to principal $G_x$-bundles $\operatorname{Spec} A_n \to \mathcal{X}_n$. This yields isomorphisms $\mathcal{X}_n \cong [\operatorname{Spec} A_n/G_x]$ and affine morphisms $\mathcal{X}_n \to \mathbf{B}G_x$. We have a closed immersion $\mathcal{X}_0 \hookrightarrow \mathcal{T}$ and we can inductively find lifts

$$\begin{tikzcd} \mathcal{X}_n \arrow[r] \arrow[d, hook] & \mathcal{T} \arrow[d] \\ \mathcal{X}_{n+1} \arrow[r] \arrow[ur, dashed] & \mathbf{B}G_x \end{tikzcd}$$

since $\mathcal{T} \to \mathbf{B}G_x$ is smooth and affine (Proposition 6.5.8). The induced morphism $\mathcal{X}_1 \to \mathcal{T}_1$ is an isomorphism since it is a morphism between deformations $\mathbf{B}G_x \hookrightarrow \mathcal{X}_1$ and $\mathbf{B}G_x \hookrightarrow \mathcal{T}_1$ of the coherent sheaf $\mathcal{I}/\mathcal{I}^2$ and any such morphism is an isomorphism (by reducing to Lemma D.1.7 by smooth descent). (In fact, both $\mathcal{X}_1$ and $\mathcal{T}_1$ are trivial deformations as they admit retractions to $\mathbf{B}G_x$.) Lemma 6.5.12(2) now implies that the maps $\mathcal{X}_n \to \mathcal{T}_n$ are isomorphisms.

Let $\pi \colon \mathcal{T} \to T = T_{\mathcal{X},x}/\!\!/G_x$ be the morphism to the GIT quotient. The fiber product $\widehat{\mathcal{T}} := \operatorname{Spec} \widehat{\mathcal{O}}_{T,\pi(0)} \times_T \mathcal{T}$ is a quotient stack of the form $[\operatorname{Spec} B/G]$ where $B$ is of finite type over the noetherian complete local $\Bbbk$-algebra $B^G = \widehat{\mathcal{O}}_{T,\pi(0)}$. Therefore $\widehat{\mathcal{T}}$ is coherently complete along $\mathcal{T}_0$ (Theorem 6.4.11) and $\operatorname{MOR}(\mathcal{T}, \mathcal{X}) \xrightarrow{\sim} \varprojlim \operatorname{MOR}(\mathcal{T}_n, \mathcal{X})$ is an equivalence by Coherent Tannaka Duality (6.4.8). It follows that the morphisms $\mathcal{X}_n \cong \mathcal{T}_n \hookrightarrow \mathcal{X}$ extend to a morphism $\widehat{\mathcal{T}} \to \mathcal{X}$ filling in the diagram

$$\begin{tikzcd} \mathcal{X}_n \cong \mathcal{T}_n \arrow[r] \arrow[rrr, bend left] & \widehat{\mathcal{T}} \arrow[r] \arrow[d] \arrow[rr, dashed, bend left] \arrow[dr, phantom, "\square"] & \mathcal{T} \arrow[d] & \mathcal{X} \\ & \operatorname{Spec} \widehat{\mathcal{O}}_{T,\pi(0)} \arrow[r] & T. & \end{tikzcd}$$

The functor parameterizing isomorphism classes of morphisms

$$F\colon \mathrm{Sch}/T \to \mathrm{Sets}, \qquad (T' \to T) \mapsto \{T' \times_T \mathcal{T} \to \mathcal{X}\}/\sim$$

is limit preserving as $\mathcal{X}$ is of finite type over $\Bbbk$ (see Exercise 3.3.31). The morphism $\widehat{\mathcal{T}} \to \mathcal{X}$ yields an element of $F$ over $\operatorname{Spec} \widehat{\mathcal{O}}_{T,\pi(0)}$. By Artin Approximation (A.10.9), there exists an étale morphism $(U,u) \to (T,0)$ where $U$ is an affine scheme with a $\Bbbk$-point $u \in U$ and a morphism $(U \times_T \mathcal{T}, (u,0)) \to (\mathcal{X}, x)$ agreeing with $(\widehat{\mathcal{T}}, 0) \to (\mathcal{X}, x)$ to first order. Since $U \times_T \mathcal{T}$ is smooth at $(u,0)$ and $\mathcal{X}$ is smooth at $x$, and since $U \times_T \mathcal{T} \to \mathcal{X}$ induces an isomorphism of tangent spaces and stabilizer groups at $(u,0)$, the morphism $U \times_T \mathcal{T} \to \mathcal{X}$ is étale at $(u,0)$. Observe that $U \times_T \mathcal{T}$ is of the form $[\operatorname{Spec} A/G_x]$ for a finitely generated $\Bbbk$-algebra $A$ such that $U = \operatorname{Spec} A^{G_x}$. We can arrange that $[\operatorname{Spec} A/G_x] \to \mathcal{X}$ is étale everywhere after replacing $U$ with an open affine subscheme and $\operatorname{Spec} A$ with its preimage. That $[\operatorname{Spec} A/G_x] \to \mathcal{X}$ can be arranged to be affine follows from Proposition 6.5.13. □

**Proposition 6.5.13.** *Let $\mathcal{X}$ be an algebraic stack of finite type over an algebraically closed field with affine diagonal. Let $f\colon [\operatorname{Spec} A/G] \to \mathcal{X}$ be a finite type morphism with $G$ linearly reductive. If $w \in \operatorname{Spec} A$ has closed $G$-orbit and $f$ induces an isomorphism of stabilizer groups at $w$, then there exists a $G$-invariant, affine, and open subscheme $U \subset \operatorname{Spec} A$ containing $w$ such that $f|_{[U/G]}$ is affine.*

*Proof.* Set $\mathcal{W} = [\operatorname{Spec} A/G]$ with $\pi\colon \mathcal{W} \to \operatorname{Spec} A^G$. Since $f\colon \mathcal{W} \to \mathcal{X}$ is quasi-finite on an open subset $\mathcal{U}$, then $\{\pi(w)\}$ and $\pi(\mathcal{W} \setminus \mathcal{U})$ are disjoint closed subspaces and choosing an affine open $V \subset \operatorname{Spec} A^G \setminus \pi(\mathcal{W} \setminus \mathcal{U})$ containing $\pi(w)$, we may replace $\mathcal{W}$ with $\pi^{-1}(V)$ and we can assume that $f\colon \mathcal{W} \to \mathcal{X}$ is quasi-finite.

Choose a smooth presentation $V = \operatorname{Spec} B \to \mathcal{X}$ and consider the fiber product

$$\begin{tikzcd} \mathcal{W}_V \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & V = \operatorname{Spec} B \arrow[d] \\ \mathcal{W} = [\operatorname{Spec} A/G] \arrow[r] & \mathcal{X}. \end{tikzcd}$$

Since $\mathcal{X}$ has affine diagonal, $\operatorname{Spec} B \to \mathcal{X}$ is affine and therefore $\mathcal{W}_V$ is cohomologically affine. As $\mathcal{W}_V$ has quasi-finite diagonal, Proposition 6.3.28 implies that $\mathcal{W}_V \to V$ is separated, and it follows from descent that $\mathcal{W} \to \mathcal{X}$ is also separated and that the relative inertia $I_{\mathcal{W}/\mathcal{X}} \to \mathcal{W}$ is finite. Since the fiber over $w \in \mathcal{W}$ is trivial, there is an open neighborhood $\mathcal{U}$ over which the relative inertia is trivial. As in the first paragraph, we may replace $\mathcal{U}$ with an open substack of the form $[\operatorname{Spec} C/G]$ containing $w$. Since $f|_{\mathcal{U}}\colon \mathcal{U} \to \mathcal{X}$ is a representable and cohomologically affine morphism, Serre's Criterion for Affineness (6.3.16) implies that $f|_{\mathcal{U}}$ is affine. □

### 6.5.4 Equivariant Artin Algebraization

The smoothness hypothesis of $x \in \mathcal{X}$ was used above to establish that $\mathcal{T}_n \cong \mathcal{X}_n$ and that $U \times_T \mathcal{T} \to \mathcal{X}$ is étale. More critically, it implied that $\varprojlim \Gamma(\mathcal{X}_n, \mathcal{O}_{\mathcal{X}_n})$, which is identified with the $G_x$-invariants of a miniversal deformation space, is the completion of a finitely generated $\Bbbk$-algebra, namely $\widehat{\mathcal{O}}_{T,0}$. If $x \in \mathcal{X}$ is not smooth, it seems difficult to directly establish that $\varprojlim \Gamma(\mathcal{X}_n, \mathcal{O}_{\mathcal{X}_n})$ is the completion of a finitely generated $\Bbbk$-algebra. Recall that we encountered a similar issue when discussing Artin Algebraization (D.6.6). When the complete local ring $R$ is known to be the completion of a finitely generated algebra, then Artin Algebraization is an easy consequence of Artin Approximation (see Remark D.6.8). To circumvent this issue in our general proof of Artin Algebraization, we wrote $R = \widehat{\mathcal{O}}_{V,v}/I$ where $V$ is a finite type $\Bbbk$-scheme and used Artin Approximation to simultaneously approximate both the given object over $R$ and the equations defining $I$. We follow a similar strategy but proceed $G$-equivariantly this time.

We will use the following extension of the notion of formal versality introduced in Definition D.3.5: for an algebraic stack $\widehat{\mathcal{T}}$ with a unique closed point $t$, a morphism $\widehat{\xi}\colon \widehat{\mathcal{T}} \to \mathcal{X}$ of prestacks over Sch is *formally versal at $t$* if every commutative diagram

$$\begin{tikzcd} \mathcal{Z} \arrow[r] \arrow[d, hook] & \widehat{\mathcal{T}} \arrow[d, "\widehat{\xi}"] \\ \mathcal{Z}' \arrow[r] \arrow[ur, dashed] & \mathcal{X} \end{tikzcd}$$

has a lift, where $\mathcal{Z} \hookrightarrow \mathcal{Z}'$ is a closed immersion of noetherian algebraic stacks with affine diagonal, $|\mathcal{Z}| = |\mathcal{Z}'|$ consists of a single point and the image of $\mathcal{Z} \to \widehat{\mathcal{T}}$ is $t$.

**Theorem 6.5.14** (Equivariant Artin Algebraization)**.** *Let $\Bbbk$ be an algebraically closed field and $R$ be a complete noetherian local $\Bbbk$-algebra with residue field $\Bbbk$. Let $\widehat{\mathcal{T}} = [\operatorname{Spec} B/G]$ be an algebraic stack of finite type over $R = B^G$, where $G$ is linearly reductive. Assume that the unique closed point $t \in \widehat{\mathcal{T}}$ has stabilizer equal to $G$. If $\mathcal{X}$ is a limit preserving prestack over $\mathrm{Sch}/\Bbbk$ and $\eta\colon \widehat{\mathcal{T}} \to \mathcal{X}$ is a morphism of prestacks formally versal at $t$, then there exists*

1. *an algebraic stack $\mathcal{W} = [\operatorname{Spec} A/G]$ of finite type over $\Bbbk$ and a closed point $w \in \mathcal{W}$;*
2. *morphisms $f\colon \mathcal{W} \to \mathcal{X}$ and $\varphi\colon \widehat{\mathcal{T}} \to \mathcal{W}$ such that in the diagram*

$$\begin{tikzcd} \widehat{\mathcal{T}} \arrow[d, "\varphi"'] \arrow[dr, "\eta"] & \\ \mathcal{W} \arrow[r, "f"] & \mathcal{X} \end{tikzcd} \tag{6.5.4}$$

*the induced morphisms $\varphi_n\colon \widehat{\mathcal{T}}_n \to \mathcal{W}_n$ between the $n$th nilpotent thickenings of $t$ and $w$ are isomorphisms, and there exists compatible 2-isomorphisms $\eta_n \xrightarrow{\sim} f_n \circ \varphi_n$.*

*Moreover, if $\mathcal{X}$ is an algebraic stack of finite type over $\Bbbk$ with affine diagonal, then it can be arranged that (6.5.4) is commutative and that $\varphi$ induces an isomorphism $\widehat{\mathcal{T}} \to \widehat{\mathcal{W}} := \mathcal{W} \times_W \operatorname{Spec} \widehat{\mathcal{O}}_{W,\pi(w)}$, where $\pi\colon \mathcal{W} \to W = \operatorname{Spec} A^G$.*

**Remark 6.5.15.** If one takes $G$ to be the trivial group, one recovers the classical version of Artin Algebraization (D.6.6).

As in the proof of Theorem D.6.6, we will apply Artin Approximation to a well-chosen integer $N$ to construct $\mathcal{W}$ such that there are isomorphisms $\mathcal{W}_n \cong \widehat{\mathcal{T}}_n$ for $n \le N$ and such that the Artin–Rees Lemma implies that there are also isomorphisms $\mathcal{W}_n \cong \widehat{\mathcal{T}}_n$ for $n > N$. To get control over the constant in the Artin–Rees Lemma, we need to generalize Definition D.6.3: for a noetherian algebraic stack $\mathcal{X}$ with a closed point $x$ defined by a a sheaf of ideals $\mathfrak{m}_x$ and an integer $c \ge 0$, we say that $(\mathsf{AR})_c$ *holds at $x$* for a map $\varphi\colon E \to F$ of coherent sheaves on $\mathcal{X}$ if

$$\varphi(E) \cap \mathfrak{m}_x^n F \subset \varphi(\mathfrak{m}_x^{n-c} E), \quad \forall n \ge c.$$

When $\mathcal{X}$ is a scheme, $(\mathsf{AR})_c$ holds for all sufficiently large $c$ by the Artin–Rees lemma, and it even holds replacing $\{x\}$ with a closed subscheme. By smooth descent, $(\mathsf{AR})_c$ also holds for algebraic stacks for $c \gg 0$.

*Proof.* The morphism $\eta\colon \widehat{\mathcal{T}} \to \mathcal{X}$ and the structure morphism $\widehat{\mathcal{T}} \to \mathbf{B}G$ induce a morphism $\widehat{\mathcal{T}} \to \mathcal{X} \times \mathbf{B}G$. We let $\widehat{T} = \operatorname{Spec} R$ be the GIT quotient of $\widehat{\mathcal{T}} = [\operatorname{Spec} B/G]$. Since $R$ is the colimit of its finitely generated $\Bbbk$-subalgebras and $\mathcal{X} \times \mathbf{B}G$ is limit preserving, limit methods (§A.6) imply that there is a commutative diagram

$$\begin{tikzcd} \widehat{\mathcal{T}} \arrow[r] \arrow[rr, bend left] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{S} \arrow[r] \arrow[d] & \mathcal{X} \times \mathbf{B}G \arrow[d] \\ \widehat{T} \arrow[r] & S \arrow[r] & \operatorname{Spec} \Bbbk \end{tikzcd}$$

where $S = \operatorname{Spec} R'$ is an affine scheme of finite type over $\Bbbk$, $\mathcal{S}$ is an algebraic stack of finite type over $S$ with affine diagonal such that $\widehat{\mathcal{T}} = \widehat{T} \times_S \mathcal{S}$, and $\widehat{\mathcal{T}} \to \mathcal{X} \times \mathbf{B}G$ factors as $\widehat{\mathcal{T}} \to \mathcal{S} \to \mathcal{X} \times \mathbf{B}G$. Moreover, we can arrange that $\mathcal{S} \to \mathbf{B}G$ is affine. Let $\widetilde{s} \in \mathcal{S}$ and $s \in S$ be the images of $t$. By possibly adding generators to $R'$ so that $R' \to R \to R/\mathfrak{m}_R^2$ is surjective, we can arrange that $\widehat{\mathcal{O}}_{S,s} \to R$ is surjective by Lemma A.10.15, or in other words that $\widehat{T} \to \widehat{S} := \operatorname{Spec} \widehat{\mathcal{O}}_{S,s}$ is a closed immersion.

Note that $\widehat{\mathcal{T}}$ is a closed substack of $\mathcal{S} \times_S \widehat{S}$. By choosing a resolution $\mathcal{O}_{\widehat{S}}^{\oplus r} \to \mathcal{O}_{\widehat{S}} \twoheadrightarrow R$ and pulling it back $\mathcal{S} \times_S \widehat{S}$, we obtain a resolution

$$\ker(\beta) \xhookrightarrow{\alpha} \mathcal{O}_{\mathcal{S} \times_S \widehat{S}}^{\oplus r} \xrightarrow{\beta} \mathcal{O}_{\mathcal{S} \times_S \widehat{S}} \twoheadrightarrow \mathcal{O}_{\widehat{\mathcal{T}}}. \tag{6.5.5}$$

Consider the functor $F\colon \mathrm{Sch}/S \to \mathrm{Sets}$ assigning $(U \to S)$ to the set of isomorphism classes of complexes

$$L \xrightarrow{\alpha} \mathcal{O}_{\mathcal{S} \times_S U}^{\oplus r} \xrightarrow{\beta} \mathcal{O}_{\mathcal{S} \times_S U}$$

of finitely presented quasi-coherent $\mathcal{O}_{\mathcal{S} \times_S U}$-modules. By standard limit arguments, $F$ is limit preserving. The complex (6.5.5) defines an element $(\alpha, \beta) \in F(\widehat{S})$ such that $\operatorname{coker}(\beta) = \mathcal{O}_{\widehat{\mathcal{T}}}$. Let $N$ be an integer such that $(\mathsf{AR})_N$ holds for $\alpha$ and $\beta$ at $(\widetilde{s}, s)$.

Artin Approximation (A.10.9) gives an étale neighborhood $(S', s') \to (S, s)$ and an element $(\alpha', \beta') \in F(S')$ such that $(\alpha, \beta) = (\alpha', \beta')$ in $F(\mathcal{O}_{S,s}/\mathfrak{m}_s^{N+1})$. We let $\mathcal{W} \hookrightarrow \mathcal{S} \times_S S'$ be the closed substack defined by $\operatorname{coker}(\beta')$ and set $w = (\widetilde{s}, s') \in \mathcal{W}$. Letting $S_n$, $S'_n$ and $\widehat{T}_n$ be the $n$th nilpotent thickenings of $S$, $S'$ and $\widehat{T}$ at the images of $t \in \mathcal{T}$, we have that $\widehat{\mathcal{T}} \times_{\widehat{T}} \widehat{T}_N$ and $\mathcal{W} \times_{S'} S'_N$ are equal as closed substacks of $\mathcal{S} \times_S S_N$. This gives (1)-(2) for $n \le N$. In particular, we have an isomorphism $\varphi_N\colon \widehat{\mathcal{T}}_N \to \mathcal{W}_N$ and we let $\psi_N\colon \mathcal{W}_N \to \widehat{\mathcal{T}}_N$ be its inverse.

Using that $\eta\colon \widehat{\mathcal{T}} \to \mathcal{X}$ is formally versal, we can inductively find compatible lifts for $n \ge N$

$$\begin{tikzcd} \mathcal{W}_n \arrow[r, "\psi_n"] \arrow[d, hook] & \widehat{\mathcal{T}} \arrow[d, "\eta"] \\ \mathcal{W}_{n+1} \arrow[r] \arrow[ur, dashed, "\psi_{n+1}"] & \mathcal{X}. \end{tikzcd}$$

On the other hand, applying Lemma D.6.4 (generalized to stacks by smooth descent) on $\mathcal{S} \times_S \widehat{S}$ with $c = N$ to the complex (6.5.5) and the restriction of the complex defined by $(\alpha', \beta')$, we obtain an isomorphism $\mathrm{Gr}_{\mathfrak{m}_t} \mathcal{O}_{\widehat{\mathcal{T}}} \cong \mathrm{Gr}_{\mathfrak{m}_w} \mathcal{O}_{\mathcal{W}}$ of graded $\mathcal{O}_{\mathbf{B}G}$-modules. By Lemma 6.5.12, the induced morphisms $\psi_n \colon \mathcal{W}_n \to \widehat{\mathcal{T}}_n$ are isomorphisms for all $n$. As $\widehat{\mathcal{T}}$ is coherently complete (Theorem 6.4.11), Coherent Tannaka Duality (6.4.8) implies that the inverses $\varphi_n = \psi_n^{-1} \colon \widehat{\mathcal{T}}_n \to \mathcal{W}_n$ effectivize to a morphism $\varphi \colon \widehat{\mathcal{T}} \to \mathcal{W}$. This completes (1)-(2).

For the final statement, when $\mathcal{X}$ is algebraic, we again apply Coherent Tannaka Duality, using the coherent completeness of both $\widehat{\mathcal{T}}$ and $\mathcal{W}$. By applying Corollary 6.4.8 to the inverses $\psi_n = \varphi_n^{-1}$, we can construct an inverse $\psi \colon \widehat{\mathcal{W}} \to \widehat{\mathcal{T}}$ of $\varphi$. Thus $\varphi \colon \widehat{\mathcal{T}} \to \widehat{\mathcal{W}}$ is an isomorphism. Using the fully faithfulness of Corollary 6.4.8, there is a 2-isomorphism $\eta \to f \circ \varphi$ extending the given 2-isomorphisms $\eta_n \xrightarrow{\sim} f_n \circ \varphi_n$ and thus $\widehat{\mathcal{T}} \to \widehat{\mathcal{W}}$ is a morphism over $\mathcal{X}$. $\square$

### 6.5.5 Proof of the Local Structure Theorem—general case

*Proof of Theorem 6.5.1.* We may assume that $x \in \mathcal{X}$ is a closed point. Let $\mathcal{T} := [T_{\mathcal{X},x}/G_x]$, let $\pi \colon \mathcal{T} \to T = T_{\mathcal{X},x}/\!\!/G_x$ be the morphism to the GIT quotient, and let $\widehat{\mathcal{T}} := \operatorname{Spec} \widehat{\mathcal{O}}_{T,\pi(0)} \times_T \mathcal{T}$. Let $\mathcal{T}_0 = \mathbf{B}G_x$ be the closed substack supported at the origin and $\mathcal{T}_n$ its $n$th nilpotent thickenings.

We will construct compatible closed immersions $\mathcal{X}_n \hookrightarrow \mathcal{T}_n$. Since $G_x$ is linearly reductive, $\mathcal{X}_0 = \mathbf{B}G_x$ is cohomologically affine. By deforming the principal $G_x$-bundle $\operatorname{Spec} \Bbbk \to \mathcal{X}_0$ using Proposition 6.5.10, we can inductively construct isomorphisms $\mathcal{X}_n \cong [\operatorname{Spec} A_n/G_x]$. By the deformation theory of the smooth and affine morphism $\widehat{\mathcal{T}} \to \mathbf{B}G_x$ (Proposition 6.5.8), we can inductively find lifts

$$\begin{tikzcd} \mathcal{X}_n \arrow[r] \arrow[d, hook] & \mathcal{T} \arrow[d] \\ \mathcal{X}_{n+1} \arrow[r] \arrow[ur, dashed] & \mathbf{B}G_x. \end{tikzcd}$$

As in the smooth case, $\mathcal{X}_1 \to \mathcal{T}_1$ is an isomorphism. By Lemma 6.5.12(1), each morphism $\mathcal{X}_n \to \mathcal{T}_n$ is a closed immersion.

If $I_n$ denotes the ideal sheaf defining $\mathcal{X}_n \hookrightarrow \mathcal{T}_n$, then $\mathcal{O}_{\mathcal{T}_n}/I_n$ is a system of coherent $\mathcal{O}_{\mathcal{T}_n}$-modules. Since $\widehat{\mathcal{T}}$ is coherently complete (Theorem 6.4.11), there exists a coherent sheaf of ideals $I \subset \mathcal{O}_{\widehat{\mathcal{T}}}$ such that the surjection $\mathcal{O}_{\widehat{\mathcal{T}}} \to \mathcal{O}_{\widehat{\mathcal{T}}}/I$ extends the surjections $\mathcal{O}_{\mathcal{T}_n} \to \mathcal{O}_{\mathcal{X}_n}$. The closed immersion $\widehat{\mathcal{X}} \hookrightarrow \widehat{\mathcal{T}}$ defined by $I$ extends the given closed immersions $\mathcal{X}_n \hookrightarrow \mathcal{T}_n$ yielding a commutative diagram

$$\begin{tikzcd} \mathcal{X}_n \arrow[r, hook] \arrow[d, hook] \arrow[rr, bend left] & \widehat{\mathcal{X}} \arrow[r, dashed, "\eta"] \arrow[d, hook] & \mathcal{X} \\ \mathcal{T}_n \arrow[r, hook] & \widehat{\mathcal{T}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{T} \arrow[d] \\ & \operatorname{Spec} \widehat{\mathcal{O}}_{T,0} \arrow[r] & T. \end{tikzcd}$$

of solid arrows. Since $\widehat{\mathcal{X}}$ is also coherently complete, Coherent Tannaka Duality (6.4.8) gives a morphism $\eta \colon \widehat{\mathcal{X}} \to \mathcal{X}$ extending the above diagram. Since $\widehat{\mathcal{X}}$ has the same nilpotent thickenings of $\widehat{\mathcal{X}}$, the morphism $\eta \colon \widehat{\mathcal{X}} \to \mathcal{X}$ is formally versal at 0.

By Equivariant Artin Algebraization (6.5.14) with $G = G_x$, we obtain a morphism $f\colon \mathcal{W} = [\operatorname{Spec} B/G_x] \to \mathcal{X}$ from an algebraic stack $\mathcal{W}$ of finite type over $\Bbbk$ with a closed point $w \in \mathcal{W}$ and a morphism $\varphi\colon \widehat{\mathcal{X}} \to \mathcal{W}$ over $\mathcal{X}$ inducing an isomorphism $\widehat{\mathcal{X}} \to \mathcal{W} \times_W \operatorname{Spec} \widehat{\mathcal{O}}_{W,\pi(w)}$ where $\pi\colon \mathcal{W} \to \operatorname{Spec} B^{G_x}$. Since $f\colon \mathcal{W} \to \mathcal{X}$ induces isomorphisms $\mathcal{W}_n \to \mathcal{X}_n$, $f$ is étale at $w$. After replacing $\mathcal{W}$ with an open substack, we can arrange that $f$ is étale everywhere. By Proposition 6.5.13, we can also arrange that $f$ is affine.

See also [AHR20, AHR19, AHLHR22]. □

### 6.5.6 The coherent completion at a point

We say that $(\mathcal{X}, x)$ is a *complete local stack* if $\mathcal{X}$ is a noetherian algebraic stack with affine stabilizers and with a unique closed point $x$ such that $\mathcal{X}$ is coherently complete along the residual gerbe $\mathcal{G}_x$. An important example is $([\operatorname{Spec} A/G], x)$ where $G$ is linearly reductive over an algebraically closed field $\Bbbk$, $A^G$ is a complete noetherian local $\Bbbk$-algebra with residue field $\Bbbk$, $A$ is of finite type over $R$, and the unique closed point $x$ is fixed by $G$ (Theorem 6.4.11). For instance, $([\mathbb{A}^n/\mathbb{G}_m], 0)$ is complete local.

The *coherent completion* of a noetherian algebraic stack $\mathcal{X}$ at a point $x$ is a complete local stack $(\widehat{\mathcal{X}}_x, \widehat{x})$ together with a morphism $\eta\colon (\widehat{\mathcal{X}}_x, \widehat{x}) \to (\mathcal{X}, x)$ inducing isomorphisms of $n$th infinitesimal neighborhoods of $\widehat{x}$ and $x$. If $\mathcal{X}$ has affine stabilizers, then the pair $(\widehat{\mathcal{X}}_x, \eta)$ is unique up to unique 2-isomorphism by Coherent Tannaka Duality (6.4.8).

**Theorem 6.5.16.** *Let $\mathcal{X}$ be an algebraic stack of finite type over an algebraically closed field $\Bbbk$ with affine diagonal. For every point $x \in \mathcal{X}(k)$ with linearly reductive stabilizer $G_x$, the coherent completion $\widehat{\mathcal{X}}_x$ exists. Moreover,*

*(1) The coherent completion is a quotient stack $\widehat{\mathcal{X}}_x = [\operatorname{Spec} B/G_x]$ such that the invariant ring $B^{G_x}$ is the completion of a finite type $\Bbbk$-algebra and $B^{G_x} \to B$ is of finite type.*

*(2) Let $f\colon (\mathcal{W}, w) \to (\mathcal{X}, x)$ be an étale morphism where $\mathcal{W} = [\operatorname{Spec} A/G_x]$, the point $w \in |\mathcal{W}|$ is closed, and $f$ induces an isomorphism of stabilizer groups at $w$. Then $\widehat{\mathcal{X}}_x = \mathcal{W} \times_W \operatorname{Spec} \widehat{\mathcal{O}}_{W,\pi(w)}$, where $\pi\colon \mathcal{W} \to W = \operatorname{Spec} A^{G_x}$ is the morphism to the GIT quotient.*

*(3) If $\pi\colon \mathcal{X} \to X$ is a good moduli space, then $\widehat{\mathcal{X}}_x = \mathcal{X} \times_X \operatorname{Spec} \widehat{\mathcal{O}}_{X,\pi(x)}$.*

*Proof.* The Local Structure Theorem (6.5.1) gives an étale morphism $f\colon (\mathcal{W}, w) \to (\mathcal{X}, x)$, where $\mathcal{W} = [\operatorname{Spec} A/G_x]$ and $f$ induces an isomorphism of stabilizer groups at the closed point $w$. The main statement and Parts (1) and (2) follow by taking $\widehat{\mathcal{X}}_x = \mathcal{W} \times_W \operatorname{Spec} \widehat{\mathcal{O}}_{W,\pi(w)}$ and $B = A \otimes_{A^{G_x}} \widehat{A^{G_x}}$. Indeed, $\widehat{\mathcal{X}}_x = [\operatorname{Spec} B/G_x]$ is coherently complete by Theorem 6.4.11. Part (3) follows from (2) using Corollary 6.5.3. □

We have the following stacky generalization of the fact that completions determine the étale local structure of finite type schemes (Corollary A.10.13).

**Theorem 6.5.17.** *Let $\mathcal{X}$ and $\mathcal{Y}$ be algebraic stacks of finite type over an algebraically closed field $\Bbbk$ with affine diagonal. Suppose $x \in \mathcal{X}$ and $y \in \mathcal{Y}$ are $\Bbbk$-points with linearly reductive stabilizer group schemes $G_x$ and $G_y$, respectively. Then the following are equivalent:*

*(1) There exist compatible isomorphisms $\mathcal{X}_n \to \mathcal{Y}_n$.*

*(2) There exists an isomorphism $\widehat{\mathcal{X}}_x \to \widehat{\mathcal{Y}}_y$.*

*(3) There exist an affine scheme* $\operatorname{Spec} A$ *with an action of* $G_x$*, a point* $w \in \operatorname{Spec} A$ *fixed by* $G_x$*, and a diagram of étale morphisms*

$$\begin{tikzcd} & {[\operatorname{Spec} A/G_x]} \arrow[dl, "f"'] \arrow[dr, "g"] & \\ \mathcal{X} & & \mathcal{Y} \end{tikzcd}$$

*such that* $f(w) = x$ *and* $g(w) = y$*, and both* $f$ *and* $g$ *induce isomorphisms of stabilizer groups at* $w$*.*

*If, in addition, the points* $x \in \mathcal{X}$ *and* $y \in \mathcal{Y}$ *are smooth, then the conditions above are equivalent to the existence of an isomorphism* $G_x \to G_y$ *of group schemes and an isomorphism* $T_{\mathcal{X},x} \to T_{\mathcal{Y},y}$ *of tangent spaces which is equivariant under* $G_x \to G_y$*.*

*Proof.* The implications (3) $\Longrightarrow$ (2) $\Longrightarrow$ (1) are immediate. We also have (1) $\Longrightarrow$ (2) by Coherent Tannaka Duality (6.4.8) To show that (2) $\Longrightarrow$ (3), let $(\mathcal{W} = [\operatorname{Spec} A/G_x], w) \to (\mathcal{X}, x)$ be an étale neighborhood as given by the Local Structure Theorem (6.5.1). Let $\pi \colon \mathcal{W} \to W = \operatorname{Spec} A^{G_x}$ denote the good moduli space. Then $\widehat{\mathcal{X}}_x = \mathcal{W} \times_W \operatorname{Spec} \widehat{\mathcal{O}}_{W,\pi(w)}$. The functor

$$F \colon \operatorname{Sch}/W \to \operatorname{Sets}, \quad (T \to W) \mapsto \operatorname{Hom}(\mathcal{W} \times_W T, \mathcal{Y})$$

is locally of finite presentation. Artin Approximation (A.10.9) applied to $F$ and $\alpha \in F(\operatorname{Spec} \widehat{\mathcal{O}}_{W,\pi(w)})$ provides an étale morphism $(W', w') \to (W, w)$ and a morphism $\varphi \colon \mathcal{W}' := \mathcal{W} \times_W W' \to \mathcal{Y}$ such that $\varphi|_{\mathcal{W}'_1} \colon \mathcal{W}'_1 \to \mathcal{Y}_1$ is an isomorphism. Since $\widehat{\mathcal{W}'}_{w'} \cong \widehat{\mathcal{X}}_x \cong \widehat{\mathcal{Y}}_y$, it follows that $\varphi$ induces an isomorphism $\widehat{\mathcal{W}'} \to \widehat{\mathcal{Y}}$ by Lemma 6.5.12. After replacing $W'$ with an open neighborhood, we thus obtain an étale morphism $(\mathcal{W}', w') \to (\mathcal{Y}, y)$. The final statement is clear from Luna's Etale Slice Theorem (6.5.4). $\square$

### 6.5.7 Applications to equivariant geometry

Sumihiro's Theorem on Torus Action (C.3.13) asserts that for a normal scheme of finite type over $\Bbbk$ with the action of a torus $T$, every $\Bbbk$-point has a $T$-invariant affine open neighborhood. If $X$ is not normal, there are not necessarily $T$-invariant affine open neighborhoods, e.g. consider nodal cubic $X$ equipped with a $\mathbb{G}_m$-action near its node $x \in X$. However, there is always a $T$-equivariant affine *étale* neighborhood.

**Theorem 6.5.18.** *Let* $X$ *be an algebraic space locally of finite type over an algebraically closed field* $\Bbbk$ *with affine diagonal. Suppose that* $X$ *has an action of an affine algebraic group* $G$*. If* $x \in X(\Bbbk)$ *has linearly reductive stabilizer, then there exists a* $G$*-equivariant étale neighborhood* $(\operatorname{Spec} A, u) \to (X, x)$ *inducing an isomorphism of stabilizer groups at* $u$*.*

*If* $G$ *is a torus, then every point has a* $G$*-invariant étale neighborhood* $(\operatorname{Spec} A, u) \to (X, x)$ *inducing an isomorphism of stabilizer groups at* $u$*.*

*Proof.* By the Local Structure Theorem (6.5.1), there is an étale neighborhood $([\operatorname{Spec} A/G_x], u) \to ([X/G], x)$ such that $w$ is a closed point and $f$ induces an isomorphism of stabilizer groups at $w$. By Proposition 6.5.13, after replacing $\operatorname{Spec} A$ with a $G_x$-invariant open affine neighborhood of $w$, we can arrange that the composition $[\operatorname{Spec} A/G_x] \to [X/G] \to \mathbf{B}G$ is affine. Therefore, $W := [\operatorname{Spec} A/G_x] \times_{[X/G]} X$ is an affine scheme and $W \to X$ is a $G$-equivariant étale neighborhood of $x$.

When $G$ is a torus, then any subgroup of $G$ and in particular each stabilizer group is linearly reductive. $\square$

## 6.6 $\mathbb{G}_m$-actions, one-parameter subgroups, and filtrations

We show that the fixed locus of a linearly reductive group action on a smooth variety is smooth (Theorem 6.6.2) and prove the representability and properties of the attractor locus with respect to a $\mathbb{G}_m$-action (Theorem 6.6.7). After establishing a general version of the Białynicki-Birula Stratification (Theorem 6.6.9), we discuss applications to computing cohomology (§6.6.13). The Cartan Decomposition (Theorem 6.6.23) is used to prove the Destabilization Theorem (Theorem 6.6.28), which will be used to prove the Hilbert–Mumford Criterion in the next section. Finally, we interpret maps from $\Theta = [\mathbb{A}^1/\mathbb{G}_m]$ into quotient stacks as the data of a one-parameter subgroup and a point whose limit exists (Proposition 6.6.30) and into a stack of coherent sheaves as a filtration of a vector bundle (Proposition 6.6.33). Many of these results will be useful for the development of both Geometric Invariant Theory in §6.7 and the existence of good moduli spaces in §6.8.

### 6.6.1 Fixed loci

**Definition 6.6.1** (Fixed locus)**.** If $X$ is an algebraic space over a field $\Bbbk$ equipped with an action of an affine algebraic group $G$, we define the *fixed locus* as the functor

$$X^G := \underline{\mathrm{Mor}}^G(\operatorname{Spec}\Bbbk, X)\colon \mathrm{Sch}/\Bbbk \to \mathrm{Sets}$$

assigning a $\Bbbk$-scheme $S$ to the set of $G$-equivariant maps from $S$ to $X$, where $S$ is endowed with the trivial $G$ action.

**Theorem 6.6.2.** *Let $X$ be an algebraic space of finite type over an algebraically closed field $\Bbbk$ with affine diagonal equipped with an action of a linearly reductive algebraic group $G$. Then*

*(1) The fixed locus $X^G$ is represented by a subscheme of $X$;*

*(2) If $G$ is a torus, then $X^G$ is a closed subscheme.*

*(3) If $X$ is smooth, so is $X^G$.*

*Proof.* If $G$ is connected and $U \to X$ is a $G$-invariant étale morphism, we claim that

$$\begin{tikzcd} U^G \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & U \arrow[d] \\ X^G \arrow[r, hook] & X \end{tikzcd} \tag{6.6.1}$$

is cartesian. Indeed, suppose $S \to U$ is a map such that $S \to U \to X$ is $G$-invariant. Let $U_S \to S$ be the base change of $U \to X$ by $S \to X$. Since $U_S \to S$ is $G$-invariant, it suffices to show that the section $j\colon S \to U_S$ is $G$-invariant. As $U \to X$ is étale, $j\colon S \to U_S$ is an open immersion. Because $G$ is connected, for each point $s \in S$, the $G$-orbit $Gj(s) \subset U_S$ is connected and thus contained in $S$.

For (1), given a fixed point $x \in X^G(\Bbbk)$, Theorem 6.5.18 produces a $G$-invariant étale neighborhood $(U, u) \to (X, x)$ with $U$ affine and $u \in U^G(\Bbbk)$. If $G$ is connected, then $U^G \to X^G$ is étale and representable by (6.6.1). Thus it suffices to show that $U^G$ is representable. Since $U$ is affine, we can choose a $G$-equivariant embedding $U \hookrightarrow \mathbb{A}(V)$ into a finite dimensional $G$-representation. In this case, $\mathbb{A}(V)^G = \mathbb{A}(V^G)$ and thus $U^G = U \cap \mathbb{A}(V)^G$ is representable. In general, let $G^0 \subset G$ be the connected component of the identity, and let $g_1, \dots, g_n \in G(\Bbbk)$ be representatives of the finitely many cosets $G(\Bbbk)/G^0(\Bbbk)$. Then $G/G_0$ acts on $X^{G_0}$ and $X^G = \bigcap_i (X^{G_0})^{g_i}$, where $(X^{G_0})^{g_i}$ is identified with the fiber product of the diagonal $X^G \to X^G \times X^G$ and the map $X^G \to X^G \times X^G$ given by $x \mapsto (x, gx)$.

For (2), every subgroup of $G$ is linearly reductive and Theorem 6.5.18 therefore produces a $G$-invariant étale surjective morphism $U \to X$ from an affine scheme. As $G$ is connected, the argument above shows that $U^G \hookrightarrow U$ is a closed immersion and thus by étale descent so is $X^G \hookrightarrow X$.

For (3), if $x \in X^G(\Bbbk)$, there is a $G$-invariant étale morphism $(U, u) \to (X, x)$ from an affine scheme and a $G$-invariant étale morphism $U \to T_{U,u}$ as in the proof of Luna's Étale Slice Theorem (see (6.5.2)). Since $T^G_{U,u}$ is a linear subspace, it is smooth. Since $U^G \to X^G$ and $U^G \to T^G_{U,u}$ are étale at $u$, the statement follows from étale descent. See also [Ive72, Prop. 1.3] and [Mil17, Thm. 13.1]. $\square$

### 6.6.2 Limits under $\mathbb{G}_m$-actions and attractor loci

**Definition 6.6.3** (Limits). Given a $\mathbb{G}_m$-action on an algebraic space $X$ over a field $\Bbbk$ and a point $u \in U(\Bbbk)$, we say that the *limit* $\lim_{t \to 0} t \cdot u$ *exists* if there exists an extension of the diagram

$$\begin{tikzcd} \mathbb{G}_m \arrow[r, "t \mapsto t \cdot u"] \arrow[d, hook] & U \\ \mathbb{A}^1 \arrow[ur, dashed] & \end{tikzcd}$$

The valuative criteria for separatedness and properness imply that the limit is unique if $X$ is separated and that there is always a unique limit if $X$ is proper.

**Definition 6.6.4** (Attractor locus). Let $X$ be a separated algebraic space of finite type over $\Bbbk$ equipped with an action of $\mathbb{G}_m$. Define the *attractor locus* as the functor

$$X^+ := \underline{\mathrm{Mor}}^{\mathbb{G}_m}(\mathbb{A}^1, X) \colon \mathrm{Sch}/\Bbbk \to \mathrm{Sets}$$

assigning a $\Bbbk$-scheme $S$ to the set of $\mathbb{G}_m$-equivariant maps from $S \times \mathbb{A}^1$ to $X$, where $\mathbb{G}_m$ acts trivially on $S$ and with the usual scaling action on $\mathbb{A}^1$

Evaluation at 0 defines a morphism of functors

$$\mathrm{ev}_0 \colon X^+ \to X^{\mathbb{G}_m}.$$

On $\Bbbk$-points, $X^+(\Bbbk)$ is the set of points $x \in X(\Bbbk)$ such that $\lim_{t \to 0} t \cdot x$ exists, and $\mathrm{ev}_0(x)$ is this limit. Since $X$ is separated, the limit is unique if it exists. If $X$ is proper, the limit always exists and $X^+(\Bbbk) = X(\Bbbk)$. The functorial definition of $X^+$ endows it with an interesting scheme-structure, e.g. when $\mathbb{G}_m$ acts on $X = \mathbb{P}^1$ via $t \cdot [x : y] = [tx : y]$, then $X^+ = \mathbb{A}^1 \coprod \{\infty\}$.

**Exercise 6.6.5** (Affine case). If $X = \operatorname{Spec} A$ is affine, then the $\mathbb{G}_m$-action induces a grading $A = \bigoplus_{d \in \mathbb{A}} A_d$. Show that the functors $X^{\mathbb{G}_m}$ and $X^+$ are representable by the closed subschemes of $X$ defined by the ideals $\sum_{d \neq 0} A_d$ and $\sum_{d < 0} A_d$.

**Example 6.6.6** (Centralizers and parabolics). Let $G$ be an affine algebraic group over an algebraically closed field $\Bbbk$. A one-parameter subgroup $\lambda \colon \mathbb{G}_m \to G$ induces a $\mathbb{G}_m$-action on $G$ via conjugation $t \cdot g := \lambda(t) g \lambda(t)^{-1}$. Under this action, the fixed locus $G^{\mathbb{G}_m} = C_\lambda$ is identified with the centralizer of $\lambda$ and the attractor locus

$G_\lambda^+ = P_\lambda$ is identified with the subgroup consisting of elements $g \in G$ such that $\lim_{t\to 0} \lambda(t) g \lambda(t)^{-1}$ exists. The unipotent subgroup $U_\lambda$ is identified with kernel of $\mathrm{ev}_0 \colon P_\lambda \to C_\lambda$. When $G$ is reductive, $P_\lambda \subset G$ is a parabolic subgroup or in other words $G/P_\lambda$ is projective. See §C.3.3 for more properties of these subgroups.

We say that a map $X \to Y$ is an *affine fibration* (resp. *Zariski-local affine fibration*) if there exists an étale (resp. Zariski) cover $\{Y_i \to Y\}$ such that $X \times_Y Y_i \cong \mathbb{A}^n_{Y_i}$ over $Y_i$. Since the transition functions are not required to be linear, this notion is more general than a vector bundle.

**Theorem 6.6.7.** *Let $X$ be a separated algebraic space of finite type over an algebraically closed field $\Bbbk$ equipped with an action of $\mathbb{G}_m$. The functor $X^+$ is representable by an algebraic space of finite type over $\Bbbk$ and $\mathrm{ev}_0 \colon X^+ \to X^{\mathbb{G}_m}$ is an affine morphism.*

*Assume in addition that $X$ is smooth (resp. smooth scheme). Then $X^{\mathbb{G}_m}$ is also smooth and $\mathrm{ev}_0 \colon X^+ \to X^{\mathbb{G}_m}$ is an affine fibration (resp. Zariski-local affine fibration). If $x \in X^{\mathbb{G}_m}$ and $T_{X,x} = T_{>0} \oplus T_0 \oplus T_{<0}$ is the $\mathbb{G}_m$-equivariant decomposition into nonnegative, zero, and positive weights, then $T_{X_i,x} = T_0 \oplus T_{>0}$, $T_{F_i,x} = T_0$, and $X_i \to F_i$ has relative dimension $\dim T_{>0}$.*

*Proof.* If $X = \operatorname{Spec} A$ is affine, then $X^{\mathbb{G}_m}$ and $X^+$ are closed subschemes of $X$ (Exercise 6.6.5). In the special case that $X = \mathbb{A}(V)$ where $V$ is a finite dimensional $G$-representation, then $X^{\mathbb{G}_m} = \mathbb{A}(V^G)$ and $X^+ = \mathbb{A}(V_{\geq 0})$ where $V_{\geq 0}$ is the direct sum of the non-negative isotypic components, and moreover $\mathrm{ev}_0 \colon X^+ \to X^{\mathbb{G}_m}$ is a relative affine space.

We claim that if $U \to X$ is a $\mathbb{G}_m$-invariant étale morphism, then the diagram

$$\begin{tikzcd} U^+ \arrow[r, "\mathrm{ev}_0"] \arrow[d] \arrow[dr, phantom, "\square"] & U^{\mathbb{G}_m} \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & U \arrow[d] \\ X^+ \arrow[r, "\mathrm{ev}_0"] & X^{\mathbb{G}_m} \arrow[r, hook] & X \end{tikzcd} \tag{6.6.2}$$

is cartesian. The right square was verified in the proof of Theorem 6.6.2. For the left square, we need to show that there exists a unique $\mathbb{G}_m$-equivariant morphism filling in a $\mathbb{G}_m$-equivariant diagram

$$\begin{tikzcd} \operatorname{Spec} \Bbbk \times S \arrow[r] \arrow[d] & U \arrow[d] \\ \mathbb{A}^1 \times S \arrow[r] \arrow[ur, dashed] & X \end{tikzcd} \tag{6.6.3}$$

where $S$ is an affine scheme of finite type over $k$, and the vertical left arrow is the inclusion of the origin. For each $n \geq 1$, the formal lifting property of étaleness yields a unique $\mathbb{G}_m$-equivariant map $\operatorname{Spec} k[x]/x^n \times S \to U$ such that

$$\begin{tikzcd} \operatorname{Spec} \Bbbk \times S \arrow[r] \arrow[d] & U \arrow[d] \\ \operatorname{Spec}(\Bbbk[x]/x^n) \times S \arrow[r] \arrow[ur, dashed] & X \end{tikzcd}$$

commutes. As $[\mathbb{A}^1/\mathbb{G}_m] \times S$ is coherently complete along $\mathbf{B}\mathbb{G}_{m,S}$ (Exercise 6.4.14), Coherent Tannaka Duality in the form of Exercise 6.4.10 yields a unique $\mathbb{G}_m$-equivariant morphism $\mathbb{A}^1 \times S \to U$ such that (6.6.3) commutes.

Choose a $\mathbb{G}_m$-invariant étale surjective morphism $U \to X$ from an affine scheme (Theorem 6.5.18). Then (6.6.2) implies that $U^+ \to X^+$ is etale and representable, and since $U^+$ is an affine scheme of finite type, it follows that $X^+$ is an algebraic space of finite type. Since $U^+ \to U^{\mathbb{G}_m}$ is affine, étale descent implies that $X^+ \to X^{\mathbb{G}_m}$ is also affine.

If $X$ is smooth, then $X^{\mathbb{G}_m}$ is smooth by Theorem 6.6.2. As $U$ is also smooth, for each $u \in U^{\mathbb{G}_m}(\Bbbk)$, there is a $\mathbb{G}_m$-equivariant morphism $U \to T_{U,u}$ étale at $u$ with $f(u) = 0$ (Lemma 6.5.6). Then $U^+ \to T^+_{U,u}$ is also étale at $u$. Let $V \subset U$ be the open locus where $U^+ \to T^+_{U,u}$ is étale. Since $V$ is $\mathbb{G}_m$-equivariant, if $v \in V^{\mathbb{G}_m}$, then $\mathrm{ev}_0^{-1}(v) \subset V$. Choosing an affine subscheme $V' \subset V^{\mathbb{G}_m}$ containing $u$ and replacing $U^+$ with $\mathrm{ev}_0^{-1}(V')$, we may assume that $U^+ \to T^+_{U,u}$ is everywhere étale. By (6.6.2), we have a cartesian diagram

$$\begin{tikzcd}
T^+_{U,u} \arrow[d] \arrow[dr, phantom, "\square"] & U^+ \arrow[l] \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X^+ \arrow[d] \\
T^{\mathbb{G}_m}_{U,u} & U^{\mathbb{G}_m} \arrow[l] \arrow[r] & X^{\mathbb{G}_m}
\end{tikzcd} \tag{6.6.4}$$

where the horizontal arrows are étale. With $T_{X,x} = T_{>0} \oplus T_0 \oplus T_{<0}$, there are identifications $T^{\mathbb{G}_m}_{X,x} = T_0$ and $T^+_{X,x} = T_{>0} \oplus T_0$. Since $T^+_{U,u} \to T^{\mathbb{G}_m}_{U,u}$ a surjection of vector spaces, $U \to U^{\mathbb{G}_m}$ is a Zariski-local affine fibration. By étale descent, $X \to X^{\mathbb{G}_m}$ is an affine fibration of relative dimension $\dim T_{>0}$.

If $X$ is a smooth scheme, then by Sumihiro's Theorem on Torus Actions (C.3.13), we may choose $U = \coprod_i U_i \to X$ such that $\{U_i\}$ is a $\mathbb{G}_m$-invariant affine open covering. Then (6.6.4) implies that $X^+ \to X^{\mathbb{G}_m}$ is a Zariski-local affine fibration.

See also [Dri13, Prop. 1.2.2, Thm. 1.4.2] and [AHR20, Thm. 5.16]. $\square$

**Remark 6.6.8.** Another approach to establish the algebraicity of $X^+$ in Theorem 6.6.7 is to show that the stack $\underline{\mathrm{Mor}}([\mathbb{A}^1/\mathbb{G}_m], \mathcal{X})$, whose objects over a $\Bbbk$-scheme $S$ are morphisms $[\mathbb{A}^1/\mathbb{G}_m]_S \to \mathcal{X}$, is algebraic when $\mathcal{X}$ has affine diagonal. This can be shown by verifying Artin's Axioms (D.7.4) where the crucial step is to verify the effectivity condition (AA$_5$): this follows from the coherent completeness of $[\mathbb{A}^1/\mathbb{G}_m]_R$, where $R$ is a noetherian local $\Bbbk$-algebra, along the unique closed point (Theorem 6.4.11) together with Coherent Tannaka Duality (6.4.8).

When $\mathcal{X} = [X/\mathbb{G}_m]$, then a $\mathbb{G}_m$-equivariant morphism $\mathbb{A}^1 \to X$ corresponds to a morphism $[\mathbb{A}^1/\mathbb{G}_m] \to [X/\mathbb{G}_m]$ over $\mathbf{B}\mathbb{G}_m$ (Exercise 3.1.15), and there is a cartesian diagram

$$\begin{tikzcd}
\underline{\mathrm{Mor}}^{\mathbb{G}_m}(\mathbb{A}^1, X) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \underline{\mathrm{Mor}}([\mathbb{A}^1/\mathbb{G}_m], [X/\mathbb{G}_m]) \arrow[d] \\
\operatorname{Spec} \Bbbk \arrow[r] & \underline{\mathrm{Mor}}([\mathbb{A}^1/\mathbb{G}_m], \mathbf{B}\mathbb{G}_m).
\end{tikzcd}$$

The algebraicity of the stacks of morphisms implies that $\underline{\mathrm{Mor}}^{\mathbb{G}_m}(\mathbb{A}^1, X)$ is an algebraic space.

### 6.6.3 The Białynicki-Birula Stratification

**Theorem 6.6.9** (Białynicki-Birula Stratification[3]). *Let $X$ be a separated algebraic space of finite type over an algebraically closed field $\Bbbk$ with an action of $\mathbb{G}_m$. Let $X^{\mathbb{G}_m} = \coprod_{i=1}^n F_i$ be the fixed locus with connected components $F_i$. There exists an affine morphism $X_i \to F_i$ for each $i$ and a monomorphism $\coprod_i X_i \to X$. Moreover,*

*(1) If $X$ is proper, then $\coprod_i X_i \to X$ is surjective.*

*(2) If $X$ is smooth (resp. smooth scheme), then $F_i$ is smooth and $X_i \to F_i$ is a (resp. Zariski-local) affine fibration. If $x \in F_i$ and $T_{X,x} = T_{x,>0} \oplus T_{x,0} \oplus T_{x,<0}$ is the $\mathbb{G}_m$-equivariant decomposition into nonnegative, zero, and positive weights, then $T_{X_i,x} = T_{x,>0} \oplus T_{x,0}$, $T_{F_i,x} = T_{x,0}$, and $X_i \to F_i$ has relative dimension* $\dim T_{x,>0}$.

*(3) The map $X_i \hookrightarrow X$ is a locally closed immersion under any of the following conditions:*

*(a) $X$ is affine,*

*(b) $X$ is a smooth scheme, or*

*(c) there exists a $\mathbb{G}_m$-equivariant locally closed immersion $X \hookrightarrow \mathbb{P}(V)$ where $V$ is a $\mathbb{G}_m$-representation (e.g., $X$ is a normal quasi-projective variety).*

*(4) If $X$ is smooth, irreducible, and quasi-projective, then the stratification $X^+ = \coprod_i X_i$ is* filterable, *i.e. there is an ordering of the indices such that $X_{\geq i} := \bigcup_{j \geq i} X_j$ is closed for each $i$. If in addition there are finitely many fixed points $\{x_1, \ldots, x_n\}$, then $T_{x_i,0} = 0$ and $X_i = \mathbb{A}(T_{x_i,>0})$ is an affine space; in particular,*

$$X^+ = X_{\geq 1} \supset X_{\geq 2} \supset \cdots \supset X_{\geq n} \supset \emptyset$$

*is a* cell decomposition, *i.e. each $X_{\geq i} \setminus X_{\geq i-1} = X_i$ is an affine space.*

*Proof.* By Theorem 6.6.7, $X^+$ is representable and there is affine morphism $\mathrm{ev}_0 \colon X^+ \to X^{\mathbb{G}_m}$ of finite type. We define $X_i$ as the preimage $\mathrm{ev}_0^{-1}(F_i)$. Since $X$ is separated, the inclusion $X^+ \hookrightarrow X$ is a monomorphism. This gives the main statement. If $X$ is proper, then $X^+ \to X$ is surjective (i.e. (1) holds) as $\lim_{t \to 0} t \cdot x$ exists for every $x \in X(\Bbbk)$. Statement (2) follows directly from Theorem 6.6.7.

For (3), if $X = \operatorname{Spec} A$ and $A = \bigoplus_d A_d$ is the grading induced by the $\mathbb{G}_m$-action, then $X^+$ is the closed subscheme defined by the ideal $\sum_{d<0} A_d$ (Exercise 6.6.5) and in particular affine. If $X$ is a smooth scheme, then there exists a $\mathbb{G}_m$-invariant affine open cover (Theorem C.3.12). For any point $x \in X^+$, let $x_0$ be the image of $x$ under $\mathrm{ev}_0 \colon X^+ \to X^0$, and choose a $\mathbb{G}_m$-invariant affine open neighborhood $U \subset X$ of $x_0$. This induces a diagram

$$\begin{tikzcd} U^+ \arrow[r, hook] \arrow[drr, hook] & \mathrm{ev}_1^{-1}(U) \arrow[r] \arrow[d, hook] & U \arrow[d, hook] \\ & X^+ \arrow[r, "\mathrm{ev}_1"] & X. \end{tikzcd} \tag{6.6.5}$$

Since $U^+ \to U$ is a closed immersion (as $U$ is affine) and $X^+ \to X$ is separated (it is a monomorphism), $U^+ \to \mathrm{ev}_1^{-1}(U)$ is a closed immersion. Since $U^+ = X^+ \times_{X^0} U^0$ (see (6.6.2)), $x \in U^+$ and $U^+ \to X^+$ is an open immersion. In particular, $U^+ \subset \mathrm{ev}_1^{-1}(U)$

[3] This is frequently referred to as the 'Białynicki-Birula Decomposition' as some authors prefer to reserve the term 'stratification' to a decomposition where each stratum has a neighborhood which is topologically locally trivial.

is an open and closed subscheme containing $x$. On the other hand, $X_i$ is smooth and connected (as $X_i \to F_i$ is an affine fibration), thus irreducible. It follows that $X_i \cap U^+ = X_i \cap \mathrm{ev}_1^{-1}(U)$ and that $X_i \cap \mathrm{ev}_1^{-1}(U) \to U$ is a closed immersion which in turn implies that $X_i \to X$ is a locally closed immersion. The final case (3)(c) easily reduces to the case of $X = \mathbb{P}(V)$ in which a direct calculation shows that each $X_i$ is of the form $\mathbb{P}(W) \setminus \mathbb{P}(W')$ for linear subspaces $W' \subset W \subset V$. See also [BB73, Thm. 4.1], [Hes81, Thm. 4.5,p. 69], [Dri13, Thm. B.0.3], [AHR20, Thm. 5.27], and [JS21, Thm. 1.5].

For (4), by Sumihiro's Theorem on Linearizations (C.3.12), we can choose a $G$-equivariant locally closed immersion $X \hookrightarrow \mathbb{P}^n$, where $\mathbb{G}_m$ acts on $\mathbb{P}^n$ via $t \cdot [x_0 : \ldots : x_n] = [t^{d_0}x_0 : \ldots : t^{d_n}x_n]$ with $d_0 \leq \cdots \leq d_n$. Let $D_1, \ldots, D_s$ be the distinct weights and set $J_i = \{j \mid d_j = D_i\}$ so that $J_1 \cup \cdots \cup J_s$ is a partition of $\{0, 1, \ldots, n\}$. Then $(\mathbb{P}^n)^{\mathbb{G}_m} = \coprod_{i=1}^s F_i$ where $F_i = V(x_j \mid j \in J_i)$. The preimage of $F_i$ under the morphism $\mathrm{ev}_0 \colon \mathbb{P}^n \to (\mathbb{P}^n)^{\mathbb{G}_m}$, given by $p \mapsto \lim_{t \to 0} t \cdot p$, is

$$P_i := \mathrm{ev}_0^{-1}(F_i) = \left\{ [x_0 : \ldots : x_n] \;\middle|\; \begin{array}{l} x_j = 0 \text{ for all } j \in J_1 \cup \cdots \cup J_{i-1} \\ x_k \neq 0 \text{ for some } k \in J_i \end{array} \right\},$$

Moreover, the union

$$P_{\geq i} := \bigcup_{j \geq i} P_j = V(x_k \mid k \in J_1 \cup \cdots \cup J_{i-1}) \subset \mathbb{P}^n$$

is closed. The fixed locus for $X$ is $X^{\mathbb{G}_m} = (\mathbb{P}^n)^{\mathbb{G}_m} \cap X = \coprod_i F_i \cap X$. For each $i$, we write $F_i \cap X = \coprod_{j=1}^{l_i} F_{ij}$ and $P_i \cap X = \coprod_{j=1}^{l_i} X_{ij}$ as the irreducible decompositions. Then $\mathrm{ev}_0 \colon \mathbb{P}^n \to (\mathbb{P}^n)^{\mathbb{G}_m}$ restricts to morphisms $\mathrm{ev}_0 \colon X_{ij} \to F_{ij}$. For $j \neq k$, the strata $X_{ij}$ and $X_{ik}$ are disjoint, and thus $\overline{X}_{ij} \cap \overline{X}_{ik} \subset P_{\geq i+1} \cap X$. It follows that

$$(P_{\geq i+1} \cap X) \cup X_{i1} \cup \cdots \cup X_{ij} \subset X$$

is closed for each $j = 1, \ldots, s$. Ordering the strata as $X_{11}, \ldots, X_{1l_1}, \ldots, X_{s1}, \ldots, X_{sl}$ establishes the claim. See also [Bir76, Thm. 3]. $\square$

**Remark 6.6.10.** It is not true in general that $X_i \hookrightarrow X$ is a locally closed immersion. Based on Hironaka's example of a proper, non-projective, smooth 3-fold, Sommese constructed a smooth algebraic space $X$ such that $X_i \hookrightarrow X$ is not a locally closed immersion [Som82]. On the other hand, Konarski provided an example of a normal proper toric variety $X$ such that $X_i \hookrightarrow X$ is not a locally closed immersion [Kon82].

**Remark 6.6.11** (Morse stratifications)**.** The Białynicki-Birula stratification of $X$ can be obtained as the Morse stratification corresponding to the non-degenerate Morse function $\mu \colon X \to \mathrm{Lie}(S^1)^\vee = \mathbb{R}$: a point $x \in X$ lies in $X_i$ if only if the limit of its forward trajectory under the gradient flow of $\mu$ lies in $F_i$. See [CS79].

**Example 6.6.12.** Suppose $\mathbb{G}_m$ acts on $X = \mathbb{P}^2$ via $t \cdot [x : y : z] = [x : ty : t^2]$. Then $X^{\mathbb{G}_m} = F_1 \coprod F_2 \coprod F_3$ where $F_1 = \{[1 : 0 : 0]\}$, $F_2 = \{[0 : 1 : 0]\}$, and $F_3 = \{[0 : 0 : 1]\}$, and $X_1 = \{x \neq 0\} = \mathbb{A}^2$, $X_2 = \{[0 : y : z] \mid y \neq 0\} = \mathbb{A}^1$ and $X_3 = F_3$.

Let $\widetilde{X}$ be the blowup $\mathrm{Bl}_p X$ at the fixed point $p = [0 : 1 : 0]$. Then $\mathbb{G}_m$ acts on the exceptional divisor $E \cong \mathbb{P}^1$ via $t \cdot [u : v] = [u : t^2 v]$ with fixed points $q_1 = [1 : 0]$ and $q_2 = [0 : 1]$. The fixed locus $\widetilde{X}^{\mathbb{G}_m}$ contains four points $\widetilde{F}_1 = \{[1 : 0 : 0]\}$, $\widetilde{F}_2 = \{q_1\}$, $\widetilde{F}_3 = \{q_2\}$, and $\widetilde{F}_4 = \{[0 : 0 : 1]\}$. We have that $\widetilde{X}_1 = X_1 \cong \mathbb{A}^2$, $\widetilde{X}_2 = X_2 \cong \mathbb{A}^1$ ,

$X_3 = E \setminus \{q_2\} \cong \mathbb{A}^1$, and $\widetilde{X}_4 = X_4 = \widetilde{F}_4$ as illustrated in Figure 6.1. Observe that $\overline{X}_3 \setminus X_3 = \{q_2\}$ is not the union of other strata.

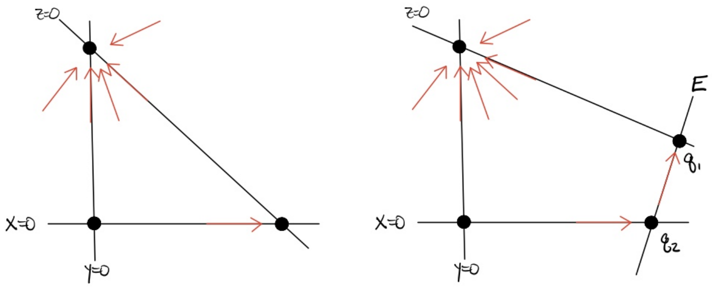



Figure 6.1: Białynicki-Birula stratifications for $\mathbb{P}^2$ (left) and $\mathrm{Bl}_p \mathbb{P}^2$ (right).

### 6.6.4 Applications of the Białynicki-Birula Stratification to cohomology

**Proposition 6.6.13.** *Let $X$ be a smooth, irreducible, and quasi-projective scheme over an algebraically closed field $\Bbbk$ with an action of $\mathbb{G}_m$ such that there are only finitely many fixed points. Then $A_i(X)$ is a free $\mathbb{Z}$-module generated by the closures of the $i$-dimensional cells. If in addition $\Bbbk = \mathbb{C}$, then the cycle map $\mathrm{CH}_i(X) \to \mathrm{H}_{2i}^{\mathrm{BM}}(X, \mathbb{Z})$ to Borel–Moore homology is an isomorphism and $\mathrm{H}_{2i+1}^{\mathrm{BM}}(X, \mathbb{Z}) = 0$.*

**Remark 6.6.14.** When $X$ is compact (e.g. projective), then $H_{2i}^{\mathrm{BM}}(X, \mathbb{Z})$ is ordinary integral singular homology.

*Proof.* The Białynicki-Birula Stratification (6.6.9(4)) implies that $X$ has a cell decomposition, and the statement follows from [Ful98, Ex. 19.1.11]. See also [Bri97, §3.2]. □

**Example 6.6.15** (Chow groups of $\mathrm{Hilb}_n(\mathbb{A}^2)$)**.** Let $X = \mathrm{Hilb}_n(\mathbb{A}^2)$ be the Hilbert scheme of $n$ points; this is a smooth irreducible scheme (see 1.5.3). The natural action of $T = \mathbb{G}_m^2$ induces a $T$-action on $X$. Under the $\mathbb{G}_m$-action induced by a one-parameter subgroup $\mathbb{G}_m \to T$ given by positive weights, the evaluation map $\mathrm{ev}_0 \colon X^+ \to X$ is surjective, and the $\mathbb{G}_m$-fixed points correspond to subschemes $Z = V(I) \subset \mathbb{A}^2$ supported at the origin where $I$ is a monomial ideal. We see that there are only finitely many $\mathbb{G}_m$-fixed points. We may therefore use Proposition 6.6.13 to compute $\mathrm{CH}^*(X)$.

For a monomial ideal $I \subset R := \Bbbk[x, y]$, for each integer $i$, define

$$a_i := \min\{j \mid x^i y^j \in I\}$$

and let $r$ be the largest integer such that $a_r > 0$. Then $a_0 \geq \ldots \geq a_r$ is a partition of $n$ and $I = (y^{a_0}, xy^{a_1}, \ldots, x^{r+1})$. We need to compute the dimension of the positive weight space $T_{I,>0}$ of the $\mathbb{G}_m$-action on the tangent space

$$T_I = \mathrm{Hom}_R(I, R/I)$$

of $X$ at the monomial ideal $I$; see Exercise 1.5.5 for the identification of the tangent space. To accomplish this, we first argue that

$$T_I = \sum_{0 \leq i \leq j \leq r} \sum_{s = a_{j+1}}^{a_j - 1} \left(\chi_1^{i-j-1} \chi_2^{a_i - s - 1} + \chi_1^{j-i} \chi_2^{s - a_i}\right), \tag{6.6.6}$$

as $T = \mathbb{G}_m^2$ representations, where $\chi_i \colon T \to \mathbb{G}_m$ denotes the one-dimensional representation giving by $(t_1, t_2) \mapsto t_i^{-1}$. There are

$$\sum_{0 \le i \le j \le r} 2(a_j - a_{j+1}) = 2 \sum_{0 \le i \le r} a_i = 2n = \dim T_I$$

one-dimensional representations appearing on the right-hand side, and they are linearly independent. It thus suffices to show that each of them occurs in $T_I$. An $R$-module map $\phi \colon I \to R/I$ is given by the values $\phi(x^i y^{a_i})$ subject to the relations

$$\phi(x^{i+1} y^{a_i}) = x\phi(x^i a_i) \quad \text{and} \quad \phi(x^i y^{a_{i-1}}) = y^{a_{i-1}-a_i}\phi(x^i a_i).$$

Let $0 \le i \le j \le r$ and $a_{j+1} \le s < a_j$. Defining

$$\begin{aligned} \phi_{i,j,s} \colon I \to R/I, \quad x^l y^{a_l} \quad &\mapsto \begin{cases} x^{l+j-i} y^{a_l+s-a_i} & \text{if } l \le i \\ 0 & \text{otherwise} \end{cases} \\ \psi_{i,j,s} \colon I \to R/I, \quad x^l y^{a_l} \quad &\mapsto \begin{cases} x^{l+i-j-1} y^{a_l+s-a_i} & \text{if } l \ge j+1 \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

one checks that $\phi_{i,j,s}$ and $\psi_{i,j,s}$ are $R$-module maps that are eigenvectors for $\chi_1^{j-i}\chi_2^{s-a_i}$ and $\chi_1^{i-j-1}\chi_2^{a_i-s-1}$. Thus (6.6.6) holds.

Choose $\lambda = (\lambda_1, \lambda_2) \colon \mathbb{G}_m \to T$ with $\lambda_1 \gg \lambda_2$. Under our sign conventions, a character $\chi_1^a \chi_2^b$ appearing in (6.6.6) has positive weight with respect to $\lambda$ if $a < 0$, or if $a = 0$ and $b < 0$. Thus

$$T_{I,>0} = \sum_{0 \le i \le j \le r} \sum_{s=a_{j+1}}^{a_j-1} \chi_1^{i-j-1}\chi_2^{a_i-s-1} + \sum_{j=0}^{r} \sum_{s=a_{j+1}}^{a_j-1} \chi_1^{j-i}\chi_2^{s-a_i}$$

and

$$\begin{aligned} \dim T_{I,>0} &= \left( \sum_{i=0}^{r} \sum_{j=i}^{r} (a_j - a_{j+1}) \right) + \left( \sum_{j=0}^{r} (a_j - a_{j+1}) \right) \\ &= \left( \sum_{i=0}^{r} a_i \right) + a_0 = n + a_0 \end{aligned}$$

Since there is a bijection between monomial ideals $I \subset R = \Bbbk[x,y]$ with $\dim_{\Bbbk} R/I = n$ and partitions $a_0 \ge \cdots \ge a_r$ of $n$, for every $d \ge 0$, the number of monomial ideals $I$ such that $\dim T_{I,>0} = d$ is equal to

$$P(2n-d, d-n) := \# \{\text{partitions } a_1 \ge \cdots \ge a_r \text{ of } 2n-d \text{ with each } a_i \le d-n\}. \tag{6.6.7}$$

It follows from Proposition 6.6.13 that

$$\dim \mathrm{CH}_d(\mathrm{Hilb}_n(\mathbb{A}^2))_{\mathbb{Q}} = P(2n-d, d-n).$$

See also [ESm87, Thm. 1.1] and [Göt94, §2.2].

**Exercise 6.6.16** (Chow groups of $\mathrm{Hilb}_n(\mathbb{P}^2)$)**.** Follow the above strategy to show that the $d$th Betti number $b_d$ of $\mathrm{Hilb}_n(\mathbb{P}^2)$ (or equivalently $\dim \mathrm{CH}_d(\mathrm{Hilb}_n(\mathbb{P}^2))$) is equal to

$$b_d = \sum_{n_0+n_1+n_2=n} \sum_{p+r=d-n_1} P(p, n_0-p)P(n_1)P(2n_2-r, r-n_2),$$

where $P(a)$ is the number of partitions of $a$ and $P(a,b)$ is defined by (6.6.7).

**Remark 6.6.17.** Göttsche used the Weil conjectures in [Göt90, Thm. 0.1] (see also [Göt94, Thm. 2.3.10]) to show that for any smooth projective surface $S$ over $\mathbb{C}$ or $\overline{\mathbb{F}}_q$ that the Poincaré polynomial $p(S^{[n]}, z) = \sum_i b_i(S^{[n]}) z^n$ of $S^{[n]} := \mathrm{Hilb}_n(S)$ satisfies

$$\sum_{n=0}^{\infty} p(S^{[n]}, z) t^n = \prod_{m=1}^{\infty} \frac{(1 + z^{2m-1} t^m)^{b_1(S)} (1 + z^{2m+1} t^m)^{b_3(S)}}{(1 - z^{2m-2} t^m)^{b_0(S)} (1 - z^{2m} t^m)^{b_2(S)} (1 - z^{2m+2} t^m)^{b_4(S)}}$$

In particular, the Betti numbers of $S^{[n]}$ only depend on the Betti numbers of $S$. While each term $p(S^{[n]}, z)$ does not admit a particularly nice expression, the generating function involving *all* $n$ does.

On the other hand, Nakajima constructed an action of the Heisenberg algebra on $\mathrm{H}_*(S^{[n]})$, which can be used to recover the Betti number formula above as well as additional properties of the cohomology ring [Nak97] (see also [Nak99b]).

We can also use the Białynicki-Birula Stratification to compute equivariant Chow rings $\mathrm{CH}^*_G(X)$ (or equivalently the Chow ring $\mathrm{CH}^*([X/G])$ of the quotient stack) as introduced in §6.1.7. The following statements can also be made in de Rham or singular cohomology (§6.1.8) where instead of the excision sequence above, one uses the Thom–Gysin long exact sequence.

We will use the following two lemmas, which we state in a generality that we can also apply to the HKKN stratification of §6.7.6.

**Lemma 6.6.18.** *Let $X$ be a smooth irreducible scheme over an algebraically closed $\Bbbk$ with an action of a smooth affine algebraic group $G$. Let $S_1, \ldots, S_r \subset X$ be nonempty, disjoint, smooth, irreducible, and locally closed $G$-invariant subschemes such that $X = \coprod_i S_i$ and such that $S_{\geq i} := \bigcup_{j \geq i} S_j$ is closed for each $i$. Let $d_i$ be the codimension of $S_i$ in $X$. If the top Chern class $c^G_{d_i}(N_{S_i/X}) \in \mathrm{CH}^*_G(S_i)_{\mathbb{Q}}$ is a nonzerodivisor for each $i$, then*

$$\dim \mathrm{CH}^k_G(X)_{\mathbb{Q}} = \sum_{i=1}^{r} \dim \mathrm{CH}^{k-d_i}_G(S_i)_{\mathbb{Q}}$$

*for each $k$.*

*Proof.* By assumption, $S_{\leq i} = \bigcup_{j \leq i} S_j$ is open for each $i$, and $S_i \subset S_{\leq i}$ is a closed subscheme with open complement $S_{<i}$. We have a commutative diagram

$$\begin{tikzcd}
\mathrm{CH}^{k-d_i}_G(S_i) \arrow[r] \arrow[dr] & \mathrm{CH}^k_G(S_{\leq i}) \arrow[r] \arrow[d] & \mathrm{CH}^k_G(S_{<i}) \arrow[r] & 0 \\
& \mathrm{CH}^k_G(S_i) & &
\end{tikzcd}$$

where the top row is the right exact excision sequence (6.1.33(3)) and the vertical downward arrow is given by intersecting with $S_i$. By the self-intersection formula (6.1.33(5)), the composition $\mathrm{CH}^{k-d_i}_G(S_i) \to \mathrm{CH}^k_G(S_i)$ is multiplication by $c^G_{d_i}(N_{S_i/X})$. By hypothesis, this map is injective after tensoring with $\mathbb{Q}$. It follows that the top row is an exact sequence after tensoring with $\mathbb{Q}$, and that

$$\dim \mathrm{CH}^k_G(S_{\leq i})_{\mathbb{Q}} = \dim \mathrm{CH}^{k-d_i}_G(S_i)_{\mathbb{Q}} + \dim \mathrm{CH}^k_G(S_{<i})_{\mathbb{Q}}.$$

The formula follows from induction. See [AB83, Prop. 1.9]. □

**Remark 6.6.19.** If $[S_i/G]$ is Deligne–Mumford, then $\mathrm{CH}^k_G(S_i)$ vanishes for $k \gg 0$ and $c^G_{d_i}(N_{S_i/X})$ is a zero divisor.

The following gives a condition for the top Chern class to be a nonzerodivisor.

**Lemma 6.6.20.** *Let $X$ be a smooth irreducible scheme over an algebraically closed $\Bbbk$ with an action of a connected, smooth, and affine algebraic group $G$, and let $N$ be a $G$-equivariant vector bundle of rank $d$ on $X$. Suppose that there is a subgroup $\mathbb{G}_m \subset G$ acting trivially on $X$ and a point $x \in X(\Bbbk)$ such that $N \otimes \kappa(x)$ contains no $\mathbb{G}_m$-invariant vectors. Then $c^G_d(N) \in \mathrm{CH}^*_G(X)_{\mathbb{Q}}$ is a nonzerodivisor.*

*Proof.* Choose a maximal torus $T$ containing $\mathbb{G}_m$ and a character $T \to \mathbb{G}_m$ such that the composition $\mathbb{G}_m \hookrightarrow T \to \mathbb{G}_m$ is given by $t \mapsto t^d$ for $d > 0$. By (6.1.33(7)), $\mathrm{CH}^*_G(X)_{\mathbb{Q}} = \mathrm{CH}^*_T(X)^W_{\mathbb{Q}}$ where $W$ is the Weyl group. Since $\mathrm{CH}^*_G(X)_{\mathbb{Q}}$ is a subring of $\mathrm{CH}^*_T(X)_{\mathbb{Q}}$, we are reduced to show that $c^T_d(N) \in \mathrm{CH}^*_T(X)_{\mathbb{Q}}$ is a nonzerodivisor. If we write $T$ as the product of the given $\mathbb{G}_m$ and a subtorus $T'$, then

$$\mathrm{CH}^*_T(X) \cong \mathrm{CH}^*_{T'}(X) \otimes \mathrm{CH}^*(\mathbf{B}\mathbb{G}_m) \cong \mathrm{CH}^*_{T'}(X)[z]$$

by (6.1.33(6)). For $x \in X(\Bbbk)$, we can write

$$\begin{aligned} c^T_d(N) &= \sum c^{T'}_i(N) \otimes c^{\mathbb{G}_m}_{d-i}(N \otimes \kappa(x)) \\ &= 1 \otimes c^{\mathbb{G}_m}_d(N \otimes \kappa(x)) + \text{higher degree terms.} \end{aligned}$$

If $a_1, \dots, a_d$ denote the $\mathbb{G}_m$-weights of $N \otimes \kappa(x)$, then by hypothesis each $a_i \neq 0$ and

$$c^{\mathbb{G}_m}_d(N \otimes \kappa(x)) = \big(\prod_i a_i\big) z^d \in \mathrm{CH}^*(\mathbf{B}\mathbb{G}_m)_{\mathbb{Q}} \cong \mathbb{Q}[z]$$

is a nonzerodivisor,and therefore $c^T_d(N)$ is also a nonzerodivisor.

See also [AB83, Prop. 13.4] and [Bri97, §3.2]. ☐

We define the *$G$-equivariant Chow–Poincaré polynomial* of a $G$-equivariant scheme $X$ as

$$p_G(X,t) = \sum_{d=0}^{\infty} \big(\dim \mathrm{CH}^d_G(X)_{\mathbb{Q}}\big) t^d.$$

We also denote $p(X,t) = \sum_{d=0}^{\infty} \big(\dim \mathrm{CH}^d(X)_{\mathbb{Q}}\big) t^d$ as the (non-equivariant) *Chow–Poincaré polynomial.*

**Proposition 6.6.21.** *Let $X$ be a smooth, irreducible, and quasi-projective scheme over an algebraically closed field $\Bbbk$ with an action of $\mathbb{G}_m$ such that $X^+ \to X$ is surjective (i.e. $X$ is projective). Let $X = \coprod_{i=1}^r X_i$ and $X^{\mathbb{G}_m} = \coprod_{i=1}^r F_i$ be the Białynicki-Birula Stratification (6.6.9), and let $d_i$ be the codimension of $X_i$ in $X$. Then*

$$p_{\mathbb{G}_m}(X,t) = \sum_{i=1}^{r} p(F_i) \cdot t^{d_i}(1-t)^{-1}.$$

*Proof.* Since each $F_i$ is smooth and $X_i \to F_i$ is a Zariski-local affine fibration (Theorem 6.6.7), the pullback map $\mathrm{CH}^*_{\mathbb{G}_m}(F_i) \xrightarrow{\sim} \mathrm{CH}^*_{\mathbb{G}_m}(X_i)$ is an isomorphism (6.1.33(2)). Under this isomorphism, $N_{X_i/X}$ is the image of its restriction $(N_{X_i/X})|_{F_i}$. For $x \in F_i$, $N_{X_i/X} \otimes \kappa(x) = T_{x,<0}$ has no $\mathbb{G}_m$-invariant vectors and thus Lemma 6.6.20

implies that $c_{d_i}^{\mathbb{G}_m}((N_{X_i/X})|_{F_i})$ is a nonzerodivisor. Lemma 6.6.18 therefore implies that $p_{\mathbb{G}_m}(X,t) = \sum_i p_{\mathbb{G}_m}(X_i,t)$. Since

$$\mathrm{CH}^*_{\mathbb{G}_m}(X_i) \cong \mathrm{CH}^*_{\mathbb{G}_m}(F_i) \cong \mathrm{CH}^*(F_i) \otimes \mathrm{CH}^*(\mathbf{B}\mathbb{G}_m) \cong \mathrm{CH}^*(F_i)[z],$$

where the second equality uses (6.1.33(6)), we have the identity $p_{\mathbb{G}_m}(F_i,t) = p(F_i)(1-t)^{-1}$ and the statement follows. □

### 6.6.5 One-parameter subgroups and the Cartan Decomposition

If $G$ is an algebraic group over a field $\Bbbk$, a *one-parameter subgroup* is a homomorphism $\lambda\colon \mathbb{G}_m \to G$ of algebraic groups. Despite the terminology, we do not require that a one-parameter subgroup is injective, e.g. $\mathbb{G}_m \to \mathbb{G}_m, t \mapsto t^2$ is a one-parameter subgroup. See §C.3.3 for more background and examples.

Given an action of $G$ on an algebraic space $X$, the limit $\lim_{t\to 0}\lambda(t)\cdot x$ exists if the multiplication map $\mathbb{G}_m \to X$, defined by $t \mapsto \lambda(t)\cdot x$ extends to a map $\mathbb{A}^1 \to X$. If $X$ is separated, then the limit is unique if it exists, while if $X$ is proper, then there is always a unique limit.

**Example 6.6.22.** If $X = \mathbb{P}(V)$ where $V$ is a finite dimensional representation of $G$ and $\lambda\colon \mathbb{G}_m \to G$ is a one-parameter subgroup, then we can choose a basis of $V$ such that $\lambda(t)\cdot(x_1,\dots,x_n) = (t^{d_1}x_1,\dots,t^{d_n}x_n)$ with $d_1 \le \cdots \le d_n$. If $d = \min\{d_i \mid x_i \neq 0\}$, then $\lim_{t\to 0}\lambda(t)\cdot[x_0 : \dots : x_n] = [x_0' : \dots : x_n']$ where $x_i' = x_i$ for all $i$ such that $d_i = d$ and is 0 otherwise.

We now state and prove the Cartan Decomposition: an element $g \in G(K)$ over the fraction field $K$ of a DVR $R$ can be multiplied on the left and right by elements of $G(R)$ such that it is induced from a one-parameter subgroup. For a one-parameter subgroup $\lambda\colon \mathbb{G}_m \to G$, we denote by $\lambda|_K \in G(K)$ the image of the composition

$$\operatorname{Spec} K \to \mathbb{G}_m \xrightarrow{\lambda} G$$

where the first map is defined by the $\Bbbk$-algebra map $\Bbbk[t]_t \to K$ taking $t$ to a uniformizer in $R$.

The Cartan Decomposition will provide the key algebraic input in the proof of the Hilbert–Mumford Criterion (6.7.13).

**Theorem 6.6.23** (Cartan Decomposition[4])**.** *Let $G$ be a reductive algebraic group. Let $R$ be a complete DVR over $\Bbbk$ with residue field $\Bbbk$ and fraction field $K$. Then for every element $g \in G(K)$, there exists $h_1, h_2 \in G(R)$ and a one-parameter subgroup $\lambda\colon \mathbb{G}_m \to G$ such that*

$$g = h_1 \lambda|_K h_2.$$

*Proof.* As our proof will utilize that $\mathbf{B}G$ is $\mathsf{S}$-complete (Definition 6.8.9), a concept developed §6.8, we postpone the proof until Proposition 6.8.45. In fact, we show not only that the theorem holds for reductive groups but that it characterizes reductivity. See also [IM65, Cor. 2.17], [Ses72, Thm. 2.1] and [BT72, §4]. □

**Remark 6.6.24** (Equivalent formulation)**.** Let $T \subset G$ be a maximal torus. The above theorem is equivalent to the identity

$$G(K) = G(R)T(K)G(R).$$

[4] This is sometimes also referred to as the Iwahori Decomposition or the Cartan–Iwahori–Matsumoto Decomposition

To see how the theorem implies the above identity, choose $h \in G(R)$ such that $h\lambda|_K h^{-1} \in T(K)$. Then

$$g = h_1 \lambda|_K h_2 = \underbrace{(h_1 h^{-1})}_{\in G(R)} \underbrace{(h\lambda|_K h^{-1})}_{\in T(K)} \underbrace{(hh_2)}_{\in G(R)}.$$

Conversely, suppose $g = h_1 t h_2$ for $h_1, h_2 \in G(R)$ and $t \in T(K)$. If we write $T \cong \mathbb{G}_m^r$ and $\pi \in R$ as the uniformizing parameter, then $t = (u_1 \pi^{d_1}, \ldots, u_r \pi^{d_r})$ for units $u_i \in R^\times$ and integers $d_i \in \mathbb{Z}$. After replacing $h_1$ with $h_1 \cdot (u_1, \ldots, u_r)$, we can write $g = h_1 \lambda|_K h_2$ where $\lambda \colon \mathbb{G}_m \to T \subset G$ is the one-parameter subgroup given by $t \mapsto (t^{d_1}, \ldots, t^{d_r})$.

**Remark 6.6.25** (Case of $\mathrm{GL}_n$)**.** The Cartan Decomposition for $\mathrm{GL}_n$ can be established by an elementary linear algebra argument. Let $g = (g_{ij}) \in \mathrm{GL}_n(K)$. After performing row and column operations, we can assume that $g_{1,1} = \pi^d$ has minimal valuation among the $g_{ij}$, where $\pi \in R$ is a uniformizer. For each $k \geq 2$, we write $g_{k,1} = u\pi^e$. Now perform the row operations where the $n$th row $r_n$ is exchanged for $r_n - u\pi^{e-d} r_1$. In this way, we can arrange that $g_{k,1} = 0$ for $k \geq 2$. By performing analogous column operations, we can also arrange that $g_{1,k} = 0$ for $k \geq 2$. The statement is thus established by induction.

**Exercise 6.6.26.** Let $\Bbbk$ be a field.

(a) Let $X \subset \mathbb{P}(V)$ be a $\mathbb{G}_m$-equivariant locally closed subscheme where $V$ is a finite dimensional $\mathbb{G}_m$-representation. Show that $[X/\mathbb{G}_m]$ is separated if and only if $X$ has no $\mathbb{G}_m$-fixed points, or in other words that the diagonal $[X/\mathbb{G}_m] \to [X/\mathbb{G}_m] \times [X/\mathbb{G}_m]$ is finite if and only if it is quasi-finite.

(b) Let $G$ be a reductive algebraic group acting on an algebraic space $X$ over $\Bbbk$. Show that $[X/G]$ is separated if and only if for every one-parameter subgroup $\lambda \colon \mathbb{G}_m \to G$, the corresponding quotient stack $[X/\mathbb{G}_m]$ is separated.

*Hint: Verify the valuative criterion by applying the Cartan Decomposition.*

**Remark 6.6.27.** Unlike the case of $\mathbb{G}_m$ in (a), it is not true $[X/G]$ is separated for an action of an affine algebraic group $G$ acting linearly on a quasi-projective scheme $X$ with finite stabilizers. See Exercise 3.9.3(d) for such an example by a free action of $\mathrm{SL}_2$ on a quasi-affine variety.

### 6.6.6 The Destabilization Theorem

**Theorem 6.6.28** (Destabilization Theorem)**.** *Let $G$ be a reductive algebraic group over an algebraically closed field $\Bbbk$ acting on an affine scheme $X$ of finite type over $\Bbbk$. Given $x \in X(\Bbbk)$, there exists a one-parameter subgroup $\lambda \colon \mathbb{G}_m \to G$ such that $x_0 := \lim_{t \to 0} \lambda(t) \cdot x$ exists and has closed $G$-orbit.*

*Proof.* Let $R = \Bbbk[[t]]$ with fraction field $K = \Bbbk((t))$. We can choose an element $g \in G(K)$ and a commutative diagram

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d, hook] & Gx \arrow[d, hook] \\ \operatorname{Spec} R \arrow[r, "\widetilde{g}"] & X \end{tikzcd}$$

where the top map is given by the composition $\operatorname{Spec} K \xrightarrow{g} G \to Gx$ and such that $y := \widetilde{g}(0)$ has closed $G$-orbit. By the Cartan decomposition, there exists

$h_1, h_2 \in G(R)$ and a one-parameter subgroup $\lambda\colon \mathbb{G}_m \to G$ such that $h_1 g = \lambda|_K h_2$. By applying the general fact that for $a \in G(R)$ and $b \in X(R)$, $(a \cdot b)(0) = a(0) \cdot b(0)$ to $h_1 \in G(R)$ and $\widetilde{g} \in X(R)$, we obtain that

$$\lim_{t \to 0} \lambda(t) h_2(t) \cdot x = \lim_{t \to 0} h_1(t) g(t) \cdot x = h_1(0) \cdot \widetilde{g}(0) = h_1(0) \cdot y \in Gy. \tag{6.6.8}$$

We claim that the related but *possibly different* limit $\lim_{t \to 0} \lambda(t) h_2(0) \cdot x$ exists and is also contained in the closed orbit $Gy$. Once this is established, the theorem would be established by using the one-parameter subgroup $h_2(0)^{-1} \lambda h_2(0)$:

$$\lim_{t \to 0} (h_2(0)^{-1} \lambda h_2(0))(t) \cdot x = h_2^{-1}(0) \cdot \lim_{t \to 0} \lambda(t) h_2(0) \cdot x \in Gy.$$

First, to see that $\lim_{t \to 0} \lambda(t) h_2(0) \cdot x$ exists, we may apply Proposition C.3.4(1) below to reduce to the case that $X = \mathbb{A}(V)$ is a $G$-representation. We may choose a basis of $V \cong \Bbbk^n$ such that the $\lambda$-action has weights $\lambda_1, \ldots, \lambda_n$. We may also write $h_2 \cdot x = (a_1, \ldots, a_n) \in X(R)$ with each $a_i \in \Bbbk[\![t]\!]$ and further decompose $a_i = a_i(0) + a_i'$ with $a_i' \in (t)$. Since

$$\lim_{t \to 0} \lambda(t) h_2(t) \cdot x = \lim_{t \to 0} (t^{\lambda_1}(a_1(0) + a_1'), \ldots, t^{\lambda_n}(a_n(0) + a_n')) \tag{6.6.9}$$

exists, we see that for each $i$ with $\lambda_i < 0$, we must have that $a_i(0) = 0$, which in turn implies that $\lim_{t \to 0} \lambda(t) h_2(0) \cdot x$ exists.

Finally, to see that this limit lies in $Gy$, we may apply Proposition C.3.4(2) to obtain a $G$-equivariant map $f\colon X \to \mathbb{A}(W)$ such that $f^{-1}(0) = Gy$. We are thus reduced to showing that $\lim_{t \to 0} \lambda(t) h_2(0) \cdot f(x) = 0$. By computing the limit $\lim_{t \to 0} \lambda(t) h_2(t) \cdot f(x)$ as in (6.6.9), the same argument shows that since $\lim_{t \to 0} \lambda(t) h_2(t) \cdot f(x) = 0$, we must also have that $\lim_{t \to 0} \lambda(t) h_2(0) \cdot f(x) = 0$. See also [GIT, p. 53] and [Kem78, Thm. 1.4]. □

**Corollary 6.6.29** (Destabilization Theorem II)**.** *Let $\mathcal{X}$ be an algebraic stack of finite type over an algebraically closed field $\Bbbk$ with affine diagonal. Let $x \rightsquigarrow x_0$ be a specialization of $\Bbbk$-points such that the stabilizer $G_{x_0}$ is linearly reductive. Then there exists a morphism $[\mathbb{A}^1/\mathbb{G}_m] \to \mathcal{X}$ representing the specialization $x \rightsquigarrow x_0$.* □

*Proof.* The Local Structure Theorem (6.5.1) yields an étale morphism $[\operatorname{Spec} A/G_{x_0}] \to \mathcal{X}$ and a point $w_0$ mapping to $x_0$. After possibly replacing $\operatorname{Spec} A$ with a $G_{x_0}$-invariant affine subscheme, we can assume that $w_0$ is a closed point. The specialization $x \rightsquigarrow x_0$ lifts a specialization $w \rightsquigarrow w_0$ in $[\operatorname{Spec} A/G_{x_0}]$, and we can choose a representative $\widetilde{w} \in \operatorname{Spec} A$ of the orbit corresponding to $w$. The Destabilization Theorem gives a one-parameter subgroup $\lambda\colon \mathbb{G}_m \to G$ such that $\widetilde{w}_0 = \lim_{t \to 0} \lambda(t) \cdot \widetilde{w}$ exists and has closed orbit. By Affine GIT (6.3.7), there is a unique closed orbit in $\overline{G\widetilde{w}}$ and thus $\widetilde{w}_0 \in \operatorname{Spec} A$ maps to $w_0$. The $\mathbb{G}_m$-equivariant extension $\mathbb{A}^1 \to X$ of $t \mapsto \lambda(t) \cdot \widetilde{w}$ defines a morphism of algebraic stacks $[\mathbb{A}^1/\mathbb{G}_m] \to [\operatorname{Spec} A/G_{x_0}]$ such that the image of the specialization $1 \rightsquigarrow 0$ is $w \rightsquigarrow w_0$. The composition $[\mathbb{A}^1/\mathbb{G}_m] \to [\operatorname{Spec} A/G_{x_0}] \to \mathcal{X}$ yields the desired map. □

### 6.6.7 Maps from $\Theta = [\mathbb{A}^1/\mathbb{G}_m]$

We define the quotient stack

$$\Theta = [\mathbb{A}^1/\mathbb{G}_m]$$

over $\operatorname{Spec}\mathbb{Z}$; when we are working over a field $\Bbbk$, we will abuse notation by also using $\Theta$ to denote $\Theta_\Bbbk = [\mathbb{A}^1_\Bbbk/\mathbb{G}_{m,\Bbbk}]$.

While a map $\mathcal{X} \to \Theta$ from an algebraic stack is classified by a line bundle and a section (Example 3.9.16), maps $\Theta \to \mathcal{X}$ *from* $\Theta$ often also have geometric significance. We provide such descriptions for maps from $\Theta$ to quotient stacks, stacks of coherent sheaves, and the stack of all curves. These descriptions will be useful to interpret the valuative criteria of $\Theta$ and $\mathsf{S}$-completness introduced in §6.8.2. In fact, we will provide descriptions of maps from $\Theta_R := \Theta \times_k R$ where $R$ is a $\Bbbk$-algebra; the reader is encouraged to consider the case that $R = \Bbbk$ on a first reading.

#### Quotient stacks

Given a quotient stack $[X/G]$, a one-parameter subgroup $\lambda\colon \mathbb{G}_m \to G$, and a point $x \in X(\Bbbk)$ such that $\lim_{t\to 0} \lambda(t)\cdot x \in X$ exists, the $\mathbb{G}_m$-equivariant extension $\mathbb{A}^1 \to X$ induces a morphism $[\mathbb{A}^1/\mathbb{G}_m] \to [X/G]$ of algebraic stacks. The next proposition asserts that the converse is also true, i.e. any map $[\mathbb{A}^1/\mathbb{G}_m] \to [X/G]$ is induced by a one-parameter subgroup $\lambda$ and a point $x \in X(\Bbbk)$.

Recall from §C.3.3 that if $\lambda\colon \mathbb{G}_m \to G$ is a one-parameter subgroup, then $P_\lambda \subset G$ denotes the subgroup of elements $g \in G$ such that $\lim_{t\to 0} \lambda(t) g \lambda(t)^{-1}$ exists. If $G$ is reductive, then $P_\lambda$ is a parabolic .

**Proposition 6.6.30.** *Let $G$ be a smooth affine algebraic group over an algebraically closed field $\Bbbk$, and let $X$ be a separated algebraic space of finite type over $\Bbbk$. For every complete noetherian local $\Bbbk$-algebra $R$ with algebraically closed residue field $\mathbb{F}$, there is an equivalence of groupoids*

$$\mathrm{MOR}_\Bbbk(\Theta_R, [X/G]) \xrightarrow{\sim} \big\{(x \in X(R), \lambda\colon \mathbb{G}_m \to G) \mid \lim_{t\to 0} \lambda(t)\cdot x \in X(R) \text{ exists}\big\};$$

*a morphism $(x,\lambda) \to (x',\lambda')$ is an isomorphism class of a pair $(g,h)$ with $g \in P_\lambda(R)$ and $h \in G(\Bbbk)$ such that $x' = hgx$ and $\lambda' = h\lambda h^{-1}$ where $(g,h) \sim (c^{-1}g, hc)$ for $c \in C_\lambda$.*

*Under this correspondence, the morphism $\Theta_R \to [X/G]$ sends $1$ to $x$ and $0$ to $\lim_{t\to 0} \lambda(t)\cdot x$.*

**Remark 6.6.31.** Observe that over $R = \Bbbk$ an isomorphism $(x,\lambda) \xrightarrow{\sim} (x',\lambda)$ is given by an element $g \in P_\lambda(\Bbbk)$ such that $x' = gx$. In particular, the automorphism group of $(x,\lambda)$ is $P_\lambda \cap G_x$.

*Proof.* Given $(x,\lambda)$, the $\mathbb{G}_m$-equivariant map $m_{x,\lambda}\colon \mathbb{G}_{m,R} \to X$ defined by $t \mapsto \lambda(t)\cdot x$ extends to a commutative diagram

$$\begin{tikzcd} \mathbb{G}_{m,R} \arrow[r, "m_{x,\lambda}"] \arrow[d, hook] & X \\ \mathbb{A}^1_R \arrow[ur, dashed, "\widetilde{m}_{x,\lambda}"'] & \end{tikzcd}$$

The extension is $\mathbb{G}_m$-equivariant and induces a morphism of quotient stacks $f_{x,\lambda}\colon \Theta_R \to [X/G]$. We will show that this defines a functor

$$\begin{aligned} \{(x,\lambda) \mid \lim_{t\to 0} \lambda(t)\cdot x \text{ exists}\} &\to \mathrm{MOR}_\Bbbk(\Theta_R, [X/G]) \\ (x,\lambda) &\mapsto f_{x,\lambda}. \end{aligned} \tag{6.6.10}$$

Given a morphism $(g,h)\colon (x,\lambda)\to(x',\lambda')$, we need to define a 2-morphism $f_{x,\lambda}\xrightarrow{\sim} f_{x',\lambda'}$. Since $h$ determines a canonical isomorphism $f_{x',\lambda'}\xrightarrow{\sim} f_{h^{-1}x',\lambda}$, it suffices to define a 2-morphism $f_{x,\lambda}\xrightarrow{\sim} f_{h^{-1}x',\lambda}$. Since $g\in P_\lambda(R)$, the map $t\mapsto \lambda(t)g\lambda(t)^{-1}$ extends to a map $\widetilde{g}\colon \mathbb{A}^1_R\to G$ such that $\widetilde{m}_{h^{-1}x',\lambda}=\widetilde{g}\cdot\widetilde{m}_{x,\lambda}$ (as $h^{-1}x'=gx$). The element $\widetilde{g}$ defines an isomorphism $f_{x,\lambda}\xrightarrow{\sim} f_{h^{-1}x',\lambda}$. For $c\in C_\lambda$, the pairs $(g,h)$ and $(c^{-1}g,hc)$ define the same isomorphism: indeed this follows from the observation that if $c\in G_x$, then $(c^{-1},c)$ defines the identify automorphism of $f_{x,\lambda}$. Conversely, any isomorphism $f_{x,\lambda}\xrightarrow{\sim} f_{x',\lambda}$ is an induced by element $\widetilde{g}\in G(\mathbb{A}^1)$ satisfying $\widetilde{m}_{x',\lambda}=\widetilde{g}\cdot\widetilde{m}_{x,\lambda}$, and we see that (6.6.10) is a fully faithful functor.

To see essential surjectivity of (6.6.10), let $f\colon \Theta_R\to[X/G]$ be a morphism. In the fiber diagram

$$\begin{tikzcd}
\mathcal{P} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X \arrow[d] \\
\Theta_R \arrow[r, "f"] & {[X/G]}
\end{tikzcd}$$

$\mathcal{P}\to\Theta_R$ is a principal $G$-bundle. The restriction $\mathcal{P}|_{\mathbf{B}\mathbb{G}_{m,\mathbb{F}}}$ along the unique closed point $0\colon \mathbf{B}\mathbb{G}_{m,\mathbb{F}}\to\Theta_R$, corresponds to a $\mathbb{G}_m$-equivariant principal $G$-bundle $P$ on $\operatorname{Spec}\mathbb{F}$. After choosing an isomorphism $P\cong G$, we see that $\mathcal{P}$ corresponds to a one-parameter subgroup $\lambda'\colon \mathbb{G}_{m,\mathbb{F}}\to G_{\mathbb{F}}$.

Choose a maximal torus $T\subset G$ over $\Bbbk$. Since all maximal tori of $G_{\mathbb{F}}$ are conjugate (Proposition C.3.5), there exists an element $q\in G(\mathbb{F})$ such that the image of $q\lambda' q^{-1}$ is contained in $T_{\mathbb{F}}$. Letting $n=\dim T$, there are equivalences

$$\operatorname{Hom}_{\Bbbk}(\mathbb{G}_m,T)\cong\mathbb{Z}^n\cong\operatorname{Hom}_{\mathbb{F}}(\mathbb{G}_{m,\mathbb{F}},T_{\mathbb{F}}),$$

where the composition is given by $\lambda\mapsto\lambda\times_{\Bbbk}\mathbb{F}$. It follows that there is a one-parameter subgroup $\lambda\colon\mathbb{G}_m\to T$ whose base change $\lambda\times_{\Bbbk}\mathbb{F}$ is conjugate to $\lambda'$. On the other hand, every one-parameter subgroup $\lambda$ induces a $\mathbb{G}_m$-action on the product $\mathbb{A}^1_R\times G$ by $t\cdot(x,g)=(tx,g\lambda(t)^{-1})$ and thus a principal $G$-bundle $\mathcal{P}_\lambda:=[(\mathbb{A}^1_R\times G)/\mathbb{G}_m]$ over $\Theta_R$. We claim that there is an isomorphism $\alpha\colon\mathcal{P}\to\mathcal{P}_\lambda$ of principal $G$-bundles. By construction, we have an isomorphism $\alpha_0\colon \mathcal{P}|_{\mathbf{B}\mathbb{G}_{m,\mathbb{F}}}\to\mathcal{P}_\lambda|_{\mathbf{B}\mathbb{G}_{m,\mathbb{F}}}$. Since $\underline{\operatorname{Isom}}_{\Theta_R}(\mathcal{P},\mathcal{P}_\lambda)\to\Theta_R$ is smooth (as it's a principal $G$-bundle and $G$ is smooth), we may use deformation theory (Proposition 6.5.8) to construct compatible isomorphisms $\alpha_n\colon\mathcal{P}|_{\mathcal{X}_n}\to\mathcal{P}_\lambda|_{\mathcal{X}_n}$ over the nilpotent thickenings $\mathcal{X}_n$ of $0\colon\mathbf{B}\mathbb{G}_{m,\mathbb{F}}\hookrightarrow\Theta_R$. Coherent Tannaka Duality (6.4.8) coupled with the coherent completeness of $\Theta_R$ along $\mathbf{B}\mathbb{G}_{m,\mathbb{F}}$ (Theorem 6.4.11) implies that the isomorphisms $\alpha_n$ extend to an isomorphism $\alpha\colon\mathcal{P}\to\mathcal{P}_\lambda$. Restricting the composition $\mathbb{A}^1_R\times G\to\mathcal{P}_\lambda\xrightarrow{\alpha^{-1}}\mathcal{P}\to X$ to the identity in $G$ yields a $\mathbb{G}_m$-equivariant morphism $\mathbb{A}^1_R\to X$. One checks that this corresponds to the given map $f\colon\Theta_R\to[X/G]$ on quotient stacks. Letting $x\in X(R)$ be the image of 1, we see that $f_{x,\lambda}$ is 2-isomorphic to $f$. $\square$

**Remark 6.6.32.** Proposition 6.6.30 can be upgraded to a description of the stack of morphisms from $[\mathbb{A}^1/\mathbb{G}_m]$ to $[X/G]$. Namely, there is a decomposition

$$\underline{\operatorname{Mor}}([\mathbb{A}^1/\mathbb{G}_m],[X/G])\cong\coprod_\lambda[X^+_\lambda/P_\lambda]$$

where $\lambda$ varies over conjugacy classes of one-parameter subgroups. The loci $X^+_\lambda$ are often locally closed subschemes (see Theorem 6.6.9(3)), and $P_\lambda\subset G$ is a parabolic subgroup if $G$ is reductive.

The algebraicity of this stack was discussed already in Remark 6.6.8. See also [HL14, Thm. 1.37].

### Stacks of coherent sheaves

Given a projective scheme $X$, let $\underline{\mathrm{Coh}}(X)$ denote the algebraic stack of coherent sheaves on $X$ (see Exercise 3.1.22).

**Proposition 6.6.33.** *Let $X$ be a projective scheme over an algebraically closed field $\Bbbk$. For a noetherian $\Bbbk$-algebra $R$, $\mathrm{MOR}_{\Bbbk}(\Theta_R, \underline{\mathrm{Coh}}(X))$ is equivalent to the groupoid of pairs $(E, E_\bullet)$ where $E$ is a coherent sheaf on $X_R$ flat over $R$ and*

$$E_\bullet : 0 \subset \cdots \subset E_{i-1} \subset E_i \subset E_{i+1} \subset \cdots \subset E$$

*is a $\mathbb{Z}$-graded filtration such that $E_i = 0$ for $i \ll 0$, $E_i = E$ for $i \gg 0$, and each factor $E_i/E_{i-1}$ is flat over $R$. A morphism $(E, E_\bullet) \to (E', E'_\bullet)$ is an isomorphism $E \to E'$ of coherent sheaves compatible with the filtration.*

*Under this correspondence, the morphism $\Theta_R \to \underline{\mathrm{Coh}}(X)$ sends 1 to $E$ and 0 to the associated graded $\operatorname{gr} E_\bullet := \bigoplus_i E_i/E_{i-1}$, and factors through $\mathrm{Bun}(X) \subset \underline{\mathrm{Coh}}(X)$ if and only if $E$ and each factor $E_i/E_{i-1}$ is a vector bundle.*

*Proof.* A morphism $\Theta_R \to \underline{\mathrm{Coh}}(X)$ corresponds to a coherent sheaf $\mathcal{F}$ on $C \times \Theta_R$ flat over $\Theta_R$. By smooth descent, this corresponds to a coherent sheaf on $C \times \mathbb{A}^1_R$ flat over $\mathbb{A}^1_R$ together with a $\mathbb{G}_m$-action. Pushing forward $\mathcal{F}$ along the affine morphism $C \times \Theta_R \to C \times \mathbf{B}\mathbb{G}_{m,R}$, we see that $\mathcal{F}$ also corresponds to a graded $\mathcal{O}_{C_R}[x]$-module flat over $R[x]$. Writing $\mathcal{F} = \bigoplus_i E_i$ with each $E_i$ a coherent sheaf on $C_R$, then multiplication by $x$ induces maps $x\colon E_i \to E_{i+1}$ which are necessarily injective as $\mathcal{F}$ is flat over $R[x]$, hence torsion free. Since $\mathcal{F}$ is finitely generated as a graded $R[x]$-module, there exists finitely many homogeneous generators with bounded degree. Thus $E_i = E$ for $i \gg 0$. On the other hand, considering the $\mathcal{O}_{C_R}[x]$-submodule $E_{\geq d} := \bigoplus_{i \geq n} E_i \subset \mathcal{F}$, the ascending chain $\cdots \subset E_{\geq d} \subset E_{\geq d-1} \subset \cdots \subset \mathcal{F}$ must terminate as $\mathcal{F}$ is noetherian. It follows that $E_i = 0$ for $i \ll 0$. Since $\mathcal{F}$ is flat as an $R[x]$-module, the quotient $\mathcal{F}/x\mathcal{F} = \bigoplus_i E_i/E_{i-1}$ is flat as an $R$-module and thus each factor $E_i/E_{i-1}$ is flat over $R$.

Conversely, given $E$ and a filtration $E_\bullet$ satisfying the above conditions, consider the graded $\mathcal{O}_{C_R}[x]$-module $\mathcal{F} := \bigoplus_i E_i$; this is frequently referred to as the 'Rees construction'. We will show by induction that $E_{\geq d} := \bigoplus_{i \geq d} E_i$ is flat and finitely generated over $R[x]$; this implies that $\mathcal{F}$ is flat and finitely generated over $R[x]$ since $E_i = 0$ for $i \ll 0$. For $d \gg 0$, $E_{\geq d}$ is isomorphic to the graded $R[x]$-module $(E \otimes_R R[x])\langle d \rangle$, where $\langle d \rangle$ denotes the grading shift, and is thus flat and finitely generated. For every $d$, we have an exact sequence

$$0 \to (E_d \otimes_R R[x])\langle d \rangle \to E_{\geq d} \to ((E_{d+1}/E_d) \otimes_R R[x])\langle d+1 \rangle \to 0.$$

The flatness of $E$ and the quotients $E_{d+1}/E_d$ implies the flatness of each $E_d$. Thus the left and right term above are flat and finitely generated as $R[x]$-modules, and thus so is the middle term. $\square$

### Stack of all curves

**Proposition 6.6.34.** *Let $\mathcal{M}_g^{\mathrm{all}}$ be the algebraic stack of all proper curves (Theorem 5.4.7) over an algebraically closed field $\Bbbk$. For every $\Bbbk$-algebra $R$, $\mathrm{MOR}_{\Bbbk}(\Theta_R, \mathcal{M}_g^{\mathrm{all}})$ is the groupoid whose objects are $\mathbb{G}_m$-equivariant families of proper curves $\mathcal{C} \to \mathbb{A}^1_R$, where $\mathbb{G}_m$ acts on $\mathbb{A}^1_R$ with the usual scaling action. Morphisms are $\mathbb{G}_m$-equivariant morphisms.*

*Proof.* The statement follows from smooth descent applied to $\mathbb{A}^1_R \to \Theta_R$. $\square$

**Remark 6.6.35.** A similar description holds for other moduli stacks of varieties. Such $\mathbb{G}_m$-equivariant maps are often called 'test configurations' in the literature.

## 6.7 Geometric Invariant Theory (GIT)

Geometric Invariant Theory (GIT) was developed by Mumford in [GIT] as a means to construct quotients and moduli spaces in algebraic geometry. For other expository accounts, we recommend [New78], [Kra84], [Dol03], [Muk03] and [Stu08].

### 6.7.1 Good quotients

Let $G$ be an affine algebraic group over an algebraically closed field $\Bbbk$ acting on an algebraic space $U$ of finite type over $\Bbbk$. In the following cases, we've already established the existence of a *geometric quotient* $U/G$ (Definition 4.3.1), i.e. a $G$-invariant map $U \to U/G$ inducing a bijection $U(\Bbbk)/G(\Bbbk) \to (U/G)(\Bbbk)$ and universal for $G$-invariant maps to algebraic spaces; in other words $[U/G] \to U/G$ is a coarse moduli space.

- If $G$ is a (reduced) finite group and the action is free (i.e. the action map $G \times U \to U \times U$ is a monomorphism), then $U/G := [U/G]$ exists as an algebraic space of finite type over $\Bbbk$ (Corollary 3.1.13). This also holds in the non-finite case: if $G$ is an algebraic group and the action is free, then $[U/G]$ is an algebraic stack (Proposition 6.2.9) such that $[U/G] \to [U/G] \times [U/G]$ is a monomorphism and therefore $U/G := [U/G]$ is an algebraic space (Theorem 4.4.10).
- If $G$ is finite and $U = \operatorname{Spec} A$ is affine, then $U/G := \operatorname{Spec} A^G$ is a geometric quotient (Theorem 4.3.6).
- If $G$ is finite and $U$ is projective (resp. quasi-projective, quasi-affine), then the quotient $U/G$ exists as a projective (resp. quasi-projective, quasi-affine) $\Bbbk$-scheme (Exercise 4.2.8).
- If $G$ is finite and $U$ is separated, then $U/G$ exists as a separated algebraic space as a consequence of the Keel–Mori Theorem (4.3.11). This also holds in the non-finite case: if $G$ is an affine algebraic group, the stabilizers of the action are finite and reduced, and the action map $G \times U \to U \times U$ is proper, then $[U/G]$ is a separated Deligne–Mumford stack (Theorem 3.6.4) and the existence of a geometric quotient follows from the Keel–Mori Theorem.

GIT studies the case where $G$ is linearly reductive[5] but not necessarily finite. GIT allows for the possibility of points $u \in U$ where the stabilizer $G_u$ may not be finite and the orbit $Gu$ may not be closed, e.g. $\mathbb{G}_m$ acting on $\mathbb{A}^1$.

In Corollary 6.3.7, we've already considered the affine case of GIT where $G$ is a linearly reductive algebraic group over an algebraically closed field $\Bbbk$ acting on an affine $\Bbbk$-scheme $\operatorname{Spec} A$. In this case, we have a commutative diagram

$$\begin{tikzcd} \operatorname{Spec} A \arrow[d] \arrow[dr, "\widetilde{\pi}"] & \\ {[\operatorname{Spec} A/G]} \arrow[r, "\pi"] & (\operatorname{Spec} A)/\!\!/G := \operatorname{Spec} A^G \end{tikzcd}$$

[5] GIT can be developed in the more general setting of actions by *reductive* algebraic groups; see Remark 6.3.10.

where $\pi\colon [\operatorname{Spec} A/G] \to \operatorname{Spec} A^G$ is a good moduli space and $\widetilde{\pi}\colon \operatorname{Spec} A \to \operatorname{Spec} A^G$ is a *good quotient*.

**Definition 6.7.1** (Good quotients)**.** Given an action of a linearly reductive algebraic group $G$ over an algebraically closed field $\Bbbk$ on an algebraic space $U$ over $\Bbbk$, a $G$-invariant map $\widetilde{\pi}\colon U \to X$ is a *good quotient* if

(1) $\mathcal{O}_X \to (\pi_*\mathcal{O}_U)^G$ is an isomorphism (where $(\pi_*\mathcal{O}_U)^G(V) = \Gamma(U_V, \mathcal{O}_{U_V})^G$ for an étale $X$-scheme $V$) and

(2) $\widetilde{\pi}$ is affine.[6]

The good quotient of $U$ by $G$ is often denoted as $U/\!\!/G = X$.

**Remark 6.7.2.** The map $\widetilde{\pi}\colon U \to X$ is a good quotient if and only if $\pi\colon [U/G] \to X$ is a good moduli space. To see the equivalence, we may assume that $X = \operatorname{Spec} B$ is affine since both properties are étale local (Lemma 6.3.20(1)). For ($\Rightarrow$), $U = \operatorname{Spec} A$ is also affine and $B = A^G$, and thus $[\operatorname{Spec} A/G] \to \operatorname{Spec} A^G$ is a good moduli space. To see ($\Leftarrow$), observe that since $U \to [U/G]$ is affine and $\pi_*$ is exact on quasi-coherent sheaves, the pushforward $\widetilde{\pi}_*$ is exact on quasi-coherent sheaves and thus $\widetilde{\pi}$ is affine by Serre's Criterion for Affineness (4.4.16).

**Proposition 6.7.3.** *Let $G$ be a linearly reductive algebraic group over an algebraically closed field $\Bbbk$ acting on an algebraic space $U$ over $\Bbbk$. If $\widetilde{\pi}\colon U \to X$ is a good quotient, then*

*(1) $\widetilde{\pi}$ is surjective and the image of a closed $G$-invariant subscheme is closed. The same holds for the base change $T \to X$ by a morphism from a scheme;*

*(2) for closed $G$-invariant closed subschemes $Z_1, Z_2 \subset U$, $\operatorname{im}(Z_1 \cap Z_2) = \operatorname{im}(Z_1) \cap \operatorname{im}(Z_2)$. In particular, for $x_1, x_2 \in X(\Bbbk)$, $\widetilde{\pi}(x_1) = \widetilde{\pi}(x_2)$ if and only if $\overline{Gx_1} \cap \overline{Gx_2} \neq \emptyset$, and $\widetilde{\pi}$ induces a bijection between closed $G$-orbits in $U$ and $\Bbbk$-points of $X$;*

*(3) if $U$ is noetherian, so is $X$. If $U$ is finite type over $\Bbbk$, then so is $X$, and for every coherent $\mathcal{O}_U$-module $F$ with a $G$-action, $(\pi_*F)^G$ is coherent; and*

*(4) $\widetilde{\pi}$ is universal for $G$-invariant maps to algebraic spaces.*

*Proof.* This follows from Theorem 6.3.5 as $[U/G] \to X$ is a good moduli space. $\square$

**Remark 6.7.4** (Semistable reduction in GIT)**.** Since $[U/G] \to X$ is universally closed (Theorem 6.3.5(1)), it satisfies the valuative criterion for universally closedness (Theorem 3.8.5). This translates into the following: for every DVR $R$ over $\Bbbk$ with fraction field $K$ and every map $\operatorname{Spec} R \to X$ with a lift $\eta\colon \operatorname{Spec} K \to U$, there exists an extension $R \to R'$ of DVRs, an element $g' \in G(K')$ over the fraction field of $R'$, and a lift in the commutative diagram

$$\begin{tikzcd}
\operatorname{Spec} K' \arrow[rr, bend left, "g'\cdot\eta|_{K'}"] \arrow[d] & \operatorname{Spec} K \arrow[r, "\eta"] \arrow[d] & U \arrow[d, "\widetilde{\pi}"] \\
\operatorname{Spec} R' \arrow[r] \arrow[urr, dashed] & \operatorname{Spec} R \arrow[r] & X.
\end{tikzcd}$$

In fact, if $R = \Bbbk[\![x]\!]$, it can be arranged that $R \to R'$ is finite; see [Mum77, Lem. 5.3] and [AHLH18, Thm. A.8].

[6] A *good quotient* is sometimes defined as an affine $G$-invariant morphism $\widetilde{\pi}\colon U \to X$ such that $\mathcal{O}_X \xrightarrow{\sim} (\pi_*\mathcal{O}_U)^G$ and properties Proposition 6.7.3(1)-(2) holds, c.f. [Ses72, Def. 1.5].

### 6.7.2 Projective GIT

Let $U$ be a projective scheme over an algebraically closed field $\Bbbk$ with an action of a linearly reductive algebraic group $G$. Suppose that there is a $G$-equivariant embedding $U \hookrightarrow \mathbb{P}(V)$, where $V$ is a finite dimensional $G$-representation; this is equivalent to giving a very ample line bundle $\mathcal{O}_U(1)$ with a $G$-action, i.e. a very ample $G$-linearization (see §C.3.4).

**Definition 6.7.5.** We define the *semistable* and *stable* locus as

$$U^{\mathrm{ss}} := \{u \in U \mid \text{there exists } f \in \Gamma(U, \mathcal{O}_U(d))^G \text{ with } d > 0 \text{ such that } f(u) \neq 0\},$$

$$U^{\mathrm{s}} := \left\{ u \in U \;\middle|\; \begin{array}{l} \text{there exists } f \in \Gamma(U, \mathcal{O}_U(d))^G \text{ with } d > 0 \text{ such that} \\ \quad -f(u) \neq 0, \\ \quad -\text{the orbit } Gu \subset U_f \text{ is closed, and} \\ \quad -\text{the function } U \to \mathbb{Z},\ x \mapsto \dim G_x \text{ is constant in an open} \\ \qquad \text{neighborhood of } u^{7} \end{array} \right\}.$$

A point $u \in U$ is called *semistable* (resp. *stable*) if $u \in U^{\mathrm{ss}}$ (resp. $u \in U^{\mathrm{s}}$).[8] The *nullcone* $\widehat{N} \subset \mathbb{A}(V)$ is by definition the affine cone over $U \setminus U^{\mathrm{ss}}$: it is set of points $u$ in the affine cone $\widehat{U} \subset \mathbb{A}(V)$ such that $f(u) = 0$ for every non-constant $G$-invariant polynomial on $\mathbb{A}(V)$.

We stress that the stable and semistable loci depend on the choice of $G$-equivariant embedding $U \hookrightarrow \mathbb{P}(V)$. When $U$ is a normal projective variety, then every line bundle $L$ has a positive tensor power $L^{\otimes n}$ that has a $G$-linearization by Sumihiro's Theorem on Linearizations (C.3.12). For example, $\mathcal{O}(1)$ on $\mathbb{P}^n$ does not have a $\mathrm{PGL}_{n+1}$-linearization, but $\mathcal{O}(n+1)$ does.

Let $R = \bigoplus_{d \geq 0} \Gamma(U, \mathcal{O}_U(d))$ be the projective coordinate ring. We consider the map

$$\widetilde{\pi} \colon U^{\mathrm{ss}} \to U^{\mathrm{ss}} /\!\!/ G := \operatorname{Proj} R^G. \tag{6.7.1}$$

Note that $U^{\mathrm{ss}}$ may be empty in which case $\operatorname{Proj} R^G$ is the empty scheme. If $U^{\mathrm{ss}}$ is non-empty, it is precisely the locus where the rational map $\operatorname{Proj} R \dashrightarrow \operatorname{Proj} R^G$ is defined.

**Theorem 6.7.6.** *Let $G$ be a linearly reductive algebraic group over an algebraically closed field $\Bbbk$. Let $U \subset \mathbb{P}(V)$ be a $G$-equivariant closed subscheme where $V$ is a finite dimensional $G$-representation. Then there is a cartesian diagram*

$$\begin{tikzcd} U^{\mathrm{s}} \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & U^{\mathrm{ss}} \arrow[r, hook] \arrow[d, "\widetilde{\pi}"] & U \\ U^{\mathrm{s}}/G \arrow[r, hook] & U^{\mathrm{ss}} /\!\!/ G & \end{tikzcd}$$

*where $U^{\mathrm{s}}/G \subset U^{\mathrm{ss}} /\!\!/ G$ is an open subscheme, the map $\widetilde{\pi}$ of (6.7.1) is a good quotient, and the restriction $\widetilde{\pi}|_{U^{\mathrm{s}}} \colon U^{\mathrm{s}} \to U^{\mathrm{s}}/G$ is a geometric quotient. Moreover, $U^{\mathrm{ss}} /\!\!/ G$ is projective with an ample line bundle $L$ such that $\widetilde{\pi}^* L \cong \mathcal{O}_U(N)$ for some $N$.*

*If in addition the action of $G$ on $U$ has generically finite stabilizers, then the action of $G$ on $U^{\mathrm{s}}$ is proper (i.e. the action map $G \times U^{\mathrm{s}} \to U^{\mathrm{s}} \times U^{\mathrm{s}}$ is proper) or in other words $[U^{\mathrm{s}}/G]$ is separated.*

[7] Since the function $x \mapsto \dim G_x$ is upper semi-continuous, this condition is automatic if $\dim G_u = 0$.

[8] In the literature, a point $u \in U$ is sometimes called 'unstable' if it is not semistable; we avoid this potentially misleading terminology.

*Proof.* Since $U$ is projective, $R = \bigoplus_{d \geq 0} \Gamma(U, \mathcal{O}_U(d))$ is finitely generated over $\Bbbk$. Thus by Corollary 6.3.7(3), $R^G$ is also finitely generated over $\Bbbk$ and $U^{\mathrm{ss}} /\!\!/ G = \operatorname{Proj} R^G$ is projective. As localization commutes with taking invariants, $(R^G)_{(f)} = (R_{(f)})^G$ for every homogeneous element $f \in R^G$ of positive degree. We thus have a cartesian diagram

$$\begin{tikzcd}
U_f = \operatorname{Spec} R_{(f)} \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & U^{\mathrm{ss}} \arrow[r, hook] \arrow[d, "\widetilde{\pi}"] & U \\
U_f /\!\!/ G = (U^{\mathrm{ss}} /\!\!/ G)_f \arrow[r, hook] & U^{\mathrm{ss}} /\!\!/ G.
\end{tikzcd}$$

Since the property of being a good quotient is Zariski local and since the loci $(U^{\mathrm{ss}} /\!\!/ G)_f$ cover $U^{\mathrm{ss}} /\!\!/ G$, we conclude that $\widetilde{\pi} \colon U^{\mathrm{ss}} \to U^{\mathrm{ss}} /\!\!/ G$ is a good quotient. By construction, $U^{\mathrm{ss}} /\!\!/ G$ is projective and there is an integer $N$ such that $L := \mathcal{O}_{U^{\mathrm{ss}} /\!\!/ G}(N)$ is an ample line bundle which pulls back to $\mathcal{O}_U(N)|_{U^{\mathrm{ss}}}$.

To show that $U^{\mathrm{s}} \to U^{\mathrm{s}}/G$ is a geometric quotient, it suffices to show that every $G$-orbit in $U^{\mathrm{s}}$ is closed. Since the dimension of the stabilizer increases under orbit degeneration, it in fact suffices to show that the dimension of the stabilizers in $U^{\mathrm{s}}$ is locally constant. Every point $u \in U^{\mathrm{s}}$ has by definition an open neighborhood $V \subset U$ such that $\dim G_v = \dim G_u$ for all $v \in V$. Since $\dim G = \dim G_v + \dim Gv$, we see that the dimension of the orbit is constant on $V$. Finally, if there is a dense open subset of $U$ which has dimension 0 stabilizers, then it follows from the definition of stability that every $u \in U^{\mathrm{s}}$ has a finite (possibly non-reduced) stabilizer. Since $[U^{\mathrm{s}}/G] \to U^s/G$ is also a good moduli space and $[U^{\mathrm{s}}/G]$ has quasi-finite diagonal, it follows from Proposition 6.3.28 that $[U^{\mathrm{s}}/G]$ is separated. $\square$

**Example 6.7.7.** Given $\mathbb{G}_m$ acting on $\mathbb{P}^2$ via $t \cdot [x : y : z] = [tx : t^{-1}y : z]$, the semistable locus is the complement of $V(xy, z) = \{[0:1:0], [1:0:0]\}$ and the good quotient is $(\mathbb{P}^2)^{\mathrm{ss}} \to \operatorname{Proj} \Bbbk[xy, z] = \mathbb{P}^1$. The fiber over $xy = 0$ is the union of three orbits and its complement is the stable locus. Observe that the restriction to $z \neq 0$ is the good quotient $\mathbb{A}^2 \to \mathbb{A}^1$, given by $(x, y) \mapsto xy$, while the fiber over $z = 0$ is the line at infinity with $[0:1:0]$ and $[1:0:0]$ removed.

**Example 6.7.8.** Consider the diagonal action of $\mathrm{SL}_2$ on $X = (\mathbb{P}^1)^4$ and the $\mathrm{SL}_2$-equivariant Segre embedding

$$(\mathbb{P}^1)^4 \to \mathbb{P}^{15}, \quad ([x_1 : y_1], \ldots, [x_4 : y_4]) \mapsto [x_1x_2x_3x_4, \ldots, y_1y_2y_3y_4].$$

This corresponds to the $\mathrm{SL}_2$-linearization of $L := \mathcal{O}(1) \boxtimes \cdots \boxtimes \mathcal{O}(1)$. The invariant ring $\bigoplus_{d \geq 0} \Gamma(X, L^{\otimes d})$ is generated in degree 1 by the *generalized cross ratios*

$$\begin{aligned}
I_1 &= (x_1y_2 - x_2y_1)(x_3y_4 - x_4y_3) \\
I_2 &= (x_1y_3 - x_3y_1)(x_2y_4 - x_4y_2) \\
I_3 &= (x_1y_4 - x_4y_1)(x_2y_3 - x_3y_2)
\end{aligned}$$

with the linear relation $I_1 - I_2 + I_3 = 0$. The invariant ring is $\Bbbk[I_1, I_2]$ and the quotient $X^{\mathrm{ss}} /\!\!/ \mathrm{SL}_2 = \mathbb{P}^1$. The semistable locus $X^{\mathrm{ss}}$ consists of tuples where at most two points are equal, while the stable locus consists of tuples of distinct points.

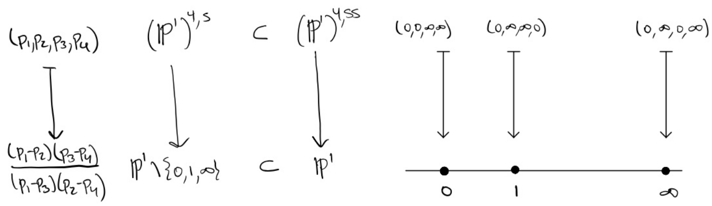



Figure 6.2: 4 unordered points up to projective equivalence

An ordered tuple $(p_1, \ldots, p_4)$ of distinct points is mapped to the *cross ratio*

$$\frac{(p_1 - p_2)(p_3 - p_4)}{(p_1 - p_3)(p_2 - p_4)}.$$

In particular, two stable tuples are projectively equivalent (i.e. in the same $\mathrm{SL}_2$ orbit) if and only if they have the same cross ratio. The complement $X^{\mathrm{ss}} \setminus X^{\mathrm{s}}$ contains 3 closed orbits: the $\mathrm{SL}_2$-orbits of $(0, 0, \infty, \infty)$, $(0, \infty, 0, \infty)$, and $(0, \infty, \infty, 0)$. Tuples such as $(0, 0, 1, \infty)$ or $(1, \infty, 0, 0)$ have non-closed $\mathrm{SL}_2$-orbits in $X^{\mathrm{ss}}$ with $\mathrm{SL}_2 \cdot (0, 0, \infty, \infty)$ in the orbit closure. See Example 6.7.17 to see the computations of the stable and semistable locus for the more general case of $n$ ordered points in $\mathbb{P}^1$.

**Remark 6.7.9** (Symplectic reduction)**.** There is an interesting connection between GIT and symplectic geometry. Let $G$ be a reductive algebraic group over $\mathbb{C}$ acting on a smooth projective variety $U \subset \mathbb{P}(V)$ where $V$ is an $n+1$ dimensional $G$-representation. Let $\omega$ be a symplectic form on $U$, and let $K \subset G$ be a maximal compact subgroup $K$ and $\mathfrak{k}$ its Lie algebra. There is a *moment map*

$$\mu \colon U \to \mathfrak{k}^\vee$$

which is $K$-equivariant with respect to the coadjoint action on $\mathfrak{k}^\vee$ and satisfies $d\mu(x)(\xi) \cdot a = \omega_x(\xi, v_x)$ for $u \in U$, $\xi \in T_x U$, and $a \in \mathfrak{k}$, where $v_x$ is the vector field on $U$ obtained by the infinitesimal action of $K$ on $U$. Then

$$u \in U \text{ is semistable } \iff \overline{Gu} \cap \mu^{-1}(0) \neq \emptyset$$

and the inclusion $\mu^{-1}(0) \hookrightarrow U$ induces a homeomorphism $\mu^{-1}(0)/K \to U^{\mathrm{ss}}/\!\!/G$. See [MFK94, §8].

**Exercise 6.7.10** (Affine GIT with respect to a character)**.** Let $U = \operatorname{Spec} A$ be a finite type scheme over an algebraically closed field $\Bbbk$ with an action of an affine algebraic group $G$ specified by a coaction $\sigma \colon A \to \Gamma(G, \mathcal{O}_G) \otimes A$. Let $\chi \colon G \to \mathbb{G}_m = \operatorname{Spec} \Bbbk[t]_t$ be a character. Define the *semistable* and *stable* locus as

$$U^{\mathrm{ss}} := \left\{ u \in U \;\middle|\; \begin{array}{l} \text{there exists } f \in A \text{ such that } f(u) \neq 0 \\ \text{and } \sigma(f) = \chi^*(t)^d \otimes f \text{ for } d > 0 \end{array} \right\}$$

$$U^{\mathrm{s}} := \left\{ u \in U \;\middle|\; \begin{array}{l} \text{there exists } f \in A \text{ such that } f(u) \neq 0,\ \sigma(f) = \chi^*(t)^d \otimes f \\ \text{for } d > 0 \text{, the orbit } Gu \subset U_f \text{ is closed, and the function} \\ x \mapsto \dim G_x \text{ is constant in an open neighborhood of } u \end{array} \right\}.$$

Defining $U^{\mathrm{ss}}/\!\!/G := \operatorname{Proj} \bigoplus_{d \geq 0} A_d$ where $A_d = \{f \in A \mid \sigma(f) = \chi^*(t)^d \otimes f\}$, show that the conclusion of Theorem 6.7.6 holds except that $U^{\mathrm{ss}}/\!\!/G$ is projective over $A^G = A_0$ (rather than $\Bbbk$).

For example, under the scaling $\mathbb{G}_m$-action on $U = \mathbb{A}^n$ and with respect to the the identity character $\chi = \mathrm{id}$, then $U^{\mathrm{ss}} = U^{\mathrm{s}} = \mathbb{A}^n \setminus 0$ and the quotient is $\mathbb{P}^{n-1}$.

**Exercise 6.7.11** (Projective GIT over an affine)**.** Let $U$ be a projective scheme over a finitely generated $\Bbbk$-algebra $B$, where $\Bbbk$ is an algebraically closed field, and let $G$ be an affine algebraic group acting on $U$. Suppose that there is a $G$-equivariant embedding $U \hookrightarrow \mathbb{P}_R(\mathcal{E})$, where $\mathcal{E}$ is a vector bundle with a $G$-action. Defining the semistable locus $U^{\mathrm{ss}}$ and stable locus $U^{\mathrm{s}}$ exactly as in Definition 6.7.5, show that the conclusion of Theorem 6.7.6 holds except that $U^{\mathrm{ss}}/\!\!/G$ is projective over $B^G$ (rather than $\Bbbk$).

### 6.7.3 Hilbert–Mumford Criterion

The stable and semistable locus can often be effectively computed using the Hilbert–Mumford Criterion. To set up the formulation, let $U \subset \mathbb{P}(V)$ be a $G$-equivariant closed subscheme where $V$ is a finite dimensional $G$-representation, and let $u \in U$ be a $\Bbbk$-point with a lift $\widetilde{u} \in \mathbb{A}(V)$. Given a one-parameter subgroup $\lambda \colon \mathbb{G}_m \to G$, we can choose a basis $V \cong \Bbbk^n$ such that $\lambda(t) \cdot (v_1, \dots, v_n) = (t^{d_1} v_1, \dots, t^{d_n} v_n)$. Define the *Hilbert–Mumford index* as

$$\mu(u, \lambda) := \max_{i, \widetilde{u}_i \neq 0} -d_i. \tag{6.7.2}$$

Equivalently, if $u_0 = \lim_{t \to 0} \lambda(t) \cdot u \in \mathbb{P}(V)$ (which exists since $\mathbb{P}(V)$ is proper), then $\mathbb{G}_m$ fixes $u_0$ and $\mu(u, \lambda)$ is the opposite of the weight of the induced $\mathbb{G}_m$-action on the line $L_{u_0} \subset V$ classified by $u_0$.

**Remark 6.7.12.** From the definition of the Hilbert–Mumford index, we see that

(a) $\lim_{t \to 0} \lambda(t) \cdot \widetilde{u}$ exists if and only if $\mu(u, \lambda) \leq 0$,

(b) $\lim_{t \to 0} t \cdot \widetilde{u} = 0$ if and only if $\mu(u, \lambda) < 0$, and

(c) $\mu(gx, g\lambda g^{-1}) = \mu(x, \lambda)$.

**Theorem 6.7.13** (Hilbert–Mumford Criterion)**.** *Let $G$ be a linearly reductive algebraic group over an algebraically closed field $\Bbbk$ acting on a $G$-equivariant closed subscheme $U \subset \mathbb{P}(V)$, where $V$ is a finite dimensional $G$-representation. Let $u \in \mathbb{P}(V)$ be a $\Bbbk$-point with a lift $\widetilde{u} \in \mathbb{A}(V)$. Then*

$$\begin{aligned} u \in U^{\mathrm{ss}} &\iff 0 \notin \overline{G\widetilde{u}} \\ &\iff \lim_{t \to 0} \lambda(t) \cdot \widetilde{u} \neq 0 \textit{ for all } \lambda \colon \mathbb{G}_m \to G \\ &\iff \mu(u, \lambda) \geq 0 \textit{ for all } \lambda \colon \mathbb{G}_m \to G. \end{aligned}$$

*If in addition the action of $G$ on $U$ has generically finite stabilizers, then*

$$\begin{aligned} u \in U^{\mathrm{s}} &\iff G\widetilde{u} \subset \mathbb{A}(V) \textit{ is closed} \\ &\iff \mu(u, \lambda) > 0 \textit{ for all non-trivial } \lambda \colon \mathbb{G}_m \to G. \end{aligned}$$

**Remark 6.7.14.** The criterion that is now referred to as the "Hilbert–Mumford Criterion" was first developed by Hilbert in [Hil1893, § 15-16] and then adapted by Mumford in [GIT, p. 53]. It holds more generally when $G$ is reductive.

*Proof.* For semistability, the first ($\Rightarrow$) implication is clear: if $0 \in \overline{G\widetilde{u}}$, then for every non-constant invariant function,we have that $f(\widetilde{u}) = f(0) = 0$; hence $u \notin U^{\mathrm{ss}}$. For the converse, if $0 \notin \overline{G\widetilde{u}}$, then $0$ and $\overline{G\widetilde{u}}$ are disjoint closed $G$-invariant subschemes of $\mathbb{A}(V)$. Therefore their images in $\mathbb{A}(V)/\!\!/G = \operatorname{Spec}(\operatorname{Sym}^* V^\vee)^G$ are disjoint (Corollary 6.3.7(2)). We may thus find an invariant function $f \in (\operatorname{Sym}^* V^\vee)^G$ with $f(0) = 0$ and $f(\widetilde{u}) \neq 0$ which we may assume to be homogeneous of positive degree, i.e $f \in \operatorname{Sym}^d V^\vee = \Gamma(\mathbb{P}(V), \mathcal{O}(d))$ for $d > 0$. In the second equivalence, $(\Rightarrow)$ is again clear: if there is a $\lambda$ such that $\lim_{t\to 0} \lambda(t) \cdot \widetilde{u} = 0$, then $0 \in \overline{G\widetilde{u}}$. Conversely, if $0 \in \overline{G\widetilde{u}}$, Theorem 6.6.28 provides a one-parameter subgroup $\lambda$ such that the limit of $u$ under $\lambda$ is 0. The third equivalence follows from the definition of the Hilbert–Mumford index (see Remark 6.7.12).

For stability, we may assume that $u \in U^{\mathrm{ss}}$; otherwise 0 is in the closure of $G\widetilde{u}$ and thus $G\widetilde{u}$ is not closed. By definition, there is an invariant section $f \in \Gamma(U, \mathcal{O}(d))^G$ of positive degree not vanishing at $u$. After possibly increasing $d$, we can arrange that $f$ extends to an invariant section $f \in \Gamma(\mathbb{P}(V), \mathcal{O}(d))^G$: this follows from the exact sequence $0 \to I_U \to \mathcal{O}_{\mathbb{P}(V)} \to \mathcal{O}_U \to 0$ using the vanishing of $\mathrm{H}^1(\mathbb{P}(V), I_U(N))$ for $N \gg 0$ and the exactness of taking invariants (i.e. the linear reductivity of $G$). We may thus view $f$ as a homogeneous polynomial of degree $d$ on $\mathbb{A}(V)$. Letting $\alpha = f(\widetilde{u})$, we have a commutative diagram

$$\begin{tikzcd} G \arrow[r, "\Psi_{\widetilde{u}}"] \arrow[dr, "\Psi_u"'] & V(f-\alpha) \arrow[r, hook] \arrow[d] & \mathbb{A}(V) \\ & \mathbb{P}(V)_f & \end{tikzcd}$$

where $\Psi_u(g) = g \cdot u$ and $\Psi_{\widetilde{u}}(g) = g \cdot \widetilde{u}$. By assumption, we have that $\dim G_u = \dim G_{\widetilde{u}} = 0$ so both stabilizers are finite, thus proper. By Exercise 3.3.15(b), $Gu \subset \mathbb{P}(V)_f$ is closed if and only if $\Psi_u$ is proper, and $G\widetilde{u} \subset \mathbb{A}(V)$ is closed if and only if $\Psi_{\widetilde{u}}$ is proper. On the other hand, $V(f-\alpha) \to \mathbb{P}(V)_f$ is proper, and thus $\Psi_u$ is proper if and only if $\Psi_{\widetilde{u}}$ is. Thus $Gu \subset U_f$ is closed if and only if $G\widetilde{u} \subset \mathbb{A}(V)$ is closed giving the first equivalence. For the second equivalence, if $G\widetilde{u}$ is not closed, then there exists a one-parameter subgroup $\lambda\colon \mathbb{G}_m \to G$ such that $\lim_{t\to 0} \lambda(t) \cdot \widetilde{u}$ exists and is not contained in $G\widetilde{u}$. This gives a non-trivial $\lambda$ with $\mu(u, \lambda) \leq 0$. Conversely, if $G\widetilde{u}$ is closed, then $\Psi_{\widetilde{u}}$ is proper and therefore for every non-trivial $\lambda$, the map $\mathbb{G}_m \to \mathbb{A}(V)$, defined by $t \mapsto \lambda(t) \cdot \widetilde{u}$, is also proper. This implies that $\lim_{t\to 0} \lambda(t)\widetilde{u}$ does not exist as otherwise the limit would define an extension $\mathbb{A}^1 \to \mathbb{A}(V)$ of $\mathbb{G}_m \to \mathbb{A}(V)$ and applying the valuative criterion

$$\begin{tikzcd} \mathbb{G}_m \arrow[r, equal] \arrow[d, hook] & \mathbb{G}_m \arrow[d] \\ \mathbb{A}^1 \arrow[r] \arrow[ur, dashed] & \mathbb{A}(V) \end{tikzcd}$$

would yield a contradiction. Since the limit doesn't exist, $\mu(u, \lambda) > 0$. $\square$

We also provide a stack-theoretic criterion for a point $u \in [U/G]$ to be *semistable*, i.e. $u$ is contained in the open substack $[U^{\mathrm{ss}}/G]$. The data of a $G$-equivariant embedding $U \subset \mathbb{P}(V)$ is classified by a line bundle $L$ on $[U/G]$ such that the pullback of $L$ under $U \to [U/G]$ is very ample. Since the stable and semistable locus are $G$-invariant, they define open substacks of $[U/G]$. The data of a point $u \in U(\Bbbk)$ and a one-parameter subgroup $\lambda\colon \mathbb{G}_m \to G$ up to conjugation is classified by a map $f_{u,\lambda}\colon [\mathbb{A}^1/\mathbb{G}_m] \to [U/G]$ such that the induced map

$$\mathbf{B}\mathbb{G}_m \overset{0}{\hookrightarrow} [\mathbb{A}^1/\mathbb{G}_m] \xrightarrow{f_{u,\lambda}} [U/G] \to \mathbf{B}G$$

corresponds $\lambda$. The Hilbert–Mumford index is $\mu(u,\lambda) = -\operatorname{wt}(f_{u,\lambda}^* L)|_{\mathbf{B}\mathbb{G}_m}$.

**Corollary 6.7.15** (Hilbert–Mumford Criterion)**.** *Let $G$ be a linearly reductive algebraic group over an algebraically closed field $\Bbbk$ acting on a projective $\Bbbk$-scheme $U$. Let $L$ be a line bundle on $[U/G]$ corresponding to a very ample $G$-linearization. Then $u \in [U/G]$ is semistable if and only if $\operatorname{wt}((f^*L)|_{B\mathbb{G}_m}) \geq 0$ for all maps*

$$f\colon [\mathbb{A}^1/\mathbb{G}_m] \to [U/G], \quad \text{with } f(1) \simeq u.$$

*If in addition the action of $G$ on $U$ has generically finite stabilizers, then $u$ is stable if and only if $\operatorname{wt}((f^*L)|_{B\mathbb{G}_m}) > 0$ for all maps $f\colon [\mathbb{A}^1/\mathbb{G}_m] \to [U/G]$ such that $f(1) \simeq u$ and the induced map $\mathbb{G}_m \to G_{f(0)}$ on stabilizers is non-trivial.* □

**Exercise 6.7.16** (Affine Hilbert-Mumford Criterion)**.** Let $G$ be a linearly reductive group over an algebraically closed field $\Bbbk$ acting on an affine scheme $U = \operatorname{Spec} A$ of finite type. Let $\chi\colon G \to \mathbb{G}_m$ be a character, and let $U^{\mathrm{ss}}$ and $U^{\mathrm{s}}$ be the semistable and stable locus with respect to $\chi$ as defined in Exercise 6.7.10. For $u \in U(\Bbbk)$, show that

$$u \in U^{\mathrm{ss}} \iff \text{for all one-parameter subgroups } \lambda\colon \mathbb{G}_m \to G \text{ such that } \lim_{t\to 0} \lambda(t)\cdot u \text{ exists, } \langle \chi, \lambda\rangle \geq 0$$

where $\langle -,-\rangle$ is the natural pairing of characters and one-parameter subgroups. If in addition the action of $G$ on $U$ has generically finite stabilizers, show that $u \in U^{\mathrm{s}}$ if and only if the same condition holds with strict inequality $\langle \chi, \lambda\rangle > 0$.

*Hint: Consider the action of $G$ on $U \times \mathbb{A}^1$ induced by $\chi$ defined by $g\cdot(u,z) = (g\cdot u, \chi(g)^{-1}\cdot z)$, and show that $u \notin U^{\mathrm{ss}}$ if and only if $\overline{G\cdot(u,1)} \cap (U\times\{0\}) \neq \emptyset$. Use the Destabilization Theorem (6.6.28) to show that this is equivalent to the existence of a one-parameter subgroup $\lambda$ such that*

$$\lim_{t\to 0}\lambda(t)\cdot(u,1) = \lim_{t\to 0}(\lambda(t)\cdot u, t^{-\langle\chi,\lambda\rangle}) \in U\times\{0\}.$$

### 6.7.4 Examples

**Example 6.7.17.** Consider the diagonal action of $\mathrm{SL}_2$ on $X = (\mathbb{P}^1)^n$, and consider the $\mathrm{SL}_2$-equivariant Segre embedding $(\mathbb{P}^1)^n \to \mathbb{P}^{2^n-1}$ (or equivalently the $\mathrm{SL}_2$-linearization $\mathcal{O}(1)\boxtimes\cdots\boxtimes\mathcal{O}(1)$). We claim that

$$\begin{aligned} X^{\mathrm{s}} &= \{(p_1,\dots,p_n) \mid \text{for all } q\in\mathbb{P}^1,\ \#\{i \mid p_i = q\} < n/2\} \\ X^{\mathrm{ss}} &= \{(p_1,\dots,p_n) \mid \text{for all } q\in\mathbb{P}^1,\ \#\{i \mid p_i = q\} \leq n/2\}. \end{aligned}$$

To see this, let $(p_1,\dots,p_n) \in X(\Bbbk)$ and $\lambda\colon \mathbb{G}_m \to \mathrm{SL}_2$ be a one-parameter subgroup. There exists $g \in \mathrm{SL}_2(\Bbbk)$ such that $g\lambda g^{-1} = \lambda_0^d$ for some $d \in \mathbb{Z}$ where $\lambda_0(t) = \begin{pmatrix} t^{-1} & 0 \\ 0 & t\end{pmatrix}$. We can assume $d \geq 0$ as the case $d < 0$ is handled similarly. Since $\mu(x,\lambda) = \mu(gx,\lambda_0^d) = d\mu(gx,\lambda_0)$, it suffices to compute $\mu(gx,\lambda_0)$. Since $\mu(-,\lambda_0)$ is symmetric with respect to the $S_n$-action, we can assume that $gx = (0,\dots,0,p_k,\dots,p_n)$ with $p_k,\dots,p_n \neq 0$. A coordinate of the Segre embedding is of the form $(\prod_{i\in\Sigma} x_i)(\prod_{i\notin\Sigma} y_i)$ for a subset $\Sigma \subset \{1,\dots,n\}$, and its weight is $n - 2(\#\Sigma)$. The coordinate where $gx$ is nonzero with the largest weight is $y_1\cdots y_k x_{k+1}\cdots x_n$ with weight $2k-n$. Thus $\mu(gx,\lambda_0) = n-2k$. Therefore, if no more than (resp. less than) $n/2$ of the points $p_i$ are the same, then $x$ is semistable (resp. stable) if and only if $n \geq 2k$ (resp. $n > 2k$). Conversely, if more than (resp. at least) $n/2$ of the same, then after translating by an element of $\mathrm{SL}_2$ and using the symmetry of the $S_n$-action, we can write $u = (0, \ldots, 0, p_k, \ldots, p_n)$ with $k > n/2$ (resp. $k \geq n/2$) and $\lambda_0 = \mathrm{diag}(t^{-1}, t)$ destabilizes $u$.

If $n$ is odd, then $X^{\mathrm{ss}} = X^{\mathrm{s}}$ and $X^{\mathrm{ss}} \to X^{\mathrm{ss}} /\!\!/ \mathrm{SL}_2$ is a geometric quotient. If $n$ is even, the map $X^{\mathrm{ss}} \to X^{\mathrm{ss}} /\!\!/ \mathrm{SL}_2$ identifies $(p_1, \ldots, p_n)$ and $(q_1, \ldots, q_n)$ if there is a subset $\Sigma \subset \{1, \ldots, n\}$ of size $n/2$ such that $p_i = p_j$ and $q_i = q_j$ for all $i, j \in \Sigma$; in this case, the unique closed orbit in fiber is the orbit of the $n$-tuple with 0's in positions in $\Sigma$ and $\infty$'s elsewhere. The complement $X^{\mathrm{ss}} \setminus X^{\mathrm{s}}$ has precisely $\frac{1}{2}\binom{n}{n/2}$ closed orbits.

A modification of the argument yields the same stable and semistable locus for the action of $\mathrm{PGL}_2$ on $(\mathbb{P}^1)^n$ under the $\mathrm{PGL}_2$-linearization $\mathcal{O}(2) \boxtimes \cdots \boxtimes \mathcal{O}(2)$. Since $\mathrm{Aut}(\mathbb{P}^1) = \mathrm{PGL}_2$, the quotient $X^{\mathrm{ss}} /\!\!/ \mathrm{SL}_2 = X^{\mathrm{ss}} /\!\!/ \mathrm{PGL}_2$ can be viewed as a compactification of the moduli of $n$ ordered points in $\mathbb{P}^1$ up to projective equivalence.

**Exercise 6.7.18.**

(a) Under the action of $\mathrm{SL}_2$ on the projectivization $\mathbb{P}(\Gamma(\mathbb{P}^1, \mathcal{O}(n))) \cong \mathbb{P}^n$ of binary forms of degree $n$, show that the semistable (resp. stable) locus consists of binary forms $f(x, y)$ such that every linear factor has multiplicity less than or equal to (resp. less than) $n/2$.

(b) Under the $\mathrm{SL}_2$-linearization $\mathcal{O}(a_1) \boxtimes \cdots \boxtimes \mathcal{O}(a_n)$ on $(\mathbb{P}^1)^n$ with each $a_i > 0$, show that the semistable (resp. stable) locus consists of tuples $(p_1, \ldots, p_n)$ such that for all $q \in \mathbb{P}^1(\Bbbk)$,

$$\sum_{p_i = q} a_i \leq (\sum_{i=1}^{n} a_i)/2$$

(resp. strict inequality holds).

(c) Under the $\mathrm{SL}_{r+1}$ action on $(\mathbb{P}^r)^n$ and the $\mathrm{SL}_{r+1}$-linearization $\mathcal{O}(a_1) \boxtimes \cdots \boxtimes \mathcal{O}(a_n)$ with each $a_i > 0$, show that the semistable (resp. stable) locus consists of tuples $(p_1, \ldots, p_n)$ such that for every linear subspace $W \subsetneq \mathbb{P}^r$

$$\sum_{p_i \in W} a_i \leq \frac{\dim W + 1}{r+1} (\sum_{i=1}^{n} a_i)$$

(resp. strict inequality holds).

**Exercise 6.7.19** (Cubic curves)**.** Consider the action of $\mathrm{SL}_3$ on the projective space $\mathbb{P}(\mathrm{H}^0(\mathbb{P}^2, \mathcal{O}(3)))$ of cubic curves in $\mathbb{P}^2$. Show that the semistable locus consists of curves with at worst nodal singularities and that the stable locus consists of smooth curves.

**Remark 6.7.20** (Quartic curves)**.** A more involved calculation shows that under the $\mathrm{SL}_3$ action on $\mathbb{P}(\mathrm{H}^0(\mathbb{P}^2, \mathcal{O}(4)))$, a quartic curve is semistable if and only if it doesn't contain a triple point and is not the union of a cubic curve and an inflection tangent line, and is stable if and only if it has at worst nodal and cuspidal singularities. See also [Mum77, §1.13].

**Remark 6.7.21** (Cubic surfaces)**.** Under the action of $\mathrm{SL}_4$ on $\mathbb{P}(\mathrm{H}^0(\mathbb{P}^3, \mathcal{O}(3)))$, a cubic surface is stable (resp. semistable) if and only if it has finitely many singular points and the singularities are ordinary double points (resp. ordinary double points or rank two double points whose axes are not contained in the surface). See [Muk03, Thm. 7.14] and [Hil1893].

**Exercise 6.7.22** (Quiver GIT)**.** A *quiver* $Q = (Q_0, Q_1)$ is a directed graph where $Q_0$ is a finite set of vertices and $Q_1$ is a finite set of arrows; there are source and target maps $s, t \colon Q_1 \to Q_0$. A $\Bbbk$-representation of $Q$ consists of a vector space $V_i$ for every $i \in Q_0$ together with linear maps $L_\alpha \colon V_i \to V_j$ for every arrow $\alpha \colon i \to j$. If each $V_i$ is finite dimensional with $d_i = \dim V_i$, we say that $d = (d_i)$ is the dimension vector of $V$.

Fix $d = (d_i)$ and consider the space

$$R(Q,d) = \prod_{\alpha \in Q_1} \mathrm{Hom}(\Bbbk^{s(\alpha)}, \Bbbk^{t(\alpha)})$$

of representations with dimension vector $d$. This inherits an action of $\prod_i \mathrm{GL}_{d_i}$ via $(g_i) \cdot (L_\alpha) = (g_{t(\alpha)} L_\alpha g_{s(\alpha)}^{-1})$. The diagonal subgroup $\mathbb{G}_m \subset \prod_i \mathrm{GL}_{d_i}$ consisting of tuples $(t \, \mathrm{id}_{\Bbbk^{d_i}})$ of scalar matrices for $t \in \mathbb{G}_m$ is normal and acts trivially. Therefore the quotient $G := (\prod_i \mathrm{GL}_{d_i})/\mathbb{G}_m$ also acts on $R(Q,d)$.

For any tuple $a = (a_i)_{i \in Q_0}$ of integers such that $\sum_i a_i d_i = 0$, consider the character

$$\chi_a \colon G \to \mathbb{G}_m, \quad (g_i) \mapsto \prod_i \det(g_i)^{a_i}.$$

Use the Affine Hilbert–Mumford Criterion (6.7.16) to show that a representation $V \in R(Q,d)$ is semistable (resp. stable) with respect to $\chi$ if and only if for every subrepresentation $W \subset V$ (i.e. subspaces $W_i \subset V_i$ such that $L_\alpha(W_{s(\alpha)}) \subset W_{t(\alpha)}$),

$$\sum_i a_i \dim W_i \geq 0$$

(resp. strict inequality holds). See also [Kin94, Prop. 3.1].

**Remark 6.7.23** (Cox construction of toric varieties)**.** Let $X = X(\Sigma)$ be a proper toric variety with fan $\Sigma \subset N_{\mathbb{R}}$ and torus $T_N$, where $N$ is a lattice with dual $M$. Letting $\Sigma(1)$ denote the rays of the fan, the divisors $D_\rho$ associated to $\rho \in \Sigma(1)$ generate the class group. There is a short exact sequence

$$0 \to M \to \mathbb{Z}^{\Sigma(1)} \to \mathrm{Cl}(X) \to 0.$$

The algebraic group $G := \mathrm{Hom}(\mathrm{Cl}(X), \mathbb{G}_m)$ is diagonalizable (hence linearly reductive) and sits in a short exact sequence

$$1 \to G \to \mathbb{G}_m^{\Sigma(1)} \to T_N \to 1$$

obtained by applying $\mathrm{Hom}(-, \mathbb{G}_m)$ to the above sequence. The group $G$ acts naturally on $\mathbb{A}^{\Sigma(1)}$.

For a cone $\sigma \in \Sigma$, let $x^\sigma := \prod_{\rho \in \sigma(1)} x_\rho$. Define the closed subset $Z \subset \mathbb{A}^{\Sigma(1)}$ by the vanishing of the ideal generated by the monomials $x^\sigma$ as $\sigma$ varies over maximal dimensional cones; this set can also be described as the union $\bigcup_C V(x_\rho \mid \rho \in C)$ where the union runs over primitive collections $C \subset \Sigma(1)$, i.e. subsets $C$ such that $C$ is not contained in $\sigma(1)$ for any $\sigma \in \Sigma$ and such that for any $C' \subsetneq C$, there exists $\sigma \in \Sigma$ with $C' \subset \sigma(1)$. This locus $Z$ is $G$-invariant.

The main theorem here is that $X$ is isomorphic to the good quotient $(\mathbb{A}^{\Sigma(1)} \setminus Z)/\!\!/G$. This is the so-called 'Cox construction of $X$', and it gives $X$ homogeneous coordinates in a similar fashion to how $\mathbb{A}^{n+1}$ gives homogeneous coordinates for $\mathbb{P}^n = (\mathbb{A}^{n+1} \setminus 0)/\mathbb{G}_m$. When $\Sigma$ is a simplicial fan, $X$ is a geometric quotient $(\mathbb{A}^{\Sigma(1)} \setminus Z)/G$. Moreover, the class group $\mathrm{Cl}(X)$ is identified with group of character $\mathbb{X}^*(G)$, and if $L$ is an ample line bundle on $X$ corresponding to a character $\chi$, then $\mathbb{A}^{\Sigma(1)} \setminus Z$ is the semistable locus for the action of $G$ on $\mathbb{A}^{\Sigma(1)}$ with respect to the character $\chi$. See [Cox95] and [CLS11, §5].

**Example 6.7.24** (Variation of GIT for $\mathbb{G}_m$-actions)**.** Consider a $\mathbb{G}_m$-action on an affine scheme $X = \operatorname{Spec} A$ of finite type over $\Bbbk$. In this example, we will consider how the GIT quotients (with respect to a character in the sense Exercise 6.7.10) vary as we vary the character of $\mathbb{G}_m$. There is a bijection $\operatorname{Hom}(\mathbb{G}_m, \mathbb{G}_m) \cong \mathbb{Z}$ and we write $\chi_d(t) = t^d$ as the character corresponding to $d \in \mathbb{Z}$.

Let $A = \bigoplus_{n \in \mathbb{Z}} A_n$ be the induced grading. There are three cases for the semistable locus $X^{\mathrm{ss}}_{\chi_d}$ with respect to the character $\chi_d$:

(1) $d = 0$: $X^{\mathrm{ss}}(0) := X^{\mathrm{ss}}_{\chi_0} = X$ and $X^{\mathrm{ss}}_{\chi_0}/\!\!/\mathbb{G}_m = \operatorname{Spec} A_0$.
(2) $d > 0$: $X^{\mathrm{ss}}(+) := X^{\mathrm{ss}}_{\chi_d} = X \setminus V(\sum_{n<0} A_n)$ and $X^{\mathrm{ss}}_{\chi_d} = \operatorname{Proj} \bigoplus_{d \geq 0} A_{nd}$ is independent of $d$; moreover $X^{\mathrm{ss}}(0)$ is identified with $X^+_{\chi_d}$ with respect to the one-parameter subgroup $\chi_d$ (Exercise 6.6.5).
(3) $d < 0$: $X^{\mathrm{ss}}(-) := X^{\mathrm{ss}}_{\chi_d} = X \setminus V(\sum_{n>0} A_n) = X^+_{\chi_d}$ and $X^{\mathrm{ss}}_{\chi_d} = \operatorname{Proj} \bigoplus_{d \geq 0} A_{-nd}$ is independent of $d$.

There is a commutative diagram

$$\begin{tikzcd}
X^{\mathrm{ss}}(+) \arrow[r, hook] \arrow[d] & X \arrow[d] & X^{\mathrm{ss}}(-) \arrow[l, hook'] \arrow[d] \\
X^{\mathrm{ss}}(+)/\!\!/\mathbb{G}_m \arrow[r] & X/\!\!/\mathbb{G}_m & X^{\mathrm{ss}}(-)/\!\!/\mathbb{G}_m \arrow[l]
\end{tikzcd}$$

where the vertical maps are good quotients, the top maps are open immersions, and the bottom maps are projective. The Affine Hilbert–Mumford Criterion (6.7.16) implies that there are identifications of the stable loci with respect to $\chi_0$, $\chi_1$, and $\chi_{-1}$: $X^{\mathrm{s}}(0) = X \setminus (X^{\mathrm{ss}}(+) \cap X^{\mathrm{ss}}(-))$, $X^{\mathrm{s}}(+) = X^{\mathrm{ss}}(+) = X \setminus X^{\mathrm{ss}}(-)$, and $X^{\mathrm{s}}(-) = X^{\mathrm{ss}}(-) = X \setminus X^{\mathrm{ss}}(+)$. Therefore, we see that if both $X^{\mathrm{ss}}(+)$ and $X^{\mathrm{ss}}(-)$ are nonempty, then $X^{\mathrm{ss}}(+)/\mathbb{G}_m \to X/\!\!/\mathbb{G}_m$ and $X^{\mathrm{ss}}(-)/\mathbb{G}_m \to X/\!\!/\mathbb{G}_m$ are isomorphisms over $X^{\mathrm{s}}(0)/\mathbb{G}_m$, and in particular birational. We also see that if the complements of $X^{\mathrm{ss}}(+)$ and $X^{\mathrm{ss}}(-)$ in $X$ each have codimension at least two, then the birational map $X^{\mathrm{ss}}(+)/\!\!/\mathbb{G}_m \dashrightarrow X^{\mathrm{ss}}(-)/\!\!/\mathbb{G}_m$ is an isomorphism in codimension 2 such that the divisor $\mathcal{O}(1)$ (which is relatively ample over $X/\!\!/\mathbb{G}_m$) pushes forward to a divisor on $X^{\mathrm{ss}}(-)/\!\!/\mathbb{G}_m$ whose dual is relatively ample, i.e. $X^{\mathrm{ss}}(+)/\!\!/\mathbb{G}_m \dashrightarrow X^{\mathrm{ss}}(-)/\!\!/\mathbb{G}_m$ is a flip with respect to $\mathcal{O}(1)$.

**Remark 6.7.25** (Variation of GIT)**.** Extending the previous example, consider a projective variety $X$ over $\Bbbk$ with an action of a linearly reductive group $G$. Two line bundles (resp. $G$-linearizations) $L_1$ and $L_2$ on $X$ are algebraically equivalent (resp. $G$-algebraically equivalent) if there is a connected variety $T$, points $t_1, t_2 \in T(\Bbbk)$, and a line bundle (resp. $G$-linearization) $\mathcal{L}$ on $X \times T$ such that $L_i = \mathcal{L}|_{X \times \{t_i\}}$. The Neron–Severi group $\mathrm{NS}(X)$ (resp. $G$-equivariant Neron–Severi group $\mathrm{NS}^G(X)$) of line bundles (resp. $G$-linearizations) on $X$ up to ($G$-)algebraic equivalence is finitely generated. The kernel of $\mathrm{NS}^G(X)_{\mathbb{R}} \to \mathrm{NS}(X)$ is identified with the rational character group $\mathbb{X}^*(G)_{\mathbb{R}}$. We let $\mathrm{Eff}^G(X) \subset \mathrm{NS}^G(X)_{\mathbb{R}}$ be the cone of $G$-effective linearizations, i.e. $G$-linearizations $L$ such that there is a nonzero section of $L^{\otimes d}$) for some $d > 0$ or in other words such that $X^{\mathrm{ss}}_L \neq \emptyset$. We also let $\mathrm{Amp}^G(X) \subset \mathrm{NS}^G(X)_{\mathbb{R}}$ be the cone of ample $G$-linearizations.

The main results of variation of GIT can be formulated as follows. The semistable locus $X^{\mathrm{ss}}_L$ only depends on the $G$-algebraic equivalence class of $L$. There is a polyhedral decomposition of the cone $\mathrm{Amp}^G(X) \cap \mathrm{Eff}^G(X)$ defined by codimension 1 walls such that the semistable locus is constant in any open chamber. If $L_0$ is on a wall while $L_+$ and $L_-$ are on opposite adjacent chambers, then there is a commutative diagram

$$\begin{tikzcd}
X^{\mathrm{ss}}_{L_+} \arrow[r, hook] \arrow[d] & X^{\mathrm{ss}}_{L_0} \arrow[d] & X^{\mathrm{ss}}_{L_-} \arrow[l, hook'] \arrow[d] \\
X^{\mathrm{ss}}_{L_+} /\!\!/ G \arrow[r] & X^{\mathrm{ss}}_{L_0} /\!\!/ G & X^{\mathrm{ss}}_{L_-} /\!\!/ G \arrow[l]
\end{tikzcd}$$

where the vertical maps are good quotients, the top maps are open immersions, and the bottom maps are projective. If $X^{\mathrm{ss}}_{L_+}$ and $X^{\mathrm{ss}}_{L_-}$ are non-empty, the bottom maps are birational; when the bottom maps are isomorphisms in codimension 2, then $X^{\mathrm{ss}}_{L_+} /\!\!/ G \dashrightarrow X^{\mathrm{ss}}_{L_-} /\!\!/ G$ is a flip with respect to the line bundle $\mathcal{O}(1)$ on $X^{\mathrm{ss}}_{L_+} /\!\!/ G$, which is relatively ample over $X^{\mathrm{ss}}_{L_0} /\!\!/ G$ and which pulls back to $L_+|_{X^{\mathrm{ss}}_{L_+}}$.

See [Tha96] and [DH98].

**Remark 6.7.26** (Mori Dream Spaces)**.** There is an interesting connection between the Mori program and variation of GIT. A normal $\mathbb{Q}$-factorial projective variety $X$ is a *Mori dream space* if (1) $\mathrm{Pic}(X)_{\mathbb{Q}} = \mathrm{NS}(X)_{\mathbb{Q}}$, (2) the cone $\mathrm{Nef}(X)$ of nef line bundles is the affine hull of finitely many semiample line bundles, and (3) there are finitely many birational maps $f_i \colon X \dashrightarrow X_i$, which are isomorphisms in codimension 1, to a $\mathbb{Q}$-factorial normal projective variety $X_i$ such that the movable cone $\mathrm{Mov}(X)$ is the union of $f_i^{-1}(\mathrm{Amp}(X_i)_{\mathbb{Q}})$; a line bundle is *movable* if its stable base locus has codimension at least 2. In other words, $X$ is a Mori dream space if $\mathrm{Mov}(X)$ has a finite wall and chamber decomposition such that the projective variety determined by the line bundle is constant within an open chamber.

Equivalently, $X$ is a Mori dream space if $\mathrm{Pic}(X)_{\mathbb{Q}} = \mathrm{NS}(X)_{\mathbb{Q}}$ and the Cox ring

$$\mathrm{Cox}(X) := \bigoplus_{(d_1,\ldots,d_n)\in\mathbb{N}^n} \Gamma(X, L_1^{d_1} \otimes \cdots \otimes L_n^{d_n})$$

is finitely generated, where $L_1, \ldots, L_n$ is a basis for $\mathrm{Pic}(X)_{\mathbb{Q}}$ such that their affine hull contains $\mathrm{Eff}(X)_{\mathbb{Q}}$. If $X$ is a Mori dream space, then $X$ along with each birational model $X_i$ is a GIT quotient of the semistable locus of $\mathrm{Spec}(\mathrm{Cox}(X))$ by the torus $\mathbb{G}_m^n$ with respect to a character. Moreover, there is an identification of the Mori chambers of $\mathrm{Mov}(X)$ with the variation of GIT chambers for the action of $\mathbb{G}_m^n$ on $\mathrm{Spec}(\mathrm{Cox}(X))$. See also [HK00].

**Example 6.7.27** (Partial desingularization)**.** If $U$ is a smooth variety and $U \to X$ is a geometric quotient by a linearly reductive group, then $X$ necessarily has finite quotient singularities; this is a consequence of the Local Structure Theorem (4.3.14). On the other hand, if $U \to X$ is a good quotient, then $X$ can have worse singularities. Nevertheless, there is a canonical procedure to partially resolve the singularities of $X$ so that they become finite quotient singularities.

Suppose that there is an open subset $X' \subset X$ such that $\pi_0(X') \to X'$ is a geometric quotient; this happens for example if $U = V^{\mathrm{ss}}$ is the semistable locus with respect to the action of $G$ on a projective variety $V \subset \mathbb{P}^N$ and the stable locus $V^{\mathrm{s}}$ is nonempty. Then there is a commutative diagram

$$\begin{tikzcd} U_n \arrow[r] \arrow[d, "\pi_n"] & U_{n-1} \arrow[r] \arrow[d, "\pi_{n-1}"] & \cdots \arrow[r] & U = U_0 \arrow[d, "\pi_0"] \\ X_n \arrow[r] & X_{n-1} \arrow[r] & \cdots \arrow[r] & X = X_0 \end{tikzcd}$$

such that:

- Each $U_{i+1}$ is a $G$-invariant open subscheme of the blow-up $\mathrm{Bl}_Z U_i$, where $Z$ is a $G$-invariant smooth closed subscheme whose stabilizers are of maximal dimension, and $U_{i+1} \subset \mathrm{Bl}_Z U_i$ is the complement of the strict transform of $\pi_i^{-1}(\pi_i(Z))$. If $U = V^{\mathrm{ss}}$ is the semistable locus of a projective variety with respect to a $G$-linearization $L$, then $U_{i+1}$ is the semistable locus with respect to $(q^*L)^{\otimes n} \otimes \mathcal{O}(-E)$ for $n \gg 0$, where $q\colon \mathrm{Bl}_Z U_i \to U_i$ and $E$ denotes the exceptional divisor.
- The maps $X_{i+1} \to X_i$ are projective birational.
- The maps $\pi_i\colon U_i \to X_i$ are good quotients by $G$, and the map $\pi_n\colon U_n \to X_n$ is a geometric quotient. In particular, $X_n$ has finite quotient singularities.

For a simple example of this procedure in action, consider the $\mathbb{G}_m$-action on $\mathbb{A}^2$ with weights 1 and $-1$. In this case, the quotient $\mathbb{A}^2/\!\!/\mathbb{G}_m \cong \mathbb{A}^1$ is smooth but not a geometric quotient. The procedure tells us to take the blow-up $\mathrm{Bl}_0 \mathbb{A}^2$ at the origin and the complement $U_1$ of the strict transform of $V(xy)$. Then $\mathbb{G}_m$-acts with finite stabilizers on $U_1$ and $U_1 \to \mathbb{A}^2$ is $\mathbb{G}_m$-invariant birational (but neither proper nor surjective) map inducing an isomorphism $U_1/\mathbb{G}_m \to \mathbb{A}^2/\!\!/\mathbb{G}_m$ on quotients.

See [Kir85], [Rei89], and [ER21].

### 6.7.5 Kempf's Optimal Destabilization Theorem

Given an algebraic group $G$ over an algebraically closed field $\Bbbk$, we define $\mathbb{X}_*(G)$ as the set of one-parameter subgroups $\mathbb{G}_m \to G$. Recall that for a torus $T \cong \mathbb{G}_m^n$, $\mathbb{X}_*(T) \cong \mathbb{Z}^n$ (see Example C.3.6).

**Definition 6.7.28.** A *length* $\|-\|$ *on* $\mathbb{X}_*(G)$ is a non-negative real-valued function on $\mathbb{X}_*(G)$ which is conjugation invariant, i.e. $\left\|g\lambda g^{-1}\right\| = \|\lambda\|$ for $\lambda \in \mathbb{X}_*(G)$ and $g \in G(\Bbbk)$, and such that for every maximal torus $T \subset G$, there is a positive definite integral-valued bilinear form $(-,-)$ on $\mathbb{X}_*(T)$ with $(\lambda,\lambda) = \|\lambda\|^2$ for $\lambda \in \mathbb{X}_*(T)$.

**Example 6.7.29.** If $G = \mathrm{GL}_n$, then any one-parameter subgroup $\lambda$ is conjugate to a one-parameter subgroup of the form $t \mapsto \mathrm{diag}(t^{d_1}, \ldots, t^{d_n})$ and we can define $\|\lambda\| = \sqrt{d_1^2 + \cdots + d_n^2}$.

**Example 6.7.30.** For every reductive algebraic group $G$, there is a length $\|-\|$ on $\mathbb{X}_*(G)$. To see this, let $T \subset G$ be a maximal torus and choose a positive definite integral-valued bilinear form $(-,-)$ on $\mathbb{X}_*(T)$, which is invariant under the conjugation action of the Weyl group $W := N(T)/T$. There is a bijection $\mathbb{X}_*(G)/G \cong \mathbb{X}_*(T)/W$ between conjugacy classes of $\mathbb{X}_*(G)$ under $G$ and conjugacy classes of $\mathbb{X}_*(T)$ under $W$. In other words, for every $\lambda \in \mathbb{X}_*(G)$ there exists $g \in G(\Bbbk)$ such that $g\lambda g^{-1} \in \mathbb{X}_*(T)$, and moreover for any other element $g' \in G(\Bbbk)$ such that $g'\lambda g'^{-1} \in \mathbb{X}_*(T)$, then $g\lambda g^{-1}$ and $g'\lambda g'^{-1}$ are conjugate under $W$. It follows that $\|\lambda\|^2 := (g\lambda g^{-1}, g\lambda g^{-1})$ is well-defined.

Let $X = \operatorname{Spec} A$ be an affine $\Bbbk$-scheme with the action of $G$ and let $x_0 \in X(\Bbbk)$ be a point with closed orbit. For every point $x \in X(\Bbbk)$ with $Gx_0 \subset \overline{Gx}$ and a one-parameter subgroup $\lambda\colon \mathbb{G}_m \to G$ such that $\lim_{t\to 0} \lambda(t)\cdot x$ exists, we define the *Hilbert–Mumford index of $x$ with respect to $\lambda$* as

$$\mu(x,\lambda) = -\deg f_{x,\lambda}^{-1}(Gx_0). \tag{6.7.3}$$

where $f_{x,\lambda}\colon \mathbb{A}^1 \to X$ is the map extending $\mathbb{G}_m \to X$, $t \mapsto \lambda(t)\cdot x$. Note that if $\lim_{t\to 0}\lambda(t)\cdot x \notin Gx_0$, then $\mu(x,\lambda)=0$.

Since $\mu(x,\lambda^n) = n\cdot\mu(x,\lambda)$, it natural to consider the *normalized Hilbert–Mumford index*

$$\frac{\mu(x,\lambda)}{\|\lambda\|}$$

as a measure of how quickly $\lambda(t)\cdot x$ approaches the closed orbit $Gx_0$. The more negative the normalized Hilbert–Mumford index is, the faster $\lambda(t)\cdot x$ approaches $Gx_0$. Kempf proved that there is a one-parameter subgroup minimizing this index and that it is unique up to conjugation.

**Theorem 6.7.31** (Kempf's Optimal Destabilization Theorem—affine version)**.** *Let $G$ be a reductive algebraic group over an algebraically closed field $\Bbbk$ with a length $\|-\|$ on $\mathbb{X}_*(G)$. Let $X = \operatorname{Spec} A$ be an affine scheme of finite type over $\Bbbk$ with an action of $G$. Let $x_0 \in X(\Bbbk)$ be a point with a closed orbit. For every point $x \in X(\Bbbk)$ with $Gx_0 \subset \overline{Gx}$, there exists a one-parameter subgroup $\lambda_0\colon \mathbb{G}_m \to G$ such that $\mu(x,\lambda_0)/\|\lambda_0\|$ achieves a minimal value $M(x)$ over all $\lambda \in \mathbb{X}_*(G)$ such that $\lim_{t\to 0}\lambda(t)\cdot x \in Gx_0$.*

*If $\lambda_0'$ is another such one-parameter subgroup, then $P(\lambda_0) = P(\lambda_0')$ and $\lambda_0' = u\lambda_0 u^{-1}$ for a unique element $x \in X(\lambda_0)$. Every maximal torus $T \subset P_{\lambda_0}$ contains a unique indivisible element achieving this minimum value.*

**Remark 6.7.32.** The subgroup $P_{\lambda_0} = \{g \in G \mid \lim_{t\to 0}\lambda_0(t) g \lambda_0(t)^{-1} \text{ exists}\}$ is the parabolic associated to $\lambda_0$ and $U_{\lambda_0} = \{g \in G \mid \lim_{t\to 0}\lambda_0(t) g\lambda_0(t)^{-1} = 1\}$ is the unipotent radical of $P_{\lambda_0}$; see §C.3.3.

In the projective case where there is a $G$-equivariant embedding $X \hookrightarrow \mathbb{P}(V)$, we have already defined the Hilbert–Mumford index $\mu(x,\lambda)$ in (6.7.2) as follows: choosing a basis of $V$ such that $\mathbb{G}_m$ acts on $\mathbb{A}(V) = \mathbb{A}^n$ with weights $d_1,\dots,d_n$ and a lift $\widehat{x} = (u_1,\dots,u_n) \in \mathbb{A}(V)$ of $x$, then $-\mu(x,\lambda)$ is defined as the smallest $d_i$ with $u_i \neq 0$. If $\lim_{t\to 0}\lambda(t)\cdot\widehat{x}$ exists, then this agrees with the definition in (6.7.3). To see this, observe that the extension $f_{\widehat{x},\lambda}\colon \mathbb{A}^1 \to \mathbb{A}^n$ of the map $t \mapsto \lambda(t)\cdot\widehat{x}$ is the map $t \mapsto (t^{d_i}u_i)$ and $f_{\widehat{x},\lambda}^{-1}(0) = \operatorname{Spec} \Bbbk[t]/(t^d)$ where $d$ is the smallest $d_i$ with $u_i \neq 0$.

The projective version below follows from applying the affine version (Theorem 6.7.31) to a lift $\widehat{x} \in \mathbb{A}(V)$ of a non-semistable point $x \in \mathbb{P}(V)$. In this case, the closed orbit in $\overline{G\widehat{x}}$ is the fixed point 0. The following theorem also holds for reductive groups, but we restrict to linearly reductive groups as we've only discussed semistability in that context.

**Theorem 6.7.33** (Kempf's Optimal Destabilization Theorem—projective version)**.** *Let $G$ be a linearly reductive algebraic group over an algebraically closed field $\Bbbk$ with a length $\|-\|$ on $\mathbb{X}_*(G)$. Let $X \subset \mathbb{P}(V)$ be a $G$-equivariant closed subscheme where $V$ is a finite dimensional $G$-representation. For every non-semistable point $x \in X(\Bbbk)$, there exists a one-parameter subgroup $\lambda_0\colon \mathbb{G}_m \to G$ such that $\mu(x,\lambda_0)/\|\lambda_0\|$ achieves a minimal value $M(x)$ over all $\lambda \in \mathbb{X}_*(G)$.*

*If $\lambda_0'$ is another such one-parameter subgroup, then $P_{\lambda_0} = P_{\lambda_0'}$ and $\lambda_0' = g\lambda_0 g^{-1}$ for a unique element $g \in X(\lambda_0)$. Every maximal torus $T \subset P_{\lambda_0}$ contains a unique indivisible element achieving this minimum value.* □

**Definition 6.7.34.** We call any $\lambda_0$ satisfying Theorem 6.7.31 or Theorem 6.7.33 a *Kempf optimal destabilizing one-parameter subgroup for $x$*, and we call $M(x)$ the *optimal normalized Hilbert–Mumford index* for $x$.

*Proof of Theorem 6.7.31.* The proof is simpler when $x_0 \in \overline{Gx}$ is a fixed point, such as in the projective version when the closed orbit is $0 \in \mathbb{A}(V)$; the reader is encouraged to keep this case in mind. By Proposition C.3.4, we may choose finite dimensional $G$-representations $V$ and $W$ along with $G$-equivariant maps

$$\begin{tikzcd} & \mathbb{A}(V) \\ X \arrow[ur, hook, "i"] \arrow[dr, "f"'] & \\ & \mathbb{A}(W), \end{tikzcd} \tag{6.7.4}$$

where $i\colon X \hookrightarrow \mathbb{A}(V)$ is a closed immersion with $i(x_0) = 0$ and $f\colon X \to \mathbb{A}(W)$ is a morphism with $f^{-1}(0) = Gx_0$. When $x_0$ is a fixed point, we can take $f = i$ in (6.7.4).

A one-parameter subgroup $\lambda\colon \mathbb{G}_m \to G$ induces $\mathbb{G}_m$-actions on $V$ and $W$, and thus gradings $V = \bigoplus_{d\in\mathbb{Z}} V_d$ and $W = \bigoplus_{d\in\mathbb{Z}} W_d$. We define

$$\begin{aligned} m(i(x), \lambda) &= \min\{d \mid \text{the projection of } i(x) \text{ to } V_d \text{ is nonzero}\}, \\ m(f(x), \lambda) &= \min\{d \mid \text{the projection of } f(x) \text{ to } W_d \text{ is nonzero}\}. \end{aligned}$$

For any $g \in G$, we have the identities $m(i(v), \lambda) = m(i(g\cdot v), g\lambda g^{-1})$ and $m(f(v), \lambda) = m(f(g\cdot v), g\lambda g^{-1})$.

It is easy to see that if $\lim_{t\to 0} \lambda(t)\cdot x$ exists, then $\mu(x, \lambda) = -m(f(x), \lambda)$, and that

$$\begin{aligned} \lim_{t\to 0} \lambda(t)\cdot x \text{ exists} &\iff m(i(x), \lambda) \geq 0, \\ \lim_{t\to 0} \lambda(t)\cdot x \in Gx_0 &\iff m(i(x), \lambda) \geq 0 \text{ and } m(f(x), \lambda) > 0. \end{aligned}$$

By the Destabilization Theorem (6.6.28), there exists $\lambda_x \in \mathbb{X}_*(G)$ such that $m(i(x), \lambda_x) \geq 0$ and $m(f(x), \lambda_x) > 0$.

*Case of a torus:* Let $T \subset G$ be a maximal torus containing $\lambda_x$. We can decompose $V = \bigoplus_{\chi\in\mathbb{X}^*(T)} V_\chi$ as a $T$-representation where $\mathbb{X}^*(T)$ denotes the set of characters of $T$. We define *the state of $i(x) \in V$ with respect to $T$* to be the set

$$\mathrm{State}_T(i(x)) = \{\chi \in \mathbb{X}^*(T) \mid \text{the projection of } i(x) \text{ to } V_\chi \text{ is nonzero}\}.$$

Likewise, we have the state $\mathrm{State}_T(f(x)) \subset \mathbb{X}^*(T)$ of $f(x) \in W$ with respect to $T$.

Let $\langle -, - \rangle$ be the natural pairing $\mathbb{X}^*(T) \times \mathbb{X}_*(T) \to \mathbb{Z}$. For a one-parameter subgroup $\lambda \in \mathbb{X}_*(T)$, we have identifications

$$m(i(x), \lambda) = \min_{\chi\in\mathrm{State}_T(i(x))} \langle \chi, \lambda\rangle \quad \text{and} \quad m(f(x), \lambda) = \min_{\chi\in\mathrm{State}_T(f(x))} \langle \chi, \lambda\rangle.$$

We claim that the function $\lambda \mapsto m(f(x), \lambda)/\|\lambda\|$ achieves a maximum value on the set $\{\lambda \neq 0 \in \mathbb{X}_*(T) \mid m(i(x), \lambda_T) \geq 0\}$ at a one-parameter subgroup $\lambda_T$, and that any other one-parameter subgroup achieving this minimum is a positive multiple of $\lambda_T$. This is precisely the conclusion of Lemma 6.7.35 below applied to the lattice $L = \mathbb{X}_*(T) \cong \mathbb{Z}^r$ and the subsets of $\mathbb{X}^*(T) \cong \mathrm{Hom}_{\mathbb{Z}}(L, \mathbb{Z})$ given by $F := \mathrm{State}_T(i(x))$ and $G := \mathrm{State}_T(f(x))$.

*General case:* If $T \subset G$ is a maximal torus and $g \in G(\Bbbk)$, then there is an identification $\mathbb{X}^*(T) \cong \mathbb{X}^*(gTg^{-1})$ given by identifying $\chi \in \mathbb{X}^*(T)$ with the character $gTg^{-1} \to \mathbb{G}_m$ defined by $gtg^{-1} \mapsto \chi(t)$. Under this identification, $\mathrm{State}_T(i(x)) = \mathrm{State}_{gTg^{-1}}(i(gx))$. Given a one-parameter subgroup $\lambda \in \mathbb{X}_*(G)$, we've seen that $m(f(x), \lambda) = m(f(gx), g\lambda g^{-1})$ for $g \in G(\Bbbk)$. We claim that in fact $m(f(x), \lambda) = m(f(x), p\lambda p^{-1})$ for $p \in P_\lambda$. By symmetry, it suffices to show that $m(f(x), \lambda) \leq m(f(x), p\lambda p^{-1})$. Interpreting $-m(f(x), \lambda)$ as the smallest integer $d$ such that $\lim_{t\to 0} t^d \lambda(t) \cdot f(x) \in \mathbb{A}(W)$ exists, we need to show that $\lim_{t\to 0} t^d p\lambda(t)p^{-1} \cdot f(x) \in \mathbb{A}(W)$ exists. This follows from the computation

$$\begin{aligned}\lim_{t\to 0}\left(t^d p\lambda(t)p^{-1} \cdot f(x)\right) &= \lim_{t\to 0}\Big(p \cdot \left(\lambda(t)p^{-1}\lambda(t)^{-1}\right) \cdot \left(t^d\lambda(t)f(x)\right)\Big) \\ &= p \cdot \left(\lim_{t\to 0}\lambda(t)p^{-1}\lambda(t)^{-1}\right) \cdot \left(\lim_{t\to 0} t^d\lambda(t)f(x)\right).\end{aligned}$$

We now show that the function $\lambda \mapsto m(f(x), \lambda)/\|\lambda\|$ achieves a minimum value on

$$\Sigma := \{\lambda \in \mathbb{X}_*(G) \mid m(i(x), \lambda) \geq 0\}.$$

If $T$ is a maximal torus, by the torus case, we know that for every $g \in G(\Bbbk)$ there is a minimum value on each *non-empty* set $\mathbb{X}_*(gTg^{-1}) \cap \Sigma$, and that the minimum is determined by the subsets of $\mathbb{X}_*(T)$ given by $\mathrm{State}_{gTg^{-1}}(i(x)) \cong \mathrm{State}_T(i(g^{-1}u))$ and $\mathrm{State}_{gTg^{-1}}(f(x)) \cong \mathrm{State}_T(f(g^{-1}u))$. Since these subsets are contained in the finite set of characters $\chi$ with $V_\chi \neq 0$ (resp. $W_\chi \neq 0$), there are only finitely many minimum values as $g$ ranges over $G(\Bbbk)$. Since the image of any $\lambda \in \mathbb{X}_*(G)$ is contained in $gTg^{-1}$ for some $g \in G(\Bbbk)$, it follows that there is a global minimum value achieved by a one-parameter subgroup $\lambda_0 \in \Sigma$. We may assume that $\lambda_0$ is indivisible, i.e. $\lambda_0$ cannot be written as a positive multiple of another one-parameter subgroup.

To establish the uniqueness, we choose a maximal torus $T \subset G$ containing $\lambda_0$. By the torus case, $\lambda_0 \in \mathbb{X}_*(T) \cap \Sigma$ is the unique indivisible one-parameter subgroup achieving the minimal value. For $p \in P_{\lambda_0}$, the conjugate one-parameter subgroup $p\lambda_0 p^{-1}$ also achieves this minimal value. Since any other maximal torus $T' \subset P_{\lambda_0}$ is $pTp^{-1}$ for some $p \in P_{\lambda_0}$, we see that $\mathbb{X}_*(T') \cap \Sigma$ also contains a unique indivisible element achieving the minimum value. Finally, let $\lambda_1 \in \mathbb{X}_*(G)$ be another indivisible element achieving the minimum value. The intersection $P_{\lambda_0} \cap P_{\lambda_1}$ contains a maximal torus $T$ of $G$ (Proposition C.3.10(d)), and we can write $\lambda_T = p_0\lambda_0 p_0^{-1} = p_1\lambda_1 p_1^{-1}$ for $p_0, p_1 \in P_{\lambda_T}$. It follows that $P_{\lambda_0} = P_{\lambda_T} = P_{\lambda_1}$, and that $\lambda_0$ and $\lambda_1$ are conjugate by a unique element element of $U_{\lambda_T}$ (Proposition C.3.10(c)).

See also [Kem78, Thm. 3.4]. □

The argument above used the following lemma in convex geometry.

**Lemma 6.7.35.** *Let $\Lambda$ be a finite dimensional lattice, and let $F$ and $G$ be non-empty finite subsets of $\Lambda^\vee = \mathrm{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{Z})$. Assume that $\Lambda_{\mathbb{R}} = \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$ has a positive definite inner product which is integral valued on $\Lambda$. Define*

$$f_{\min}\colon \Lambda_{\mathbb{R}} \to \mathbb{R},\ \lambda \mapsto \min_{f\in F} f(\lambda) \quad \textit{and} \quad g_{\min}\colon \Lambda_{\mathbb{R}} \to \mathbb{R},\ \lambda \mapsto \min_{g\in G} g(\lambda).$$

*Suppose that there exists $\lambda \in \Lambda_{\mathbb{R}}$ such that $f_{\min}(\lambda) \geq 0$ and $g_{\min}(\lambda) > 0$. Then the function*

$$C_F := \{\lambda \neq 0 \in \Lambda_{\mathbb{R}} \mid f_{\min}(\lambda) \geq 0\} \to \mathbb{R},$$
$$\lambda \mapsto g_{\min}(\lambda)/\|\lambda\|$$

*obtains a maximum value $M$, and there exists a unique element $\lambda_0 \in C_F \cap \Lambda$ such that $M = g_{\min}(\lambda_0)/\|\lambda_0\|$ and such that any other element $\lambda \in C_F \cap \Lambda$ with $M = g_{\min}(\lambda)/\|\lambda\|$ is an integral multiple of $\lambda_0$.*

*Proof.* The set $\{\lambda \in C_F \mid g_{\min}(\lambda) \geq 1\}$ is closed and convex, and therefore contains a unique point $\lambda'$ closest to the origin. Since $g_{\min}(\alpha\lambda') = \alpha g_{\min}(\lambda')$ for $\alpha \in \mathbb{R}$, we must have that $g_{\min}(\lambda') = 1$ and that $\lambda' \in C_F$ is the unique point with $g_{\min}(\lambda') = 1$ and $g_{\min}(\lambda')/\|\lambda'\| = M$.

We now argue that the ray spanned by $\lambda'$ contains an integral point. If $\lambda'$ is in the interior of $\{\lambda \in C_F \mid g_{\min}(\lambda) \geq 1\}$, i.e. $f(\lambda') > 0$ for all $f \in F$ and there is a unique $g \in G$ with $g(\lambda') = 1$, then $\lambda'$ is the closed point to the origin on the affine plane defined by $g = 1$. We claim that $\lambda' = g^*/\langle g^*, g^*\rangle$ where $g^* \in \Lambda_{\mathbb{R}}$ is the unique point such that $\langle g^*, \lambda\rangle = g(\lambda)$ for all $\lambda \in \Lambda_{\mathbb{R}}$. Indeed, the point $\lambda'$ is contained in the plane $g = 1$, and for any other point $\lambda$ on this plane, we have that $\langle \lambda', \lambda\rangle = 1/\langle g^*, g^*\rangle = \langle \lambda', \lambda'\rangle$ and the Cauchy–Schwarz inequality implies that $\langle \lambda', \lambda'\rangle^2 = \langle \lambda', \lambda\rangle^2 \leq \langle \lambda', \lambda'\rangle\langle \lambda, \lambda\rangle$ so that $\langle \lambda', \lambda'\rangle \leq \langle \lambda, \lambda\rangle$. Since the inner product and $g$ take integral values, $g^* \in \Lambda$. We then take $\lambda_0$ to be the unique indivisible element in the ray spanned by $g^*$.

To reduce to this case, let $f_1, \ldots, f_t \in F$ be the functions satisfying $f_i(\lambda') = 0$, and let $g_1, \ldots, g_s \in G$ be the functions satisfying $g_i(\lambda') = g_{\min}(\lambda')$. Since each $f_i$ and $g_j$ take integral values, we may restrict to the subspace

$$W := \left\{\lambda \in \Lambda_{\mathbb{R}} \;\middle|\; \begin{array}{l} f_1(\lambda) = \cdots = f_t(\lambda) = 0 \\ g_1(\lambda) = \cdots = g_s(\lambda) \end{array}\right\},$$

and the lattice $W \cap \Lambda$. Then $\lambda'$ is in the interior of $\{\lambda \in C_F \cap W \mid g_{\min}(\lambda) \geq 1\}$ and thus is the closest point to the origin contained in the affine plane define by $g_1 = 1$. □

**Corollary 6.7.36.** *In the setting of Theorem 6.7.31 or Theorem 6.7.33, there is a unique morphism $f\colon [\mathbb{A}^1/\mathbb{G}_m] \to [X/G]$ with $f(1) \simeq x$ and $f(0) \simeq x_0$.*

*Proof.* By Proposition 6.6.30, a morphism $[\mathbb{A}^1/\mathbb{G}_m] \to [X/G]$ is determined by a one-parameter subgroup $\lambda$ such that $\lim_{t\to 0} \lambda(t)x \in Gx_0$, and that $\lambda$ is unique up to conjugation by $P_\lambda$. Since any two of Kempf's worst one-parameter subgroups are conjugate under $U_\lambda$ (and thus $P_\lambda$), the statement follows. □

**Example 6.7.37.** We revisit the $\mathrm{SL}_2$ action on $(\mathbb{P}^1)^n$ with the linearization given by the Segre embedding $(\mathbb{P}^1)^n \hookrightarrow \mathbb{P}^{2^n-1}$ (Example 6.7.17). The non-semistable consists of tuples $x = (p_1, \ldots, p_n)$ where more than $n/2$ points are equal. Suppose that precisely $k > n/2$ points are equal. Since the Hilbert–Mumford index is symmetric, we can assume that the first $k$ are equal. If $\lambda\colon \mathbb{G}_m \to \mathrm{SL}_2$ is a one-parameter subgroup, we can choose $g \in \mathrm{SL}_2(\Bbbk)$ with $g\lambda g^{-1} = \lambda_0^d$ where $d \in \mathbb{Z}$ and $\lambda_0(t) = \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix}$. After rescaling the norm, we can assume that $\|\lambda_0\| = 1$. We also assume that $d \geq 0$ as the $d < 0$ case can be handled similarly. Then

$$\frac{\mu(x, \lambda)}{\|\lambda\|} = \frac{\mu(gx, g\lambda g^{-1})}{\|g\lambda g^{-1}\|} = \mu(gx, \lambda_0)$$

This index is negative if and only if $gx = \{0, \ldots, 0, p_{k+1}, \ldots, p_n\}$ in which case $\mu(gx, \lambda_0) = n - 2k$. It follows that $\lambda_0$ (resp. $g^{-1}\lambda_0 g$) is a Kempf optimal destabilizing one parameter subgroup for $gx$ (resp $x$). Observe that the parabolic $P_{\lambda_0} \subset \mathrm{SL}_2$ of lower triangular matrices is also the stabilizer of $0 \in \mathbb{P}^1$, and thus $G_{gx} \subset P_{\lambda_0}$. For any $h \in P_{\lambda_0}$, $h^{-1}\lambda_0 h$ (resp. $(hg)^{-1}\lambda_0 hg$) is also a Kempf optimal destabilizing subgroup for $gx$ (resp. $x$).

**Exercise 6.7.38.** Let $G$ be a reductive algebraic group over an algebraically closed field $\Bbbk$ with a length $\|-\|$ on $\mathbb{X}_*(G)$. Let $X = \operatorname{Spec} A$ be an affine scheme of finite type over $\Bbbk$ with an action of $G$. Let $x_0 \in X(\Bbbk)$ have closed $G$-orbit. Let $x \in X(\Bbbk)$ be a point such that $Gx_0 \subset \overline{Gx}$, and let $P_x$ be the parabolic determined by Kempf's Optimal Destabilization Theorem (6.7.31).

(a) Show that for all $g \in G(\Bbbk)$ that $gP_xg^{-1} = P_{gx}$.

*Hint: Show that if $P_x = P_\lambda$ for a one-parameter subgroup $\lambda$, then $P_{gx} = P_{g\lambda g^{-1}}$.*

(b) Show that $G_x \subset P_x$.

*Hint: Use that for a parabolic $P$, $N_G(P) = P$ (Proposition C.3.10).*

The following criterion can sometimes be used to check stability/semistability by computing Hilbert–Mumford indices only for one-parameter subgroups in a fixed maximal torus.

**Exercise 6.7.39** (Kempf–Morrison Criterion)**.** Let $G = \mathrm{GL}(W)$ or $\mathrm{SL}(W)$, where $W$ is finite dimensional vector space over an algebraically closed field $\Bbbk$ of characteristic 0. Let $X \subset \mathbb{P}(V)$ be a $G$-invariant closed subscheme, where $V$ is a finite dimensional $G$-representation. Let $x \in X(\Bbbk)$. Assume that there is a linearly reductive subgroup $H \subset G_x$ such that $W$ decomposes as a direct sum of distinct $H$-representations. Let $T \subset G$ be a maximal torus compatible with this decomposition. Show that

$$x \in X^{\mathrm{ss}} \iff \mu(x, \lambda) \leq 0 \text{ for all } \lambda \colon \mathbb{G}_m \to T,$$
$$x \in X^{\mathrm{s}} \iff \mu(x, \lambda) < 0 \text{ for all } \lambda \colon \mathbb{G}_m \to T.$$

*Hint: If $u \notin X^{\mathrm{ss}}$, let $\lambda_0 \colon \mathbb{G}_m \to G$ be a Kempf optimal destabilization one-parameter subgroup and $0 = V_0 \subset V_1 \subset \cdots \subset V_k = V$ be the filtration induced by the parabolic $P_{\lambda_0}$. Use Exercise 6.7.38 to conclude that each $V_i$ is $H$-invariant, and use the hypothesis on the $H$-representation $V$ to show that each $V_i$ is $T$-invariant; thus $T \subset P_{\lambda_0}$. Apply Kempf's Optimal Destabilization Theorem again to find $\lambda$ in $T$ with $\mu(x, \lambda) < 0$. If $u \notin X^{\mathrm{s}}$, letting $\widehat{x} \in \mathbb{A}(V)$ be a lift of $x$ and $\widehat{x}_0 \in \overline{G\widehat{x}}$ be a point with closed orbit, repeat the above argument using the affine version of Kempf's Optimal Destabilization Theorem.*

**Exercise 6.7.40** (Existence of destabilizing one-parameter subgroups over a perfect field)**.** Let $X$ be an affine scheme of finite type over a perfect field $\Bbbk$, and let $G$ be a reductive algebraic group over $\Bbbk$ acting on $X$. This exercise will show that for every point $x \in X(\Bbbk)$, there exists a one-parameter subgroup $\lambda \colon \mathbb{G}_m \to G$ *defined over* $\Bbbk$ such that $\lim_{t \to 0} \lambda(t) \cdot x$ has closed $G$-orbit. See also [Kem78, §4].

(1) Show that if $\mathrm{Gal} := \mathrm{Gal}(\overline{\Bbbk}/\Bbbk)$ is the geometric Galois group, then Gal acts on the set $\mathbb{X}_*(G_{\Bbbk})$ of one-parameters subgroups such that $\mathbb{X}_*(G) = \mathbb{X}_*(G_{\overline{\Bbbk}})^{\mathrm{Gal}}$.

(2) Show that there exists a length $\|-\|$ on $\mathbb{X}_*(G_{\overline{\Bbbk}})$ which is invariant under the action of Gal.

(3) Show that the subsets $\{\lambda \in \mathbb{X}_*(G_{\overline{\Bbbk}}) \mid \lim_{t \to 0} \lambda(t) \cdot x \in X(\overline{\Bbbk}) \text{ exists}\}$ and $\{\lambda \in \mathbb{X}_*(G_{\overline{\Bbbk}}) \mid \lim_{t \to 0} \lambda(t) \cdot x \in G_{\overline{\Bbbk}} x_0\}$ are Gal-invariant where $G_{\overline{\Bbbk}} x_0$ is

the unique closed orbit in $\overline{G_{\bar{\Bbbk}} u}$. Moreover, show that if $V$ and $W$ are $G$-representations as in (6.7.4), then the functions $m(i(x), \lambda)$ and $m(f(x), \lambda)$ are Gal-invariant.

(4) Generalize Theorem 6.7.31 and Theorem 6.7.33 to the case when $\Bbbk$ is a perfect field and $x \in X(\Bbbk)$.

In particular, if $G$ has no non-trivial one-parameter subgroups defined over $\Bbbk$, then the $G$-orbit of any $\Bbbk$-point is closed.

Finally, we record the following consequence of *the proof* of Kempf's Optimal Destabilization Theorem (6.7.31). This will play a key role in the proof of the HKKN Stratification (6.7.42).

**Proposition 6.7.41.** *Let $G$ be a reductive algebraic group over an algebraically closed field $\Bbbk$ with a length $\|-\|$ on $\mathbb{X}_*(G)$. Let $X = \operatorname{Spec} A$ be an affine scheme of finite type over $\Bbbk$ with an action of $G$ with a unique closed orbit $Gx_0$. Fix a maximal torus $T \subset G$. There are finitely many one-parameter subgroups $\lambda_1, \ldots, \lambda_n \in \mathbb{X}_*(T)$ and numbers $M_1, \ldots, M_n \in \mathbb{R}_{<0}$ such that for every point $x \in X(\Bbbk)$, there exists a unique $i = 1, \ldots, n$ such that $\lambda_i$ is an optimal Kempf one-parameter subgroup for $gx$ for some $g \in G$, and such that $M_i = \mu(x, \lambda_i)/\,\|\lambda_i\|$.*

*Proof.* We will use the notation of the proof of Theorem 6.7.31. For $x \in X(\Bbbk)$, the unique parabolic subgroup of a Kempf optimal destabilization one-parameter subgroup is determined by the subsets $\operatorname{State}_{gTg^{-1}}(i(x)) \cong \operatorname{State}_T(i(gx)) \subset \mathbb{X}_*(T)$ and $\operatorname{State}_{gTg^{-1}}(f(x)) \cong \operatorname{State}_T(f(gx)) \subset \mathbb{X}_*(T)$ as $g$ ranges over $G(\Bbbk)$. These subsets are contained in the finite subset of characters $\chi \in \mathbb{X}_*(T)$ with $V_\chi \neq 0$ or $W_\chi \neq 0$. Thus there are only finitely many possibilities for an optimal destabilizing subgroup of $T$, and the statement follows. $\square$

### 6.7.6 The Hesselink–Kempf–Kirwan–Ness Stratification

For an action of a reductive group $G$ on a projective variety $X \subset \mathbb{P}^n$, we show that the non-semistable locus admits a stratification into locally closed subschemes according to the normalized Hilbert–Mumford index

$$M(x) := \mu(x, \lambda)/\,\|\lambda\| \in \mathbb{R}_{<0}$$

of a Kempf optimal destabilizing one-parameter subgroup $\lambda$ of a point $x \in X \setminus X^{\mathrm{ss}}$. The more negative the index $M(x)$ is, the more non-semistable (or 'unstable') the point $x$ is. The strata will be indexed by pairs $(\lambda, M)$ where $\lambda \in \mathbb{X}_*(G)$ and $M \in \mathbb{R}_{<0}$.

Recall from the Białynicki-Birula decomposition that for a one-parameter subgroup $\lambda \colon \mathbb{G}_m \to G$, the attractor locus $X_\lambda^+ = \operatorname{Mor}^{\mathbb{G}_m}(\mathbb{A}^1, X)$ for the induced $\mathbb{G}_m$-action is a disjoint union of locally closed subschemes.

**Theorem 6.7.42** (The HKKN Statification)**.** *Let $G$ be a linearly reductive algebraic group over an algebraically closed field $\Bbbk$ with a maximal torus $T$ and a length $\|-\|$ on $\mathbb{X}_*(G)$. Let $X \subset \mathbb{P}(V)$ be a $G$-equivariant closed subscheme where $V$ is a finite dimensional $G$-representation. There is a finite subset $\Sigma \subset \mathbb{X}_*(T) \times \mathbb{R}_{<0}$ and a stratification of the non-semistable locus into $G$-invariant locally closed subschemes*

$$X \setminus X^{\mathrm{ss}} = \coprod_{(\lambda, M) \in \Sigma} S_{\lambda, M}$$

*such that for each $(\lambda, M) \in \Sigma$,* *(1)* $X^+_{\lambda,M} := \{x \in X^+_\lambda \mid M(x) = M\}$ *is a $P_\lambda$-invariant locally closed subscheme of $X$ consisting of points $x$ such that $\lambda$ is a Kempf optimal destabilizing one-parameter subgroup for $x$, and $S_{\lambda,M} = G \cdot X^+_{\lambda,M}$;*

*(2) a point $x \in X^+_{\lambda,M}$ if and only if $\mathrm{ev}_0(x) = \lim_{t\to 0} \lambda(t)\cdot x \in X^+_{\lambda,M} \cap X^\lambda$; thus $Z_{\lambda,M} := \{x \in X^\lambda \mid M(x) = M\}$ is a $C_\lambda$-invariant closed subscheme of $X^+_{\lambda,M}$ such that $X^+_{\lambda,M} = \mathrm{ev}_0^{-1}(Z_{\lambda,M})$.*

*(3) the natural map $G \times^{P_\lambda} X^+_{\lambda,M} \to S_{\lambda,M}$ is finite, surjective, and universally injective; if $\mathrm{char}(\Bbbk) = 0$, then $G \times^{P_\lambda} X^+_{\lambda,M} \to S_{\lambda,M}$ is an isomorphism.*

*(4) the locus*

$$\bigcup_{(\lambda',M')\in\Sigma, M'\le M} S_{\lambda',M'}$$

*is closed and in particular contains $\overline{S}_{\lambda,M}$;*

*(5) if $X$ is smooth, then so is each $X^+_{\lambda,M}$; in $\mathrm{char}(\Bbbk) = 0$, the strata $S_{\lambda,M}$ is also smooth.*

**Remark 6.7.43.** The locus $S_{\lambda,M}$ is called a *stratum* while $X^+_{\lambda,M}$ and $Z_{\lambda,M}$ are sometimes called a *blade* and *center* of the stratum. In characteristic 0, we have stack-theoretic equivalences $[X^+_{\lambda,M}/P_\lambda] \cong [S_{\lambda,M}/G]$ and a stratification

$$[(X \setminus X^{\mathrm{ss}})/G] = \coprod_{(\lambda,M)\in\Sigma} [S_{\lambda,M}/G].$$

For each $(\lambda, M)$, there is a diagram

$$\begin{tikzcd} {[Z_{\lambda,M}/C_\lambda]} \arrow[r, "i"'] & {[X^+_{\lambda,M}/P_\lambda]} \arrow[l, bend right, "\mathrm{ev}_0"'] \cong {[S_{\lambda,M}/G]} \arrow[r, hook] & {[X/G]} \end{tikzcd} \tag{6.7.5}$$

such that $\mathrm{ev}_0 \circ i = \mathrm{id}$.

*Proof.* Let $\widehat{X} \subset \mathbb{A}(V)$ be the affine cone of $X$, and let $\widehat{N} \subset \widehat{X}$ be the nullcone, i.e. the affine cone of $X \setminus X^{\mathrm{ss}}$. Then $0 \in \widehat{N}$ is the unique closed $G$-orbit. Applying Proposition 6.7.41 to the nullcone $\widehat{N} \subset \mathbb{A}(V)$, there is a finite subset $\Sigma \subset \mathbb{X}_*(T)\times\mathbb{R}_{<0}$ such that for every point $\widehat{x} \in \widehat{N} \setminus 0$, there is a unique $(\lambda, M) \in \Sigma$ such that $\lambda$ is a Kempf optimal destabilizing one-parameter subgroup for $\widehat{x}$ with $M = \mu(\widehat{x}, \lambda)/\|\lambda\|$.

Since $\widehat{N}$ is affine, the locus $\widehat{N}^+_\lambda \subset \widehat{N}$ is a closed subscheme for each $(\lambda, M) \in \Sigma$ (Exercise 6.6.5). Since $G$ is reductive, $P_\lambda \subset G$ is parabolic and

$$[\widehat{N}^+_\lambda/P_\lambda] \cong [G \times^{P_\lambda} \widehat{N}^+_\lambda/G] \to [\widehat{N}/G]$$

is projective. The image of this morphism is a closed substack corresponding to a closed $G$-invariant subscheme $\widehat{S}_\lambda$ such that $\widehat{S}_\lambda = G \cdot \widehat{N}^+_\lambda$. The loci $\widehat{N}^+_\lambda$ and $\widehat{S}_\lambda$ are invariant under scaling and are thus the affine cones over closed subschemes $N_\lambda$ and $S_\lambda$ of $X \setminus X^{\mathrm{ss}}$ such that $S_\lambda = G \cdot N_\lambda$.

The locus $X^+_{\lambda,M} := \{x \in X^+_\lambda \mid M(x) = M\}$ is identified with the points $x \in N_\lambda$ with $M(x) = M$. Moreover, $S_{\lambda,M} := \{x \in S_\lambda \mid M(x) = M\}$ is identified with $G \cdot X^+_{\lambda,M}$. There are identifications

$$X^+_{\lambda,M} = X^+_\lambda \setminus \bigcup_{(\lambda',M'), M'<M} X^+_{\lambda',M'} \quad \text{and} \quad S_\lambda = S_{\lambda,M} \setminus \bigcup_{(\lambda',M'), M'<M} S_{\lambda',M'}.$$

Thus $X^+_{\lambda,M}$ and $S_{\lambda,M}$ are open in $X^+_\lambda$ and $S_\lambda$, and each are locally closed in $U \setminus U^{\mathrm{ss}}$. From the conclusion of Proposition 6.7.41, the loci $S_{\lambda,M}$ are disjoint and cover $U \setminus U^{\mathrm{ss}}$. This gives (1).

For (2), if $x \in X \subset \mathbb{P}(V)$, then the limit $x_0 = \lim_{t\to 0} \lambda(t)\cdot x$ is the projection onto the subspace $W = \oplus V_\chi$ ranging over characters $\chi \in \mathbb{X}^*(T)$ such that the projection $\mathrm{proj}_\chi(x)$ of $x$ to $V_\chi$ is nonzero and $\langle \chi, \lambda\rangle = -\mu(x,\lambda)$. By Lemma 6.7.35, $\lambda$ lies on the ray spanned by the unique point closest to the origin in the closed convex set of $C_x = \{\lambda \in \mathbb{X}_*(T)_\mathbb{R} \mid \langle \chi, \lambda\rangle \geq 1, \mathrm{proj}_\chi(x) \neq 0\}$. It follows that $\lambda$ is also the closest point to the origin in the analogously defined set $C_{x_0}$. Alternatively, one can check that if $\lambda_0 \in \mathbb{X}_*(T)$ is a optimal destabilizing one-parameter subgroup for $x_0$, then $\mu(x_0,\lambda_0)/\left\|\lambda_0\right\| \leq \mu(x,\lambda)/\left\|\lambda\right\|$ (giving the implication $x_0 \in X^+_{\lambda,M} \Rightarrow x \in X^+_{\lambda,M}$) and $\mu(x,\lambda^N\lambda_0)/\left\|\lambda^N\lambda_0\right\| \leq \mu(x,\lambda)/\left\|\lambda\right\|$ for $N \gg 0$ (giving the implication $x \in X^+_{\lambda,M} \Rightarrow x_0 \in X^+_{\lambda,M}$).

For (3), for $x \in X^+_{\lambda,M}$ we claim that

$$P_\lambda = \{g \in G(\Bbbk) \mid gx \in X^+_{\lambda,M}\}. \tag{6.7.6}$$

Since $X^+_{\lambda,M}$ is $P_\lambda$-invariant, we have the inclusion '$\subset$'. Conversely, if $gx \in X^+_{\lambda,M}$, then both $\lambda$ and $g\lambda g^{-1}$ are optimal destabilization one-parameter subgroups for $x$. By Kempf's Optimal Destabilization Theorem (6.7.31), the parabolics $P_\lambda$ and $P_{g\lambda g^{-1}} = gP_\lambda g^{-1}$ are equal. Since $N_G(P_\lambda) = P_\lambda$ (Proposition C.3.10), we conclude that $g \in P_\lambda$. Since $[X^+_\lambda/P_\lambda] \to [X/G]$ is proper, so is $[X^+_{\lambda,m}/P_\lambda] \to [S_{\lambda,M}/G]$. The map $[X^+_{\lambda,m}/P_\lambda] \to [S_{\lambda,M}/G]$ is surjective by construction, and injective on $\Bbbk$-points by (6.7.6); it is thus finite, surjective, and universally injective, and moreover an isomorphism if $\mathrm{char}(\Bbbk) = 0$.

For (4), given $M < 0$, assume by induction that $\bigcup_{(\lambda',M')\in\Sigma, M'<M} S_{\lambda',M'}$ is closed. Then for each $(\lambda, M)$, we have that

$$S_{\lambda,M} = S_\lambda \setminus \bigcup_{(\lambda',M')\in\Sigma, M'<M} S_{\lambda',M'}$$

and it follows that $\bigcup_{(\lambda',M')\in\Sigma, M'\leq M} S_{\lambda',M'}$ is closed.

For (5), if $X$ is smooth, then each $X^+_\lambda$ is smooth (Theorem 6.6.7). Since $X^+_{\lambda,M} \subset X^+_\lambda$ is open, $X^+_{\lambda,M}$ is also smooth. In $\mathrm{char}(\Bbbk) = 0$, $S_{\lambda,M} = G \times^{P_\lambda} X^+_{\lambda,M}$ by Part (3) and thus also smooth.

See also [Hes81, §3], [Hes79, §4] and [Kir84, §12-13]. □

**Remark 6.7.44.** When $X$ is a smooth projective variety over $\mathbb{C}$, the HKKN stratification coincides with the Morse stratification of the square-norm of the moment map $\|-\|^2 : X \to \mathbb{R}$. Given $x \in X$, the optimal destabilizing one-parameter subgroup corresponds to the path of steepest descent starting from $x$. The centers $Z_{\lambda,M}$ correspond to the set of critical values of $\|-\|^2$ while the strata $S_{\lambda,M}$ are the locally closed submanifolds consisting of points which flow to $Z_{\lambda,M}$. See [Kir84, §6] and [Nes84].

**Example 6.7.45.** Let $\mathbb{G}_m$ act linearly on $X = \mathbb{P}^2$ with weights $-1, 2, 3$. Letting $\lambda = \mathrm{id}$ be the identity one-parameter subgroup, the non-semistable locus is $V(x^2y, x^3z)$ has the stratification $S_{\lambda^{-1},-1} \cup S_{\lambda,-2} \cup S_{\lambda,-3}$ where $S_{\lambda^{-1},-1} = \{[1:0:0]\}$, $S_{\lambda,-2} = \{[0:y:z] \mid y \neq 0\}$, and $S_{\lambda,-3} = \{[0:0:1]\}$.

**Example 6.7.46.** Revisiting the action of $\mathrm{SL}_2$ on $X = (\mathbb{P}^1)^n$ with the Segre linearization (Example 6.7.37), let $\lambda_0 \colon \mathbb{G}_m \to \mathrm{SL}_2$ be the one-parameter subgroup defined by $\lambda_0(t) = \mathrm{diag}(t^{-1}, t)$. The strata are indexed by $(\lambda_0, -1), (\lambda_0, -3), \ldots, (\lambda_0, -n)$ if $n$ is odd and by $(\lambda_0, -2), (\lambda_0, -4), \ldots, (\lambda_0, -n)$ if $n$ is even. The strata $S_{\lambda_0, n-2k}$ consists of tuples with precisely $k > n/2$ points in common and has codimension $k-1$. The blade $X^+_{\lambda_0, n-2k}$ consists of tuples where precisely $k$ points are 0 while the center $Z_{\lambda_0, n-2k}$ is the set of $\mathbb{G}_m$-fixed points where $k$ points are 0 and $n-k$ points are $\infty$.

**Remark 6.7.47** ($\Theta$-stratifications)**.** As indicated in Remark 6.6.32, there is an identification

$$\underline{\mathrm{Mor}}([\mathbb{A}^1/\mathbb{G}_m], [X/G]) = \coprod_{\lambda \in \mathbb{X}_*(G)/\sim} [X^+_\lambda / P_\lambda],$$

where $\mathbb{X}_*(G)/\sim$ represents the set of one-parameter subgroups up to conjugation.

A $\Theta$-*stratification* of an algebraic stack $\mathcal{X}$ locally of finite type over $\Bbbk$ is the data of a totally ordered set $\Sigma$ with a minimal element $0 \in \Sigma$ and a stratification into locally closed substacks

$$\mathcal{X} = \coprod_{\lambda \in \Sigma} \mathcal{S}_\lambda$$

such that:

(1) for each $\lambda \in \Sigma$, $\mathcal{X}_{\leq \lambda} := \bigcup_{\rho \leq \lambda} \mathcal{S}_\rho$ is an open substack of $\mathcal{X}$,

(2) for each $\lambda \in \Sigma$, there is a union of connected components (called a $\Theta$-*stratum* of $\mathcal{X}_{\leq\lambda}$)

$$\mathcal{S}'_\lambda \subset \underline{\mathrm{Mor}}([\mathbb{A}^1/\mathbb{G}_m], \mathcal{X}_{\leq \lambda})$$

such that $\mathrm{ev}_0 \colon \mathcal{S}'_\lambda \to \mathcal{X}_{\leq\lambda}$ is a closed immersion mapping isomorphically onto $\mathcal{S}_\lambda$, and

(3) for every $x \in |\mathcal{X}|$, the set $\{\lambda \in \Sigma \mid x \in |\mathcal{X}_{\leq \lambda}\}$ has a minimal element.

See [HL14]. The semistable locus $\mathcal{X}^{\mathrm{ss}}$ is by definition the open substack $\mathcal{X}_{\leq 0} = \mathcal{S}_0$. Let $\mathcal{Z}'_\lambda$ be the preimage of $\mathcal{S}'_\lambda$ under the map

$$i \colon \underline{\mathrm{Mor}}(\mathbf{B}\mathbb{G}_m, [X/G]) \to \underline{\mathrm{Mor}}([\mathbb{A}^1/\mathbb{G}_m], \mathcal{X}).$$

The map $\mathrm{ev}_0 \colon \underline{\mathrm{Mor}}([\mathbb{A}^1/\mathbb{G}_m], [X/G]) \to \underline{\mathrm{Mor}}(B\mathbb{G}_m, \mathcal{X})$ obtained by restricting to 0 is a section of $i$, and there is a diagram analogous to (6.7.5)

$$\begin{tikzcd} \mathcal{Z}'_\lambda \arrow[r, "i"] & \mathcal{S}'_\lambda \arrow[l, bend right, "\mathrm{ev}_0"'] \arrow[r, hook] & \mathcal{X}. \end{tikzcd}$$

In characteristic 0, the HKKN stratification is an example of a $\Theta$-stratification, where one orders the indices $(\lambda, M)$ first by $-M$ and then arbitrarily by $\lambda$. In the next chapter, we will see that the moduli stack $\mathrm{Bun}_{r,d}(C)$ has a $\Theta$-stratification called the Harder–Narasimhan–Shatz stratification.

Recall that the *Chow–Poincare polynomial* of a $G$-equivariant scheme $X$ is $p_G(X,t) = \sum_{d=0}^\infty \big(\dim \mathrm{CH}^d_G(U)_{\mathbb{Q}}\big) t^d$.

**Proposition 6.7.48** (Kirwan Surjectivity)**.** *Under the hypotheses of Theorem 6.7.42, assume further assume that $X$ is smooth and irreducible, and that* $\mathrm{char}(\Bbbk) = 0$*. Suppose that for all $(\lambda, M)$, the stratum $S_{\lambda,M}$ is equidimensional of codimension $d_{\lambda,M}$. Then*

$$\dim \mathrm{CH}^k_G(X)_{\mathbb{Q}} = \dim \mathrm{CH}^k_G(X^{\mathrm{ss}})_{\mathbb{Q}} + \sum_{(\lambda,M)} \dim \mathrm{CH}^{k-d_{\lambda,M}}_{C_\lambda}(Z_{\lambda,M})_{\mathbb{Q}}$$

*and*

$$p_G(X,t) = p_G(X^{\mathrm{ss}},t) + \sum_{(\lambda,M)} p_{C_\lambda}(Z_{\lambda,M},t)t^{d_{\lambda,M}}.$$

*Proof.* From Theorem 6.7.42, we know that $[S_{\lambda,M}/G] \cong [X^+_{\lambda,M}/P_\lambda]$. From Theorem 6.6.7, we know that $\mathrm{ev}_0 \colon X^+_{\lambda,M} \to Z_{\lambda,M}$ sending a point to its limit is a Zariski-local affine fibration and equivariant with respect to $P_\lambda \to C_\lambda$. We claim that $[X^+_{\lambda,M}/P_\lambda] \to [Z_{\lambda,M}/C_\lambda]$ induces an isomorphism

$$\mathrm{CH}^*_{C_\lambda}(Z_{\lambda,M}) \to \mathrm{CH}^*_{P_\lambda}(X^+_{\lambda,M}). \tag{6.7.7}$$

By the definition of the equivariant Chow groups, $\mathrm{CH}^i_{P_\lambda}(X^+_{\lambda,M})$ is identified by $\mathrm{CH}^i(X^+_{\lambda,M} \times^{P_\lambda} V)$ where $V$ is an open subspace $\mathbb{A}(W)$ of a $P_\lambda$-representation such that $P_\lambda$ acts freely on $V$ and $\mathbb{A}(W) \setminus V$ has sufficiently high codimension. On the other hand, $\mathrm{CH}^i_{C_\lambda}(Z_{\lambda,M})$ is identified with $\mathrm{CH}^i(Z_{\lambda,M} \times^{C_\lambda} V)$ and the map (6.7.7) corresponds to the pullback map on Chow induced from the composition

$$X^+_{\lambda,M} \times^{P_\lambda} V \to Z_{\lambda,M} \times^{P_\lambda} V \to Z_{\lambda,M} \times^{C_\lambda} V.$$

The first map is a Zariski local affine fibration and the second map is a principal bundle under $U_\lambda = \ker(P_\lambda \to C_\lambda)$. Since $U_\lambda$ is unipotent, $U_\lambda$ is isomorphic to affine space and principal $U_\lambda$-bundles are locally trivial in the Zariski topology (see §C.3.2). We conclude that (6.7.7) is an isomorphism.

We also claim that $c_{d_{\lambda,M}}(N_{S_{\lambda,M}/X}) \in \mathrm{CH}^*_G(S_{\lambda,M})$ is a nonzerodivisor. Since $N_{S_{\lambda,M}/X}|_{Z_{\lambda,M}}$ is identified with $N_{S_{\lambda,M}/X}$ under $\mathrm{CH}^*_{C_\lambda}(Z_{\lambda,M}) \cong \mathrm{CH}^*_G(S_{\lambda,M})$, it suffices to show that $c_{d_{\lambda,M}}((N_{S_{\lambda,M}/X})|_{Z_{\lambda,M}}) \in \mathrm{CH}^*_{C_\lambda}(Z_{\lambda,M})_{\mathbb{Q}}$ is a nonzerodivisor where $d = d_{\lambda,M}$. By Theorem 6.6.7, $\lambda$ acts on a fiber of the normal bundle with nonzero weights. Thus Lemma 6.6.20 implies that $c_{d_{\lambda,M}}((N_{S_{\lambda,M}/X})|_{Z_{\lambda,M}})$ is a nonzerodivisor.

We therefore can apply Lemma 6.6.18 with the strata $S_{\lambda,M}$ ordered first by $-M$ and then with any ordering of the $\lambda$'s; the semistable locus $U^{\mathrm{ss}}$ is viewed as a stratum with the smallest index. This yields

$$\begin{aligned}\dim \mathrm{CH}^k_G(X)_{\mathbb{Q}} &= \dim \mathrm{CH}^k_G(X^{\mathrm{ss}})_{\mathbb{Q}} + \sum_{(\lambda,M)} \dim \mathrm{CH}^{k-d_{\lambda,M}}_G(S_{\lambda,M})_{\mathbb{Q}} \\ &= \dim \mathrm{CH}^k_G(X^{\mathrm{ss}})_{\mathbb{Q}} + \sum_{(\lambda,M)} \dim \mathrm{CH}^{k-d_{\lambda,M}}_{C_{\lambda,M}}(Z_{\lambda,M})_{\mathbb{Q}}.\end{aligned}$$

□

**Remark 6.7.49.** This formula was established for de Rham cohomology in [Kir84, Thm. 5.4]. Instead of the excision sequence

$$\mathrm{CH}^{k-d_{\lambda,M}}_G(S_{\lambda,M}) \to \mathrm{CH}^k_G(S_{\le(\lambda,M)}) \to \mathrm{CH}^k_G(S_{<(\lambda,M)}) \to 0,$$

one uses the Thom–Gysin long exact sequence

$$\cdots \to \mathrm{H}^{k-d_{\lambda,M}}_G(S_{\lambda,M}) \to \mathrm{H}^k_G(S_{\le(\lambda,M)}) \to \mathrm{H}^k_G(S_{<(\lambda,M)}) \to \cdots.$$

In this case, the surjectivity of the right map for all $(\lambda, M)$ is equivalent to the injectivity of the left map for all $(\lambda, M)$, and the latter condition is verified as above by the showing the top Chern class of the normal bundle is a nonzerodivisor.

**Example 6.7.50.** As an application, we can compute the dimension of the rational Chow groups of $[(\mathbb{P}^1)^{n,\mathrm{ss}}/\operatorname{SL}_2]$ using the computation of the stratification in Example 6.7.46. When $n$ is odd, this also gives the dimension of the rational Chow groups of the GIT quotient $(\mathbb{P}^1)^{n,\mathrm{ss}}/\operatorname{SL}_2$ by Properties 6.1.33(4).

Since $[(\mathbb{P}^1)^n/\operatorname{SL}_2] \to \mathbf{B}\operatorname{SL}_2$ is an iterated $\mathbb{P}^1$-bundle and $\operatorname{CH}^*(\operatorname{SL}_2) \cong \mathbb{Z}[T]$ generated in degree 2,

$$\begin{aligned}\operatorname{CH}^*([(\mathbb{P}^1)^n/\operatorname{SL}_2]) &\cong \operatorname{CH}^*((\mathbb{P}^1)^n) \otimes \operatorname{CH}^*(\mathbf{B}\operatorname{SL}_2)\\ &\cong \mathbb{Z}[H_1,\dots,H_n]/(H_1,\dots,H_n)^2 \otimes \mathbb{Z}[T]\end{aligned}$$

and the Chow–Poincare polynomial is $p_{\operatorname{SL}_2}((\mathbb{P}^1)^n, t) = (1+t)^n(1-t^2)^{-1}$. On the other hand, the strata $S_{\lambda,n-2k}$ where precisely $k$ points are the same has codimension $k-1$ and its center $Z_{\lambda,n-2k}$ consists of $\binom{n}{k}$ $\mathbb{G}_m$-fixed points. Thus $p_{\mathbb{G}_m}(Z_{\lambda,n-2k}, t) = \binom{n}{k}(1-t)^{-1}$ and

$$\begin{aligned}p_G((\mathbb{P}^1)^{\mathrm{ss}}, t) &= (1+t)^n(1-t)^{-1} - \sum_{k>n/2}\binom{n}{k}t^{k-1}(1-t)^{-1}\\ &= 1 + nt + \cdots + \left(1 + (n-1) + \binom{n-1}{2} + \cdots + \binom{n-1}{\min(d, n-3-d)}\right)t^d\\ &+ \cdots + nt^{n-4} + t^{n-3}.\end{aligned}$$

See also [Kir84, §16.1].

## 6.8 Existence of good moduli spaces

In this section, we provide necessary and sufficient conditions for the existence of a separated good moduli space in characteristic 0.

**Theorem 6.8.1** (Existence Theorem of Good Moduli Spaces)**.** *Let $\mathcal{X}$ be an algebraic stack, of finite type over an algebraic closed field $\Bbbk$ of characteristic 0, with affine diagonal. There exists a good moduli space $\pi\colon \mathcal{X} \to X$ with $X$ a separated algebraic space if and only if $\mathcal{X}$ is $\Theta$-complete (Definition 6.8.7) and S-complete (Definition 6.8.9).*

*Moreover, $X$ is proper if and only $\mathcal{X}$ satisfies the existence part of the valuative criterion for properness.*

The conditions of $\Theta$-completeness and S-completeness are defined and discussed in detail in §6.8.2.

### 6.8.1 Strategy for constructing good moduli spaces

We first explain how the Local Structure Theorem for Algebraic Stacks (6.5.1) gives us a natural strategy to construct the good moduli space $X$. Namely, for each closed point $x \in \mathcal{X}$, we have an étale quotient presentation

$$\begin{array}{ccc}\mathcal{W} = [\operatorname{Spec} A/G_x] & \xrightarrow{f} & \mathcal{X}\\ \downarrow & & \\ W = \operatorname{Spec} A^{G_x} & & \end{array}$$

where $f$ is affine étale, and there is a preimage $w \in \mathcal{W}$ of $x$ such that $f$ induces an isomorphism of stabilizer groups at $w$. We want to show that the GIT quotients $W = \operatorname{Spec} A^{G_x}$ as $x$ ranges over closed points provide étale models that can be glued to a good moduli space of $\mathcal{X}$. To this end, we need to construct an étale equivalence relation on $W$. Since $f$ is affine, the fiber product $\mathcal{R} := \mathcal{W} \times_{\mathcal{X}} \mathcal{W}$ is isomorphic to a quotient stack $[\operatorname{Spec} B/G_x]$ and we have a diagram

$$\begin{tikzcd} \mathcal{R} \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] \arrow[d] & \mathcal{W} \arrow[r, "f"] \arrow[d] & \mathcal{X} \\ R \arrow[r, shift left, "q_1"] \arrow[r, shift right, "q_2"'] & W & \end{tikzcd}$$

where $R = \operatorname{Spec} B^{G_x}$. If $q_1, q_2 \colon R \rightrightarrows W$ defines an étale equivalence relation, the algebraic space quotient $W/R$ gives a candidate for a good moduli space of $f(\mathcal{W}) \subset \mathcal{X}$.

Luna's Fundamental Lemma (6.3.26) provides condition on when $q_1, q_2 \colon R \rightrightarrows W$ are étale: we need that for all closed points $r \in \mathcal{R}$ that

(a) $p_1(r), p_2(r) \in \mathcal{W}$ are closed points; and

(b) $p_1$ and $p_2$ induce isomorphisms of stabilizer groups at $r$.

On the other hand, we know that $f(w) \in \mathcal{X}$ is closed and $f$ induces an isomorphism of stabilizer groups at the given preimage $w$ of $x$. We want to show that there is an open neighborhood $\mathcal{U}$ of $w$ such that the restriction $f|_{\mathcal{U}}$ satisfies: (a) $f|_{\mathcal{U}}$ sends closed points map to closed points and (b) $f|_{\mathcal{U}}$ induces isomorphisms of stabilizer groups at closed points, and moreover that these conditions are stable under base change. While property (a) is stable under base change, property (b) is not, and we will introduce a stronger condition below—called $\Theta$-surjectivity (Definition 6.8.27)—which is stable under base change and implies (b).

The role of $\Theta$-completeness and S-completeness in the construction of the good moduli space is the following: the $\Theta$-completeness of $\mathcal{X}$ implies that $\Theta$-surjectivity holds (and thus condition (a) and its base changes hold) in an open neighborhood of $w$ (Proposition 6.8.31) while S-completeness implies that condition (b) holds in an open neighborhood of $w$ (Proposition 6.8.37).

### Counterexamples

The following examples do not admit good moduli spaces. We will explain why the approach outlined above fails and then later explain how they violate the conditions of $\Theta$-completeness and S-completeness. We work over an algebraically closed field $\Bbbk$.

**Example 6.8.2.** Consider the action of $\mathbb{G}_m$ on $\mathbb{P}^1$ given by $t \cdot [x : y] = [tx : y]$. The quotient stack $\mathcal{X} = [\mathbb{P}^1/\mathbb{G}_m]$ does not admit a good moduli space. Note that Theorem 6.3.5(2) implies that every $\Bbbk$-point has a unique closed point in its closure. Here we see that $[1 : 1]$ specializes to two closed points $[1 : 0]$ and $[0 : 1]$. Alternatively, if there were a good moduli space, it would have to be $\mathcal{X} \to \operatorname{Spec} \Bbbk$ (which is universal for maps to algebraic spaces), but then the composition $\mathbb{P}^1 \to \mathcal{X} \to \operatorname{Spec} \Bbbk$ would be affine by Serre's Criterion for Affineness (4.4.16), a contradiction.

There are two open substacks $\mathcal{U}_1, \mathcal{U}_2 \subset [\mathbb{P}^1/\mathbb{G}_m]$ isomorphic to $[\mathbb{A}^1/\mathbb{G}_m]$ each which admits a good moduli space $\pi_i \colon \mathcal{U}_i \to \operatorname{Spec} \Bbbk$ but they do not glue to a good moduli space of $\mathcal{X}$: the intersection $\mathcal{U}_1 \cap \mathcal{U}_2$ is the open point in both $\mathcal{U}_1$ and $\mathcal{U}_2$ and not the preimage of an open subscheme under $\pi_i$. To see how the approach above fails, observe that the étale presentation $f\colon \mathcal{W} := \mathcal{U}_1 \coprod \mathcal{U}_2 \to \mathcal{X}$ satisfies (a) and (b) but the base changes $p_1, p_2\colon \mathcal{W} \times_{\mathcal{X}} \mathcal{W} = \mathcal{U}_1 \coprod \mathcal{U}_2 \coprod \mathcal{U}_1 \cap \mathcal{U}_2 \to \mathcal{W}$ fails (b), i.e. the closed point in $\mathcal{U}_1 \cap \mathcal{U}_2$ is mapped to a non-closed point under either projection.

**Example 6.8.3.** For a related example, let $C$ be the projective nodal cubic with its $\mathbb{G}_m$-action. The quotient $\mathcal{X} = [C/\mathbb{G}_m]$ has two points—one open and one closed—but while there is no topological obstruction as above, $\mathcal{X}$ again does not admit a good moduli space because $C$ is projective, not affine. Viewing the nodal cubic as the quotient of nodal union $X'$ of two $\mathbb{P}^1$'s along 0 and $\infty$ modulo the rotation action of $\mathbb{Z}/2$, we have a finite étale cover $[X'/\mathbb{G}_m] \to [X/\mathbb{G}_m]$. Removing one of the origins, we have an affine étale cover $\mathcal{W} = [\operatorname{Spec}(k[x,y]/xy)/\mathbb{G}_m] \to \mathcal{X}$ where $\mathbb{G}_m$ acts via $t \cdot (x,y) = (tx, t^{-1}y)$. Again, this map sends closed points to closed points, but the projections $\mathcal{W} \times_{\mathcal{X}} \mathcal{W} \rightrightarrows \mathcal{W}$ do not.

**Example 6.8.4.** Let $\mathbb{G}_m$ act on $\mathbb{A}^2$ via $t \cdot (x,y) = (tx, y)$ and set $\mathcal{X} = [\mathbb{A}^2/\mathbb{G}_m] \smallsetminus 0$. The point $p = (1,0) \in \mathcal{X}$ is closed with trivial stabilizer, and the open immersion $f\colon \mathbb{A}^1 \hookrightarrow \mathcal{X}$, sending $z$ to $(z,1)$, is an étale quotient presentation. Note that while $f(0)$ is closed, the image $f(z)$ is not closed for $z \neq 0$. The map $\mathcal{X} \to \mathbb{A}^1$ defined by $(x,y) \mapsto y$ is not a good moduli space as $\mathbb{A}^2 \smallsetminus 0$ is not affine.

We will see in the next section that the previous examples violate $\Theta$-completeness. Similar phenomena can naturally occur in moduli, e.g. by removing a single polystable but not stable vector bundle from $\operatorname{Bun}_{r,d}(C)^{\mathrm{ss}}$. The next examples violate $\mathsf{S}$-completeness.

**Example 6.8.5.** Suppose $\operatorname{char}(\Bbbk) \neq 2$ and let $G = \mathbb{Z}/2$ act on the non-separated union $U = \mathbb{A}^1 \bigcup_{x \neq 0} \mathbb{A}^1$ by exchanging the copies of $\mathbb{A}^1$. The quotient stack $[U/G]$ has a $\mathbb{Z}/2$ stabilizer everywhere except at the origin. This is a Deligne–Mumford stack with quasi-finite but not finite inertia; in fact we've seen this before in Exercise 4.3.19 to illustrate the necessity of the separatedness condition in the Keel–Mori Theorem (4.3.11). By precomposing by the inclusion of one of the $\mathbb{A}^1$'s, we have an affine étale morphism $\mathbb{A}^1 \to [U/G]$ which is stabilizer preserving at 0 but not in any open neighborhood of 0.

For a related example, the Deligne–Mumford locus $\mathcal{X}^{\mathrm{DM}}$ in the moduli stack $\mathcal{X} = [\operatorname{Sym}^4 \mathbb{P}^1 / \operatorname{PGL}_2]$ of four unordered points in $\mathbb{P}^1$ is not separated (see Example 4.3.20). Note however that the stable locus $\mathcal{X}^{\mathrm{s}}$ consisting of four distinct points is separated and the semistable locus $\mathcal{X}^{\mathrm{ss}} = \mathcal{X}^{\mathrm{DM}} \cup \{[0 : 0 : \infty : \infty]\}$ has a projective good moduli space.

**Example 6.8.6.** Consider the action of $G = \mathbb{G}_m \rtimes \mathbb{Z}/2$ on $X = \mathbb{A}^2 \smallsetminus 0$ via $t \cdot (a,b) = (ta, t^{-1}b)$ and $-1 \cdot (a,b) = (b,a)$. Note that every point $(a,b) \in X$ with $ab \neq 0$ is fixed by the order 2 element $(a/b, -1) \in G$. The quotient stack $[X/G]$ is a non-separated Deligne–Mumford stack which does not admit a good moduli space; note however that $[\mathbb{A}^2/G] \to \operatorname{Spec} \Bbbk[xy]$ is a good moduli space.

### 6.8.2 The valuative criteria: $\Theta$- and S-completeness

We define the stack 'Theta' as

$$\Theta := [\mathbb{A}^1/\mathbb{G}_m]$$

over $\operatorname{Spec}\mathbb{Z}$.[9] If $R$ is a DVR with fraction field $K$ and residue field $\kappa$, we define $\Theta_R := \Theta \times \operatorname{Spec} R$ and set $0 \in \Theta_R$ to be the unique closed point. Observe that $\Theta_R$

[9]The symbol $\Theta$ is used as it resembles the picture of the two orbits of $\mathbb{G}_m$ on the complex plane.

is a local model of the quotient stack $[\mathbb{A}^2/\mathbb{G}_m]$ with weights $0, 1$ as it is identified with the base change of the good moduli space $[\mathbb{A}^2/\mathbb{G}_m] \to \operatorname{Spec} \Bbbk[x]$ along the map $\operatorname{Spec} R \to \operatorname{Spec} \Bbbk[x]$ where $x$ maps to a uniformizer $\pi$ in $R$.

The following cartesian diagram gives a schematic picture of $\Theta_R$ (where $x$ is the coordinate on $\mathbb{A}^1$ and $\pi \in R$ is the uniformizer).

$$\begin{tikzcd}
 & \operatorname{Spec} R \arrow[dr, hook, "x \neq 0"] & & \mathbf{B}\mathbb{G}_{m,R} \arrow[dl, hook', "x=0"'] & \\
\operatorname{Spec} K \arrow[ur, hook] \arrow[dr, hook] & & \Theta_R & & \mathbf{B}\mathbb{G}_{m,\kappa} \arrow[ul, hook'] \arrow[dl, hook'] \\
 & \Theta_K \arrow[ur, hook, "\pi \neq 0"'] & & \Theta_\kappa \arrow[ul, hook', "\pi = 0"] &
\end{tikzcd} \tag{6.8.1}$$

where the maps to the left are open immersions and to the right are closed immersions.

In particular, a morphism $\Theta_R \smallsetminus 0 = \operatorname{Spec} R \bigcup_{\operatorname{Spec} K} \Theta_K \to \mathcal{X}$ to an algebraic stack is the data of morphisms $\operatorname{Spec} R \to \mathcal{X}$ and $\Theta_K \to \mathcal{X}$ together with an isomorphism of their restrictions to $\operatorname{Spec} K$.

**Definition 6.8.7.** A noetherian algebraic stack $\mathcal{X}$ is $\Theta$-*complete*[10] if for every DVR $R$, every commutative diagram

$$\begin{tikzcd}
\Theta_R \smallsetminus 0 \arrow[r] \arrow[d] & \mathcal{X} \\
\Theta_R \arrow[ur, dashed] &
\end{tikzcd} \tag{6.8.2}$$

of solid arrows can be uniquely filled in.

**Remark 6.8.8.** We can state an equivalent formulation using the stack $\underline{\mathrm{Mor}}(\Theta, \mathcal{X})$ classifying morphisms $\Theta \to \mathcal{X}$. Evaluation at 1 gives a morphism

$$\mathrm{ev}_1 \colon \underline{\mathrm{Mor}}(\Theta, \mathcal{X}) \to \mathcal{X}, \quad f \mapsto f(1)$$

of stacks, and the $\Theta$-completeness of $\mathcal{X}$ is equivalent to the morphism $\mathrm{ev}_1$ satisfying the valuative criterion for properness. If $\mathcal{X}$ is of finite type over an algebraically closed field $\Bbbk$, then the stack $\underline{\mathrm{Mor}}(\Theta, \mathcal{X})$ is an algebraic stack *locally* of finite type over $\Bbbk$; see Remark 6.6.32 where an explicit description is given when $\mathcal{X}$ is a quotient stack. The stack $\underline{\mathrm{Mor}}(\Theta, \mathcal{X})$ is however rarely quasi-compact, e.g. for $\mathcal{X} = \mathbf{B}\mathbb{G}_m$, and $\mathrm{ev}_1$ is thus rarely proper.

For a DVR $R$ with fraction field $K$, residue field $\kappa$, and uniformizer $\pi$, we define

$$\phi_R := [\operatorname{Spec}\big(R[s,t]/(st - \pi)\big)/\mathbb{G}_m], \tag{6.8.3}$$

where $s$ and $t$ have $\mathbb{G}_m$-weights 1 and $-1$ respectively.[11] The quotient $\phi_R$ is a local model of the quotient stack $[\mathbb{A}^2/\mathbb{G}_m]$ with weights $1, -1$ as it is identified with the base change of the good moduli space $[\mathbb{A}^2/\mathbb{G}_m] \to \operatorname{Spec} \Bbbk[xy]$ along the map $\operatorname{Spec} R \to \operatorname{Spec} \Bbbk[xy]$ given by $xy \mapsto \pi$.

[10] In the literature, the term '$\Theta$-reductive' is often used.

[11] The symbol $\phi$ is used because it looks like the non-separated affine line with an additional origin. In the literature, $\overline{\mathrm{ST}}_R$ is used as it is a compactification of $\mathrm{ST}_R = \overline{\mathrm{ST}}_R \smallsetminus 0 = \operatorname{Spec} R \bigcup_{\operatorname{Spec} K} \operatorname{Spec} R$, which is the 'standard test' scheme for separatedness.

The locus where $s \neq 0$ in $\phi_R$ is isomorphic to $[\operatorname{Spec}\big(R[s,t]_s/(t-\pi/s)\big)/\mathbb{G}_m] \cong [\operatorname{Spec}(R[s]_s)/\mathbb{G}_m] \cong \operatorname{Spec} R$ and the locus where $t \neq 0$ has a similar description. We thus have cartesian diagrams analogous to (6.8.1)

$$
\begin{tikzcd}
 & \operatorname{Spec} R \arrow[dr, hook, "s\neq 0"] & & \Theta_\kappa \arrow[dl, hook', "s=0"'] & \\
\operatorname{Spec} K \arrow[ur, hook] \arrow[dr, hook] & & \phi_R & & \mathbf{B}\mathbb{G}_{m,\kappa} \arrow[ul, hook'] \arrow[dl, hook] \\
 & \operatorname{Spec} R \arrow[ur, hook, "t\neq 0"'] & & \Theta_\kappa \arrow[ul, hook', "t=0"] &
\end{tikzcd}
\tag{6.8.4}
$$

where the maps to the left are open immersions and to the right are closed immersions.

In particular, a morphism $\phi_R \smallsetminus 0 = \operatorname{Spec} R \bigcup_{\operatorname{Spec} K} \operatorname{Spec} R \to \mathcal{X}$ to an algebraic stack is the data of two morphisms $\xi, \xi' \colon \operatorname{Spec} R \to \mathcal{X}$ together with an isomorphism $\xi_K \simeq \xi'_K$ over $\operatorname{Spec} K$.

**Definition 6.8.9.** A noetherian algebraic stack $\mathcal{X}$ is *$\mathsf{S}$-complete* if for every DVR $R$, every commutative diagram

$$
\begin{tikzcd}
\phi_R \smallsetminus 0 \arrow[r] \arrow[d] & \mathcal{X} \\
\phi_R \arrow[ur, dashed] &
\end{tikzcd}
\tag{6.8.5}
$$

of solid arrows can be uniquely filled in.[12]

**Remark 6.8.10.** There are obvious extensions of the definition of $\Theta$-completeness and $\mathsf{S}$-completeness to morphisms $f \colon \mathcal{X} \to \mathcal{Y}$ but we will not need such notions.

**Lemma 6.8.11.** *A noetherian algebraic stack with affine diagonal is $\Theta$-complete (resp. $\mathsf{S}$-complete), if and only if every diagram* (6.8.2) *(resp.* (6.8.5)*), there exists a lift after an extension of DVRs $R \subset R'$. In particular, $\Theta$-completeness and $\mathsf{S}$-completeness can be verified on complete DVRs with algebraically closed residue fields.*

*Proof.* We begin with the observation that if $\mathcal{X} \to \mathcal{Y}$ has affine diagonal and $j \colon \mathcal{U} \to \mathcal{T}$ is an open immersion of algebraic stacks over $\mathcal{Y}$ with $j_*\mathcal{O}_{\mathcal{U}} = \mathcal{O}_{\mathcal{T}}$, then two extensions $f_1, f_2 \colon \mathcal{T} \to \mathcal{X}$ of a $\mathcal{Y}$-morphism $\mathcal{U} \to \mathcal{X}$ are canonically 2-isomorphic. Indeed, since $\operatorname{Isom}_{\mathcal{T}}(f_1, f_2) \to \mathcal{T}$ is affine, the section over $\mathcal{U}$ induced by the 2-isomorphism $f_1|_{\mathcal{U}} \xrightarrow{\sim} f_2|_{\mathcal{U}}$ extends uniquely to a section of $\mathcal{T}$.

Consider a diagram (6.8.2), an extension of DVRs $R \subset R'$, and a lifting $\Theta_{R'} \to \mathcal{X}$. The open immersion $j \colon \Theta_R \smallsetminus 0 \to \Theta_R$ satisfies $j_*\mathcal{O}_{\Theta_R \smallsetminus 0} = \mathcal{O}_{\Theta_R}$ and by flat base change, the same property holds for the morphisms obtained by base changing $j$ along $\Theta_{R'} \to \Theta_R$, $\Theta_{R'} \times_{\Theta_R} \Theta_{R'} \to \Theta_R$, and $\Theta_{R'} \times_{\Theta_R} \Theta_{R'} \times_{\Theta_R} \Theta_{R'} \to \Theta_R$. By the above observation, there exists a canonical 2-isomorphism between the two extensions $\Theta_{R'} \times_{\Theta_R} \Theta_{R'} \rightrightarrows \Theta_{R'} \to \mathcal{X}$ which necessarily satisfies the cocycle condition. By fpqc descent, the lifting $\Theta_{R'} \to \mathcal{X}$ descends to a lifting $\Theta_R \to \mathcal{X}$. The same argument works for $\mathsf{S}$-completeness. $\square$

[12] The 'S' stands for 'Seshadri' as $\mathsf{S}$-completeness is a geometric property reminiscent of how the S-equivalence relation on sheaves implies separatedness of the moduli space.

**Remark 6.8.12.** It is even true that when $\mathcal{X}$ is of finite type over $\Bbbk$, these criteria can be verified on DVRs essentially of finite type over $\Bbbk$; see [AHLH18, §4]. We will not use this fact.

**Lemma 6.8.13.** *Let $f\colon \mathcal{X} \to \mathcal{Y}$ be an affine morphism of noetherian algebraic stacks. If $\mathcal{Y}$ is $\Theta$-complete (resp. $\mathsf{S}$-complete), so is $\mathcal{X}$.*

*Proof.* Since $\Theta_R$ is regular and $0 \in \Theta_R$ is codimension 2, the pushforward of the structure sheaf along $\Theta_R \setminus 0 \to \Theta_R$ is the structure sheaf. We therefore have canonical equivalences

$$\begin{aligned}\mathrm{MOR}_{\mathcal{Y}}(\Theta_R \setminus 0, \mathcal{X}) &\cong \mathrm{MOR}_{\mathcal{O}_{\mathcal{Y}}-\mathrm{alg}}(f_*\mathcal{O}_{\mathcal{X}}, (\Theta_R \setminus 0 \to \mathcal{Y})_*\mathcal{O}_{\Theta_R \setminus 0})\\ &\cong \mathrm{MOR}_{\mathcal{O}_{\mathcal{Y}}-\mathrm{alg}}(f_*\mathcal{O}_{\mathcal{X}}, (\Theta_R \to \mathcal{Y})_*\mathcal{O}_{\Theta_R})\\ &\cong \mathrm{MOR}_{\mathcal{Y}}(\Theta_R, \mathcal{X}).\end{aligned}$$

The case of $\mathsf{S}$-completeness is identical. $\square$

**Proposition 6.8.14.** *If $G$ is a reductive group over an algebraically closed field $\Bbbk$, then every quotient stack* $[\operatorname{Spec} A/G]$ *is $\Theta$-complete and $\mathsf{S}$-complete.*

*Proof.* We first show that $\mathbf{B}\,\mathrm{GL}_n$ is $\Theta$-complete. A morphism $\Theta_R \setminus 0 \to \mathcal{X}$ corresponds to a vector bundle $E$ on $\Theta_R \setminus 0$. The algebraic stack $\Theta_R$ is regular and $0 \in \Theta_R$ is a codimension 2 point. If $\widetilde{E}$ is a coherent sheaf on $\Theta_R$ extending $E$, then the double dual $\widetilde{E}^{\vee\vee}$ is a vector bundle extending $E$. (In fact the pushforward of $E$ along $\Theta_R \setminus 0 \hookrightarrow \Theta_R$ is a vector bundle.) This provides the desired extension $\Theta_R \to \mathcal{X}$. As $G$ is affine, we can choose a faithful representation $G \subset \mathrm{GL}_n$. As $G$ is reductive, the quotient $\mathrm{GL}_n/G$ is affine by Matushima's Theorem (C.4.9). Using the cartesian diagram

$$\begin{array}{ccc}\mathrm{GL}_n/G & \longrightarrow & \operatorname{Spec}\Bbbk\\ \big\downarrow & \square & \big\downarrow\\ \mathbf{B}G & \longrightarrow & \mathbf{B}\,\mathrm{GL}_n\end{array}$$

and smooth descent, we see that $\mathbf{B}G \to \mathbf{B}\,\mathrm{GL}_n$ is affine. We conclude that $\mathbf{B}G$ and $[\operatorname{Spec} A/G]$ are $\Theta$-complete by Lemma 6.8.13. $\square$

As a result, we see that the hypotheses of $\Theta$-completeness and $\mathsf{S}$-completeness in Theorem 6.8.1 are necessary.

**Corollary 6.8.15.** *Let $\mathcal{X}$ be an algebraic stack of finite type over an algebraically closed field $\Bbbk$ with affine diagonal. If $\pi\colon \mathcal{X} \to X$ be a good moduli space, then $\mathcal{X}$ is $\Theta$-complete. Moreover, $\mathcal{X}$ is $\mathsf{S}$-complete if and only if $X$ is separated.*

*Proof.* For a $\Bbbk$-algebra $A$, the map $\Theta_A \to \operatorname{Spec} A$ is a good moduli space, and thus every map $\Theta_A \to X$ factors through $\operatorname{Spec} A$ by the universality of good moduli spaces (Theorem 6.3.5(4)). If $R$ is a DVR with fraction field $K$, then every map $\Theta_R \to X$ (resp. $\Theta_K \to X$) factors through $\operatorname{Spec} R$ (resp. $\operatorname{Spec} K$). To see that $\mathcal{X}$ is $\Theta$-complete, it therefore suffices to find a lift of every commutative diagram

$$\begin{array}{ccc}\Theta_R \setminus 0 & \longrightarrow & \mathcal{X}\\ \big\downarrow & \nearrow & \big\downarrow{\scriptstyle \pi}\\ \operatorname{Spec} R & \longrightarrow & X\end{array}$$

of solid arrows. By the Local Structure for Good Moduli Spaces (6.5.3), there exists an étale morphism $\operatorname{Spec} B \to X$ containing the image of $\operatorname{Spec} R$ such that $\mathcal{X} \times_X \operatorname{Spec} B \cong [\operatorname{Spec} A/G]$ with $G$ linearly reductive and $B = A^G$. Since $\operatorname{Spec} R \to X$ lifts to $\operatorname{Spec} B$ after an extension of DVRs and since $\Theta$-completeness can be checked after an extension (Lemma 6.8.11), we are reduced to the case of $[\operatorname{Spec} A/G]$. This is Proposition 6.8.14.

If $X$ is separated, then $X$ is $\mathsf{S}$-complete as $\phi_R \smallsetminus 0 = \operatorname{Spec} R \cup_{\operatorname{Spec} K} \operatorname{Spec} R \to X$ factors through $\operatorname{Spec} R$ by the valuative criterion for separatedness. The above argument can be repeated to show that $\mathcal{X}$ is $\mathsf{S}$-complete. Conversely, suppose $f, g\colon \operatorname{Spec} R \to X$ are two maps such that $f|_K = g|_K$. After possibly an extension of $R$, we may choose a lift $\operatorname{Spec} K \to \mathcal{X}$ of $f|_K = g|_K$. Since $\mathcal{X} \to X$ is universally closed (Theorem 6.3.5(1)), after possibly further extensions of $R$, we may choose lifts $\widetilde{f}, \widetilde{g}\colon \operatorname{Spec} R \to \mathcal{X}$ of $f, g$ such that $\widetilde{f}|_K \cong \widetilde{g}|_K$ by the Valuative Criterion for Universal Closedness (3.8.5). Since $\mathcal{X}$ is $\mathsf{S}$-complete, we can extend $\widetilde{f}$ and $\widetilde{g}$ to a morphism $\phi_R \to \mathcal{X}$. As $\phi_R \to \operatorname{Spec} R$ is a good moduli space and hence universal for maps to algebraic spaces, the morphism $\phi_R \to \mathcal{X}$ descends to a unique morphism $\operatorname{Spec} R \to X$ which necessarily must be equal to both $f$ and $g$. We conclude that $X$ is separated by the Valuative Criterion for Separatedness. $\square$

**Lemma 6.8.16.** *Let $\mathcal{X}$ be a noetherian algebraic stack with affine and quasi-finite diagonal. If $R$ is a complete DVR, every map $\Theta_R \to \mathcal{X}$ (resp. $\phi_R \to \mathcal{X}$) factors through $\Theta_R \to \operatorname{Spec} R$ (resp. $\phi_R \to \operatorname{Spec} R$).*

*Proof.* Since good moduli spaces are universal for maps to algebraic spaces, we already know the claim when $\mathcal{X}$ is an algebraic space. In fact, we will reduce to the case when $\mathcal{X}$ is affine, in which case the factorizations follow easily from the fact that $\Gamma(\Theta_R, \mathcal{O}_{\Theta_R}) = \Gamma(\phi_R, \mathcal{O}_{\phi_R}) = R$.

Let $x \in \mathcal{X}(\kappa)$ be the image of $0 \in \Theta_R$. Since $\mathbb{G}_m$ has no nontrivial finite quotients, the induced map $\mathbb{G}_m \to G_x$ on stabilizers is trivial. By Proposition 4.2.14, we may find a smooth presentation $U \to \mathcal{X}$ from an affine scheme together with a lift $u \in U(\kappa)$ of $x$. The map $\mathbf{B}\mathbb{G}_{m,\kappa} \to \mathcal{X}$ factors through $u\colon \operatorname{Spec} \kappa \to U$ and thus lifts to a map $\mathbf{B}\mathbb{G}_{m,\kappa} \to \operatorname{Spec} \kappa \xrightarrow{u} U$. Letting $\mathcal{T}_n$ be the nth nilpotent thickening of $\mathbf{B}\mathbb{G}_{m,\kappa} \hookrightarrow \Theta_R$, deformation theory (Proposition 6.5.8) implies that we may find compatible lifts $\mathcal{T}_n \to U$ of $\mathcal{T}_n \hookrightarrow \Theta_R \to \mathcal{X}$. By Coherent Tannaka Duality (6.4.8), there is an extension $\Theta_R \to U$. Since $\Theta_R \to U$ factors through $\operatorname{Spec} R$, so does $\Theta_R \to \mathcal{X}$. $\square$

**Proposition 6.8.17.** *Every noetherian algebraic stack $\mathcal{X}$ with affine and quasi-finite diagonal (e.g. a Deligne–Mumford stack with affine diagonal) is $\Theta$-complete. Moreover, $\mathcal{X}$ is $\mathsf{S}$-complete if and only if it is separated.*

*Proof.* By Lemma 6.8.11, $\Theta$-completeness and $\mathsf{S}$-completeness can be tested on a complete DVR $R$. Lemma 6.8.16 implies that that $\mathcal{X}$ is $\Theta$-complete and also implies that $\mathcal{X}$ is $\mathsf{S}$-complete if only if every diagram

$$\begin{tikzcd} \operatorname{Spec} R \bigcup_{\operatorname{Spec} K} \operatorname{Spec} R \arrow[r] \arrow[d] & \mathcal{X} \\ \operatorname{Spec} R \arrow[ur, dashed] & \end{tikzcd}$$

has a lift, which is the usual valuative criterion for separatedness. $\square$

**Example 6.8.18.** The examples in Examples 6.8.5 and 6.8.6 of non-separated Deligne–Mumford stacks are not $\mathsf{S}$-complete.

### 6.8.3 Examples of $\Theta$- and S-completeness

We discuss the valuative criteria of $\Theta$-completeness and $\mathsf{S}$-completeness for quotient stacks, stacks of coherent sheaves, and the stack of all curves.

#### Quotient stacks

By Proposition 6.6.30, a map $\Theta \to [U/G]$ is classified by a point $u \in U$ and a one-parameter subgroup $\lambda\colon \mathbb{G}_m \to G$ such that $\lim_{t\to 0} \lambda(t) \cdot u$ exists. We apply this to provide a geometric characterization of $\Theta$-completeness for quotient stacks. Recall that the attractor locus $U_\lambda^+$ represents the functor $\mathrm{Mor}_{\Bbbk}^{\mathbb{G}_m}(\mathbb{A}^1, U)$ (Theorem 6.6.7). The evaluation map $\mathrm{ev}_1\colon U_\lambda^+ \to U$ is defined by sending $f\colon \mathbb{A}^1 \to U$ to $f(1)$.

**Proposition 6.8.19.** *Let $G$ be a smooth linearly reductive group over an algebraically closed field $\Bbbk$, and $U$ be a separated algebraic space of finite type over $\Bbbk$ with an action of $G$. Then*

| | | |
|---|---|---|
| *$[U/G]$ is $\Theta$-complete* | $\iff$ | *for every map $u\colon \mathrm{Spec}\, R \to U$ from a complete DVR over $\Bbbk$ with algebraically closed residue field and one-parameter subgroup $\lambda\colon \mathbb{G}_m \to G$ such that $\lim_{t\to 0} \lambda(t) \cdot u_K \in U(K)$ exists, then $\lim_{t\to 0} \lambda(t) \cdot u \in U(R)$ also exists;* |
| | $\iff$ | *for every one-parameter subgroup $\lambda\colon \mathbb{G}_m \to G$, the morphism $\mathrm{ev}_1\colon U_\lambda^+ \to U$ is a closed immersion.* |

*Proof.* Since $G$ is linearly reductive, $\mathbf{B}G$ is $\Theta$-complete (Proposition 6.8.14). Therefore $\Theta$-completeness of $[U/G]$ is equivalent to the existence of a lift in every diagram

$$\begin{tikzcd} \Theta_R \setminus 0 \arrow[r] \arrow[d, hook] & {[U/G]} \arrow[d] \\ \Theta_R \arrow[r] \arrow[ur, dashed] & \mathbf{B}G \end{tikzcd} \tag{6.8.6}$$

where $R$ is a complete DVR with algebraically closed residue field (Lemma 6.8.11). By Proposition 6.6.30, the map $\Theta_R \to \mathbf{B}G$ corresponds to a one-parameter subgroup $\lambda\colon \mathbb{G}_m \to G$ while $\Theta_R \setminus 0 \to [U/G]$ corresponds to a map $u\colon \mathrm{Spec}\, R \to U$ such that $\lim_{t\to 0} \lambda(t) \cdot u_K \in U(K)$ exists. In other words, we have a commutative diagram

$$\begin{tikzcd} \mathrm{Spec}\, K \arrow[r] \arrow[d] & U_\lambda^+ \arrow[d, "\mathrm{ev}_1"] \\ \mathrm{Spec}\, R \arrow[r, "u"] \arrow[ur, dashed] & U \end{tikzcd} \tag{6.8.7}$$

of solid arrows. A lift of (6.8.6) corresponds to the existence of $\lim_{t\to 0} \lambda(t) \cdot u \in U(R)$ or equivalently to a lift of (6.8.7). Since $\mathrm{ev}_1\colon U_\lambda^+ \to U$ is a monomorphism of finite type, it is closed immersion if and only if it is proper or equivalently satisfies the existence part of the valuative criterion. $\square$

**Example 6.8.20.** When $U = \mathrm{Spec}\, A$ is affine, a one-parameter subgroup $\lambda\colon \mathbb{G}_m \to G$ induces a grading $A = \bigoplus_{d\in\mathbb{Z}} A_d$, and $U_\lambda^+$ is represented by $V(\sum_{d<0} A_d)$ (Exercise 6.6.5). We see thus that $\mathrm{ev}_1\colon U_\lambda^+ \hookrightarrow U$ is a closed immersion; this recovers the fact that $[U/G]$ is $\Theta$-complete (Proposition 6.8.14).

**Example 6.8.21.** We can use this criterion to see that Examples 6.8.2 to 6.8.4 are not $\Theta$-complete. For $[\mathbb{P}^1/\mathbb{G}_m]$ with action $t \cdot [x : y] = [tx : y]$, taking $\lambda = \mathrm{id}$ we have that $(\mathbb{P}^1)^+_\lambda = \mathbb{A}^1 \coprod \{\infty\}$. For the quotient $[C/\mathbb{G}_m]$ of the nodal cubic $C$ with normalization $\mathbb{P}^1 \to C$ identifying 0 and $\infty$, then $C^+_\infty = \mathbb{P}^1 \setminus \infty$ for $\lambda = \mathrm{id}$. Finally, for $[X/\mathbb{G}_m]$ with $X = \mathbb{A}^2 \smallsetminus 0$ with action $t \cdot (x, y) = (tx, y)$, then $X^+_\lambda = \{y \neq 0\}$ for $\lambda = \mathrm{id}$.

**Example 6.8.22.** We can also provide a interpretation using the algebraic stack $\underline{\mathrm{Mor}}(\Theta, [X/G])$ of morphisms which decomposes as a disjoint union $\coprod_\lambda [X^+_\lambda/P_\lambda]$ where $\lambda$ varies over conjugation classes of one-parameter subgroups $\lambda \colon \mathbb{G}_m \to G$ (Remark 6.6.32). The evaluation morphism $\mathrm{ev}_0 \colon [X^+_\lambda/P_\lambda] \to [X/G]$ is induced by the inclusion $X^+_\lambda \to X$. The $\Theta$-completeness of $[X/G]$ corresponds to the properness of the maps $[X^+_\lambda/P_\lambda] \to [X/G]$.

One can also give a criteria for when $[U/G]$ is $\mathsf{S}$-complete in terms of one-parameter subgroups $\lambda \colon \mathbb{G}_m \to G$ and properties of the morphism $\mathrm{Mor}^{\mathbb{G}_m}_{\mathbb{A}^1}(\mathbb{A}^2, U \times \mathbb{A}^1) \to U \times U \times \mathbb{A}^1$, where the maps to $U$ are obtained by restricting along the two maps $\mathbb{A}^1 \to \mathbb{A}^2$ given by $x \mapsto (x, 1)$ and $x \mapsto (1, x)$.

### Stacks of coherent sheaves

Given a projective scheme $X$, let $\underline{\mathrm{Coh}}(X)$ denote the algebraic stack of coherent sheaves on $X$ (see Exercise 3.1.22). Recall that maps $\Theta \to \underline{\mathrm{Coh}}(X)$ correspond to filtrations (Proposition 6.6.33).

**Proposition 6.8.23.** *For every projective scheme $X$ over an algebraically closed field $\Bbbk$, the algebraic stack $\underline{\mathrm{Coh}}(X)$ is $\Theta$-complete and $\mathsf{S}$-complete.*

*Proof.* Given a DVR $R$, Proposition 6.6.33 implies that a map $\Theta_R \smallsetminus 0 \to \underline{\mathrm{Coh}}(X)$ corresponds to a coherent sheaf $E$ on $X_R$ flat over $R$ and a $\mathbb{Z}$-graded filtration $F_\bullet \colon \cdots F_{i-1} \subset F_i \subset \cdots \subset E_K$ such that $F_i = E_K$ for $i \gg 0$, $F_i = 0$ for $i \ll 0$, and $F_i/F_{i-1}$ is flat over $R$. Viewing $E$ is a subsheaf of $E_K$, we define $E_i := F_i \cap E$ as the intersection in $E_K$. Since $E_i/E_{i-1}$ is a subsheaf of $F_i/F_{i-1}$, it is torsion free, hence flat as an $R$-module. The filtration $E_\bullet$ defines an extension $\Theta_R \to \underline{\mathrm{Coh}}(X)$. (Aside: this is exactly the argument for the valuative criterion of properness of the Quot scheme (Proposition 1.4.2). Note also that if we let $j \colon \Theta_R \smallsetminus 0 \hookrightarrow \Theta_R$, let $j_x \colon \operatorname{Spec} R \hookrightarrow \Theta_R$ and let $j_\pi \colon \Theta_K \to \Theta_R$ denote the open immersions and we let $\mathcal{E}$ be the coherent sheaf on $C \times (\Theta_R \setminus 0)$ denoting the union of $E$ and $F_\bullet$, then the extension is given by $(\mathrm{id} \times j)_* \mathcal{E} = j_{x,*}E \cap j_{\pi,*}F_\bullet = E[x^{\pm 1}] \cap F_\bullet = E_\bullet$, where $E[x^{\pm 1}]$ is the $\mathbb{Z}$-graded filtration given by placing $E$ in every degree.)

For $\mathsf{S}$-completeness, suppose we are given a map $\phi_R \smallsetminus 0 \to \underline{\mathrm{Coh}}(X)$ corresponding to coherent sheaves $E$ and $F$ flat over $R$ and an isomorphism $\alpha \colon E_K \to F_K$. Recalling the quotient presentation $\phi_R = [\operatorname{Spec}(R[s,t]/(st - \pi))/\mathbb{G}_m]$, we have several natural open immersions: $j \colon \phi_R \smallsetminus 0 \hookrightarrow \phi_R$, $j_s, j_t \colon \operatorname{Spec} R \hookrightarrow \phi_R$ (with $s \neq 0$ and $t \neq 0$), and $j_{st} \colon \operatorname{Spec} K \to \phi_R$ (with $st \neq 0$). We compute the pushforward as the equalizer

$$\begin{array}{ccccccc} 0 \longrightarrow (\mathrm{id} \times j)_* \mathcal{E} & \longrightarrow & (\mathrm{id} \times j_s)_* E \oplus (\mathrm{id} \times j_t)_* F & \longrightarrow & (\mathrm{id} \times j_{st})_* F_K \\ & & (a, b) & \longmapsto & a - \alpha(b). \end{array}$$

The pushforwards can be computed as graded modules over $R[s,t]/(st-\pi)$:

$$\begin{aligned}(\mathrm{id}\times j_{st})_*F_K &= F_K \otimes_R R[t^{\pm 1}] = \bigoplus_{n\in\mathbb{Z}} F_K t^n,\\ (\mathrm{id}\times j_s)_*E &= E\otimes_R R[t^{\pm 1}] = \bigoplus_{n\in\mathbb{Z}} Et^n,\\ (\mathrm{id}\times j_t)_*F &= F\otimes_R R[s^{\pm 1}] \cong \bigoplus_{n\in\mathbb{Z}} (\pi^{-n}\cdot F)t^n \subset (\mathrm{id}\times j_{st})_*F_K\end{aligned}$$

where we've used that $s = t^{-1}\pi$. Thus

$$j_*\mathcal{E} \cong \bigoplus_{n\in\mathbb{Z}} \big(E\cap(\pi^{-n}\cdot F)\big)t^n \subset (\mathrm{id}\times j_{st})_*F_K.$$

Each $R$-module $E\cap(\pi^{-n}\cdot F)\subset E$ is finitely generated since $E$ is. Moreover, the ascending chain $\cdots\subset E\cap(\pi^{-n}\cdot F)\subset E\cap(\pi^{-n-1}\cdot F)\subset\cdots$ terminates to $E$ and it follows that $j_*\mathcal{E}$ is coherent. To show that $j_*\mathcal{E}$ is flat over $\phi_R$, we only need to check that it is flat over 0. By the Local Criterion for Flatness (Theorem A.2.5), we need to show that $\mathrm{Tor}_1^A(A/\mathfrak{m}, j_*\mathcal{E}) = 0$ where $A = R[s,t]/(st-\pi)$ and $\mathfrak{m} = (s,t)$. The Koszul complex gives a resolution of the residue field $\kappa = A/\mathfrak{m} = R/\pi$:

$$0\to A\xrightarrow{(t,-s)} A\oplus A\xrightarrow{(s,t)} A\to\kappa\to 0.$$

Tensoring with $j_*\mathcal{E}$ yields a complex

$$0\to j_*\mathcal{E}\xrightarrow{(t,-s)} j_*\mathcal{E}\oplus j_*\mathcal{E}\xrightarrow{(s,t)} j_*\mathcal{E}. \tag{6.8.8}$$

The pushforward of the exact sequence

$$0\to\mathcal{O}_{\phi_R\smallsetminus 0}\xrightarrow{(t,-s)}\mathcal{O}_{\phi_R\smallsetminus 0}\oplus\mathcal{O}_{\phi_R\smallsetminus 0}\xrightarrow{(s,t)}\mathcal{O}_{\phi_R\smallsetminus 0}\to 0$$

along $\mathrm{id}\times j\colon C\times\phi_R\smallsetminus 0\hookrightarrow C\times\phi_R$ is a left exact sequence of vector bundles and tensoring with $j_*\mathcal{E}$ yields a left exact sequence which identified with (6.8.8). Thus $\mathrm{Tor}_1^A(A/\mathfrak{m}, j_*\mathcal{E}) = 0$. □

The description in Proposition 6.6.33 interpreting maps from $\Theta$ as filtrations allows us to prove simple criteria for an open substack $\mathcal{U}\subset\underline{\mathrm{Coh}}(X)$ to be $\Theta$-complete or $\mathsf{S}$-complete. We call two $\mathbb{Z}$-graded filtrations

$$E_\bullet : 0\subset\cdots\subset E_{i-1}\subset E_i\subset E_{i+1}\subset\cdots\subset E$$

and

$$F^\bullet : F\supset\cdots\supset F^{i-1}\supset F^i\supset F^{i+1}\supset\cdots\supset 0$$

are *opposite* if $E_i/E_{i-1}\cong F^i/F^{i+1}$ for all $i$. Observe that $F_\bullet$ defined by $F_i = F^{-i}$ is a $\mathbb{Z}$-graded filtration with the same indexing as $E_\bullet$ and being opposite means that $\mathrm{gr}\,E_\bullet$ is isomorphic as a $\mathbb{Z}$-graded sheaf to $\mathrm{gr}\,F_\bullet$ with the opposite grading. A map $[(\mathrm{Spec}\,\Bbbk[x,y]/xy)/\mathbb{G}_m]\to\underline{\mathrm{Coh}}(X)$, where $t\cdot(x,y) = (tx, t^{-1}y)$, is the same data as two opposite filtration $E_\bullet$ and $F^\bullet$ such that $E_i = 0$ and $F^i = F$ for $i\ll 0$, and $E_i = E$ and $F^i = 0$ for $i\gg 0$; in this case, under this map $(1,0)\mapsto E$, $(0,1)\mapsto F$, and $(0,0)\mapsto\mathrm{gr}\,E_\bullet$.

**Proposition 6.8.24.** *Let $C$ be a smooth, connected, and projective scheme over an algebraically closed field $\Bbbk$, and let $\mathcal{U}\subset\underline{\mathrm{Coh}}(C)$ be an open substack.*

(1) *The substack $\mathcal{U}$ is $\Theta$-complete if and only if for every DVR $R$ (with fraction field $K$ and residue field $\kappa$), coherent sheaf $E$ on $C_R$ flat over $R$, and $\mathbb{Z}$-graded filtration $E_\bullet$ with $E_i = 0$ for $i \ll 0$, $E_i = E$ for $i \gg 0$ and with each $E_i/E_{i-1}$ flat over $R$, then if $E$ and $\operatorname{gr}(E_\bullet|_K)$ are in $\mathcal{U}$, so is $\operatorname{gr}(E_\bullet|_\kappa)$.*

(2) *The substack is $\mathsf{S}$-complete if and only if for every pair of opposite filtrations $E_\bullet$ and $F^\bullet$ of $E, F \in \mathcal{U}(\Bbbk)$, the associated graded $\operatorname{gr} E_\bullet$ is in $\mathcal{U}$.*

**Remark 6.8.25.** For a projective scheme of arbitrary dimension, Part (1) and the ($\Leftarrow$) implication in (2) hold with the same proof.

*Proof.* Since we already know that $\underline{\mathrm{Coh}}(C)$ is $\mathsf{S}$-complete and $\Theta$-complete, the valuative criteria for $\mathcal{U}$ are equivalent to the existence of lifts for all commutative diagrams

$$\begin{tikzcd} \Theta_R \smallsetminus 0 \arrow[r] \arrow[d, hook] & \mathcal{U} \arrow[d, hook] \\ \Theta_R \arrow[r] \arrow[ur, dashed] & \underline{\mathrm{Coh}}(C) \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \phi_R \smallsetminus 0 \arrow[r] \arrow[d, hook] & \mathcal{U} \arrow[d, hook] \\ \phi_R \arrow[r] \arrow[ur, dashed] & \underline{\mathrm{Coh}}(C) \end{tikzcd}$$

where $R$ is a DVR. In other words, we need to show that the images of 0 under the unique fillings $\Theta_R \to \underline{\mathrm{Coh}}(C)$ and $\phi_R \to \underline{\mathrm{Coh}}(C)$ are contained in $\mathcal{U}$. Therefore (1) holds as the image of 0 under $\Theta_R \to \underline{\mathrm{Coh}}(C)$ is $\operatorname{gr}(E_\bullet|_\kappa)$.

For the ($\Leftarrow$) implication in (2), the restriction of $\phi_R \to \underline{\mathrm{Coh}}(C)$ along $\pi = 0$ yields a map $[\operatorname{Spec}(\Bbbk[x,y]/xy)/\mathbb{G}_m] \to \underline{\mathrm{Coh}}(C)$ corresponding to opposite filtrations $E_\bullet$ and $F^\bullet$. If we $\operatorname{gr} E_\bullet \in \mathcal{U}(\Bbbk)$, then the image of $\phi_R \to \underline{\mathrm{Coh}}(C)$ is contained in $\mathcal{U}$. Conversely, let $[\operatorname{Spec}(\Bbbk[x,y]/xy)/\mathbb{G}_m] \to \underline{\mathrm{Coh}}(X)$ be a map such that the images of $(1,0)$ and $(0,1)$ are in $\mathcal{U}$ but the image of $(0,0)$ is not in $\mathcal{U}$. Let $\mathcal{X}_n$ be the $n$th nilpotent thickening of the closed immersion $[\operatorname{Spec}(\Bbbk[x,y]/xy)/\mathbb{G}_m] \hookrightarrow \phi_R$. Since the obstruction to lifting a coherent sheaf $E$ lies in the second coherent cohomology of $\mathcal{X}_0$ and since $\mathcal{X}_0$ is cohomologically affine, deformation theory and Coherent Tannaka Duality (6.4.8) yield an extension $\phi_R \to \underline{\mathrm{Coh}}(X)$ with the image of $\phi_R \smallsetminus 0$ contained in $\mathcal{U}$. $\square$

**Remark 6.8.26.** If the genus of $C$ is at least 2, then the stack of vector bundles $\operatorname{Bun}(C)$ is not $\Theta$-complete nor $\mathsf{S}$-complete. Let $p \in C$ be a point defined by the vanishing of a section $s \in \Gamma(C, \mathcal{O}(p))$, and let $I \subset \mathcal{O}_{C_R}$ be the ideal sheaf of $(p, 0) \in C \times \operatorname{Spec} R$. The injection $(s, -\pi)\colon \mathcal{O}_{C_R}(-p) \hookrightarrow \mathcal{O}_{C_R} \oplus \mathcal{O}_{C_R}(p)$ has quotient $I$, which is torsion free, hence flat over $R$, but is not a vector bundle. By Proposition 6.8.24, we see that $\operatorname{Bun}(C)$ is not $\Theta$-complete.

Let $L$ and $M$ be line bundles on $C$, and let $p \in C$ be a point such that $\operatorname{Ext}^1_{\mathcal{O}_C}(M, L(p))$ and $\operatorname{Ext}^1_{\mathcal{O}_C}(L, M(p))$ are nonzero; if $L$ and $M$ have the same degree, then a Riemann–Roch calculation shows that both $\operatorname{Ext}^1$ groups are nonzero. Let $Q$ (resp. $Q'$) be a nontrivial extension of $M$ by $L(p)$ (resp. $L$ by $M(p)$). Then

$$E_\bullet : 0 \subset L \subset L(p) \subset Q \quad \text{and} \quad F^\bullet : Q' \supset M(p) \supset M \supset 0$$

define opposite filtrations where $E_0 = L$ and $F^0 = Q'$. The associated graded $\operatorname{gr} E_\bullet = L \oplus \kappa(p) \oplus M$ is not a vector bundle, and thus $\operatorname{Bun}(C)$ is not $\mathsf{S}$-complete by Proposition 6.8.24.

We will apply the above criteria later to verify that the stack $\operatorname{Bun}^{\mathrm{ss}}_{r,d}(C)$ of semistable vector bundles on a smooth, connected, and projective curve is both $\Theta$-complete and $\mathsf{S}$-complete.

**Stack of all curves**

The stacks $\mathcal{M}_g$ and $\overline{\mathcal{M}}_g$ of smooth and stable curves are both $\Theta$-complete and $\mathsf{S}$-complete as they are separated Deligne–Mumford stacks. While maps from $\Theta$ to the stack of all curves correspond to test configurations (Proposition 6.6.34), there is unfortunately no known simple criteria—similar to the above criteria for quotient stacks and stacks of coherent sheaves—to verify whether a given substack of the stack $\mathcal{M}_g^{\mathrm{all}}$ of all curves is $\Theta$-complete or $\mathsf{S}$-complete.

### 6.8.4 $\Theta$-completeness and $\Theta$-surjectivity

The property that a morphism $\mathcal{X} \to \mathcal{Y}$ sends closed points to closed points is not stable under base change (see Examples 6.8.2 and 6.8.3). We introduce a stronger and better behaved property called $\Theta$-surjectivity. The main result of this section is that an étale quotient presentation $([\operatorname{Spec} A/G_x], w) \to (\mathcal{X}, x)$ is $\Theta$-surjective in an open neighborhood of $w$ as long as $\mathcal{X}$ is $\Theta$-complete (prop:theta-surjective-in-open-neighborhood). As motivated in §6.8.1, this result will be crucial in proving the main existence theorem (Theorem 6.8.1) of this section.

**Definition 6.8.27.** Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks and $x \in \mathcal{X}(\Bbbk)$ be a geometric point. We say that $f$ is *$\Theta$-surjective at $x$* if every diagram

$$\begin{tikzcd} \operatorname{Spec}\Bbbk \arrow[r, "x"] \arrow[d, hook, "1"] & \mathcal{X} \arrow[d, "f"] \\ \Theta_{\Bbbk} \arrow[r] \arrow[ur, dashed] & \mathcal{Y} \end{tikzcd} \tag{6.8.9}$$

has a lift. We say that $f$ is *$\Theta$-surjective* if it is $\Theta$-surjective at every geometric point.

This notion is clearly stable under base change. Every morphism $f\colon \mathcal{X} \to \mathcal{Y}$ of noetherian algebraic stacks where $\mathcal{Y}$ has affine and quasi-finite diagonal is $\Theta$-surjective since in this case every map $\Theta_{\Bbbk} \to \mathcal{Y}$ factors through $\operatorname{Spec}\Bbbk$ (Lemma 6.8.16). The next lemma gives conditions for when the lift is unique and when the definition is independent of the choice of geometric point.

**Lemma 6.8.28.** *Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a separated, representable, and finite type morphism of noetherian algebraic stacks.*

*(1) Every lift of (6.8.9) is unique.*

*(2) If $f$ is $\Theta$-surjective at a geometric point $x \in \mathcal{X}(\Bbbk)$, then $f$ is $\Theta$-surjective at every other geometric point $x' \in \mathcal{X}(\Bbbk')$ representing the same point in $|\mathcal{X}|$ as $x$.*

*Proof.* Part (1) follows from descent and the valuative criterion for separatedness. To show (2), it suffices to show that given an extension $\Bbbk \to \Bbbk'$ of algebraically closed fields, a lift $\Theta_{\Bbbk'} \to \mathcal{X}$ implies the existence of a lift $\Theta_{\Bbbk} \to \mathcal{X}$. We write $\Bbbk' = \bigcup_\lambda A_\lambda$ as a union of finitely generated $\Bbbk$-subalgebras. By Limit Methods (§A.6), there exists a lift $\Theta_{A_\lambda} \to \mathcal{X}$ of $\operatorname{Spec} A_\lambda \to \mathcal{X}$. Restricting along a closed point of $\operatorname{Spec} A_\lambda$ provides a lift over $\Bbbk$. $\square$

**Proposition 6.8.29.** *Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks, each of finite type over an algebraically closed field $\Bbbk$ with affine diagonal. Suppose that the closed points of $\mathcal{Y}$ have linearly reductive stabilizer. If $f$ is $\Theta$-surjective, then $f$ sends closed points to closed points.*

*Proof.* Let $x \in \mathcal{X}$ be a closed point. Let $f(x) \rightsquigarrow y_0$ be a specialization to a closed point. By Corollary 6.6.29, this specialization can be realized by a map $\Theta \to \mathcal{Y}$. Since $f$ is $\Theta$-surjective, this can be lifted to a map $g\colon \Theta \to \mathcal{X}$ with $g(1) = x$. But $x \in \mathcal{X}$ is a closed point, so this lift must correspond to the trivial specialization $x \rightsquigarrow x$. It follows that $f(x) = y_0$ is a closed point. $\square$

**Remark 6.8.30.** The converse is not true. In Example 6.8.3, where $C$ is the nodal cubic with $\mathbb{G}_m$-action, the étale morphism $[\operatorname{Spec}(\Bbbk[x,y]/(xy))/\mathbb{G}_m] \to [C/\mathbb{G}_m]$ sends closed points to closed points but is not $\Theta$-surjective.

**Proposition 6.8.31.** *Let $\mathcal{X}$ be an algebraic stack of finite type over an algebraically closed field $\Bbbk$ with affine diagonal such that the closed points of $\mathcal{X}$ have linearly reductive stabilizers. Let $x \in \mathcal{X}$ be a closed point, let $f\colon ([\operatorname{Spec} A/G_x], w) \to (\mathcal{X}, x)$ be an affine étale morphism inducing an isomorphism of stabilizer groups at $w$, and let $\pi\colon [\operatorname{Spec} A/G_x] \to \operatorname{Spec} A^{G_x}$. If $\mathcal{X}$ is $\Theta$-complete, there exists an open affine neighborhood $U \subset \operatorname{Spec} A^{G_x}$ of $\pi(w)$ such that $f|_{\pi^{-1}(U)}\colon \pi^{-1}(U) \to \mathcal{X}$ is $\Theta$-surjective.*

*Proof.* Let $\mathcal{W} = [\operatorname{Spec} A/G_x]$ and define $\Sigma_f \subset |\mathcal{W}|$ as the set of points $y \in \mathcal{W}$ such that $f$ is $\Theta$-surjective at $y$. We first show that $\Sigma_f \subset \mathcal{W}$ is open if $\mathcal{X} \cong [\operatorname{Spec} B/G]$ with $G$ linearly reductive. Zariski's Main Theorem (6.1.10) provides a factorization

$$f\colon \mathcal{W} \overset{j}{\hookrightarrow} \widetilde{\mathcal{X}} \xrightarrow{\nu} \mathcal{X}$$

where $j$ is an open immersion and $\nu$ is a finite morphism. By Lemma 6.8.13, $[\operatorname{Spec} B/G]$ is $\Theta$-complete, and by Proposition 6.8.14, $\widetilde{\mathcal{X}}$ is also $\Theta$-complete. As $\nu$ is finite, $\Sigma_j = \Sigma_f$ and we may assume that $f$ is an open immersion. Let $\mathcal{Z} \subset \mathcal{X}$ be the reduced complement of $\mathcal{W}$ and let $\pi\colon \mathcal{X} \to \operatorname{Spec} B^G$ denote the good moduli space. We claim that $|\mathcal{W}| \smallsetminus \Sigma_f = \pi^{-1}(\pi(|\mathcal{Z}|)) \cap |\mathcal{W}|$. The inclusion "$\subset$" is clear: the morphism $\mathcal{X} \smallsetminus \pi^{-1}(\pi(|\mathcal{Z}|)) \hookrightarrow \mathcal{X}$ is the base change of the $\Theta$-surjective morphism $X \smallsetminus \pi(|\mathcal{Z}|) \hookrightarrow X$ of algebraic spaces. Conversely, let $y \in \pi^{-1}(\pi(|\mathcal{Z}|)) \cap |\mathcal{W}|$ represented by a geometric point $\operatorname{Spec} \mathbb{F} \to \mathcal{X}$. Let $z \in |\mathcal{Z}_{\mathbb{F}}|$ be the unique closed point in the closure of $y \in |\Theta_{\mathbb{F}}|$ and let $\Theta_{\mathbb{F}} \to \mathcal{X}_{\mathbb{F}}$ be a morphism representing the specialization $y \rightsquigarrow z$ (Corollary 6.6.29). Since $\Theta_{\mathbb{F}} \to \mathcal{X}$ does not lift to $\mathcal{W}$, $y \notin \Sigma_f$.

We now claim that $\Sigma_f \subset \mathcal{W}$ is constructible. Use the Local Structure Theorem (6.5.1) to choose an affine, étale, and surjective morphism $g\colon \mathcal{X}' = [\operatorname{Spec} B/G] \to \mathcal{X}$ with $G$ linearly reductive. Let $\mathcal{W}' = \mathcal{W} \times_{\mathcal{X}} \mathcal{X}'$ with projections $g'\colon \mathcal{W}' \to \mathcal{W}$ and $f'\colon \mathcal{W}' \to \mathcal{X}'$. Since we already know that $\Sigma_{f'}$ is open, the claim follows from Chevalley's Theorem (3.3.29) once we show that $\mathcal{W} \setminus \Sigma_f = g'(\mathcal{W}' \setminus \Sigma_{f'})$. To see this, it suffices to show that for an algebraically closed field $K$, every map $h\colon \Theta_{\mathbb{F}} \to \mathcal{X}$ lifts to a map $h'\colon \Theta_{\mathbb{F}} \to \mathcal{X}'$. Let $x' \in \mathcal{X}'(\mathbb{F})$ be a preimage of $h(0) \in \mathcal{X}(\mathbb{F})$. Since $g$ is representable and étale, the induced map $G_{x'} \to G_{h(0)}$ on stabilizers is injective with finite cokernel. Thus the map $\mathbb{G}_{m,\mathbb{F}} \to G_{h(0)}$ on stabilizers induced by $h\colon \Theta_{\mathbb{F}} \to \mathcal{X}$ factors through $G_{x'}$. We may therefore lift the map $h|_{\mathbf{B}\mathbb{G}_{m,\mathbb{F}}}$ to a map $\mathbf{B}\mathbb{G}_{m,\mathbb{F}} \to \mathcal{X}'$. Letting $\mathcal{X}_n$ be the $n$th nilpotent thickening of $\mathbf{B}\mathbb{G}_{m,\mathbb{F}} \hookrightarrow \Theta_{\mathbb{F}}$, there are compatible lifts $\mathcal{X}_n \to \mathcal{X}'$ of $\mathcal{X}_n \to \mathcal{X}$ by deformation theory (Proposition 6.5.8) which extends to a lift $\Theta_{\mathbb{F}} \to \mathcal{X}'$ by Coherent Tannaka Duality (6.4.8).

Since $\Sigma_f \subset \mathcal{W}$ is constructible and $w \in \Sigma_f$, to show that $\Sigma_f$ is open, it suffices to show that for every generization $\xi \rightsquigarrow w$ of $w$ is contained in $\Sigma_f$. Let $h\colon \operatorname{Spec} R \to \mathcal{W}$ be a morphism from a complete DVR representing the specialization $\xi \rightsquigarrow w$. Letting $K$ and $\kappa$ be the fraction and residue field of $R$, we claim that there exists a lift

(necessarily unique as $f$ is separated)

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r, "h_{\mathbb{F}}"] \arrow[d, hook] & \mathcal{W} \arrow[d, "f"] \\ \Theta_K \arrow[r, "g"] \arrow[ur, dashed, "\widetilde{g}"] & \mathcal{X}. \end{tikzcd} \tag{6.8.10}$$

This claim implies that $f$ is $\Theta$-surjective at $\xi$, i.e. $\xi \in \Sigma_f$. To show the claim, we first apply the $\Theta$-completeness of $\mathcal{X}$ to construct a lift

$$\begin{tikzcd} \Theta_R \smallsetminus 0 \arrow[r, "(f\circ h)\cup g"] \arrow[d, hook] & \mathcal{X} \\ \Theta_R. \arrow[ur, dashed, "q"'] & \end{tikzcd}$$

Since $\mathcal{W} \to \mathcal{X}$ is stabilizer preserving at $w$, we have a lift $\mathbf{B}\mathbb{G}_{m,\kappa} \to \mathcal{W}$ of $q|_{\mathbf{B}\mathbb{G}_{m,\kappa}}$. Since $\Theta_R$ is coherently complete along $\mathbf{B}\mathbb{G}_{m,\kappa}$ (6.4.11), we may apply deformation theory (Proposition 6.5.8) and Coherent Tannaka Duality (6.4.8) to construct a lift

$$\begin{tikzcd} \mathbf{B}\mathbb{G}_{m,\kappa} \arrow[r] \arrow[d, hook] & \mathcal{W} \arrow[d, "f"] \\ \Theta_R \arrow[r, "q"] \arrow[ur, dashed, "\widetilde{q}"] & \mathcal{X} \end{tikzcd}$$

The restriction $\widetilde{q}|_{\operatorname{Spec} R}$ is 2-isomorphic to $h$ since it agrees at the closed point and $f$ is étale. It follows that $\widetilde{g} := \widetilde{q}|_{\Theta_K}$ is a lift of (6.8.10). $\square$

The topology of $\Bbbk$-points of $\Theta$-complete stacks is analogous to the topology of quotient stacks arising from GIT.

**Proposition 6.8.32.** *Let $\mathcal{X}$ be an algebraic stack of finite type over an algebraically closed field $\Bbbk$ with affine diagonal. Assume that $\mathcal{X}$ is $\Theta$-complete and that the closed points of $\mathcal{X}$ have linearly reductive stabilizer. Then the closure of every $\Bbbk$-point contains a unique closed point.*

*Proof.* Assume that $x$ and $x'$ are two closed points in the closure of $p \in \mathcal{X}(\Bbbk)$. By Corollary 6.6.29, there are maps $f, f' \colon \Theta \to \mathcal{X}$ realizing the specializations $p \rightsquigarrow x$ and $p \rightsquigarrow x'$. Under the action of $\mathbb{G}_m^2$ on $\mathbb{A}^2$ given by $(t_1, t_2) \cdot (y_1, y_2) = (t_1 y_1, t_2 y_2)$, the maps $f$ and $f'$ glue to define a map $[\mathbb{A}^2/\mathbb{G}_m^2] \smallsetminus 0 \to \mathcal{X}$. By considering only the diagonal $\mathbb{G}_m$-action, the map $[\mathbb{A}^2/\mathbb{G}_m] \smallsetminus 0 \to \mathcal{X}$ extends to $\Psi \colon [\mathbb{A}^2/\mathbb{G}_m] \to \mathcal{X}$ by the $\Theta$-completeness of $\mathcal{X}$. Then $\Psi(0,0)$ is a common specialization of $x = \Psi(1,0)$ and $x' = \Psi(0,1)$. Since $x$ and $x'$ are closed points, we have that $x = \Psi(0,0) = x'$. $\square$

**Exercise 6.8.33.** With the hypotheses of Proposition 6.8.32, show that if in addition $\mathcal{X}$ has a unique closed point, then $\mathcal{X} \cong [\operatorname{Spec}(A)/G_x]$ such that $A^{G_x}$ is an artinian local $\Bbbk$-algebra with residue field $\Bbbk$.

### 6.8.5 Unpunctured inertia

We prove that an $\mathsf{S}$-complete stack $\mathcal{X}$ has 'unpunctured inertia' (Theorem 6.8.40) and the consequence that an étale quotient presentation $f \colon ([\operatorname{Spec} A/G_x], w) \to (\mathcal{X}, x)$ is stabilizer preserving in an open neighborhood of $w$ (Proposition 6.8.37).

**Definition 6.8.34.** We say that a noetherian algebraic stack has *unpunctured inertia* if for every closed point $x \in |\mathcal{X}|$ and every formally versal morphism $p\colon (T, t) \to (\mathcal{X}, x)$ where $T$ is the spectrum of a local ring with closed point $t$, every connected component of the inertia group scheme $\underline{\mathrm{Aut}}_{\mathcal{X}}(p) \to T$ has non-empty intersection with the fiber over $t$.

**Remark 6.8.35.** Here $(T, t) \to (\mathcal{X}, x)$ is formally versal if the map $\widehat{T} \to \mathcal{X}$ from the completion is is formally versal at $t$ as in Definition D.3.5.

**Remark 6.8.36.** Unpuncturedness is related to the purity of the morphism $\underline{\mathrm{Aut}}_{\mathcal{X}}(p) \to T$ as defined in [RG71, §3.3] (see also [SP, Tag 0CV5]). If $T$ is the spectrum of a strictly henselian local ring, then purity requires that if $s \in T$ is an *arbitrary* point and $\gamma$ is an associated point in the fiber $\underline{\mathrm{Aut}}_{\mathcal{X}}(p)_s$, then the closure of $\gamma$ in $\underline{\mathrm{Aut}}_{\mathcal{X}}(p)$ has non-empty intersection with the fiber over the closed point $t$ of $T$.

**Proposition 6.8.37.** *Let $\mathcal{X}$ be an algebraic stack of finite type over an algebraically closed field $\Bbbk$ with affine diagonal. Let $x \in \mathcal{X}$ be a closed point with linearly reductive stabilizer. Let $f\colon ([\operatorname{Spec} A/G_x], w) \to (\mathcal{X}, x)$ be an affine étale morphism inducing an isomorphism of stabilizer groups at $w$, and let $\pi\colon [\operatorname{Spec} A/G_x] \to \operatorname{Spec} A^{G_x}$. If $\mathcal{X}$ has unpunctured inertia, there exists an open affine neighborhood $U \subset \operatorname{Spec} A^{G_x}$ of $\pi(w)$ such that $f|_{\pi^{-1}(U)}\colon \pi^{-1}(U) \to \mathcal{X}$ induces isomorphisms of stabilizer groups at all points.*

*Proof.* Set $\mathcal{W} = [\operatorname{Spec} A/G_x]$. It suffices to find an open neighborhood $\mathcal{U} \subset \mathcal{W}$ of $w$ such that $f|_{\mathcal{U}}\colon \mathcal{U} \to \mathcal{X}$ induces an isomorphism $I_{\mathcal{U}} \to \mathcal{U} \times_{\mathcal{X}} I_{\mathcal{X}}$. Consider the cartesian diagram

$$\begin{tikzcd} I_{\mathcal{W}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{W} \times_{\mathcal{X}} I_{\mathcal{X}} \arrow[d] \\ \mathcal{W} \arrow[r] & \mathcal{W} \times_{\mathcal{X}} \mathcal{W}; \end{tikzcd}$$

see Exercise 3.2.13. Since $f$ is separated and étale, the morphism $I_{\mathcal{W}} \to \mathcal{W} \times_{\mathcal{X}} I_{\mathcal{X}}$ is finite and étale. We set $\mathcal{Z} \subset \mathcal{W} \times_{\mathcal{X}} I_{\mathcal{X}}$ to be the open and closed substack over which $I_{\mathcal{W}} \to \mathcal{W} \times_{\mathcal{X}} I_{\mathcal{X}}$ is *not* an isomorphism. Since $f$ is stabilizer preserving at $w$, the point $w$ is not contained in the image of $\mathcal{Z}$ under $p_1\colon \mathcal{W} \times_{\mathcal{X}} I_{\mathcal{X}} \to \mathcal{W}$.

Consider a formally smooth morphism $(T, t) \to (\mathcal{X}, x)$ from the spectrum of a local ring with closed point $t$. Since $\mathcal{X}$ has unpunctured inertia, the preimage of $\mathcal{Z}$ in $\mathcal{W} \times_{\mathcal{X}} I_{\mathcal{X}} \times_{\mathcal{X}} T$ is empty; indeed, if there were a non-empty connected component of this preimage, it must intersect the fiber over $t$ non-trivially contradicting that $w \notin p_1(\mathcal{Z})$. This in turn implies that $w \notin \overline{p_1(\mathcal{Z})}$. Therefore, if we set $\mathcal{U} = \mathcal{W} \smallsetminus \overline{p_1(\mathcal{Z})}$, the induced morphism $I_{\mathcal{U}} \to \mathcal{U} \times_{\mathcal{X}} I_{\mathcal{X}}$ is an isomorphism. $\square$

**Proposition 6.8.38.** *Let $\mathcal{X}$ be a noetherian algebraic stack.*

*(1) If $\mathcal{X}$ has quasi-finite inertia, then $\mathcal{X}$ has unpunctured inertia if and only if $\mathcal{X}$ has finite inertia.*

*(2) If $\mathcal{X}$ has connected stabilizer groups, then $\mathcal{X}$ has unpunctured inertia.*

*Proof.* If $\mathcal{X}$ has finite inertia, then $\underline{\mathrm{Aut}}_{\mathcal{X}}(p) \to T$ is finite, so clearly the image of each connected component contains the unique closed point $t \in T$. For the converse, we may assume that $T$ is the spectrum of a Henselian local ring, in which case $\underline{\mathrm{Aut}}_{\mathcal{X}}(p) = G \coprod H$ where $G \to T$ finite and the fiber of $H \to T$ over $t$ is empty (Proposition A.9.3). If $T$ is nonempty (i.e. $\underline{\mathrm{Aut}}_{\mathcal{X}}(p) \to T$ is not finite), then any connected component of $T$ doesn't meet the central fiber and thus $\mathcal{X}$ does not have unpunctured inertia.

For (2), by definition, all fibers of $\underline{\mathrm{Aut}}_{\mathcal{X}}(p) \to U$ are connected, so every connected component of $\underline{\mathrm{Aut}}_{\mathcal{X}}(p)$ intersects the component containing the identity section. $\square$

**Remark 6.8.39.** For algebraic stacks with connected stabilizer groups (e.g. the moduli stack $\mathrm{Bun}^{\mathrm{ss}}_{r,d}(C)$ of semistable vector bundles on a curve), Proposition 6.8.38(2) implies unpunctured inertia. The deeper result below (Theorem 6.8.40) is therefore unneeded in the proof of the existence of a good moduli space of $\mathrm{Bun}^{\mathrm{ss}}_{r,d}(C)$.

The rest of this section is dedicated to proving the following theorem.

**Theorem 6.8.40.** *Let $\mathcal{X}$ be an algebraic stack of finite type over an algebraically closed field $\Bbbk$ with affine diagonal. Assume that the closed points have linearly reductive stabilizers. If $\mathcal{X}$ is $\mathsf{S}$-complete, then $\mathcal{X}$ has unpunctured inertia.*

*Proof.* Let $x \in |\mathcal{X}|$ be a closed point, let $p\colon (U,u) \to (\mathcal{X}, x)$ be a formally smooth morphism from the spectrum of a local ring, and let $H \subset \underline{\mathrm{Aut}}_{\mathcal{X}}(p)$ be a connected component. The image of the projection $H \to U$ is a constructible set whose closure contains $u$. It follows that we can find a DVR $R$ with residue field $\Bbbk$ and a map $\operatorname{Spec} R \to U$ whose special point maps to $u$ and whose generic point lies in the image of $H \to U$. Let $\xi\colon \operatorname{Spec} R \to U \xrightarrow{p} \mathcal{X}$ denote the composition. After a residually-trivial extension of DVRs, we may assume that the generic point $\operatorname{Spec} K \to U$ lifts to $H$. This gives a commutative diagram

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & \operatorname{Spec} R \arrow[d] \arrow[dr, "\xi"] & \\ H \arrow[r] & U \arrow[r, "p"] & \mathcal{X}. \end{tikzcd}$$

Let $H_K$ be the base change of $H \to U$ along $\operatorname{Spec} K \to U$. We claim we can choose a finite type point $g \in H_K$ of finite order. If $g \in H_K$ is a finite type point, then after replacing $K$ with a finite field extension, we can decompose $g = g_s g_u$ under the Jordan decomposition, where $g_s$ is semisimple and $g_u$ is unipotent (see §C.3.2). Now consider the reduced Zariski closed $K$-subgroup $H' \subset \underline{\mathrm{Aut}}_{\mathcal{X}}(p)_K$ generated by $g_s$. Because $g_s$ is semisimple, $H'$ is a diagonalizable group scheme over $K$, and we may replace $g_s$ with a finite order element in $H'$ which still commutes with $g_u$. If $\mathrm{char}(K) > 0$, then $g_u$ has finite order and we are finished. If $\mathrm{char}(K) = 0$, then $g_u$ lies in the identity component of $G$, so $g$ lies on the same component as the finite order element $g_s$. This gives the desired element.

We claim that after replacing $R$ with a residually-trivial extension, there is a map $\xi'\colon \operatorname{Spec} R \to \mathcal{X}$ such that $\xi'_K \simeq \xi_K$ and $g \in H_K$ extends to an automorphism of $\xi'$. This would finish the proof: since the closure of $g$ meets the fiber of $\underline{\mathrm{Aut}}_{\mathcal{X}}(p) \to U$ over $u$, the component $H$ must also meet the central fiber.

If $\mathcal{X} \cong [\operatorname{Spec} A/\operatorname{GL}_n]$, then this claim is precisely the content of Proposition 6.8.41 below. We will use the Local Structure Theorem (6.5.1) to reduce to this case: let $f\colon ([\operatorname{Spec} A/G_x], w) \to (\mathcal{X}, x)$ be an étale quotient presentation. After replacing $R$ with a residually-trivial extension, we may lift $\xi$ to a map $\widetilde{\xi}\colon \operatorname{Spec} R \to [\operatorname{Spec} A/G_x]$ such that $\widetilde{\xi}(0) = w$. To show that $g$ lifts to an element $\widetilde{g} \in \mathrm{Aut}(\widetilde{\xi}_K)$, we will use $\mathsf{S}$-completeness. We may glue $\xi$ to itself along $g$ to define a morphism

$$\operatorname{Spec} R \bigcup_{\operatorname{Spec} K} \operatorname{Spec} R = \phi_R \setminus 0 \to \mathcal{X}.$$

Since $\mathcal{X}$ is $\mathsf{S}$-complete, this map extends to a morphism $h\colon \phi_R \to \mathcal{X}$. Since $\xi(0) = x$ and $x$ is a closed point, the image $h(0)$ of $0 \in \phi_R$ is also $x$. Since $f$ is stabilizer preserving at $w$, we may lift $h|_{\mathbf{B}\mathbb{G}_m}$ to a map $\widetilde{h}_0\colon \mathbf{B}\mathbb{G}_m \to [\operatorname{Spec} A/G_x]$ with image $w$. By Deformation Theory (6.5.8), we may find compatible lifts to $[\operatorname{Spec} A/G_x]$ of the restrictions of $h$ to the nilpotent thickenings of $\phi_R$ along 0, and by Coherent Tannaka Duality (6.4.8), we may find construct a lift $\widetilde{h}$ below

$$\begin{tikzcd}
\mathbf{B}\mathbb{G}_m \arrow[r, "\widetilde{h}_0"] \arrow[d, hook] & {[\operatorname{Spec} A/G_x]} \arrow[d, "f"] \\
\phi_R \arrow[r, "h"] \arrow[ur, dashed, "\widetilde{h}"] & \mathcal{X}.
\end{tikzcd}$$

Since $f$ is affine and étale, both restrictions $\widetilde{h}|_{s\neq 0}$ and $\widetilde{h}|_{t\neq 0}$ to $\operatorname{Spec} R$ are isomorphic to $\widetilde{\xi}$ and thus $\widetilde{h}|_{\phi_R \smallsetminus 0}$ gives a lift $\widetilde{g} \in \operatorname{Aut}(\widetilde{\xi}_K)$ of $g$. Finally, we apply Proposition 6.8.41 to construct a map $\widetilde{\xi}'\colon \operatorname{Spec} R \to [\operatorname{Spec} A/G_x]$ with $\widetilde{\xi}'(0) = w$ such that $\widetilde{\xi}_K \simeq \widetilde{\xi}'_K$ and $\widetilde{g}$ extends to an automorphism of $\widetilde{\xi}'$. The composition $\xi'' := f \circ \widetilde{\xi}''\colon \operatorname{Spec} R \to \mathcal{X}$ then satisfies the claim.

See also [AHLH18, Thm. 5.2]. □

Our proof used the following valuative criterion for a quotient stack.

**Proposition 6.8.41.** *Let $\mathcal{X} = [\operatorname{Spec} A/G]$ where $\operatorname{Spec} A$ is an affine scheme of finite type over an algebraically closed field $\Bbbk$ equipped with an action by a linearly reductive group $G$. Let $x \in \mathcal{X}$ be a closed point. Then $\mathcal{X}$ satisfies the following property:*

($\star$) *For every DVR $R$ with residue field $\Bbbk$ and fraction field $K$, for every morphism $\xi\colon \operatorname{Spec} R \to \mathcal{X}$ with $\xi(0) \simeq x$, and for every $K$-pint $g \in \operatorname{Aut}_{\mathcal{X}}(\xi_K)$ of finite order, there is an extension $R \to R'$ of DVRs (with $K' = \operatorname{Frac}(R')$) and a morphism $\xi'\colon \operatorname{Spec} R' \to \mathcal{X}$ such that $\xi'(0) \simeq x$, $\xi'_{K'} \simeq \xi_{K'}$ and $g|_{K'}$ extends to an automorphism of $\xi'$.*

**Remark 6.8.42.** In other words, for every map $\xi\colon \operatorname{Spec} R \to \operatorname{Spec} A$ and element $g \in G_{\xi_K} \subset G(K)$ of finite order, there exists after an extension $R \subset R'$ of DVRs and an element $h \in G(K')$ such that $h \cdot \xi_{K'}$ extends to a map $\xi'\colon \operatorname{Spec} R' \to \operatorname{Spec} A$ with $\xi'(0) \in Gx$ and such that $h^{-1} g|_{K'} h$ extends to an $R'$-point of $G$.

To illustrate this criterion, consider the the action of $G = \mathbb{G}_m \rtimes \mathbb{Z}/2$ on $\mathbb{A}^2$ via $t\cdot(a,b) = (ta, t^{-1}b)$ and $-1\cdot(a,b) = (b,a)$. Note that every point $(a,b) \in \mathbb{A}^2$ with $ab \neq 0$ is fixed by the order 2 element $(a/b, -1) \in G$. Consider $\xi\colon \operatorname{Spec} R = \Bbbk[\![z]\!] \to \mathbb{A}^2$ via $z \mapsto (z^2, z)$. The element $g = (z^{-1}, -1) \in G(\Bbbk(\!(z)\!))$ stabilizes $\xi_K$ but does not extend to $G(\Bbbk[\![z]\!])$. However, we may take the degree 2 ramified extension $\Bbbk[\![z]\!] \to \Bbbk[\![\sqrt{z}]\!]$ and define $\xi'\colon \operatorname{Spec} \Bbbk[\![\sqrt{z}]\!] \to \mathbb{A}^2$ by $\sqrt{z} \mapsto ((\sqrt{z})^3, (\sqrt{z})^3)$. Over the generic point, there is an isomorphism $\xi'_{\Bbbk(\!(\sqrt{z})\!)} \simeq \xi_{\Bbbk(\!(\sqrt{z})\!)}$ given by $h = (\sqrt{z}, -1) \in G(\Bbbk(\!(\sqrt{z})\!))$ and the element $g|_{\Bbbk(\!(\sqrt{z})\!)} = (\sqrt{z}, -1)^{-1} \cdot g|_{K'} \cdot (\sqrt{z}, -1) = (1, -1) \in G(\Bbbk(\!(\sqrt{z})\!))$ extends to an element of $G(\Bbbk[\![\sqrt{z}]\!])$-point.

*Proof.* After choosing an embedding $G \hookrightarrow \mathrm{GL}_n$ and replacing $[\operatorname{Spec} A/G]$ with $[(\operatorname{Spec} A \times^{G_x} \mathrm{GL}_n)/\mathrm{GL}_n]$, we may assume that $G = \mathrm{GL}_n$.

We first verify ($\star$) for quotient stacks $[\operatorname{Spec} A/G] = \operatorname{Spec} A \times \mathbf{B}G$ with a trivial action. As $R$ is local and $G = \mathrm{GL}_n$, the composition $\operatorname{Spec} R \to [\operatorname{Spec} A/G] \to \mathbf{B}G$ corresponds to the trivial $G$-bundle. We need to prove that every finite order element $g \in G(K)$ is conjugate to an element of $G(R)$ after passing to an extension of the

DVR $R$. We can conjugate $g$ to its Jordan canonical form (after an extension of $R$). Since $g$ has finite order, the diagonal entries of the resulting matrix are $r$th roots of unity for some $r$. Because the group $\mu_r$ of $r^{th}$ roots of unity is a finite group scheme over $\operatorname{Spec} R$, the entries of the Jordan canonical form must lie in $R$.

If $\mathcal{X} = [X/G]$ with $X$ proper over $\Bbbk$, we show that $(\star)$ holds except that $\xi'(0)$ may not be isomorphic to $x$. Since $p\colon \mathcal{X} \to \mathbf{B}G$ is proper and representable, for every morphism $\xi\colon \operatorname{Spec} R \to \mathcal{X}$ from a DVR, we have a closed immersion $\underline{\mathrm{Aut}}_{\mathcal{X}}(\xi) \hookrightarrow \underline{\mathrm{Aut}}_{\mathbf{B}G}(p \circ \xi)$ of group schemes over $\operatorname{Spec} R$. Moreover, any lift of the generic point of a morphism $\operatorname{Spec} R \to \mathbf{B}G$ to $[X/G]$ extends to a unique morphism $\operatorname{Spec} R \to \mathbf{B}G$. Therefore, given an element $g \in \operatorname{Aut}_{\mathcal{X}}(\xi_K)$, we use that $(\star)$ holds for $\mathbf{B}G$ to find (after replacing $R$ with an extension) a morphism $\eta\colon \operatorname{Spec} R \to \mathbf{B}G$ such that $\eta_K \simeq (p \circ \xi)_K$ and $g|_K$ extends to a $R$-point of $\operatorname{Aut}_{\mathbf{B}G}(\eta)$. If we lift $\eta$ to a morphism $\xi'\colon \operatorname{Spec} R \to [X/G]$ such that $\xi'_K \simeq \xi_K$, then the element $g|_K$ extends to an automorphism of $\xi'$.

In verifying $(\star)$ for $[\operatorname{Spec} A/G]$, we may assume that $A$ is reduced. Viewing $[\operatorname{Spec} A/G]$ as an algebraic stack which is affine and of finite type over $\operatorname{Spec} A^G \times \mathbf{B}G$, we can choose a vector bundle $\mathcal{E}$ on $\operatorname{Spec} A^G \times \mathbf{B}G$ and a $G$-equivariant embedding $\operatorname{Spec} A \hookrightarrow \mathbb{A}_{A^G}(\mathcal{E})$ over $A^G$. Viewing $\mathbb{A}_{A^G}(\mathcal{E})$ as an open subscheme of $\mathbb{P}_{A^G}(\mathcal{E} \oplus \mathcal{O})$, we let $X$ be the closure of $\operatorname{Spec} A$ in $\mathbb{P}_{A^G}(\mathcal{E} \oplus \mathcal{O})$. This gives a $G$-equivariant diagram

$$\begin{tikzcd} \operatorname{Spec} A \arrow[r, hook] \arrow[dr] & X \arrow[d] \\ & \operatorname{Spec} A^G \end{tikzcd} \tag{6.8.11}$$

where $X$ is a reduced projective scheme and the complement $X \setminus \operatorname{Spec} A$ is the support of an ample $G$-invariant Cartier divisor $E$. We also claim that $\operatorname{Spec} A$ is precisely the semistable locus of $X$ with respect to $\mathcal{O}_X(E)$ in the sense of Exercise 6.7.11. Indeed the tautological invariant section $s\colon \mathcal{O}_X \to \mathcal{O}_X(E)$ restricts to an isomorphism over $\operatorname{Spec} A$ and thus $\operatorname{Spec} A \subset X^{\mathrm{ss}}$. Conversely, $s^n$ defines an isomorphism

$$A^G \xrightarrow{\sim} \Gamma(X, \mathcal{O}_X(nE))^G$$

for all $n \geq 0$. Under this isomorphism, for every invariant global section $f \in \Gamma(X, \mathcal{O}_X(nE))^G$, the restriction $f|_{\operatorname{Spec} A}$ agrees with a section of the form $gs^n$, where $g$ is the pullback of a function under the map $X \to \operatorname{Spec} A^G$. It follows that $f = g \cdot s^n$ because $X$ is reduced. This shows that $X^{\mathrm{ss}} \subset \operatorname{Spec} A$.

We now verify that $(\star)$ holds for $[\operatorname{Spec} A/G]$. Let $\xi\colon \operatorname{Spec} R \to [\operatorname{Spec} A/G]$ be a map with $\xi(0) \simeq x$, and let $g \in \operatorname{Aut}_{\mathcal{X}}(\xi_K)$ be a finite order $K$-point. By applying the above result to $[X/G]$, there exists (after an extension of $R$) a map $\xi'\colon \operatorname{Spec} R \to [X/G]$ such that $\xi'_K \simeq \xi_K$ and $g$ extends to an element of $\operatorname{Aut}_{\mathcal{X}}(\xi')$ but where $\xi'(0)$ may not be isomorphic to $x$. The stabilizer group scheme $\operatorname{Stab}_G(X) \subset X \times G$ is a closed subscheme equivariant with respect to the product action of $G$ on $X \times G$ where $G$ acts on itself via conjugation. The pair $(\xi', g)$ defines a morphism

$$\eta\colon \operatorname{Spec} R \to [\operatorname{Stab}_G(X)/G].$$

We will show that after an extension of $R$, there is a map $\eta'\colon \operatorname{Spec} R \to [\operatorname{Stab}_G(\operatorname{Spec} A)/G]$ with $\eta'_K \simeq \eta_K$. Similar to (6.8.11), we have a $G$-equivariant diagram

$$\begin{tikzcd} \operatorname{Stab}_G(\operatorname{Spec} A) \arrow[r, hook] \arrow[dr] & \operatorname{Stab}_G(X) \arrow[d] \\ & \operatorname{Spec} A^G \times G \end{tikzcd}$$

with $\mathrm{Stab}_G(X)$ projective over $\operatorname{Spec} A^G \times G$. We claim that the semistable locus of $\mathrm{Stab}_G(X)$ for the action of $G$ with respect to the pullback of $\mathcal{O}_X(E)$ is precisely $\mathrm{Stab}_G(\operatorname{Spec} A)$ in the sense of Exercise 6.7.11. The invariant section $s \in \Gamma(X, \mathcal{O}_X(E))^G$ pulls back to an invariant section on $\mathrm{Stab}_G(X)$ and thus $\mathrm{Stab}_G(\operatorname{Spec} A) \subset \mathrm{Stab}_G(X)^{\mathrm{ss}}$. To see the converse, suppose that $(y,h) \in \mathrm{Stab}_G(X)$ with $y \notin X^{\mathrm{ss}} = \operatorname{Spec} A$. Applying Kempf's Optimal Destabilizing Theorem (6.7.33) to a lift $\widehat{y}$ of $y$ to the affine cone $\widehat{X} \to \operatorname{Spec} A^G$ of $X$ yields a one-parameter subgroup $\lambda\colon \mathbb{G}_m \to G$ such that $\lim_{t\to 0} \lambda(t)\cdot \widehat{y} \in \widehat{X}$ exists and is contained in the zero section $\operatorname{Spec} A^G$. Moreover, since $G_y \subset P_\lambda$ (Exercise 6.7.38), $\lim_{t \to 0} \lambda(t) \cdot (\widehat{y}, h)$ also exists and is contained in the zero section of the affine cone over of $\mathrm{Stab}_G(X)$; thus $(y,h)$ is not semistable.

The induced morphism $\mathrm{Stab}_G(\operatorname{Spec} A)/\!\!/G \to (\operatorname{Spec} A^G \times G)/\!\!/G$ of GIT quotients is proper, and the good moduli space $[\mathrm{Stab}_G(\operatorname{Spec} A)/G] \to \mathrm{Stab}_G(\operatorname{Spec} A)/\!\!/G$ is universally closed. By the valuative criterion, the composition

$$\operatorname{Spec} R \xrightarrow{\eta} \mathrm{Stab}_G(X) \to \operatorname{Spec} A^G \times G \to (\operatorname{Spec} A^G \times G)/\!\!/G$$

lifts a morphism $\chi\colon \operatorname{Spec} R \to [\mathrm{Stab}_G(\operatorname{Spec} A)/G]$ such that $\chi_K \simeq \xi_K$ after an extension of $R$. The composition $\xi'\colon \operatorname{Spec} R \xrightarrow{\chi} [\mathrm{Stab}_G(\operatorname{Spec} A)/G] \to [\operatorname{Spec} A/G]$ has the property that $\xi'_K \simeq \xi_K$ and that $g$ extends to an element of $\mathrm{Aut}_{\mathcal{X}}(\xi')$. To arrange that $\xi'(0) \simeq x$, we apply Lemma 6.8.43 below. □

**Lemma 6.8.43.** *Let $\mathcal{X} = [\operatorname{Spec} A/G]$ where $\operatorname{Spec} A$ is an affine scheme of finite type over an algebraically closed field $\Bbbk$ equipped with an action of a reductive group $G$. Let $\xi, \xi'\colon \operatorname{Spec} R \to \mathcal{X}$ be morphisms from a DVR with residue field $\Bbbk$ such that $\xi_K \simeq \xi'_K$ and $\xi(0) \in \mathcal{X}$ is a closed point. For every element $g \in \mathrm{Aut}_{\mathcal{X}}(\xi')$, there exists (after replacing $R$ with an extension) a morphism $\xi''\colon \operatorname{Spec} R \to \mathcal{X}$ such that $\xi''_K \simeq \xi'_K$, $g|_K$ extends to an automorphism of $\xi''$, and $\xi''(0) \simeq \xi(0)$.*

*Proof.* Since $\xi(0)$ and $\xi'(0)$ lie in the same fiber of $\mathcal{X} \to \operatorname{Spec} A^G$, the closure of $\xi'(0)$ in $|\mathcal{X}|$ must contain $\xi(0)$. Kempf's Criterion (6.7.31) yields a canonical map $f\colon \Theta \to [\operatorname{Spec} A/G]$ with $f(1) \simeq \xi'(0)$ and $f(0) \simeq \xi(0)$. Since $f$ is canonical, every automorphism of $f(1)$ extends to an automorphism of the map $f$. In particular the restriction of $g \in \mathrm{Aut}_{\mathcal{X}}(\xi')$ to $f(1) = \xi'(0)$ extends uniquely to an automorphism $g_f$ of $f$.

We now apply the Strange Gluing Lemma (6.8.44), which after replacing $R$ with $R[\pi^{1/N}]$ and precomposing $f$ with the map $\Theta \to \Theta$ defined by $x \mapsto x^N$ for $N \gg 0$, yields a *unique* map $\gamma\colon \phi_R \to \mathcal{X}$, such that $\gamma|_{s=0} \simeq f$ and $\gamma|_{t\neq 0} \simeq \xi'$. The uniqueness $\gamma$ guarantees that the automorphism $g$ of $\xi'$ and $g_f$ of $f$ extends uniquely to an automorphism $g_\gamma$ of $\gamma$. Finally, we construct the desired map $\xi''$ as the composition

$$\xi''\colon \operatorname{Spec}(R[\sqrt{\pi}]) \xrightarrow{q} \phi_R \xrightarrow{\gamma} \mathcal{X},$$

where in $(s,t,\pi)$ coordinates the first map $q$ is defined by $(\sqrt{\pi}, \sqrt{\pi}, \pi)$. Under $q$, the special point of $\operatorname{Spec}(R[\sqrt{\pi}])$ maps to the point $0 \in \phi_R$. By construction, $\xi''(0) \simeq \xi(0)$ and the automorphism $g$ of $\gamma$ restricts to an automorphism of $\xi''$ extending $g|_{K(\sqrt{\pi})}$. □

**Lemma 6.8.44** (Strange Gluing Lemma)**.** *Let $\mathcal{X}$ be an algebraic stack of finite type over an algebraically closed field $\Bbbk$ with affine diagonal. Let $R$ be a DVR with residue field $\Bbbk$. Let $f\colon \Theta \to \mathcal{X}$ and $\xi\colon \operatorname{Spec} R \to \mathcal{X}$ be morphisms with an isomorphism $f(1) \simeq \xi(0)$. For $N \gg 0$, after replacing $R$ with $R[\pi^{1/N}]$ and $f$ with the composition*

$\Theta \xrightarrow{N} \Theta \xrightarrow{f} \mathcal{X}$, *there is a unique morphism* $\gamma\colon \phi_R \to \mathcal{X}$ *such that* $\gamma|_{s=0} \simeq f$ *and* $\gamma|_{t\neq 0} \simeq \xi$.

*Proof.* For $n > 0$, define

$$\phi_R^{n,1} = [\operatorname{Spec}(R[s,t]/(st^n - \pi))/\mathbb{G}_m]$$

where the $\mathbb{G}_m$-acts with weight $n$ on $s$ and $-1$ on $t$. We have a closed immersion $\Theta \hookrightarrow \phi_R^{n,1}$ defined by $s = 0$ and an open immersion $\operatorname{Spec} R \hookrightarrow \phi_R^{n,1}$ defined by $t \neq 0$. Note that any morphism $\phi_R^{n,1} \to \mathcal{X}$ restricts to morphisms $f\colon \Theta \to \mathcal{X}$ and $\xi\colon \operatorname{Spec} R \to \mathcal{X}$ along with an isomorphism $\xi(0) \simeq f(1)$. We will show conversely that for $n \gg 0$, any $f\colon \Theta \to \mathcal{X}$ and $\xi\colon \operatorname{Spec} R \to \mathcal{X}$ with $\xi(0) \simeq f(1)$ extends canonically to a map $\phi_R^{n,1} \to \mathcal{X}$.

Letting $C = R[t, \pi/t, \pi/t^2, \ldots] \subset R[t]_t$, the diagram

$$\begin{tikzcd}
\operatorname{Spec} \Bbbk[t]_t \arrow[r] \arrow[d] & \operatorname{Spec} \Bbbk[t] \arrow[d] \\
\operatorname{Spec} R[t]_t \arrow[r] & \operatorname{Spec} C
\end{tikzcd}$$

is a pushout in the category of schemes (Theorem A.8.1). This diagram is $\mathbb{G}_m$-equivariant, and the diagram obtained by taking the fiber product with $\mathbb{G}_m$ is also a pushout. It follows from Corollary A.8.6 that taking quotients by $\mathbb{G}_m$ yields a diagram

$$\begin{tikzcd}
\operatorname{Spec} \Bbbk \arrow[r] \arrow[d] & \Theta \arrow[d] \arrow[ddr, bend left, "f"] & \\
\operatorname{Spec} R \arrow[r] \arrow[drr, bend right, "\xi"'] & {[\operatorname{Spec}(C)/\mathbb{G}_m]} \arrow[dr, dashed, "\Psi"] & \\
 & & \mathcal{X}
\end{tikzcd} \tag{6.8.12}$$

where the square is a pushout in the category of algebraic stacks with affine diagonal; this induces the dotted arrow $\Psi$. We can write $C$ as a union $C = \bigcup C_n$ where $C_n := R[t, \pi/t^n] \subset R[t]_t$. Note that $C_n \cong R[s,t]/(st^n - \pi)$ so in particular $[\operatorname{Spec}(C_n)/\mathbb{G}_m] \cong \phi_R^{n,1}$. As $\mathcal{X} \to S$ is locally of finite presentation, for $n \gg 0$ the morphism $\Psi$ factors uniquely as $[\operatorname{Spec}(C)/\mathbb{G}_m] \to \phi_R^{n,1} \to \mathcal{X}$ (Exercise 3.3.31).

To finish the proof, compose the uniquely defined map $\phi_R^{n,1} \to \mathcal{X}$ with the canonical map $\phi_{R[\pi^{1/n}]} \to \mathcal{X}$ induced by the map of graded algebras $R[s,t]/(st^n - \pi) \to R[\pi^{1/n}][s^{1/n}, t]/(s^{1/n}t - \pi)$, where $s^{1/n}$ has weight 1. $\square$

### 6.8.6 S-completeness and reductivity

We've already seen that S-completeness characterizes separatedness (Proposition 6.8.17 and Corollary 6.8.15). We've also seen that it implies unpunctured inertia (Theorem 6.8.40) and therefore implies the existence of stabilizer preserving local quotient presentations (Proposition 6.8.37). We now prove a third remarkable property of S-completeness: it characterizes reductivity. More precisely, a smooth affine algebraic group $G$ is reductive if and only if $\mathbf{B}G$ is S-complete if and only if $G$ has Cartan Decompositions (Proposition 6.8.45). This also completes the proof of Theorem 6.6.23

**Proposition 6.8.45.** *Let $G$ be a smooth affine algebraic group over an algebraically closed field $\Bbbk$. The following are equivalent:*

*(1) $G$ is reductive,*

*(2) $\mathbf{B}G$ is S-complete, and*

*(3) $G$ satisfies the Cartan Decomposition: for every complete DVR $R$ over $\Bbbk$ with residue field $\Bbbk$ and fraction field $K$ and for every element $g \in G(K)$, there exists elements $h_1, h_2 \in G(R)$ and a one-parameter subgroup $\lambda\colon \mathbb{G}_m \to G$ such that*

$$g = h_1 \lambda|_K h_2.$$

*In particular, if $\mathcal{X}$ is an S-complete algebraic stack and $x \in \mathcal{X}$ is a closed point with smooth affine stabilizer $G_x$, then $G_x$ is reductive.*

*Proof.* For (2) $\Rightarrow$ (3), observe that since $\phi_R \setminus 0 = \operatorname{Spec} R \bigcup_{\operatorname{Spec} K} \operatorname{Spec} R$, an element $g \in G(K)$ determines a morphism

$$\rho_g\colon \phi_R \setminus 0 \to \mathbf{B}G$$

by gluing two trivial $G$-torsors over $\operatorname{Spec} R$ via the isomorphism induced by $g$ of their restrictions to $\operatorname{Spec} K$. Since $\mathbf{B}G$ is S-complete, we have a lift

$$\begin{tikzcd} \phi_R \setminus 0 \arrow[r, "\rho_g"] \arrow[d] & \mathbf{B}G \\ \phi_R. \arrow[ur, dashed, "h"] & \end{tikzcd} \tag{6.8.13}$$

Restricting $h$ to the origin gives a map $\mathbf{B}\mathbb{G}_m \hookrightarrow \phi_R \xrightarrow{h} \mathbf{B}G$ which corresponds to a map $\lambda\colon \mathbb{G}_m \to G$ (up to conjugation); this provides us with our candidate one-parameter subgroup. We make two observations:

- If $g, g' \in G(K)$ are elements, the morphisms $\rho_g, \rho_{g'}\colon \phi_R \setminus 0 \to \mathbf{B}G$ are isomorphic if and only if there are elements $h, h' \in G(R)$ such that $hg = g'h'$.
- If $\lambda\colon \mathbb{G}_m \to G$ is a one-parameter subgroup and $\lambda|_{\phi_R \setminus 0}$ denotes the composition $\phi_R \setminus 0 \hookrightarrow \phi_R \to \mathbf{B}\mathbb{G}_m \xrightarrow{\lambda} \mathbf{B}G$, then $\lambda|_{\phi_R \setminus 0}$ and $\rho_{g'}$, where $g' = \lambda|_K$, are isomorphic.

It therefore suffices to show that the extension $h$ in (6.8.13) is isomorphic to $\lambda|_{\phi_R}\colon \phi_R \to \mathbf{B}\mathbb{G}_m \xrightarrow{\lambda} \mathbf{B}G$. To see this, let $\mathcal{P}$ and $\mathcal{P}'$ denote the principal $G$-bundles over $\phi_R$ classifying $h$ and $\lambda|_{\phi_R}$. Since $G$ is smooth and affine, $\underline{\operatorname{Isom}}_{\phi_R}(\mathcal{P}, \mathcal{P}') \to \phi_R$ is smooth and affine. We have a section over the inclusion $\mathcal{X}_0 := \mathbf{B}\mathbb{G}_m \hookrightarrow \phi_R$ of 0. Letting $\mathcal{X}_n$ denote the $n$th nilpotent thickening, deformation theory (Proposition 6.5.8) and the cohomological affineness of $\mathcal{X}_n$ implies that we may find compatible sections over $\mathcal{X}_n$. Coherent Tannaka Duality (6.4.8) and the coherent completeness of $\phi_R$ along $\mathbf{B}\mathbb{G}_m$ (Theorem 6.4.11) implies that the map

$$\operatorname{MOR}_{\phi_R}(\phi_R, \underline{\operatorname{Isom}}_{\phi_R}(\mathcal{P}, \mathcal{P}')) \to \varprojlim \operatorname{MOR}_{\phi_R}(\mathcal{X}_n, \underline{\operatorname{Isom}}_{\phi_R}(\mathcal{P}, \mathcal{P}'))$$

is an equivalence. We thus obtain a section of $\underline{\operatorname{Isom}}_{\phi_R}(\mathcal{P}, \mathcal{P}') \to \phi_R$, i.e. an isomorphism between $\mathcal{P}$ and $\mathcal{P}'$.

To see that (3) $\Rightarrow$ (2), it suffices to show that every map $\phi_R \setminus 0 \to \mathbf{B}G$ extends to a map $\phi_R \to \mathbf{B}G$ where $R$ is a complete DVR over $\Bbbk$ with residue field $\Bbbk$

(Lemma 6.8.11). Since every principal $G$-bundle over $\operatorname{Spec} R$ is trivial, the map $\phi_R \smallsetminus 0 \to \mathbf{B}G$ is isomorphic to $\rho_g$ for some element $g \in G(K)$. Writing $g = h_1 \lambda|_K h_2$, the two observations above show that $\phi_R \to \mathbf{B}\mathbb{G}_m \to \mathbf{B}G$ is an extension.

We've already seen that (1) $\Rightarrow$ (2) in Proposition 6.8.14. Conversely, if $G$ is not reductive, there is a normal subgroup $\mathbb{G}_a \triangleleft R_u(G)$ of the unipotent radical. As $G/R_u(G)$ and $R_u(G)/\mathbb{G}_a$ are both affine, the composition $\mathbf{B}\mathbb{G}_a \to \mathbf{B}R_u G \to \mathbf{B}G$ is affine. By Lemma 6.8.13, this would imply that $\mathbf{B}\mathbb{G}_a$ is $\mathsf{S}$-complete but this is a contradiction: taking $R = \Bbbk[\![x]\!]$ and $K = \Bbbk(\!(x)\!)$ the element $x \in \mathbb{G}_a(K)$ cannot be written as $h_1 \lambda|_K h_2$.

See also [AHHL21, Thm. A]. □

### 6.8.7 Proof of the Existence Theorem of Good Moduli Spaces

The necessity of $\Theta$-completeness and $\mathsf{S}$-completeness for the existence of a good moduli space was established in Corollary 6.8.15. We now establish the sufficiency following the strategy outlined in §6.8.1.

*Proof of Theorem 6.8.1.* Since $\mathcal{X}$ is $\mathsf{S}$-complete and $\operatorname{char}(\Bbbk) = 0$, the stabilizer $G_x$ of every closed point $x \in \mathcal{X}$ is linearly reductive (Proposition 6.8.45). By the Local Structure Theorem (6.5.1), there exists an affine étale morphism $f : ([\operatorname{Spec} A/G_x], w) \to (\mathcal{X}, x)$ inducing an isomorphism of stabilizer groups at $x$. Since $\mathcal{X}$ is $\Theta$-complete and $\mathsf{S}$-complete, we may assume that $f$ is $\Theta$-surjective and stabilizer preserving at all points after replacing $[\operatorname{Spec} A/G_x]$ with an open neighborhood of $x$ (Propositions 6.8.31 and 6.8.37). Since $\mathcal{X}$ is quasi-compact, there exists finitely many closed points $x_i \in \mathcal{X}$ and morphisms $f_i : [\operatorname{Spec} A_i/G_{x_i}] \to \mathcal{X}$ as above whose images cover $\mathcal{X}$. Choosing embeddings $G_{x_i} \hookrightarrow \mathrm{GL}_n$ for some $n$, there are equivalence $[\operatorname{Spec} A_i/G_{x_i}] \cong [(\operatorname{Spec} A_i \times^{G_{x_i}} \mathrm{GL}_n)/\mathrm{GL}_n]$. Setting $A = \prod_i (A_i \times^{G_{x_i}} \mathrm{GL}_n)$, there is an surjective, affine, and étale morphism

$$f : \mathcal{X}_1 := [\operatorname{Spec} A/\mathrm{GL}_n] \to \mathcal{X}$$

which is $\Theta$-surjective and stabilizer preserving at all points. Since $\operatorname{char}(\Bbbk) = 0$, there is a good moduli space $\mathcal{X}_1 \to X_1 := \operatorname{Spec} A^{\mathrm{GL}_N}$.

Set $\mathcal{X}_2 = \mathcal{X}_1 \times_{\mathcal{X}} \mathcal{X}_1$. The projections $p_1, p_2 : \mathcal{X}_2 \to \mathcal{X}_1$ are also affine, étale, $\Theta$-surjective, and stabilizer preserving. Since $f$ is affine, $\mathcal{X}_2 \cong [\operatorname{Spec} B/\mathrm{GL}_n]$ and there is a good moduli space $\mathcal{X}_2 \to X_2 := \operatorname{Spec} B^{\mathrm{GL}_n}$. This provides a commutative diagram

$$\begin{tikzcd} \mathcal{X}_2 \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] \arrow[d] & \mathcal{X}_1 \arrow[r, "f"] \arrow[d] & \mathcal{X} \arrow[d, dashed] \\ X_2 \arrow[r, shift left, "q_1"] \arrow[r, shift right, "q_2"'] & X_1 \arrow[r, dashed] & X \end{tikzcd} \tag{6.8.14}$$

which each square on the left is cartesian by Luna's Fundamental Lemma (6.3.26). Moreover, by the universality of good moduli spaces (Theorem 6.3.5(4)), the étale groupoid structure on $\mathcal{X}_2 \rightrightarrows \mathcal{X}_1$ induces a étale groupoid structure on $X_2 \rightrightarrows X_1$.

We claim that $X_2 \rightrightarrows X_1$ is an étale equivalence relation, i.e. that the quotient stack $[X_1/X_2]$ is an algebraic space. By the Characterization of Algebraic Spaces (3.6.5), it suffices to show that if $x_1 \in X_1$ is a $\Bbbk$-point, then $(x_1, x_1)$ has a unique preimage under $(q_1, q_2) : X_2 \to X_1 \times X_1$. Let $x_2, x_2' \in X_2$ be two points mapping to $(x_1, x_1) \in X_1 \times X_1$, and let $\widetilde{x}_2, \widetilde{x}_2' \in \mathcal{X}_2$ be the unique closed points in their preimages. Since $f$ is $\Theta$-surjective, the images $p_1(\widetilde{x}_2)$, $p_2(\widetilde{x}_2)$, $p_1(\widetilde{x}'_2)$, and $p_2(\widetilde{x}'_2)$ are all closed points of $\mathcal{X}_1$ over $x_1$, and therefore they are all identified with the unique closed point $\widetilde{x}_1$ over $x_1$. On the other hand, since $f$ is stabilizer preserving, the stabilizer groups of $\widetilde{x}_2$ and $\widetilde{x}'_2$ are the same as the stabilizer groups of $\widetilde{x}_1$ and of its image in $\mathcal{X}$. Let's denote this stabilizer group by $G$. It follows that the fiber product of $(p_1, p_2)\colon \mathcal{X}_2 \to \mathcal{X}_1 \times \mathcal{X}_1$ along the inclusion of the residual gerbe $\mathcal{G}_{(\widetilde{x}_1, \widetilde{x}_1)} = \mathbf{B}G \times \mathbf{B}G \to \mathcal{X}_1 \times \mathcal{X}_1$ is isomorphic to $\mathbf{B}G$ and thus identified with the residual gerbe of a *unique* closed point. Therefore $x_2 = x'_2$.

Since $X_2 \rightrightarrows X_1$ is an étale equivalence relation, the quotient $X = X_1/X_2$ is an algebraic space. From étale descent, there is a morphism $\mathcal{X} \to X$ which pulls back under $X_1 \to X$ to the good moduli space $\mathcal{X}_1 \to X_1$. By descent of good moduli spaces (Lemma 6.3.20(2)), $\mathcal{X} \to X$ is a good moduli space. Finally, we use that $\mathcal{X}$ is $\mathsf{S}$-complete to conclude that $X$ is separated (Corollary 6.8.15). $\square$

# Chapter 7

# Moduli of semistable vector bundles on a curve

TO BE WRITTEN

## 7.1 Semistable vector bundles

### 7.1.1 Review of properties of coherent sheaves

### 7.1.2 Definition of semistability and basic properties

### 7.1.3 Families of vector bundles

### 7.1.4 Openness of semistability

### 7.1.5 Boundedness of semistability

### 7.1.6 Vector bundles with fixed determinant

### 7.1.7 Existence of stable bundles

## 7.2 Filtrations, stratifications, and existence of moduli spaces

### 7.2.1 Jordan–Hölder filtrations

### 7.2.2 Harder–Narasimhan filtrations

### 7.2.3 The Schatz Stratification

### 7.2.4 $\Theta$- and $\mathsf{S}$-completeness of $\mathrm{Bun}_{r,d}^{\mathrm{ss}}$

### 7.2.5 Existence of a separated moduli space

## 7.3 Semistable reduction: Langton's Theorem

### 7.3.1 Elementary modifications

### 7.3.2 Proof of Langton's Theorem

## 7.4 Geometric properties: irreducibility, unirationality, and more

### 7.4.1 Smoothness and dimension

### 7.4.2 Properties of stable locus

### 7.4.3 Descending the universal vector bundle

### 7.4.4 Irreducibility

### 7.4.5 Unirationality

### 7.4.6 Picard groups

## 7.5 Projectivity

### 7.5.1 Projectivity of the Picard scheme

### 7.5.2 Determinantal line bundles

### 7.5.3 A characterization of semistability and the semiampleness of $L_V$

### 7.5.4 Ampleness of $L_V$

### 7.5.5 A GIT construction

# Chapter 8

# Glimpse of other moduli

TO BE WRITTEN

## 8.1 Moduli of varieties

### 8.1.1 Weighted pointed stable curves

### 8.1.2 Abelian varieties

### 8.1.3 Stable maps

### 8.1.4 Stable varieties

### 8.1.5 Stable pairs

### 8.1.6 K-stability

## 8.2 Moduli of sheaves and bundles

### 8.2.1 Higgs bundles

### 8.2.2 Sheaves on higher dimensional varieties

### 8.2.3 $G$-bundles

### 8.2.4 Complexes and Bridgeland stability

# Appendix A

# Morphisms of schemes

In this appendix, we recall definitions and summarize certain properties of morphisms of schemes: locally of finite presentation, flat, smooth, étale, and unramified.

We pay particular attention to properties that can be described *functorially*, i.e. properties of schemes and their morphisms that can be characterized in terms of their functors. For instance, the properties of being separated, universally closed, proper, locally of finite presentation, smooth, étale, and unramified can be characterized functorially. Such descriptions are particularly advantageous for us since we systematically study moduli problems via functors and stacks. For example, the valuative criterion for properness for $\overline{\mathcal{M}}_g$ amounts to checking that every family of curves over a punctured curve (i.e. over the generic point of a DVR) can be extended uniquely (after possibly a finite extension) to the entire curve (i.e. DVR). Similarly, the smoothness of $\overline{\mathcal{M}}_g$ can be shown by using the functorial formal lifting criterion.

## A.1 Morphisms locally of finite presentation

A morphism of schemes $f\colon X \to Y$ is *locally of finite type* (resp. *locally of finite presentation*) if for all affine open subschemes $\operatorname{Spec} B \subset Y$ and $\operatorname{Spec} A \subset f^{-1}(\operatorname{Spec} B)$, there is surjection $A[x_1, \ldots, x_n] \to B$ of $A$-algebras (resp. a surjection $A[x_1, \ldots, x_n] \to B$ with finitely generated kernel). If in addition $f$ is quasi-compact (resp. quasi-compact and quasi-separated), we say that $f$ is *of finite type* (resp. *of finite presentation*).

**Remark A.1.1.** When $Y$ is locally noetherian, these two notions coincide. However, in the non-noetherian setting, even closed immersions may not be locally of finite presentation, e.g. $\operatorname{Spec} \mathbb{C} \hookrightarrow \operatorname{Spec} \mathbb{C}[x_1, x_2, \ldots]$. Since functors and stacks are defined in these notes on the entire category of schemes, it is often necessary to work with non-noetherian schemes. In particular, when defining a moduli functor or stack, we need to specify what families of objects are over possibly non-noetherian schemes. Morphisms of finite presentation are better behaved than morphisms of finite type, so we often use the former condition. For example, when defining a family of smooth curves $\pi\colon \mathcal{C} \to S$, we require not only that $\pi$ is proper and smooth but also of finite presentation.

Before stating the functorial characterization of locally of finite presentation morphism, we recall the notion of systems.

**Definition A.1.2.** A *directed system* (resp. *inverse system*) in a category $\mathcal{C}$ is a partially ordered set $(I, \geq)$ which is directed (i.e. for every $i, j \in I$ there exists $k \in I$ such that $k \geq i$ and $k \geq j$) together with a covariant (resp. contravariant) $I \to \mathcal{C}$.

**Proposition A.1.3.** *A morphism $f\colon X \to Y$ of schemes is locally of finite presentation if and only if for every inverse system $\{\operatorname{Spec} A_\lambda\}_{\lambda \in I}$ of affine schemes over $Y$, the natural map*

$$\operatorname{colim}_\lambda \operatorname{Mor}_Y(\operatorname{Spec} A_\lambda, X) \to \operatorname{Mor}_Y(\operatorname{Spec}(\operatorname{colim}_\lambda A_\lambda), X) \tag{A.1.1}$$

*is bijective.*

There's a conceptual reason for this: every ring $A$ (e.g. $\mathbb{C}[x_1, x_2, \ldots]$) is the union (or colimit) of its finitely generated subalgebras $A_\lambda$. The condition that every map $\operatorname{Spec} A \to X$ factors through some $\operatorname{Spec} A_\lambda \to X$ can be viewed as the condition that specifying $\operatorname{Spec} A \to X$ over $Y$ depends on only finite data (i.e. local generators and relations for the ring maps defining $X \to Y$) and therefore translates to a finiteness condition on $X$ over $Y$. We encourage the reader to verify the proposition, especially in the case of a morphism of affine schemes. See [EGA, IV.8.14.2] or [SP, Tag 01ZC].

## A.2 Flatness

You can't get very far in moduli theory without flatness. While its definition is seemingly abstract and algebraic, it is a magical geometric property of a morphism $X \to Y$ that ensures that fibers $X_y$ 'vary nicely' as $y \in Y$ varies. This principle is nicely evidenced by Flatness via the Hilbert Polynomial (A.2.4). It is the reason why we define objects of our moduli stacks as flat families.

### A.2.1 Flatness criteria

A module $M$ over a ring $A$ is *flat* if the functor

$$- \otimes_A M\colon \operatorname{Mod}(A) \to \operatorname{Mod}(A)$$

is exact. We recall the following criteria:

(1) (Stalk Criterion) $M$ is flat over $A$ if and only if $M_{\mathfrak{p}}$ is flat over $A_{\mathfrak{p}}$ for every prime (equivalently maximal) ideal $\mathfrak{p}$. More generally, if $A \to B$ is a ring map, a $B$-module $N$ is flat if and only if for every prime $\mathfrak{q} \subset B$ with preimage $\mathfrak{p} \subset A$, $N_{\mathfrak{q}}$ is flat over $A_{\mathfrak{p}}$.

(2) (Ideal Criterion) $M$ is flat if and only if for every finitely generated ideal $I \subset A$, the map $I \otimes_A M \to M$ is injective [Eis95, Prop. 6.1]. (When $A$ is a PID, this implies that $M$ is flat if and only if $M$ is torsion free.)

(3) (Tor Criterion) $M$ is flat if and only if $\operatorname{Tor}_1^A(A/I, M) = 0$ for all finitely generated ideals $I \subset A$ [Eis95, Prop. 6.1].

(4) (Finitely Presented Criterion) $M$ is finitely presented and flat over $A$ if and only if $M$ is finite and projective if and only if $M$ is finite and locally free (i.e. $M_{\mathfrak{p}}$ is finite and free for all prime—or equivalently maximal—ideals $\mathfrak{p}$); see [SP, Tag 00NX]. (Without the finitely presented hypothesis, Lazard's Theorem states that $M$ is flat over $A$ if and only if $M$ can be written as a directed colimit $\operatorname{colim}_{i \in I} M_i$ of free finite $A$-modules $M_i$; see [Eis95, A6.6] or [SP, Tag 058G].)

(5) (Equational Criterion) $M$ is flat if and only if for every relation $\sum_{i=1}^{n} a_i m_i = 0$ with $a_i \in A$ and $m_i \in M$, there exists $m'_j \in M$ for $j = 1, \ldots, r$ and $a'_{ij} \in A$ such that $\sum_{j=1}^{r} a'_{ij} m'_j = m_i$ for all $i$ and $\sum_{i=1}^{n} a'_{ij} a_i = 0$ for all $j$ [Eis95, Cor. 6.5].

If $f: X \to Y$ is a morphism of schemes, then a quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ is *flat* if for all affine opens $\operatorname{Spec} B \subset Y$ and $\operatorname{Spec} A \subset f^{-1}(\operatorname{Spec} B)$, the $B$-module $\Gamma(\operatorname{Spec} A, \mathcal{F})$ is a flat.

**Proposition A.2.1** (Flat Equivalences)**.** *Let $f: X \to Y$ be a morphism of schemes and $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. The following are equivalent:*

*(1) $\mathcal{F}$ is flat over $Y$;*

*(2) There exists a Zariski-cover $\{\operatorname{Spec} B_i\}$ of $Y$ and $\{\operatorname{Spec} A_{ij}\}$ of $f^{-1}(\operatorname{Spec} B_i)$ such that $\Gamma(\operatorname{Spec} A_{ij}, \mathcal{F})$ is flat as an $B_i$-module under the ring map $B_i \to A_{ij}$;*

*(3) For all $x \in X$, the $\mathcal{O}_{X,x}$-module $\mathcal{F}_x$ is flat as an $\mathcal{O}_{Y,y}$-module.*

*(4) The functor*

$$\operatorname{QCoh}(Y) \to \operatorname{QCoh}(X), \qquad \mathcal{G} \mapsto f^*\mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{F}$$

*is exact.*

*Proof.* See [Har77, §III.9] or [SP, Tag 01U2]. □

We say that a morphism $f: X \to Y$ of schemes is *flat at $x \in X$* (resp. a quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ is *flat at $x \in X$*) if there exists a Zariski open neighborhood $U \subset X$ containing $x$ such that $f|_U$ (resp. $\mathcal{F}|_U$) is flat over $Y$. This is equivalent to the flatness of $\mathcal{O}_{X,x}$ (resp. $\mathcal{F}_x$) as an $\mathcal{O}_{Y,y}$-module.

**Proposition A.2.2** (Flatness Criterion over Smooth Curves)**.** *Let $C$ be an integral and regular scheme of dimension 1 (e.g. the spectrum of a DVR or a smooth connected curve over a field), and let $X \to C$ be a morphism of schemes. A quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ is flat over $C$ if and only if every associated point of $\mathcal{F}$ maps to the generic point of $C$.*

*Proof.* A short argument shows that this follows from the fact that a module over a DVR is flat if and only if it is torsion free; see [Har77, III.9.7]. □

Over higher dimensional bases, it is sometimes possible to check flatness by reducing to the above criterion over a smooth curve. This is called the valuative criterion for flatness: if $f: X \to S$ is a finite type morphism of noetherian schemes over a reduced scheme $S$ and $\mathcal{F}$ is a coherent $\mathcal{O}_X$-module, then $\mathcal{F}$ is flat at $x \in X$ if and only if for every map $(\operatorname{Spec} R, 0) \to (S, f(x))$ from a DVR, the restriction $\mathcal{F}|_{X_R}$ is flat over $R$ at all points in $X_R := X \times_S \operatorname{Spec} R$ over 0 and $x$ [EGA, IV.11.8.1]. Despite providing a conceptual geometric criterion for flatness, it is surprisingly rarely used in moduli theory.

**Proposition A.2.3** (Flatness Criterion over Artinian Rings)**.** *A module over an artinian ring is flat if and only if it is free if and only if it is projective.*

*Proof.* See [SP, Tag 051E]. □

Recall that if $X \subset \mathbb{P}^n_K$ is a subscheme and $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_X$-module, the *Hilbert polynomial* of $\mathcal{F}$ is $P_{\mathcal{F}}(z) = \chi(X, \mathcal{F}(z)) \in \mathbb{Q}[z]$.

**Proposition A.2.4** (Flatness via the Hilbert Polynomial)**.** *Let $S$ be a connected, reduced, and noetherian scheme, and let $X \subset \mathbb{P}^n_S$ be a closed subscheme. A coherent $\mathcal{O}_X$-module $\mathcal{F}$ is flat over $S$ if and only if the function*

$$S \to \mathbb{Q}[z], \qquad s \mapsto P_{\mathcal{F}|_{X_s}}$$

*assigning a point $s \in S$ to the Hilbert polynomial of the restriction $\mathcal{F}|_{X_s}$ to the fiber $X_s \subset \mathbb{P}^n_{\kappa(s)}$ is constant.*

*Proof.* See [Har77, Thm. 9.9]. $\square$

**Theorem A.2.5** (Local and Infinitesimal Criteria for Flatness)**.** *Let $A \to B$ be a local homomorphism of noetherian local rings, and let $M$ be a finite $B$-module. The following are equivalent:*

*(1) $M$ is flat over $A$,*

*(2) (Local Criterion) $\operatorname{Tor}_1^A(A/\mathfrak{m}_A, M) = 0$, and*

*(3) (Infinitesimal Criterion) $M/\mathfrak{m}_A^n M$ is flat over $A/\mathfrak{m}_A^n$ for every $n \geq 1$.*

*Proof.* See [Eis95, Thm. 6.8, Exer. 6.5] and [SP, Tag 00MK]. $\square$

The following consequence of the Local Criterion for Flatness is particularly useful in deformation theory.

**Corollary A.2.6.** *Let $A \twoheadrightarrow A_0$ be a surjective homomorphism of noetherian rings with kernel $I$ such that $I^2 = 0$. An $A$-module $M$ is flat over $A$ if and only if*

*(1) $M_0 := M \otimes_A A_0$ is flat over $A_0$, and*

*(2) the map $M_0 \otimes_{A_0} I \to M$ is injective.*

*Proof.* For ($\Rightarrow$), condition (1) holds by base change and condition (2) holds by tensoring the exact sequence $0 \to I \to A \to A_0 \to 0$ with $M$ and using the identification $M \otimes_A I \cong M_0 \otimes_{A_0} I$. For ($\Leftarrow$), by the Local Criterion for Flatness (A.2.5) it suffices to show that $\operatorname{Tor}_1^A(A/\mathfrak{p}, M) = 0$ for all prime ideals $\mathfrak{p} \subset A$. Let $\mathfrak{p}_0 := \mathfrak{p}/I \subset A$. Consider the following diagram which is obtained by tensoring the exact sequences $0 \to I \to \mathfrak{p} \to \mathfrak{p}_0 \to 0$ and $0 \to I \to A \to A_0 \to 0$ with $M$:

$$\begin{tikzcd}
 & & \operatorname{Tor}_1^A(M, A/\mathfrak{p}) \arrow[r] \arrow[d] & \operatorname{Tor}_1^{A_0}(M_0, A_0/\mathfrak{p}_0) \arrow[d] & \\
0 \arrow[r] & M \otimes_A I \arrow[r] \arrow[d, equal] & M \otimes_A \mathfrak{p} \arrow[r] \arrow[d] & M_0 \otimes_{A_0} \mathfrak{p}_0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M \otimes_A I \arrow[r] & M \arrow[r] \arrow[d, two heads] & M_0 \arrow[r] \arrow[d, two heads] & 0 \\
 & & M \otimes_A A/\mathfrak{p} \arrow[r] & M_0 \otimes_{A_0} A_0/\mathfrak{p}_0 &
\end{tikzcd}$$

Condition (2) implies that the second row is exact, and it follows that the first row is also exact, where we've used the identification $M \otimes_A \mathfrak{p}_0 \cong M_0 \otimes_{A_0} \mathfrak{p}_0$. Condition (1) implies that $\operatorname{Tor}_1^{A_0}(M_0, A_0/\mathfrak{p}_0) = 0$ and it follows from the snake lemma that $\operatorname{Tor}_1^A(M, A/\mathfrak{p}) = 0$. See also [Har10, Prop. 2.2]. $\square$

**Remark A.2.7.** Applying this with $A = \Bbbk[\epsilon]/(\epsilon^2)$ being the dual numbers and $A' = \Bbbk$, we recover the fact that an $A$-module $M$ is flat if and only if $M \otimes_{\Bbbk[\epsilon]/(\epsilon^2)} \Bbbk \xrightarrow{\epsilon} M$ is injective. This also follows from the fact that a module $N$ over a ring $B$ is flat if and only if for every ideal $I \subset B$, the map $I \otimes_B M \to M$ is injective, and using that the only ideal in $\Bbbk[\epsilon]/(\epsilon^2)$ is $(\epsilon)$.

The Local Criterion for Flatness also provides the following useful criterion for when slicing preserves flatness.

**Corollary A.2.8.** *Let $f\colon X \to S$ be a morphism locally of finite presentation, and let $x \in X$ be a point with image $s \in S$. If $f$ is flat at $x$ and the image of $h \in \mathfrak{m}_x \subset \mathcal{O}_{X,x}$ in the local ring $\mathcal{O}_{X_s,x}$ of the fiber is a nonzerodivisor, then there exists an open neighborhood $U \subset X$ of $x$ such that $h$ extends to a global function on $U$ and the composition $V(h) \hookrightarrow U \to S$ is locally of finite presentation and flat at $x$.*

*Proof.* The noetherian case is a consequence of the following algebraic statement: if $A \to B$ is a flat local ring homomorphism of noetherian local rings and $f \in \mathfrak{m}_B$ is a nonzerodivisor in $B \otimes_A A/\mathfrak{m}_A$, then $A \to B/(f)$ is flat. It suffices to show that $f$ is also a nonzerodivisor in $B$. Indeed, in this case $0 \to B \xrightarrow{f} B \to B/(f) \to 0$ is an exact sequence which implies that $\operatorname{Tor}_1^A(A/\mathfrak{m}_A, B/(f)) = 0$ and thus $B/(f)$ is flat over $A$ by the Local Criterion for Flatness (A.2.5). Let $b \in B$ and suppose that $fb = 0$. We already know that $b \in \mathfrak{m}_A B$ and we claim that $b \in \mathfrak{m}_A^n B$ for $n > 0$ implies that $b \in \mathfrak{m}_A^{n+1} B$. Given this claim, $b \in \bigcap_{n>0} \mathfrak{m}_A^n B$ and thus $b = 0$ by Krull's Intersection Theorem. Let $a_1, \ldots, a_r$ be minimal generators of $\mathfrak{m}_A^n$ as an $A$-module. Write $b = \sum_i a_i b_i$ for $b_i \in B$. Then $0 = fb = \sum_i a_i(fb_i)$. Since $B$ is $A$-flat, the Equational Criterion implies that there exists $m'_j \in B$ and $a'_{ij} \in A$ such that $\sum_j a'_{ij} a'_j = fb_i$ for all $i$ and $\sum_i a'_{ij} a_i = 0$ for all $j$. By Nakayama's Lemma and our choice of the $a_i$'s, each $a'_{ij}$ is in $\mathfrak{m}_A$ (as otherwise the images $\overline{a}_i \in \mathfrak{m}_A^n/\mathfrak{m}_A^{n+1}$ would be linearly dependent over $A/\mathfrak{m}_A$). This implies that $fb_i \in \mathfrak{m}_A B$. As $f$ is a nonzerodivisor in $B \otimes_A A/\mathfrak{m}_A$, we see that $b_i \in \mathfrak{m}_A B$ for each $i$ and thus $b \in \mathfrak{m}_A^{n+1}$.

The general case can be reduced to the noetherian case using limit methods (A.6). See also [Mat89, Thm. 22.5] or [SP, Tag 056X]. ∎

**Theorem A.2.9** (Fibral Flatness Criterion)**.** *Consider a commutative diagram*

$$\begin{tikzcd} X \arrow[rr] \arrow[dr] & & Y \arrow[dl] \\ & S & \end{tikzcd}$$

*of schemes, and let $F$ be a quasi-coherent $\mathcal{O}_X$-module of finite presentation. Assume that $X \to S$ is locally of finite presentation and $Y \to S$ is locally of finite type. Let $x \in X$ with images $y \in Y$ and $s \in S$. If the stalk $\mathcal{F}_x$ is nonzero, then the following are equivalent:*

*(1) $F$ is flat over $S$ at $x$, and $\mathcal{F}_s := \mathcal{F}|_{X_s}$ is flat over $Y_s$ at $x$, and*

*(2) $Y$ is flat over $S$ at $y$ and $\mathcal{F}$ is flat over $Y$ at $x$.*

*Proof.* See [SP, Tag 039A] ∎

If $A \to B$ is a local ring map of noetherian local rings, then $\dim B = \dim A + \dim B/\mathfrak{m}_A B$. The following is a partial converse.

**Theorem A.2.10** (Miracle Flatness)**.** *Let $A \to B$ be a local homomorphism of noetherian local rings. Assume that*

1. $A$ *is regular,*
2. $B$ *is Cohen–Macaulay, and*
3. $\dim B = \dim A + \dim B/\mathfrak{m}_A B$.

*Then* $A \to B$ *is flat.*

*Proof.* See [Nag62, Thm. 25.16] or [SP, Tag 00R4]. □

### A.2.2 Properties of flatness

**Theorem A.2.11** (Generic Flatness). *Let* $f\colon X \to S$ *be a finite type morphism of schemes and* $\mathcal{F}$ *be a finite type quasi-coherent* $\mathcal{O}_X$*-module. If* $S$ *is reduced, there exists a dense open subscheme* $U \subset S$ *such that* $X_U \to U$ *is flat and of presentation and such that* $\mathcal{F}|_{X_U}$ *is flat over* $U$ *and of finite presentation as on* $\mathcal{O}_{X_U}$*-module.*

*Proof.* See [SP, Tag 052B]. □

**Proposition A.2.12** (Fppf Morphisms are Open). *Let* $f\colon X \to Y$ *be a morphism of schemes. If* $f$ *is flat and locally of finite presentation, then for every open subset* $U \subset X$*, the image* $f(U) \subset Y$ *is open.*

*Proof.* See [SP, Tag 01UA]. □

**Proposition A.2.13.** *A flat monomorphism locally of finite presentation (e.g. an étale monomorphism) is an open immersion.*

**Theorem A.2.14** (Existence of Flattening Stratifications). *Let* $X \to S$ *be a projective morphism of noetherian schemes,* $\mathcal{O}_X(1)$ *be a relatively ample line bundle and* $\mathcal{F}$ *be a coherent sheaf on* $X$*. For each polynomial* $P \in \mathbb{Q}[z]$*, there exists a locally closed subscheme* $S_P \hookrightarrow S$ *such that a morphism* $T \to S$ *factors through* $S_P$ *if and only if the pullback* $\mathcal{F}_T$ *of* $\mathcal{F}$ *to* $X_T$ *is flat over* $T$ *and for every* $t \in T$*, the pullback* $\mathcal{F}_{\kappa(t)}$ *to* $X_{\kappa(t)}$ *has Hilbert polynomial* $P$*.*

*Moreover, there exists a finite indexing set* $I$ *of polynomials such that* $S = \coprod_{P \in I} S_P$ *set-theoretically. The closure of* $S_P$ *in* $S$ *is contained set-theoretically in the union* $\bigcup_{P \leq Q} S_Q$*, where* $P \leq Q$ *if and only if* $P(z) \leq Q(z)$ *for* $z \gg 0$*.*

*Proof.* See [$\mathrm{FGA_{IV}}$] or [Mum66, §8]. □

**Remark A.2.15.** When $X \to S$ is only proper, there is a *universal flattening*, i.e. an algebraic space $S'$ and a morphism $S' \to S$ such that a map $T \to S$ factors through $S' \to S$ if and only if the pullback $\mathcal{F}|_{X_T}$ to $X_T := X \times_S T$ is flat over $T$ [SP, Tag 05UG]. In general, $S'$ may not be a disjoint union of locally closed subschemes of $S$; see [Kre13].

**Theorem A.2.16** (Raynaud-Gruson Flatification). *Let* $Y$ *be a quasi-compact and quasi-separated scheme and* $X \to Y$ *be a finitely presented morphism which is flat over a quasi-compact open subscheme* $U \subset Y$*. Then there is a commutative diagram*

$$\begin{tikzcd} \widetilde{X} \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ Y' \arrow[r, "p"] & Y \end{tikzcd}$$

*where* $p\colon Y' \to Y$ *is a blow-up of a finitely presented closed subscheme* $Z \subset Y$ *disjoint from* $U$ *and the strict transform* $\widetilde{X}$ *of* $X$ *is flat over* $Y'$*.*

The *strict transform* $\widetilde{X}$ above is by definition the closure of $(Y' \setminus p^{-1}(Z)) \times_Y X$ in the base change $Y' \times_Y X$.

*Proof.* See [RG71, Thm. I.5.2.2] or [SP, Tag 0815]. □

### A.2.3 Faithful flatness

A module $M$ over a ring $A$ is *faithfully flat* if the functor $-\otimes_A M\colon \mathrm{Mod}(A) \to \mathrm{Mod}(A)$ is faithfully exact, i.e. a sequence $N' \to N \to N''$ of $A$-modules is exact if and only if $N' \otimes_A M \to N \otimes_A M \to N'' \otimes_A M$ is exact

for every nonzero map $\phi\colon N \to N'$ of $A$-modules, the induced map $\phi \otimes_A M\colon N \otimes_A M \to N' \otimes_A M$ is also nonzero.

**Proposition A.2.17** (Faithfully Flat Equivalences)**.** *Let $A$ be a ring and $M$ be a* flat *$A$-module. The following are equivalent:*

*(1) $M$ is faithfully flat;*

*(2) for every nonzero map $\phi\colon N \to N'$ of $A$-modules, the induced map $\phi \otimes_A M\colon N \otimes_A M \to N' \otimes_A M$ is also nonzero;*

*(3) for every nonzero $A$-module $N$, the tensor product $N \otimes_A M$ is nonzero;*

*(4) for every prime ideal $\mathfrak{p} \subset A$, the tensor product $M \otimes_A \kappa(\mathfrak{p})$ is nonzero; and*

*(5) for every maximal ideal $\mathfrak{m} \subset A$, the tensor product $M \otimes_A \kappa(\mathfrak{m}) \cong M/\mathfrak{m}M$ is nonzero.*

*Proof.* See [SP, Tag 00H9]. □

When $M = B$ is an $A$-algebra, then by (4) a flat ring map $A \to B$ is faithfully flat if $\operatorname{Spec} B \to \operatorname{Spec} A$ is surjective, or equivalently by (5) every maximal ideal of $A$ is in the image of $\operatorname{Spec} B \to \operatorname{Spec} A$. The latter equivalence implies that any flat local ring map is faithfully flat.

A morphism $f\colon X \to Y$ of schemes is *faithfully flat* if $f$ is flat and surjective. This is equivalent to the condition that $f^*\colon \mathrm{QCoh}(Y) \to \mathrm{QCoh}(X)$ is faithfully exact. It is also equivalent to the condition that a quasi-coherent $\mathcal{O}_Y$-module (resp. a morphism of quasi-coherent $\mathcal{O}_Y$-modules) is zero if and only if its pullback is. Faithfully flat morphisms play an important role in descent theory; see §B.

### A.2.4 Fppf and fpqc morphisms

Fppf and fpqc morphisms are acronyms for 'fidèlement plate de présentation finie' and 'fidèlement plat et quasi-compact,' respectively. Despite this terminology, an fpqc morphism is more general than a faithfully flat and quasi-compact map.

**Definition A.2.18.** We define a morphism $f\colon X \to Y$ of schemes to be:

(1) *fppf* if $f$ is faithfully flat and locally of finite presentation, and

(2) *fpqc* is $f$ is faithfully flat and every quasi-compact open subset of $Y$ is the image of a quasi-compact open subset of $X$.

**Remark A.2.19.** A quasi-compact and faithfully flat morphism is fpqc. In addition, an open and faithfully flat morphism is fpqc: for a quasi-compact open subset $V \subset Y$, we can write $f^{-1}(V) = \bigcup_i U_i$ as a union of affines, and since each $f(U_i) \subset V$ is open and $V$ is quasi-compact, we see that $V$ is the image of finitely many of the $U_i$'s. In particular, since every fppf morphism is open (Proposition A.2.12), an fppf morphism is also fpqc.

An fppf (resp. fpqc) cover $\{X_i \to X\}$ is a collection of morphisms such that $\coprod_i X_i \to X$ is fppf (resp. fpqc).

### A.2.5 Universally injective homomorphisms

The defining characteristic of a flat module is that it preserves every injection under tensoring. The dual notion of an injection of modules, which is preserved under tensoring, is also a very useful property.

**Definition A.2.20.** A homomorphism $M \to N$ of $A$-modules if *universally injective* if for every $A$-module $P$, the map $M \otimes_A P \to N \otimes_A P$ is injective. A ring map $A \to B$ is *universally injective* if it is as a map of $A$-modules.

We will use this notion in a fundamental way in our proof of Coherent Tannaka Duality (Theorem 6.4.1). To this end, the following properties will be used:

**Proposition A.2.21.**

*(1) A faithfully flat ring map $A \to B$ is universally injective.*

*(2) A split injective $M \to N$ of $A$-modules is universally injective. The converse is true if $N/M$ is finitely presented.*

*(3) If $A \to A'$ is faithfully flat, then a map $M \to N$ of $A$-modules is universally injective if and only if $M \otimes_A A' \to N \otimes_A A'$ is.*

*(4) If $A \to B$ is universally injective and $B \to B \otimes_A B, b \mapsto b \otimes 1$ is faithfully flat, then $A \to B$ is faithfully flat.*

*Proof.* For (1), (2) and (1), see [SP, Tags 08WP, 058L and 08XD]. Part (3) follows directly from the faithful exactness of $- \otimes_A A'$. See also [Laz69] and [Lam99, §4J]. □

**Remark A.2.22.** It is also true that an $A$-module map $M \to N$ is universally injective if and only if $\mathrm{Hom}_A(P, N) \to \mathrm{Hom}_A(P, N/M)$ is surjective for all finitely presented $A$-modules $P$ [SP, Tag 058F]. If $M/N$ is flat, then $M \to N$ is universally injective; if in addition $N$ is flat, then the converse is true (in which case $M$ is also flat [SP, Tag 058P].

Remarkably universally injective ring maps are precisely those maps that satisfy effective descent for modules; see Remark B.1.5.

## A.3 Étale, smooth, unramified, and syntomic morphisms

We discuss étale, smooth, unramified, and syntomic morphisms of schemes. We will have the following implications:

$$\text{unramified} \Longleftarrow \text{étale} \Longrightarrow \text{smooth} \Longrightarrow \text{syntomic} \Longrightarrow \text{fppf} \Longrightarrow \text{fpqc}.$$

### A.3.1 Smooth morphisms

A morphism $f \colon X \to Y$ of schemes is *smooth* if $f$ is locally of finite presentation, flat, and for every $y \in Y$ the geometric fiber $X_{\overline{\kappa(y)}} = X \times_Y \operatorname{Spec} \overline{\kappa(y)}$ is regular.

**Smooth Equivalences A.3.1.** Let $f \colon X \to Y$ be morphism of (resp. noetherian) schemes locally of finite presentation. The following are equivalent:

(1) $f$ is smooth;

(2) $f$ satisfies the *Infinitesimal Lifting Criterion for Smoothness* (sometimes referred to as *formal smoothness*): for every surjection $A \to A_0$ of rings with nilpotent kernel (resp. surjection $A \to A_0$ of local artinian rings whose kernel is isomorphic to the residue field $A/\mathfrak{m}_A$) and every commutative diagram

$$\begin{tikzcd} \operatorname{Spec} A_0 \arrow[r] \arrow[d, hook] & X \arrow[d, "f"] \\ \operatorname{Spec} A \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

of solid arrows, there exists a dotted arrow filling in the diagram;

(3) $f$ satisfies the *Jacobi Criterion for Smoothness*: for every point $x \in X$, there exists affine open neighborhoods $\operatorname{Spec} B$ of $f(x)$ and $\operatorname{Spec} A \subset f^{-1}(\operatorname{Spec} B)$ of $x$ and an $A$-algebra isomorphism

$$B \cong \big(A[x_1, \ldots, x_n]/(f_1, \ldots, f_r)\big)_g$$

for some $f_1, \ldots, f_r, g \in A[x_1, \ldots, x_n]$ with $r \leq n$ such that the determinant $\det(\frac{\delta f_j}{\delta x_i})_{1 \leq i,j \leq r} \in B$ of the Jacobi matrix, defined by the partial derivatives with respect to the *first* $r$ $x_i$'s, is a unit.

If in addition $X$ and $Y$ are locally of finite type over an algebraically closed field $\Bbbk$, then the above are equivalent to:

(4) for all $x \in X(\Bbbk)$, there is an isomorphism $\widehat{\mathcal{O}}_{X,x} \cong \widehat{\mathcal{O}}_{Y,y}[\![x_1, \ldots, x_r]\!]$ of $\widehat{\mathcal{O}}_{Y,y}$-algebras.

We say that a morphism $f\colon X \to Y$ of schemes is *smooth at* $x \in X$ if there exists an open neighborhood $U \subset X$ of $x$ such that $f|_U\colon U \to Y$ is smooth.

If $f\colon X \to Y$ is a smooth morphism of schemes, then $\Omega_{X/Y}$ is a locally free $\mathcal{O}_X$-module of finite rank. If $Y$ is connected, the rank of $\Omega_{X/Y}$ is the dimension of any fiber.

### A.3.2 Étale morphisms

A morphism $f\colon X \to Y$ of schemes is *étale* if $f$ is smooth of relative dimension 0 (i.e. $f$ is smooth and $\dim X_y = 0$ for all $y \in Y$).

**Étale Equivalences A.3.2.** Let $f\colon X \to Y$ be morphism of (resp. noetherian) schemes locally of finite presentation. The following are equivalent:

(1) $f$ is étale;

(2) $f$ is smooth and $\Omega_{X/Y} = 0$;

(3) $f$ is flat and for all $y \in Y$, the fiber $X_y$ is isomorphic to a disjoint union $\coprod_i \operatorname{Spec} K_i$ where each $K_i$ is separable field extension of $\kappa(y)$; (This is exactly the condition that $f$ is flat and unramified; see §A.3.3.)

(4) $f$ satisfies the *Infinitesimal Lifting Criterion for Étaleness*: for every surjection $A \to A_0$ of rings with nilpotent kernel (resp. surjection $A \to A_0$ of local artinian rings whose kernel is isomorphic to the residue field $A/\mathfrak{m}_A$) and every commutative diagram

$$\begin{tikzcd} \operatorname{Spec} A_0 \arrow[r] \arrow[d, hook] & X \arrow[d, "f"] \\ \operatorname{Spec} A \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

of solid arrows, there exists a unique dotted arrow filling in the diagram;

(5) $f$ satisfies the *Jacobi Criterion for Étaleness*: for every point $x \in X$, there exists affine open neighborhoods $\operatorname{Spec} B$ of $f(x)$ and $\operatorname{Spec} A \subset f^{-1}(\operatorname{Spec} B)$ of $x$ and an $A$-algebra isomorphism

$$B \cong \big(A[x_1, \ldots, x_n]/(f_1, \ldots, f_n)\big)_g$$

for some $f_1, \ldots, f_n, g \in A[x_1, \ldots, x_n]$ such that the determinant $\det(\frac{\delta f_j}{\delta x_i})_{1 \le i,j \le n} \in B$ is a unit.

If in addition $X$ and $Y$ are locally of finite type over an algebraically closed field $\Bbbk$, then the above are equivalent to:

(6) for all $x \in X(\Bbbk)$, the induced map $\widehat{\mathcal{O}}_{Y,y} \to \widehat{\mathcal{O}}_{X,x}$ on completions is an isomorphism

If in addition $X$ and $Y$ are smooth over $\Bbbk$, then the above are equivalent to:

(7) for all $x \in X(\Bbbk)$, the induced map $T_{X,x} \to T_{Y,y}$ on tangent spaces is an isomorphism.

We say that a morphism $f \colon X \to Y$ of schemes is *étale at* $x \in X$ if there exists an open neighborhood $U \subset X$ of $x$ such that $f|_U \colon U \to Y$ is étale.

### A.3.3 Unramified morphisms

A morphism $f \colon X \to Y$ of schemes is *unramified* if $f$ is locally of finite type and every geometric fiber is discrete and reduced. Note that this second condition is equivalent to requiring that for all $y \in Y$, the fiber $X_y$ is isomorphic to a disjoint union $\coprod_i \operatorname{Spec} K_i$ where each $K_i$ is a separable field extension of $\kappa(y)$.

**Caution A.3.3.** We are following the conventions of [RG71] and [SP] rather than [EGA] as we only require that $f$ is locally of finite type rather than locally of finite presentation.

**Unramified Equivalences A.3.4.** Let $f \colon X \to Y$ be morphism of schemes locally of finite type. The following are equivalent:

(1) $f$ is unramified;

(2) $\Omega_{X/Y} = 0$;

(3) $f$ satisfies the *Infinitesimal Lifting Criterion for Unramifiedness*: for every surjection $A \to A_0$ of rings with nilpotent kernel (resp. surjection $A \to A_0$ of local artinian rings whose kernel is isomorphic to the residue field $A/\mathfrak{m}_A$) and every commutative diagram

$$\begin{tikzcd} \operatorname{Spec} A_0 \arrow[r] \arrow[d, hook] & X \arrow[d, "f"] \\ \operatorname{Spec} A \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

of solid arrows, there exists at most one dotted arrow filling in the diagram.

If in addition $X$ and $Y$ are locally of finite type over an algebraically closed field $\Bbbk$, then the above are equivalent to:

(4) for all $x \in X(\Bbbk)$, the induced map $\widehat{\mathcal{O}}_{Y,y} \to \widehat{\mathcal{O}}_{X,x}$ on completions is surjective.

We say that a morphism $f \colon X \to Y$ of schemes is *unramified at* $x \in X$ if there exists an open neighborhood $U \subset X$ of $x$ such that $f|_U \colon U \to Y$ is unramified.

### A.3.4 Étale-local structure of smooth, étale, and unramified morphisms

Every smooth morphism looks étale-locally like relative affine space $\mathbb{A}^n_B \to \operatorname{Spec} B$.

**Proposition A.3.5.** *Let $X \to Y$ be a morphism of schemes smooth at $x \in X$. There exists affine open subschemes $\operatorname{Spec} A \subset X$ and $\operatorname{Spec} B \subset Y$ with $x \in \operatorname{Spec} A$, and a commutative diagram*

$$\begin{tikzcd} X \arrow[d] & \operatorname{Spec} A \arrow[l, hook', "\text{op}"'] \arrow[r, "\text{ét}"] \arrow[d] & \mathbb{A}^n_B \arrow[dl] \\ Y & \operatorname{Spec} B \arrow[l, hook', "\text{op}"'] & \end{tikzcd}$$

*where $\operatorname{Spec} A \to \mathbb{A}^n_B$ is étale.*

*Proof.* See [SP, Tag 039P]. □

**Corollary A.3.6.** *Let $f\colon X \to Y$ be a morphism of schemes smooth at $x \in X$. Then there exists an étale neighborhood $Y' \to Y$ of $f(x)$ such that $X \times_Y Y' \to Y'$ has a section.*

*Proof.* We apply the proposition. The morphism $\mathbb{A}^n_B \to \operatorname{Spec} B$ admits the zero section $\operatorname{Spec} B \to \mathbb{A}^n_B$ and we let $Y' := \operatorname{Spec} B \times_{\mathbb{A}^n_B} \operatorname{Spec} A$. Then $Y' \to \operatorname{Spec} B$ is étale and $Y' \to \operatorname{Spec} A \hookrightarrow X$ defines a section $Y' \to X \times_Y Y'$ of $X \times_Y Y' \to Y'$. □

Every étale (resp. unramified) morphism is étale-locally an isomorphism (resp. closed immersion).

**Proposition A.3.7.** *Let $f\colon X \to S$ be a separated morphism of schemes étale at $x \in X$. Then there exists an étale neighborhood $(U, u) \to (S, f(x))$ and a finite disjoint union decomposition*

$$X_U = W \amalg \coprod_i V_i$$

*such that each $V_i \to U$ is an isomorphism and the the fiber $W_u$ contains no point over $x$.*

*Proof.* See [SP, Tags 04HM and 04HG]. □

### A.3.5 Further properties

**Proposition A.3.8** (Fiberwise Criteria for Étaleness/Smoothness/Unramifiedness)**.** *Consider a diagram*

$$\begin{tikzcd} X \arrow[rr] \arrow[dr] & & Y \arrow[dl] \\ & S & \end{tikzcd}$$

*of schemes where $X \to S$ and $Y \to S$ are locally of finite presentation.*

(1) *$X \to Y$ is unramified if and only if $X_s \to Y_s$ is for all $s \in S$.*

(2) *If $X \to S$ is flat, then $X \to Y$ is étale (resp. smooth) if and only if $X_s \to Y_s$ is for all $s \in S$.*

**Remark A.3.9.** With the same hypotheses, let $x \in X$ be a point with image $s \in S$. Then $X \to Y$ is étale (resp. smooth, unramified) at $x \in X$ if and only if $X_s \to Y_s$ is at $x$.

**Corollary A.3.10.** *If $f\colon X \to Y$ is a proper flat morphism of finite presentation, then the set $\Sigma_f$, consisting of points $y \in Y$ where $X_y \to \operatorname{Spec} \kappa(y)$ is smooth, is open.*

*Proof.* By Remark A.3.9, if $y \in Y$ is a point such that $X_y \to \operatorname{Spec} \kappa(y)$ is smooth, then $f\colon X \to Y$ is smooth in an open neighborhood of $X_y$. If $Z \subset X$ is the closed locus where $f\colon X \to Y$ is not smooth, then $f(Z) \subset Y$ is precisely the locus where the fibers of $f$ are not smooth. Since $f$ is proper, $f(Z)$ is closed. □

**Proposition A.3.11.** *Let $X \to Y$ be a smooth morphism of noetherian schemes. For every point $x \in X$ with image $y \in Y$,*

$$\dim_x(X) = \dim_y(Y) + \dim_x(X_y).$$

**Proposition A.3.12.** *If $X \to Y$ is a finite étale morphism, there exists a finite étale cover $Y' \to Y$ such that $X \times_Y Y' \to Y'$ is a trivial covering, i.e. $X \times_Y Y'$ is isomorphism to $\coprod_i Y'$ over $Y'$.*

*Proof.* We may assume that the degree $d$ of $X \to Y$ is constant. The scheme $(X/Y)^d = \underbrace{X \times_Y \cdots \times_Y X}_{d}$ represents the functor on $Sch/Y$ assigning a $Y$-scheme $T$ to the set of $d$ sections of $X \times_Y T \to T$. Each pairwise diagonal $(X/Y)^{d-1} \to (X/Y)^d$ is an open and closed immersion and we set $(X/Y)^d_0 \subset (X/Y)^d$ to be the complement of all pairwise diagonals. The projection morphism $(X/Y)^d_0 \to Y$ is finite étale and the functorial description gives $d$ disjoint sections of $X \times_Y (X/Y)^d_0 \to (X/Y)^d_0$. □

**Proposition A.3.13.** *A dominant unramified morphism $X \to Y$ of schemes with $Y$ normal and $X$ connected is étale.*

*Proof.* See [SGA1, Cor. I.9.11]. □

### A.3.6 Fitting ideals and the singular locus

For background references on fitting ideals, we recommend [SP, Tag 07Z6] and [Eis95, §20]. If $R$ is a noetherian ring and $M$ is a finitely generated $R$-module, the *$k$th fitting ideal* $\operatorname{Fit}_k(M)$ *of $M$* is the ideal generated by the $(n-k) \times (n-k)$ minors of a matrix $A$ defining a presentation

$$\bigoplus_{i \in I} R \xrightarrow{A} R^n \to M \to 0.$$

Of course, when $M$ is finitely presented (e.g. $R$ is noetherian), then the left-hand term can be assumed to be a finite free module $R^m$, in which case $A$ is an $m \times n$ matrix and $\operatorname{Fit}_k(M)$ is a finitely generated ideal. The fitting ideal is independent of the choice of presentation. This defines an increasing sequence of ideals

$$0 = \operatorname{Fit}_{-1}(M) \subset \operatorname{Fit}_0(M) \subset \operatorname{Fit}_1(M) \subset \cdots$$

such that $\operatorname{Fit}_k(M) = R$ if $M$ can be generated by $k$ elements. The $R$-module $M$ is locally free of rank $r$ if and only if $\operatorname{Fit}_{r-1}(M) = 0$ and $\operatorname{Fit}_r(M) = R$, and in this case $\operatorname{Fit}_k(M) = 0$ for all $k < r$. There is an identification $\operatorname{Fit}_k(M \otimes_R S) = \operatorname{Fit}_k(M)S$ for a ring map $R \to S$. In particular, $\operatorname{Fit}_k(M_f) = \operatorname{Fit}_k(M)_f$ for $f \in R$, and $\operatorname{Fit}_k(M_{\mathfrak{p}}) = \operatorname{Fit}_k(M)_{\mathfrak{p}}$ for a prime ideal $\mathfrak{p} \subset R$; moreover $\operatorname{Fit}_k(M) \otimes_R \widehat{R} = \operatorname{Fit}_k(\widehat{M})$ if $R$ is a noetherian local ring.

If $X$ is a scheme and $F$ is a finitely generated quasi-coherent sheaf on $X$, the *$k$th fitting ideal sheaf* $\operatorname{Fit}_k(F)$ *of* $F$ is the quasi-coherent sheaf of ideals defined by the property that for an affine open $U \subset X$, $\Gamma(U, \operatorname{Fit}_k(F)) = \operatorname{Fit}_k(\Gamma(F, U))$.

Fitting ideals allow us to define a scheme structure on the singular locus.

**Definition A.3.14.** If $X$ is a noetherian scheme of pure dimension $d$ over a field $\Bbbk$ , we define the *singular locus of* $X$ as the subscheme $\operatorname{Sing}(X) := V(\operatorname{Fit}_d(\Omega_{X/\Bbbk}))$ defined by the $d$th fitting ideal of of $\Omega_{X/\Bbbk}$.

More generally, if $X \to S$ is an fppf morphism such that every fiber has pure dimension $d$, we define the *relative singular locus* as the subscheme $\operatorname{Sing}(X/S) := V(\operatorname{Fit}_d(\Omega_{X/S}))$.

For example, if $X = \operatorname{Spec} R$ with $R = \Bbbk[x_1, \dots, x_n]/I$ and $I = (f_1, \dots, f_m)$, then using the exact sequence $I/I^2 \to \Omega_{\mathbb{A}^n/\Bbbk}|_X \to \Omega_{X/\Bbbk} \to 0$, we see that there is a resolution

$$\mathcal{O}_X^m \xrightarrow{J} \mathcal{O}_X^n \to \Omega_{X/\Bbbk} \to 0 \quad \text{with } J = \left(\frac{\partial f_j}{\partial x_i}\right),$$

and $\operatorname{Sing}(X)$ is defined by all $(n-d) \times (n-d)$ minors of the $n \times m$ Jacobian matrix $J$.

### A.3.7 Local complete intersections

A scheme $X$ locally of finite type over a field $\Bbbk$ is a *local complete intersection at* $p \in X$ if there exists an affine open neighborhood $p \in \operatorname{Spec} A \subset A$ such that $A$ is a global complete intersection over $\Bbbk$, i.e. $A \cong \Bbbk[x_1, \dots, x_n]/(f_1, \dots, f_c)$ with $\dim A = n - c$. The scheme $X$ is a *local complete intersection* if it is at every point.

**Proposition A.3.15.** *For a scheme $X$ locally of finite type over a field $\Bbbk$ and a point $p \in X$, the following are equivalent:*

*(1) $X$ is a local complete intersection at $p$,*

*(2) the local ring $\mathcal{O}_{X,x} \cong R/(f_1, \dots, f_c)$ where $R$ is a regular local ring and $f_1, \dots, f_c \in R$ is a regular sequence, and*

*(3) the completion $\widehat{\mathcal{O}}_{X,x} \cong R/(f_1, \dots, f_c)$ where $R$ is a regular local ring and $f_1, \dots, f_c \in R$ is a regular sequence.*

*Proof.* See [SP, Tags 00S8 and 09PY]. □

For a scheme locally of finite type over a field $\Bbbk$, we have the following implications:

smooth $\implies$ local complete intersection $\implies$ Gorenstein $\implies$ Cohen–Macaulay.

### A.3.8 Syntomic morphisms

There is also a well-behaved relative notion of local complete intersections: a morphism of schemes $f \colon X \to S$ is *syntomic* (or a *flat local complete intersection morphism*) if $f$ is fppf and every fiber is a local complete intersection. We say that

$f\colon X \to S$ is *syntomic at* $x \in X$ if there is an open neighborhood $U$ of $x$ such that $f|_U$ is syntomic; this is equivalent to $f$ being fppf in an open neighborhood of $x$ and the fiber $X_s$ being a local complete intersection at $x$, where $s = f(x)$. Moreover, syntomic morphisms have a local structure analogous to local complete intersections: a morphism $f\colon X \to S$ is syntomic at $x \in X$ if and only if there are affine open neighborhood $x \in \operatorname{Spec} A \subset X$ and $\operatorname{Spec} B \subset Y$ with $f(\operatorname{Spec} A) \subset \operatorname{Spec} B$ such that $A \cong B[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ and every nonempty fiber of $\operatorname{Spec} A \to \operatorname{Spec} B$ has dimension $n - c$. See [EGA, IV §19.3], [SGA6, VIII §1] and [SP, Tag 01UB].

### A.3.9 Lifting étale, smooth, and syntomic morphisms along closed immersions

The following fact is sometimes convenient.

**Proposition A.3.16.** *Consider a diagram*

$$\begin{tikzcd} \operatorname{Spec} A_0 \arrow[r, hook, dashed] \arrow[d] \arrow[dr, phantom, "\square"] & \operatorname{Spec} A \arrow[d, dashed] \\ \operatorname{Spec} B_0 \arrow[r, hook] & \operatorname{Spec} B \end{tikzcd}$$

*of solid arrows where* $\operatorname{Spec} B \hookrightarrow \operatorname{Spec} B_0$ *is a closed immersion. If* $\operatorname{Spec} A_0 \to \operatorname{Spec} B_0$ *is étale (resp. smooth, syntomic), then there exists an étale (resp. smooth, syntomic) morphism* $\operatorname{Spec} A \to \operatorname{Spec} B$ *making the above diagram cartesian.*

*Proof.* See [SP, Tags 04D1 and 07M8]. □

## A.4 Properness and the valuative criterion

One of the most important applications of the valuative criterion is in moduli theory, where it can be applied for instance to show that $\overline{\mathcal{M}}_g$ is proper and $\mathrm{Bun}_C^{\mathrm{ss}}$ is universally closed. As we generalize the criterion to algebraic stacks, we quickly recap how it's established for schemes.

### A.4.1 Preliminaries

The starting point is the following lifting criterion for quasi-compact morphisms to be closed.

**Lemma A.4.1.** *A quasi-compact morphism* $f\colon X \to Y$ *of schemes is closed if and only if for every point* $x \in X$*, every specialization* $f(x) \rightsquigarrow y_0$ *lifts to a specialization* $x \rightsquigarrow x_0$*:*

$$\begin{tikzcd} X \arrow[d, "f"] & x \arrow[r, squiggly, dashed] \arrow[d, mapsto] & x_0 \arrow[d, mapsto, dashed] \\ Y & f(x) \arrow[r, squiggly] & y_0. \end{tikzcd}$$

*Proof.* The implication ($\Rightarrow$) is clear as $f(\overline{\{x\}}) \subset Y$ is closed. For the converse, after replacing $X$ with a closed subscheme, it suffices to show that $f(X)$ is closed. We can assume that $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$ are affine (since $f$ is quasi-compact) and reduced (since the question is topological). The scheme-theoretic image of $\operatorname{Spec} A \to \operatorname{Spec} B$ is defined by $I := \ker(B \to A)$. By replacing $B$ with $B/I$, we can assume that $B \to A$ is injective. For every minimal prime $\mathfrak{p} \in \operatorname{Spec} B$, the localization $B_{\mathfrak{p}} \to A_{\mathfrak{p}}$ is injective and thus $A_{\mathfrak{p}} \neq 0$ and the fiber $f^{-1}(\mathfrak{p}) = \operatorname{Spec} A_{\mathfrak{p}}$ is non-empty. Since $f(X)$ contains all the minimal primes and is closed under specialization, $f(X) = Y$ is closed. $\square$

The noetherian valuative criterion depends on the following algebraic fact:

**Proposition A.4.2.** *Let $(A, \mathfrak{m}_A)$ be a noetherian local domain with fraction field $K$ such that $A$ is not a field. If $K \to L$ is a finitely generated field extension, then there exists a DVR $R$ with fraction field $L$ dominating $A$ (i.e. $A \subset R$ and $\mathfrak{m}_A \cap K = \mathfrak{m}_R$ is the maximal ideal of $R$).*

*Proof.* We first reduce to the case that $K \to L$ is a finite field extension. To this end, choose a transcendence basis $x_1, \ldots, x_n \in L$ over $K$ and replace $A$ with $A[x_1, \ldots, x_n]_{\mathfrak{n}}$ where $\mathfrak{n} = \mathfrak{m}_A A[x_1, \ldots, x_n] + (x_1, \ldots, x_n)$.

Let $X = \operatorname{Spec} A$ with closed point $x = \mathfrak{m}_A$. Let $B = \operatorname{Bl}_x X$ be the blow-up of $X$ at $x$ with exceptional divisor $E$. If $\xi \in E$ is a generic point, then $\mathcal{O}_{B,\xi}$ is a noetherian domain of dimension 1 (by Krull's Hauptidealsatz) with fraction field $K$. We now let $R \subset L$ be the integral closure of $\mathcal{O}_{B,\xi}$ in $L$. By Krull–Akizuki (Proposition A.4.3), $R$ is noetherian. Since $R$ is also normal of dimension 1, it is a DVR. $\square$

**Proposition A.4.3** (Krull–Akizuki)**.** *Let $R$ be a noetherian domain of dimension 1 with fraction field $K$. If $K \to L$ is a finite extension of fields, then every ring $A$ with $R \subset A \subset L$ is noetherian.*

*Proof.* See [Nag62, p. 115] or [SP, Tag 00PG]. $\square$

Krull–Akizuki has the following geometric implication:

**Proposition A.4.4.** *If $f\colon X \to Y$ is a finite type morphism of noetherian schemes, $x \in X$ and $f(x) \rightsquigarrow y_0$ is a specialization, then there exists a commutative diagram*

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & X \arrow[d, "f"] & & x \arrow[d, mapsto] \\ \operatorname{Spec} R \arrow[r] & Y & & f(x) \arrow[r, rightsquigarrow] & y_0. \end{tikzcd}$$

*where $R$ is a DVR with fraction field $K$, the image of $\operatorname{Spec} K \to X$ is $x$ and $\operatorname{Spec} R \to Y$ realizes the specialization $f(x) \rightsquigarrow y_0$. In particular, every specialization $x \rightsquigarrow x_0$ in a noetherian scheme is realized by a map $\operatorname{Spec} R \to X$ from a DVR.*

*Proof.* After replacing $X$ with $\overline{\{f(x)\}}$ and $Y$ with $\overline{\{x\}}$, we may assume that $X$ and $Y$ are integral with generic points $x$ and $f(x)$. Then $\mathcal{O}_{Y,y_0}$ is a noetherian local domain with fraction field $\kappa(f(x))$. By applying Proposition A.4.2 to the field extension $\kappa(f(x)) \to \kappa(x)$, we obtain a DVR $R$ with fraction field $\kappa(x)$ dominating $\mathcal{O}_{Y,y_0}$ which yields the desired diagram. $\square$

### A.4.2 The Valuative Criteria

**Theorem A.4.5** (Valuative Criteria for Proper/Separated/Universally Closed Morphisms)**.** *Let $f\colon X \to Y$ be a finite type morphism of noetherian schemes. Consider* *a commutative diagram*

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \operatorname{Spec} R \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd} \tag{A.4.1}$$

*of solid arrows where $R$ is a DVR with fraction field $K$. Then*

*(1) $f$ is proper if and only if for every diagram* (A.4.1)*, there exists a unique lift.*

*(2) $f$ is separated if and only if for every diagram* (A.4.1)*, any two lifts are equal.*

*(3) $f$ is universally closed if and only if for every diagram* (A.4.1)*, there exists a lift.*

*Proof.* We first claim that it suffices to handle the universally closed case. Indeed, a morphism $X \to Y$ is separated if and only if the diagonal $X \to X \times_Y X$ is universally closed, and the equality of two lifts in the valuative criterion for $X \to Y$ corresponds to the existence of a lift in the valuative criterion for $X \to X \times_Y X$.

Suppose that $X \to Y$ satisfies the valuative criterion for universally closedness. To show that $X \to Y$ is universally closed, we claim that it suffices to check that for every *finite type* morphism $T \to Y$, the base change $X_T \to T$ is closed. Indeed, suppose that for an arbitrary morphism $T \to Y$ of schemes, the base change $f_T \colon X_T \to T$ is not closed. By Lemma A.4.1, there exists $x \in X_T$ and a specialization $f_T(x) \rightsquigarrow t_0$ which doesn't lift to a specialization $x \rightsquigarrow x_0$. This implies that $Z = \overline{\{x\}} \subset X_T$ has trivial intersection with the fiber $(X_T)_{t_0}$. Applying Lemma A.4.6 (with its notation) yields, after replacing $T$ with an open neighborhood of $t_0$, a commutative diagram

$$\begin{tikzcd} & x \arrow[d, mapsto] & X_T \arrow[r] \arrow[d, "f_T"] & X_{T'} \arrow[r] \arrow[d, "f_{T'}"] & X \arrow[d, "f"] \\ f_T(x) \arrow[r, squiggly] & t_0 & T \arrow[r, "g"] & T' \arrow[r] & Y \end{tikzcd}$$

where $T' \to Y$ is finite type and a closed subscheme $Z' \subset X_{T'}$ such that $f_{T'}(Z')$ contains $g(f_T(x))$ but not $g(t_0)$. This shows that $f_{T'} \colon X_{T'} \to T'$ is not closed.

Since the valuative criterion holds for $X \to Y$, it also holds for the morphism $X_T \to T$ of *noetherian* schemes. It therefore suffices to show that $X \to Y$ is closed. By Lemma A.4.1, it suffices to show that given $x \in X$, every specialization $f(x) \rightsquigarrow y_0$ lifts to a specialization $x \rightsquigarrow x_0$. By Proposition A.4.4, there exists a diagram (A.4.1) such that $\operatorname{Spec} R \to Y$ realizes $f(x) \rightsquigarrow y_0$ with a lift $\operatorname{Spec} K \to X$ whose image is $x$. The valuative criterion implies the existence of a lift $\operatorname{Spec} R \to X$, which in turn yields a specialization $x \rightsquigarrow x_0$ lifting $f(x) \rightsquigarrow y_0$.

Conversely, assume that $f \colon X \to Y$ is universally closed and that we are given a diagram (A.4.1). By replacing $Y$ with $\operatorname{Spec} R$ and $X$ with $X \times_Y \operatorname{Spec} R$, we may assume that $Y = \operatorname{Spec} R$ and that we have a diagram

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r, "x"] \arrow[d] & X \arrow[dl] \\ \operatorname{Spec} R & \end{tikzcd}$$

By replacing $X$ with $\overline{\{x\}}$, we may assume that $X$ is integral with generic point $x$. Since $X \to \operatorname{Spec} R$ is closed, there exists a specialization $x \rightsquigarrow x_0$ mapping to the specialization of the generic point to the closed point in $\operatorname{Spec} R$. This gives an inclusion of local rings $R \hookrightarrow \mathcal{O}_{X,x_0}$ in $K$. Since $R$ is a valuation ring with fraction field $K$ (i.e. is maximal among local rings properly contained in $K$), we see that $R = \mathcal{O}_{X,x_0}$ and the inclusion $\operatorname{Spec} \mathcal{O}_{X,x_0} \to X$ gives the desired lift. $\square$

**Lemma A.4.6.** *Let $f\colon X \to Y$ be a quasi-compact morphism of schemes. Let $T \to Y$ be a morphism of schemes, $t_0 \in T$ be a point and $Z \subset X_T$ a closed subscheme with trivial intersection with the fiber $(X_T)_{t_0}$. Then after replacing $T$ with an open neighborhood of $t_0$, there exists a finite type morphism $T' \to Y$ of schemes with a factorization $T \xrightarrow{g} T' \to Y$ and a closed subscheme $Z' \subset X_{T'}$ with trivial intersection with the fiber $(X_{T'})_{g(t_0)}$ such that $\operatorname{im}(Z \hookrightarrow X_T \to X_{T'}) \subset Z'$.*

*Proof.* See [SP, Tag 05BD]. $\square$

### A.4.3 Universally submersive morphisms

A morphism $f\colon X \to Y$ of schemes is *submersive* if $f$ is surjective and $Y$ has the quotient topology, i.e. a subset $U \subset Y$ is open if and only if $f^{-1}(U)$ is open, and $f\colon X \to Y$ is *universally submersive* if for every map $Y' \to Y$, the base change $X \times_Y Y' \to Y'$ is submersive.

**Exercise A.4.7.** (1) Show that a morphism $f\colon X \to Y$ of noetherian schemes is universally submersive if and only if for every map $\operatorname{Spec} R \to Y$ from a DVR, there is a commutative diagram

$$\begin{tikzcd} \operatorname{Spec} R' \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \operatorname{Spec} R \arrow[r] & Y \end{tikzcd}$$

where $R \to R'$ is a local homomorphism of DVRs.

(2) Show that every fppf morphism or universally closed morphism of noetherian schemes is universally submersive.

## A.5 Quasi-finite morphisms and Zariski's Main Theorem

We say that a locally of finite type morphism $x\colon X \to Y$ of schemes is *quasi-finite at* $x \in X$ if $x$ is isolated in the fiber $X_{f(x)} = X \times_Y \operatorname{Spec} \kappa(k(y))$. When $f\colon X \to Y$ is quasi-compact, then this is equivalent to $f^{-1}(f(x))$ being a finite set. We say that $f\colon X \to Y$ is *locally quasi-finite* if $f$ is locally of finite type and quasi-finite at every point, and *quasi-finite* if $f$ is of finite type and quasi-finite.

**Theorem A.5.1** (Étale Localization of Quasi-finite Morphisms)**.** *Let $f\colon X \to S$ be a separated and finite type morphism of schemes which is quasi-finite at $x \in X$. There exists an étale neighborhood $(S', s') \to (S, f(x))$ with $\kappa(s') = \kappa(f(x))$ and a decomposition $X \times_S S' = F \sqcup W$ into open and closed subschemes such that $F \to S'$ is finite and the fiber $W_{s'}$ is empty.*

*Proof.* See [EGA, IV.8.12.3] or [SP, Tag 02LP]. $\square$

**Proposition A.5.2.** *A separated and quasi-finite morphism $f\colon X \to Y$ of schemes factors as*

$$f\colon X \to \mathcal{S}\mathrm{pec}_Y\, f_*\mathcal{O}_X \to Y$$

*where $X \hookrightarrow \mathcal{S}\mathrm{pec}_Y\, f_*\mathcal{O}_X$ is an open immersion and $\mathcal{S}\mathrm{pec}_Y\, f_*\mathcal{O}_X \to Y$ is affine.*

*Proof.* As $f_*\mathcal{O}_X$ commutes with étale (even flat) base change on $Y$, so does the above factorization of $f\colon X \to Y$. Therefore, it suffices to show that every point $y \in Y$ has an étale neighborhood where the proposition holds. By Theorem A.5.1 we may assume that $X = X_1 \sqcup X_2$ with $X_1$ finite over $Y$ and $(X_2)_y = \emptyset$. After replacing $Y$ with $\mathcal{S}\mathrm{pec}_Y\, f_*\mathcal{O}_X$, we may also assume that $f_*\mathcal{O}_X = \mathcal{O}_Y$. As $\mathcal{O}_X = \mathcal{A}_1 \times \mathcal{A}_2$ is the product of quasi-coherent $\mathcal{O}_X$-algebras, $\mathcal{O}_Y = f_*\mathcal{O}_X = f_*\mathcal{A}_1 \times f_*\mathcal{A}_2$ and thus $Y$ decomposes as $Y_1 \sqcup Y_2$ such that $y \in Y_1$ and $f(X_i) \subset Y_i$ for $i = 1, 2$. After replacing $Y$ with $Y_1$, we see that $X \to Y$ is finite. Thus $X$ is affine and $X = Y = \mathcal{S}\mathrm{pec}_Y\, f_*\mathcal{O}_X$. □

In the above factorization, $f_*\mathcal{O}_Y$ may not be a finite type $\mathcal{O}_Y$-algebra (even if $Y$ is a noetherian affine scheme, then $\Gamma(X, \mathcal{O}_X)$ may not be a noetherian ring; see [Ols16, Ex. 7.2.15]). However, we may modify the factorization to arrange that $X \to Y$ factors as an open immersion followed by a *finite* morphism.

**Theorem A.5.3** (Zariski's Main Theorem)**.** *A separated and quasi-finite morphism $f\colon X \to Y$ of schemes factors as the composition of a dense open immersion $X \hookrightarrow \widetilde{Y}$ and a finite morphism $\widetilde{X} \to X$. In particular, $f$ is quasi-affine.*

*Proof.* If $\mathcal{A} \subset f_*\mathcal{O}_X$ denotes the integral closure of $\mathcal{O}_Y \to f_*\mathcal{O}_X$, there is a factorization $X \xrightarrow{j} \mathcal{S}\mathrm{pec}_Y\, \mathcal{A} \to Y$. We claim that $j\colon X \to \mathcal{S}\mathrm{pec}_Y\, \mathcal{A}$ is an open immersion. To show this claim, it suffices to show that for every point $x \in X$, there is an open neighborhood $V \subset \mathcal{S}\mathrm{pec}_Y\, \mathcal{A}$ of $j(x)$ such that $j^{-1}(V) \to V$ is an isomorphism. Since normalization commutes with étale base change (Proposition A.5.4) and being an open immersion at a point is an étale local property, we are free to replace $Y$ by an étale neighborhood of $f(x)$. By Theorem A.5.1, we can assume that $X = F \sqcup W$ with $F$ finite over $Y$ and $x \in F$. In this case, the normalization $\mathcal{S}\mathrm{pec}_Y\, \mathcal{A}$ of $Y$ in $X$ is $F \sqcup \widetilde{W}$ where $\widetilde{W}$ is the normalization of $Y$ in $W$, and the claim follows.

By construction, $\mathcal{S}\mathrm{pec}_Y\, \mathcal{A} \to Y$ is integral. We can write $\mathcal{A} = \operatorname{colim} \mathcal{A}_\lambda$ as the colimit of finite type $\mathcal{O}_Y$-algebras and since open immersions descent under limits (Proposition A.6.7), we see that $X \to \mathcal{S}\mathrm{pec}_Y\, \mathcal{A}_\lambda$ is an open immersion for $\lambda \gg 0$. Since $\mathcal{S}\mathrm{pec}_Y\, \mathcal{A}_\lambda \to Y$ is integral and finite type, it is finite.

See also [EGA, IV.8.12.6] or [SP, Tag 05K0]. □

The following algebra result was used above and will be useful to generalize Zariski's Main Theorem to algebraic spaces (Theorem 4.4.9) and stacks (Theorem 4.4.9).

**Proposition A.5.4.** *Let $Y$ be a scheme, $\mathcal{B}$ be a quasi-coherent $\mathcal{O}_Y$-algebra and $\widetilde{\mathcal{B}}$ be the integral closure of $\mathcal{O}_Y$ in $\mathcal{B}$. If $f\colon X \to Y$ is a smooth morphism, then $f^*\widetilde{\mathcal{B}}$ is identified with the integral closure of $\mathcal{O}_X$ in $f^*\mathcal{B}$.*

*Proof.* See [SP, Tag 03GG] or [LMB00, Prop. 16.2]. □

Zariski's Main Theorem has some useful corollaries.

**Corollary A.5.5.** *A quasi-finite and proper morphism (resp. proper monomorphism) of schemes is finite (resp. a closed immersion).* □

*Proof.* If $f\colon X \to Y$ is a quasi-finite and proper, Zariski's Main Theorem (A.5.3) gives a factorization $f\colon X \hookrightarrow \widetilde{X} \to Y$ and the dense open immersion $X \hookrightarrow \widetilde{X}$ is also closed, thus an isomorphism. On the other hand, if $f\colon X \to Y$ is a proper monomorphism, then it is also quasi-finite, thus finite. The statement reduces to the algebra fact that a finite epimorphism of rings is surjective (c.f. [SP, Tag 04VT]). $\square$

**Remark A.5.6.** Every universally closed morphism is necessarily quasi-compact [SP, Tag 04XU]. It follows that every morphism, which is universally closed, locally of finite type, and a monomorphism, is a closed immersion; see also [SP, Tag 04XV].

## A.6 Limits of schemes

In moduli theory, we often need to deal with non-noetherian rings for the simple reason that moduli functors and stacks are defined over the category Sch of all schemes. Working instead with the category of locally noetherian schemes has the limitation that it is not closed under fiber products while working instead with the category of schemes finite type over a field or $\mathbb{Z}$ doesn't contain local rings of schemes or their completions.

In any case, using the limit methods presented in this section, it is usually straightforward to reduce properties of schemes and their morphisms to the noetherian case.

### A.6.1 Limits of schemes

The first result states that a limit exists for an inverse system $(S_\lambda, f_{\lambda\mu})_{\lambda\in\Lambda}$ of schemes over a directed set $\Lambda$ (see Definition A.1.2) where the transition map $f_{\lambda\mu}\colon S_\lambda \to S_\mu$ for every $\lambda \geq \mu$ is affine.

**Proposition A.6.1** (Existence of Limits)**.** *If $(S_\lambda, f_{\lambda\mu})_{\lambda\in\Lambda}$ is an inverse system of schemes with affine transition maps, then the limit $S = \lim_\lambda S_\lambda$ exists in the category of schemes such that each morphism $f_\lambda\colon S \to S_\lambda$ is affine.*

*Proof.* If each $S_\lambda = \operatorname{Spec} A_\lambda$ is affine, one takes $S = \operatorname{Spec}(\operatorname{colim}_\lambda A_\lambda)$. In general, choose an element $0 \in I$ and set $S = \mathcal{S}\mathrm{pec}_{S_0}(\operatorname{colim}_{\lambda\geq 0} f_{\lambda 0,*}\mathcal{O}_{S_\lambda})$. Details can be found in [EGA, IV.8.2] and [SP, Tag 01YX]. $\square$

A morphism $f\colon X \to Y$ of schemes is locally of finite presentation if and only if for every inverse system $(S_\lambda, f_{\lambda\mu})$ of affine schemes over $Y$, the map $\operatorname{colim}_\lambda \operatorname{Mor}_Y(S_\lambda, X) \to \operatorname{Mor}_Y(\lim_\lambda S_\lambda, X)$ is bijective (Proposition A.1.3) The same holds for inverse systems of quasi-compact and quasi-separated schemes over $S$ with affine transition maps; see [EGA, IV.8.14.1] and [SP, Tag 01ZC].

### A.6.2 Noetherian approximation

Every affine scheme $\operatorname{Spec} A$ is the limit of affine schemes $\operatorname{Spec} A_\lambda$ of finite type over $\mathbb{Z}$. This follows from the fact that the ring $A$ is the union of its finitely generated $\mathbb{Z}$-subalgebras. More generally, we have:

**Proposition A.6.2** (Relative Noetherian Approximation)**.** *Let $X \to S$ be a morphism of schemes with $X$ quasi-compact and quasi-separated and with $S$ quasi-separated. Then $X = \lim_\lambda X_\lambda$ is a limit of an inverse system $(X_\lambda, f_{\lambda\mu})$ of schemes of finite presentation over $S$ with affine transition maps over $S$.*

*Proof.* See [SP, Tag 09MV]. □

When $S = \operatorname{Spec} \mathbb{Z}$, this is often referred to as *Absolute Noetherian Approximation* and was first established in [TT90, Thm. C.9].

### A.6.3 Descending properties under limits

**Proposition A.6.3** (Descending Properties of Schemes under Limits)**.** *Let $S = \lim_\lambda S_\lambda$ be a limit of an inverse system of quasi-compact and quasi-separated schemes with affine transition maps. If $S$ is affine (resp. quasi-affine, separated), then so is $S_\lambda$ for $\lambda \gg 0$.*

*Proof.* See [SP, Tags 01Z6, 086Q and 01Z5] and [TT90, Props C.6-7]. □

**Proposition A.6.4** (Descending Morphisms under Limits)**.** *Let $S = \lim_\lambda S_{\lambda \in \Lambda}$ be a limit of an inverse system of quasi-compact and quasi-separated schemes with affine transition maps.*

*(1) For a finitely presented morphism $X \to S$ of schemes, there exists an index $0 \in \Lambda$ and a finitely presented morphism $X_0 \to S_0$ of schemes such that $X \cong X_0 \times_{S_0} S$. Moreover, if we define $X_\lambda := X_0 \times_{S_0} S_\lambda$ for $\lambda > 0$, then $X = \lim_{\lambda \geq 0} X_\lambda$ is the limit of the inverse system $(X_\lambda, f_{\lambda\mu})$ where the (affine) transition map $f_{\lambda\mu} \colon X_\lambda \to X_\mu$ is the base change of $S_\lambda \to S_\mu$ for $\lambda \geq \mu$.*

*(2) Let $X_0$ and $Y_0$ be finitely presented schemes over $S_0$ for some index $0 \in \Lambda$. For $\lambda > 0$, set $X_\lambda = X_0 \times_{S_0} S_\lambda$ and $Y_\lambda = Y_0 \times_{S_0} S_\lambda$, and let $X = \lim_\lambda X_\lambda$ and $Y = \lim_\lambda Y_\lambda$ be the limits (Proposition A.6.1). Then the natural map*

$$\operatorname{colim}_{\lambda \geq 0} \operatorname{Mor}_{S_\lambda}(X_\lambda, Y_\lambda) \to \operatorname{Mor}_S(X, Y)$$

*is bijective.*

**Remark A.6.5.** In other words, the category of schemes finitely presented over $S$ is the colimit of the categories of schemes finitely presented over $S_\lambda$.

*Proof.* See [EGA, IV.8.8] and [SP, Tag 01ZM]. □

**Definition A.6.6.** We say that a property $\mathcal{P}$ of morphisms of schemes *descends under limits* if the following holds for every limit $S = \lim_{\lambda \in \Lambda} S_\lambda$ of an inverse system of quasi-compact and quasi-separated schemes with affine transition maps: for every index $0 \in \Lambda$, and for every morphism $g_0 \colon X_0 \to Y_0$ of quasi-compact and quasi-separated schemes with base changes $g_\lambda \colon X_\lambda \to Y_\lambda$ over $S_\lambda$ and $g \colon X \to Y$ over $S$, we require that if $g$ has $\mathcal{P}$, then $g_\lambda$ has $\mathcal{P}$ for $\lambda \gg 0$.

**Proposition A.6.7** (Descending Properties of Morphisms under Limits)**.** *The following properties of morphisms of schemes descend under limits: isomorphism, closed immersion, open immersion, affine, quasi-affine, finite, quasi-finite, proper, projective, quasi-projective, separated, monomorphism, surjective, flat, locally of finite presentation, unramified, étale, smooth, syntomic, and for any integer $d$ the property that every fiber is connected and has pure dimension $d$.*

*Proof.* See [EGA, IV 8.10.5] and [SP, Tags 081C and 05M5]. □

Suppose $S = \operatorname{colim} S_\lambda$ is the colimit of an inverse system $(S_\lambda, f_{\lambda\mu})$ of quasi-compact and quasi-separated schemes with affine transition maps. If $F_0$ is a quasi-coherent sheaf on $S_0$ for an index $0 \in \Lambda$, and $F = f_\lambda^* F$ and $F_\lambda = f_{\lambda 0}^* F_0$ are the pullbacks to $S$ and $S_\lambda$ for $\lambda \geq 0$, then $\Gamma(S, F) = \operatorname{colim}_{\lambda \geq 0} \Gamma(S_\lambda, F_\lambda)$ [SP, Tag 01Z0]. Moreover, a quasi-coherent sheaf on $S$ and its properties often descend to some $S_\lambda$.

**Proposition A.6.8** (Descending Sheaves under Limits)**.** *Let $(S_\lambda, f_{\lambda\mu})$ be an inverse system of quasi-compact and quasi-separated schemes with affine transition maps and limit $S = \lim_{\lambda\in\Lambda} S_\lambda$. Denote the projection maps by $f_\lambda \colon S \to S_\lambda$.*

*(1) If $F$ is an $\mathcal{O}_S$-module of finite presentation (resp. vector bundle, line bundle), then there exists an index $i \in \Lambda$ and an $\mathcal{O}_{S_\lambda}$ module $F_\lambda$ of finite presentation (resp. vector bundle, line bundle) such that $F \cong f_\lambda^* F_\lambda$.*

*(2) For an index $0 \in \Lambda$, let $F_0$ and $G_0$ be $\mathcal{O}_{S_0}$-modules of finite presentation. The natural map*

$$\operatorname{colim}_{\lambda\geq 0} \operatorname{Hom}_{\mathcal{O}_{S_\lambda}}(f_{\lambda 0}^* F_0, f_{\lambda 0}^* G_0) \to \operatorname{Hom}_{\mathcal{O}_S}(f_0^* F_0, f_0^* G_0)$$

*is bijective.*

*(3) For an index $0 \in \Lambda$, let $f_0 \colon X_0 \to Y_0$ be a finitely presented morphism of schemes over $S_0$ and let $F_0$ be a quasi-coherent sheaf on $X_0$ of finite presentation. If the pullback of $F_0$ under $X_0 \times_{S_0} S \to X_0$ is flat over $Y_0 \times_{S_0} S$, then the pullback of $F_0$ under $X_0 \times_{S_0} S_\lambda \to X_0$ is flat over $Y_0 \times_{S_0} S_\lambda$ for $\lambda \gg 0$.*

**Remark A.6.9.** In other words, the category of finitely presented modules over $S$ is the colimit of the categories of finitely presented modules over $S_\lambda$.

*Proof.* See [EGA, IV.8.5.2] and [SP, Tags 01ZR, 0B8W and 05LY]. □

### A.6.4 Application

For a typical application of noetherian approximation in moduli theory, we illustrate here how properties of an arbitrary family of curves can be reduced to a family over a noetherian base.

**Proposition A.6.10.** *Let $S$ be a quasi-compact and quasi-separated scheme (e.g. an affine scheme), and let $\mathcal{C} \to S$ be a proper flat and finitely presented morphism of schemes such that every geometric fiber has dimension at most 1. Then there exists a cartesian diagram*

$$\begin{tikzcd} \mathcal{C} \arrow[r] \arrow[d] & \mathcal{C}' \arrow[d] \\ S \arrow[r] & S' \end{tikzcd}$$

*where $S'$ is a scheme of finite type over $\mathbb{Z}$ and $\mathcal{C}' \to S'$ is a proper flat morphism of schemes such that every geometric fiber has dimension at most 1. Moreover, if $\mathcal{C} \to S$ is smooth, then $\mathcal{C}' \to S'$ can also be arranged to be smooth.*

**Remark A.6.11.** The upshot is that we can now establish properties of the morphism $\mathcal{C}_\lambda \to S_\lambda$ of *noetherian* schemes and then deduce properties of $\mathcal{C} \to S$ by base change. In Lemma 5.2.17, we show if $\mathcal{C} \to S$ is a nodal family of curves, then $\mathcal{C}' \to S'$ can be arranged to be nodal.

*Proof.* Write $S = \lim_{\lambda\in\Lambda} S_\lambda$ as a limit of an inverse system of schemes of finite type over $\mathbb{Z}$ (Proposition A.6.1). Note that each $S_\lambda$ is quasi-compact and quasi-separated. Since $\mathcal{C} \to S$ is finitely presented, there exists an index $0 \in \Lambda$ and a finitely presented morphism $\mathcal{C}_0 \to S_0$ such that $\mathcal{C} \cong \mathcal{C}_0 \times_{S_0} S$ (Proposition A.6.4). For each $\lambda > 0$, we can define $\mathcal{C}_\lambda = \mathcal{C}_0 \times_{S_0} S_\lambda$ and we have a cartesian diagram

$$\begin{tikzcd} \mathcal{C} \arrow[r] \arrow[d] & \mathcal{C}_\lambda \arrow[d] \\ S \arrow[r] & S_\lambda. \end{tikzcd}$$

Since $\mathcal{C} \to S$ is flat and proper with fiber of dimension at most 1 (resp. smooth), then there exists $\lambda_0 \in I$ such that the same is true for $\mathcal{C}_\lambda \to S_\lambda$ for all $\lambda \geq \lambda_0$ (Proposition A.6.7). We now take $S' = S_\lambda$ and $\mathcal{C}' = \mathcal{C}_\lambda$ for every $\lambda \geq \lambda_0$. $\square$

## A.7 Cohomology and base change

Given a proper morphism $f\colon X \to Y$ of noetherian schemes and a coherent sheaf $F$ on $X$, we would like to know:

(a) When is $\mathrm{R}^i f_* F$ a vector bundle on $Y$?

(b) For a morphism of schemes $Y' \to Y$ inducing a cartesian diagram

$$\begin{tikzcd} X_{Y'} \arrow[r, "g'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y, \end{tikzcd}$$

when is the comparison map

$$\phi^i_{Y'}\colon g^* \mathrm{R}^i f_* F \to \mathrm{R}^i f'_* g'^* F \tag{A.7.1}$$

an isomorphism?

When $f\colon X \to Y$ is flat, Flat Base Change tells us that (A.7.1) is always an isomorphism. Cohomology and Base Change provides an answer when $F$ is flat over $Y$.

Cohomology and Base Change is an essential tool in moduli theory. It can be applied to verify properties of families of objects and construct vector bundles on moduli spaces. For instance, for a family $\pi\colon \mathcal{C} \to S$ of smooth curves, we can verify that $\pi_* \Omega^{\otimes k}_{\mathcal{C}/S}$ is a vector bundle for $k > 0$ whose construction commutes with base change on $S$ and that $\mathcal{C}$ embeds canonically into $\mathbb{P}_S(\pi_* \Omega^{\otimes k}_{\mathcal{C}/S})$ for $k \geq 3$ (Proposition 5.1.9). These properties are used for instance to verify the algebraicity of $\mathcal{M}_g$ (Theorem 3.1.16). On the other hand, applying this result to the universal family $\pi\colon \mathcal{U}_g \to \mathcal{M}_g$ yields vector bundles $\pi_* \Omega^{\otimes k}_{\mathcal{U}_g/\mathcal{M}_g}$ on $\mathcal{M}_g$; when $k = 1$, this is a vector bundle of rank $g$ called the *Hodge bundle*.

### A.7.1 Algebraic input

The key algebraic input to Cohomology and Base Change is the following:

**Theorem A.7.1.** *Let $X \to \operatorname{Spec} A$ be a proper morphism of noetherian schemes and $F$ be a coherent sheaf on $X$ which is flat over $A$. There is a complex*

$$K^\bullet\colon 0 \to K^0 \to K^1 \to \cdots \to K^n \to 0$$

*of finitely generated, projective $A$-modules such that $\mathrm{H}^i(X, F) = \mathrm{H}^i(K^\bullet)$ for all $i$.*

*Moreover, for every $A$-module $M$, $\mathrm{H}^i(X, F \otimes_A M) = H^i(K^\bullet \otimes_A M)$. In particular, for a morphism $\operatorname{Spec} B \to \operatorname{Spec} A$ of schemes, $\mathrm{H}^i(X_B, F_B) = \mathrm{H}^i(K^\bullet \otimes_A B)$ where $X_B := X \times_{\operatorname{Spec} A} \operatorname{Spec} B$ and $F_B$ is the pullback of $F$ to $X_B$.*

*Proof.* This is established by choosing a finite affine cover $\{U_i\}$ of $X$ and considering the corresponding alternating Čech complex $C^\bullet$ on $\{U_i\}$ with coefficients in $F$.

Then $C^\bullet$ is a finite complex of free (but not finitely generated) $A$-modules and $\mathrm{H}^i(X, F) = \mathrm{H}^i(C^\bullet)$. The result is then obtained by inductively refining $C^\bullet$ to build a finite complex $K^\bullet$ of finitely generated, projective $A$-modules which is quasi-isomorphic to $C^\bullet$.

See [Mum70, p.46] where the last statement is established for $A$-algebras $B$ but the argument goes through for every $A$-module $M$. See also [SP, Tag 07VK] or [Vak17, 28.2.1]. □

**Remark A.7.2** (Perfect complexes)**.** A bounded complex $K^\bullet$ of coherent sheaves on a noetherian scheme $X$ is *perfect* if there is an affine cover $X = \bigcup_i U_i$ such that each $K^\bullet|_{U_i}$ is quasi-isomorphic to a bounded complex of vector bundles on $U_i$. (By a vector bundle, we mean a locally free sheaf of finite rank—this is equivalent to the corresponding module on $\Gamma(U_i, \mathcal{O}_{U_i})$ to be finitely generated and projective.) If $X$ is affine (or more generally has the resolution property, i.e. every coherent sheaf is the quotient of a vector bundle), then $K^\bullet$ is perfect if and only if it is quasi-isomorphic to a bounded complex of vector bundles on $X$ [SP, Tags 066Y and 0F8F]). Moreover, the compact objects in $D_{\mathrm{QCoh}}(X)$ are precisely the perfect complexes [SP, Tag 09M8].

With this terminology in place, Theorem A.7.1 has the following translation: $\mathrm{R}f_*F \in D^b_{\mathrm{Coh}}(\operatorname{Spec} A)$ is perfect [SP, Tag 07VK]. More generally, if $F^\bullet$ is a perfect complex on $X$, then $\mathrm{R}f_*F^\bullet$ is also perfect [SP, Tag 0A1H].

### A.7.2 Theorems of Semicontinuity, Grauert, and Cohomology and Base Change

Theorem A.7.1 tells us for a proper morphism $X \to \operatorname{Spec} A$ and coherent sheaf $F$ on $X$ flat over $A$, the cohomology $\mathrm{H}^i(X, F)$ can be computed using a perfect complex $K^\bullet$. Since Zariski-locally on the base, the complex $K^\bullet$ is a finite complex of *free* objects, this reduces cohomological questions to linear algebra.

**Theorem A.7.3** (Semicontinuity Theorem)**.** *Let $f\colon X \to Y$ be a proper morphism of noetherian schemes and $F$ be a coherent sheaf on $X$ which is flat over $Y$.*

*(1) For each $i \geq 0$, the function*

$$Y \to \mathbb{Z}, \quad y \mapsto \mathrm{h}^i(X_y, F_y)$$

*is upper semicontinuous.*

*(2) The function*

$$Y \to \mathbb{Z}, \quad y \mapsto \chi(X_y, F_y) = \sum_{i=0}^{\infty} (-1)^i \mathrm{h}^i(X_y, F_y)$$

*is locally constant.*

*Proof.* This is a direct consequence of Theorem A.7.1; see also [Mum70, p. 47], [Har77, Thm. 12.8] or [Vak17, 28.2.4]. □

When the base scheme is reduced, Grauert's Theorem provides a criterion for when the higher pushforward sheaves $\mathrm{R}^i f_* F$ are vector bundles.

**Theorem A.7.4** (Grauert's Theorem)**.** *Let $f\colon X \to Y$ be a proper morphism of noetherian schemes and let $F$ be a coherent sheaf on $X$ which is flat over $Y$. Assume that $Y$ is reduced and connected. For each integer $i$, the following are equivalent:*

*(1) the function $y \mapsto \mathrm{h}^i(X_y, F_y)$ is constant; and*

*(2) $\mathrm{R}^i f_* F$ is a vector bundle and the comparison map*

$$\phi^i_y \colon \mathrm{R}^i f_* F \otimes \kappa(y) \to \mathrm{H}^i(X_y, F_y)$$

*is an isomorphism for all $y \in Y$.*

*If these conditions hold, then we have the following additional properties:*

*(a) for all maps $g \colon Y' \to Y$ of schemes, the comparison map $\phi^i_{Y'} \colon g^* \mathrm{R}^p f_* F \to \mathrm{R}^i f'_* g'^* F$ is an isomorphism; and*

*(b) The comparison map $\phi^{i-1}_y \colon \mathrm{R}^{i-1} f_* F \otimes \kappa(y) \to \mathrm{H}^{i-1}(X_y, F_y)$ is an isomorphism.*

*Proof.* See [Mum70, p.51-2], [Har77, Cor. 12.9] and [Vak17, 28.1.5]. □

Grauert's Theorem is proved by using that $\mathrm{R} f_* F$ is a perfect complex and a linear algebra argument to show that $\mathrm{R}^i f_* F \otimes \kappa(y)$ has constant dimension. Since $Y$ is reduced, $\mathrm{R}^i f_* F$ is a vector bundle. When $Y$ is not reduced, the local criterion for flatness can be leveraged to provide the following useful criteria.

**Theorem A.7.5** (Cohomology and Base Change)**.** *Let $f \colon X \to Y$ be a proper and finitely presented morphism of schemes, and let $F$ be a finitely presented sheaf on $X$ which is flat over $Y$. Suppose that for a point $y \in Y$ and integer $i$, the comparison map $\phi^i_y \colon \mathrm{R}^i f_* F \otimes \kappa(y) \to \mathrm{H}^i(X_y, F_y)$ is surjective. Then the following hold*

*(a) There is an open neighborhood $V \subset Y$ of $y$ such that for every morphism $Y' \to V$ of schemes, the comparison map $\phi^i_{Y'} \colon g^* \mathrm{R}^p f_* F \to \mathrm{R}^i f'_* g'^* F$ is an isomorphism. In particular, $\phi^i_y$ is an isomorphism.*

*(b) $\phi^{i-1}_y$ is surjective if and only if $\mathrm{R}^i f_* F$ is a vector bundle in an open neighborhood of $y$.*

*Proof.* See [EGA, III.7.7.5, III.7.7.10, III.7.8.4], [Har77, Thm. 12.11] and [Vak17, 28.1.6]. □

**Remark A.7.6.** For moduli-theoretic applications, it is sometimes convenient to apply Cohomology and Base Change in the non-noetherian setting. Using the methods of Noetherian Approximation from §A.6, it is not hard to see how the general statement follows from the noetherian version. Since the statement is local on $Y$, we can assume $Y$ is affine and we can write $Y = \lim_{\lambda \in \Lambda} Y_\lambda$ as a limit of affine schemes of finite type over $\mathbb{Z}$. Since $X \to Y$ is finitely presented, there exists an index $0 \in \Lambda$ and a finitely presented morphism $X_0 \to Y_0$ such that $X \cong X_0 \times_{Y_0} Y$ (Proposition A.6.4). For each $\lambda > 0$, we can define $X_\lambda = X_0 \times_{Y_0} Y_\lambda$ and we have $X \cong X_\lambda \times_{Y_\lambda} Y$. By Proposition A.6.7, $X_\lambda \to Y_\lambda$ is proper for $\lambda \gg 0$. By Proposition A.6.8(1), there exists an index $\mu \in \Lambda$ and a coherent sheaf $F_\mu$ on $X_\mu$ that pulls back to $F$ under $X \to X_\mu$. For $\lambda > \mu$, set $F_\lambda$ to be the pullback of $F_\mu$ under $X_\lambda \to X_\mu$. By Proposition A.6.8(3), $F_\lambda$ is flat over $Y_\lambda$ for $\lambda \gg 0$. We may now apply noetherian Cohomology and Base Change to the data of $X_\lambda \to Y_\lambda$ and $F_\lambda$ for $\lambda \gg 0$, and we may deduce the same properties for $X \to Y$ and $F$ under the base change $Y \to Y_\lambda$.

**Corollary A.7.7.** *Let $f \colon X \to Y$ be a proper morphism of noetherian schemes and let $F$ be a coherent sheaf on $X$ which is flat over $Y$. The following are equivalent:*

*(1) $\mathrm{H}^i(X_y, F_y) = 0$ for all $y \in Y$ and $i > 0$; and*

*(2)* $\mathrm{R}^i f_*F = 0$ *for all* $i > 0$*, and* $f_*F$ *is a vector bundle whose construction commutes with base change on* $Y$ *(i.e. for all morphisms* $g\colon Y' \to Y$ *of schemes, the comparison map* $\phi^0_{Y'}\colon g^*f_*F \to f'_*g'^*F$ *is an isomorphism).*

*Proof.* The implication (2) $\Rightarrow$ (1) follows from choosing $N > \dim X_y$ so that $\mathrm{H}^N(X_y, F_y) = 0$ and $\phi^N_y$ is surjective, and then inductively applying Cohomology and Base Change (A.7.5(b)) to conclude $\phi^i_y$ is surjective for all $i \le N$.

For the converse, since $\phi^i_y\colon \mathrm{R}^i f_*F \otimes \kappa(y) \to \mathrm{H}^i(X_y, F_y) = 0$ is surjective for all $y \in Y$ and $i > 0$, A.7.5(a) implies that each $\phi^i_y$ is an isomorphism and therefore $\mathrm{R}^i f_*F = 0$ for $i > 0$. We now apply Cohomology and Base Change three more times: A.7.5(b) with $i = 1$ implies that $\phi^0_y$ is surjective for all $y \in Y$, A.7.5(b) with $i = 0$ (as $\phi^{-1}_y$ is necessarily surjective) implies that $f_*F$ is a vector bundle, and A.7.5(a) with $i = 0$ implies that the construction of $f_*F$ commutes with base change on $Y$. $\square$

### A.7.3 Applications to moduli theory

Here is a typical application of Cohomology and Base Change to moduli theory. The following proposition is used to establish properties of smooth families of curves (Proposition 5.1.9), and its argument applies in the same way to families of stable curves (Proposition 5.3.9).

**Proposition A.7.8.** *Let* $\pi\colon \mathcal{C} \to S$ *be a family of smooth curves of genus* $g \ge 2$ *(i.e.* $\mathcal{C} \to S$ *is a smooth, proper morphism of schemes such that every geometric fiber is a connected curve of genus* $g$*). Then*

*(1)* $\pi_*\mathcal{O}_{\mathcal{C}} = \mathcal{O}_S$*;*

*(2) For* $k > 1$*, the pushforward* $\pi_*(\Omega^{\otimes k}_{\mathcal{C}/S})$ *is a vector bundle of rank* $(2k-1)(g-1)$ *whose construction commutes with base change on* $S$ *and* $\mathrm{R}^i\pi_*(\Omega^{\otimes k}_{\mathcal{C}/S}) = 0$ *for* $i > 0$*.*

*(3) The pushforward* $\pi_*(\Omega_{\mathcal{C}/S})$ *is a vector bundle of rank* $g$ *whose construction commutes with base change on* $S$ *and* $\mathrm{R}^1\pi_*(\Omega_{\mathcal{C}/S}) \cong \mathcal{O}_S$ *while* $\mathrm{R}^i\pi_*(\Omega_{\mathcal{C}/S}) = 0$ *for* $i \ge 2$*.*

*Proof.* To see (1), observe that $\mathrm{H}^0(\mathcal{C}_s, \mathcal{O}_{\mathcal{C}_s}) = \kappa(s)$ for all $s \in S$ since $\mathcal{C}_s$ is proper and geometrically connected. It follows that $\phi^0_s\colon \pi_*\mathcal{O}_{\mathcal{C}} \otimes \kappa(s) \to \mathrm{H}^0(\mathcal{C}_s, \mathcal{O}_{\mathcal{C}_s})$ is surjective. Cohomology and Base Change (A.7.5(a)–(b) with $i = 0$) implies that $\phi^0_s$ is an isomorphism and that $\pi_*\mathcal{O}_{\mathcal{C}}$ is a line bundle. On a fiber over $s \in S$, the natural map $\mathcal{O}_S \to \pi_*\mathcal{O}_{\mathcal{C}}$ induces a surjective map $\kappa(s) \to \pi_*\mathcal{O}_{\mathcal{C}} \otimes \kappa(s)$ (as post-composing with $\phi^0_s\colon \pi_*\mathcal{O}_{\mathcal{C}} \otimes \kappa(s) \to \mathrm{H}^0(\mathcal{C}_s, \mathcal{O}_{\mathcal{C}_s}) = \kappa(s)$ is the identity). Thus $\mathcal{O}_S \to \pi_*\mathcal{O}_{\mathcal{C}}$ is a surjective morphism of line bundles, hence an isomorphism.

For (2) with $k > 1$, $\mathrm{H}^1(\mathcal{C}_s, \Omega^{\otimes k}_{\mathcal{C}_s/\kappa(s)}) = \mathrm{H}^0(\mathcal{C}_s, \Omega^{\otimes(1-k)}_{\mathcal{C}_s/\kappa(s)})$ for all $s \in S$ by Serre Duality (5.1.2) and this vanishes as $\deg(\Omega^{\otimes(1-k)}_{\mathcal{C}_s/\kappa(s)}) < 0$. Note that we also have the vanishing of $\mathrm{H}^i(\mathcal{C}_s, \Omega^{\otimes k}_{\mathcal{C}_s/\kappa(s)})$ for $i \ge 2$ since $\dim \mathcal{C}_s = 1$. Cohomology and Base Change (A.7.5(a)) gives the vanishing of the higher pushforward $\mathrm{R}^i\pi_*(\Omega^{\otimes k}_{\mathcal{C}/S}) = 0$ for $i > 0$. On the other hand, $\mathrm{h}^0(\mathcal{C}_s, \Omega^{\otimes k}_{\mathcal{C}_s/\kappa(s)}) = \deg(\Omega^{\otimes k}_{\mathcal{C}_s/\kappa(s)}) + 1 - g = (2k-1)(g-1)$ by Riemann–Roch (5.1.1). Corollary A.7.7 implies that $\pi_*(\Omega^{\otimes k}_{\mathcal{C}/S})$ is a vector bundle of rank $(2k-1)(g-1)$.

For (3), since $\Omega_{\mathcal{C}/S}$ is a relative dualizing sheaf (see [Liu02, §6.4]), Grothendieck–Serre Duality implies that $\mathrm{R}^1\pi_*\Omega_{\mathcal{C}/S} \cong \pi_*\mathcal{O}_{\mathcal{C}}$ and this is identified with $\mathcal{O}_S$ by (1). For $i \ge 2$, $\mathrm{H}^i(\mathcal{C}_s, \Omega_{\mathcal{C}/S} \otimes \kappa(s)) = 0$ and A.7.5(a) implies that $\mathrm{R}^i\pi_*\Omega_{\mathcal{C}/S} = 0$.

Applying A.7.5(b) with $i = 2$ yields that $\phi^1_s\colon \mathrm{R}^1\pi_*\Omega_{\mathcal{C}/S} \otimes \kappa(s) \to \mathrm{H}^1(\mathcal{C}_s, \Omega_{\mathcal{C}_s/\kappa(s)})$ is surjective for every $s \in S$ and thus an isomorphism (A.7.5(a) with $i = 1$). Since $\mathrm{R}^1\pi_*\Omega_{\mathcal{C}/S} \cong \pi_*\mathcal{O}_{\mathcal{C}} \cong \mathcal{O}_S$ is a line bundle, applying A.7.5(b) with $i = 1$ implies that $\phi^0_s\colon \pi_*\Omega_{\mathcal{C}/S} \otimes \kappa(s) \to \mathrm{H}^0(\mathcal{C}_s, \Omega_{\mathcal{C}_s/\kappa(s)})$ is surjective, and applying A.7.5(a)–(b) with $i = 0$ implies that $\pi_*\Omega_{\mathcal{C}/S}$ is a vector bundle of rank $\mathrm{h}^0(\mathcal{C}_s, \Omega_{\mathcal{C}_s/\kappa(s)}) = g$ whose construction commutes with base change. □

The following proposition is useful to verify the algebraicity of stacks of coherent sheaves and vector bundles (Theorem 3.1.20).

**Proposition A.7.9.** *Let $p\colon X \to S$ be a proper morphism of schemes and $F$ be a finitely presented sheaf on $X$ of finite presentation and flat over $S$. Suppose that $\dim X_s \leq d$ for all $s \in S$. The subset $S'$ of points $s \in S$ such that $\mathrm{H}^j(X_s, F_s) = 0$ for all $j > 0$ is open. Denoting $X' = p^{-1}(S')$, $p' := p|_{X'}\colon X' \to S$ and $F' = F|_{X'}$, we have that $\mathrm{R}^j p'_* F' = 0$ for all $j > 0$ and that $p'_*F'$ is a vector bundle whose construction commutes with base change.*

*Proof.* For each $j = 1, \dots, d$, A.7.5(a) implies that the locus of points $s \in S$ such that $\mathrm{H}^j(X_s, F_s) = 0$ is open and the comparison map $\phi^j_s\colon \mathrm{R}^j p_* F \otimes \kappa(s) \to \mathrm{H}^j(X_s, F_s)$ is an isomorphism. It follows that $\mathrm{R}^j p'_* F = 0$ which allows us to apply A.7.5(b) (with $i = 1$) to conclude that $\phi^0_s\colon p'_*F' \otimes \kappa(s) \to \mathrm{H}^0(X_s, F_s)$ is surjective. Apply A.7.5(a)-(b) (with $i = 0$) now gives the final statement. □

For the following proposition (specialized to $n = 1$) is convenient to define determinantal line bundles on $\mathrm{Bun}_{C,r,d}$.

**Proposition A.7.10.** *Let $f\colon X \to S$ be a smooth projective morphism of relative dimension $n$ between noetherian schemes. If $F$ is a coherent sheaf on $X$ flat over $S$, there is a locally free resolution*

$$0 \to F_n \to F_{n-1} \to \cdots \to F_0 \to F$$

*such that*

- $\mathrm{R}^i f_* F_d = 0$ *for* $i \neq n$ *and* $d = 0, \dots, n$,
- $\mathrm{R}^n f_* F_d$ *is locally free for* $d = 0, \dots, n$,
- $\mathrm{R}^i f_* F$ *is the* $(n-i)$*th homology* $H_{n-i}(\mathrm{R}^n f_* F_\bullet)$ *of the complex* $\mathrm{R}^n f_* F_\bullet$.
- *the determinant*
$$\det \mathrm{R} f_* F := \bigotimes_i (\det \mathrm{R}^i f_* F)^{(-1)^i}$$
*is identified with* $\bigotimes_d (\det \mathrm{R}^n f_* F_d)^{(-1)^{n-d}}$.

*Moreover, the construction is compatible with base change on $S$.*

*Proof.* See [HL10, Prop. 2.1.10]. □

### A.7.4 Applications to line bundles

Given a proper flat morphism $f\colon X \to Y$, when is a line bundle $L$ on $X$ the pullback of a line bundle on $Y$? More generally, is there a largest subscheme $Z \subset Y$ where $L_Z$ on $X_Z = X \times_Y Z$ is the pullback of a line bundle on $Z$? In this section, we provide three answers in increasing complexity.

As we must impose conditions on the fibers $X_y$, we first discuss relationships between various conditions.

**Lemma A.7.11.** *Let $f\colon X \to Y$ be a proper flat morphism of noetherian schemes. Consider the following conditions:*

*(1) the geometric fibers of $f\colon X \to Y$ are non-empty, connected and reduced;*

*(2) $\mathrm{h}^0(X_y, \mathcal{O}_{X_y}) = 1$ for all $y \in Y$; and*

*(3) $\mathcal{O}_Y = f_*\mathcal{O}_X$ and this holds after arbitrary base change (i.e. $\mathcal{O}_T = f_{T,*}\mathcal{O}_{X_T}$ for a morphism $T \to Y$ of schemes).*

*Then (1) $\Rightarrow$ (2) $\iff$ (3).*

*Proof.* If (1) holds, then $\mathrm{H}^0(X_y, \mathcal{O}_{X_y}) \otimes_{\kappa(y)} \overline{\kappa(y)} = \mathrm{H}^0(X \times_Y \overline{\kappa(y)}, \mathcal{O}_{X \times_Y \overline{\kappa(y)}}) = \overline{\kappa(y)}$ by Flat Base Change and the fact that a connected, reduced and proper scheme over an algebraically closed field has only constant functions. This gives (2).

If (2) holds, then the comparison map $\phi_y^0\colon f_*\mathcal{O}_X \otimes \kappa(y) \to \mathrm{H}^0(X_y, \mathcal{O}_{X_y}) = \kappa(y)$ is necessarily surjective as we have the global section $1 \in \mathrm{H}^0(Y, f_*\mathcal{O}_X)$. Theorem A.7.5 (with $i = 0$) implies that $f_*\mathcal{O}_X$ is a line bundle and that $\mathcal{O}_Y \to f_*\mathcal{O}_X$ is a surjection of line bundles, hence an isomorphism. Since the same argument applies to the base change $X_T \to T$, this gives (3). The converse (3) $\Rightarrow$ (2) follows by consider the base change $T = \operatorname{Spec} \kappa(y) \to Y$. □

When $Y$ is reduced, Grauert's Theorem provides a complete answer to when a line bundle is a pullback.

**Proposition A.7.12** (Version 1)**.** *Let $f\colon X \to Y$ be a proper flat morphism of noetherian schemes such that $\mathrm{h}^0(X_y, \mathcal{O}_{X_y}) = 1$ for all $y \in Y$. Let $L$ be a line bundle on $X$. If $Y$ is reduced, then $L = f^*M$ for a line bundle $M$ on $Y$ if and only if $L_y$ is trivial for all $y \in Y$. Moreover, if these conditions hold, then $M = f_*L$ and the adjunction morphism $f^*f_*L \to L$ is an isomorphism.*

*Proof.* The condition on geometric fibers implies that $\mathrm{h}^0(X_y, L_y) = 1$ and Grauert's Theorem (A.7.4) implies that $f_*L$ is a line bundle and that $f_*L \otimes \kappa(y) \xrightarrow{\sim} \mathrm{H}^0(X_y, L_y)$ is an isomorphism. We claim that $f^*f_*L \to L$ is surjective. It suffices to show that $(f^*f_*L)|_{X_y} \to L|_{X_y}$ is surjective. Denoting $f_y\colon X_y \to \operatorname{Spec} \kappa(y)$, we have identifications $(f^*f_*L)|_{X_y} = f_y^*(f_*L \otimes \kappa(y)) = f_y^*(\mathrm{H}^0(X_y, L_y)) = \mathcal{O}_{X_y}$ and the claim follows. Since $f^*f_*L \to L$ is a surjection of line bundles, it is an isomorphism. □

**Exercise A.7.13.** Show that if $Y$ is a connected and reduced noetherian scheme and $E$ is a vector bundle, then $\operatorname{Pic}(\mathbb{P}_Y(E)) = \operatorname{Pic}(Y) \times \mathbb{Z}$. See also [Har77, Exer. III.12.5].

**Proposition A.7.14** (Version 2)**.** *Let $f\colon X \to Y$ be a proper flat morphism of noetherian schemes with integral geometric fibers. For a line bundle $L$ on $X$, the locus*

$$\{y \in Y \mid L_y \text{ is trivial}\} \subset Y$$

*is closed.*

*Proof.* The important observation here is that for a geometrically integral and proper scheme $Z$ over field $\Bbbk$, a line bundle $M$ is trivial if and only if $\mathrm{h}^0(Z, M) > 0$ and $\mathrm{h}^0(Z, M^\vee) > 0$. To see that the latter condition is sufficient, observe that we have nonzero homomorphisms $\mathcal{O}_Z \to M$ and $\mathcal{O}_Z \to M^\vee$, the latter of which dualizes to a nonzero map $M \to \mathcal{O}_Z$. Since $Z$ is integral, the composition $\mathcal{O}_Z \to M \to \mathcal{O}_Z$ is also nonzero and is defined by a constant in $\mathrm{H}^0(Z, \mathcal{O}_Z) = \Bbbk$. It follows that $M \to \mathcal{O}_Z$ is a surjective map of line bundles, hence an isomorphism. By the Semicontinuity Theorem (A.7.3) the condition that $\mathrm{h}^0(X_y, L_y) > 0$ and $\mathrm{h}^0(X_y, L_y^\vee) > 0$ is closed, and the statement follows. See also [Mum70, p. 51]. □

**Remark A.7.15.** If the geometric fibers are only connected and reduced, the locus may fail to be closed. For example, consider a family of smooth curves $f\colon X \to Y$ where $Y$ is a curve and $X$ is a smooth surface. For a closed point $x \in X$, consider the blow-up $\mathrm{Bl}_x X \to X$ and let $E$ be the exceptional divisor. Then $\mathrm{Bl}_x X \to Y$ is a proper flat morphism, and the fiber over $f(x) \in Y$ is connected and reduced but reducible. The line bundle $L = \mathcal{O}_{\mathrm{Bl}_x X}(E)$ has the property that the fiber $L_y$ is trivial if and only if $y \neq f(x)$.

The two versions above can be combined into the following powerful statement for a proper flat morphism $X \to Y$. For moduli-theoretic applications, it is essential that we allow the possibility that $Y$ is non-reduced and that the fibers $X_y$ be reducible. The proposition will be applied in the proof of Theorem 3.1.16 to show that the locus of curves $C$ in a Hilbert scheme $\mathrm{Hilb}^P(\mathbb{P}^{5g-6}_{\mathbb{Z}}/\mathbb{Z})$ which are tri-canonically is a closed condition.

**Proposition A.7.16** (Version 3)**.** *Let $f\colon X \to Y$ be a proper flat morphism of noetherian schemes such that $\mathrm{h}^0(X_y, \mathcal{O}_{X_y}) = 1$ for all $y \in Y$ (resp. the geometric fibers are integral). For a line bundle $L$ on $X$, there is a unique locally closed (resp. closed) subscheme $Z \subset Y$ such that*

*(1) $L_Z$ on $X_Z = X \times_Y Z$ is the pullback of a line bundle on $Z$; and*

*(2) if $T \to Y$ is a morphism of schemes such that $L_T$ on $X_T$ is the pullback of a line bundle on $T$, then $T \to Y$ factors through $Z$.*

**Remark A.7.17.** In other words, the functor

$$\mathrm{Sch}/Y \to \mathrm{Sets},$$
$$(T \to Y) \mapsto \begin{cases} \{*\} & \text{if } L_T \text{ is the pullback of a line bundle on } T \\ \emptyset & \text{otherwise} \end{cases}$$

is representable by a locally closed (resp. closed) subscheme of $Y$.

*Proof.* By the Semicontinuity Theorem (A.7.3), the locus $V = \{y \in Y \mid \mathrm{h}^0(X_y, L_y) \leq 1\}$ is open. Since for points $y \notin V$, $L_y$ is not trivial, we may replace $Y$ with $V$ and assume that $\mathrm{h}^0(X_y, L_y) \leq 1$ for all $y \in Y$.

Observe that if $L = f^*M$ for a line bundle $M$ on $Y$, then by using the projection formula and the fact that $\mathcal{O}_Y = f_*\mathcal{O}_X$ (Lemma A.7.11), we see that $f_*L \cong f_*f^*M \cong f_*f^*\mathcal{O}_Y \otimes M \cong f_*\mathcal{O}_X \otimes M \cong M$ is a line bundle and that the adjunction map $f^*f_*L \to L$ is an isomorphism. The same holds for the base change $X_T \to T$, and we conclude that $L_T$ is a pullback of a line bundle on $T$ if and only if $f_{T,*}L$ is a line bundle and $f_T^*f_{T,*}L \to L$ is an isomorphism. We see therefore that the question is Zariski-local on $Y$ and $T$. We will show that every point $y \in Y$ has an open neighborhood where the proposition holds.

By applying Theorem A.7.1 and after replacing $Y$ with an open affine neighborhood of $y$, we may assume that there is a homomorphism $d\colon A^{r_0} \xrightarrow{d} A^{r_1}$ of finitely generated and free $A$-modules such that for every ring map $A \to B$, $\mathrm{H}^0(X_B, L_B) = \ker(d \otimes B)$. Consider the dual $d^\vee$ of $d$, we define $M$ as the cokernel in the sequence

$$A^{r_1} \xrightarrow{d^\vee} A^{r_0} \to M \to 0.$$

Tensoring over $A \to B$ yields a right exact sequence

$$B^{r_1} \xrightarrow{d^\vee \otimes B} B^{r_0} \to M \otimes_A B \to 0$$

which after applying the contravariant left-exact functor $\mathrm{Hom}_B(-, B)$ becomes

$$0 \to \mathrm{Hom}_B(M \otimes_A B, B) \to B^{r_0} \xrightarrow{d \otimes_A B} B^{r_1}.$$

We conclude that

$$\mathrm{H}^0(X_B, L_B) = \mathrm{Hom}_B(M \otimes_A B, B) = \mathrm{Hom}_A(M, B). \tag{A.7.2}$$

Applying this to $A \to \kappa(y)$ for every point $y \in \operatorname{Spec} A$, we have $\mathrm{H}^0(X_y, L_y) = \mathrm{Hom}_A(M, \kappa(y)) = M \otimes_A \kappa(y)$.

If $\mathrm{h}^0(X_y, L_y) = 0$, then $L_y$ is not trivial and there is an open neighborhood $U$ of $y$ such that $\widetilde{M}|_U = 0$. The proposition holds over $U$ since there are no morphisms $T \to U$ from a non-empty scheme such that $L_T$ is a pullback.

If $\mathrm{h}^0(X_y, L_y) = 1$, then $M \otimes_A \kappa(y) = \kappa(y)$ and by Nakayama's lemma, after replacing $Y$ with an open affine neighborhood of $y$, there is a surjection $A \to M$. Write $M = A/I$ for an ideal $I$ and define the closed subscheme $Z = V(I) \subset Y$. Observe that $\mathrm{H}^0(Z, L_Z) = \mathrm{Hom}_A(A/I, A/I) = A/I$ so that $f_{Z,*}L_Z$ is the trivial line bundle. For an $A/I$-algebra $B$, we have that $\mathrm{H}^0(X_B, L_B) = \mathrm{Hom}_A(A/I, B) = B$. It follows that the comparison map $\mathrm{H}^0(X_Z, L_Z) \otimes_{A/I} B \to \mathrm{H}^0(X_B, L_B)$ is an isomorphism, or in other words the construction of $f_{Z,*}L_Z$ commutes with base change. We claim that $T \to Y$ factors through $Z$ if and only if $f_{T,*}L_T$ is a line bundle. This question is Zariski-local on $T$ so we may assume $T = \operatorname{Spec} B$ is affine. If $f_{T,*}L_T$ is a line bundle, we may assume $f_{T,*}L_T = \mathcal{O}_T$ is trivial since the question is local on $T$. Then $B = \mathrm{Hom}_A(A/I, B)$ implies that $I \subset \ker(A \to B)$ or in other words that $A \to B$ factors as $A \to A/I \to B$. Finally, considering the adjunction morphism $\lambda\colon f_Z^* f_{Z,*} L_Z \to L_Z$ on $X_Z$, we claim that for $y \in Z$, $L_y$ is trivial if and only if $\lambda|_{X_y}$ is surjective. If $\lambda|_{X_y}$ is surjective, then using that $f_{Z,*}L_Z = \mathcal{O}_Z$, we have a surjection $\mathcal{O}_{X_y} \to L_y$ of line bundles, hence an isomorphism. For converse, observe that since $f_{Z,*}L_Z$ commutes with base change, the comparison map $f_{Z,*}L_Z \otimes \kappa(y) = \mathrm{H}^0(X_y, L_y)$ is an isomorphism. Denoting $f_y\colon X_y \to \operatorname{Spec} \kappa(y)$, we have identifications $(f_Z^* f_{Z,*} L_Z)|_{X_y} = f_y^*(f_{Z,*}L_Z \otimes \kappa(y)) = f_y^* f_{y,*} L_y$ under which $\lambda|_{X_y}$ corresponds to the adjunction map $f_y^* f_{y,*} L_y \to L_y$ which is an isomorphism. Replacing $Z$ with $Z \setminus \operatorname{Supp}(\operatorname{coker}(\lambda))$ establishes the proposition in the case that $\mathrm{h}^0(X_y, \mathcal{O}_{X_y}) = 1$ for all $y \in Y$. If the fibers are geometrically integral, then Proposition A.7.14 implies that $Z$ is closed.

See also [Mum70, p. 90], [Vie95, Lem. 1.19] and [SP, Tags 0BEZ and 0BF0]. □

**Remark A.7.18.** Note that to prove the strongest version, we needed the strongest version of our various cohomology and base change results, namely Theorem A.7.1.

**Remark A.7.19.** For a proper flat morphism $X \to S$, define the *Picard functor* as

$$\mathrm{Pic}_{X/S}\colon \mathrm{Sch}/S \to \mathrm{Sets}, \quad T \mapsto \mathrm{Pic}(X_T)/f_T^* \mathrm{Pic}(T).$$

If $f\colon X \to S$ has geometrically integral fibers, then the existence of a closed subscheme $Z \subset Y$ characterized by Proposition A.7.16 is equivalent to the diagonal morphism $\mathrm{Pic}_{X/S} \to \mathrm{Pic}_{X/S} \times_S \mathrm{Pic}_{X/S}$ of presheaves over $\mathrm{Sch}/S$ being representable by closed immersions, i.e. $\mathrm{Pic}_{X/S}$ is separated over $S$.

## A.8 Pushouts

### A.8.1 Existence of pushouts

Pushouts are the dual notion of fiber products. Unlike fiber products, pushouts may not always exist. However, Ferrand showed that they often exist when one of the maps is a closed immersion and the other is an affine morphism.

**Theorem A.8.1** (Ferrand's Theorem on the Existence of Pushouts)**.** *Consider a diagram*

$$\begin{tikzcd} X_0 \arrow[r, hook, "i"] \arrow[d, "f_0"'] & X \arrow[d, dashed, "f"] \\ Y_0 \arrow[r, hook, dashed, "j"] & Y \end{tikzcd} \tag{A.8.1}$$

*of schemes where $i\colon X_0 \hookrightarrow X$ is a closed immersion and $f_0\colon X_0 \to Y_0$ is affine. If*

($\star$) *for every point $y_0 \in Y_0$, the subspace $f_0^{-1}(\operatorname{Spec} \mathcal{O}_{Y_0,y_0}) \subset X_0$ has a basis of open affine neighborhoods of $X$,*

*then there exists a closed immersion $j\colon Y_0 \hookrightarrow Y$ and an affine morphism $f\colon X \to Y$ of schemes such that (A.8.1) is cocartesian (i.e. a pushout). Moreover, we have the following properties:*

*(a) the square (A.8.1) is cartesian, $X \to Y$ restricts to an isomorphism $X \setminus X_0 \to Y \setminus Y_0$ and the induced map $X \amalg Y_0 \to Y$ is universally submersive;*

*(b) the induced map*

$$\mathcal{O}_Y \to j_* \mathcal{O}_{Y_0} \times_{(j \circ f_0)_* \mathcal{O}_{X_0}} f_* \mathcal{O}_X$$

*is an isomorphism of sheaves; and*

*(c) if $f_0$ is finite (resp. integral), then so is $f$. In this case, Condition ($\star$) can be replaced with the condition that every finite set of points in $X_0$ and $Y$ is contained in an open affine (resp. for every $y_0 \in Y_0$, $f_0^{-1}(y_0)$ is contained in an open affine). Finally, if $X_0$, $X$, and $Y_0$ are of finite type over a noetherian scheme, then so is $Y$.*

*Proof.* See [Fer03, Thm. 5.4 and 7.1] and [SP, Tag 0ECH]. □

**Example A.8.2** (Affine case)**.** In the affine case where $X = \operatorname{Spec} A$, $X_0 = \operatorname{Spec} A_0$, $Y_0 = \operatorname{Spec} B_0$, then $\operatorname{Spec}(A \times_{A_0} B_0)$ is the pushout $X \amalg_{X_0} Y_0$.

**Example A.8.3** (Gluing and pinching)**.** If $X_0 \hookrightarrow X$ and $X_0 \hookrightarrow Y_0$ are closed immersions, the pushout $X \amalg_{X_0} Y_0$ can be viewed as the gluing of $X$ and $Y_0$ along $X_0$. For example, the nodal curve $\operatorname{Spec} k[x,y]/xy$ is the union of $\mathbb{A}^1$ and $\mathbb{A}^1$ along their origins. If $X_0 = Z \sqcup Z$ is the union of two isomorphic disjoint subschemes of $X$ and $X_0 \to Z$ is the projection, then the pushout $X \amalg_{Z \sqcup Z} Z$ can be viewed as the pinching of the two copies of $Z$ in $X$. For example, the nodal cubic curve is the pinching of 0 and $\infty$ in $\mathbb{P}^1$.

**Example A.8.4** (Non-noetherianess)**.** When $f_0\colon X_0 \to Y_0$ is affine but not finite, the pushout $X \amalg_{X_0} Y_0$ is often not noetherian. For example, if $X_0 = V(x) \subset X = \mathbb{A}^2_{\Bbbk}$ and $f_0\colon X_0 \to \operatorname{Spec} \Bbbk$, the pushout is the non-noetherian affine scheme defined by

$$k[x,y] \times_{k[x]} k = k[x, xy, xy^2, xy^3, \ldots] \subset k[x,y].$$

On the other hand, we wouldn't expect a finite type pushout as one cannot contract the $y$-axis in $\mathbb{A}^2_{\Bbbk}$.

### A.8.2 Properties of pushouts

Given a fiber product diagram of rings

$$\begin{tikzcd}
B \arrow[r] \arrow[d, two heads] & A \arrow[d, two heads] \\
B_0 \arrow[r] & A_0
\end{tikzcd}$$

with $A \twoheadrightarrow A_0$ surjective and $B := B_0 \times_{A_0} A$, the fiber product $\mathrm{Mod}(B_0) \times_{\mathrm{Mod}(A_0)} \mathrm{Mod}(A)$ is the category of triples $(N_0, M, \alpha)$ consisting of a $B_0$-module $N_0$, an $A$-module $M$ and isomorphism $\alpha\colon N_0 \otimes_{B_0} A_0 \xrightarrow{\sim} M \otimes_A A_0$. Equivalently, an object is a diagram

$$\begin{tikzcd}
 & & M \arrow[dl] \arrow[d, dash] \\
N_0 \arrow[r] \arrow[d, dash] & M_0 \arrow[d, dash] & A \arrow[dl] \\
B_0 \arrow[r] & A_0 &
\end{tikzcd} \tag{A.8.2}$$

where $N_0$, $M_0$ and $M$ are modules over $B_0$, $A_0$ and $A$, and the maps $N_0 \to M_0$ and $M \to M_0$ are morphisms of $B_0$ and $A$-modules inducing isomorphisms $N_0 \otimes_{B_0} A_0 \to M_0$ and $M \otimes_A A_0 \to M_0$.

We define functors

$$\mathrm{Mod}(B) \underset{L}{\overset{R}{\leftrightarrows}} \mathrm{Mod}(B_0) \times_{\mathrm{Mod}(A_0)} \mathrm{Mod}(A) \tag{A.8.3}$$

where for a $B$-module $N$, $L(N) := (N \otimes_B B_0, N \otimes_B A, \alpha)$ with $\alpha$ being the canonical isomorphism $(N \otimes_B B_0) \otimes_{B_0} A_0 \xrightarrow{\sim} (N \otimes_B A) \otimes_A A_0$. For an object $(N_0, M, \alpha)$ corresponding to a diagram (A.8.2), we define $R(N_0, M, \alpha) := N_0 \times_{M_0} M$, which we can view as:

$$\begin{tikzcd}[row sep=small]
 & N_0 \times_{M_0} M \arrow[rr] \arrow[dl] \arrow[dd, dash] & & M \arrow[dl] \arrow[dd, dash] \\
N_0 \arrow[rr] \arrow[dd, dash] & & M_0 \arrow[dd, dash] & \\
 & B_0 \times_{A_0} A \arrow[rr] \arrow[dl] & & A \arrow[dl] \\
B_0 \arrow[rr] & & A_0. &
\end{tikzcd}$$

**Proposition A.8.5.** *The functors $L$ and $R$ restrict to an equivalence on the full subcategories of flat (resp. finite) modules.*

*Proof.* See [Fer03, Thm. 2.2], [Sch68, Lem. 3.4] and [SP, Tag 0D2G] where it is established more generally that $R$ is the right adjoint of $L$, the adjunction morphism $L \circ R \to \mathrm{id}$ is an isomorphism, and for a $B$-module $N$, the adjunction map $N \to R \circ L(N)$ is surjective whose kernel is annihilated by $I = \ker(B \to B_0)$ and contained in $IM$. □

**Corollary A.8.6.** *Consider a commutative cube of schemes*

$$\begin{tikzcd}[row sep=small]
 & X_0' \arrow[rr, hook] \arrow[dl] \arrow[dd] & & X' \arrow[dl] \arrow[dd] \\
X_0 \arrow[rr, hook] \arrow[dd] & & X \arrow[dd] & \\
 & Y_0' \arrow[rr, hook] \arrow[dl] & & Y' \arrow[dl] \\
Y_0 \arrow[rr, hook] & & Y &
\end{tikzcd}$$

*of schemes where $X_0 \hookrightarrow X$ is a closed immersion and $X_0 \to Y_0$ is affine.*

*(1) Assume that $Y' \to Y$ is a flat morphism of schemes and $X'_0$, $Y'_0$, and $X'$ are the base changes under $Y' \to Y$ (i.e. the bottom, left, top and right faces are cartesian).*

*(a) If the front face is a pushout, then so is the back face and the natural functor*

$$\mathrm{QCoh}(Y') \to \mathrm{QCoh}(Y'_0) \times_{\mathrm{QCoh}(X'_0)} \mathrm{QCoh}(X'),$$

*restricts to an equivalence on the full subcategories of $\mathrm{QCoh}(Y')$, $\mathrm{QCoh}(Y'_0)$ and $\mathrm{QCoh}(X')$ containing finitely presented $\mathcal{O}$-modules flat over $Y'$, $Y'_0$ and $X'$.*

*(b) If in addition $Y' \to Y$ is faithfully flat and locally of finite presentation, then the back face being a pushout implies that the front face is as well.*

*(2) If the top and left faces are cartesian, and the front and back faces are pushouts, then all faces are cartesian. Moreover, if $Y'_0 \to Y_0$ and $X' \to X$ are étale (resp. smooth, flat), so is $Y' \to Y$.*

## A.9 Henselizations

### A.9.1 Henselian and strictly henselian local rings

Let $(R, \mathfrak{m})$ be a local ring with residue field $\kappa$. We will denote the image of $a \in R$ (resp. $f \in R[x]$) as $\overline{a} \in \kappa$ (resp. $\overline{f} \in \kappa[x]$). If $f \in R[t]$, we denote its derivative by $f' \in R[t]$. Note that $\overline{f'} = \overline{f}'$.

**Definition A.9.1.** Let $(R, \mathfrak{m})$ be a local ring with residue field $\kappa$.

(1) We say that $R$ is *henselian* if for a monic polynomial $f \in R[t]$, every root $\alpha_0 \in \kappa$ of $\overline{f}$ with $\overline{f'}(\alpha_0) \neq 0$ lifts to a root $\alpha \in R$ of $f$.

(2) We say that $R$ is *strictly henselian* if $R$ is henselian and $\kappa$ is separably closed.

**Remark A.9.2.** Hensel's lemma states that a complete DVR $R$ (e.g. $\mathbb{Z}_p$) is henselian.

**Proposition A.9.3** (Henselian Equivalences)**.** *The following are equivalent for a local ring $(R, \mathfrak{m})$ with residue field $\kappa$:*

*(1) $R$ is henselian;*

*(2) for a polynomial $f \in R[t]$, every factorization $\overline{f} = g_0 h_0$ with $\gcd(g_0, h_0) = 1$ lifts to a factorization $f = gh$ with $\overline{g} = g_0$ and $\overline{h} = h_0$;*

*(3) every finite $R$-algebra is a finite product of local rings finite over $R$;*

*(4) every quasi-finite $R$-algebra $A$ is isomorphic to a product $A \cong B \times C$ where $B$ is a finite over $R$ and $C \otimes_R \kappa = 0$;*

*(5) every étale ring homomorphism $\phi \colon R \to A$ and a prime $\mathfrak{p} \subset A$ with $\phi^{-1}(\mathfrak{p}) = \mathfrak{m}$ and $\kappa = \kappa(\mathfrak{p})$ has a unique section $s \colon A \to R$ with $s^{-1}(\mathfrak{m}) = \mathfrak{p}$.*

*Moreover $R$ is strictly henselian if and only if for every étale ring homomorphism $\phi \colon R \to A$ and prime $\mathfrak{p} \subset A$ with $\phi^{-1}(\mathfrak{p}) = \mathfrak{m}$, there is a unique section $s \colon A \to R$ with $s^{-1}(\mathfrak{m}) = \mathfrak{p}$.*

*Proof.* See [EGA, IV.18.5.11], [Mil80, Thm. I.4.2] and [SP, Tag 04GG]. □

**Remark A.9.4.** The following stronger version of (4) holds for a henselian ring $R$: for every quasi-finite and separated morphism $X \to \operatorname{Spec} R$ of schemes, $X \cong F \sqcup W$ with $F$ finite over $R$ and $W \times_R \kappa = \emptyset$. This is a reformulation of Étale Localization of Quasi-finite Morphisms (Theorem A.5.1); see Exercise A.9.10.

**Remark A.9.5.** Both property (5) and the analogous property of strictly henselian rings generalize to étale morphisms $X \to \operatorname{Spec} A$ of schemes: every section over $\operatorname{Spec} A/\mathfrak{m}$

$$\begin{tikzcd} & X \arrow[d] \\ \operatorname{Spec} A/\mathfrak{m} \arrow[ur] \arrow[r, hook] & \operatorname{Spec} A \arrow[u, bend right, dashed] \end{tikzcd}$$

extends to a global section.

**Proposition A.9.6.** *Let $(R, \mathfrak{m})$ be a henselian (resp. strictly henselian) local ring with residue field $\kappa$.*

1. *Every finite $R$-algebra is a product of finite henselian local (resp. strictly henselian) $R$-algebras.*
2. *Every complete local ring is henselian.*
3. *The functor $A \mapsto A \otimes_R \kappa$ gives an equivalence of categories between finite étale $A$-algebras and finite étale $\kappa$-algebras.*

*Proof.* See [EGA, IV.18.5.10, IV.18.5.14-15], [Mil80, 4.3-4.5] and [SP, Tag 04GE]. □

**Remark A.9.7.** The more general notion of henselian pairs is sometimes useful (although it won't be used in these notes). A pair $(X, X_0)$ consisting of a scheme $X$ and a closed subscheme $X_0 \subset X$ is *henselian* if every finite morphism $f \colon U \to X$ induces a bijection $\operatorname{ClOpen}(U) \to \operatorname{ClOpen}(f^{-1}(X_0))$ between open and closed subschemes of $U$ and those of $f^{-1}(X_0)$. If $(R, \mathfrak{m})$ is a henselian local ring, then $(\operatorname{Spec} R, \operatorname{Spec}(R/\mathfrak{m}))$ is a henselian pair by Proposition A.9.3(3). See [EGA, IV.18.5.5] or [SP, Tag 09XD] for further discussion and equivalences.

### A.9.2 Henselizations and strict henselizations

**Definition A.9.8.** Let $(R, \mathfrak{m})$ be a local ring with residue field $\kappa$. The *henselization of $R$* is a local homomorphism $R \to R^{\mathrm{h}}$ into a henselian local ring $R^{\mathrm{h}}$ such that every other local homomorphism $R \to A$ into a henselian local ring factors uniquely through $R \to R^{\mathrm{h}}$.

Given a separable closure $\kappa \to \kappa^s$, the *strict henselization of $R$ with respect to $\kappa \to \kappa^s$* is a local homomorphism $R \to R^{\mathrm{sh}}$ into a strictly henselian local ring $(R^{\mathrm{sh}}, \mathfrak{m}^{\mathrm{sh}})$ inducing $\kappa \to \kappa^s$ on residue fields such that every other local homomorphism $R \to A$ into a strictly henselian local ring $(A, \mathfrak{m}_A)$ factors through $R \to R^{\mathrm{sh}}$ and the factorization is uniquely determined by the inclusion $R^{\mathrm{sh}}/\mathfrak{m}^{\mathrm{sh}} \to A/\mathfrak{m}_A$ of residue fields.

**Proposition A.9.9.** *Let $(R, \mathfrak{m})$ be a local ring with residue field $\kappa$ and let $\kappa \to \kappa^s$ be a separable closure. The henselization $R \to R^{\mathrm{h}}$ and strict henselization $R \to R^{\mathrm{sh}}$*

*exist and can be constructed as*

$$R^{\mathrm{h}} = \operatorname*{colim}_{(R,\mathfrak{m}) \xrightarrow{\text{ét}} (A,\mathfrak{m}_A),\ \kappa = A/\mathfrak{m}_A} A$$

$$R^{\mathrm{sh}} = \operatorname*{colim}_{\substack{(R,\mathfrak{m}) \xrightarrow{\text{ét}} (A,\mathfrak{m}_A) \\ \downarrow \quad \swarrow \\ \kappa^s}} A$$

*where the colimits are taken over the directed system of étale $R$-algebras $A$ and maximal ideals $\mathfrak{m}_A$ over $\mathfrak{m}$; in the henselian case, we require that $\kappa = R/\mathfrak{m}_A$ while in the strictly henselian case, the data includes a homomorphism $A \to \kappa^s$ over $R$. Moreover,*

*(a) the residue fields of $R^{\mathrm{h}}$ and $R^{\mathrm{sh}}$ are $\kappa$ and $\kappa^s$, respectively;*

*(b) the maps $R \to R^{\mathrm{h}}$ and $R \to R^{\mathrm{sh}}$ are faithfully flat; and*

*(c) if $R$ is noetherian, then so is $R^{\mathrm{h}}$ and $R^{\mathrm{sh}}$.*

*Proof.* See [EGA, IV.18.5-8], [Mil80, I.4] and [SP, Tags 0BSK and 07QL]. □

For a scheme $X$ and a point $x \in X$ with a choice of separable closure $\kappa(x) \to \kappa^s$, the henselization $\mathcal{O}^{\mathrm{h}}_{X,x}$ and strict henselization $\mathcal{O}^{\mathrm{h}}_{X,x}$ are the colimits of $\Gamma(U, \mathcal{O}_U)$ taken over diagrams

$$\begin{tikzcd} \operatorname{Spec}\kappa \arrow[r, "x"] \arrow[dr] & U \arrow[d, "\text{ét}"] \\ & X \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \operatorname{Spec}\kappa^s \arrow[r] \arrow[dr] & U \arrow[d, "\text{ét}"] \\ & X, \end{tikzcd}$$

respectively, where $U \to X$ is an étale morphism of schemes and $\operatorname{Spec}\kappa^s \to U$ is a lift of $\operatorname{Spec}\kappa^s \to \operatorname{Spec}\kappa(x) \xrightarrow{x} X$. Both $\mathcal{O}^{\mathrm{h}}_{X,x}$ and $\mathcal{O}^{\mathrm{sh}}_{X,x}$ can be thought of as local rings in étale topology.

**Exercise A.9.10.** Show that Étale Localization of Quasi-finite Morphisms (Theorem A.5.1) follows from the case when $S$ is the spectrum of a henselian ring (see Remark A.9.4).

*Hint: Use limit methods (Propositions A.6.4 and A.6.7) to extend a decomposition $X \times_S \operatorname{Spec}\mathcal{O}^{\mathrm{h}}_{S,s} \cong F^{\mathrm{h}} \sqcup W^{\mathrm{h}}$ to an étale neighborhood of $s$.*

## A.10 Artin Approximation

In this section, we discuss the deep result of Artin Approximation (Theorem A.10.9) which can be vaguely expressed as the following principle:

algebraic properties that hold for the completion $\widehat{\mathcal{O}}_{S,s}$ of the local ring of a scheme $S$ at a point $s$ also hold in an étale neighborhood $(S', s') \to (S, s)$.

Artin Approximation is related to another equally deep and powerful result known as Néron–Popescu Desingularization (Theorem A.10.4). Both Artin Approximation and Néron–Popescu are difficult theorems that we will not attempt to prove here. However, we will show at least how Artin Approximation easily follows from Néron–Popescu Desingularization.

### A.10.1 Néron–Popescu Desingularization

**Definition A.10.1.** A ring homomorphism $A \to B$ of noetherian rings is called *geometrically regular* if $A \to B$ is flat and for every prime ideal $\mathfrak{p} \subset A$ and every finite field extension $\kappa(\mathfrak{p}) \to \kappa'$ (where $\kappa(\mathfrak{p}) = A_\mathfrak{p}/\mathfrak{p}$), the fiber $B \otimes_A \kappa'$ is regular.

**Remark A.10.2.** It is important to note that $A \to B$ is *not* assumed to be of finite type. If it is, then $A \to B$ is geometrically regular if and only if $A \to B$ is smooth.

**Remark A.10.3.** It can be shown that it is equivalent to require that each geometric fiber $B \otimes_A \overline{\kappa(\mathfrak{p})}$ is regular, or equivalently that $B \otimes_A \kappa'$ is regular for every *inseparable* finite field extensions $\kappa(\mathfrak{p}) \to \kappa'$. In particular, in characteristic 0, $A \to B$ is geometrically regular if it is flat, and every fiber $B \otimes_A \kappa(\mathfrak{p})$ is regular.

**Theorem A.10.4** (Néron–Popescu Desingularization)**.** *A homomorphism $A \to B$ of noetherian rings is geometrically regular if and only if there is a directed system $B_\lambda$ of smooth $A$-algebras over a directed set $\Lambda$ such that $B = \operatorname{colim} B_{\lambda \in \Lambda}$.*

*Proof.* This result was proved by Néron in [Nér64] in the case that $A$ and $B$ are DVRs and in general by Popescu in [Pop85], [Pop86] and [Pop90]. We recommend [Swa98] and [SP, Tag 07GC] for an exposition of this result. □

**Example A.10.5.** A field extension $k \to l$ is geometrically regular if and only if it is separable. When $k \to l$ is algebraic, $l$ is the colimit of separable finite (i.e. étale) extensions.

**Definition A.10.6.** A noetherian local ring $A$ is called a *$G$-ring* if the homomorphism $A \to \widehat{A}$ is geometrically regular.

**Remark A.10.7.** One of the defining features of excellent schemes is that their local rings are $G$-rings.

To apply Néron–Popescu Desingularization, we will need that local rings of schemes are frequently $G$-rings.

**Theorem A.10.8.** *If $A$ is the localization of a finitely generated algebra over a field or $\mathbb{Z}$, then $A$ is a $G$-ring.*

*Proof.* While substantially easier than Néron–Popescu Desingularization, this result nevertheless requires some effort. See [EGA, IV.7.4.4] or [SP, Tag 07PX]. □

### A.10.2 Artin Approximation

Let $S$ be a scheme and consider a contravariant functor

$$F \colon \mathrm{Sch}/S \to \mathrm{Sets}$$

where $\mathrm{Sch}/S$ denotes the category of schemes over $S$. We say that $F$ is *limit preserving* (or *locally of finite presentation*) if for system of $\mathcal{O}_S$-algebras $A_\lambda$ (i.e. each $\operatorname{Spec} A_\lambda$ is an $S$-scheme), the natural map

$$\operatorname{colim} F(B_\lambda) \to F(\operatorname{colim} B_\lambda)$$

is bijective. When $F$ is a functor $\mathrm{Mor}_S(-, X)$ representable by a scheme $X$ over $S$, then this equivalent to $X \to S$ being locally of finite presentation (Proposition A.1.3).

**Theorem A.10.9** (Artin Approximation). *Let $S$ be a scheme and $s \in S$ a point such that $\mathcal{O}_{S,s}$ is a G-ring (Definition A.10.6), e.g. a scheme of finite type over a field or $\mathbb{Z}$. Let*

$$F\colon \mathrm{Sch}/S \to \mathrm{Sets}$$

*be a limit preserving contravariant functor and $\widehat{\xi} \in F(\operatorname{Spec} \widehat{\mathcal{O}}_{S,s})$. For every integer $N \geq 0$, there exists an étale morphism*

$$(S', s') \to (S, s) \qquad \textit{and} \qquad \xi' \in F(S')$$

*with $\kappa(s) = \kappa(s')$ such that the restrictions of $\widehat{\xi}$ and $\xi'$ to $\operatorname{Spec}(\mathcal{O}_{S,s}/\mathfrak{m}_s^{N+1})$ are equal.*

**Remark A.10.10.** To make sense of the restriction $\xi'$ to $\operatorname{Spec}(\mathcal{O}_{S,s}/\mathfrak{m}_s^{N+1})$, note that since $(S', s') \to (S, s)$ is a residually-trivial étale morphism, there are compatible identifications $\mathcal{O}_{S,s}/\mathfrak{m}_s^{N+1} \cong \mathcal{O}_{S',s'}/\mathfrak{m}_{s'}^{N+1}$.

**Remark A.10.11.** It is not possible in general to find $\xi' \in F(S')$ restricting to $\widehat{\xi}$ or even such that the restrictions of $\xi'$ and $\widehat{\xi}$ to $\operatorname{Spec} \mathcal{O}_{S,s}/\mathfrak{m}_s^{n+1}$ agree for all $n \geq 0$. For instance, $F$ could be the functor $\mathrm{Mor}(-, \mathbb{A}^1)$ representing the affine line $\mathbb{A}^1$ and $\widehat{\xi} \in \widehat{\mathcal{O}}_{S,s}$ could be a non-algebraic power series.

*Proof.* The theorem was originally proven in [Art69a, Cor. 2.2] in the case that $S$ is of finite type over a field or an excellent dedekind domain. We also recommend [BLR90, §3.6] for an accessible account of the case of excellent and henselian DVRs. We will show how Artin Approximation follows from Néron–Popescu Desingularization (Theorem A.10.4).

Néron–Popescu Desingularization implies that $\widehat{\mathcal{O}}_{S,s} = \operatorname{colim}_{\lambda \in \Lambda} B_\lambda$ is a directed colimit of smooth $\mathcal{O}_{S,s}$-algebras. Since $F$ is limit preserving, there exists $\lambda \in \Lambda$, a factorization $\mathcal{O}_{S,s} \to B_\lambda \to \widehat{\mathcal{O}}_{S,s}$ and an element $\xi_\lambda \in F(\operatorname{Spec} B_\lambda)$ whose restriction to $F(\operatorname{Spec} \widehat{\mathcal{O}}_{S,s})$ is $\widehat{\xi}$. Letting $B = B_\lambda$ and $\xi = \xi_\lambda$, we have a commutative diagram

$$\begin{tikzcd}
\operatorname{Spec} \widehat{\mathcal{O}}_{S,s} \arrow[r, "g"] \arrow[dr] \arrow[rr, bend left, "\widehat{\xi}"] & \operatorname{Spec} B \arrow[r, "\xi"] \arrow[d] & F \\
& \operatorname{Spec} \mathcal{O}_{S,s} &
\end{tikzcd}$$

where $\operatorname{Spec} B \to \operatorname{Spec} \mathcal{O}_{S,s}$ is smooth. We claim that we can find a commutative diagram

$$\begin{tikzcd}
S' \arrow[r, hook] \arrow[dr] & \operatorname{Spec} B \arrow[d] \\
& \operatorname{Spec} \mathcal{O}_{S,s}
\end{tikzcd} \tag{A.10.1}$$

where $S' \hookrightarrow \operatorname{Spec} B$ is a closed immersion, $(S', s') \to (\operatorname{Spec} \mathcal{O}_{S,s}, s)$ is étale, and the composition $\operatorname{Spec} \mathcal{O}_{S,s}/\mathfrak{m}_s^{N+1} \to S' \to \operatorname{Spec} B$ agrees with the restriction of $g\colon \operatorname{Spec} \widehat{\mathcal{O}}_{S,s} \to \operatorname{Spec} B$.[1] To see this, since $\Omega_{B/\mathcal{O}_{S,s}}$ is a locally free $B$-module,

[1] This is where the approximation occurs. It is not possible to find a morphism $S' \to \operatorname{Spec} B \to \operatorname{Spec} \mathcal{O}_{S,s}$ which is étale at a point $s'$ over $s$ such that the composition $\operatorname{Spec} \widehat{\mathcal{O}}_{S,s} \to S' \to \operatorname{Spec} B$ is equal to $g$.

after replacing $\operatorname{Spec} B$ with an affine open neighborhood of $g(s)$, we may assume that $\Omega_{B/\mathcal{O}_{S,s}}$ is free with basis $db_1, \ldots, db_n$. This induces a homomorphism $\mathcal{O}_{S,s}[x_1, \ldots, x_n] \to B$ defined by $x_i \mapsto b_i$ and provides a factorization

$$\operatorname{Spec} B \to \mathbb{A}^n_{\mathcal{O}_{S,s}} \to \operatorname{Spec} \mathcal{O}_{S,s}$$

where $\operatorname{Spec} B \to \mathbb{A}^n_{\mathcal{O}_{S,s}}$ is étale. Choosing a lift of the composition

$$\begin{tikzcd}
 & & & \mathcal{O}_{S,s} \arrow[d] \\
\mathcal{O}_{S,s}[x_1, \ldots, x_n] \arrow[r] \arrow[urrr, dashed] & B \arrow[r] & \widehat{\mathcal{O}}_{S,s} \arrow[r] & \mathcal{O}_{S,s}/\mathfrak{m}_s^{N+1}
\end{tikzcd}$$

defines a section $s\colon \operatorname{Spec} \mathcal{O}_{S,s} \to \mathbb{A}^n_{\mathcal{O}_{S,s}}$ and we define $S'$ as the fibered product

$$\begin{tikzcd}
S' \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\square"] & \operatorname{Spec} \mathcal{O}_{S,s} \arrow[d, hook, "s"] \\
\operatorname{Spec} B \arrow[r] & \mathbb{A}^n_{\mathcal{O}_{S,s}}.
\end{tikzcd}$$

This gives the desired diagram (A.10.1), and the composition $\xi'\colon S' \to \operatorname{Spec} B \xrightarrow{\xi} F$ is an element which agrees with $\widehat{\xi}$ up to order $N$.

Finally, we must explain how to "smear out" the étale morphism $(S', s') \to (\operatorname{Spec} \mathcal{O}_{S,s}, s)$ and the element $\xi' \in F(S')$ to an étale morphism $(S'', s'') \to (S, s)$ and an element $\xi'' \in F(S'')$. Writing $\mathcal{O}_{S,s} = \operatorname{colim}_{g \notin \mathfrak{m}_s} A_g$, we may apply Propositions A.6.4, A.6.7 and B.4.4 (or a direct argument) to find an element $g \notin \mathfrak{m}_s$ and an affine scheme $S'' = \operatorname{Spec}(A_g[y_1, \ldots, y_n]/(f''_1, \ldots, f''_m))$ such that $S'' \times_{A_g} A_{\mathfrak{m}_s} \cong S'$ and such that $S'' \to \operatorname{Spec} A_g$ is étale. As $F$ is limit preserving and $\Gamma(S'', \mathcal{O}_{S''}) = \operatorname{colim}_{h \notin \mathfrak{m}_s} A_{hg}[y_1, \ldots, y_n]/(f'_1, \ldots, f'_m)$, after replacing $g$ with $hg$, we can find an element $\xi'' \in F(S'')$ restricting to $\xi'$ and, in particular, agreeing with $\widehat{\xi}$ up to order $N$. $\square$

**Exercise A.10.12** (Alternative formulations)**.** Let $(A, \mathfrak{m})$ be a henselian local $G$-ring.

(1) Let $F = \operatorname{Hom}(-, X)\colon \mathrm{Sch}/S \to \mathrm{Sets}$ be the contravariant functor of an affine scheme $X = \operatorname{Spec} A[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$ of finite type over $A$. Note that for an $A$-algebra $B$,

$$F(B) = \{a = (a_1, \ldots, a_n) \in B^{\oplus n} \mid f_i(a) = 0 \text{ for all } i\}.$$

If $\widehat{a} = (\widehat{a}_1, \ldots, \widehat{a}_n) \in \widehat{A}_{\mathfrak{m}}$ is a solution to the equations $f_1(x) = \cdots = f_m(x) = 0$, show that Artin Approximation implies that for every $N \geq 0$, there is a solution $a = (a_1, \ldots, a_n) \in A^{\oplus n}$ to the equations $f_1(x) = \cdots = f_m(x) = 0$ such that $a \cong \widehat{a} \mod \mathfrak{m}^{N+1}$.

(2) Show that (1) implies Artin Approximation.

*Hint: Use that $F$ is limit preserving to find a finitely generated $A$-subalgebra $B \subset \widehat{\mathcal{O}}_{S,s}$ and an element $\xi \in F(B)$ restricting to $\widehat{\xi}$.*

### A.10.3 A first application of Artin Approximation

The next corollary states an important fact you may have taken for granted: if two schemes are formally isomorphic at two points, they are isomorphic in the étale topology.

**Corollary A.10.13.** *Let $X$ and $Y$ be schemes of finite type over a scheme $S$ and let $s \in S$ be a point such that $\mathcal{O}_{S,s}$ is a G-ring. If $x \in X$ and $y \in Y$ are points over $s$ such that $\widehat{\mathcal{O}}_{X,x}$ and $\widehat{\mathcal{O}}_{Y,y}$ are isomorphic as $\mathcal{O}_S$-algebras, then there exists étale morphisms*

$$
\begin{tikzcd}
 & (U,u) \arrow[dl] \arrow[dr] & \\
(X,x) & & (Y,y)
\end{tikzcd}
\tag{A.10.2}
$$

*inducing isomorphisms $\kappa(x) \xrightarrow{\sim} \kappa(u)$ and $\kappa(y) \xrightarrow{\sim} \kappa(u)$ on residue fields.*

*Proof.* The functor

$$F\colon \mathrm{Sch}/X \to \mathrm{Sets}, \quad (T \to X) \mapsto \mathrm{Mor}(T, Y)$$

is limit preserving as it can be identified with the representable functor $\mathrm{Mor}_X(-, Y \times X)$ corresponding to the finite type morphism $Y \times X \to X$. The isomorphism $\widehat{\mathcal{O}}_{X,x} \cong \widehat{\mathcal{O}}_{Y,y}$ provides an element of $F(\operatorname{Spec} \widehat{\mathcal{O}}_{X,x})$. By applying Artin Approximation with $N = 1$, we obtain a diagram as in (A.10.2) with $U \to X$ étale at $u$ with $\kappa(x) \xrightarrow{\sim} \kappa(u)$ and such that $\mathcal{O}_{Y,y}/\mathfrak{m}_y^2 \to \mathcal{O}_{U,u}/\mathfrak{m}_u^2$ is an isomorphism. Since $\widehat{\mathcal{O}}_{U,u}$ is abstractly isomorphic to $\widehat{\mathcal{O}}_{Y,y}$, Lemma A.10.15 implies that $\widehat{\mathcal{O}}_{Y,y} \to \widehat{\mathcal{O}}_{U,u}$ is an isomorphism and therefore that $(U,u) \to (Y,y)$ is étale. □

**Remark A.10.14.** If $\phi\colon \widehat{\mathcal{O}}_{Y,y} \xrightarrow{\sim} \widehat{\mathcal{O}}_{X,x}$ is the specified isomorphism, it is not always possible to arrange that the induced diagram

$$
\begin{tikzcd}
 & \widehat{\mathcal{O}}_{U,u} & \\
\widehat{\mathcal{O}}_{X,x} \arrow[ur] & & \widehat{\mathcal{O}}_{Y,y} \arrow[ul] \arrow[ll, "\phi"']
\end{tikzcd}
$$

is commutative, but the proof (using Artin Approximation with a given $N \geq 1$) shows that we can arrange that the diagram commutes modulo $\mathfrak{m}_y^{N+1}$. See also [SP, Tag 0CAV].

**Lemma A.10.15.** *Let $(A, \mathfrak{m}_A) \to (B, \mathfrak{m}_B)$ be a local homomorphism of noetherian complete local rings. If $A/\mathfrak{m}_A^2 \to B/\mathfrak{m}_B^2$ is surjective, so is $A \to B$. If in addition $A = B$, then $A \to B$ is an isomorphism.*

*Proof.* This follows from the following version of Nakayama's lemma for noetherian complete local rings $(A, \mathfrak{m})$: if $M$ is a (not-necessarily finitely generated) $A$-module such that $\bigcap_k \mathfrak{m}^k M = 0$ and $m_1, \ldots, m_n \in F$ generate $M/\mathfrak{m}M$, then $m_1, \ldots, m_n$ also generate $M$ (see [Eis95, Exercise 7.2]). The final statement follows from the fact that a surjective endomorphism of a noetherian ring is an isomorphism. □

# Appendix B

# Descent

It is hard to overstate the importance of descent in moduli theory. The central idea of descent is as simple as it is powerful. You already know that many properties of schemes and their morphisms can be checked on a Zariski cover, and descent theory implies that they can also be checked on étale covers and often even faithfully flat covers. For example, if $Y' \to Y$ is étale and surjective, then a morphism $X \to Y$ is proper if and only if $X \times_Y Y' \to Y'$ is.

The applications of descent reach far beyond moduli theory. For instance, it can be used to reduce statements about schemes over a field $k$ to the case when $k$ is algebraically closed since $k \to \overline{k}$ is faithfully flat, or reduce statements over a noetherian local ring $A$ to its completion $\widehat{A}$ since $A \to \widehat{A}$ is faithfully flat.

References: [BLR90, Ch.6], [Vis05], [Ols16, Ch. 4], [SP, Tag 0238], [EGA, §IV.2], and [SGA1, §VIII.7] (other descent results are scattered throughout EGA and SGA).

## B.1 Descending quasi-coherent sheaves

Descent theory rests on the following algebraic fact.

**Proposition B.1.1.** *If $\phi\colon A \to B$ is a faithfully flat ring map, then the sequence*

$$0 \longrightarrow A \xrightarrow{\ \phi\ } B \underset{b \mapsto 1 \otimes b}{\overset{b \mapsto b \otimes 1}{\rightrightarrows}} B \otimes_A B$$

*is exact. More generally, if $M$ is an $A$-module, the sequence*

$$0 \longrightarrow M \xrightarrow{m \mapsto m \otimes 1} M \otimes_A B \underset{m \otimes b \mapsto m \otimes 1 \otimes b}{\overset{m \otimes b \mapsto m \otimes b \otimes 1}{\rightrightarrows}} M \otimes_A B \otimes_A B \qquad \text{(B.1.1)}$$

*is exact.*

*Proof.* Note that $A \to B$ and $M \to M \otimes_A B$ are necessarily injective by Proposition A.2.17. Since $A \to B$ is faithfully flat, the sequence (B.1.1) is exact if and only if the sequence

$$M \otimes_A B \xrightarrow{m \otimes b' \mapsto m \otimes 1 \otimes b'} M \otimes_A B \otimes_A B \underset{m \otimes b \otimes b' \mapsto m \otimes 1 \otimes b \otimes b'}{\overset{m \otimes b \otimes b' \mapsto m \otimes b \otimes 1 \otimes b'}{\rightrightarrows}} M \otimes_A B \otimes_A B \otimes_A B$$

is exact. The above sequence can be rewritten as

$$M \otimes_A B \xrightarrow{x \mapsto x\otimes 1} (M \otimes_A B) \otimes_B (B \otimes_A B) \underset{x\otimes y \mapsto x \otimes 1 \otimes y}{\overset{x\otimes y \mapsto x\otimes y \otimes 1}{\rightrightarrows}} (M \otimes_A B) \otimes_B (B \otimes_A B) \otimes_B (B \otimes_A B)$$

which is precisely sequence (B.1.1) applied to ring $B \to B \otimes_A B$ given by $b \mapsto 1 \otimes b$ and the $B$-module $M \otimes_A B$. Since this ring map has a section $B \otimes_A B \to B$ given by $b \otimes b' \mapsto bb'$, we are reduced to proving the proposition when $\phi\colon A \to B$ has a section $s\colon B \to A$ with $s \circ \phi = \mathrm{id}_A$.

Let $x = \sum_i m_i \otimes b_i \in M \otimes_A B$ such that $\sum_i m_i \otimes b_i \otimes 1 = \sum_i m_i \otimes 1 \otimes b_i$. Applying $\mathrm{id}_M \otimes \mathrm{id}_B \otimes s\colon M \otimes_A B \otimes_A B \to M \otimes_A B \otimes_A A \cong M \otimes_A B$ to this identity shows that $x = \sum_i m_i \otimes \phi(s(b_i)) = \sum_i \phi(s(b_i)) m_i \otimes 1$ is in the image of $M \to M \otimes_A B$. $\square$

**Exercise B.1.2.** Denoting $(B/A)^{\otimes n}$ as the $n$-fold tensor product $B \otimes_A \cdots \otimes_A B$, show that the short exact sequence (B.1.1) extends to a long exact sequence $0 \to M \to M \otimes_A (B/A)^{\otimes 1} \to M \otimes_A (B/A)^{\otimes 2} \to \cdots$ with differentials

$$\begin{aligned} d\colon M \otimes_A (B/A)^{\otimes n} &\to M \otimes_A (B/A)^{\otimes (n+1)} \\ m \otimes b_1 \otimes \cdots b_n &\mapsto \sum_{i=0}^{n+1} (-1)^i m \otimes b_1 \otimes \cdots \otimes b_{i-1} \otimes 1 \otimes b_{i+1} \otimes \cdots b_n. \end{aligned}$$

**Proposition B.1.3.** *Let $f\colon X \to Y$ be an fpqc morphism of schemes.*

*(1) Let $G$ and $G'$ be quasi-coherent $\mathcal{O}_Y$-modules. Let $p_1, p_2$ denote the two projections $X \times_Y X \to X$ and $q$ denote the composition $X \times_Y X \xrightarrow{p_i} X \xrightarrow{f} Y$. Then the sequence*

$$\mathrm{Hom}_{\mathcal{O}_Y}(G, G') \xrightarrow{f^*} \mathrm{Hom}_{\mathcal{O}_X}(f^*G, f^*G') \underset{p_2^*}{\overset{p_1^*}{\rightrightarrows}} \mathrm{Hom}_{\mathcal{O}_{X\times_Y X}}(q^*G, q^*G')$$

*is exact.*

*(2) Let $F$ be a quasi-coherent $\mathcal{O}_X$-module and $\alpha\colon p_1^*F \to p_2^*F$ an isomorphism of $\mathcal{O}_{X\times_Y X}$-modules satisfying the cocycle condition $p_{23}^*\alpha \circ p_{12}^*\alpha = p_{13}^*\alpha$ on $X \times_Y X \times_X Y$. Then there exists a quasi-coherent $\mathcal{O}_Y$-module $G$ and an isomorphism $\phi\colon F \to f^*G$ such that $p_1^*\phi = p_2^*\phi \circ \alpha$ on $X \times_Y X$. The data $(F, \phi)$ is unique up to unique isomorphism.*

**Remark B.1.4.** The following diagram may be useful to internalize (2)

$$\begin{tikzcd} p_{23}^*\alpha \circ p_{12}^*\alpha = p_{13}^*\alpha & p_1^*F \xrightarrow{\alpha} p_2^*F & F \arrow[d, dash] & G \arrow[d, dashed] \\ X \times_Y X \times_Y X \arrow[r, shift left=2, "p_{12}"] \arrow[r, "p_{13}"] \arrow[r, shift right=2, "p_{23}"'] & X \times_Y X \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & X \arrow[r, "f"] & Y \end{tikzcd}$$

Keep in mind the special case that $\{U_i\}$ is an open covering of $Y$ and $X = \coprod_i U_i$ in which case the above fiber products correspond to intersections.

The cocycle condition $p_{23}^*\alpha \circ p_{12}^*\alpha = p_{13}^*\alpha$ and the condition that $p_1^*\phi = p_2^*\phi \circ \alpha$ should be understood as the commutativity of

$$\begin{tikzcd} p_{12}^*p_1^*F \arrow[r, "p_{12}^*\alpha"] \arrow[d, equal] & p_{12}^*p_2^*F \arrow[r, equal] & p_{23}^*p_1^*F \arrow[d, "p_{23}^*\alpha"] \\ p_{13}^*p_1^*F \arrow[r, "p_{13}^*\alpha"] & p_{13}^*p_2^*F \arrow[r, equal] & p_{23}^*p_2^*F \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} p_1^*F \arrow[r, "p_1^*\phi"] \arrow[d, "\alpha"] & p_1^*f^*G \arrow[d, equal] \\ p_2^*F \arrow[r, "p_2^*\phi"] & p_2^*f^*G. \end{tikzcd}$$

Proposition B.1.3 can be reformulated as the statement that the category $\mathrm{QCoh}(Y)$ is equivalent to the *category of descent datum for* $X \to Y$, denoted by $\mathrm{QCoh}(X \to Y)$. Here the objects of $\mathrm{QCoh}(X \to Y)$ are pairs $(F, \alpha)$ consisting of a quasi-coherent $\mathcal{O}_X$-module $F$ and an isomorphism $\alpha\colon p_1^*F \to p_2^*F$ satisfying the cocycle condition. A morphism $(F', \alpha') \to (F, \alpha)$ is a morphism $\beta\colon F' \to F$ such that $\alpha \circ p_1^*\beta = p_2^*\beta \circ \alpha'$.

*Proof.* If $X = \operatorname{Spec} B$ and $Y = \operatorname{Spec} A$ are affine, write $G = \widetilde{M}$ and $G' = \widetilde{M'}$. Proposition B.1.1 implies that $0 \to M' \to M' \otimes_A B \rightrightarrows M' \otimes_A B \otimes_A B$ is exact. Applying $\operatorname{Hom}_A(M, -)$ and using tensor-hom adjunction yields (1). For (2), writing $F = \widetilde{M}$, one defines $N$ as the equalizers of two maps $M \rightrightarrows M \otimes_A B$ defined by $m \mapsto m \otimes 1$ and $m \mapsto \alpha(m \otimes 1)$.

The general case is handled by first reducing to the case that $Y$ is affine. Since $f$ is fpqc, $Y$ is the image of a quasi-compact open subset $U \subset X$. By choosing a finite affine cover $\{U_i\}$ of $U$ and replacing $X$ with the affine scheme $\coprod_i U_i$, we have reduced to the case that $X$ is affine. We leave the details to the reader. $\square$

**Remark B.1.5.** It turns out that effective descent for modules holds for a class of ring maps $A \to B$ larger than just faithfully flat maps. Namely, it holds for universally injective maps (see Definition A.2.20), and moreover the converse is true! More precisely, $A \to B$ is universally injective if and only if the functor

$$\begin{aligned} \mathrm{Mod}_A \quad &\to \{(N, \alpha) \mid \begin{array}{l} N \in \mathrm{Mod}_B, \alpha\colon N \otimes_{B,p_1} (B \otimes_A B) \xrightarrow{\sim} N \otimes_{B,p_2} (B \otimes_A B) \\ \text{satisfying the cocycle condition } p_{23}^*\alpha \circ p_{12}^*\alpha = p_{13}^*\alpha \} \end{array} \\ M &\mapsto (M \otimes_A B, \mathrm{can}) \end{aligned}$$

to the category of descent data is an equivalence of categories. See [Mes00] or [SP, Tag 08XA].

## B.2 Descending morphisms

**Proposition B.2.1.** *Let $f\colon X \to Y$ be an fpqc morphism of schemes. If $g\colon X \to Z$ is a morphism to a scheme such that $p_1 \circ g = p_2 \circ g$ on $X \times_Y X$, then there exists a unique morphism $h\colon Y \to Z$ filling in the commutative diagram*

$$\begin{tikzcd} X \times_Y X \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & X \arrow[r, "f"] \arrow[dr, "g"'] & Y \arrow[d, dashed, "h"] \\ & & Z \end{tikzcd}$$

*of solid arrows.*

This result implies that every scheme is a sheaf in the fpqc topology; see Proposition 2.2.6.

## B.3 Descending schemes

We will use the following notation: if $f\colon X \to Y$ and $W \to Y$ are morphisms of schemes, we denote $f^*W$ as the fiber product $X \times_Y W$.

**Proposition B.3.1** (Effective Descent)**.** *Let $f\colon X \to Y$ be an fpqc morphism of schemes. Let $\mathcal{P}$ be one of the following properties of morphisms of schemes: open immersion, closed immersion, locally closed immersion, affine, quasi-affine, or separated and locally quasi-finite. If $Z \to X$ is $\mathcal{P}$-morphism of schemes and $\alpha\colon {p_1}^*(Z) \xrightarrow{\sim} p_2^*(Z)$ is an isomorphism over $X \times_Y X$ satisfying $p_{23}^*\alpha \circ p_{12}^*\alpha = p_{13}^*\alpha$, then there exists $\mathcal{P}$-morphism $W \to Y$ of schemes and an isomorphism $\phi\colon Z \to f^*(W)$ such that $p_1^*\phi = p_2^*\phi \circ \alpha$.*

**Remark B.3.2.** In the case of an open or closed immersion $Z \hookrightarrow X$, then the existence of $\alpha$ translates to the equality $p_1^{-1}(Z) = p_2^{-1}(Z)$ as subschemes of $X \times_Y X$, and there is no need for a cocycle condition.

It may be helpful to interpret the above statement using the diagram

$$\begin{tikzcd}
p_{23}^*\alpha \circ p_{12}^*\alpha = p_{13}^*\alpha & p_1^*Z \arrow[r, "\alpha", "\sim"'] \arrow[d] & p_2^*Z \arrow[dl] & Z \arrow[r, dashed] \arrow[d] & W \arrow[d, dashed] \\
X \times_Y X \times_Y X \arrow[r, shift left=2, "p_{12}"] \arrow[r, "p_{13}"] \arrow[r, shift right=2, "p_{23}"'] & X \times_Y X \arrow[rr, shift left, "p_1"] \arrow[rr, shift right, "p_2"'] & & X \arrow[r, "f"] & Y.
\end{tikzcd}$$

*Proof.* If $Z \hookrightarrow X$ is a closed immersion defined by an ideal sheaf $I_X \subset \mathcal{O}_X$, then one can apply Proposition A.6.8 to descend $I_X$ to a quasi-coherent sheaf $I_Y$ on $Y$ and to descend the inclusion $I_X \hookrightarrow \mathcal{O}_X$ to an inclusion $I_Y \hookrightarrow \mathcal{O}_Y$. Then $W = V(I_Y) \hookrightarrow Y$ is the descended scheme. The case of an open immersion can be handled by considering the reduced complement.

If $Z = \mathcal{S}\mathrm{pec}_X \mathcal{A}_X$ is affine over $X$, then Proposition A.6.8 allows us to first descend $\mathcal{A}_X$ to a quasi-coherent sheaf $\mathcal{A}_Y$ on $Y$ and then the multiplication $\mathcal{A}_X \otimes_{\mathcal{O}_X} \mathcal{A}_X \to \mathcal{A}_X$ to a morphism $\mathcal{A}_Y \otimes_{\mathcal{O}_Y} \mathcal{A}_Y \to \mathcal{A}_Y$ which will necessarily satisfy the axioms making $\mathcal{A}_Y$ into a quasi-coherent $\mathcal{O}_Y$-algebra. Then one takes $W = \mathcal{S}\mathrm{pec}_Y \mathcal{A}_Y$. The case of quasi-affine morphisms is handled by combining the affine and open immersion cases.

If $Z \to X$ is separated and locally quasi-finite, then working locally on $Z$ one reduces to the quasi-compact case, in which case $Z \to X$ is quasi-affine by Proposition A.5.2. $\square$

We will often apply Effective Descent to show that a given sheaf in the big étale or fppf topology is representable by a scheme; see Proposition 2.2.11.

## B.4 Descending properties

*This is currently an incomplete list of the descent results needed.*

### B.4.1 Descending properties of morphisms

**Proposition B.4.1** (Properties fpqc local on the target)**.** *Let $Y' \to Y$ be an fpqc morphism of schemes. Let $\mathcal{P}$ be one of the following properties of a morphism of schemes:*

*(i) isomorphism;*

*(ii) closed immersion;*

*(iii) open immersion;*

*(iv) surjective;*

(v) *proper;*

(vi) *flat;*

(vii) *smooth;*

(viii) *étale;*

(ix) *unramified;*

(x) *syntomic.*

*Then $X \to Y$ has $\mathcal{P}$ if and only if $X \times_Y Y' \to Y'$ does.*

**Proposition B.4.2** (Properties local on the source)**.** *Let $X' \to X$ be an fppf morphism of schemes. Let $\mathcal{P}$ be one of the following properties of a morphism of schemes:*

(i) *surjective;*

(ii) *fppf;*

(iii) *smooth;*

*Then $X \to Y$ has $\mathcal{P}$ if and only if $X' \to X \to Y$ does.*

*If $X' \to X$ is étale and surjective, then $X \to Y$ is étale if and only if $X' \to X \to Y$ is.*

**Proposition B.4.3** (Fpqc-local properties of quasi-coherent sheaves)**.** *Let $f\colon X \to Y$ be an fpqc morphism of schemes. Let $\mathcal{P} \in \{$finite type, finite presentation, flat, vector bundle, line bundle$\}$ be a property of quasi-coherent sheaves. If $G$ is a quasi-coherent $\mathcal{O}_Y$-module, then $G$ has $\mathcal{P}$ if and only if $f^*G$ does. If $X$ and $Y$ are noetherian, then the same holds for the property of coherence.*

*Proof.* This reduces to the algebra statement: if $A \to B$ is a faithfully flat ring map, then an $A$-module $M$ is finitely generated (resp. finitely presented, flat, locally free of finite rank) if and only if $M \otimes_A B$ is. The $(\Rightarrow)$ implications are clear. Conversely, if $M \otimes_A B$ is finitely generated, then let $y_1, \ldots, y_m \in M \otimes_A B$ be generators and write $y_i = \sum x_i \otimes b_i$. Since $(x_1, \ldots, x_n)\colon A^n \to M$ base changes to a surjective map, it is surjective. Repeating this argument to the kernel, we see that the property of being finite presentation descends. For flatness, suppose that $M \otimes_A B$ is flat. By faithful flatness, the exactness of $M \otimes_A -$ is equivalent to the exactness of $(M \otimes_A B) \otimes_B (- \otimes_A B)$, which follows from the flatness of $A \to B$ and the flatness of the $B$-module $M \otimes_A B$. As being locally free of finite rank is equivalent to being finitely presented and flat, the final statement also follows.

See also [SP, Tag 05AY]. □

**Proposition B.4.4** (Descending properties of schemes)**.** *Let $X \to Y$ be an fpqc morphism of schemes. Suppose $X$ has one of the following properties: locally noetherian, quasi-compact, noetherian, integral, reduced, normal, and regular. Then $Y$ has the same property.*

*Proof.* First, note that quasi-compactness descends under any surjective map. Therefore, this reduces to the following statements in algebra: if $A \to B$ is a faithfully flat ring map and $B$ is noetherian (resp. a domain, reduced, normal, or regular), then so is $A$. The map $A \to B$ is injective and $I = IB \cap A$ for every ideal $I \subset A$. For noetherianess, if $I_1 \subset I_2 \subset \cdots$ is an ascending chain of ideals, then since $I_1 B \subset I_2 B \subset \cdots$ terminates, so does $I_1 = I_1 B \cap A \subset I_2 = I_2 B \cap A \subset \cdots$. By injectivity of $A \to B$, the 'domain' and 'reduced' cases are clear.

For normality and regularity, we can assume that $A \to B$ is a local ring map. If $B$ is a normal domain and $a/b$ is integral over $A$ where $a, b \in A$, then $a/b \in B$

as $B$ is normal. This implies that $a$ is the ideal of $B$ generated by $b$, and thus $a\colon B \to B/bB$ is the zero map. As this map is the base change of $a\colon A \to A/bA$, faithful flatness implies that $a\colon A \to A/bA$ is the zero map and thus $a \in (b)$ and $a/b \in A$. For regularity, we will appeal to the fact that a noetherian local ring $A$ of dimension $d$ is regular if and only if every finitely generated $A$-module $M$ has a resolution $0 \to A^{k_d} \to \cdots \to A^{k_1} \to A^{k_0} \to M \to 0$, moreover if this holds and

$$0 \to K \to \cdots \to A^{k_1} \to A^{k_0} \to M \to 0 \tag{B.4.1}$$

is any exact sequence, then $K$ is free; see [Eis95, Thm. 19.12] and [SP, Tag 00OC]. If $M$ is a finitely generated $A$-module, choose an exact sequence (B.4.1). Since $B$ is regular, $K \otimes_A B$ is free. By Proposition B.4.3, $K$ is free and thus $A$ is regular.

See also [SP, Tags 033D, 034B and 06QL]. $\square$

**Remark B.4.5.** For example, if $A$ is a noetherian local ring, then the map $A \to \widehat{A}$ to its completion is faithfully flat. If the completion $\widehat{A}$ is reduced (resp. normal, regular), then the above result implies that the same holds for $A$. While the converse holds for regularity, it does not hold in general for reducedness and normality. However, if $A$ is essentially of finite type over a field (or more generally excellent), then $A$ is reduced (resp. normal) if and only if $\widehat{A}$ is, and moreover in this case the normalization commutes with completion. See [SP, Tags 07NZ and 0C23].

The property of being locally noetherian is fppf local, but the other properties are not. For instance, there are finite type $\Bbbk$-schemes that are non-reduced, non-normal, and non-regular, but any such scheme is flat over $\Bbbk$. However, reducedness, normality, and regularity are smooth-local properties.

**Proposition B.4.6** (Smooth local properties of schemes)**.** *Let $X \to Y$ be a smooth and surjective morphism of schemes. Let $\mathcal{P}$ be one of the following properties: locally noetherian, reduced, normal, and regularity. Then $X$ has $\mathcal{P}$ if and only if $Y$ has $\mathcal{P}$.*

*Proof.* The ($\Rightarrow$) implications follows from Proposition B.4.4. If $Y$ is locally noetherian, so is $X$ by Hilbert's Basis Theorem. The remaining properties follow from the algebraic statement that if $A \to B$ is a smooth ring map and $A$ is reduced (resp. normal, regular), then so is $B$ [SP, Tags 033B, 033C and 036D]. See also [SP, Tag 034D]. $\square$

The property of being a domain is however not smooth local (nor even étale local), e.g. there is a reducible étale neighborhood of the nodal cubic (see Example 0.5.2).

**Proposition B.4.7** (Descending ampleness)**.** *Let $f\colon X \to Y$ be a morphism of schemes and $L$ be a line bundle on $X$. If $Y' \to Y$ is an fpqc morphism of schemes, then $L$ is relatively ample over $Y$ if and only if the pullback of $L$ to $X \times_Y Y'$ is relatively ample over $Y'$.*

*Proof.* See [SP, Tag 0D3C]. $\square$

# Appendix C

# Algebraic groups and actions

## C.1 Group schemes and actions

### C.1.1 Group schemes

**Definition C.1.1.** A *group scheme over a scheme $S$* is a morphism $\pi\colon G \to S$ of schemes together with a multiplication morphism $m\colon G \times_S G \to G$, an inverse morphism $\iota\colon G \to G$ and an identity morphism $e\colon S \to G$ (with each morphism over $S$) such that the following diagrams commute:

$$\begin{tikzcd} G \times_S G \times_S G \arrow[r, "\mathrm{id}_G \times m"] \arrow[d, "m \times \mathrm{id}_G"'] & G \times_S G \arrow[d, "m"] \\ G \times_S G \arrow[r, "m"] & G \end{tikzcd} \qquad \begin{tikzcd} G \arrow[r, "(\mathrm{id}_G, \iota)"] \arrow[d, "(\iota, \mathrm{id}_G)"'] \arrow[dr, "e \circ \pi"] & G \times_S G \arrow[d, "m"] \\ G \times_S G \arrow[r, "m"] & G \end{tikzcd} \qquad \begin{tikzcd} G \arrow[r, "(e\circ\pi, \mathrm{id}_G)"] \arrow[d, "(\mathrm{id}_G, e\circ\pi)"'] \arrow[dr, "\mathrm{id}_G"] & G \times_S G \arrow[d, "m"] \\ G \times_S G \arrow[r, "m"] & G \end{tikzcd}$$

Associativity — Law of inverse — Law of identity

For group schemes $H$ and $G$ over $S$, a *morphism of group schemes* is a morphism $\phi\colon H \to G$ schemes over $S$ such that $m_G \circ (\phi \times \phi) = \phi \circ m_H$. A *(closed) subgroup* of $G$ is a (closed) subscheme $H \subset G$ such that $H \to G \xrightarrow{m_G} G \times G$ factors through $H \times H$.

**Remark C.1.2.** If $G$ and $S$ are affine, then by reversing the arrows above gives $\Gamma(G, \mathcal{O}_G)$ the structure of a *Hopf algebra* over $\Gamma(S, \mathcal{O}_S)$.

**Exercise C.1.3.** Show that a group scheme over $S$ is equivalently defined as a scheme $G$ over $S$ together with a factorization

$$\begin{tikzcd} \mathrm{Sch}/S \arrow[r, dashed] \arrow[dr, "\mathrm{Mor}_S(-,G)"'] & \mathrm{Gps} \arrow[d] \\ & \mathrm{Sets} \end{tikzcd}$$

where Gps $\to$ Sets is the forgetful functor.
(*We are not requiring that there exists a factorization; the factorization is part of the data. Indeed, the same scheme can have multiple structures as a group scheme, e.g.* $\mathbb{Z}/4$ *and* $\mathbb{Z}/2 \times \mathbb{Z}/2$ *over* $\mathbb{C}$.)

**Example C.1.4.** The following examples of group schemes are the most relevant for us. Let $S = \operatorname{Spec} R$.

(1) The *multiplicative group scheme over $R$* is $\mathbb{G}_{m,R} = \operatorname{Spec} R[t]_t$ with comultiplication $m^*\colon R[t]_t \to R[t]_t \otimes_R R[t']_{t'}$ given by $t \mapsto tt'$ while the *group scheme of $n$th roots of unity* is $\boldsymbol{\mu}_{n,R} = \operatorname{Spec} R[t]/(t^n - 1)$ with comultiplication also defined by $t \mapsto tt'$.

(2) The *additive group scheme over $R$* is $\mathbb{G}_{a,R} = \operatorname{Spec} R[t]$ with comultiplication $m^*\colon R[t] \to R[t] \otimes_R R[t']$ given by $t \mapsto t + t'$.

Let $V$ be a free $R$-module of finite rank.

(3) The *general linear group on $V$* is

$$\mathrm{GL}(V) = \operatorname{Spec}(\operatorname{Sym}^*(\operatorname{End}(V))_{\det})$$

with the comultiplication $m^*\colon \operatorname{Sym}^*(\operatorname{End}(V)) \to \operatorname{Sym}^*(\operatorname{End}(V)) \otimes_R \operatorname{Sym}^*(\operatorname{End}(V))$ defined as following: for a basis $v_1, \ldots, v_n$ of $V$, then for $i, j = 1, \ldots, n$, the endomorphisms $x_{ij}\colon V \to V$ defined by $v_i \mapsto v_j$ and $v_k \mapsto 0$ if $k \neq i$ define a basis of $\operatorname{End}(V)$, and we define $m^*(x_{ij}) = x_{i1}x'_{1j} + \cdots + x_{in}x'_{nj}$.

(4) The *special linear group on* $V$ is $\mathrm{SL}(V)$ is the closed subgroup of $\mathrm{GL}(V)$ defined by $\det = 1$.

(5) The *projective linear group* $\mathrm{PGL}(V)$ is the affine group scheme

$$\operatorname{Proj}(\operatorname{Sym}^*(\operatorname{End}(V)))_{\det}$$

with the comultiplication defined similarly to $\mathrm{GL}(V)$.

We write $\mathrm{GL}_{n,R} = \mathrm{GL}(R^n)$, $\mathrm{SL}_{n,R} = \mathrm{GL}(R^n)$ and $\mathrm{PGL}_{n,R} = \mathrm{PGL}(R^n)$. We often simply write $\mathbb{G}_m$, $\mathrm{GL}_n$, $\mathrm{SL}_n$ and $\mathrm{PGL}_n$ when the base ring is understood.

**Exercise C.1.5.**

(a) Provide functorial descriptions of each of the group schemes above.

(b) Show that every abstract group $G$ can be given the structure of a group scheme $\amalg_{g \in G} S$ over a base scheme $S$. Provide both explicit and functorial descriptions.

**Example C.1.6** (Diagonalizable group schemes)**.** Let $R$ be a field and $A$ be a finitely generated abelian group. We let $R[A]$ be the free $R$-module generated by elements of $A$. The $R$-module $R[A]$ has the structure of an $R$-algebra with multiplication on generators induced from multiplication in $A$. The comultiplication $R[A] \to R[A] \otimes_R R[A']$ defined by $a \mapsto a \otimes a'$ defines a group scheme $D_R(A) = \operatorname{Spec} R[A]$ over $\operatorname{Spec} R$. A group scheme $G$ over $\operatorname{Spec} R$ is *diagonalizable* if $G \cong D_R(A)$ for some $A$.

The group scheme $D_R(\mathbb{Z}^r) = \mathbb{G}_{m,R}^r$ is the $r$-dimensional split torus while $D_R(\mathbb{Z}/n) = \boldsymbol{\mu}_{n,R} = \operatorname{Spec} R[t]/(t^n - 1)$ is the group of $n$th roots of unity. The classification of finitely generated abelian groups implies that every diagonalizable group scheme is a product of $\mathbb{G}_m^r \times \boldsymbol{\mu}_{n_1} \times \cdots \times \boldsymbol{\mu}_{n_k}$.

A group scheme $G \to S$ is of *multiplicative type* if it becomes diagonalizable after étale cover of $S$.

**Exercise C.1.7.** Describe $D_{\Bbbk}(A)$ as a functor $\mathrm{Sch}/R \to \mathrm{Gps}$.

We recall the following general properties of group schemes.

**Proposition C.1.8.** *Let $G \to S$ be a locally of finite type group scheme.*

*(1) The function*

$$S \to \mathbb{Z}, \quad s \mapsto \dim G_s$$

*is upper semi-continuous.*

(2) *$G \to S$ is trivial if and only if the fiber $G_s$ is trivial for each $s \in S$.*

(3) *$G \to S$ is separated (resp. quasi-separated) if and only if the identity section $S \to G$ is a closed immersion (resp. quasi-compact).*

*Proof.* For (1), recall that for any finite type morphism $\pi\colon G \to S$, the function $G \to \mathbb{Z}$, defined by $g \mapsto G_{\pi(g)}$, is upper semi-continuos [EGA, IV.13.1.3]. As $G \to S$ is a group scheme, there is a section $S \to G$ and the composition $S \to G \to \mathbb{Z}$ is also upper semi-continuous. For (2) $G \to S$ is unramified as every fiber is. Therefore $\Omega_{G/S} = 0$ and the diagonal $G \to G \times_S G$ is an open immersion. It follows that the identity section $S \to G$ is a surjective open immersion, thus an isomorphism. For (3), see [SP, Tag 047G]. $\square$

### C.1.2 Group actions

**Definition C.1.9.** Let $G \to S$ be a group scheme with multiplication $m$ and identity $e$. An *action* of $G$ on a scheme $p\colon X \to S$ is a morphism $a\colon G \times_S X \to X$ over $S$ such that the following diagrams commute:

$$\begin{tikzcd}
G \times_S G \times_S X \arrow[r, "\mathrm{id}_G \times a"] \arrow[d, "m \times \mathrm{id}_X"'] & G \times_S X \arrow[d, "a"] \\
G \times_S X \arrow[r, "a"] & G
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "(e \circ p, \mathrm{id}_X)"] \arrow[dr, "\mathrm{id}_X"'] & G \times_S X \arrow[d, "a"] \\
 & X
\end{tikzcd}$$

If $X \to S$ and $Y \to S$ are schemes with actions of $G \to S$, a morphism $f\colon X \to Y$ of schemes over $S$ is *$G$-equivariant* if $a_Y \circ (\mathrm{id} \times f) = f \circ a_X$, and is *$G$-invariant* if $G$-equivariant and $Y$ has the trivial $G$-action.

**Exercise C.1.10.** Show that giving a group action of $G \to S$ on $X \to S$ is the same as giving an action of the functor $\mathrm{Mor}_S(-, G)\colon \mathrm{Sch}/S \to \mathrm{Gps}$ on the functor $\mathrm{Mor}_S(-, X)\colon \mathrm{Sch}/S \to \mathrm{Sets}$.
(*This requires first spelling out what it means for a functor to groups to act on a functor to sets.*)

### C.1.3 Representations

Let $S = \operatorname{Spec} R$ be an affine scheme, and let $G \to S$ be a group scheme with multiplication $m$ and identity $e$. A *representation* (or *comodule*) of $G$ is an $R$-module $V$ together with a homomorphism $\sigma\colon V \to \Gamma(G, \mathcal{O}_G) \otimes_R V$ of $R$-modules (referred to as a *coaction*) such that the following diagrams commute:

$$\begin{tikzcd}
V \arrow[r, "\sigma"] \arrow[d, "\sigma"'] & \Gamma(G, \mathcal{O}_G) \otimes_R V \arrow[d, "\mathrm{id}_G \otimes \sigma"] \\
\Gamma(G, \mathcal{O}_G) \otimes_R V \arrow[r, "m^* \otimes \mathrm{id}_V"] & \Gamma(G, \mathcal{O}_G) \otimes_R \Gamma(G, \mathcal{O}_G) \otimes_R V
\end{tikzcd}
\qquad
\begin{tikzcd}
V \arrow[r, "\sigma"] \arrow[dr, "\mathrm{id}_V"'] & \Gamma(G, \mathcal{O}_G) \otimes_R V \arrow[d, "e^* \otimes \mathrm{id}_V"] \\
 & V
\end{tikzcd}$$

**Example C.1.11.**

(1) Given an $R$-module $V$, the *trivial representation* on $V$ is defined using the coaction $\sigma(v) = 1 \otimes v$.

(2) The *regular representation* on $\Gamma(G, \mathcal{O}_G)$ is defined using the comultiplication $m^*\colon \Gamma(G, \mathcal{O}_G) \to \Gamma(G, \mathcal{O}_G) \otimes_R \Gamma(G, \mathcal{O}_G)$.

(3) The *standard representation* of $\mathrm{GL}_{n,R} = \operatorname{Spec} R[x_{ij}]_{\det}$ (or a subgroup scheme of $\mathrm{GL}_{n,R}$) on $V = R^n$ is given by the coaction $\sigma\colon V \to \Gamma(\mathrm{GL}_{n,R}) \otimes_R V$ defined by $\sigma(e_i) = \sum_{j=1}^n x_{ij} \otimes e_j$ where $(e_1, \ldots, e_n)$ is the standard basis of $V$.

A representation $V$ of $G$ induces an action of $G$ on $\mathbb{A}(V) = \operatorname{Spec} \operatorname{Sym}^* V$, which we refer to as a *linear action*. Morphisms of representations and subrepresentations are defined in the obvious way.

**Exercise C.1.12.** If $R = \Bbbk$ is a field and $V$ is a finite dimensional vector space, show that giving $V$ the structure as a representation is the same as giving a homomorphism $G \to \mathrm{GL}(V)$ of group schemes.

**Proposition C.1.13.** *Let $G = D_{\Bbbk}(A)$ be a diagonalizable group scheme over a field $\Bbbk$. Every representation of $G$ is a direct sum of one-dimensional representations.*

*Proof.* Let $G = D_{\Bbbk}(A)$ and let $V$ be a free representation of $G$ with coaction $\sigma\colon V \to \Bbbk[A] \otimes_{\Bbbk} V$. Each $a \in A$ defines a one-dimensional representation $W_a = A$ of $D_{\Bbbk}(A)$ defined by the coaction $W_a \to \Bbbk[A] \otimes_{\Bbbk} W_a$ defined by $1 \mapsto a \otimes 1$. For $a \in A$, the subspace

$$V_a := \{v \in V \mid \sigma(v) = a \otimes v\}$$

is isomorphic to $W_a \otimes V_a$ as $G$-representations, where $V_a$ is viewed as the trivial representation (so that in the case $V_a$ is finite dimensional, $V_a \cong W_a^{\dim V_a}$). (Note that when $a = 0$, $W_a$ is the trivial one-dimensional representation and $V^G = V_0$.) We leave the reader to check that $V \cong \oplus_{a \in A} V_a$ as $G$-representations. See also [Mil17, Thm 12.30] and [SGA3$_{\mathrm{I}}$, 5.3.3]. □

## C.2 Principal $G$-bundles

The concept of a principal $G$-bundle is an algebraic formulation of a topological fiber bundle $P \to T$ where $G$ acts freely and transitively on $P$ with quotient $T = P/G$.

### C.2.1 Definition and equivalences

**Definition C.2.1.** Let $G \to S$ be an fppf affine group scheme. A *principal $G$-bundle over an $S$-scheme $T$* is a scheme $P$ with an action of $G$ via $\sigma\colon G \times_S P \to P$ such that $P \to T$ is a $G$-invariant fppf morphism and

$$(\sigma, p_2)\colon G \times_S P \to P \times_T P, \qquad (g, p) \mapsto (gp, p)$$

is an isomorphism.

*Morphisms of principal $G$-bundles* are $G$-equivariant morphisms of schemes. A principal $G$-bundle $P \to T$ is *trivial* if there is an $G$-equivariant isomorphism $P \cong G \times_S T$, where $G$ acts on $G \times_S T$ via multiplication on the first factor.

A principal $G$-bundle $P \to T$ are examples of *$G$-torsors* (Definition 6.2.12) over $(\mathrm{Sch}/T)_{\mathrm{fppf}}$ by viewing $P$ as a sheaf in the big fppf topology over $T$ (see Example 6.2.16). In these notes, we will always distinguish between these two notions, but in conversation or the literature, they are often conflated.

**Exercise C.2.2.** Show that $P \to T$ is a principal $G$-bundle over an $S$-scheme $T$ if and only if $P \to T$ is a principal $G \times_S T$-bundle.

**Exercise C.2.3.** Show that a morphism of principal $G$-bundles is necessarily an isomorphism.

Principal $G$-bundles can be trivialized fppf-locally, or even étale locally if $G$ is smooth.

**Proposition C.2.4.** *Let $G \to S$ be an fppf group scheme and $P \to T$ be a $G$-equivariant morphism of $S$-schemes where $T$ has the trivial action. Then $P \to T$ is a principal $G$-bundle if and only if there exists an fppf morphism $T' \to T$ such that $P \times_T T'$ is the trivial principal $G$-bundle over $T'$. Moreover, if $G \to S$ is smooth, we can arrange that $T' \to T$ is surjective and étale.*

*Proof.* The ($\Rightarrow$) direction follows from the definition by taking $T' = P \to T$. For ($\Leftarrow$), after base changing $G \to S$ by $T \to S$, we assume that $G$ is defined over $T$ (see Exercise C.2.2). Let $G_{T'}$ and $P_{T'}$ be the base changes of $G$ and $P$ along $T' \to T$. The base change of the action map $(\sigma, p_2)\colon G \times_T P \to P \times_T P$ along $T' \to T$ is the action map $G_{T'} \times_{T'} P_{T'} \to P_{T'} \times_{T'} P_{T'}$ of $G_{T'}$ acting on $P_{T'}$ over $T'$. Since $P_{T'}$ is trivial, this latter action map is an isomorphism. Since the property of being an isomorphism descends under fppf morphisms (Proposition B.4.1), we conclude that $(\sigma, p_2)\colon G \times_T P \to P \times_T P$ is an isomorphism.

If $G$ is smooth, then $T' = P \to S$ is a surjective smooth morphism such that $P_{T'}$ is trivial. Since there is a section of $T' \to S$ after a surjective étale morphism $S' \to S$ (Corollary A.3.6), $P_{S'}$ is also trivial. □

**Proposition C.2.5** (Effective Descent for Principal $G$-bundles)**.** *Let $G \to S$ be an fppf affine group scheme. Let $f\colon X \to Y$ be an fpqc morphism of schemes over $S$. If $P \to X$ is a principal $G$-bundle and $\alpha\colon {p_1}^*(P) \xrightarrow{\sim} p_2^*(P)$ is an isomorphism of principal $G$-bundles over $X \times_Y X$ satisfying $p_{12}^*\alpha \circ p_{23}^*\alpha = p_{13}^*\alpha$, then there exists a principal $G$-bundle $Q \to Y$ and an isomorphism $\phi\colon P \to f^*(Q)$ of principal $G$-bundles such that $p_1^*\phi = p_2^*\phi \circ \alpha$.*

*Proof.* By Effective Descent (Proposition B.3.1) for affine morphisms, there is a scheme $Q$ affine over $Y$ and an isomorphism $\phi\colon P \to f^*(Q)$ of schemes. By applying descent for morphisms (Proposition A.6.4), we can descend the action $G \times_S P \to P$ to an action $G \times_S Q \to Q$ giving $Q$ the structure of a principal $G$-bundle and making $\phi\colon P \to f^*(Q)$ a $G$-equivariant isomorphism. □

## C.2.2 Examples of principal $G$-bundles

**Exercise C.2.6.** Let $L/K$ be a finite Galois extension and $G = \mathrm{Gal}(L/K)$ be its Galois group viewed as a finite group scheme over $\operatorname{Spec} K$. Show that $\operatorname{Spec} L \to \operatorname{Spec} K$ is a principal $G$-bundle.

**Exercise C.2.7.** If $T$ is a scheme, show that there there is an equivalence of categories

$$\{\text{line bundles on } T\} \xrightarrow{\sim} \{\text{principal } \mathbb{G}_m\text{-bundles on } T\}$$
$$L \mapsto \mathbb{A}(L) \setminus T$$

between the *groupoids* of line bundles on $T$ (where the only morphisms allowed are isomorphisms) and $\mathbb{G}_m$-torsors on $T$. If $L$ is a line bundle (i.e. invertible $\mathcal{O}_T$-module), then $\mathbb{A}(L)$ denotes the total space $\operatorname{Spec}\operatorname{Sym}^* L^\vee$ of $L$ and $T \subset \mathbb{A}(L)$ denotes the image of the zero section $T \to \mathbb{A}(L)$.

**Exercise C.2.8.** If $T$ is a scheme and $d \geq 1$, show that there there is an equivalence of *groupoids*

$$\begin{aligned} \{\text{finite, étale, and degree } d \text{ covers of } T\} &\xrightarrow{\sim} \{\text{principal } S_d\text{-bundles over } T\} \\ (Y \to T) &\mapsto (\underbrace{Y \times_T \cdots \times_T Y}_{d \text{ times}} \setminus \Delta \to T) \\ (P/S_{d-1} \to T) &\leftarrow\!\shortmid (P \to T). \end{aligned}$$

For the rightward map, the symmetric group $S_d$ acts on the $d$-fold fiber product $Y \times_T \cdots \times_T Y$ by permutation, and $\Delta$ denotes the $S_d$-equivariant closed locus of $d$-tuples where at least two points coincide. Alternatively, $Y \times_T \cdots \times_T Y \setminus \Delta$ can be identified with the scheme $\underline{\mathrm{Isom}}_T(T \times \{1, \ldots, d\}, Y)$ parameterizing isomorphisms of the trivial finite étale cover of degree $d$ and $Y$. For the leftward map, $S_{d-1} \subset S_d$ denotes the subgroup of permutations fixing the $d$th index, and $P/S_{d-1}$ denotes the quotient scheme of this free action (see Exercise 4.2.8).

**Exercise C.2.9.**

(a) Show that the standard projection $\mathbb{A}^{n+1} \setminus 0 \to \mathbb{P}^n$ is a principal $\mathbb{G}_m$-bundle.

(b) For each line bundle $\mathcal{O}(d)$ on $\mathbb{P}^n$, explicitly determine the corresponding principal $\mathbb{G}_m$-bundle. In particular, which $\mathcal{O}(d)$ correspond to the principal $\mathbb{G}_m$-bundle of (a)?

**Exercise C.2.10.** Let $G \to S$ be a smooth affine group scheme. Let $P \to T$ and $Q \to T$ be principal $G$-bundles. Show that the functor

$$\underline{\mathrm{Isom}}_T(P, Q) \colon \mathrm{Sch}/T \to \mathrm{Sets},$$

assigning a $T$-scheme $T'$ to the set of isomorphisms of the principal $G$-bundles $P \times_T T'$ and $Q \times_T T'$, is representable by a scheme which is also a principal $G$-bundle over $T$.

**Exercise C.2.11** (Principal $\mathrm{GL}_n$-bundles)**.** Let $T$ be a scheme.

(a) If $E$ is a vector bundle over $T$ of rank $n$, the *frame bundle* $\mathrm{Frame}_T(E)$ is defined as the functor $\underline{\mathrm{Isom}}_T(\mathcal{O}_T^n, E)$ on $\mathrm{Sch}/T$, i.e

$$\begin{aligned} \mathrm{Frame}_T(E) \colon \mathrm{Sch}/T &\to \mathrm{Sets} \\ (T' \to T) &\mapsto \{\text{trivializations } \alpha \colon \mathcal{O}_T^n \xrightarrow{\sim} f^*E\}. \end{aligned}$$

Show that $\mathrm{Frame}_T(E)$ is representable by a scheme and that $\mathrm{Frame}_T(E) \to T$ is a principal $\mathrm{GL}_n$-bundle.

(b) If $P \to T$ is a principal $\mathrm{GL}_n$-bundle, then define $P \times^{\mathrm{GL}_n} \mathbb{A}^n := (P \times \mathbb{A}^n)/\mathrm{GL}_n$ where $\mathrm{GL}_n$ acts diagonally via its given action on $P$ and the standard action on $\mathbb{A}^n$. (The action is free and the quotient $(P \times \mathbb{A}^n)/\mathrm{GL}_n$ can be interpreted as the sheafification of the quotient presheaf $\mathrm{Sch}/T \to \mathrm{Sets}$ taking $T \mapsto (P \times \mathbb{A}^n)(T)/\mathrm{GL}_n(T)$ in the big Zariski (or big étale) topology or equivalently as the algebraic space quotient (Corollary 4.4.12)). Show that $(P \times \mathbb{A}^n)/\mathrm{GL}_n$ is representable by a scheme and is the total space of a vector bundle over $T$.

*Hint: Use Effective Descent for Principal $G$-bundles (C.2.5).*

(c) Conclude that

$$\begin{aligned} \{\text{vector bundles over } T\} &\to \{\text{principal } \mathrm{GL}_n\text{-bundles over } T\} \\ E &\mapsto \mathrm{Frame}_T(E) \\ (P \times \mathbb{A}^n)/\mathrm{GL}_n &\leftarrow\!\shortmid P \end{aligned}$$

defines an equivalence of categories between the *groupoids* of vector bundles over $T$ and principal $\mathrm{GL}_n$-bundles over $T$.

**Exercise C.2.12** (Principal $\mathrm{SL}_n$-bundles)**.** Show that the groupoid of principal $\mathrm{SL}_n$-bundles over a scheme $T$ is equivalent to the groupoid of pairs $(V, \alpha)$ where $V$ is a vector bundle on $T$ of rank $n$ and $\alpha\colon \mathcal{O}_T \xrightarrow{\sim} \det V$ is a trivialization and a morphism $(V', \alpha') \to (V, \alpha)$ of pairs is an isomorphism $\phi\colon V' \to V$ such that $\alpha' = \alpha \circ \det \phi$.

**Exercise C.2.13** (Brauer–Severi schemes)**.** A morphism $X \to T$ of schemes is a *Brauer–Severi scheme of relative dimension* $r$ if there exists an étale cover $T' \to T$ and an isomorphism $X \times_T T' \cong \mathbb{P}^r_{T'}$. An example of a non-trivial Brauer-Severi scheme is $\operatorname{Proj} \mathbb{R}[x, y, z]/(x^2 + y^2 + z^2) \to \operatorname{Spec} \mathbb{R}$. Show that

$$\begin{aligned} \{\text{Brauer–Severi schemes of rel. dim. } r \text{ over } T\} &\to \{\text{principal } \mathrm{PGL}_r\text{-bundles over } T\} \\ X &\mapsto \underline{\mathrm{Isom}}_T(\mathbb{P}^r_T, X) \\ (P \times \mathbb{P}^r)/\mathrm{PGL}_r &\leftarrow\!\shortmid P \end{aligned}$$

defines an equivalence of groupoids.

**Exercise C.2.14.** Let $X \to S$ be a proper, flat, and finitely presented morphism of schemes. Assume that for every geometric point $\operatorname{Spec} \Bbbk \to S$, the geometric fiber $X \times_S \Bbbk$ is isomorphic to $\mathbb{P}^1_{\Bbbk}$. Show that $X \to S$ is a Brauer–Severi scheme of relative dimension 1 following one of the approaches below.

*Approach 1 (local-to-global): Show that for every point $s \in S$, there is a finite and separable field extension $\kappa(s) \to K$ such that $X \times_S K \cong \mathbb{P}^1_K$. Show that there an étale neighborhood $(S', s') \to (S, s)$ such that $K \cong \kappa(s')$ over $\kappa(s)$. Assuming now that $X \times_s \kappa(s) \cong \mathbb{P}^1_{\kappa(s)}$, use deformation theory (Proposition D.2.6) to show that there are compatible isomorphisms $X \times_S \mathcal{O}_{S,s}/\mathfrak{m}_s^n \cong \mathbb{P}^1_{\mathcal{O}_{S,s}/\mathfrak{m}_s^n}$ for $n > 0$. Use Grothendieck's Existence Theorem (D.4.4) to show that $X \times_S \widehat{\mathcal{O}}_{S,s} \cong \mathbb{P}^1_{\widehat{\mathcal{O}}_{S,s}}$. Apply Artin Approximation (A.10.9) to show that there is an étale neighborhood $(S', s') \to (S, s)$ such that $X \times_S S' \cong \mathbb{P}^1_{S'}$.*

*Approach 2 (direct): Assuming that there is a section $\sigma\colon S \to X$ of $\pi\colon X \to S$, show that every point $s \in S$ has an open neighborhood $U \subset S$ such that $X \times_S U \cong \mathbb{P}^1_U$. Letting $L$ be the line bundle on $X$ corresponding to the Cartier divisor $\sigma$, use Cohomology and Base Change (A.7.5) to show that $\mathcal{E} := \pi_*\mathcal{L}$ is a rank 2 vector bundle on $S$, that $\pi^*\mathcal{E} \twoheadrightarrow \mathcal{L}$ is surjective, and that $X \cong \mathbb{P}(\mathcal{E})$ over $S$. Conclude by choosing an open neighborhood of $s \in S$ where $\mathcal{E}$ is trivial. Returning to the general case, show that there is an effective divisor $D$ associated to $\Omega^\vee_{X/S}$ such that $D \to S$ is étale. Reduce to the case where $X \to S$ has a section by base changing by $D \to S$. See also [Har77, Prop. 25.3 and Exer. 25.2].*

**Exercise C.2.15** (Azumaya algebras)**.** Let $T$ be a noetherian scheme. An *Azumaya algebra of rank* $r^2$ over $T$ is a (possibly non-commutative) associative $\mathcal{O}_T$-algebra $A$ which is coherent as an $\mathcal{O}_T$-module and such that there is an étale covering $T' \to T$ where $A \otimes_{\mathcal{O}_T} \mathcal{O}_{T'}$ is isomorphic to the matrix algebra $M_r(\mathcal{O}_T)$; see [Mil80, §IV.2]. An Azumaya algebra over a field $\Bbbk$ is a central simple algebra (i.e. a finite dimensional associative $\Bbbk$-algebra which is simple and whose center is $\Bbbk$); the quaternions is an example of a central simple algebra over $\mathbb{R}$.

Show that the assignment

$$A \mapsto \underline{\mathrm{Isom}}_T(M_r(\mathcal{O}_T), A)$$

defines a bijection between Azumaya algebras of rank $r^2$ over $T$ and $\mathrm{PGL}_n$-torsors over $T$.

**Remark C.2.16.** Exercise C.2.13 and Exercise C.2.15 provide bijections

$$\{\text{Azumaya algebras of rank } r^2\} \simeq \{\text{principal } \mathrm{PGL}_n\text{-torsors}\}$$
$$\simeq \{\text{Brauer–Severi schemes of relative dimension } r\}$$

on sets of isomorphism classes of objects over $T$. The composition takes an Azumaya algebra $\mathcal{A}$ over $T$ to the Brauer–Severi scheme defined as the closed subscheme $X \subset \mathrm{Gr}_T(r, \mathcal{A})$ classifying rank $r$ right ideals. In Exercises 6.2.44 and 6.2.44 and remark 6.2.43, Azumaya algebras are used to define the *Brauer group* $\mathrm{Br}(T)$ and we discuss the relationship to $\mathbb{G}_m$-gerbes.

**Exercise C.2.17** (Orthogonal group)**.** Let $\Bbbk$ be a field with $\mathrm{char}(\Bbbk) \neq 2$, and let $V$ be an $n$ dimensional vector space with a non-degenerate quadratic form $q$. Let $O(q) \subset \mathrm{GL}(V)$ be the subgroup of invertible matrices preserving the quadratic form. If $q = x_1^2 + \cdots + x_n^2$ is the diagonalized quadratic form, then $O_n = O(q)$ is the set of orthogonal matrices $A$ (i.e. $AA^\top = I$).

Show that there is a bijection between principal $O(q)$-bundles over a $\Bbbk$-scheme $T$ and vector bundles of rank $n$ on $T$ with a non-degenerate quadratic form.

# C.3 Algebraic groups

An *algebraic group over a field* $\Bbbk$ is a group scheme $G$ of finite type over $\Bbbk$.

## C.3.1 Properties of algebraic groups

**Proposition C.3.1.** *Let $G$ be a group scheme locally of finite type over field $\Bbbk$ (e.g. $G$ is an algebraic group). Let $G^0$ be the connected component containing the identity element.*

*(1) $G$ is separated.*

*(2) (Cartier's Theorem) If $\mathrm{char}(\Bbbk) = 0$, then $G$ is smooth.*

*(3) If $\Bbbk$ is perfect, then $G$ is smooth if and only if $G$ is reduced if and only if $G$ is geometrically reduced, and moreover $G_{\mathrm{red}} \subset G$ is a subgroup scheme.*

*(4) $G^0$ is an open and closed irreducible subgroup scheme of finite type that commutes with field extensions of $\Bbbk$, and $G/G^0$ is an etale algebraic group over $\Bbbk$.*

*(5) Any subgroup $H \subset G$ is closed.*

*(6) If $G$ acts on a finite type $\Bbbk$-scheme $X$ and $x \in X$ is a closed point, the orbit $Gu$, defined set-theoretically as the image of $G \to X, g \mapsto g \cdot x$, is open in its closure $\overline{Gx}$. Moreover,*

$$\dim G = \dim Gx + \dim G_x,$$

*and the function $x \mapsto \dim G_x$ is upper semicontinuous while $x \mapsto \dim Gx$ is lower semicontinuous.*

*(7) If $G$ is of finite type and $H \subset G$ is a subgroup, then $G/H$ is quasi-projective.*

*(8) (Barsotti–Chevalley's Structure Theorem) If $G$ is smooth and connected, then there is a unique connected, affine, and normal subgroup $H \triangleleft G$, which is smooth if $\Bbbk$ is perfect, such that $G/H$ is an abelian variety.*

**Remark C.3.2.** In (7), the quotient $G/H$ represents the sheafification of the quotient presheaf

$$(G/H)^{\mathrm{pre}} \colon \mathrm{Sch}/\Bbbk \to \mathrm{Sets}, \quad S \mapsto G(S)/H(S)$$

in the big étale topology $(\mathrm{Sch}/\Bbbk)_{\text{ét}}$. When we say that $1 \to K \to G \to Q \to 0$ is an *exact sequence of algebraic groups*, we mean that $K \subset G$ is a subgroup and $Q \cong G/K$.

In (8), an *abelian variety* by definition is a smooth proper algebraic group over a field. It is necessarily projective and has a commutative group law [Mum70, pgs. 39, 59]. An *elliptic curve* is an abelian variety of dimension 1.

*Proof.* Proposition C.1.8(3) implies (1) since any $\Bbbk$-point of a locally of finite type $\Bbbk$-scheme is closed. For (2), see [Car62, §15], [Oor66], [Mum66, p.167], [Mil17, Thm. 3.23 and Cor. 8.39], and [SP, Tag 047N]. For (3), see [Mil17, Prop. 1.26, Cor. 1.39] and [SP, Tags 047P and 047R].

For (4), see [Mil17, Prop. 1.34] and [SP, Tag 0B7R]. What may seem surprising here is that $G^0$ is automatically quasi-compact.[1] This follows from a simple argument: reduce to the case that $\Bbbk$ is algebraically closed and choose a nonempty open affine subscheme $U \subset G$. After shrinking, we may assume that $U$ is closed under taking inverses. The quasi-compactness of $G$ follows from the surjectivity of the multiplication map $U \times U \to G$ is surjective. If $g \in G(\Bbbk)$, then since $U$ is dense, the intersection $U \cap gU$ contains an element $h$. If we write $h = gu$, then $g = hu^{-1}$.

See [Mil17, Prop. 1.68] for (5) and [Mil17, Props. 1.65] for the first part of (6). The identity $\dim G = \dim Gx + \dim G_x$ follows from the identification $Gx \cong G/G_x$, while the semicontinuity statements follow from Proposition C.1.8(1) applied to the stabilizer group scheme $S_X$ over $X$, defined as the fiber product of the action map $G \times X \to X \times X$ and the diagonal $X \to X \times X$.

For the existence of a projective embedding of $G/H$ in (7), see [Ray70, Cor VI.2.6], [Bri17, Thm. 5.2.2], and [Mil17, Thm. 8.4.4]. Chevalley announced a proof of (8) in 1953, but a proof did not appear until [Che60]. In the meantime, Barsotti provided an independent proof [Bar55a], [Bar55b]. Rosenlicht provided a more elementary argument in [Ros56]. See [Con02], [Bri17, Thms. 1 and 2], and [Mil17, Thm. 8.27] for modern expositions. □

### C.3.2 Properties of affine algebraic groups

We will mostly be concerned with algebraic groups that are affine. These are sometimes also called *linear algebraic groups* (as justified by Part (2) below).

**Proposition C.3.3.** *Let $G$ be an affine algebraic group over a field $\Bbbk$.*

*(1) Every representation $V$ of $G$ is a union of its finite dimensional subrepresentations.*

*(2) There exists a finite dimensional representation $V$ and a closed immersion $G \hookrightarrow \mathrm{GL}(V)$ of group schemes.*

*Proof.* For (1), let $\sigma \colon V \to \Gamma(G, \mathcal{O}_G) \otimes_{\Bbbk} V$ be the coaction. It suffices to show that every finite dimensional subspace $W \subset V$ is contained in a finite dimensional sub-$G$-representation $W' \subset V$. If $w_1, \dots, w_n$ is a basis of $W$ and $\sigma(w_i) = \sum_j f_{ij} \otimes v_{ij}$, then one checks that the subspace generated by $v_{ij}$ is $G$-invariant and contains $W$. For (2), we consider the regular representation $\Gamma(G, \mathcal{O}_G)$ of $G$ and apply (1) to

[1] We use this in the text to show the boundedness of $\underline{\mathrm{Pic}}^0_X$; see Theorem 6.2.55.

construct a finite dimensional subrepresentation $V$ containing $\Bbbk$-algebra generators. One checks that this gives a closed immersion $G \hookrightarrow \mathrm{GL}(V)$. See [Mil17, Prop. 4.7, Cor. 4.10]. $\square$

We will repeatedly use the following simple consequence of Proposition C.3.3(1).

**Proposition C.3.4.** *Let $G$ be an affine algebraic group over a field $\Bbbk$. Let $X$ be an affine scheme of finite type over $\Bbbk$ with an action of $G$.*

*(1) There exists a $G$-equivariant closed immersion $X \hookrightarrow \mathbb{A}(V)$ where $V$ is a finite dimensional $G$-representation.*

*(2) For every $G$-invariant closed subscheme $Z \subset X$, there exists a $G$-equivariant morphism $f: X \to \mathbb{A}(W)$, where $W$ is a finite dimensional $G$-representation, such that $f^{-1}(0) = Z$.*

*Proof.* Write $X = \operatorname{Spec} A$ and let $f_1, \ldots, f_n$ be $\Bbbk$-algebra generators. By C.3.3(1) there is a finite dimensional $G$-invariant subspace $V \subset A$ containing each $f_i$. The surjection $\operatorname{Sym}^* V \to A$ induces a $G$-equivariant embedding $X \hookrightarrow \mathbb{A}(V)$. For (2), let $Z = \operatorname{Spec} A/I$ and let $g_1, \ldots, g_m \in I$ be generators. Letting $W \subset I$ be a finite dimensional $G$-invariant subspace containing each $g_i$, we see that $f: X \to \mathbb{A}(W)$ is a $G$-equivariant map with $f^{-1}(0) = Z$. $\square$

A subgroup $T \subset G$ of an affine algebraic group over a field $\Bbbk$ is called a *torus* if $T_{\overline{\Bbbk}} \cong \mathbb{G}^n_{m,\overline{\Bbbk}}$ and a *maximal torus* if it not contained in a larger torus. The set of diagonal matrices in $\mathrm{GL}_n$ is a maximal torus.

**Proposition C.3.5.** *Let $G$ be an affine algebraic group over a field $\Bbbk$.*

*(1) $G$ contains a maximal torus $T$ such that $T_{\Bbbk'} \subset G_{\Bbbk'}$ is a maximal torus for every field extension $\Bbbk \to \Bbbk'$.*

*(2) If $\Bbbk$ is algebraically closed, all maximal tori are conjugate.*

*Proof.* See [Mil17, Thms. 17.82 and 17.105]. $\square$

There is of course much more to the theory of affine algebraic groups. We quickly mention a few facts that will be useful in the text.

### Jordan decompositions

Recall that an element $g \in \mathrm{GL}_n(\Bbbk)$ is *semisimple* if it becomes diagonalizable after an extension of $\Bbbk$ and *unipotent* if $g - 1$ is nilpotent, i.e. $(1 - g)^n = 0$ for some $n$. If $G$ is an affine algebraic group over a perfect field $\Bbbk$, then for every element $g \in G(\Bbbk)$, there are unique elements $g_s, g_u \in G(\Bbbk)$ such that $g = g_s g_u = g_u g_s$ such that the images of $g_s$ and $g_u$ under any representation $G \to \mathrm{GL}_n$ are semisimple and unipotent, respectively; see [Mil17, Thm. 9.17]. We call $g_s$ and $g_u$ the *semisimple* and *unipotent parts* of $g$.

### Unipotent groups

An affine algebraic group $G$ over a field $\Bbbk$ is *unipotent* if there is a faithful representation $V$ and a basis $V \cong \Bbbk^n$ such that the image of the induced map $G \hookrightarrow \mathrm{GL}(V) \cong \mathrm{GL}_n$ is contained in the subgroup $\mathbb{U}_n$ of upper triangle matrices with 1's along the diagonal. An example of a unipotent group is $\mathbb{G}_a$. We have the following equivalences:

- $G$ is unipotent,

- $G$ has a filtration of normal subgroups $1 = G_0 \triangleleft G_1 \triangleleft \cdots \triangleleft G_n = G$ where $G_i/G_{i-1} \cong \mathbb{G}_a$,
- for every nonzero representation $V$ of $G$, $V^G \neq 0$, and
- every element $g \in G$ is unipotent, i.e. $g = g_u$.

**Special groups**

An affine algebraic group $G$ over a field $\Bbbk$ is *special* if every principal $G$-bundle $P \to T$ is Zariski-locally trivial. For example, $\mathrm{GL}_n$ (e.g. $\mathbb{G}_m = \mathrm{GL}_n$)is a special group as principal $\mathrm{GL}_n$-bundles correspond to vector bundles (Exercise C.2.11). It is also true that $\mathbb{G}_a$ is special. One way to see this is to use that quasi-coherent cohomology can be computed either in the Zariski or étale-site (Proposition 4.1.33). Viewing the structure sheaf $\mathcal{O}_T$ as $\mathbb{G}_a$, this implies that $\mathrm{H}^1(T, \mathbb{G}_a) = \mathrm{H}^1(T_{\text{ét}}, \mathbb{G}_a)$ and there is thus a bijective correspondence between $\mathbb{G}_a$-torsors in the Zariski and small étale topology.

If $1 \to K \to G \to Q \to 1$ is an exact sequence of affine algebraic groups and both $K$ and $Q$ are special, it is not hard to see that $G$ is also special. It follows from the characterization of unipotent groups as extensions of $\mathbb{G}_a$ that a unipotent group is special.

### C.3.3 One-parameter subgroups, centralizers, and parabolics

Let $G$ be a smooth affine algebraic group over an algebraically closed field $\Bbbk$. A *one-parameter subgroup* is a homomorphism $\lambda\colon \mathbb{G}_m \to G$ of algebraic groups (which is not necessarily a subgroup). We let $\mathbb{X}_*(G)$ be the set of one-parameter subgroups. On the other hand, a *character* is a homomorphism $\chi\colon G \to \mathbb{G}_m$ and we let $\mathbb{X}^*(G)$ be the group of characters. Since any character $\mathbb{G}_m \to \mathbb{G}_m$ is given by $t \mapsto t^d$ for some $d \in \mathbb{Z}$, there is a pairing

$$\langle -, - \rangle \colon \mathbb{X}_*(G) \times \mathbb{X}^*(G) \to \mathbb{X}^*(\mathbb{G}_m) \cong \mathbb{Z}, \quad (\lambda, \chi) \mapsto \chi \circ \lambda.$$

**Example C.3.6** (Tori)**.** If $T \cong \mathbb{G}_m^n$ is an $n$-dimensional torus, then any one-parameter subgroup $\lambda\colon \mathbb{G}_m \to T$ is given by $t \mapsto (t^{\lambda_1}, \cdots, t^{\lambda_n})$ for integers $\lambda_i$ while a character of $T$ is given by $(t_1, \ldots, t_n) \mapsto t_1^{\chi_1} \cdots t_n^{\chi_n}$ for integers $\chi_i$. We thus have bijections $\mathbb{X}_*(T) \cong \mathbb{Z}^n$ and $\mathbb{X}^*(T) \cong \mathbb{Z}^n$ where the pairing $\langle -, - \rangle \colon \mathbb{X}_*(T) \times \mathbb{X}^*(T) \to \mathbb{Z}$ is identified with the standard inner product.

**Example C.3.7** ($\mathrm{GL}_n$)**.** Every one-parameter subgroup $\lambda$ is contained in a maximal torus, and since maximal tori are conjugate (Proposition C.3.5), there exists $g \in G(\Bbbk)$ such that $g\lambda g^{-1}$ is contained in the maximal torus consisting of diagonal matrices. The structure of $\mathbb{X}_*(\mathrm{GL}_n)$ is more complicated than in the torus case; see Remark C.3.11.

**Definition C.3.8** (Centralizers, parabolics, and unipotents)**.** Given a one-parameter subgroup $\lambda\colon \mathbb{G}_m \to G$ of an algebraic group, we define the subgroups:

$$\begin{array}{lll} C_\lambda = & \{g \in G \mid \lambda(t)g = g\lambda(t) \text{ for all } t\} & \text{(centralizer)} \\ P_\lambda = & \{g \in G \mid \lim_{t\to 0} \lambda(t)g\lambda(t)^{-1} \text{ exists}\} & \text{(parabolic)} \\ U_\lambda = & \{g \in G \mid \lim_{t\to 0} \lambda(t)g\lambda(t)^{-1} = 1\} & \text{(unipotent).} \end{array}$$

Functorially, for a $\Bbbk$-algebra $R$, $C_\lambda(R)$ (resp. $P_\lambda(R)$, $U_\lambda(R)$) consist of elements $g \in G(R)$ such that $\lambda_R = g^{-1}\lambda_R g$ (resp. $\lim_{t\to 0} \lambda(t)g\lambda(t)^{-1}$ exists, $\lim_{t\to 0} \lambda(t)g\lambda(t)^{-1} =$

1). Note that under the conjugation action of $\lambda$ on $G$, $C_\lambda$ is the fixed locus while while $P_\lambda$ is the attractor locus $G^+_\lambda$ as defined in §6.6.1. There is a homomorphism $P_\lambda \to C_\lambda$ defined by $g \mapsto \lim_{t\to 0} \lambda(t) g \lambda(t)^{-1}$ which is the identity on $C_\lambda$ yielding a split short exact sequence

$$1 \to U_\lambda \to P_\lambda \to C_\lambda \to 1.$$

**Example C.3.9.** Let $\lambda\colon \mathbb{G}_m \to \mathrm{GL}_n$ be a one-parameter subgroup. After a change of basis, we can assume that $\lambda(t) = \mathrm{diag}(t^{\lambda_1}, \cdots, t^{\lambda_n})$ with $\lambda_1 \le \cdots \le \lambda_n$. Given $(g_{ij}) \in \mathrm{GL}_n$, one has that

$$\lambda(t)\big(g_{ij}\big)\lambda(t)^{-1} = \big(t^{\lambda_i - \lambda_j} g_{ij}\big).$$

If $n_1, \ldots, n_s$ are integers with $\sum_i n_i = n$ such that

$$\lambda_1 = \cdots = \lambda_{n_1} < \lambda_{n_1+1} = \cdots = \lambda_{n_1+n_2} < \cdots < \lambda_{n-n_s+1} = \cdots = \lambda_n,$$

then $C_\lambda = \mathrm{GL}_{n_1} \times \cdots \times \mathrm{GL}_{n_s}$ is the subgroup of block diagonal matrices while $P_\lambda$ is the subgroup of block upper triangular matrices.

For example, if $\lambda(t) = (t^{-1}, t^2, t^2, t^7)$, then

$$U_\lambda = \begin{pmatrix} 1 & * & * & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad P_\lambda = \begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & 0 & 0 & * \end{pmatrix}, \quad C_\lambda = \begin{pmatrix} * & 0 & 0 & 0 \\ 0 & * & * & 0 \\ 0 & * & * & 0 \\ 0 & 0 & 0 & * \end{pmatrix}.$$

We record the following properties of parabolic subgroups. Reductive groups are discussed in more detail in §C.4.2.

**Proposition C.3.10.** *Let $G$ be a connected reductive algebraic group over an algebraically closed field $\Bbbk$, and let $\lambda\colon \mathbb{G}_m \to G$ be a one-parameter subgroup.*

*(a) The centralizer $C_\lambda$ is connected and reductive.*

*(b) The subgroup $P_\lambda$ is connected and parabolic, i.e. $G/P_\lambda$ is projective, and $N_G(P_\lambda) = P_\lambda$.*

*(c) The subgroup $U_\lambda$ is the unipotent radical of $P_\lambda$, and it acts freely and transitively on the set of one-parameter subgroups of $P_\lambda$ which are conjugate (under $P_\lambda$) to $\lambda$.*

*(d) If $\lambda, \lambda'\colon \mathbb{G}_m \to G$ are one-parameter subgroups, the intersection $P_\lambda \cap P_{\lambda'}$ contains a maximal torus of $G$.*

*Proof.* See [Con14, Thm. 4.1.7, Cor. 5.2.8] for (a)-(c) and [Bor91, Prop. 20.7] for (d). □

**Remark C.3.11** (Spherical buildings)**.** The set of one-parameter subgroups of a reductive group can be given the structure of an "ungainly but remarkable metric space" (as described by Mumford in [GIT, p.55]), first introduced by J. Tits, after introducing the equivalence relation where $\lambda \sim \rho$ if there exists $g \in P_\lambda(\Bbbk)$ such that $\rho(t^m) = g^{-1}\lambda(t^m)g$ for integers $n, m$.

### C.3.4 Line bundles with $G$-actions

If $G$ is an algebraic group over a field $\Bbbk$ acting on a $\Bbbk$-scheme $U$ via $\sigma\colon G \times U \to U$, then a *line bundle with a $G$-action* or a *$G$-linearization* is a line bundle $L$ on $U$

together with an isomorphism $\alpha\colon \sigma^*L \xrightarrow{\sim} p_2^*L$ satisfying the cocycle condition: the diagram

$$\begin{tikzcd}
& (\sigma \circ (\mathrm{id}_G \times \sigma))^*L \arrow[r, "(\mathrm{id}_G \times \sigma)^*\alpha"] & (p_2 \circ (\mathrm{id}_G \times \sigma))^*L \arrow[dr, equal] & \\
(\sigma \circ (\mu \times \mathrm{id}_U))^*L \arrow[ur, equal] \arrow[dr, "(\mu\times\mathrm{id}_U)^*\alpha"] & & & (\sigma \circ p_{23})^*L \arrow[dl, "p_{23}^*\alpha"] \\
& (p_2 \circ (\mu \times \mathrm{id}_U))^*L \arrow[r, equal] & (p_2 \circ p_{23})^*L &
\end{tikzcd}$$

commutes where $\mu\colon G \times G \to G$ denotes multiplication. When $U$ is projective, a very ample line bundle $L$ with a $G$-action corresponds to a finite dimensional $G$-representation $V = \mathrm{H}^0(U, L)$ and a $G$-equivariant closed immersion $U \hookrightarrow \mathbb{P}(V)$.

**Theorem C.3.12** (Sumihiro's Theorem on Linearizations)**.** *Let $G$ be a connected, smooth, and affine algebraic group over an algebraically closed field. Let $U$ be a normal scheme over $\Bbbk$ with an action of $G$.*

*(1) If $L$ is a line bundle on $U$, then there exists an integer $n > 0$ such that $L^{\otimes n}$ admits a $G$-action.*

*(2) If $U$ is quasi-projective, there exists a locally closed embedding $U \hookrightarrow \mathbb{P}(V)$ where $V$ is a finite dimensional $G$-representation.*

*(3) Every point $u \in U$ has a $G$-invariant quasi-projective open neighborhood.*

*Proof.* For (1), see [Sum74, Thm. 1], [Sum75, Lem. 1.2], and [KKLV89, Prop. 2.4]. Part (2) is a direct consequence of (1). For (3), see [Sum74, Lem. 8] and [Sum75, Thm. 3.8]. □

When $G$ is a torus, we have the stronger result that $U$ has a $G$-invariant affine cover.

**Theorem C.3.13** (Sumihiro's Theorem on Torus Actions)**.** *Let $U$ be a normal scheme over an algebraically closed field $\Bbbk$ with an action of a torus $T$. Then any point $u \in U$ has a $T$-invariant affine open neighborhood.*

*Proof.* See [Sum74, Cor. 2] and [Sum75, Cor. 3.11]. □

**Remark C.3.14.** Theorems C.3.12 and C.3.13 can fail if $U$ is not normal, e.g. the plane nodal cubic curve has a $\mathbb{G}_m$-action and no $\mathbb{G}_m$-invariant neighborhood of the origin can be embedded $\mathbb{G}_m$-equivariantly into projective space.

## C.4 Reductivity

We denote by $\mathrm{Rep}(G)$ the category of representations of an algebraic group $G$. If $V$ is a $G$-representation with coaction $\sigma\colon V \to \Gamma(G, \mathcal{O}_G) \otimes V$, then the *invariants* are $V^G := \{v \in V \mid \sigma(v) = 1 \otimes v\}$. A representation $V$ of $G$ is *irreducible* if every subrepresentation $W \subset V$ is either 0 or $V$.

### C.4.1 Linear reductive groups

There are various notions of reductivity but the one most central to this book is linear reductivity.

**Definition C.4.1.** An affine algebraic group $G$ over a field $\Bbbk$ is *linearly reductive* if the functor $\mathrm{Rep}(G) \to \mathrm{Vect}_{\Bbbk}$, taking a $G$-representation $V$ to its $G$-invariants $V^G$, is exact.

**Proposition C.4.2.** *Let $G$ be an affine algebraic group over a field $\Bbbk$. The following are equivalent:*

*(1) $G$ is linearly reductive;*

*(1′) The functor $\mathrm{Rep}^{\mathrm{fd}}(G) \to \mathrm{Vect}_{\Bbbk}$, $V \mapsto V^G$, on the category of finite dimensional representations is exact;*

*(2) Every $G$-representation (resp. finite dimensional $G$-representation) is a direct sum of irreducible representations.*

*(3) Given a $G$-representation (resp. finite dimensional $G$-representation) $V$ and a $G$-invariant subspace $W \subset V$, there exists a $G$-invariant subspace $W' \subset V$ such that $V = W \oplus W'$.*

*(4) For every finite dimensional representation $V$ and fixed $\Bbbk$-point $x \in \mathbb{P}(V)^G$, there exists a $G$-invariant linear function $f \in \Gamma(\mathbb{P}(V), \mathcal{O}(1))^G$ such that $f(x) \neq 0$.*

*Proof.* TO ADD □

**Remark C.4.3.** In the notation introduced in §6.3, $G$ is linearly reductive if and only if $\mathbf{B}G \to \operatorname{Spec} k$ is cohomologically affine or equivalently a good moduli space.

It is not hard to see that for a field extension $\Bbbk \to \Bbbk'$, $G$ is linearly reductive if and only if $G_{\Bbbk'}$ is and that linearly reductive groups are closed under extension Linearly reductive algebraic group are also closed under extension. See Lemma 6.3.15 and Proposition 6.3.17.

**Proposition C.4.4** (Maschke's Theorem)**.** *Let $G$ be a finite group whose order is prime to* $\mathrm{char}(\Bbbk)$*. Then $G$ is linearly reductive.*

*Proof.* If $V$ is a $G$-representation, averaging over translates gives a $G$-equivariant $\Bbbk$-linear

$$R_V \colon V \to V^G, \qquad v \mapsto \frac{1}{|G|} \sum_{g \in G} g \cdot v, \tag{C.4.1}$$

which is on the identity on $V^G$ and compatible with a map $f \colon V \to W$ of $G$-representations, i.e. $R_W \circ f = f \circ R_V$. It follows that a surjection $V \to W$ of $G$-representations induces a surjection $V^G \to W^G$ on invariants. □

**Example C.4.5.** In characteristic $p$, there is a 2-dimensional representation $V$ of $G = \mathbb{Z}/p$ where a generator acts via the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. The surjection $V \to \Bbbk$ onto the first component is a surjection of $G$-representations. The induced map $V^G \to \Bbbk$ on invariants is the zero map. Note however that the element $e_1^p \in \mathrm{Sym}^p V$ is $G$-invariant and maps to 1. Geometrically, this gives an action of $G$ on $\mathbb{A}^2 = \mathbb{A}(V)$ where $(1, 0)$ is $G$-invariant; the invariant hypersurface $x^p$ doesn't contain $p$, but there is no such hyperplane.

**Remark C.4.6** (Reynolds operator)**.** The map (C.4.1) is called a *Reynolds operator* for the action of $G$ on $V$. If $G$ is linearly reductive, the canonical projections $R_V \colon V \to V^G$ are Reynold operators, i.e. $\Bbbk$-linear maps which are the identity on $V^G$ and compatible with maps of $G$-representations. For an action of $G$ on a $\Bbbk$-scheme $\operatorname{Spec} A$ with dual action $A \to \Gamma(G, \mathcal{O}_G) \otimes A$, there is a projection $R_A \colon A \to A^G$. This is not a ring map, but since multiplication $A^G \otimes A \to A$ is a map of $G$-representations commuting with the Reynold operators, we have that

$$R_A(xy) = xR_A(y) \quad \text{for } x \in A^G,\ y \in A.$$

This is called the *Reynolds identity* and shows that $A \to A^G$ is an $A^G$-algebra homomorphism.

In Remark 6.3.9, the Reynolds operator was applied to show that $A^G$ is finitely generated whenever $A$ is. It can also be used to show that $\operatorname{Spec} A \to \operatorname{Spec} A^G$ separates $G$-orbits and has the properties of affine GIT quotients (Corollary 6.3.7); see [GIT, §1.2]. As we will see in the proof of Theorem C.4.7, one technique to prove that a given group is linearly reductive is to construct Reynolds operators.

### C.4.2 Reductive groups

An affine algebraic group $G$ over an algebraically closed field $\Bbbk$ is called *reductive* if every smooth, connected, unipotent, and normal subgroup of $G$ is trivial. [2] Over $\mathbb{C}$, an affine algebraic group is reductive if and only if it is the complexification of any maximal compact subgroup [Hoc65, XVII.5]. Reductive groups are a particularly nice class of algebraic groups appearing in many branches of mathematics and can be completely classified in terms of their root datum.

For a smooth affine algebraic group $G$, there are subgroups $R(G)$ and $R_u(G)$ of $G$, called the *radical* and *unipotent radical*, which are maximal among connected, normal and solvable (resp. connected normal unipotent) subgroups. Over an algebraically closed field $\Bbbk$, $G$ is reductive if $R_u(G)$ is trivial and called *semisimple* if $R(G)$ is trivial. The center $Z(G)$ of a reductive group $G$ is diagonalizable and contains the radical $R(G) \subset Z(G)$ as its largest subtorus, and the quotient $G/R(G)$ is semisimple. For a smooth affine algebraic group $G$, the quotient $G/R_u(G)$ is reductive. Over an arbitrary field $\Bbbk$, $G$ is called reductive if $G_{\overline{\Bbbk}}$ is. The unipotent radical $R_u(G)$ commutes with separable field extensions, and so over a perfect field $\Bbbk$, $G$ is reductive if and only if $R_u(G)$ is trivial. See [Bor91, Hum75, Spr98, Mil17].

The classical algebraic groups of $\mathrm{GL}_n$, $\mathrm{PGL}_n$, $\mathrm{SL}_n$, or $\mathrm{SP}_{2n}$ are reductive in every characteristic. We develop GIT in this book for actions by linearly reductive groups, and it is therefore imperative to know that these classical groups are linearly reductive in characteristic 0.

**Theorem C.4.7.** *In characteristic* 0*, a reductive algebraic group is linearly reductive. The converse is true in every characteristic for smooth algebraic groups.*

*Proof.* In [Hil1890], Hilbert established the linearly reductivity for $\mathrm{SL}_{n,\mathbb{C}}$ and $\mathrm{GL}_{n,\mathbb{C}}$ using a explicit differential operator well-known to 19th century invariant theorists: the $\Omega$*-process*. We will sketch the argument for $G = \mathrm{GL}_n$ over $\mathbb{C}$. Write $\Gamma(\mathrm{GL}_{n,\mathbb{C}}, \mathcal{O}_{\mathrm{GL}_n}) = \mathbb{C}[X_{ij}]_{\det}$. Let $V$ be a finite dimensional $\mathrm{GL}_n$-representation such

[2] Sometimes $G$ is also assumed to be connected. For a *reductive group scheme* $G \to S$, there is no such ambiguity in the literature: $G$ is smooth and affine over $S$ with connected and reductive geometric fibers [SGA3$_{\mathrm{III}}$, Exp. XIX, Defn. 2.7].

that the scalar matrices act with weight $k$, and let $\sigma\colon \mathbb{C}[V] \to \mathbb{C}[X_{ij}]_{\det} \otimes \mathbb{C}[V]$ be the dual action on the coordinate ring $\mathbb{C}[V]$ of $\mathbb{A}(V^\vee)$. The differential operator

$$\Omega := \det\big(\frac{\partial}{\partial X_{ij}}\big)$$

acts linearly on $\mathbb{C}[X_{ij}]_{\det}$ and also on $\mathbb{C}[X_{ij}]_{\det} \otimes \mathbb{C}[V]$. One checks that the map

$$V \to V^{\mathrm{GL}_n}, \quad f \mapsto \frac{1}{\Omega^k\big(\det(X_{ij})^k\big)}\Omega^k\big(\det(X_{ij})^k\sigma(f)\big)$$

defines a Reynolds operator. As with the averaging operator (C.4.1) in Maschke's Theorem (C.4.4), this shows that $\mathrm{GL}_n$ is linearly reductive and a variant of the argument shows that $\mathrm{SL}_n$ is also linearly reductive. The argument is algebraic and works over every field of characteristic 0. See also [Stu08, §4.3], [Dol03, §2.1] and [DK15, §4.5.3].

Extending an integral procedure developed by Hurwitz and Schur and ideas of Cartan, Weyl [Wey26, Wey25] showed that every reductive algebraic group over $\mathbb{C}$ is linearly reductive. The technique is now referred to as 'Weyl's unitarian trick'. A Lie group $G$ has a left $G$-invariant measure $\mu$, called the *left Haar measure*. When $G$ is compact, this measure is finite, and for a finite dimensional $G$-representation $V$, averaging gives a $\Bbbk$-linear map

$$V \to V^G, \quad v \mapsto \frac{1}{\int_G d\mu(g)}\int_G (g\cdot v)d\mu(g)$$

constant on $V^G$ and compatible with maps of $G$-representations. This is a Reynolds operator (Remark C.4.6) exactly as the averaging map in Maschke's Theorem (C.4.4) and implies that $V \mapsto V^G$ is exact. For every reductive algebraic group $G$ over $\mathbb{C}$, there is a real Lie subgroup $K \subset G(\mathbb{C})$ which is dense in the Zariski topology and compact in the Euclidean topology; for $\mathrm{GL}_{n,\mathbb{C}}$, $K = U_n$ is the subgroup of unitary matrices (hence the name 'unitarian trick'). Then for a finite dimensional $G$-representation $V$, there is an identification $V^K = V^G$, and since the functor taking $K$-invariant is exact, so is the functor taking $G$-invariants. See also [Dol03, §3.2], [Bum13, Thm. 14.3].[3]

There is also an algebraic argument using the *Casimir operator*. First, one reduces to the case that $G$ is semisimple because every reductive group is an extension of a torus by a semisimple group. Given a representation $\rho\colon \mathfrak{g} \to V$ of the Lie algebra, there is a symmetric bilinear form on $\mathfrak{g}$ defined by $\langle x, y\rangle = \mathrm{Tr}(\rho(x)\circ\rho(y))$. Letting $\{e_i\}$ be a basis of $\mathfrak{g}$ and $\{e'_i\}$ be a dual basis with respect to $\langle -,-\rangle$, the *Casimir operator* is the $\mathfrak{g}$-endomorphism $c_V := \sum_{i=1}^n \rho(e_i)\circ\rho(e'_i)$ on $V$. To show that $G$ is linearly reductive, it suffices to find a complement of any codimension 1 irreducible subspace $W \subset V$. As $G$ is semisimple, $G$ acts trivial on $V/W$ and therefore so does $\mathfrak{g}$. It follows that $\mathfrak{g}$ takes $V$ into $W$ and therefore so does $c_V$, i.e. $c_V(V) \subset W$. On the other hand, since $W$ is irreducible, $c_V$ acts on $W$ by multiplication by a scalar (Schur's lemma). It follows that $\ker(c_V) \subset V$ is a complement of $W$. See also [Mil17, Thm. 22.42], [Muk03, §4.3], [Hum78, §6.2] and [DK15, §4.5.2].

[3]This analytic argument suffices to show the linear reductivity of a reductive group $G$ over every characteristic 0 field by the limit methods of §A.6 (or by using the classification theorem of reductive groups): there is a subfield $\Bbbk' \subset \Bbbk$ of finite transcendence degree over $\mathbb{Q}$ and a group scheme $G' \to \operatorname{Spec}\Bbbk'$ such that $G'_{\Bbbk} = G$. Choosing an embedding $\Bbbk' \hookrightarrow \mathbb{C}$ and using the fact that both the notions of reductivity and linear reductivity are insensitive to separable field extensions, we see that the linear reductivity of $G'_{\mathbb{C}}$ implies the linear reductivity of $G$.

For the converse, we need to show that for a linearly reductive group $G$, the unipotent radical $R_u(G)$ is trivial. Since $G/R_u(G)$ is affine, Matsushima's Theorem (6.3.19) implies that $R_u(G)$ is linearly reductive. However, a non-trivial unipotent group is not linearly reductive. Indeed, it suffices to show this for $\mathbb{G}_a$. Let $V = \Bbbk^2$ be the two-dimensional representation given by $t \mapsto \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$. The projection $V \to \Bbbk, (x, y) \mapsto x$ is a surjection of $\mathbb{G}_a$-representations with no complement, i.e. there is no invariant $(x, y) \in V^{\mathbb{G}_a}$ with $x \neq 0$ (in fact there is no invariant $f \in (\mathrm{Sym}^d V)^{\mathbb{G}_a}$ with $d > 0$ and $f(1, 0)$ nonzero. See also [NM64]. $\square$

**Example C.4.8.** The algebraic groups such as $\mathrm{GL}_n$, $\mathrm{PGL}_n$, $\mathrm{SL}_n$, or $\mathrm{SP}_{2n}$ are not linearly reductive in characteristic $p$. For example, in characteristic 2, consider the action of $\mathrm{SL}_2$ acts on the space $V = \mathrm{Sym}^2(\Bbbk^\vee) = \{Ax^2 + Bxy + Cy^2\}$ of degree 2 binary forms. The subspace $W$ consisting of squares $L^2$ of linear forms is a $\mathrm{GL}_2$-invariant subspace with no complement; the quotient $V \to V/W = \mathbb{C}$ is given by $(A, B, C) \mapsto B$. While there is no invariant *linear* function not vanishing at $(0, 1, 0)$, the discriminant $\Delta = B^2 \in \mathrm{Sym}^2 V^\vee$ is an invariant function nonzero at $(0, 1, 0)$ (verifying the geometric reductivity condition given below).

**Theorem C.4.9** (Matsushima's Theorem)**.** *Let $G$ be a reductive group over a field $\Bbbk$. Then a subgroup $H \subset G$ is reductive if and only if $G/H$ is affine.*

*Proof.* See Proposition 6.3.19 for a proof when $G$ is linearly reductive. The general case can be proven in a similar way relying on the a generalization of Serre's Criterion for Affineness: an algebraic space $U$, satisfying the property that for a surjection $\mathcal{A} \to \mathcal{B}$ of $\mathcal{O}_U$-algebras every global section of $\mathcal{B}$ has a positive power that lifts, is affine. See also [Mat60], [BB63], [Ric77], [FS82], [Alp13, Thm. 12.5] and [Alp14, Thm 9.4.1]. $\square$

### C.4.3 Geometrically reductive groups

An affine algebraic group $G$ is called *geometrically reductive* (or sometimes called *semi-reductive*) if for every surjection $V \to W$ of $G$-representations and $w \in W^G$, there exists $n > 0$ such that $w^{p^n}$ is in the image of $\mathrm{Sym}^{p^n} V \to \mathrm{Sym}^{p^n} W$. It suffices to take $V$ finite dimensional and $W$ the trivial representation, in which case the condition translates to a geometric property that for a fixed $\Bbbk$-point $x \in \mathbb{P}(V)^G$, there is an invariant homogenous polynomial $f \in \Gamma(\mathbb{P}(V), \mathcal{O}(p^n))^G$ for $n > 0$ with $f(x) \neq 0$ (analogous to Proposition C.4.2(4) except that $f$ need not be linear).

Geometrically reductive groups appear in the context of GIT (§6.7) as their defining property can be used to show that the quotient morphisms $\operatorname{Spec} A \to \operatorname{Spec} A^G$ have desirable properties (e.g. separates closed orbits and $A^G$ is finitely generated). In an effort to extend GIT to actions by reductive groups such as $\mathrm{SL}_n$ and $\mathrm{GL}_n$ in positive characteristic, Mumford conjectured in [GIT, preface] a reductive group is geometrically reductive. This conjecture was resolved by Haboush [Hab75]; see also [SS11], [Ses69] and [Oda64]. Conversely, a smooth geometrically reductive group is reductive. In fact, an affine algebraic group $G$ is geometrically reductive if and only if $G_{\mathrm{red}}$ is reductive.

On the other hand, a linearly reductive group is clearly geometrically reductive. The converse is true in characteristic 0 [Alp14, Lem. 9.2.8].

We thus have the implications:

$$\text{linearly reductive} \underset{\overset{\text{char}=0}{\Longleftarrow}}{\Longrightarrow} \text{geometrically reductive} \underset{\overset{G \text{ smooth}}{\Longrightarrow}}{\Longleftarrow} \text{reductive.}$$

A smooth algebraic group $G$ in characteristic $p$ is linearly reductive if and only if the connected component $G^0$ is a torus and the order of $G/G^0$ is prime to $p$ [Nag62]. Every finite (possibly non-reduced) group scheme $G$ is geometrically reductive but is linearly reductive if and only if $G^0$ is diagonalizable and $G/G^0$ has order prime to $p$ [HR15, Thm. 1.2]. A commutative algebraic group $G$ is reductive if and only if it's diagonalizable.

We also point out that a smooth algebraic group $G$ satisfying the property that $A^G$ is finitely generated for every coaction on a finitely generated $\Bbbk$-algebra, then $G$ is necessarily reductive.

# Appendix D

# Deformation Theory

Deformation theory is the study of the local geometry of a moduli space $\mathcal{M}$ near an object $E_0 \in \mathcal{M}(\Bbbk)$. We focus primarily on the following three deformation problems:

(A) Embedded deformations $Z_0 \subset X$ of a closed subscheme $Z_0$ in a fixed projective scheme $X$ over $\Bbbk$. Here the moduli problem is the Hilbert functor $\mathrm{Hilb}^P(X)$ and $E_0 = [Z_0 \subset X] \in \mathrm{Hilb}^P(X)(\Bbbk)$.

(B) Deformations of a scheme $E_0$ over $\Bbbk$. In this section, the main example for us is when $E_0$ is a smooth curve, in which case the moduli problem is $\mathcal{M}_g$ and $[E_0] \in \mathcal{M}_g(\Bbbk)$.

(C) Deformations of a coherent sheaf $E_0$ on a fixed projective scheme $C$ over $\Bbbk$. The main example for us is when $C$ is a smooth curve and $E_0$ is a vector bundle, in which case the moduli problem is $\mathrm{Bun}_C$ and $[E_0] \in \mathrm{Bun}_C(\Bbbk)$.

In this chapter, we sketch the local-to-global approach of deformation theory by zooming in around $E_0 \in \mathcal{M}(\Bbbk)$ and studying successively first-order neighborhoods of $\mathcal{M}$ at $E_0$, higher-order deformations of $E_0$, formal neighborhoods of $E_0$ and eventually étale or smooth neighborhoods of $E_0$.

(1) A *first-order deformation* of $E_0$ is an object $E \in \mathcal{M}(\Bbbk[\epsilon])$ over the dual numbers $\Bbbk[\epsilon] := \Bbbk[\epsilon]/(\epsilon^2)$ together with an isomorphism $\alpha\colon E_0 \to E|_{\mathrm{Spec}\,\Bbbk}$, or in other words a commutative diagram

$$\begin{tikzcd} \mathrm{Spec}\,\Bbbk \arrow[r, hook] \arrow[dr, "{[E_0]}"'] & \mathrm{Spec}\,\Bbbk[\epsilon] \arrow[d, dashed, "{[E]}"] \\ & \mathcal{M} \end{tikzcd}$$

allowing us to view $E$ as a tangent vector of $\mathcal{M}$ at $E_0$. We classify first-order deformations of Problems (A)–(C) in §D.1.

(2) Given a surjection $A' \twoheadrightarrow A$ of artinian local $\Bbbk$-algebras with residue field $\Bbbk$ and an object object $E \in \mathcal{M}(A)$ with an isomorphism $E_0 \to E|_{\mathrm{Spec}\,\Bbbk}$, a *deformation of $E$ over $A'$* is an object $E' \in \mathcal{M}(A')$ with an isomorphism $\alpha\colon E \to E'|_{\mathrm{Spec}\,A}$. Pictorially, this corresponds to a commutative diagram

$$\begin{tikzcd} \mathrm{Spec}\,\Bbbk \arrow[r, hook] \arrow[drr, "{[E_0]}"'] & \mathrm{Spec}\,A \arrow[r, hook] \arrow[dr, "{[E]}"] & \mathrm{Spec}\,A' \arrow[d, dashed, "{[E']}"] \\ & & \mathcal{M}. \end{tikzcd}$$

For Problems (A)–(C), we determine when a deformation $E'$ of $E$ over $A'$ exists and we classify them in §D.2

(3) Given a noetherian complete local $\Bbbk$-algebra $(R, \mathfrak{m})$, a *formal deformation of* $E_0$ *over* $R$ is a compatible collection of deformations $E_n \in \mathcal{M}(R/\mathfrak{m}^{n+1})$ of $E_0$, and a formal deformation $\{E_n\}$ is *versal* if every other deformation factors through one of the $E_n$ (see Definition D.3.5 for a precise definition). Rim–Schlessinger's Criteria (Theorem D.3.11) provides criteria for the existence of a versal deformation $\{E_n\}$ of $E_0$, and in §D.3 we verify the criteria for the Problems (A)–(C).

(4) A formal deformation $\{E_n\}$ over $(R, \mathfrak{m})$ is *effective* if there exists an object $\widehat{E} \in \mathcal{M}(R)$ extending the $\{E_n\}$, or in other words there exists a commutative diagram

$$\begin{tikzcd}
\operatorname{Spec} R/\mathfrak{m} \arrow[r, hook] \arrow[rrrrd, "{[E_0]}"'] & \operatorname{Spec} R/\mathfrak{m}^2 \arrow[r, hook] \arrow[rrrd, "{[E_1]}"'] & \operatorname{Spec} R/\mathfrak{m}^3 \arrow[r, hook] \arrow[rrd, "{[E_2]}"] & \cdots \arrow[r, hook] & \operatorname{Spec} R \arrow[d, dashed, "{[\widehat{E}]}"] \\
& & & & \mathcal{M}.
\end{tikzcd}$$

In §D.4, we show how Grothendieck's Existence Theorem (D.4.4) implies that formal deformations are effective for Problems (A)–(C).

(5) In §D.5, we take a detour from the local-to-global approach to provide a glimpse into the role of the cotangent complex in deformation theory.

(6) Given an effective versal formal deformation $\widehat{E}$ over $R$, Artin Algebraization (Theorem D.6.6) ensures the existence of a *finite type* $\Bbbk$-scheme $U$ with a point $u \in U(\Bbbk)$ and an object $E \in \mathcal{M}(U)$ such that $R \cong \widehat{\mathcal{O}}_{U,u}$ and $\widehat{E}|_{\operatorname{Spec} R/\mathfrak{m}^{n+1}} \cong E|_{\operatorname{Spec} R/\mathfrak{m}^{n+1}}$ for all $n$.

(7) Artin's Axioms for Algebraicity (Theorems D.7.1 and D.7.4) provides criteria to verify the algebraicity of a moduli problem $\mathcal{M}$. Namely, it provides conditions to ensure that the morphism $[E]\colon U \to \mathcal{M}$ constructed above is a smooth morphism in an open neighborhood of $E_0$.

In this chapter, $\Bbbk$ will denote an algebraically closed field. In §D.3, §D.6 and §D.7, we work over the category of $\Bbbk$-schemes for convenience but the results hold more generally.

## D.1 First order deformations

Denote the *dual numbers* by $\Bbbk[\epsilon] := \Bbbk[\epsilon]/(\epsilon^2)$.

### D.1.1 First order embedded deformations

**Definition D.1.1.** Let $X$ be a projective scheme over a $\Bbbk$ and $Z_0 \subset X$ be a closed subscheme. A *first-order deformation of* $Z_0 \subset X$ is a closed subscheme $Z \subset X_{\Bbbk[\epsilon]} := X \times_{\Bbbk} \Bbbk[\epsilon]$ flat over $\Bbbk[\epsilon]$ such that $Z_0 = Z \times_{\Bbbk[\epsilon]} \Bbbk$. Pictorially, a first-order deformation is a filling of the diagram

$$\begin{tikzcd}
& X \arrow[rr, hook] \arrow[dd] & & X_{\Bbbk[\epsilon]} \arrow[dd] \\
Z_0 \arrow[ur, hook, "\mathrm{cl}"] \arrow[rr, hook, dashed] \arrow[dr] & & Z \arrow[ur, hook, dashed, "\mathrm{cl}"] \arrow[dr, dashed, "\mathrm{flat}"] & \\
& \operatorname{Spec} \Bbbk \arrow[rr, hook] & & \operatorname{Spec} \Bbbk[\epsilon].
\end{tikzcd}$$

with a scheme $Z$ and dotted arrows making the diagram cartesian.

We say that $Z \subset X_{\Bbbk[\epsilon]}$ is *trivial* if $Z = Z_0 \times_{\Bbbk} \Bbbk[\epsilon]$.

**Remark D.1.2.** Since $Z_0$ and the central fiber $Z \times_{\Bbbk[\epsilon]} \Bbbk$ of $Z$ are embedded in $X$, it makes sense to require that they are equal.

**Remark D.1.3.** The closed subscheme $Z_0 \subset X$ defines a $\Bbbk$-point $[Z_0 \subset X] \in \operatorname{Hilb}^P(X)$ of the Hilbert scheme where $P$ is the Hilbert polynomial of $Z_0$ with respect to a fixed ample line bundle on $X$. A first-order deformation corresponds to a commutative diagram

$$\begin{tikzcd}
\operatorname{Spec} \Bbbk \arrow[r, "{[Z_0 \subset X]}"] \arrow[d, hook] & \operatorname{Hilb}^P(X) \\
\operatorname{Spec} \Bbbk[\epsilon], \arrow[ur, "{[Z \subset X_{\Bbbk[\epsilon]}]}"'] &
\end{tikzcd}$$

or in other words a tangent vector $[Z \subset X_{\Bbbk[\epsilon]}] \in T_{\operatorname{Hilb}^P(X),[Z_0 \subset X]}$.

**Proposition D.1.4.** *Let $X$ be a projective scheme over a $\Bbbk$ and $Z_0 \subset X$ be a closed subscheme defined by a sheaf of ideals $I_0 \subset \mathcal{O}_X$. There is a bijection*

$$\{\textit{first-order deformations } Z \subset X_{\Bbbk[\epsilon]}\} \cong \mathrm{H}^0(Z_0, N_{Z_0/X})$$

*where $N_{Z_0/X} = \mathscr{H}om_{\mathcal{O}_{Z_0}}(I_0/I_0^2, \mathcal{O}_{Z_0})$ is the normal sheaf. Under this correspondence, the trivial deformation corresponds to $0 \in \mathrm{H}^0(Z_0, N_{Z_0/X})$.*

**Remark D.1.5.** In light of Remark D.1.3, this proposition gives a bijection $T_{\operatorname{Hilb}^P(X),[Z_0 \subset X]} \cong \mathrm{H}^0(Z_0, N_{Z_0/X})$.

*Proof.* We sketch the bijection and point the reader to [Har10, Prop. 2.3] and [Ser06, Prop. 3.2.1] for details. After reducing to the affine case $X = \operatorname{Spec} B$ and $Z_0 = \operatorname{Spec} B/I_0$, we need to show that the set of first-order deformations is bijective to

$$\mathrm{H}^0(Z_0, N_{Z_0/X}) \cong \operatorname{Hom}_{B/I_0}(I_0/I_0^2, B/I_0) \cong \operatorname{Hom}_B(I_0, B/I_0).$$

Given a first-order deformation $Z = \operatorname{Spec} B[\epsilon]/I$, the flatness of $Z$ over $\Bbbk[\epsilon]$ ensures that tensoring the exact sequence $0 \to I \to B[\epsilon] \to B[\epsilon]/I \to 0$ of $\Bbbk[\epsilon]$-modules with $\Bbbk = \Bbbk[\epsilon]/(\epsilon)$ yields an exact sequence $0 \to I_0 \to B \to B/I_0 \to 0$. We define $\alpha \colon I_0 \to B/I_0$ as follows: for $x_0 \in I_0$, choose a preimage $x = a + b\epsilon \in I$ and set $\alpha(x_0) := \overline{b} \in B/I_0$. Conversely, given a $B$-module homomorphism $\alpha \colon I_0 \to B/I_0$, we define

$$I = \{a + b\epsilon \mid a \in I_0, b \in B \text{ such that } \overline{b} = \alpha(a) \in B/I_0\} \subset B[\epsilon].$$

Then $(B[\epsilon]/I) \otimes_{\Bbbk[\epsilon]} \Bbbk = B/I_0$. To see that $B[\epsilon]/I$ is flat over $\Bbbk[\epsilon]$, we need to check that the map $B/I_0 \xrightarrow{\epsilon} B[\epsilon]/I$ is injective (see Remark A.2.7): given $b \in B$ with $\epsilon b \in I$, then $b \in I_0$ by the definition of $I$. Thus $Z = \operatorname{Spec} B[\epsilon]/I$ defines a first-order deformation of $Z_0$. ∎

### D.1.2 Locally trivial first-order deformations of schemes

**Definition D.1.6.** Let $X_0$ be a scheme over a $\Bbbk$. A *first-order deformation of* $X_0$ is a scheme $X$ flat over $\Bbbk[\epsilon]$ together with an isomorphism $\alpha\colon X_0 \to X \times_{\Bbbk[\epsilon]} \Bbbk$, or in other words a cartesian diagram

$$\begin{tikzcd} X_0 \arrow[r, hook, dashed] \arrow[d] \arrow[dr, phantom, "\square"] & X \arrow[d, dashed, "\text{flat}"] \\ \operatorname{Spec}\Bbbk \arrow[r, hook] & \operatorname{Spec}\Bbbk[\epsilon]. \end{tikzcd} \tag{D.1.1}$$

A *morphism of first-order deformations* $(X, \alpha)$ *and* $(X', \alpha')$ is a morphism $\beta\colon X \to X'$ of schemes over $\Bbbk[\epsilon]$ such that $(\beta \times_{\Bbbk[\epsilon]} \Bbbk) \circ \alpha = \alpha'$, or in other words considering $X$ and $X'$ in cartesian diagrams (D.1.1), we require the restriction of $\beta$ to central fiber $X_0$ to be the identity.

We say that $X$ is *trivial* if $X$ is isomorphic as first-order deformations to $X \times_{\Bbbk} \Bbbk[\epsilon]$, and *locally trivial* if there exists a Zariski-cover $X = \bigcup_i U_i$ such that $U_i$ is a trivial first-order deformation of $U_i \times_{\Bbbk[\epsilon]} \Bbbk \subset X_0$.

Every morphism of deformations is necessarily an isomorphism. This is a consequence of the following algebraic fact.

**Lemma D.1.7.** *Let $A$ be a ring, $\mathfrak{m} \subset A$ be a nilpotent ideal (e.g. $(A, \mathfrak{m})$ is an artinian local ring), and $M \to N$ be a homomorphism of $A$-modules. Assume that $N$ is flat over $A$. If $M/\mathfrak{m}M \to N/\mathfrak{m}N$ is an isomorphism, so is $M \to N$.*

*Proof.* The right exact sequence $M \to N \to C \to 0$ becomes $M/\mathfrak{m}M \to N/\mathfrak{m}N \to C/\mathfrak{m}C \to 0$ after modding out by $\mathfrak{m}$, and we see that $C = \mathfrak{m}C$. As $\mathfrak{m}^n = 0$ for some $n$, we obtain that $C = \mathfrak{m}C = \mathfrak{m}^2C = \cdots = \mathfrak{m}^nC = 0$. Considering now the exact sequence $0 \to K \to M \to N \to 0$, the flatness of $N$ implies that we get an exact sequence $0 \to K/\mathfrak{m}K \to M/\mathfrak{m}M \to N/\mathfrak{m}N \to 0$. Thus $K = \mathfrak{m}K = \cdots = \mathfrak{m}^nK = 0$, and we see that $M \to N$ is an isomorphism. $\square$

**Proposition D.1.8.** *Every first-order deformation of a smooth affine scheme $X_0$ over $\Bbbk$ is trivial. In other words, $X_0$ is rigid.*

*Proof.* Let $X$ be a first-order deformation of $X_0$. Since $X_0 \to \operatorname{Spec}\Bbbk$ is smooth, we may apply the Infinitesimal Lifting Criterion for Smoothness (A.3.1) to construct a lift $X \to X_0$ making the diagram

$$\begin{tikzcd} X_0 \arrow[r, "\text{id}"] \arrow[d, hook] & X_0 \arrow[d, "\text{smooth}"] \\ X \arrow[r] \arrow[ur, dashed] & \operatorname{Spec}\Bbbk \end{tikzcd}$$

commute. This induces a morphism $X \to X_0 \times_{\Bbbk} \Bbbk[\epsilon]$ over $\Bbbk[\epsilon]$ which restricts to the identity on $X_0$, and thus is an isomorphism by Lemma D.1.7.

See also [Har77, Exer. II.8.7]. $\square$

**Remark D.1.9.** If $X_0$ is not smooth or affine, then first-order deformations are not necessarily trivial. For example, if $X_0 = \operatorname{Spec}\Bbbk[x, y]/(xy)$ is the nodal affine plane curve, then $X = \operatorname{Spec}\Bbbk[x, y, \epsilon]/(xy - \epsilon) \to \operatorname{Spec}\Bbbk[\epsilon]$ is a non-trivial first-order deformation.

On the other hand, consider an elliptic curve $E_0 = V(y^2z - x(x-z)(x-2z)) \subset \mathbb{P}^2$ is a elliptic curve over $\Bbbk$ with $\mathrm{char}(\Bbbk) \neq 2, 3$. It is easy to write down global deformations by deforming the coefficients of the defining equations: $\mathcal{E} = V(y^2z - (x-\lambda z)(x-z)(x-2z)) \subset \mathbb{P}^2 \times \mathbb{A}^1$ (where $\mathbb{A}^1$ has coordinate $\lambda$) defines a flat projective morphism $\mathcal{E} \to \mathbb{A}^1$ such the central fiber $\mathcal{E}_0$ is isomorphic to $E_0$. Restricting $\mathcal{E}$ to the family $E := \mathcal{E} \times_{\mathbb{A}^1} \operatorname{Spec} \Bbbk[\lambda]/\lambda^2$ over the dual numbers defines a *non-trivial* first-order deformation. For an affine open $U_0 \subset E_0$ and setting $U \subset E$ to be the open subscheme with the same topological space as $U_0$, then there is an isomorphism $U \xrightarrow{\sim} U_0 \times_{\Bbbk} \Bbbk[\epsilon]$. These local isomorphisms however do not glue to a global isomorphism $E \xrightarrow{\sim} E_0 \times_{\Bbbk} \Bbbk[\epsilon]$. Since every deformation of a smooth scheme is obtained by gluing together trivial deformations, we need to understand automorphisms of trivial deformations to classify global deformations.

**Lemma D.1.10.** *If $X_0 = \operatorname{Spec} A$ is an affine scheme over $\Bbbk$ and $X = \operatorname{Spec} A[\epsilon]$ is the trivial first-order deformation, then there are identifications*

$$\{\textit{automorphisms } X \to X \textit{ of first-order defs}\} \cong \operatorname{Der}_{\Bbbk}(A, A) \cong \operatorname{Hom}_A(\Omega_{A/\Bbbk}, A).$$

*Proof.* The second equivalence is given by the universal property of the module of differentials. An automorphism of the trivial first-order deformation corresponds to a $\Bbbk[\epsilon]$-algebra isomorphism $\phi\colon A \oplus A\epsilon \to A \oplus A\epsilon$ which is the identity modulo $\epsilon$. Therefore, $\phi$ is determined by the images $\phi(a) = a + d(a)\epsilon$ of elements $a \in A$ where $d\colon A \to A$ is $\Bbbk$-linear map. Since $\phi$ is a ring homomorphism, for elements $a, a' \in A$, we must have that $aa' + d(aa')\epsilon = (a + d(a)\epsilon)(a' + d(a')\epsilon) = aa' + (ad(a') + a'd(a))\epsilon$ and we see that $d\colon A \to A$ is a $\Bbbk$-derivation. □

For a scheme $X_0$ over $\Bbbk$, let $\operatorname{Def}(X_0)$ and $\operatorname{Def}^{\mathrm{lt}}(X_0)$ denote the sets of isomorphism classes of first-order and locally trivial first-order deformations.

**Proposition D.1.11.** *For a scheme $X_0$ of finite type over $\Bbbk$ with affine diagonal, there is a bijection*

$$\operatorname{Def}^{\mathrm{lt}}(X_0) \cong \mathrm{H}^1(X_0, T_{X_0}),$$

*where $T_{X_0} = \mathscr{H}om_{\mathcal{O}_{X_0}}(\Omega_{X_0/\Bbbk}, \mathcal{O}_{X_0})$. The trivial deformation corresponds to $0 \in \mathrm{H}^1(X_0, T_{X_0})$.*

*In particular, if in addition $X_0$ is smooth over $\Bbbk$, then there is a bijection*

$$\operatorname{Def}(X_0) \cong \mathrm{H}^1(X_0, T_{X_0}).$$

*Proof.* Let $X \to \operatorname{Spec} \Bbbk[\epsilon]$ be a locally trivial first-order deformation of $X_0$. Choose an affine cover $\{U_i\}$ of $X_0$ and isomorphisms $\phi_{ij} := \phi_i\colon U_i \times_{\Bbbk} \Bbbk[\epsilon] \xrightarrow{\sim} X|_{U_i}$, where $X|_{U_i} \subset X$ denotes the open subscheme with the same topological space as $U_i$. Letting $U_{ij} = U_i \cap U_j$, we have automorphisms $\phi_j^{-1}|_{U_{ij} \times_{\Bbbk} \Bbbk[\epsilon]} \circ \phi_i|_{U_{ij} \times_{\Bbbk} \Bbbk[\epsilon]}$ of the trivial deformation $U_{ij} \times_{\Bbbk} \Bbbk[\epsilon]$ which by Lemma D.1.10 corresponds to elements $\phi_{ij} \in \operatorname{Hom}_{\mathcal{O}_{U_{ij}}}(\Omega_{U_{ij}/\Bbbk}, \mathcal{O}_{U_{ij}})$. Since $\phi_{ij} \circ \phi_{jk} = \phi_{ik}$ on $U_{ijk} := U_i \cap U_j \cap U_k$, we have that $\phi_{ij} + \phi_{jk} = \phi_{ik} \in T_{X_0}(U_{ijk})$. Recall that $\mathrm{H}^1(X_0, T_{X_0})$ can be computed using the Céch complex

$$\begin{array}{c} 0 \longrightarrow \bigoplus_i T_{X_0}(U_i) \xrightarrow{d_0} \bigoplus_{i,j} T_{X_0}(U_{ij}) \xrightarrow{\quad d_1 \quad} \bigoplus_{i,j,k} T_{X_0}(U_{ijk}) \\ (s_{ij}) \longmapsto (s_{ij}|_{U_{ijk}} - s_{ik}|_{U_{ijk}} + s_{jk}|_{U_{ijk}})_{ijk} \end{array}$$

As $\{\phi_{ij}\} \in \bigoplus_{i,j} T_{X_0}(U_{ij})$ is in the kernel of $d_1$, it defines an element of $\mathrm{H}^1(X_0, T_{X_0}) = \ker(d_1)/\operatorname{im}(d_0)$. Conversely, given an element of $\mathrm{H}^1(X_0, T_{X_0})$ and a choice of representative $\{\phi_{ij}\} \in \ker(d_1)$, then viewing each $\phi_{ij}$ as an automorphism $\phi_{ij}$ of the trivial deformation of $U_{ij}$, we may glue together the trivial deformations $U_i \times_{\Bbbk} \Bbbk[\epsilon]$ along $U_{ij} \times_{\Bbbk} \Bbbk[\epsilon]$ via $\phi_{ij}$ to construct a global first-order deformation $X$ of $X_0$.

For the final statement, observe that since $X_0$ is smooth over $\Bbbk$, every first-order deformation is locally trivial by Proposition D.1.8. See also [Har77, Exer. III.4.10 and Ex. III.9.13.2]. □

**Example D.1.12.** If $C$ is a smooth projective curve of genus $g \geq 2$, then we've computed that

$$T_{\mathcal{M}_g,[C]} = \mathrm{H}^1(C, T_C) \stackrel{\mathrm{SD}}{=} \mathrm{H}^0(C, \Omega_{C/\Bbbk}^{\otimes 2})$$

which by Riemann–Roch is a $3g-3$ dimensional vector space.

**Exercise D.1.13.** Use the Euler exact sequence to show that $\mathrm{H}^1(\mathbb{P}^n, T_{\mathbb{P}^n}) = 0$ and conclude that every first-order deformation of $\mathbb{P}^n$ is trivial, i.e. $\mathbb{P}^n$ is rigid.

### D.1.3 First order deformations of vector bundles and coherent sheaves

**Definition D.1.14.** Let $X$ be a projective scheme over $\Bbbk$ and $E_0$ be a coherent sheaf. A *first-order deformation of* $E_0$ is a pair $(E, \alpha)$ where $E$ is a coherent sheaf on $X \times_{\Bbbk} \Bbbk[\epsilon]$ flat over $\Bbbk[\epsilon]$ and $\alpha \colon E_0 \xrightarrow{\sim} E|_X$ is an isomorphism. Pictorially, we have

$$\begin{tikzcd} E_0 \arrow[d, dash] & E \arrow[d, dash, "\mathrm{flat}/\Bbbk[\epsilon]"] \\ X \arrow[r, hook] & X_{\Bbbk[\epsilon]}. \end{tikzcd}$$

A *morphism* $(E, \alpha) \to (E', \alpha')$ *of first-order deformations* is a morphism $\beta \colon E \to E'$ (equivalently an isomorphism by Lemma D.1.7) of coherent sheaves on $X'$ such that $\alpha' = \beta|_X \circ \alpha$.

We say that $(E, \alpha)$ is *trivial* if it isomorphic as first-order deformations to $(p^* E_0, \mathrm{id})$ where $p \colon X_{\Bbbk[\epsilon]} \to X$.

**Proposition D.1.15.** *Let $X$ be a scheme over $\Bbbk$ and $E_0$ be a coherent sheaf. There is a bijection*

$$\{\textit{first-order deformations } (E, \alpha) \textit{ of } E_0\}/\!\sim \;\; \cong \mathrm{Ext}^1_{\mathcal{O}_X}(E_0, E_0)$$

*Under this correspondence, the trivial deformation corresponds to $0 \in \mathrm{Ext}^1_{\mathcal{O}_X}(E_0, E_0)$.*

*If in addition $E_0$ is a vector bundle (resp. line bundle), then the set of isomorphism classes of first-order deformations of $E_0$ is bijective to $\mathrm{H}^1(X, \mathscr{E}nd_{\mathcal{O}_X}(E_0))$ (resp. $\mathrm{H}^1(X, \mathcal{O}_X)$).*

*Proof.* If $(E, \alpha)$ is a first-order deformation then since $E$ is flat over $\Bbbk[\epsilon]$, we may tensor the exact sequence $0 \to \Bbbk \xrightarrow{\epsilon} \Bbbk[\epsilon] \to \Bbbk \to 0$ of $\Bbbk[\epsilon]$-modules with $E$ to obtain an exact sequence $0 \to E_0 \xrightarrow{\epsilon} E \to E_0 \to 0$ (after identifying $E \otimes_{\Bbbk[\epsilon]} \Bbbk$ with $E_0$ via $\alpha$). Since $\mathrm{Ext}^1_{\mathcal{O}_X}(E_0, E_0)$ parameterizes isomorphism classes of extensions [Har77, Exer. III.6.1], we have constructed an element of $\mathrm{Ext}^1_{\mathcal{O}_X}(E_0, E_0)$. Conversely, given an exact sequence $0 \to E_0 \xrightarrow{\alpha} E \to E_0 \to 0$, then $E$ is a coherent sheaf on $X_{\Bbbk[\epsilon]}$ and is flat over $\Bbbk[\epsilon]$ by the flatness criterion over the dual numbers (see Remark A.2.7). The restriction $E|_X$ is isomorphic to $E_0$ via $\alpha$.

See also [Har10, Thm. 2.7]. □

**Remark D.1.16.** The classifications of Propositions D.2.2, D.2.6 and D.2.15 give a vector space structure to the set of isomorphism classes of first-order deformations. This vector space structure can also be witnessed as a consequence of Rim–Schlessinger's homogeneity condition; see Lemma D.3.13.

## D.2 Higher-order deformations and obstructions

Let $\mathcal{M}$ be a moduli problem and $E \in \mathcal{M}(A)$ be an object defined over a ring $A$. If $A' \twoheadrightarrow A$ is a surjection of rings with square-zero kernel, this section addresses the following two questions:

(1) Does $E$ deform to an object $E' \in \mathcal{M}(A')$?

(2) If so, can we classify all such deformations?

Pictorially, we have:

$$\begin{tikzcd} E \arrow[d, no head] & E' \arrow[d, no head, dashed] \\ \operatorname{Spec} A \arrow[r, hook] & \operatorname{Spec} A'. \end{tikzcd}$$

where $\operatorname{Spec} A \hookrightarrow \operatorname{Spec} A'$ is a closed immersion of schemes with the same topological space. Note that since $J = \ker(A' \to A)$ is square-zero, $J = J/J^2$ is naturally a module over $A = A'/J$. In other words, Question (1) asks whether there is some "obstruction" to the existence of a deformation $E'$ while (2) seeks to classify all higher-order deformations given that there is no obstruction.

An interesting case is when $A$ and $A'$ are local artinian algebras with residue field $\Bbbk$ and the kernel $J = \ker(A' \to A)$ satisfies $\mathfrak{m}_{A'} J = 0$ (which implies that $J^2 = 0$). In this case, $J = J/\mathfrak{m}_{A'} J$ is naturally a vector space over $\Bbbk = A'/\mathfrak{m}_{A'}$. Setting $E_0 := E|_{\Bbbk} \in \mathcal{M}(\Bbbk)$, we can view $E$ as a deformation over $E_0$ over $A$, and we are attempting to classify the higher-order deformations over $A'$. If there are no obstructions to deforming, then the Infinitesimal Lifting Criterion for Smoothness (3.7.1) implies that $\mathcal{M}$ is smooth at $[E_0]$.

The previous section studied the specific case when $A = \Bbbk$ and $A' = \Bbbk[\epsilon]$ in which case deformations of an object $E_0 \in \mathcal{M}(\Bbbk)$ over $A'$ correspond to first-order deformations. In this case, the obstruction vanishes as there is always the trivial deformation (i.e. the pullback of $E_0$ along $\operatorname{Spec} \Bbbk[\epsilon] \to \operatorname{Spec} \Bbbk$). Other examples of $A' \to A$ to keep in mind are $\Bbbk[x]/x^{n+1} \twoheadrightarrow \Bbbk[x]/x^n$ and $\mathbb{Z}/p^{n+1} \twoheadrightarrow \mathbb{Z}/p^n$ where we inductively attempt to deform $E_0$ over the nilpotent thickenings $\operatorname{Spec} \Bbbk[x]/x^{n+1} \hookrightarrow \mathbb{A}^1$ and $\operatorname{Spec} \mathbb{Z}/p^{n+1} \hookrightarrow \operatorname{Spec} \mathbb{Z}$.

### D.2.1 Higher order embedded deformations

**Definition D.2.1.** Let $A' \twoheadrightarrow A$ be a surjection of noetherian rings with square-zero kernel. Let $X'$ be a scheme over $A'$ and set $X := X' \times_{A'} A$. Let $Z \subset X$ be a closed subscheme flat over $A$. A *deformation of* $Z \subset X$ *over* $A'$ is a closed subscheme $Z' \subset X'$ flat over $A'$ such that $Z' \times_{A'} A = Z$ as closed subschemes of $X$. Pictorially, a deformation is a filling of the cartesian diagram

$$\begin{tikzcd}
& X \arrow[rr, hook] \arrow[dd] & & X' \arrow[dd] \\
Z \arrow[ur, hook, "\mathrm{cl}"] \arrow[rr, hook, dashed] \arrow[dr, "\mathrm{flat}"'] & & Z' \arrow[ur, hook, dashed, "\mathrm{cl}"] \arrow[dr, dashed, "\mathrm{flat}"'] & \\
& \operatorname{Spec} A \arrow[rr, hook] & & \operatorname{Spec} A'.
\end{tikzcd}$$

The formulation of the next proposition uses the following notion: a *torsor of a group* $G$ is a transitive and free action of $G$ on a set.

**Proposition D.2.2.** *Let $A' \twoheadrightarrow A$ be a surjection of noetherian rings with square-zero kernel $J$. Let $X'$ be a scheme over $A'$ with affine diagonal (e.g. separated) and $Z \subset X := X' \times_{\Bbbk} A$ be a closed subscheme flat over $A$ defined by a sheaf of ideals $I \subset \mathcal{O}_X$. Then*

*(1) If there exists a deformation $Z' \subset X'$ of $Z \subset X$ over $A'$, then the set of such deformations is a torsor under $\mathrm{H}^0(Z, N_{Z/X} \otimes_A J) = \mathrm{Ext}^0_{\mathcal{O}_Z}(I/I^2, J)$.*

*(2) There exists an element $\mathrm{ob}_Z \in \mathrm{Ext}^1_{\mathcal{O}_Z}(I/I^2, J)$ (depending on $Z$ and $A' \twoheadrightarrow A$) such that there exists a deformation of $Z \subset X$ over $A'$ if and only if $\mathrm{ob}_Z = 0$.*

**Remark D.2.3.** An interesting example is when $X = X_0 \times_{\Bbbk} A$ and $X' = X_0 \times_{\Bbbk} A'$ are the base changes of a $\Bbbk$-scheme $X_0$. If the closed subscheme $Z \subset X$ has constant Hilbert polynomial $P$ (i.e. for each $s \in \operatorname{Spec} A$, the Hilbert polynomial of $Z_s \subset X_0 \times_{\Bbbk} \kappa(s)$, with respect to a fixed ample line bundle on $X_0$, is independent of $s$), then we have an object $[Z \subset X] \in \mathrm{Hilb}^P(X_0)(A)$ of the Hilbert scheme. In this case, a deformation of $Z \subset X$ over $A'$ is an object $[Z' \subset X'] \in \mathrm{Hilb}^P(X_0)(A')$ which restricts to $[Z \subset X]$. Note that when $A' \twoheadrightarrow A$ is a surjection of local artinian $\Bbbk$-algebras with $\mathfrak{m}_{A'} J = 0$, then there is an identification $\mathrm{H}^0(Z, N_{Z/X} \otimes_A J) = \mathrm{H}^0(Z_0, N_{Z_0/X_0} \otimes_{\Bbbk} J)$ where $Z_0 = Z \times_A \Bbbk$.

**Remark D.2.4.** In the case that deformations of $Z \subset X$ over $A'$ exist Zariski-locally on $X$, then there is an obstruction element $\mathrm{ob}_Z \in H^1(Z, N_{Z/X} \otimes_A J)$.

*Proof.* Suppose first that $X' = \operatorname{Spec} B'$, $X = \operatorname{Spec} B$ where $B = B' \otimes_{A'} A$ and $Z = \operatorname{Spec} B/I$. If there exists a deformation $Z' = \operatorname{Spec} B'/I'$, then there is an exact diagram

$$\begin{tikzcd}
& 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & I \otimes_A J \arrow[r] \arrow[d] & I' \arrow[r] \arrow[d] & I \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & B \otimes_A J \arrow[r] \arrow[d] & B' \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & (B/I) \otimes_A J \arrow[r] \arrow[d] & B'/I' \arrow[r] \arrow[d] & B/I \arrow[r] \arrow[d] & 0 \\
& 0 & 0 & 0 &
\end{tikzcd}$$

The exactness of the bottom row (resp. middle row) is equivalent to the flatness of $B'/I'$ (resp. $B'$) over $A'$ by the Local Criterion of Flatness (Corollary A.2.6), while the exactness of the left column follows from the flatness of $B/I$ over $A$. Conversely, an exact diagram defines a deformation $Z' = \operatorname{Spec} B'/I'$.

We will define an action $\operatorname{Hom}_B(I, (B/I) \otimes_A J)$ on the set of deformations as follows: given $\phi \in \operatorname{Hom}_B(I, (B/I) \otimes_A J)$ and a deformation $Z' = \operatorname{Spec} B'/I'$, define $I'' \subset B'$ as the set of elements $x'' \in B'$ such that its image $\overline{x}'' \in B$ lies in $I$ and such that a lifting $x' \in I'$ of $\overline{x}'' \in I$ satisfies $x'' - x' = \phi(\overline{x''}) \in (B/I) \otimes_A J$ (noting that this condition is independent of the choice of lifting $x'$). One checks that $\operatorname{Spec} B'/I''$ is another deformation.

On the other hand, given two deformations defined by ideals $I'$ and $I''$, we define $\phi\colon I \to (B/I) \otimes_A J$ by $\phi(x) = \overline{x' - x''}$ where $x' \in I'$ and $x'' \in I''$ are lifts of $x$ (which forces $x' - x'' \in B \otimes_A J$). One checks that this is a $B$-module homomorphism providing an inverse to the above construction. We have natural identifications

$$\operatorname{Hom}_B(I, (B/I) \otimes_A J) = \operatorname{Hom}_{B/I}(I/I^2, B/I \otimes_A J) = \mathrm{H}^0(Z, N_{Z/X} \otimes_A J).$$

The above constructions globalize to $X$ and establish (1).

For (2), we will assume that there is an open cover $\{U_i\}$ of $X$ such that there exists deformations $Z_i' \subset X' \cap U_i$ of $Z \cap U_i \subset X \cap U_i$ (noting that $X$ and $X'$ are homeomorphic). On $U_{ij} = U_i \cap U_j$, the two deformations $Z_i'|_{U_{ij}}$ and $Z_j'|_{U_{ij}}$ defines an element $\phi_{ij} \in \mathrm{H}^0(U_{ij}, N_{Z/X} \otimes_A J)$ which in turn defines a Céch 1-cocycle $(\phi_{ij}) \in \mathrm{H}^1(X, N_{Z/X} \otimes_A J)$. We leave the reader to check that the vanishing of $(\phi_{ij})$ characterizes whether there is a deformation of $Z \subset X$ over $A'$.

See also [Har10, Thm. 6.2]. □

### D.2.2 Higher-order deformations of schemes

In this chapter, we discuss higher-order deformations and obstructions for schemes that are smooth or more generally local complete intersections.

**Definition D.2.5.** Let $A' \twoheadrightarrow A$ be a surjection of noetherian rings with square-zero kernel and $X \to \operatorname{Spec} A$ be a flat morphism of schemes. A *deformation of $X \to \operatorname{Spec} A$ over $A'$* is a flat morphism $X' \to \operatorname{Spec} A'$ together with an isomorphism $\alpha\colon X \xrightarrow{\sim} X' \times_{A'} A$ over $A$, or in other words a cartesian diagram

$$\begin{tikzcd} X \arrow[r, hook, dashed] \arrow[d, "\text{flat}"] & X' \arrow[d, dashed, "\text{flat}"] \\ \operatorname{Spec} A \arrow[r, hook] & \operatorname{Spec} A'. \end{tikzcd} \tag{D.2.1}$$

A *morphism of deformations over $A'$* is a morphism of schemes over $A'$ restricting to the identity on $X$. By Lemma D.1.7, every morphism of deformations is an isomorphism.

**Proposition D.2.6** (Higher-order deformations of Smooth Schemes)**.** *Let $A' \twoheadrightarrow A$ be a surjection of noetherian rings with square-zero kernel $J$. If $X \to \operatorname{Spec} A$ is a smooth morphism of schemes where $X$ has affine diagonal (e.g. separated), then*

1. *The group of automorphisms of a deformation $X' \to \operatorname{Spec} A'$ of $X \to \operatorname{Spec} A$ over $A'$ is bijective to $\mathrm{H}^0(X, T_{X/A} \otimes_A J)$.*
2. *If there exists a deformation of $X \to \operatorname{Spec} A$ over $A'$, then the set of isomorphism classes of all such deformations is a torsor under $\mathrm{H}^1(X, T_{X/A} \otimes_A J)$.*
3. *There is an element $\mathrm{ob}_X \in \mathrm{H}^2(X, T_{X/A} \otimes_A J)$ with the property that there exists a deformation of $X \to \operatorname{Spec} A$ over $A'$ if and only if $\mathrm{ob}_X = 0$.*

**Remark D.2.7.** If $A$ and $A'$ are local artinian rings with residue field $\Bbbk$ such that $\mathfrak{m}_{A'}J = 0$ and we set $X_0 := X \times_A \Bbbk$, then automorphisms, deformations and obstructions are classified by $\mathrm{H}^i(X_0, T_{X_0} \otimes_{\Bbbk} J)$ for $i = 0, 1, 2$.

*Proof.* When $X = \operatorname{Spec} A$ is an affine scheme, the same argument of Lemma D.1.10 shows that group of automorphisms of $X'$ is identified with $\operatorname{Hom}_A(\Omega_{A/\Bbbk}, J) = \mathrm{H}^0(X, T_{X/A} \otimes J)$. Since $T_{X/A} \otimes J$ and the assignment of an open subscheme $U$ to the group of automorphisms of the first-order deformation $X'|_U$ are sheaves, Part (1) follows.

Let $\{U_i\}$ be an affine cover of $X$. Part (2) follows from a similar argument to Proposition D.1.11: Fix a deformation $X' \to \operatorname{Spec} A$ of $X$. For every other deformation $X'' \to \operatorname{Spec} A$, we know that over each affine $U_i$, there is an isomorphism $\phi_i \colon X'|_{U_i} \to X''|_{U_i}$ and we let $\phi_{ij} = \phi_j^{-1}|_{X'|_{U_{ij}}} \circ \phi_i|_{X'|_{U_{ij}}}$ viewed as an element of $\mathrm{H}^0(U_i, T_{X/A} \otimes J)$. The Céch 1-cycle $(\phi_{ij})$ defines an element in $\mathrm{H}^1(X, T_{X/A} \otimes J)$.

For (3), again using that $U_i$ is affine, we can choose a deformation $U_i' \to \operatorname{Spec} A'$ of $U_i$. We can also choose isomorphisms $\phi_{ij} \colon U_i'|_{U_{ij}} \to U_j'|_{U_{ij}}$. This defines gluing data for a deformation $X'$ if $\phi_{jk} \circ \phi_{ij} = \phi_{ik}$ on the triple intersections $U_{ijk}$. The automorphism $\Psi_{ijk} = \phi_{ik}^{-1} \circ \phi_{jk} \circ \phi_{ij}$ restricts to the identity on $U_{ijk}$ and thus defines an element of $\mathrm{H}^0(U_{ijk}, T_{X/A} \otimes J)$. Consider the Céch complex with $F = T_{X/A} \otimes J$

$$\begin{array}{ccccc}
\bigoplus_{i,j} F(U_{ij}) & \xrightarrow{d_1} & \bigoplus_{i,j,k} F(U_{ijk}) & \xrightarrow{d_2} & \bigoplus_{i,j,k,l} F(U_{ijkl}) \\
(s_{ij}) & \longmapsto & (s_{ij} - s_{ik} + s_{jk})_{ijk} & & \\
 & & (s_{ijk}) & \longmapsto & (s_{ijk} - s_{ijl} + s_{ikl} - s_{jkl})_{ijkl}.
\end{array}$$

One checks that $d_2(\Psi_{ijk}) = 0$ and that if $\phi'_{ij}$ is a different choice of isomorphisms then the corresponding element $(\Psi'_{ijk})$ differs from $(\Psi_{ijk})$ by an element in the image of $d_1$. Thus $(\Psi_{ijk})$ is a well-defined element of $\mathrm{H}^2(X, T_{X/A} \otimes J)$.

See also [Har10, Cor. 10.3]. $\square$

**Exercise D.2.8** (Interpretation of deformations and obstruction using gerbes)**.** With the hypotheses of Proposition D.2.6, consider the category $\mathcal{G}$ over $\mathrm{Sch}/X$ whose objects over $S \to X$ are cartesian diagrams

$$\begin{tikzcd}
S \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & S' \arrow[d] \\
\operatorname{Spec} A \arrow[r, hook] & \operatorname{Spec} A'
\end{tikzcd}$$

where $S \to \operatorname{Spec} A$ is the composition $S \to X \to \operatorname{Spec} A$. A morphism $(S \to X, S \hookrightarrow S' \to \operatorname{Spec} A') \to (T \to X, T \hookrightarrow T' \to \operatorname{Spec} A')$ is the data of a morphism $\phi \colon S' \to T'$ over $A'$ such that $\phi$ restricts to a morphism $S \to T$ over $X$.

(a) Show that $\mathcal{G}$ is a gerbe banded by the sheaf of groups $T_{X/A} \otimes_A J$ on $X$. (*Hint: Use Lemma D.1.7 to show it is a prestack. See Definition 6.2.21 for the definition of a banded gerbe.*)

(b) Give an alternate proof of Proposition D.2.6. (*Hint:* For part (3), use Exercise 6.2.38.)

**Exercise D.2.9** (Deformations of principal $G$-bundles)**.** Let $G$ be a smooth affine algebraic group over a field $\Bbbk$ with Lie algebra $\mathfrak{g}$. Let $X \hookrightarrow X'$ be a closed immersion of finite type $\Bbbk$-schemes defined by a square-zero sheaf of ideals $J$ and assume that $X$ has affine diagonal. If $P \to X$ is a principal $G$-bundle, one can define deformations over $X'$ and automorphisms of deformations analogous to the case of smooth morphisms. Show that

(1) The group of automorphisms of a deformation $P' \to X'$ of $P \to X$ is bijective to $\mathrm{H}^0(X, \mathfrak{g} \otimes J)$.

(2) If there exists a deformation over $X'$, then the set of isomorphism classes of all such deformations is a torsor under $\mathrm{H}^1(X, \mathfrak{g} \otimes J)$.

(3) There is an element $\mathrm{ob}_X \in \mathrm{H}^2(X, \mathfrak{g} \otimes J)$ with the property that there exists a deformation over $X'$ if and only if $\mathrm{ob}_X = 0$.

**Example D.2.10** (Abelian varieties)**.** If $X_0$ is an abelian variety over $\mathbb{C}$ of dimension $n$, then it turns out that deforming $X_0$ as an abstract scheme is equivalent to deforming it as an abelian variety, and that obstructions to deforming $X_0$ as an abelian variety also live in $\mathrm{H}^2(X_0, T_{X_0})$. Using that $\Omega_{X_0} = \mathcal{O}_{X_0}^n$ is trivial and the Hodge symmetries, we see that $\mathrm{H}^2(X_0, T_{X_0}) = \mathrm{H}^2(X_0, \mathcal{O}_{X_0})^{\oplus n} = \mathrm{H}^0(X_0, \bigwedge^2 \mathcal{O}_{X_0}^n)^{\oplus n}$ is nonzero. Nevertheless, Grothendieck and Mumford showed that given every deformation problem as in (D.2.1), the obstruction $\mathrm{ob}_X \in \mathrm{H}^2(X, T_{X/A} \otimes_A J)$ vanishes! This shows that abelian varieties are unobstructed, and their moduli space is formally smooth. See [Oor71].

**Proposition D.2.11** (Higher-order deformations of Complete Intersections)**.** *Let $X_0$ be a scheme of finite type over a field $\Bbbk$ such that $X_0$ is generically smooth and a local complete intersection. Let $A' \twoheadrightarrow A$ be a surjection of local noetherian rings with residue field $\Bbbk$. Assume that the kernel $J := \ker(A' \to A)$ satisfies $\mathfrak{m}_{A'} J = 0$. If $X \to \operatorname{Spec} A$ is a flat morphism of schemes with central fiber $X \times_A \Bbbk \cong X_0$, then*

*(1) The group of automorphisms of a deformation $X' \to \operatorname{Spec} A'$ of $X \to \operatorname{Spec} A$ over $A'$ is bijective to $\mathrm{Ext}^0_{\mathcal{O}_{X_0}}(\Omega_{X_0}, J)$.*

*(2) If there exists a deformation of $X \to \operatorname{Spec} A$ over $A'$, then the set of isomorphism classes of all such deformations is a torsor under $\mathrm{Ext}^1_{\mathcal{O}_{X_0}}(\Omega_{X_0}, J)$.*

*(3) There is an element $\mathrm{ob}_X \in \mathrm{Ext}^2_{\mathcal{O}_{X_0}}(\Omega_{X_0}, J)$ with the property that there exists a deformation of $X \to \operatorname{Spec} A$ over $A'$ if and only if $\mathrm{ob}_X = 0$.*

*Proof.* See [Vis97, Thm. 4.4] for an explicit argument. Alternatively, since $X_0$ is generically smooth and a local complete intersection, the cotangent complex $L_{X_0}$ is quasi-isomorphic to $\Omega_{X_0}$ (see Example D.5.11) and thus the result follows from the fact that the cotangent complex controls deformations (Theorem D.5.10). □

**Exercise D.2.12.**

(1) Show that under the bijection $\mathrm{Def}(\Bbbk[\![x, y]\!]/(xy)) \cong \Bbbk$ an element $t \in \Bbbk$ corresponds to the first-order deformation $\operatorname{Spec} \Bbbk[\![x, y, \epsilon]\!]/(xy - t\epsilon)$.

(2) Classify first-order deformations of the $A_k$-singularity $\Bbbk[\![x, y]\!]/(y^2 - x^{k+1})$.

**Exercise D.2.13** (Higher-order deformations of Morphisms)**.** Let $\Bbbk$ be a field and $A' \twoheadrightarrow A$ be a surjection of local artinian rings with residue field $\Bbbk$. Let $f\colon X \to Y$ be a morphism of schemes over $A$. A *deformation of $f\colon X \to Y$ over $A'$* is a morphism $f'\colon X' \to Y'$ of schemes over $\operatorname{Spec} A'$ together with isomorphisms $\alpha'\colon X \xrightarrow{\sim} X' \times_{A'} A$ and $\beta'\colon Y \xrightarrow{\sim} Y' \times_{A'} A$ such that both $X'$ and $Y'$ are flat over $A'$ and such that the base change of $f'$ to $A$ is equal to $f$ under the isomorphisms $\alpha$ and $\beta$. In other words, a deformation is a cartesian diagram

$$
\begin{tikzcd}
X \arrow[r, hook] \arrow[d, "f"] \arrow[dr, phantom, "\square"] & X' \arrow[d, "f'"] \\
Y \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & Y' \arrow[d] \\
\operatorname{Spec} A \arrow[r, hook] & \operatorname{Spec} A'.
\end{tikzcd}
$$

A *morphism* of deformations $(X' \to Y', \alpha', \beta') \to (X'' \to Y'', \alpha'', \beta'')$ consists of morphisms $X' \to X''$ and $Y' \to Y''$ over $A'$ compatible with the given isomorphisms.

Assume that $X$ and $Y$ are proper $A$, and that $f_*\mathcal{O}_X = \mathcal{O}_Y$ and $\mathrm{R}^1 f_*\mathcal{O}_X = 0$. Show that the functor taking a deformation $f' : X' \to Y'$ of $f : X \to Y$ over $A'$ to the deformation $X'$ over $X$ over $A'$ induces an isomorphism of categories.

*Hint: Given a deformation $X'$ over $X$, define $Y'$ as the ringed space $(Y, f_*\mathcal{O}_{X'})$ (using that $X$ and $X'$ have the same topological space). Use the conditions of $f$ and the flatness of $X'$ over $A'$ to show that $Y'$ is a scheme flat over $A'$. See also [Vak06, §5.3], [Ran89, Thm. 3.3], and [SP, Tag 0E3X]. (For additional properties of deformations of morphisms, see [Ser06, §3.4].)*

### D.2.3 Higher-order deformations of vector bundles

**Definition D.2.14.** Let $A' \twoheadrightarrow A$ be a surjection of noetherian rings with square-zero kernel $J$. Let $X' \to \operatorname{Spec} A'$ be finite type morphism of schemes and set $X := X' \times_{A'} A$. Given a coherent sheaf $E$ on $X$ flat over $A$, a *deformation of $E$ over $A' \to A$* is a pair $(E', \alpha)$ where $E'$ is a coherent sheaf on $X'$ flat over $A'$ and $\alpha : E \to E'|_X$ is an isomorphism. Pictorially, we have

$$
\begin{tikzcd}
E \arrow[d, dash, "\text{flat}/A"] & E' \arrow[d, dash, dashed, "\text{flat}/A'"] \\
X \arrow[r, hook] & X'.
\end{tikzcd}
$$

A *morphism* $(E, \alpha) \to (E', \alpha')$ *of deformations* is a morphism $\beta : E \to E'$ of coherent sheaves on $X_{A'}$ such that $\alpha' = \beta|_X \circ \alpha$. By Lemma D.1.7, every morphism of deformations is an isomorphism.

**Proposition D.2.15.** *Let $A' \twoheadrightarrow A$ be a surjection of noetherian rings with square-zero kernel $J$. Let $X' \to \operatorname{Spec} A'$ be a flat and finite type morphism of schemes such that $X'$ has affine diagonal (e.g. separated) and set $X := X' \times_{A'} A$. Let $E$ be a vector bundle on $X$ over $A$.*

1. *(1) The group of automorphisms of a deformation $E'$ of $E$ over $A'$ is bijective to $\mathrm{H}^0(X, \mathscr{E}nd_{\mathcal{O}_X}(E) \otimes_A J)$.*
2. *(2) If there exists a deformation of $E$ over $A'$, then the set of isomorphism classes of all such deformations is a torsor under $\mathrm{H}^1(X, \mathscr{E}nd_{\mathcal{O}_X}(E) \otimes_A J)$.*
3. *(3) There is an element $\mathrm{ob}_E \in \mathrm{H}^2(X, \mathscr{E}nd_{\mathcal{O}_X}(E) \otimes_A J)$ with the property that there exists a deformation of $E$ over $A'$ if and only if $\mathrm{ob}_E = 0$.*

**Remark D.2.16.** If $X$ and $X'$ are base changes of a finite type $\Bbbk$-scheme $X_0$ with affine diagonal, and $A$ and $A'$ are local artinian rings with residue field $\Bbbk$ such that

$\mathfrak{m}_{A'}J = 0$, then automorphisms, deformations and obstructions are classified by $\mathrm{H}^2(X_0, \mathscr{E}nd_{\mathcal{O}_{X_0}}(E_0) \otimes_{\Bbbk} J)$ for $i = 0, 1, 2$ where $E_0 = E|_{X_0}$.

*Proof.* See [Har10, Thm. 7.1]. □

**Exercise D.2.17.** Give an alternative proof of Proposition D.2.15 using the technique outlined in Exercise D.2.8.

## D.3 Versal formal deformations and Rim–Schlessinger's Criteria

### D.3.1 Functors of artin rings

For an algebraically closed field $\Bbbk$, let $\mathrm{Art}_{\Bbbk}$ denote the category of artinian local $\Bbbk$-algebras with residue field $\Bbbk$. The opposite category $\mathrm{Art}_{\Bbbk}^{\mathrm{op}}$ is equivalent to the category of local artinian $\Bbbk$-schemes $(S, s)$ with $\kappa(s) = \Bbbk$.

**Definition D.3.1.** We say that a covariant functor $F\colon \mathrm{Art}_{\Bbbk} \to \mathrm{Sets}$ is *pro-representable* if there exists a noetherian complete local $\Bbbk$-algebra $R$ such that for all $A \in \mathrm{Art}_{\Bbbk}$, there is a isomorphism $F \xrightarrow{\sim} h_R$ where $h_R := \mathrm{Hom}_{\Bbbk-\mathrm{alg}}(R, -)$.

**Remark D.3.2.** If $F\colon \mathrm{Sch}/\Bbbk \to \mathrm{Sets}$ is a contravariant functor and $x_0 \in F(\Bbbk)$, then we can consider the induced functor of artin rings

$$F_{x_0}\colon \mathrm{Art}_{\Bbbk} \to \mathrm{Sets}, \quad A \mapsto \{x \in F(A) \mid x|_{\Bbbk} = x_0 \in F(\Bbbk)\}$$

where $x|_{\Bbbk}$ denotes the image of $x$ under $F(A) \to F(A/\mathfrak{m}_A)$. If $F$ is representable by a scheme $X$ and $x \in X$ is the $\Bbbk$-point corresponding to $x_0$, then $F_{x_0}$ is pro-representable by $\widehat{\mathcal{O}}_{X,x}$.

**Exercise D.3.3.** Provide an example of a non-representable contravariant functor $F\colon \mathrm{Sch}/\Bbbk \to \mathrm{Sets}$ and an object $x_0 \in F(\Bbbk)$ such that $F_{x_0}$ is pro-representable.

Many functors of artin rings are not pro-representable. For example, if $C_0$ is a smooth, connected, and projective curve with a non-trivial automorphism group, then the covariant functor $F_{C_0}\colon \mathrm{Art}_{\Bbbk} \to \mathrm{Sets}$ where $F_{C_0}(A)$ consists of isomorphism classes of smooth proper families of curves $\mathcal{C} \to \operatorname{Spec} A$ such that $\mathcal{C} \times_A A/\mathfrak{m}_A$ is isomorphic to $C_0$, is not pro-representable. Nevertheless, many moduli functors admit *versal* deformations.

**Remark D.3.4.** To work over a more general base (e.g. of mixed characteristic), one can consider instead the following setup: let $\Lambda$ be a noetherian complete local ring with residual field $\Bbbk$ (not necessarily algebraically closed) and $\mathrm{Art}_{\Lambda}$ be the category of artinian local $\Lambda$-algebras $(A, \mathfrak{m})$ with an identification $\Bbbk \xrightarrow{\sim} A/\mathfrak{m}$. Rim–Schlessinger's Criteria (Theorem D.3.11) holds after replacing $\mathrm{Art}_{\Bbbk}$ with $\mathrm{Art}_{\Lambda}$. More generally, one can consider the setup where $A \to \Bbbk$ is a finite morphism to a field, not assumed to be the residue field.

Setting $\Lambda = \Bbbk$ recovers our setup, but in many applications, it is often useful to take $\Lambda$ to be a ring of Witt vectors, e.g. $\Lambda = \mathbb{Z}_p$. In this way, one can consider deforming an object $E_0$ over $\mathbb{F}_p$ inductively along extensions $\mathbb{Z}/p^{n+1} \twoheadrightarrow \mathbb{Z}/p^n$ with the hope of applying Rim–Schlessinger's Criteria (Theorem D.3.11) and Grothendieck's Existence Theorem (D.4.4) to deform $E_0$ to an object $\widehat{E}$ over the *characteristic zero* ring $\mathbb{Z}_p$; see Section D.4.1

### D.3.2 Versal deformations

As it's important to keep track of automorphisms, we will present Rim–Schlessinger's Criteria, a generalization of Schlessinger's Criterion from functors to prestacks. Therefore we will formulate the definition of versality for prestacks $\mathcal{X}$ over $\mathrm{Art}_{\Bbbk}^{\mathrm{op}}$. We will assume that $\mathcal{X}(\Bbbk)$ is equivalent to a set consisting of a single object, i.e. there is a unique morphism between any two objects in $\mathcal{X}(\Bbbk)$.

**Definition D.3.5.** Let $\mathcal{X}$ be a prestack over $\mathrm{Art}_{\Bbbk}^{\mathrm{op}}$ such that the groupoid $\mathcal{X}(\Bbbk)$ is equivalent to the set $\{x_0\}$.

(1) A *formal deformation* $(R, \{x_n\})$ *of* $x_0$ is the data of a noetherian complete local $\Bbbk$-algebra $(R, \mathfrak{m}_R)$ together with objects $x_n \in \mathcal{X}(R/\mathfrak{m}_R^{n+1})$ and morphisms $x_{n-1} \to x_n$ over $\operatorname{Spec} R/\mathfrak{m}_R^n \to \operatorname{Spec} R/\mathfrak{m}_R^{n+1}$, or in other words an element of $\varprojlim \mathcal{X}(R/\mathfrak{m}^n)$. When $\mathcal{X} = F$ is a covariant functor $\mathrm{Art}_{\Bbbk} \to \mathrm{Sets}$, a formal deformation is a compatible sequence of elements $x_n \in F(R/\mathfrak{m}_R^{n+1})$.

(2) A formal deformation $(R, \{x_n\})$ is *versal* if for every surjection $A \twoheadrightarrow A_0$ in $\mathrm{Art}_{\Bbbk}$ with $\mathfrak{m}_A^{n+1} = 0$, object $\eta \in \mathcal{X}(A)$ and $\Bbbk$-algebra homomorphism $\phi_0 \colon R/\mathfrak{m}_R^{n+1} \to A_0$ with an isomorphism $\alpha_0 \colon x_n|_{A_0} \xrightarrow{\sim} \eta|_{A_0}$ in $\mathcal{X}(A_0)$, there exists a $\Bbbk$-algebra homomorphism $\phi \colon R/\mathfrak{m}_R^{n+1} \to A$ and an isomorphism $\alpha \colon x_n|_A \xrightarrow{\sim} \eta$ in $\mathcal{X}(A)$ such that $\phi_0$ is the composition $R/\mathfrak{m}_R^{n+1} \xrightarrow{\phi} A \twoheadrightarrow A_0$ $\alpha|_{A_0} = \alpha_0$.

(3) A versal formal deformation $(R, \{x_n\})$ is *miniversal* (or *a pro-representable hull*) if the induced map $\mathrm{Hom}_{\Bbbk-\mathrm{alg}}(R, \Bbbk[\epsilon]) \to \mathcal{X}(\Bbbk[\epsilon])/\sim$ on isomorphism classes, defined by $(R \to R/\mathfrak{m}_R^2 \xrightarrow{\phi} \Bbbk[\epsilon]) \mapsto \phi(x_1)$, is bijective.

**Remark D.3.6.** The deformation $x_n \in \mathcal{X}(R/\mathfrak{m}_R^{n+1})$ can be viewed via Yoneda's 2-Lemma as a morphism $\operatorname{Spec} R/\mathfrak{m}_R^{n+1} \to \mathcal{X}$ or more precisely as $h_{R/\mathfrak{m}_R^{n+1}} \to \mathcal{X}$. Likewise, we can view a formal deformation as a morphism $\{x_n\} \colon h_R \to \mathcal{X}$ where $h_R = \mathrm{Hom}_{\Bbbk-\mathrm{alg}}(R, -)$ (see Exercise D.3.8). With this terminology, $\{x_n\}$ is versal if there exists a lift for every commutative diagram

$$\begin{tikzcd} \operatorname{Spec} A_0 \arrow[r] \arrow[d, hook] & h_R \arrow[d, "\{x_n\}"] \\ \operatorname{Spec} A \arrow[r, "\eta"] \arrow[ur, dashed] & \mathcal{X} \end{tikzcd} \tag{D.3.1}$$

of solid arrows where $A \twoheadrightarrow A_0$ is a surjection in $\mathrm{Art}_{\Bbbk}$. In this way, we see that a versal formal deformation corresponds to the Infinitesimal Lifting Criterion for Smoothness (see Smooth Equivalences A.3.1(2) and Theorem 3.7.1) with respect to artinian local $\Bbbk$-algebras. Meanwhile, a miniversal deformation is a versal formal deformation inducing an isomorphism on tangent spaces $h_R(\Bbbk[\epsilon]) \to \mathcal{X}(\Bbbk[\epsilon])/\sim$.

**Remark D.3.7.** The condition of versality can be checked on surjections $A \twoheadrightarrow A_0$ with $\ker(A \to A_0) \cong \Bbbk$. Indeed, the kernel of every surjection $A \twoheadrightarrow A_0$ in $\mathrm{Art}_{\Bbbk}$ is a finite dimensional $\Bbbk$-vector space, and $A \to A_0$ can be factored into surjections where each kernel is one-dimensional.

**Exercise D.3.8.** Let $R$ be a noetherian complete local $\Bbbk$-algebra and let $h_R = \mathrm{Hom}_{\Bbbk-\mathrm{alg}}(R, -)$ be the covariant functor $\mathrm{Art}_{\Bbbk} \to \mathrm{Sets}$ which we can also view as a prestack over $\mathrm{Art}_{\Bbbk}^{\mathrm{op}}$. If $\mathcal{X}$ is a prestack over $\mathrm{Art}_{\Bbbk}^{\mathrm{op}}$, show that giving a formal deformation $(R, \{x_n\})$ is equivalent to giving a morphism $h_R \to \mathcal{X}$ of prestacks.

**Remark D.3.9.** If $F$ is pro-representable by $R$, then letting $x_n \in F(R/\mathfrak{m}_R^{n+1})$ correspond to the surjection $(R \twoheadrightarrow R/\mathfrak{m}_R^{n+1}) \in h_R(R/\mathfrak{m}_R^{n+1})$, it is easy to see that $\{x_n\}$ is a versal formal deformation. In this case, there is a unique lift in (D.3.1)

**Remark D.3.10** (Global prestacks to local deformation prestacks)**.** If $\mathcal{X}$ is a prestack over $\mathrm{Sch}/\Bbbk$ and $x_0 \in \mathcal{X}(\Bbbk)$, we can consider the *local deformation prestack* $\mathcal{X}_{x_0}$ *at* $x_0$ as the prestack of morphisms $x_0 \to x$ over $\mathrm{Art}_{\Bbbk}^{\mathrm{op}}$ where a morphism $(x_0 \xrightarrow{\alpha} x) \to (x_0 \xrightarrow{\alpha'} x')$ is a morphism $\beta\colon x \to x'$ such that $\alpha' = \alpha \circ \beta$. In other words, an object of $\mathcal{X}_{x_0}$ is a pair $(x, \alpha)$ where $x \in \mathcal{X}(A)$ and $\alpha\colon x_0 \to x|_{\Bbbk}$ is an isomorphism. Note that the fiber category $\mathcal{X}_{x_0}(\Bbbk)$ is equivalent to the set $\{x_0 \xrightarrow{\mathrm{id}} x_0\}$.

If $\mathcal{X}$ is algebraic with a smooth presentation $U \to \mathcal{X}$ from a scheme and $u \in U(\Bbbk)$ is a point mapping to $x_0$, then we may set $x_n \in \mathcal{X}(\mathcal{O}_{U,u}/\mathfrak{m}_u^{n+1})$ to be the composition $\operatorname{Spec} \mathcal{O}_{U,u}/\mathfrak{m}_u^{n+1} \hookrightarrow U \to \mathcal{X}$. Then $\{x_n\}$ is a versal formal deformation.

On the other hand, if $\mathcal{X}$ is not yet known to be algebraic, one can sometimes verify the existence of versal formal deformation via Rim–Schlessinger's Criteria (Theorem D.3.11) as a first step to verifying the algebraicity of $\mathcal{X}$ via Artin's Axioms for Algebraicity (Theorem D.7.1).

### D.3.3 Rim–Schlessinger's Criteria

Rim–Schlessinger's Criteria provides necessary and sufficient conditions for a prestack $\mathcal{X}$ over $\mathrm{Art}_{\Bbbk}^{\mathrm{op}}$ or covariant functor $F\colon \mathrm{Art}_{\Bbbk} \to \mathrm{Sets}$ to admit a versal formal deformation.

**Theorem D.3.11** (Rim–Schlessinger's Criteria)**.** *Let $\mathcal{X}$ be a prestack over $\mathrm{Art}_{\Bbbk}^{\mathrm{op}}$ such that the groupoid $\mathcal{X}(\Bbbk)$ is equivalent to the set $\{x_0\}$. For morphisms $B_0 \to A_0$ and $A \to A_0$ in $\mathrm{Art}_{\Bbbk}$, consider the natural functor*

$$\mathcal{X}(B_0 \times_{A_0} A) \to \mathcal{X}(B_0) \times_{\mathcal{X}(A_0)} \mathcal{X}(A) \tag{D.3.2}$$

*Then $\mathcal{X}$ admits a miniversal formal deformation if and only if*

(RS$_1$) *the functor (D.3.2) is essentially surjective whenever $A \twoheadrightarrow A_0$ is surjection with kernel $\Bbbk$;*

(RS$_2$) *the map (D.3.2) is essentially surjective when $A_0 = \Bbbk$ and $A = \Bbbk[\epsilon]$, and given two commutative diagrams*

$$\begin{tikzcd} x_0 \arrow[r] \arrow[d] & y_0 \arrow[d, "\alpha_1"'] \arrow[dr, "\alpha_2"] & \\ x \arrow[r] \arrow[rr, bend right] & y \arrow[r, dashed, "\beta"] & y' \end{tikzcd} \qquad \textit{over} \qquad \begin{tikzcd} \operatorname{Spec} \Bbbk \arrow[r, hook] \arrow[d] & \operatorname{Spec} B_0 \arrow[d] \\ \operatorname{Spec} \Bbbk[\epsilon] \arrow[r, hook] & \operatorname{Spec}(\Bbbk[\epsilon] \times_{\Bbbk} B_0) \end{tikzcd}$$

*there exists an isomorphism $\beta\colon y \to y'$ in $\mathcal{X}(\Bbbk[\epsilon] \times_{\Bbbk} B_0)$ such that $\alpha_2 = \beta \circ \alpha_1$.*

(RS$_3$) $\dim_{\Bbbk} T_{\mathcal{X}} < \infty$ *where $T_{\mathcal{X}} := \mathcal{X}(\Bbbk[\epsilon])/\!\sim$.*

*Moreover, $\mathcal{X}$ is prorepresentable if and only if $\mathcal{X}$ is equivalent to a functor and*

(RS$_4$) *the map (D.3.2) is an equivalence whenever $A \twoheadrightarrow A_0$ is a surjection with kernel $\Bbbk$.*

Conditions (RS$_2$)–(RS$_3$) (sometimes referred to as *semi-homogeneity*) may be difficult to parse[1] but in practice, it is almost always just as easy to verify the stronger

[1]The second part of (RS$_2$) is slightly stronger than requiring that two objects in $\mathcal{X}(\Bbbk[\epsilon] \times_{\Bbbk} B_0)$ are isomorphic if and only if their images are. Note that (RS$_2$) does not require the isomorphism $\beta\colon y \to y'$ to be compatible with the given morphisms $x \to y$ and $x \to y'$.

condition (RS$_4$) (called *homogeneity*), and in fact the even stronger condition (RS$_4^*$) (called *strong homogeneity*) introduced in §D.3.4.

**Remark D.3.12** (Schlessinger's Criteria)**.** When $\mathcal{X}$ is a covariant functor $F\colon \mathrm{Art}_{\Bbbk} \to \mathrm{Sets}$ with $F(\Bbbk) = \{x_0\}$, then (RS$_1$)–(RS$_4$) translate into Schlessinger's conditions as introduced in [Sch68]:

(H$_1$) the map (D.3.2) is surjective whenever $A \twoheadrightarrow A_0$ is a surjection with kernel $\Bbbk$;

(H$_2$) the map (D.3.2) is bijective when $A_0 = \Bbbk$ and $A = \Bbbk[\epsilon]$;

(H$_3$) $\dim_{\Bbbk} F(\Bbbk[\epsilon]) < \infty$; and

(H$_4$) the map (D.3.2) is bijective whenever $A \twoheadrightarrow A_0$ is a surjection with kernel $\Bbbk$.

The functor $F$ admits a miniversal formal deformation if (H$_1$)–(H$_3$) hold and is pro-representable if (H$_3$)–(H$_4$) hold.

If $\mathcal{X}$ satisfies (RS$_1$)–(RS$_3$), then the functor $F_{\mathcal{X}}\colon \mathrm{Sch}/\Bbbk \to \mathrm{Sets}$ parameterizing isomorphism classes of objects satisfies (H$_1$)–(H$_3$) but the converse does not always hold. On the other hand, the essential surjectivity of $\mathcal{X}(B_0 \times_{A_0} A) \to \mathcal{X}(B_0) \times_{\mathcal{X}(A_0)} \mathcal{X}(A)$ implies the surjectivity of $F_{\mathcal{X}}(B_0 \times_{A_0} A) \to F_{\mathcal{X}}(B_0) \times_{F_{\mathcal{X}}(A_0)} F_{\mathcal{X}}(A)$ and the fully faithfulness for $\mathcal{X}$ implies the injectivity of $F_{\mathcal{X}}$ as long as $\mathrm{Aut}_{\mathcal{X}(B_0)}(y_0) \to \mathrm{Aut}_{\mathcal{X}(A_0)}(y_0|_{A_0})$ is surjective for an object $y_0 \in \mathcal{X}(B_0)$. This latter condition holds in the case when $F_{\mathcal{X}}(A_0)$ is a set, e.g. when $A_0 = \Bbbk$. If $\mathcal{X}$ is the local deformation prestack arising from an object $x_0 \in \widetilde{\mathcal{X}}(\Bbbk)$ of an algebraic stack $\widetilde{\mathcal{X}}$ over $\mathrm{Sch}/\Bbbk$ as in Remark D.3.10, then the surjectivity condition on automorphisms translates to the inertia stack $I_{\mathcal{X}} \to \mathcal{X}$ being smooth at $e(x_0)$, where $e\colon \mathcal{X} \to I_{\mathcal{X}}$ is the identity section, by the Infinitesimal Lifting Criterion for Smoothness (Theorem 3.7.1).

While the existence of a miniversal formal deformation of $F_{\mathcal{X}}$ suffices for many applications, for moduli problems with automorphisms it is more natural to ask for the existence of a miniversal formal deformation of $\mathcal{X}$ and this generality is needed for some applications, e.g. Artin's Algebraization (Theorem D.6.6) and Artin's Axioms for Algebraicity (Theorem D.7.4).

Before proceeding to the proof, we first show that (RS$_1$)–(RS$_2$) yield natural structures on sets of deformations. In particular, they induce a vector space structure on the tangent space $T_{\mathcal{X}} = \mathcal{X}(\Bbbk[\epsilon])/{\sim}$, which allows us to make sense of condition (RS$_3$).

**Lemma D.3.13.** *Let $\mathcal{X}$ be a prestack over $\mathrm{Art}_{\Bbbk}^{\mathrm{op}}$ such that the groupoid $\mathcal{X}(\Bbbk)$ is equivalent to the set $\{x_0\}$, and let $F_{\mathcal{X}}\colon \mathrm{Art}_{\Bbbk} \to \mathrm{Sets}$ be the covariant functor assigning $A \in \mathrm{Art}_{\Bbbk}$ to the set of isomorphism classes $\mathcal{X}(A)/{\sim}$. Assume that Condition (RS$_2$) holds for $\mathcal{X}$.*

1. *The tangent space $T_{\mathcal{X}} = F_{\mathcal{X}}(\Bbbk[\epsilon])$ has a natural structure of a $\Bbbk$-vector space. More generally, for every finite dimensional $\Bbbk$-vector space $V$, denoting $\Bbbk[V]$ as the $\Bbbk$-algebra $\Bbbk \oplus V$ defined by $V^2 = 0$, the set $F_{\mathcal{X}}(\Bbbk[V])$ has a natural structure of a $\Bbbk$-vector space and there is a functorial bijection $F_{\mathcal{X}}(\Bbbk[V]) = T_{\mathcal{X}} \otimes_{\Bbbk} V$.*
2. *Consider a surjection $B \twoheadrightarrow A$ in $\mathrm{Art}_{\Bbbk}$ with square-zero kernel $I$ and an element $x \in \mathcal{X}(A)$, and let $\mathrm{Lift}_x(B)$ be the set of morphisms $x \to y$ over $\mathrm{Spec}\, A \to \mathrm{Spec}\, B$ where $x \xrightarrow{\alpha} y$ is declared equivalent to $x \xrightarrow{\alpha'} y'$ if there is an isomorphism $\beta\colon y \to y'$ such that $\alpha' = \beta \circ \alpha$. There is an action of $T_{\mathcal{X}} \otimes I$ on $\mathrm{Lift}_x(B)$ which is functorial in $\mathcal{X}$. Assuming $\mathrm{Lift}_x(B)$ is non-empty, this action is transitive if Condition (RS$_1$) holds for $\mathcal{X}$ and free and transitive (i.e. $\mathrm{Lift}_x(B)$ is a torsor under $T_{\mathcal{X}} \otimes I$) if Condition (RS$_4$) holds for $\mathcal{X}$.*

*Proof.* We first note if $V$ is a finite dimensional vector space, then $\Bbbk[V] = \Bbbk[\epsilon] \times_{\Bbbk} \cdots \times_{\Bbbk} \Bbbk[\epsilon]$ and by applying (RS$_2$) inductively, we see that the statement of Condition (RS$_2$) also holds for $A_0 = \Bbbk$ and $A = \Bbbk[V]$. For $B_0 \in \mathrm{Art}_{\Bbbk}$, the first part of (RS$_2$) implies that $F_{\mathcal{X}}(B_0 \times_{\Bbbk} \Bbbk[V]) \xrightarrow{\sim} F_{\mathcal{X}}(B_0) \times F_{\mathcal{X}}(\Bbbk[V])$ is a bijection. In particular, $F_{\mathcal{X}}(\Bbbk[V] \times_{\Bbbk} \Bbbk[W]) \xrightarrow{\sim} F_{\mathcal{X}}(\Bbbk[V]) \times F_{\mathcal{X}}(\Bbbk[W])$ is bijective for every pair of finite dimensional vector spaces, or in other words the functor $V \mapsto F_{\mathcal{X}}(\Bbbk[V])$ commutes with finite products.

The vector space structure of $T_{\mathcal{X}} = F_{\mathcal{X}}(\Bbbk[\epsilon])$ follows from the bijectivity of

$$F_{\mathcal{X}}(\Bbbk[\epsilon] \times_{\Bbbk} \Bbbk[\epsilon']) \xrightarrow{\sim} F_{\mathcal{X}}(\Bbbk[\epsilon]) \times F_{\mathcal{X}}(\Bbbk[\epsilon']). \tag{D.3.3}$$

Indeed, addition of $\tau_1, \tau_2 \in F_{\mathcal{X}}(\Bbbk[\epsilon])$ is defined by using the above identification to view $(\tau_1, \tau_2) \in F_{\mathcal{X}}(\Bbbk[\epsilon] \times_{\Bbbk} \Bbbk[\epsilon'])$ and then taking its image under $F(\Bbbk[\epsilon] \times_{\Bbbk} \Bbbk[\epsilon']) \to F(\Bbbk[\epsilon])$ induced by the ring map $\Bbbk[\epsilon] \times_{\Bbbk} \Bbbk[\epsilon'] \to \Bbbk[\epsilon]$ taking $(\epsilon, 0)$ and $(0, \epsilon')$ to $\epsilon$. Scalar multiplication of $c \in \Bbbk$ on $\tau \in F_{\mathcal{X}}(\Bbbk[\epsilon])$ is defined by taking the image of $\tau$ under $F_{\mathcal{X}}(\Bbbk[\epsilon]) \to F_{\mathcal{X}}(\Bbbk[\epsilon])$ induced by the map $\Bbbk[\epsilon] \to \Bbbk[\epsilon]$ taking $\epsilon$ to $c\epsilon$.

The same argument gives $F_{\mathcal{X}}(\Bbbk[V])$ the structure of a vector space such that the assignment $V \mapsto F_{\mathcal{X}}(\Bbbk[V])$ is a $\Bbbk$-linear functor $\mathrm{Vect}_{\Bbbk}^{\mathrm{fd}} \to \mathrm{Vect}_{\Bbbk}$ defined on finitely dimensional $\Bbbk$-vector spaces. The natural map

$$F_{\mathcal{X}}(\Bbbk[\epsilon]) \times \mathrm{Hom}_{\Bbbk}(\Bbbk[\epsilon], \Bbbk[V]) \to F_{\mathcal{X}}(\Bbbk[V]), \qquad (\tau, \phi) \mapsto \phi^* \tau$$

is $\Bbbk$-bilinear and under the equivalences $T_{\mathcal{X}} = F_{\mathcal{X}}(\Bbbk[\epsilon])$ and $V = \mathrm{Hom}_{\Bbbk}(\Bbbk[\epsilon], \Bbbk[V])$ corresponds to a linear map $T_{\mathcal{X}} \otimes V \to F_{\mathcal{X}}(\Bbbk[V])$, which is an isomorphism. This finishes the proof of (1).

For (2), we observe that the natural map

$$B \times_A B \to \Bbbk[I] \times_{\Bbbk} B, \qquad (b_1, b_2) \mapsto (\overline{b_1} + b_2 - b_1, b_1)$$

is an isomorphism. We therefore have a diagram

$$\mathcal{X}(\Bbbk[I]) \times \mathcal{X}(B) \leftarrow \mathcal{X}(\Bbbk[I] \times_{\Bbbk} B) \cong \mathcal{X}(B \times_A B) \xrightarrow{p_1^*} \mathcal{X}(B) \tag{D.3.4}$$

where the left functor is essentially surjective by the first part of (RS$_2$). Given $\tau \in T_{\mathcal{X}} \otimes I = F_{\mathcal{X}}(\Bbbk[I])$ (with a choice of representative in $\mathcal{X}(\Bbbk[I])$) and $(x \xrightarrow{\alpha} y) \in \mathrm{Lift}_x(B)$, we would like to define $\tau \cdot (x \xrightarrow{\alpha} y)$ as the image under $p_1^*$ of a choice of preimage of $(\tau, y)$. To see that this is well-defined, consider two elements $z, z' \in \mathcal{X}(\Bbbk[I] \times_{\Bbbk} B)$ whose images in $\mathcal{X}(\Bbbk[I]) \times \mathcal{X}(B)$ are isomorphic to $(\tau, y)$. This yields a diagram

$$\begin{tikzcd} x_0 \arrow[r] \arrow[d] & y \arrow[d, "\alpha_1"'] \arrow[dr, "\alpha_2"] & \\ \tau \arrow[r] \arrow[rr, bend right] & z \arrow[r, dashed, "\beta"] & z' \end{tikzcd} \quad \text{over} \quad \begin{tikzcd} \operatorname{Spec} \Bbbk \arrow[r, hook] \arrow[d] & \operatorname{Spec} B \arrow[d] \\ \operatorname{Spec} \Bbbk[I] \arrow[r, hook] & \operatorname{Spec}(\Bbbk[I] \times_{\Bbbk} B) \end{tikzcd}$$

and by the second part of (RS$_2$), there exists a dotted arrow $\beta$ such that $\alpha_2 = \beta \circ \alpha_1$. Therefore choices of pullbacks $p_1^* z$ and $p_1^* z'$ in $\mathcal{X}(B)$ defines the same element in $\mathrm{Lift}_x(B)$. If (RS$_1$) holds, then the statement of Condition (RS$_1$) also holds for every surjection $A \mapsto A_0$ (since we may factor it as a composition of surjections whose kernels are $\Bbbk$). Therefore if (RS$_1$) holds (resp. (RS$_4$) holds), then $\mathcal{X}(B \times_A B) \to \mathcal{X}(B) \times_{\mathcal{X}(A)} \mathcal{X}(B)$ is essentially surjective (resp. an equivalence), and we see that the action is transitive (resp. free and transitive). $\square$

*Proof Theorem D.3.11.* The details of the necessity of these conditions are left to the reader. We will establish the sufficiency. The tangent space $T_{\mathcal{X}} := \mathcal{X}(\Bbbk[\epsilon])/\!\sim$ has the structure of a vector space by Lemma D.3.13(1) and is finite dimensional by (RS$_2$). Let $N = \dim_{\Bbbk} T_{\mathcal{X}}$ with basis $x_1, \ldots, x_N$ and define $S = \Bbbk[\![x_1, \ldots, x_N]\!]$. We will construct inductively a decreasing sequence of ideals $J_0 \supset J_1 \supset \cdots$ and objects $\eta_n \in \mathcal{X}(S/J_n)$ together with morphisms $\eta_n \to \eta_{n+1}$ over $\operatorname{Spec} S/J_n \hookrightarrow \operatorname{Spec} S/J_{n+1}$. We set $J_0 = \mathfrak{m}_S$ and $\eta_0 = x_0 \in \mathcal{X}(\Bbbk)$. We also set $J_1 = \mathfrak{m}_S^2$ so that $S/J_1 \cong \Bbbk[T_{\mathcal{X}}]$. Using the bijection $F_{\mathcal{X}}(\Bbbk[T_{\mathcal{X}}]) \cong T_{\mathcal{X}} \otimes_{\Bbbk} T_{\mathcal{X}}$ of Lemma D.3.13(1), the element $\sum_i x_i \otimes x_i$ defines an isomorphism class of an object $\eta_1 \in \mathcal{X}(S/J_1)$ such that the induced map $\operatorname{Spec} S/J_1 \to \mathcal{X}$ induces a bijection on tangent spaces. By construction, we have a morphism $\eta_0 \to \eta_1$ over $\operatorname{Spec} \Bbbk \hookrightarrow \operatorname{Spec} S/J_1$.

Suppose we've constructed $J_n$ and $\eta_{n-1} \to \eta_n$. We claim that the set of ideals

$$\begin{aligned} \Sigma = \{J \subset S \mid \mathfrak{m}_S J_n \subset J \subset J_n \text{ and there exists } \eta_n \to \eta \\ \text{over } \operatorname{Spec} S/J_n \hookrightarrow \operatorname{Spec} S/J\} \end{aligned} \tag{D.3.5}$$

has a minimal element. Indeed, it is non-empty since $J_n \in \Sigma$ and given $J, K \in \Sigma$, we must check that $J \cap K \in \Sigma$. To achieve this, choose an ideal $J' \subset S$ satisfying $J \subset J' \subset I$ with $J \cap K = J' \cap K$ and $J' + K = I$. Then $A/(J' \cap K) \cong A/J' \times_{A/I} A/K$. Letting $\eta_J \in \mathcal{X}(S/J)$ and $\eta_K \in \mathcal{X}(S/K)$ be the objects corresponding to $J$ and $K$, the data of $(\eta_J|_{S/J'} \leftarrow \eta_n \to \eta_K)$ defines an object of $\mathcal{X}(S/J') \times_{\mathcal{X}(S/I)} \mathcal{X}(S/K)$. The functor $\mathcal{X}(A/(J \cap K)) \to \mathcal{X}(S/J') \times_{\mathcal{X}(S/I)} \mathcal{X}(S/K)$ is essentially surjective by (RS$_1$) and the existence of preimage of $(\eta_J|_{S/J'} \leftarrow \eta_n \to \eta_K)$ shows that $J \cap K \in \Sigma$.

Setting $J = \bigcap_n J_n$, then $R = S/J$ is a noetherian complete local $\Bbbk$-algebra with ideals $I_n := J_n/J$. Since $\mathfrak{m}_S J_n \subset J_{n+1}$, we have that $\mathfrak{m}_R^{n+1} \subset I_n$ and thus $\xi_n := \eta_n|_{R/\mathfrak{m}_R^{n+1}}$ defines a formal deformation of $x_0$ over $R$.

We must check that $\xi := \{\xi_n\}$ is versal. Suppose $B \twoheadrightarrow A$ is a surjection in $\mathrm{Art}_{\Bbbk}$ with kernel $\Bbbk$ and that we have a diagram

$$\begin{tikzcd} x \arrow[r] \arrow[d] & \xi \\ y & \end{tikzcd} \quad \text{over} \quad \begin{tikzcd} \operatorname{Spec} A \arrow[r, "g"] \arrow[d, hook] & h_R \\ \operatorname{Spec} B. \arrow[ur, dashed, "\widetilde{g}"'] & \end{tikzcd}$$

We must construct a morphism $y \to \xi$ extending $x \to \xi$. We claim that it suffices to construct a morphism $\widetilde{g}\colon \operatorname{Spec} B \to h_R$ (i.e. a ring map $R \to B$) extending $g$. Since $h_R(\Bbbk[\epsilon]) \to T_{\mathcal{X}}$ is bijective, Lemma D.3.13(2) implies that there are actions of $T_{\mathcal{X}}$ on the sets $\operatorname{Lift}_x(B)$ and $\operatorname{Lift}_g(B)$ of isomorphism classes of lifts of $x$ and $g$ to objects in $\mathcal{X}(B)$ and $h_R(B)$ which are compatible with the map $\operatorname{Lift}_g(B) \to \operatorname{Lift}_x(B)$ where $\widetilde{g} \mapsto \widetilde{g}^*\xi$. Thus, we can find $\tau \in T_{\mathcal{X}}$ such that $y = \tau \cdot (\widetilde{g}^*\xi) = (\tau \cdot \widetilde{g})^*\xi$. This gives an arrow $y \to \xi$ over $\tau \cdot \widetilde{g}\colon \operatorname{Spec} B \to h_R$.

To construct $\widetilde{g}$, choose $n$ such that $R \to A$ factors as $R \to R/I_n = S/J_n \to A$. It suffices to show that $\operatorname{Spec} A \to \operatorname{Spec} S/J_n$ extends to a map $\operatorname{Spec} B \to \operatorname{Spec} S/J_{n+1}$ and for this, it suffices to show the existence of a dotted arrow making the diagram

$$\begin{tikzcd} \operatorname{Spec} A \arrow[r] \arrow[d, hook] & \operatorname{Spec} S/J_n \arrow[d, hook] \arrow[ddr, hook] & \\ \operatorname{Spec} B \arrow[r] & \operatorname{Spec} B \times_A (S/J_n) \arrow[dr, dashed] & \\ & & \operatorname{Spec} S/J_{n+1} \end{tikzcd}$$

commutative. As $S = \Bbbk[\![x_1, \ldots, x_n]\!]$, we may choose an extension $S \to B$ of $S \to S/J_n \to A$. Then $B \times_A (S/J_n) = S/K$ where $K$ is the kernel of the induced map $S \to B \times_A (S/J_n)$. The kernel $K$ lies in the set of ideals defined in (D.3.5): the inclusion $K \subset J_n$ is clear, the inclusion $\mathfrak{m}_S J_n \subset K$ is implied by the equality $\ker(B \to A) = \Bbbk$, and the existence of $\eta_n \to \eta$ over $\operatorname{Spec} S/J_n \hookrightarrow \operatorname{Spec} S/K$ follows from applying (RS$_1$) to the above square. Thus $J_{n+1} \subset K$ and we have a ring map $S/J_{n+1} \to S/K = B \times_A (S/J_n)$ inducing the desired dotted arrow.

Finally, we must show that if $\mathcal{X}$ is equivalent to a functor $F$ and (RS$_4$) holds, then $F$ is prorepresentable by $\xi = \{\xi_n\}$. Given a surjection $B \to A$ with kernel $\Bbbk$ and $x \in F(A)$, it suffices to show the existence of a *unique* lift in every diagram

$$\begin{tikzcd} \operatorname{Spec} A \arrow[r, "g"] \arrow[d, hook] & h_R \arrow[d] \\ \operatorname{Spec} B \arrow[r] \arrow[ur, dashed] & F. \end{tikzcd}$$

This holds because the map $\operatorname{Lift}_g(B) \to \operatorname{Lift}_x(B)$ is bijective by Lemma D.3.13(2) as both are torsors under $T_{\mathcal{X}}$.

See also [Sch68, Thm. 2.11], [SGA7-I, Thm. VI.1.11] and [SP, Tag 06IX], where the result is established more generally for prestacks over the category $\operatorname{Art}_\Lambda$ introduced in Remark D.3.4. □

### D.3.4 Verifying Rim–Schlessinger's Conditions

Consider the following *strong homogeneity* condition:

(RS$_4^*$) $\mathcal{X}(B_0 \times_{A_0} A) \to \mathcal{X}(B_0) \times_{\mathcal{X}(A_0)} \mathcal{X}(A)$ is an equivalence for every map $B_0 \to A_0$ and surjection $A \twoheadrightarrow A_0$ of rings with square-zero kernel (where the rings are not necessarily local artinian);

If $\mathcal{X}$ is a prestack over $(\mathrm{Sch}/\Bbbk)$ satisfying (RS$_4^*$), then the local deformation prestack $\mathcal{X}_{x_0}$ at $x_0$ (see Remark D.3.10) is easily checked to satisfy (RS$_4$). On the other hand, it turns out that every algebraic stack satisfies (RS$_4^*$); see [SP, Tag 07WN]. In other words, the Ferrand pushout $\operatorname{Spec}(B_0 \times_{A_0} A)$ is a pushout in the category of algebraic stacks. Condition (RS$_4^*$) will appear in our second version of Artin's Axioms for Algebraicity (Theorem D.7.4) as it will be useful to verify openness of versality (in addition to implying (RS$_2$)–(RS$_3$) ensuring the existence of versal formal deformations).

For a moduli problem $\mathcal{M}$, it is often possible to verify (RS$_4^*$) (and thus (RS$_4$) as well as (RS$_1$)–(RS$_2$)) as a consequence of Proposition A.8.5: for a ring map $B_0 \to A_0$ and surjection $A \twoheadrightarrow A_0$, the functor $\operatorname{Mod}(B_0 \times_{A_0} A) \to \operatorname{Mod}(B_0) \times_{\operatorname{Mod}(A_0)} \operatorname{Mod}(A)$ restricts to an equivalence on flat modules. When $B_0$, $A_0$, and $A$ are artinian, there is an elementary argument for this fact since flatness translates to freeness for modules over an artinian ring (Proposition A.2.3).

We say that a prestack $\mathcal{X}$ over $\mathrm{Sch}/\Bbbk$ *admits formal versal deformations* if for every $\Bbbk$-point $x_0$, the local deformation prestack $\mathcal{X}_{x_0}$ (Remark D.3.10) admits a formal versal deformation.

**Proposition D.3.14.** *Each of the moduli problems* $\operatorname{Hilb}^P(X)$, $\mathcal{M}_g$ *(with* $g \geq 2$*) and* $\operatorname{Bun}_C$ *over* $\Bbbk$ *satisfy* (RS$_3$) *and* (RS$_4^*$)*, and therefore admit formal versal deformations.*

*Proof.* To check (RS$_3$) for objects $[Z_0 \subset X]$, $C_0$ and $E_0$ of $\mathcal{X} = \operatorname{Hilb}^P(X)$, $\mathcal{M}_g$ and $\operatorname{Bun}_C$ defined over $\Bbbk$, we have identifications of the tangent spaces $T_{\mathcal{X}}$ with the *finite* *dimensional* $\Bbbk$-vector spaces $\mathrm{H}^0(Z_0, N_{Z_0/X})$, $H^1(C_0, T_{C_0})$ and $\mathrm{H}^1(X, \mathscr{E}nd_{\mathcal{O}_X}(E_0))$ by Propositions D.1.4, D.1.11 and D.1.15.

For (RS$_4^*$), let $B_0 \to A_0$ be a ring map and $A \twoheadrightarrow A_0$ be a surjection with square-zero kernel. Set $B = B_0 \times_{A_0} A$. For $\mathrm{Hilb}^P(X)$, Corollary A.8.6(1)–(2) implies that the diagram

$$\begin{tikzcd} X_{A_0} \arrow[r, hook] \arrow[d] & X_A \arrow[d] \\ X_{B_0} \arrow[r, hook] & X_B \end{tikzcd}$$

is a pushout, and that the functor

$$\mathrm{QCoh}(X_B) \to \mathrm{QCoh}(X_{B_0}) \times_{\mathrm{QCoh}(X_{A_0})} \mathrm{QCoh}(X_A) \tag{D.3.6}$$

restricts to an equivalence between the full subcategory of finitely presented $\mathcal{O}_{X_B}$-modules flat over $B$ and the fiber product of the full subcategories of finitely presented $\mathcal{O}$-modules flat over $B_0$ and $A$. This implies the desired equivalence $\mathrm{Hilb}^P(X)(B) \to \mathrm{Hilb}^P(X)(B_0) \times_{\mathrm{Hilb}^P(X)(A_0)} \mathrm{Hilb}^P(X)(A)$ between closed subschemes flat over the base.

For $\mathcal{M}_g$, the essential surjectivity of $\mathcal{M}_g(B) \to \mathcal{M}_g(B_0) \times_{\mathcal{M}_g(A_0)} \mathcal{M}_g(A)$ translates into the existence of an extension

$$\begin{tikzcd}[column sep=small] & \mathcal{C}_0 \arrow[rr, hook] \arrow[dl, hook] \arrow[dd] & & \mathcal{C} \arrow[dl, dashed] \arrow[dd] \\ \mathcal{D}_0 \arrow[rr, dashed, hook] \arrow[dd] & & \mathcal{D} \arrow[dd, dashed] & \\ & \operatorname{Spec} A_0 \arrow[rr, hook] \arrow[dl] & & \operatorname{Spec} A \arrow[dl] \\ \operatorname{Spec} B_0 \arrow[rr, hook] & & \operatorname{Spec} B & \end{tikzcd}$$

of smooth families of curves. The existence of $\mathcal{D}$ as a pushout of the top face follows from Theorem A.8.1. The fact that $\mathcal{D}$ is smooth over $B$ follows from Corollary A.8.6(2). The properness of $\mathcal{D} \to \operatorname{Spec} B$ follows from the properness of $\mathcal{D}_0 \to \operatorname{Spec} B_0$. The fully faithfulness translates to the bijectivity of

$$\mathrm{Aut}(\mathcal{D}/B) \to \mathrm{Aut}(\mathcal{D}_0/B_0) \times_{\mathrm{Aut}(\mathcal{C}_0/A_0)} \mathrm{Aut}(\mathcal{C}/A)$$

and follows directly from the fact that $\mathcal{D}$ is a pushout of the top face. Alternatively, one can replicate the above argument for $\mathrm{Hilb}^P(X)$ using the tricanonical embedding.

For $\mathrm{Bun}_C$, Corollary A.8.6(1) implies that the functor (D.3.6) restricts to an equivalence on finitely presented $\mathcal{O}$-modules flat over the base and therefore also on vector bundles. □

## D.4 Effective formal deformations and Grothendieck's Existence Theorem

We often would like to know when a formal deformation is effective.

**Definition D.4.1.** Let $\mathcal{X}$ be a prestack (or functor) over $(\mathrm{Sch}/\Bbbk)$. Let $x_0 \in \mathcal{X}(\Bbbk)$ and consider a formal deformation $(R, \{x_n\})$ of $x_0$ (or more precisely a formal deformation of the deformation stack $\mathcal{X}_{x_0}$ at $x_0$ as defined in Remark D.3.10). We say that $\{x_n\}$ is *effective* if there exists an object $\widehat{x} \in \mathcal{X}(R)$ and compatible isomorphisms $x_n \xrightarrow{\sim} \widehat{x}|_{\operatorname{Spec} R/\mathfrak{m}^{n+1}}$.

**Remark D.4.2.** A formal deformation $(R, \{x_n\})$ is effective if it is in the essential image of the natural functor $\mathcal{X}(R) \to \varprojlim \mathcal{X}(R/\mathfrak{m}^n)$ or in other words if there exists a dotted arrow making the diagram

$$\begin{tikzcd}
\operatorname{Spec} R/\mathfrak{m} \arrow[r, hook] \arrow[drrrr, "x_0"'] & \operatorname{Spec} R/\mathfrak{m}^2 \arrow[r, hook] \arrow[drrr, "x_1"] & \operatorname{Spec} R/\mathfrak{m}^3 \arrow[r, hook] \arrow[drr, "x_2"] & \cdots \arrow[r, hook] & \operatorname{Spec} R \arrow[d, dashed, "\widehat{x}"] \\
& & & & \mathcal{X}
\end{tikzcd}$$

commutative.

**Example D.4.3.** If $F\colon \mathrm{Sch}/\Bbbk \to \mathrm{Sets}$ is a contravariant functor representable by a scheme $X$ over $\Bbbk$, then every formal deformation $(R, \{x_n\})$ is effective. Indeed, $x_n$ corresponds to a morphism $\operatorname{Spec} R/\mathfrak{m}^{n+1} \to X$ with image $x \in X(\Bbbk)$ and thus to a $\Bbbk$-algebra homomorphism $\phi_n\colon \widehat{\mathcal{O}}_{X,x} \to R/\mathfrak{m}^{n+1}$. By taking the inverse image of $\phi_n$, we have a local homomorphism $\widehat{\mathcal{O}}_{X,x} \to R$ which in turn defines a morphism $\widehat{x}\colon \operatorname{Spec} R \to X$ extending $\{x_n\}$.

Grothendieck's Existence Theorem—sometimes referred to as Formal GAGA—can often be applied to show that formal deformations are effective.

**Theorem D.4.4** (Grothendieck's Existence Theorem)**.** *Let $X \to \operatorname{Spec} R$ be a proper morphism of schemes where $(R, \mathfrak{m})$ is a noetherian complete local ring. Set $X_n := X \times_R R/\mathfrak{m}^{n+1}$ The functor*

$$\operatorname{Coh}(X) \to \varprojlim \operatorname{Coh}(X_n), \qquad E \mapsto \{E_n\}, \tag{D.4.1}$$

*where $E_n$ is the pullback of $E$ along $X_n \to X$, is an equivalence of categories.*

*Proof.* See [EGA, III.5.1.4], [FGI+05, Thm. 8.4.2] and [SP, Tag 088E]. □

**Remark D.4.5.** The essential surjectivity of (D.4.1) translates to an extension of the diagram

$$\begin{tikzcd}
E_0 \arrow[d, no head] & E_1 \arrow[d, no head] & E_2 \arrow[d, no head] & & E \arrow[d, dashed, no head] \\
X_0 \arrow[r, hook] \arrow[d] & X_1 \arrow[r, hook] \arrow[d] & X_2 \arrow[r, hook] \arrow[d] & \cdots \arrow[r, hook] & X \arrow[d] \\
\operatorname{Spec} R/\mathfrak{m} \arrow[r, hook] & \operatorname{Spec} R/\mathfrak{m}^2 \arrow[r, hook] & \operatorname{Spec} R/\mathfrak{m}^3 \arrow[r, hook] & \cdots \arrow[r, hook] & \operatorname{Spec} R
\end{tikzcd}$$

while the fully faithfulness of (D.4.1) translates to the bijectivity of the natural map $\operatorname{Hom}_{\mathcal{O}_X}(E, F) \to \varprojlim \operatorname{Hom}_{\mathcal{O}_{X_n}}(E_n, F_n)$ for coherent sheaves $E$ and $F$ on $X$.

Using the language of formal schemes and setting $\widehat{X} = X \times_{\operatorname{Spec} R} \operatorname{Spf} R$ to be the $\mathfrak{m}$-adic completion of $X$, then Grothendieck's Existence Theorem asserts that the functor $\operatorname{Coh}(X) \to \operatorname{Coh}(\widehat{X})$, defined by $E \mapsto \widehat{E}$, is an equivalence.

**Corollary D.4.6.** *Let $(R, \mathfrak{m})$ be a noetherian complete local ring and $X_n \to \operatorname{Spec} R/\mathfrak{m}^{n+1}$ be a sequence of proper morphisms such that $X_n \times_{R/\mathfrak{m}^{n+1}} R/\mathfrak{m}^n \cong X_{n-1}$. If $L_n$ is a compatible sequence of line bundles on $X_n$ such that $L_0$ is ample, then there exists a projective morphism $X \to \operatorname{Spec} R$ and an ample line bundle $L$ on $X$ and compatible isomorphisms $X_n \cong X \times_R R/\mathfrak{m}^{n+1}$ and $L_n \xrightarrow{\sim} L|_{X_n}$.*

**Remark D.4.7.** It follows that there is an extension in the cartesian diagram

$$\begin{tikzcd}
X_0 \arrow[r, hook] \arrow[d] & X_1 \arrow[r, hook] \arrow[d] & X_2 \arrow[r, hook] \arrow[d] & \cdots \arrow[r, hook, dashed] & X \arrow[d, dashed] \\
\operatorname{Spec} R/\mathfrak{m} \arrow[r, hook] & \operatorname{Spec} R/\mathfrak{m}^2 \arrow[r, hook] & \operatorname{Spec} R/\mathfrak{m}^3 \arrow[r, hook] & \cdots \arrow[r, hook] & \operatorname{Spec} R
\end{tikzcd}$$

such that $X$ is projective over $R$. We say that the formal deformation $\{X_n \to \operatorname{Spec} R/\mathfrak{m}^{n+1}\}$ of $X_0$ is *effective* (which is sometimes referred to as *algebraizable*.

*Proof.* We sketch how this follows from Grothendieck's Existence Theorem. Consider the finitely generated graded $\Bbbk$-algebra $B = \bigoplus \mathfrak{m}^n/\mathfrak{m}^{n+1}$ and the quasi-coherent graded $\mathcal{O}_{X_0}$-algebra $\mathcal{A} = B \otimes_{\Bbbk} \mathcal{O}_{X_0}$. By applying Serre's vanishing theorem to $\operatorname{Spec}_{X_0} \mathcal{A}$ and the ample line bundle $L_0 \otimes_{\mathcal{O}_{X_0}} \mathcal{O}_{X_0'}$, we see that $\mathrm{H}^1(X_0, \mathcal{A} \otimes L_0^{\otimes d}) = 0$ for $d \gg 0$. We have a closed immersion $X_0 \hookrightarrow \mathbb{P}^N$ defined by a basis $s_{0,0}, \ldots, s_{0,N}$ of $\mathrm{H}^0(X_0, L_0^{\otimes d})$. Noting that $\mathfrak{m}^n\mathcal{O}_{X_{n+1}}/\mathfrak{m}^{n+1}\mathcal{O}_{X_{n+1}}$ is identified with $\ker(\mathcal{O}_{X_{n+1}} \to \mathcal{O}_{X_n})$, we may tensor the corresponding short exact sequence by $L_{n+1}^{\otimes d}$ to obtain a short exact sequence

$$0 \to (\mathfrak{m}^n\mathcal{O}_{X_{n+1}}/\mathfrak{m}^{n+1}\mathcal{O}_{X_{n+1}}) \otimes L_0^{\otimes d} \to L_{n+1}^{\otimes d} \to L_n^{\otimes d} \to 0,$$

where we've used that $(\mathfrak{m}^n\mathcal{O}_{X_{n+1}}/\mathfrak{m}^{n+1}\mathcal{O}_{X_{n+1}}) \otimes L_{n+1}^{\otimes d}$ is supported on $X_0$ along with the identifications $L_{n+1} \otimes \mathcal{O}_{X_m} \cong L_m$ for $m \le n$. The vanishing of $\mathrm{H}^1(X_0, \mathcal{A} \otimes L_0^{\otimes d})$ implies that we may lift the sections $s_{0,0}, \ldots, s_{0,N}$ inductively to compatible sections $s_{n,0}, \ldots, s_{n,N}$ of $\mathrm{H}^0(X_n, L_n^{\otimes d})$. By Nakayama's Lemma, the induced morphisms $X_n \hookrightarrow \mathbb{P}^N_{R/\mathfrak{m}^{n+1}}$ are closed immersions giving a commutative diagram

$$\begin{tikzcd}
& \mathbb{P}^N \arrow[rr, hook] \arrow[dd] & & \mathbb{P}^n_{R/\mathfrak{m}_R^2} \arrow[r, hook] \arrow[dd] & \cdots \arrow[rr, hook] & & \mathbb{P}^N_R \arrow[dd] \\
X_0 \arrow[ur, hook, "\mathrm{cl}"] \arrow[rr, hook] \arrow[dr] & & X_1 \arrow[ur, hook, "\mathrm{cl}"] \arrow[rr, hook] \arrow[dr] & & \cdots \arrow[r, hook, dashed] & X \arrow[ur, hook, dashed, "\mathrm{cl}"] \arrow[dr, dashed] & \\
& \operatorname{Spec} \Bbbk \arrow[rr, hook] & & \operatorname{Spec} R/\mathfrak{m}^2 \arrow[r, hook] & \cdots \arrow[rr, hook] & & \operatorname{Spec} R
\end{tikzcd}$$

Grothendieck's Existence Theorem (D.4.4) gives an equivalence $\operatorname{Coh}(\mathbb{P}^N_R) \to \varprojlim \operatorname{Coh}(\mathbb{P}^N_{R/\mathfrak{m}^{n+1}})$. Essential surjectivity gives a coherent sheaf $E$ on $\mathbb{P}^N_R$ extending $\{\mathcal{O}_{X_n}\}$ and full faithfulness gives a surjection $\mathcal{O}_{\mathbb{P}^N_R} \to E$ extending $\mathcal{O}_{\mathbb{P}^N_{R/\mathfrak{m}^{n+1}}} \to \mathcal{O}_{X_n}$. We take $X \subset \mathbb{P}^N_R$ to be the closed subscheme defined by $\ker(\mathcal{O}_{\mathbb{P}^N_R} \to E)$.

See also [EGA, III.5.4.5], [FGI+05, Thm. 8.4.10] and [SP, Tag 089A]. □

**Remark D.4.8.** Suppose that $X$ is flat over $R$ and that we are only given an ample line bundle $L_0$ on $X_0$ (but not the line bundles $L_n$). Then the obstruction to deforming $L_{n-1}$ to $L_n$ is an element $\mathrm{ob}_{L_{n-1}} \in \mathrm{H}^2(X, \mathcal{O}_X \otimes_{\Bbbk} \mathfrak{m}^n)$ by Proposition D.2.15. If these cohomology groups vanish (e.g. if $X$ is of dimension 1), then there exists compatible extensions $L_n$, and thus the formal deformation $\{X_n \to \operatorname{Spec} R/\mathfrak{m}^{n+1}\}$ are effective.

Without the existence of deformations $L_n$ of $L_0$, it is not necessarily true that formal deformations are effective. For instance, there is a projective K3 surface $(X_0, L_0)$ and a first-order deformation $X_1 \to \operatorname{Spec} \Bbbk[\epsilon]$ which is not projective (so $L_0$ does not deform to $X_1$), and a formal deformation which is not effective; see [Har10, Ex. 21.2.1]. Similarly, formal deformations of abelian varieties may not be effective. Note that for both K3 surfaces and abelian varieties, Rim–Schlessinger's Criteria applies to construct versal formal deformations.

**Corollary D.4.9.** *For each of the moduli problems* $\mathrm{Hilb}^P(X)$*,* $\mathcal{M}_g$ *(with* $g \geq 2$*) and* $\mathrm{Bun}_C$ *over* $\Bbbk$*, every formal deformation is effective. In particular, there exists effective versal formal deformations.*

*Proof.* For $\mathrm{Hilb}^P(X)$, we show the effectivity of a formal deformation $\{Z_n \subset X_{R/\mathfrak{m}^{n+1}}\}$ by following the argument at the end of the proof of Corollary D.4.6 (with $X_n \subset \mathbb{P}^N_{R/\mathfrak{m}^{n+1}}$ replaced with $Z_n \subset X_{R/\mathfrak{m}^{n+1}}$): Grothendieck's Existence Theorem (D.4.4) implies the existence of a coherent sheaf $E$ on $X_R$ extending $\{\mathcal{O}_{Z_n}\}$ and a surjection $\mathcal{O}_{X_R} \to E$ extending $\{\mathcal{O}_{X_n} \to \mathcal{O}_{Z_n}\}$. We take $Z \subset X_R$ defined by $\ker(\mathcal{O}_{X_R} \to E)$.

For $\mathcal{M}_g$, the effectivity of a formal deformation $\{C_n \to \operatorname{Spec} R/\mathfrak{m}^{n+1}\}$ follows from Corollary D.4.6 by taking $L_n$ be the ample bundle $\Omega_{C_n/(R/\mathfrak{m}^{n+1})}$, or by taking $L_0$ to be any ample bundle on $C_0$ and using Proposition D.2.15 and the vanishing of $\mathrm{H}^2(C_0, \mathcal{O}_{C_0})$ to inductively deform $L_0$ to a compatible sequence of line bundle $L_n$ on $C_n$.

For $\mathrm{Bun}_C$, the effectivity of a formal deformation of vector bundles $E_n$ on $C_{R/\mathfrak{m}^{n+1}}$ follows directly from Grothendieck's Existence Theorem (D.4.4) noting that the coherent extension is necessarily a vector bundle.

The last statement follows from the existence of versal formal deformations of these moduli problems (Proposition D.3.14). □

**Exercise D.4.10.** If $\mathcal{X}$ is an algebraic stack locally of finite type over $\Bbbk$ and $(R, \mathfrak{m})$ is a noetherian complete local ring with residue field $\Bbbk$, show that the functor

$$\mathcal{X}(R) \to \varprojlim \mathcal{X}(R/\mathfrak{m}^{n+1})$$

is an equivalence of categories. In particular, every formal deformation is effective.

### D.4.1 Lifting to characteristic 0

One striking application of deformation theory is to "lift" a smooth variety $X_0$ over a field $\Bbbk$ of $\mathrm{char}(\Bbbk) = p$ to characteristic 0. We say that $X_0$ is *liftable to characteristic* 0 if there exists a noetherian complete local ring $(R, \mathfrak{m})$ of characteristic 0 such that $R/\mathfrak{m} = \Bbbk$ and a smooth scheme $X \to \operatorname{Spec} R$ such that $X_0 \cong X \times_R \Bbbk$.[2] One can hope to then use characteristic 0 techniques (e.g. Hodge theory) on $X$ and deduce properties of $X_0$. The strategy to lift a variety $X_0$ is to inductively deform $X_0$ to smooth schemes $X_n$ over $R/\mathfrak{m}^{n+1}$ and then apply Grothendieck's Existence Theorem to effective the formal deformation. Note however that to achieve this, we must work over a mixed characteristic base as in Remark D.3.4 rather than over a fixed field $\Bbbk$.

Smooth curves are liftable as obstructions to deforming both the curve and the ample line bundle both vanish. Serre produced an example of a non-liftable projective threefold (see [Har10, Thm. 22.4]), which Mumford extended to a non-liftable projective surface (see [FGI+05, Cor. 8.6.7]). On the other hand, Mumford showed that principally polarized abelian varieties are liftable [Mum69] while Deligne showed that K3 surfaces are liftable [Del81]. These examples are quite interesting as, in both cases, formal deformations are not necessarily effective (see Remark D.4.8) and additional techniques are needed.

[2] There are some variants to this definition, e.g. when $R$ is already given as a complete DVR with residue field $\Bbbk$.

## D.5 Cotangent complex

In this chapter, we summarize properties of the cotangent complex of a morphism of schemes as introduced in [Ill71] globalizing work of André [And67] and Quillen [Qui68, Qui70] on the cotangent complex of a ring homomorphism. One advantage of the cotangent complex is that it allows us to describe the deformations and obstruction of singular schemes; see Theorem D.5.10.

### D.5.1 Properties of the cotangent complex

**Theorem D.5.1.** *For every morphism $f\colon X \to Y$ of schemes (resp. finite type morphism of noetherian schemes), there exists a complex*

$$L_{X/Y}\colon \cdots \to L^{-1}_{X/Y} \to L^0_{X/Y} \to 0$$

*of flat $\mathcal{O}_X$-modules with quasi-coherent (resp. coherent) cohomology, whose image in $D^-_{\mathrm{QCoh}}(\mathcal{O}_X)$ (resp. $D^-_{\mathrm{Coh}}(\mathcal{O}_X)$) is also denoted by $L_{X/Y}$. It satisfies the following properties:*

*(1)* $\mathrm{H}^0(X, L_{X/Y}) \cong \Omega_{X/Y}$*;*

*(2) $f$ is smooth if and only if $f$ is locally of finite presentation and $L_{X/Y}$ is a perfect complex supported in degree $0$. In this case $L_{X/Y}$ is quasi-isomorphic to the complex where the vector bundle $\Omega_{X/Y}$ sits in degree 0;*

*(3) If $f$ is flat and finitely presented, then $f$ is syntomic if and only if $L_{X/Y}$ is a perfect complex supported in degrees $[-1, 0]$. Explicitly, if $f$ factors as a local complete intersection $X \hookrightarrow \widetilde{Y}$ defined by a sheaf of ideals $I$ and a smooth morphism $\widetilde{Y} \to Y$, then $L_{X/Y}$ is quasi-isomorphic to $0 \to I/I^2 \xrightarrow{d} \Omega_{X/Y} \to 0$ (with $\Omega_{X/Y}$ in degree $0$);*

*(4) If*

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

*is a cartesian diagram with either $f$ or $g$ flat (or more generally $f$ and $g$ are tor-independent), then there is a quasi-isomorphism $g'^* L_{X/Y} \to L_{X'/Y'}$. (Note that without any flatness condition $g'^*\Omega_{X/Y} \cong \Omega_{X'/Y'}$.)*

*(5) If $X \xrightarrow{f} Y \to Z$ is a composition of morphisms of schemes, then there is an exact triangle in $D^-_{\mathrm{QCoh}}(\mathcal{O}_X)$*

$$f^* L_{Y/Z} \to L_{X/Z} \to L_{X/Y} \to f^* L_{Y/Z}[1].$$

*This induces a long exact sequence on cohomology*

$$\begin{tikzcd} & \cdots \arrow[r] & \mathrm{H}^{-2}(L_{X/Y}) \arrow[dll] \\ \mathrm{H}^{-1}(f^* L_{X/Z}) \arrow[r] & \mathrm{H}^{-1}(L_{X/Z}) \arrow[r] & \mathrm{H}^{-1}(L_{X/Y}) \arrow[dll] \\ f^*\Omega_{Y/Z} \arrow[r] & \Omega_{X/Z} \arrow[r] & \Omega_{X/Y} \arrow[r] & 0 \end{tikzcd}$$

*extending the usual right exact sequence on differentials [Har77, II.8.12]. (Note that if $f$ is smooth, then $H^{-1}(L_{X/Y}) = 0$ and $f^*\Omega_{Y/Z} \to \Omega_{X/Z}$ is injective.)*

*Proof.* See [Ill71, II.1.2.3], [SP, Tag 08T2] for the definition of the cotangent complex of a morphism of schemes (and more generally for morphisms of ringed topoi). For (1)–(5), see [Ill71, II.1.2.4.2, II.3.1.2, II.3.2.6, II.2.2.3 and II.2.1.2] and [SP, Tags 08UV, 0D0N, 0FK3, 08QQ and 08T4] (noting that [SP, Tag 08RB] relates the *naive cotangent complex* $NL_{X/Y}$ to $L_{X/Y}$). ☐

### D.5.2 Truncations of the cotangent complex

The definition of the cotangent complex relies on simplicial techniques and we won't attempt an exposition here. We will however give an explicit description of its truncation, which often suffice for applications.

First, if $X \to Y$ factors as a closed immersion $X \hookrightarrow P$ defined by a sheaf of ideals $I$ and a smooth morphism $P \to Y$, then the truncation $\tau_{\geq -1}(L_{X/Y})$ of $L_{X/Y}$ in degrees $[-1, 0]$ is quasi-isomorphic to $0 \to I/I^2 \xrightarrow{d} \Omega_{X/Y} \to 0$ (with $\Omega_{X/Y}$ in degree 0). In the case that $X \to Y$ is smooth or syntomic, then $X \hookrightarrow \widetilde{Y}$ is a regular immersion, $I/I^2$ is a vector bundle and $L_{X/Y}$ is quasi-isomorphic to $\tau_{\geq -1}(L_{X/Y})$ (Theorem D.5.1(3)).

For a morphism $X = \operatorname{Spec} A \to \operatorname{Spec} B = Y$ of affine schemes, Lichtenbaum–Schlessinger [LS67] offer an explicit description of $\tau_{\geq -2}(L_{X/Y})$. Choose a polynomial ring $P = B[x_i]$ (with possibly infinitely many generators) and a surjection $P \twoheadrightarrow A$ as $B$-algebras with kernel $I$. Choose a free $P$-module $F = \oplus_{\lambda \in \Lambda} P$ and a surjection $p \colon F \twoheadrightarrow I$ of $P$-modules with kernel $K = \ker(p)$. Let $K' \subset K$ be the submodule generated by $p(x)y - p(y)x$ for $x, y \in F$. Then the truncation $\tau_{\geq -2}(L_{X/Y})$ (or rather $\tau_{\geq -2}(L_{B/A})$) is quasi-isomorphic to the complex of $A$-modules

$$K/K' \to F \otimes_P A \to \Omega_{P/B} \otimes_P A \tag{D.5.1}$$

with the last term in degree 0; see [SP, Tag 09CG].

One defines the $T^i$ *functors* on the category of $A$-modules by

$$T^i(A/B, -) := \mathrm{H}^i(\operatorname{Hom}_A(L_{A/B}, -)),$$

which can be used for instance to describe deformations of schemes (see Example D.5.11). See also [LS67, §2.3] and [Har10, §1.3].

### D.5.3 Extensions of algebras and schemes

**Definition D.5.2.** An *extension* of a ring homomorphism $R \to A$ by an $A$-module $J$ is an exact sequence of $R$-modules

$$0 \to J \to A' \to A \to 0$$

where $A' \to A$ is an $R$-algebra homomorphism and $J \subset A'$ is an ideal with $J^2 = 0$. (Note that since $J^2 = 0$, $J = J/J^2$ is a module over $A = A'/J$.) The *trivial extension* is $A[J] := A \oplus J$ where multiplication is defined by $J^2 = 0$.

A morphism of extensions is a morphism of short exact sequences which is the identity on $J$ and $A$. By the five lemma, a morphism of extensions is necessarily an isomorphism. We let $\underline{\mathrm{Exal}}_R(A, J)$ be the groupoid of extensions of $R \to A$ by $J$, and $\mathrm{Exal}_R(A, J)$ the set of isomorphism classes.

**Remark D.5.3.** Geometrically, an extension is a commutative diagram of schemes

$$\begin{tikzcd} \operatorname{Spec} A \arrow[r, hook] \arrow[d] & \operatorname{Spec} A' \arrow[dl] \\ \operatorname{Spec} R & \end{tikzcd}$$

such that $J \cong \ker(A' \to A)$ and $J^2 = 0$.

The set of extensions $\mathrm{Exal}_R(A, J)$ is functorial with respect to $A$ and $J$:

(a) Given a map $B \to A$ of $R$-algebras, there is a map $\mathrm{Exal}_R(A, J) \to \mathrm{Exal}_R(B, J)$ given by mapping a complex $0 \to J \to A' \to A \to 0$ to $0 \to J \to A' \times_A B \to B \to 0$.

(b) Given an $A$-module map $\alpha\colon J \to J$, there is a map $\alpha_*\colon \mathrm{Exal}_R(A, J) \to \mathrm{Exal}_R(A, J)$ given by mapping a complex $0 \to J \to A' \to A \to 0$ to $0 \to J \to (A' \oplus J)/\{(-x, \alpha(x)), x \in J\} \to A \to 0$.

(c) Given modules $J$ and $K$, the natural map $(p_{1,*}, p_{2,*})\colon \mathrm{Exal}_R(A, J \oplus K) \to \mathrm{Exal}_R(A, J) \oplus \mathrm{Exal}_R(A, K)$, induced from (b) by the projections $p_1\colon J \oplus K \to J$ and $p_2\colon J \oplus K \to K$, is a bijection.

Moreover, $\mathrm{Exal}_R(A, J)$ naturally has the structure of an $A$-module: scalar multiplication by $x \in A$ is defined using (b) with $x\colon J \to J$ and addition is defined by $\mathrm{Exal}_R(A, J) \times \mathrm{Exal}_R(A, J) \cong \mathrm{Exal}_R(A, J \oplus J) \xrightarrow{\Sigma_*} \mathrm{Exal}_R(A, J)$ using the bijection in (c) and the map $\Sigma_*$ of (b) where $\Sigma\colon J \oplus J \to J$ is addition. The maps (a)–(c) are in fact maps of $A$-modules. See [Ill71, §III.1.1] for details.

**Proposition D.5.4.** *Let $R$ be a ring.*

*(1) Given a $R$-algebra $A$ and an exact sequence $0 \to J' \to J \to J'' \to 0$ of $A$-modules, there is an exact sequence*

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Der}_R(A, J') \arrow[r] & \mathrm{Der}_R(A, J) \arrow[r] & \mathrm{Der}_R(A, J'') \arrow[dll, out=0, in=180] \\ & \mathrm{Exal}_R(A, J) \arrow[r] & \mathrm{Exal}_R(A, J) \arrow[r] & \mathrm{Exal}_R(A, J'') \end{tikzcd}$$

*of $A$-modules.*

*(2) Given a homomorphism $B \to A$ of $R$-algebras, there is an exact sequence*

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Der}_B(A, J) \arrow[r] & \mathrm{Der}_R(A, J) \arrow[r] & \mathrm{Der}_R(B, J) \arrow[dll, out=0, in=180] \\ & \mathrm{Exal}_B(A, J) \arrow[r] & \mathrm{Exal}_R(A, J) \arrow[r] & \mathrm{Exal}_R(B, J) \end{tikzcd}$$

*of $A$-modules.*

*Proof.* See [EGA, 0.20.2.3] and [Ill71, III.1.2.4.3, III.1.2.5.4]. □

**Remark D.5.5.** The top row of (D.5.4) can be realized using the right exact sequence $\Omega_{B/R} \otimes_B A \to \Omega_{A/R} \to \Omega_{A/B} \to 0$. Namely, apply $\mathrm{Hom}_A(-, J)$ and use the identities $\mathrm{Hom}_A(\Omega_{A/B}, J) = \mathrm{Der}_B(A, J)$, $\mathrm{Hom}_A(\Omega_{A/R}, J) = \mathrm{Der}_R(A, J)$ and $\mathrm{Hom}_A(\Omega_{B/R} \otimes_B A, J) = \mathrm{Hom}_B(\Omega_{B/R}, J) = \mathrm{Der}_R(B, J)$.

The cotangent complex can be applied to extend these sequences to long exact sequences; see Remark D.5.8.

The definition of Exal extends naturally to schemes (and more generally to ringed topoi).

**Definition D.5.6.** An *extension* of a morphism $X \to S$ of schemes by a quasi-coherent $\mathcal{O}_X$-module $J$ is a short exact sequence

$$0 \to J \to \mathcal{O}_{X'} \to \mathcal{O}_X \to 0$$

where $X \hookrightarrow X'$ is a closed immersion of schemes defined by the sheaf of ideals $J \subset \mathcal{O}_{X'}$ with $J^2 = 0$. (Note that the condition $J^2 = 0$ implies that the $J \subset \mathcal{O}_{X'}$ is naturally a $\mathcal{O}_X$-module.) The *trivial extension* is $X[J] := (X, \mathcal{O}_X \oplus J)$ where the ring structure is defined by $J^2 = 0$.

A *morphism of extensions* is a morphism of short exact sequences which is the identity on $J$ and $\mathcal{O}_X$. We let $\underline{\mathrm{Exal}}_S(X, J)$ be the category of extensions of $X \to S$ by $J$, and $\mathrm{Exal}_S(X, J)$ be the set of isomorphism classes.

The set $\mathrm{Exal}_S(X, J)$ is naturally an $\mathcal{O}_X$-module and is functorial in $X$ and $J$. In fact, the groupoid $\underline{\mathrm{Exal}}_S(X, J)$ is a *Picard category*, and the prestack over $\mathrm{Sch}/S$ whose fiber category over $f\colon T \to S$ is $\underline{\mathrm{Exal}}_T(X_T, f^*J)$ is a *Picard stack*; see [Ill71, III.1.1.5] and [SGA4, XVIII.1.4].

### D.5.4 The cotangent complex and deformation theory

**Theorem D.5.7.** *If $X \to Y$ is a morphism of schemes and $J$ is a quasi-coherent $\mathcal{O}_Y$-module, there is a natural isomorphism*

$$\mathrm{Exal}_Y(X, J) \cong \mathrm{Ext}^1_{\mathcal{O}_X}(L_{X/Y}, J).$$

*Proof.* See [Ill71, III.1.2.3]. □

**Remark D.5.8.** This identification allows us to use the cotangent complex to extend the 6-term left exact sequences of Proposition D.5.4 to long exact sequences. Namely, applying $\mathrm{Hom}_{\mathcal{O}_X}(L_{X/Y}, -)$ to the exact sequence $0 \to J' \to J \to J''$ extends D.5.4(1) and applying $\mathrm{Hom}_{\mathcal{O}_X}(-, J)$ to the exact triangle $f^*L_{Y/Z} \to L_{X/Z} \to L_{X/Y}$ extends D.5.4(2).

When $X = \operatorname{Spec} A \to \operatorname{Spec} B = Y$ is a morphism of affine schemes, using the $T^i$ functors of §D.5.2, the above equivalence translates to $\mathrm{Exal}_B(A, J) = T^1(A/B, J)$. This can be established using the explicit description of the Lichtenbaum–Schlessinger truncated cotangent complex (D.5.1); see [LS67, 4.2.2] and [Har10, Thm. 5.1]. The $T^i$ functors can also be used to extend the 6-term sequences of Proposition D.5.4 to 9-term sequences; see [LS67, 2.3.5-6] and [Har10, Thms. 3.4-5].

**Remark D.5.9.** More generally, there is an equivalence between the groupoid $\underline{\mathrm{Exal}}_Y(X, J)$ and the groupoid obtained from the 2-term complex

$$[C^{-1} \xrightarrow{d} C^0] := \tau_{\leq 0}(\mathrm{RHom}_{\mathcal{O}_X}(\tau_{\geq -1}L_{X/Y}, J)[1])$$

where objects are elements of $C^0$ and $\mathrm{Mor}(c, c') = d^{-1}(c - c')$; see [Ill71, III.1.2.2].

**Theorem D.5.10.** *Consider the following deformation problem*

$$\begin{tikzcd} X \arrow[r, hook, dashed] \arrow[d, "f"] & X' \arrow[d, dashed, "f'"] \\ Y \arrow[r, hook, "i"] & Y' \end{tikzcd}$$

*where $f\colon X \to Y$ is a morphism of schemes and $i\colon Y \hookrightarrow Y'$ is a closed immersion of schemes defined by an ideal sheaf $J \subset \mathcal{O}_{Y'}$ with $J^2 = 0$. A* deformation *is a morphism $f'\colon X' \to Y'$ making the above diagram cartesian, and a morphism of deformations is a morphism over $Y'$ restricting to the identity on $X$.*

*(1) The group of automorphisms of a deformation $f'\colon X' \to Y'$ is isomorphic to $\operatorname{Ext}^0_{\mathcal{O}_X}(L_{X/Y}, f^*J)$.*

*(2) If there exists a deformation, then the set of deformations is a torsor under $\operatorname{Ext}^1_{\mathcal{O}_X}(L_{X/Y}, f^*J)$.*

*(3) There exists an element $\mathrm{ob}_X \in \operatorname{Ext}^2_{\mathcal{O}_X}(L_{X/Y}, f^*J)$ with the property that there exists a deformation if and only if $\mathrm{ob}_X = 0$.*

*Proof.* See [Ill71, III.2.1.7] and [SP, Tag 08UZ]. See also [LS67, 4.2.5] and [Har10, Thm. 10.1] for descriptions in the affine case using the truncated cotangent complex. □

**Example D.5.11.** As a reality check, let's first consider a smooth morphism $f\colon X \to \operatorname{Spec} A$ is smooth and a surjection $A' \twoheadrightarrow A$ of noetherian rings with square-zero kernel $J$. By the identification

$$\operatorname{Ext}^i_{\mathcal{O}_X}(L_{X/A}, f^*J) = \mathrm{H}^i(X, T_{X/A} \otimes_A J),$$

we recover Proposition D.2.6.

Second, let's consider a scheme $X_0$ locally of finite type over a field $\Bbbk$ which is generically smooth and a local complete intersection. In this case, every point of $X_0$ has an open neighborhood $U$ such that $U = V(I) \subset \mathbb{A}^n$ where $I$ is generated by a regular sequence. We always have a right exact sequence

$$I/I^2 \xrightarrow{d} \Omega_{\mathbb{A}^n}|_U \to \Omega_U \to 0, \tag{D.5.2}$$

and by properties of the cotangent complex (D.5.1(3)), we have that $L_U = [I/I^2 \to \Omega_{\mathbb{A}^n}|_U]$ and supported in degrees $[-1, 0]$. On the other hand, since $U$ is generally smooth, $I/I^2 \to \Omega_{\mathbb{A}^n}|_U$ is generically injective. But as $I/I^2$ is a vector bundle (as $I$ is generated by a regular sequence), it can have no torsion subsheaves. It follows that the sequence D.5.2 is also left exact and that $\Omega_U$ is quasi-isomorphic to $L_U$. Thus, $\Omega_{X_0}$ is also quasi-isomorphic to $L_{X_0}$, and automorphisms, deformations, and obstructions are classified by $\operatorname{Ext}^i_{\mathcal{O}_X}(\Omega_{X_0}, J)$ recovering Proposition D.2.11.

One major advantage of the cotangent complex is that for $\operatorname{Ext}^i_{\mathcal{O}_X}(L_{X/A}, f^*J)$ for $i = 0, 1, 2$ classifies automorphisms, deformations, and obstructions for an arbitrary morphism. Moreover, the truncated cotangent complex $\tau_{\geq -2}L_{X/A}$ suffices to compute automorphisms, deformations and obstructions; for instance when $X = \operatorname{Spec} B$ is affine, we get equivalent descriptions using the $T^i$ functors $T^i(L_{B/A}, f^*J)$ as defined in §D.5.2.

**Remark D.5.12.** There are analogous results for other deformation problems. For instance, for the deformation problem

$$\begin{tikzcd}
X \arrow[r, hook] \arrow[dr, "f"] \arrow[ddr] & X' \arrow[dr, dashed] \arrow[ddr] & \\
& Y \arrow[r, hook] \arrow[d] & Y' \arrow[d] \\
& Z \arrow[r, hook] & Z',
\end{tikzcd}$$

where the horizontal morphisms are closed immersions defined by square-zero ideal sheaves $J_X$, $J_Y$ and $J_Z$, then automorphisms, deformations, and obstructions are classified by $\mathrm{Ext}^i_{\mathcal{O}_X}(f^*L_{Y/Z}, J_X)$ for $i = -1, 0, 1$ [Ill71, III.2.2.4]. An important special case is when $Y = Y'$ and $Z = Z'$.

## D.6 Artin Algebraization

Artin Algebraization is a procedure to "algebraize" or extend an effective versal formal deformation $\xi \in \mathcal{M}(R)$ to an object $\eta \in \mathcal{M}(U)$ over a *finite type* $\Bbbk$-scheme $U$. In this section, we show how Artin Algebraization follows from Artin Approximation following the ideas of Conrad and de Jong [CJ02].

### D.6.1 Limit preserving prestacks

Extending the definition of a limit preserving functor §A.10.2, we say that a prestack $\mathcal{X}$ over $\mathrm{Sch}/\Bbbk$ is *limit preserving* (or *locally of finite presentation*) if for every system $B_\lambda$ of $\Bbbk$-algebras, the natural functor

$$\operatorname{colim} \mathcal{X}(B_\lambda) \to \mathcal{X}(\operatorname{colim} B_\lambda)$$

is an equivalence of categories. When $\mathcal{X}$ is an algebraic stack over $\Bbbk$, then this equivalent to the morphism $\mathcal{X} \to \operatorname{Spec} \Bbbk$ being locally of finite presentation; see Exercise 3.3.31).

**Lemma D.6.1.** *Each of the prestacks* $\mathrm{Hilb}^P(X)$, $\mathcal{M}_g$ *(with* $g \geq 2$*) and* $\mathrm{Bun}_C$ *over* $(Sch/\Bbbk)$ *are limit preserving.*

*Proof.* To add. □

### D.6.2 Conrad–de Jong Approximation

In Artin Approximation (Theorem A.10.9), the initial data is an object over a noetherian complete local $\Bbbk$-algebra $\widehat{\mathcal{O}}_{S,s}$ which is assumed to be the completion of a finitely generated $\Bbbk$-algebra at a maximal ideal. We will now see that a similar approximation result still holds if this latter hypothesis is dropped and one approximates *both* the complete local ring and the object.

Recall also that if $(A, \mathfrak{m})$ is a local ring and $M$ is an $A$-module, then the *associated graded module of* $M$ is defined as $\mathrm{Gr}_{\mathfrak{m}}(M) = \bigoplus_{n \geq 0} \mathfrak{m}^n M/\mathfrak{m}^{n+1}M$; it is a graded module over the graded ring $\mathrm{Gr}_{\mathfrak{m}}(A)$.

**Theorem D.6.2** (Conrad–de Jong Approximation)**.** *Let* $\mathcal{X}$ *be a limit preserving prestack over* $\mathrm{Sch}/\Bbbk$*. Let* $(R, \mathfrak{m}_R)$ *be a noetherian complete local* $\Bbbk$*-algebra and let* $\xi \in \mathcal{X}(R)$*. Then for every integer* $N \geq 0$*, there exist*

(1) *an affine scheme* $\operatorname{Spec} A$ *of finite type over* $\Bbbk$ *and a* $\Bbbk$*-point* $u \in \operatorname{Spec} A$*,*

(2) *an object* $\eta \in \mathcal{X}(A)$*,*

(3) *an isomorphism* $\phi_{N+1}\colon R/\mathfrak{m}_R^{N+1} \cong A/\mathfrak{m}_u^{N+1}$*,*

(4) *an isomorphism of* $\xi|_{R/\mathfrak{m}_R^{N+1}}$ *and* $\eta|_{A/\mathfrak{m}_u^{N+1}}$ *via* $\phi_N$*, and*

(5) *an isomorphism* $\mathrm{Gr}_{\mathfrak{m}_R}(R) \cong \mathrm{Gr}_{\mathfrak{m}_u}(A)$ *of graded* $\Bbbk$*-algebras.*

The proof of this theorem will proceed by simultaneously approximating equations and relations defining $R$ and the object $\xi$. The statements (1)–(4) will be easily obtained as a consequence of Artin Approximation. A nice insight of Conrad and de Jong is that condition (5) can be ensured by Artin Approximation, and moreover that this condition suffices to imply the isomorphism of complete local $\Bbbk$-algebras in Artin Algebraization. As such, condition (5) takes the most work to establish.

We will need some preparatory results controlling the constant appearing in the Artin–Rees lemma.

**Definition D.6.3** (Artin–Rees Condition)**.** Let $(A, \mathfrak{m})$ be a noetherian local ring. Let $\varphi\colon M \to N$ be a morphism of finite $A$-modules. Let $c \geq 0$ be an integer. We say that $(\mathsf{AR})_c$ *holds for* $\varphi$ if

$$\varphi(M) \cap \mathfrak{m}^n N \subset \varphi(\mathfrak{m}^{n-c} M), \quad \forall n \geq c.$$

The Artin–Rees lemma implies that $(\mathsf{AR})_c$ holds for $\varphi$ if $c$ is sufficiently large; see [AM69, Prop. 10.9] or [Eis95, Lem. 5.1].

**Lemma D.6.4.** *Let $(A, \mathfrak{m})$ be a noetherian local ring. Let*

$$L \xrightarrow{\alpha} M \xrightarrow{\beta} N \qquad \text{and} \qquad L' \xrightarrow{\alpha'} M \xrightarrow{\beta'} N$$

*be two complexes of finite $A$-modules. Let $c$ be a positive integer. Assume that*

*(a) the first sequence is exact,*

*(b) the complexes are isomorphic modulo $\mathfrak{m}^{c+1}$, and*

*(c) $(\mathsf{AR})_c$ holds for $\alpha$ and $\beta$.*

*Then there exists an isomorphism $\mathrm{Gr}_{\mathfrak{m}}(\operatorname{coker}\beta) \to \mathrm{Gr}_{\mathfrak{m}}(\operatorname{coker}\beta')$ of graded $\mathrm{Gr}_{\mathfrak{m}}(A)$-modules.*

*Proof.* The proof, while technical, is rather straightforward. First, by taking free presentations of $L$ and $L'$, we can assume that $L = L'$. One shows that $(\mathsf{AR})_c$ holds for $\beta'$ and that the second sequence is exact. Then one establishes the equality

$$\mathfrak{m}^{n+1}N + \beta(M) \cap \mathfrak{m}^n N = \mathfrak{m}^{n+1}N + \beta'(M) \cap \mathfrak{m}^n N$$

by using that $(\mathsf{AR})_c$ holds for $\beta$ to show the containment "$\subset$" and then using $(\mathsf{AR})_c$ holds for $\beta'$ to get the other containment. The statement then follows from the description $\mathrm{Gr}_{\mathfrak{m}}(\operatorname{coker}\beta)_n = \mathfrak{m}^n N/(\mathfrak{m}^{n+1}N + \beta(M) \cap \mathfrak{m}^n N)$ and the similar description of $\mathrm{Gr}_{\mathfrak{m}}(\operatorname{coker}\beta')_n$. For details, see [CJ02, §3] and [SP, Tag 07VF]. □

*Proof of Conrad–de Jong Approximation (Theorem D.6.2).* Since $\mathcal{X}$ is limit preserving and $R$ is the colimit of its finitely generated $\Bbbk$-subalgebras, there is an affine scheme $V = \operatorname{Spec} B$ of finite type over $\Bbbk$ and an object $\gamma$ of $\mathcal{X}$ over $V$ together with a 2-commutative diagram

$$\begin{tikzcd} \operatorname{Spec} R \arrow[r] \arrow[rr, bend left, "\xi"] & V \arrow[r, "\gamma"] & \mathcal{X}. \end{tikzcd}$$

Let $v \in V$ be the image of the maximal ideal $\mathfrak{m} \subset R$. After adding generators to the ring $B$ if necessary, we can assume that the composition $\widehat{\mathcal{O}}_{V,v} \to R \to R/\mathfrak{m}^2$ is surjective. This implies that $\widehat{\mathcal{O}}_{V,v} \to R$ is surjective by Lemma A.10.15. The goal now is to simultaneously approximate over $V$ the equations and relations defining the closed immersion $\operatorname{Spec} R \hookrightarrow \operatorname{Spec} \widehat{\mathcal{O}}_{V,v}$ and the object $\xi$. To accomplish this goal, we choose a resolution

$$\widehat{\mathcal{O}}_{V,v}^{\oplus r} \xrightarrow{\widehat{\alpha}} \widehat{\mathcal{O}}_{V,v}^{\oplus s} \xrightarrow{\widehat{\beta}} \widehat{\mathcal{O}}_{V,v} \to R \to 0 \tag{D.6.1}$$

as $\widehat{\mathcal{O}}_{V,v}$-modules and consider the functor

$$\begin{aligned} F\colon (\mathrm{Sch}/V) &\to \mathrm{Sets} \\ (T \to V) &\mapsto \big\{\text{complexes } \mathcal{O}_T^{\oplus r} \xrightarrow{\alpha} \mathcal{O}_T^{\oplus s} \xrightarrow{\beta} \mathcal{O}_T\big\}. \end{aligned}$$

It is not hard to check that this functor is limit preserving. The resolution in (D.6.1) yields an element of $F(\widehat{\mathcal{O}}_{V,v})$. Applying Artin Approximation (Theorem A.10.9) yields an étale morphism $(V' = \operatorname{Spec} B', v') \to (V, v)$ and an element

$$(B'^{\oplus r} \xrightarrow{\alpha'} B'^{\oplus s} \xrightarrow{\beta'} B') \in F(V') \tag{D.6.2}$$

such that $\alpha', \beta'$ are equal to $\widehat{\alpha}, \widehat{\beta}$ modulo $\mathfrak{m}^{N+1}$.

Let $U = \operatorname{Spec} A \hookrightarrow \operatorname{Spec} B' = V'$ be the closed subscheme defined by $\operatorname{im} \beta'$ and let $u = v' \in U$. Consider the composition

$$\eta\colon U \hookrightarrow V' \to V \xrightarrow{\gamma} \mathcal{X}$$

As $R = \operatorname{coker} \widehat{\beta}$ and $A = \operatorname{coker} \beta'$, we have an isomorphism $R/\mathfrak{m}^{N+1} \cong A/\mathfrak{m}_u^{N+1}$ together with an isomorphism of $\xi|_{R/\mathfrak{m}^{N+1}}$ and $\eta|_{A/\mathfrak{m}_u^{N+1}}$. This gives statements (1)–(4).

To establish (5), we need to show that there are isomorphisms $\mathfrak{m}^n/\mathfrak{m}^{n+1} \cong \mathfrak{m}_u^n/\mathfrak{m}_u^{n+1}$. For $n \leq N$, this is guaranteed by the isomorphism $R/\mathfrak{m}^{N+1} \cong A/\mathfrak{m}_u^{N+1}$. On the other hand, for $n \gg 0$, this can be seen to be a consequence of the Artin–Rees lemma. To handle the middle range of $n$, we need to control the constant appearing in the Artin–Rees lemma. First, note that before we applied Artin Approximation, we could have increased $N$ to ensure that $(\mathsf{AR})_N$ holds for $\widehat{\alpha}$ and $\widehat{\beta}$. We are thus free to assume this. Now statement (5) follows directly if we apply Lemma D.6.4 to the exact complex $\widehat{\mathcal{O}}_{V,v}^{\oplus r} \xrightarrow{\widehat{\alpha}} \widehat{\mathcal{O}}_{V,v}^{\oplus s} \xrightarrow{\widehat{\beta}} \widehat{\mathcal{O}}_{V,v}$ of (D.6.1) and the complex $\widehat{\mathcal{O}}_{V,v}^{\oplus r} \xrightarrow{\widehat{\alpha}'} \widehat{\mathcal{O}}_{V,v}^{\oplus s} \xrightarrow{\widehat{\beta}'} \widehat{\mathcal{O}}_{V,v}$ obtained by restricting (D.6.2) to $F(\widehat{\mathcal{O}}_{V,v})$.

See also [CJ02] and [SP, Tag 07XB]. □

**Exercise D.6.5.** Show that Conrad–de Jong Approximation implies Artin Approximation.

### D.6.3 Artin Algebraization

Artin Algebraization has a stronger conclusion than Artin Approximation or Conrad–de Jong Approximation in that no approximation is necessary. It guarantees the existence of an object $\eta$ over a pointed affine scheme $(\operatorname{Spec} A, u)$ of finite type over $\Bbbk$, which agrees with the given effective formal deformation $\xi$ to all orders. To ensure this, we need to impose that $\xi$ is *versal at* $u$, i.e. that the restrictions $\xi_n = \xi|_{A/\mathfrak{m}_u^{n+1}}$ define a versal formal deformation $\{\xi_n\}$ over $A$ (Definition D.3.5).

**Theorem D.6.6** (Artin Algebraization)**.** *Let $\mathcal{X}$ be a limit preserving prestack over* $\mathrm{Sch}/\Bbbk$*. Let $(R, \mathfrak{m})$ be a noetherian complete local $\Bbbk$-algebra and $\xi \in \mathcal{X}(R)$ be an effective versal formal deformation. There exist*

*(1) an affine scheme* $\operatorname{Spec} A$ *of finite type over $\Bbbk$ and a $\Bbbk$-point $u \in$* $\operatorname{Spec} A$*;*

*(2) an object $\eta \in \mathcal{X}(A)$;*

*(3) an isomorphism $\alpha\colon R \xrightarrow{\sim} \widehat{A}_{\mathfrak{m}_u}$ of $\Bbbk$-algebras; and*

*(4) a compatible family of isomorphisms $\xi|_{R/\mathfrak{m}^{n+1}} \cong \eta|_{A/\mathfrak{m}_u^{n+1}}$ (under the identification $R/\mathfrak{m}^{n+1} \cong A/\mathfrak{m}_u^{n+1}$) for $n \geq 0$.*

**Remark D.6.7.** If $\mathcal{X}$ is an algebraic stack locally of finite type over $\Bbbk$, then there exists an isomorphism $\xi \cong \eta|_{\widehat{A}_{\mathfrak{m}_u}}$.

**Remark D.6.8.** If $R$ is known to be the completion of a finitely generated $\Bbbk$-algebra, this theorem can be viewed as an easy consequence of Artin Approximation. Indeed, one applies Artin Approximation with $N = 1$ and then uses versality to obtain compatible maps $R \to A/\mathfrak{m}_u^{n+1}$ and therefore a map $R \to \widehat{A}_{\mathfrak{m}_u}$ which is an isomorphism modulo $\mathfrak{m}^2$. As $R$ and $\widehat{A}_{\mathfrak{m}_u}$ are abstractly isomorphic, the homomorphism $R \to \widehat{A}_{\mathfrak{m}_u}$ is an isomorphism (Lemma A.10.15) and the statement follows. The argument in the general case is analogous, except we use Conrad–de Jong Approximation instead of Artin Approximation.

*Proof of Artin Algebraization (Theorem D.6.6).* Applying Conrad–de Jong Approximation (Theorem D.6.2) with $N = 1$, we obtain an affine scheme $\operatorname{Spec} A$ of finite type over $\Bbbk$ with a $\Bbbk$-point $u \in \operatorname{Spec} A$, an object $\eta \in \mathcal{X}(A)$, an isomorphism $\phi_2 \colon \operatorname{Spec} A/\mathfrak{m}_u^2 \to \operatorname{Spec} R/\mathfrak{m}^2$, an isomorphism $\alpha_2 \colon \xi|_{R/\mathfrak{m}^2} \to \eta|_{A/\mathfrak{m}_u^2}$, and an isomorphism $\operatorname{Gr}_{\mathfrak{m}}(R) \cong \operatorname{Gr}_{\mathfrak{m}_u}(A)$ of graded $\Bbbk$-algebras. We claim that $\phi_2$ and $\alpha_2$ can be extended inductively to a compatible family of morphisms $\phi_n \colon \operatorname{Spec} A/\mathfrak{m}_u^{n+1} \to \operatorname{Spec} R$ and isomorphisms $\alpha_n \colon \xi|_{A/\mathfrak{m}_u^{n+1}} \to \eta|_{A/\mathfrak{m}_u^{n+1}}$. Given $\phi_n$ and $\alpha_n$, versality of $\xi$ implies that there is a lift $\phi_{n+1}$ filling in the commutative diagram

$$\begin{tikzcd}
\operatorname{Spec} A/\mathfrak{m}_u^n \arrow[r, "\phi_n"] \arrow[d] & \operatorname{Spec} R \arrow[d, "\xi"] \\
\operatorname{Spec} A/\mathfrak{m}_u^{n+1} \arrow[r, "\eta|_{A/\mathfrak{m}_u^{n+1}}"'] \arrow[ur, dashed, "\phi_{n+1}"'] & \mathcal{X},
\end{tikzcd}$$

which establishes the claim. By taking the limit, we have a homomorphism $\widehat{\phi} \colon R \to \widehat{A}_{\mathfrak{m}_u}$ which is surjective since $\phi_2$ is surjective (Lemma A.10.15). On the other hand, for each $n$ the $\Bbbk$-vector spaces $\mathfrak{m}^N/\mathfrak{m}^{N+1}$ and $\mathfrak{m}_u^N/\mathfrak{m}_u^{N+1}$ have the same dimension. This implies that $\widehat{\phi}$ is an isomorphism.

See also [Art69b, Thm. 1.6] and [CJ02, §4], where the statement is established more generally when $\mathcal{X}$ is defined over a scheme $S$ whose local rings are $G$-rings where it is required that $\operatorname{Spec} R/\mathfrak{m} \xrightarrow{\xi_0} \mathcal{X} \to S$ be of finite type. □

## D.7 Artin's Axioms for Algebraicity

A spectacular application of Artin Algebraization is a criterion—often verifiable in practice—ensuring that a given stack is algebraic. This is called Artin's Axioms for Algebraicity and we provide two versions below Theorems D.7.1 and D.7.4. This foundational result was proved by Artin in the very same paper [Art74] where he introduced algebraic stacks.

The first version can be proved easily using Artin Algebraization.

**Theorem D.7.1.** *(Artin's Axioms for Algebraicity—first version) Let $\mathcal{X}$ be a stack over $\Bbbk$. Then $\mathcal{X}$ is an algebraic stack locally of finite type over $\Bbbk$ if and only if the following conditions hold:*

(1) *(Limit preserving) The stack $\mathcal{X}$ is limit preserving over* $\mathrm{Sch}/\Bbbk$, *i.e. for every system $B_\lambda$ of $\Bbbk$-algebras, the functor*

$$\operatorname{colim} \mathcal{X}(B_\lambda) \to \mathcal{X}(\operatorname{colim} B_\lambda)$$

*is an equivalence of categories.*

(2) *(Representability of the diagonal) The diagonal $\mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is representable.*

(3) *(Existence of versal formal deformations) Every $x_0 \in \mathcal{X}(\Bbbk)$ has a versal formal deformation $\{x_n\}$ over a noetherian complete local $\Bbbk$-algebra $(R, \mathfrak{m})$ with residue field $\Bbbk$.*

(4) *(Effectivity) For every noetherian complete local $\Bbbk$-algebra $(R, \mathfrak{m})$ with residue field $\Bbbk$, the natural functor*

$$\mathcal{X}(\operatorname{Spec} R) \to \varprojlim \mathcal{X}(\operatorname{Spec} R/\mathfrak{m}^n)$$

*is an equivalence of categories.*

(5) *(Openness of versality) For every morphism $U \to \mathcal{X}$ from a finite type $\Bbbk$-scheme which is versal at $u \in U(\Bbbk)$ (i.e. the formal deformation $\{\operatorname{Spec} \widehat{\mathcal{O}}_{U,u}/\mathfrak{m}_u^{n+1} \to \mathcal{X}\}$ is versal), there exists an open neighborhood $V$ of $u$ such that $U \to \mathcal{X}$ is versal at every $\Bbbk$-point of $V$.*

*Proof.* We first note that for a representable and locally of finite type morphism $U \to \mathcal{X}$ from a finite type $\Bbbk$-scheme $U$, the Infinitesimal Lifting Criterion for Smoothness (Smooth Equivalences A.3.1, Theorem 3.7.1) implies that $U \to \mathcal{X}$ is smooth if and only if it is versal at all $\Bbbk$-points $u \in U$. Indeed, this is clear when $U \to \mathcal{X}$ is representable by schemes, and the general case follows as one can see that both properties are étale-local on $U$.

For ($\Rightarrow$), (1) holds by Exercise 3.3.31, (2) holds by Theorem 3.2.1 and (4) holds by Exercise D.4.10. If $U \to \mathcal{X}$ is a morphism from a finite type $\Bbbk$-scheme, then it is necessarily representable and locally of finite type. By using the above equivalence between versality and smoothness, (3) holds by choosing a smooth presentation $U \to \mathcal{X}$ and a preimage $u \in U(\Bbbk)$ of $x_0$ and taking the formal deformation $\{\operatorname{Spec} \mathcal{O}_{U,u}/\mathfrak{m}_u^{n+1} \to \mathcal{X}\}$, and (5) holds by openness of smoothness.

For the converse, we first note that representability of the diagonal, i.e. condition (2), implies that every morphism $U \to \mathcal{X}$ from a scheme $U$ is representable and the limit preserving property (1) implies that $U \to \mathcal{X}$ is locally of finite type. For every object $x_0 \in \mathcal{X}(\Bbbk)$, we will construct a smooth morphism $U \to \mathcal{X}$ from a scheme and a preimage $u \in U(\Bbbk)$ of $x_0$. Conditions (3)–(4) guarantee that there exists an effective versal formal deformation $\widehat{x}\colon \operatorname{Spec} R \to \mathcal{X}$ of $x_0$ where $(R, \mathfrak{m})$ is a noetherian complete local $\Bbbk$-algebra with residue field $\Bbbk$. By Artin Algebraization (Theorem D.6.6), there exists a finite type $\Bbbk$-scheme $U$, a point $u \in U(\Bbbk)$, a morphism $p\colon U \to \mathcal{X}$, an isomorphism $R \cong \widehat{\mathcal{O}}_{U,u}$ and compatible isomorphisms $p|_{R/\mathfrak{m}^{n+1}} \xrightarrow{\sim} \widehat{x}|_{R/\mathfrak{m}^{n+1}}$. By (5), we can replace $U$ with an open neighborhood of $u$ so that $U \to \mathcal{X}$ is versal at every $\Bbbk$-point of $U$. By the equivalence in the first paragraph, we have obtained a smooth morphism $(U, u) \to (\mathcal{X}, x_0)$.

See also [Art74], [LMB00, Cor. 10.11] and [SP, Tag 07Y4] where the result is established more generally. $\square$

**Remark D.7.2.** In practice, condition (1)–(4) are often easy to verify directly with (3) a consequence of Rim–Schlessinger's Criteria (Theorem D.3.11) and (4) a consequence of Grothendieck's Existence Theorem (D.4.4). Also note that (2) can sometimes be established by applying the theorem to the diagonal $\mathcal{X} \to \mathcal{X} \times \mathcal{X}$, i.e. to the Isom sheaves $\mathrm{Isom}_T(x, y)$ of objects $x, y \in \mathcal{X}(T)$ over a scheme $T$. In some cases, Condition (5) can be checked directly, while for more general moduli problems, it is often a consequence of a well-behaved deformation and obstruction theory, as will be explained in the next section.

### D.7.1 Refinements of Artin's Axioms

We state a refinement of Artin's Axioms for Algebraicity that is often easier to verify in practice. To formulate the statements, we will need a bit of notation. Let $\xi \in \mathcal{X}(A)$ be an object over a finitely generated $\Bbbk$-algebra $A$. Let $M$ be a finite $A$-module and denote by $A[M]$ the ring $A \oplus M$ defined by $M^2 = 0$. Let $\mathrm{Def}_\xi(M)$ the set of isomorphism classes of diagrams

$$\begin{tikzcd} \operatorname{Spec} A \arrow[r, "\xi"] \arrow[d, hook] & \mathcal{X} \\ \operatorname{Spec} A[M], \arrow[ur, dashed, "\eta"] & \end{tikzcd}$$

where an isomorphism of two extensions $\eta, \eta' \colon \operatorname{Spec} A[M] \to \mathcal{X}$ is by definition an isomorphism $\eta \xrightarrow{\sim} \eta'$ in $\mathcal{X}(A[M])$ restricting to the identity on $\xi$. Let $\mathrm{Aut}_\xi(M)$ be the group of automorphisms of the trivial deformation $\xi' \colon \operatorname{Spec} A[M] \to \operatorname{Spec} A \to \mathcal{X}$. Note that when $\xi \in \mathcal{X}(\Bbbk)$, then $\mathrm{Def}_\xi(\Bbbk)$ is precisely the tangent space of $\mathcal{X}$ at $\xi$ and is identified with $T\mathcal{X}_\xi = \mathcal{X}_\xi(\Bbbk[\epsilon])/\sim$ of the local deformation prestack at $\xi$ while $\mathrm{Aut}_\xi(\Bbbk)$ is the group of infinitesimal automorphism of $\xi$ and is identified with the kernel $\mathrm{Aut}_{\mathcal{X}(\Bbbk[\epsilon])}(\xi') \to \mathrm{Aut}_{\mathcal{X}(\Bbbk)}(\xi)$.

**Lemma D.7.3.** *Suppose that $\mathcal{X}$ is a prestack over* $\mathrm{Sch}/\Bbbk$ *satisfying the strong homogeneity condition* ($\mathrm{RS}_4^*$)*. Let $\xi \in \mathcal{X}(A)$ be an object over a finitely generated $\Bbbk$-algebra $A$.*

*(1) For every $A$-module $M$,* $\mathrm{Def}_\xi(M)$ *and* $\mathrm{Aut}_\xi(M)$ *are naturally $A$-modules, and the functors*

$$\mathrm{Aut}_\xi(-)\colon \mathrm{Mod}(A) \to \mathrm{Mod}(A)$$
$$\mathrm{Def}_\xi(-)\colon \mathrm{Mod}(A) \to \mathrm{Mod}(A)$$

*are $A$-linear.*

*(2) Consider a surjection $B \twoheadrightarrow A$ in* $\mathrm{Art}_\Bbbk$ *with square-zero kernel $I$, and let* $\mathrm{Lift}_\xi(B)$ *be the set of morphisms $\xi \to \eta$ over* $\operatorname{Spec} A \to \operatorname{Spec} B$ *where $\xi \xrightarrow{\alpha} \eta$ is declared equivalent to $\xi \xrightarrow{\alpha'} \eta'$ if there is an isomorphism $\beta\colon \eta \to \eta'$ such that $\alpha' = \beta \circ \alpha$. There is an action of* $\mathrm{Def}_\xi(I)$ *on* $\mathrm{Lift}_\xi(B)$ *which is functorial in $B$ and $I$. Assuming* $\mathrm{Lift}_\xi(B)$ *is non-empty, this action is free and transitive.*

*Proof.* This can be established by arguing as in Lemma D.3.13. For instance, scalar multiplication by $x \in A$ is defined by pulling back along the morphism $\operatorname{Spec} A[M] \to \operatorname{Spec} A[M]$ induced by the $A$-algebra homomorphism $A[M] \to A[M], a+m \mapsto a+xm$. Condition ($\mathrm{RS}_4^*$) implies that the functor $\mathcal{X}(A[M\oplus M]) \to \mathcal{X}(A[M]) \times_{\mathcal{X}(A)} \mathcal{X}(A[M])$ is an equivalence. Addition $M \oplus M \to M$ induces an $A$-algebra homomorphism $A[M \oplus M] \to A[M]$ and thus a functor

$$\mathcal{X}(A[M]) \times_{\mathcal{X}(A)} \mathcal{X}(A[M]) \cong \mathcal{X}(A[M \oplus M]) \to \mathcal{X}(A[M])$$

which defines addition on $\mathrm{Def}_\xi(M)$ and $\mathrm{Aut}_\xi(M)$. $\square$

**Theorem D.7.4** (Artin's Axioms for Algebraicity—second version)**.** *A stack $\mathcal{X}$ over $(\mathrm{Sch}/\Bbbk)_{\text{ét}}$ is an algebraic stack locally of finite type over $\Bbbk$ if the following conditions hold:*

($\mathrm{AA}_1$) *(Limit preserving) The stack $\mathcal{X}$ is limit preserving;*

($\mathrm{AA}_2$) *(Representability of the diagonal) The diagonal $\mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is representable;*

($\mathrm{AA}_3$) *(Finiteness of tangent spaces) For every object $\xi \colon \operatorname{Spec} \Bbbk \to \mathcal{X}$, $\mathrm{Def}_\xi(\Bbbk)$ is a finite dimensional $\Bbbk$-vector space;*

($\mathrm{AA}_4$) *(Strong homogeneity) For every $\Bbbk$-algebra homomorphism $B_0 \to A_0$ and surjection $A \twoheadrightarrow A_0$ of $\Bbbk$-algebras with square-zero kernel, the functor*

$$\mathcal{X}(B_0 \times_{A_0} A) \to \mathcal{X}(B_0) \times_{\mathcal{X}(A_0)} \mathcal{X}(A)$$

*is an equivalence, i.e. Condition ($\mathrm{RS}_4^*$) holds;*

($\mathrm{AA}_5$) *(Effectivity) For every noetherian complete local $\Bbbk$-algebra $(R, \mathfrak{m})$, the natural functor*

$$\mathcal{X}(\operatorname{Spec} R) \to \varprojlim \mathcal{X}(\operatorname{Spec} R/\mathfrak{m}^n)$$

*is an equivalence of categories;*

($\mathrm{AA}_6$) *(Existence of an obstruction theory) For every object $\xi \in \mathcal{X}(A)$ over a finitely generated $\Bbbk$-algebra $A$, there exists the following data*

*(a) there is an $A$-linear functor*

$$\mathrm{Ob}_\xi(-) \colon \mathrm{Mod}(A) \to \mathrm{Mod}(A),$$

*and for every surjection $B \to A$ with square-zero kernel $I$, there is an element $\mathrm{ob}_\xi(B) \in \mathrm{Ob}_\xi(I)$ such that there is an extension*

$$\begin{tikzcd} \operatorname{Spec} A \arrow[r, "\xi"] \arrow[d, hook] & \mathcal{X} \\ \operatorname{Spec} B \arrow[ur, dashed] & \end{tikzcd}$$

*if and only if $\mathrm{ob}_\xi(B) = 0$, and*

*(b) for every composition $B \to B' \to A$ of $\Bbbk$-algebras such that $B \twoheadrightarrow A$ and $B' \twoheadrightarrow A$ are surjective with square-zero kernels $I$ and $I'$, the image of $\mathrm{ob}_\xi(B)$ under $\mathrm{Ob}_\xi(I) \to \mathrm{Ob}_\xi(I')$ is $\mathrm{ob}_\xi(B')$; and*

($\mathrm{AA}_7$) *(Coherent deformation theory) For every object $\xi \in \mathcal{X}(A)$ over a $\Bbbk$-algebra $A$, the functors $\mathrm{Def}_\xi(-)$ and $\mathrm{Ob}_\xi(-)$ commute with products.*

*Moreover ($\mathrm{AA}_2$) can be removed if we replace ($\mathrm{AA}_3$) and ($\mathrm{AA}_7$) with:*

($\mathrm{AA}_{3'}$) *For every object $\xi \colon \operatorname{Spec} \Bbbk \to \mathcal{X}$, $\mathrm{Aut}_\xi(\Bbbk)$ and $\mathrm{Def}_\xi(\Bbbk)$ are finite dimensional $\Bbbk$-vector spaces; and*

($\mathrm{AA}_{7'}$) *For every object $\xi \in \mathcal{X}(A)$ over a $\Bbbk$-algebra $A$, the functors $\mathrm{Aut}_\xi(-)$, $\mathrm{Def}_\xi(-)$ and $\mathrm{Ob}_\xi(-)$ commute with products.*

*Proof.* Conditions ($\mathrm{AA}_3$)–($\mathrm{AA}_4$) above allow us to apply Rim–Schlessinger's Criteria (Theorem D.3.11) to deduce the existence of versal formal deformations, i.e. Condition D.7.1(3) holds. It remains to check openness of versality, i.e. Condition D.7.1(5), in order to apply the first version (Theorem D.7.1) to establish this version.

Let $\xi_0 \colon U_0 \to \mathcal{X}$ be a morphism from an affine scheme $U_0 = \operatorname{Spec} B_0$ of finite type over $\Bbbk$ which is versal at a point $u_0 \in U_0(\Bbbk)$. By (AA$_1$)–(AA$_2$), the morphism $\xi_0 \colon U_0 \to \mathcal{X}$ is representable and locally of finite type. Let $\Sigma = \{u \in U_0(\Bbbk) \mid \xi_0 \colon U_0 \to \mathcal{X}$ is not versal at $u\}$. If openness of versality does not hold, then $u_0 \in \overline{\Sigma}$ and there exists a countably infinite subset $\Sigma' = \{u_1, u_2, \ldots\} \subset \Sigma$ of distinct points with $u_0 \in \overline{\Sigma'}$.

*Step 1.* We claim that there exists a commutative diagram

$$\begin{tikzcd}
U_0 \arrow[r, hook] \arrow[d, "\xi_0"'] & U_1 \arrow[r, hook] \arrow[dl, "\xi_1"] & U_2 \arrow[r, hook] \arrow[dll, "\xi_2"] & \cdots \\
\mathcal{X}
\end{tikzcd}$$

where each closed immersion $U_{n-1} \hookrightarrow U_n$ is defined by a short exact sequence

$$0 \to \kappa(u_n) \to \mathcal{O}_{U_n} \to \mathcal{O}_{U_{n-1}} \to 0,$$

and for each $n$ and open neighborhood $W \subset U_n$ of $u_n$, the restriction $\xi_n|_W$ is not the trivial deformation of $\xi_0|_{W \cap U_0}$, i.e. there is no morphism $r \colon \xi_n|_W \to \xi_0|_{W \cap U_0}$ such that $\xi_n|_W \xrightarrow{r} \xi_0|_{W \cap U_0} \to \xi_n|_W$ is the identity. Note that for each $m \geq n$, $U_n \hookrightarrow U_m$ is a closed immersion which is square-zero (i.e. $\ker(\mathcal{O}_{U_m} \to \mathcal{O}_{U_n})$ is square-zero). We will inductively construct $U_n = \operatorname{Spec} B_n$ and $\xi_n \in \mathcal{X}(U_n)$, Since $\xi_0 \colon U_0 \to \mathcal{X}$ and $\xi_{n-1} \colon U_{n-1} \to \mathcal{X}$ are isomorphic in an open neighborhood of $u_n$, the morphism $\xi_{n-1} \colon U_{n-1} \to \mathcal{X}$ is also not versal at $u_n$. By definition of versality (using Remark D.3.7) there exists a surjection $A \to A_0$ in $\mathrm{Art}_\Bbbk$ with $\ker(A \to A_0) = \Bbbk$ and a commutative diagram

$$\begin{tikzcd}
\operatorname{Spec} A_0 \arrow[r] \arrow[d, hook] & U_{n-1} \arrow[d, "\xi_{n-1}"] \\
\operatorname{Spec} A \arrow[r] \arrow[ur, dashed, "\nexists"] & \mathcal{X},
\end{tikzcd} \tag{D.7.1}$$

such that $u_n$ is the image of $\operatorname{Spec} A_0 \to U_{n-1}$, which does not admit a lift $\operatorname{Spec} A \to U_{n-1}$. Using strong homogeneity (AA$_4$), there exists an extension of the commutative diagram

$$\begin{tikzcd}
\operatorname{Spec} A_0 \arrow[r] \arrow[d, hook] & U_{n-1} = \operatorname{Spec} B_{n-1} \arrow[d, hook] \arrow[ddr, bend left, "\xi_{n-1}"] & \\
\operatorname{Spec} A \arrow[r] \arrow[drr, bend right] & U_n = \operatorname{Spec}(A \times_{A_0} B_{n-1}) \arrow[dr, dashed, "\xi_n"] & \\
 & & \mathcal{X}
\end{tikzcd}$$

yielding an object $\xi_n$ over $U_n = \operatorname{Spec} B_n$ with $B_n := A \times_{A_0} B_{n-1}$. If $\xi_n$ were the trivial deformation of $\xi_0$ in an open neighborhood of $u_n$, then $\operatorname{Spec} A \to \mathcal{X}$ would be the trivial deformation of $\operatorname{Spec} A_0$ contradicting the obstruction to a lift of (D.7.1). Finally note that $\ker(B_n \to B_{n-1}) = \Bbbk$ since $\ker(A \to A_0) = \Bbbk$. This establishes the claim.

*Step 2.* Letting $\widehat{B} = \varprojlim B_n$ and $\widehat{U} = \operatorname{Spec} \widehat{B}$, we claim that there exists an object $\widehat{\xi} \in \mathcal{X}(\widehat{U})$ extending each $\xi_n \in \mathcal{X}(U_n)$. Let $M_n = \ker(B_n \to B_0)$ (noting that

$M_0 = 0$). Since $M_n^2 = 0$, we can view $M_n$ as a $B_0$-module. The $\Bbbk$-algebra

$$\widetilde{B} := \{(b_0, b_1, \ldots) \in \prod_{n\geq 0} B_n \mid \text{the image of each } b_n \text{ under } B_n \to B_0 \text{ is } b_0\}$$

has the following properties:

- The surjective $\Bbbk$-algebra homomorphism $\widetilde{B} \to B_0$ defined by $(b_i) \mapsto b_0$ has kernel $M := \prod_{n\geq 0} M_n$;
- The map $\widetilde{B} \to B_0[M]$ defined by $(b_0, b_1, b_2, \ldots) \mapsto (b_0, b_1-b_0, b_2-b_1, b_3-b_2, \ldots)$ is a surjective $\Bbbk$-algebra homomorphism with square-zero kernel;
- The composition $\widehat{B} \to \widetilde{B} \to B_0[M]$ induces a short exact sequence

$$\begin{tikzcd}
0 \arrow[r] & \ker(\widehat{B} \to B_0) \arrow[r] \arrow[d, equal] & \ker(\widetilde{B} \to B_0) \arrow[r] \arrow[d, equal] & \ker(B_0[M] \to B_0) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \varprojlim_{n\geq 0} M_n \arrow[r] & \prod_{n\geq 0} M_n \arrow[r] & \prod_{n\geq 0} M_n \arrow[r] & 0
\end{tikzcd}$$

$$(b_0, b_1, \ldots) \longmapsto (b_1 - b_0, b_2 - b_1, \ldots)$$

- There is an identification $\widehat{B} = \widetilde{B} \times_{B_0[M]} B_0$.

Since the lift $\xi_n \in \mathcal{X}(B_n)$ of $\xi_0$ exists for each $n$, $\mathrm{ob}_\xi(B_n) = 0 \in \mathrm{Ob}_\xi(M_n)$. By (AA$_6$)(b), the element $\mathrm{ob}_\xi(\widetilde{B})$ maps to $\mathrm{ob}_\xi(B_n)$ under $\mathrm{Ob}_\xi(M) \to \mathrm{Ob}_\xi(M_n)$. By (AA$_7$), the map $\mathrm{Ob}_\xi(M) \hookrightarrow \prod_n \mathrm{Ob}_\xi(M_n)$ is injective[3] and thus $\mathrm{ob}_\xi(\widetilde{B}) = 0 \in \mathrm{Ob}_\xi(M)$ which shows that there exists a lift $\widetilde{\xi} \in \mathcal{X}(\widetilde{B})$ of $\xi_0$.

The restrictions $\widetilde{\xi}|_{B_n}$ are not necessarily isomorphic to $\xi_n$. However, we may use the free and transitive action $\mathrm{Def}_\xi(M_n) = \mathrm{Lift}_\xi(B_0[M_n])$ on the non-empty set of liftings $\mathrm{Lift}_\xi(\widetilde{B_n})$ to find elements $t_n \in \mathrm{Def}_\xi(M_n)$ such that $\xi_n = t_n \cdot \widetilde{\xi}|_{B_n}$ (Lemma D.7.3). Since $\mathrm{Def}_\xi(M) \xrightarrow{\sim} \prod_n \mathrm{Def}_\xi(M_n)$ by (AA$_7$), there exists $\widetilde{t} \in \mathrm{Def}_\xi(M)$ mapping to $(t_n)$. After replacing $\widetilde{\xi}$ with $\widetilde{t} \cdot \widetilde{\xi}$, we can arrange that $\widetilde{\xi}|_{B_n}$ and $\xi_n$ are isomorphic for each $n$.

We now show that each restriction $\widetilde{\xi}|_{B_0[M_n]} \in \mathrm{Def}_\xi(M_n)$ under the composition $\widetilde{B} \to B_0[M] \to B_0[M_n]$ is the trivial deformation. Indeed, the map $M = \ker(\widetilde{B} \to B_0) \to \ker(B_0[M_n] \to B_0) = M_n$ induces a map $\mathrm{Def}_\xi(M) \to \mathrm{Def}_\xi(M_n)$ on deformation modules which under the identification $\mathrm{Def}_\xi(M) \xrightarrow{\sim} \prod_n \mathrm{Def}_\xi(M_n)$ of (AA$_7$) sends an element $(\eta_0, \eta_1, \ldots)$ to $(\eta_{n+1}|_{B_n} - \eta_n)$. The ring map $\widetilde{B} \to B_0[M_n]$ also induces a map $\mathrm{Lift}_\xi(\widetilde{B}) \to \mathrm{Lift}_\xi(B_0[M_n])$ which is equivariant with respect to $\mathrm{Def}_\xi(M) \to \mathrm{Def}_\xi(M_n)$. It follows that the image of $\widetilde{\xi}$ in $\mathrm{Lift}_\xi(B_0[M_n]) = \mathrm{Def}_\xi(M_n)$ is $\xi_{n+1}|_{B_n} - \xi_n = 0$.

The existence of $\widehat{\xi} \in \mathcal{X}(\widehat{B})$ extending $(\xi_n) \in \varprojlim \mathcal{X}(B_n)$ now follows from applying

[3]The hypotheses of (AA$_7$) can be weakened to only require the injectivity of $\mathrm{Ob}_\xi(M) \hookrightarrow \prod_n \mathrm{Ob}_\xi(M_n)$ although in practice one usually verifies that this map is bijective.

the identity $\widehat{B} = \widetilde{B} \times_{B_0[M]} B_0$ and strong homogeneity (AA$_4$) to the diagram

$$\begin{tikzcd}
\operatorname{Spec} B_0[M] \arrow[r] \arrow[d, hook] & \operatorname{Spec} B_0 \arrow[d, hook] \arrow[ddr, bend left, "\xi_0"] & \\
\operatorname{Spec} \widetilde{B} \arrow[r] \arrow[drr, bend right, "\widetilde{\xi}"'] & \operatorname{Spec} \widehat{B} \arrow[dr, dashed, "\widehat{\xi}"] & \\
& & \mathcal{X}
\end{tikzcd}$$

*Step 3.* We now use the versality of $\xi_0 \colon U_0 \to \mathcal{X}$ at $u_0$ to arrive at a contradiction. Since $\mathcal{X}$ is limit preserving (AA$_1$), there exists a finitely generated $\Bbbk$-subalgebra $B' \subset \widehat{B}$ and an object $\xi' \in \mathcal{X}(B')$ together with an isomorphism $\widehat{\xi} \xrightarrow{\sim} \xi|_{\widehat{B}}$. After possibly enlarging $B'$, we may assume that the composition $B' \hookrightarrow \widehat{B} \to B_0$ is surjective. There is thus a closed immersion $U_0 \hookrightarrow U' := \operatorname{Spec} B'$, and we can consider the commutative diagram

$$\begin{tikzcd}
U_0 \arrow[dr, "i"] \arrow[drr, hook, bend left] \arrow[ddr, bend right, "\mathrm{id}"'] & & \\
& U_0 \times_{\mathcal{X}} U' \arrow[r] \arrow[d] & U' = \operatorname{Spec} B' \arrow[d, "\xi'"] \\
& U_0 \arrow[r, "\xi_0"] & \mathcal{X}.
\end{tikzcd}$$

where the fiber product $U_0 \times_{\mathcal{X}} U'$ is an algebraic space locally of finite type over $\Bbbk$. Since $\xi_0 \colon U_0 \to \mathcal{X}$ is versal at $u_0$, it follows from the artinian version of the Infinitesimal Lifting Criterion for Smoothness (Smooth Equivalences A.3.1) that $U_0 \times_{\mathcal{X}} U' \to U'$ is smooth at $i(u_0)$. After replacing $U_0$ with an open affine neighborhood of $u_0$, $U'$ with the corresponding open and $\{u_1, u_2, \ldots\}$ with an infinite subsequence contained in this open, we can arrange that $U_0 \times_{\mathcal{X}} U' \to U'$ is smooth. The non-artinian version of the Infinitesimal Lifting Criterion for Smoothness implies the section of $U_0 \times_{\mathcal{X}} U' \to U'$ over $U_0$ extends to a global section $U' \to U_0 \times_{\mathcal{X}} U'$. This implies that $\xi'$ is the trivial deformation of $\xi_0$, contradicting our choice of $\xi' \colon U' \to \mathcal{X}$.

Our exposition follows [SP, Tag 0CYF] and [Hal17, Thm. A]. See also [Art74, Thm. 5.3] and [HR19a, Main Thm.]. □

**Remark D.7.5.** The converse of the theorem also holds. For the necessity of the conditions, we only need to check (AA$_3$), (AA$_4$), (AA$_6$) and (AA$_7$). Condition (AA$_3$) (finiteness of the tangent spaces) holds as $\mathcal{X}$ is of finite type over $\Bbbk$. The strong homogeneity condition (AA$_4$) holds by [SP, Tag 07WN]. Condition (AA$_6$) (existence of an obstruction theory) follows from the existence of a cotangent complex $L_{\mathcal{X}/\Bbbk}$ for $\mathcal{X}$ satisfying properties analogous to Theorem D.5.1; see [Ols06]. If $\xi \colon \operatorname{Spec} A \to \mathcal{X}$ is a morphism from a finitely generated $\Bbbk$-algebra $A$ and $I$ is an $A$-module, then we set $\operatorname{Ob}_\xi(I) := \operatorname{Ext}^1_A(\xi^* L_{\mathcal{X}/\Bbbk}, I)$. Property (AA$_6$)(b) holds as a consequence of [Ols06, Thm. 1.5], a generalization of [Ill71, III.2.2.4] (which was discussed in Remark D.5.12) from morphisms of schemes to representable morphisms of algebraic stacks. Finally, Condition (AA$_7$) ($\operatorname{Def}_\xi(-)$ and $\operatorname{Ob}_\xi(-)$ commutes with products) follows from cohomology and base change.

### D.7.2 Verifying Artin's Axioms

**Theorem D.7.6.** *Each of the stacks* $\mathrm{Hilb}^P(X)$, $\mathcal{M}_g$ *(with* $g \geq 2$*) and* $\mathrm{Bun}_C$ *over* $(Sch/\Bbbk)_{\text{ét}}$ *are algebraic stacks locally of finite type over* $\Bbbk$.

*Proof.* We check the conditions of Theorem D.7.4. Condition $(\mathrm{AA}_1)$ (limit preserving) was verified in Lemma D.6.1. For $(\mathrm{AA}_{3'})$, the finite dimensionality of the vector spaces $\mathrm{Def}_\xi(\Bbbk)$ and $\mathrm{Aut}_\xi(\Bbbk)$ for an object $\xi \in \mathcal{X}(\Bbbk)$ can be identified with:

- $\mathrm{H}^0(Z, N_{Z/X})$ and $\{0\}$ for $\xi = [Z \subset X] \in \mathrm{Hilb}^P(X)(\Bbbk)$ (Proposition D.1.4),
- $\mathrm{H}^1(C, T_C)$ and $\mathrm{H}^0(C, T_C)$ for $\xi = [C] \in \mathcal{M}_g(\Bbbk)$ (Lemma D.1.10 and Proposition D.1.11) and
- $\mathrm{Ext}^1_{\mathcal{O}_C}(E, E)$ and $\mathrm{Ext}^0_{\mathcal{O}_C}(E, E)$ for $\xi = [E] \in \mathrm{Bun}_C(\Bbbk)$ (Proposition D.1.15).

Condition $(\mathrm{AA}_4)$ (the strong homogeneity condition of $(\mathrm{RS}_4^*)$) was checked in Proposition D.3.14. Condition $(\mathrm{AA}_5)$ (effectivity) was checked in Corollary D.4.9 as a consequence of Grothendieck's Existence Theorem. For Condition $(\mathrm{AA}_6)$, we define obstruction theories as follows: for a finitely generated $\Bbbk$-algebra $A$ and an $A$-module $M$, we set

- $\mathrm{Ob}_\xi(M) := \mathrm{Ext}^1_{\mathcal{O}_Z}(I_Z/I_Z^2, M)$ for $\xi = [Z \subset X_A] \in \mathrm{Hilb}^P(X)(A)$ where $I_Z \subset \mathcal{O}_{X_A}$ is the sheaf of ideal defining $Z$.
- $\mathrm{Ob}_\xi(M) := \mathrm{H}^2(\mathcal{C}, T_{\mathcal{C}/A} \otimes_A M) = 0$ for $\xi = [\mathcal{C} \to \operatorname{Spec} A] \in \mathcal{M}_g(A)$, and
- $\mathrm{Ob}_\xi(M) := \mathrm{H}^2(C_A, \mathscr{E}nd_{\mathcal{O}_{C_A}}(E) \otimes_A M) = 0$ for $\xi = [E] \in \mathrm{Bun}_C(A)$.

Property $(\mathrm{AA}_6)$(a) holds for these obstruction theories as a consequence of Propositions D.2.2, D.2.6 and D.2.15; these results also show that $\mathrm{Aut}_\xi(M)$ and $\mathrm{Def}_\xi(M)$ are identified with the analogous cohomology groups. Condition $(\mathrm{AA}_{7'})$ ($\mathrm{Aut}_\xi(-)$, $\mathrm{Def}_\xi(-)$ and $\mathrm{Ob}_\xi(-)$ commutes with products) follows from Lemma D.7.7. □

**Lemma D.7.7.** *Let* $X \to \operatorname{Spec} A$ *be a proper morphism of noetherian schemes. Let* $E$ *and* $F$ *be coherent sheaves on* $X$ *with* $F$ *flat over* $A$. *Then the functors*

$$\begin{aligned} \mathrm{H}^i(X, F \otimes_A -)\colon \ \mathrm{Mod}(A) &\to \mathrm{Mod}(A) \qquad \textit{and} \\ \mathrm{Ext}^i_{\mathcal{O}_X}(E, F \otimes_A -)\colon \ \mathrm{Mod}(A) &\to \mathrm{Mod}(A) \end{aligned}$$

*commute with products.*

*Proof.* Since $F$ is flat over $A$, there is a perfect complex $K^\bullet$ of $A$-modules such that $\mathrm{H}^i(X, F \otimes_A -) \cong \mathrm{H}^i(K^\bullet \otimes_A -)$ (Theorem A.7.1). Write $K^d = A^{\oplus r_d}$. For every set of $A$-modules $\{M_\alpha\}$ we have an identification of complexes

$$\begin{tikzcd} 0 \arrow[r] & \prod_\alpha M_\alpha^{\oplus r_0} \arrow[r] \arrow[d, equal] & \prod_\alpha M_\alpha^{\oplus r_1} \arrow[r] \arrow[d, equal] & \cdots \arrow[r] & \prod_\alpha M_\alpha^{\oplus r_n} \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & (\prod_\alpha M_\alpha)^{\oplus r_0} \arrow[r] & (\prod_\alpha M_\alpha)^{\oplus r_1} \arrow[r] & \cdots \arrow[r] & (\prod_\alpha M_\alpha)^{\oplus r_n} \arrow[r] & 0. \end{tikzcd}$$

The top row is the product of the complexes $K^\bullet \otimes_A M_\alpha$ and its cohomology is identified with $\prod_\alpha \mathrm{H}^i(X, F \otimes_A M_\alpha)$ while the bottom row is $K^\bullet \otimes_A (\prod_\alpha M_\alpha)$ with cohomology groups $\mathrm{H}^i(X, F \otimes_A (\prod_\alpha M_\alpha))$. For the second statement, one needs to apply alternative versions of cohomology and base change (see [EGA, III.7.7.5], [SP, Tag 08JR] and [Hal14, Thm. E]). □

**Exercise D.7.8.** If $X \to S$ is a proper flat morphism of noetherian schemes, show that the stack $\underline{\mathrm{Coh}}(X/S)$, whose objects over an $S$-scheme $T$ are finitely presented, quasi-coherent sheaves on $X_T$ flat over $T$, is an algebraic stack locally of finite type over $S$.

# Appendix E

# Birational Geometry

## E.1 Birational geometry of surfaces

By a *surface*, we mean an integral scheme $X$ of pure dimension 2, which is either of finite type over an algebraically closed field $\Bbbk$ or of finite type over a complete DVR $R$ with algebraically closed residue field $\Bbbk$. In the latter case, we say that $X$ is smooth (resp. projective) if the structure morphism $X \to \operatorname{Spec} R$ is, and a curve $X_0 \subset X$ is by definition a curve contained in the central fiber.

**Theorem E.1.1** (Minimal Resolutions)**.** *Let $X$ be a surface. There exists a unique projective birational morphism $\pi\colon \widetilde{X} \to X$ from a smooth surface such that every other resolution $Y \to X$ factors as $Y \to \widetilde{X} \to X$ (or equivalently such that $K_{\widetilde{X}} \cdot E \geq 0$ for every $\pi$-exceptional curve $E$).*

*Proof.* See [Kol07, Thm. 2.16]. □

**Theorem E.1.2** (Embedded Resolutions of Curves in Surfaces)**.** *Let $X$ be a surface and $X_0 \subset X$ be a curve. There is a finite sequence of blow-ups at reduced points of $X_0$ yielding a projective birational morphism $\widetilde{X} \to X$ such that $\widetilde{X}$ is smooth and such that the preimage $\widetilde{X}_0$ of $X_0$ has set-theoretic normal crossings, i.e. $(\widetilde{X}_0)_{\mathrm{red}}$ is nodal.*

*Proof.* See [Har77, Thm V.3.9] and [Kol07, Thm. 1.47]. □

**Theorem E.1.3** (Structure Theorem of Birational Morphisms of Surfaces)**.** *Every projective birational morphism $f\colon X \to Y$ of smooth surfaces is the composition of blowing up smooth points.*

*Proof.* See [Har77, Thm V.5.5] and [Kol07, Thm 2.13]. □

**Theorem E.1.4** (Hodge Index Theorem for Exceptional Curves)**.** *Let $f\colon X \to Y$ be a projective and generically finite morphism of surfaces with $X$ smooth and $Y$ quasi-projective. Let $E_1, \ldots, E_n$ be the exceptional curves. Then the intersection form matrix $(E_i \cdot E_j)$ is negative-definite. In particular, $E_i^2 < 0$ for each $i$.*

*Proof.* See [Kol07, Thm 2.12]. □

**Theorem E.1.5** (Castelnuovo's Contraction Theorem)**.** *Let $X$ be a smooth projective surface and $E$ a smooth rational curve with $E^2 < 0$. Then there is a projective morphism $X \to Y$ to a projective surface and a point $y \in Y$ such that $f^{-1}(y) = E$ and $X \setminus E \to Y \setminus \{y\}$ is an isomorphism. If $E^2 < 0$, then $Y$ is smooth.*

*Proof.* See [Har77, Thm. V.5.7, Exer. V.5.2] and [Kol07, Thm. 2.14, Rmk. 2.15]. □

One can show that the process of repeatedly contracting smooth rational $-1$ curves in a smooth projective surface terminates (see [Har77, Thm 5.8]). Thus by applying Castelnuovo's Contractibility Criterion a finite number of times, one obtains:

**Corollary E.1.6** (Existence of Minimal Models)**.** *A smooth surface $X$ admits a projective birational morphism $X \to X_{\min}$ to a smooth surface such that every projective birational morphism $X_{\min} \to Y$ to a smooth surface is an isomorphism. In particular, $X_{\min}$ has no smooth rational $-1$ curves.*

## E.2 Positivity

The standard reference for this material is [Laz04a, Laz04b].

### E.2.1 Ample line bundles

Let $X$ be a proper scheme over an algebraically closed field $\Bbbk$. A line bundle $L$ on $X$ is *ample* if for some $m > 0$, $L^{\otimes m}$ is very ample, i.e. defines a closed embedding $|L^{\otimes m}|: X \hookrightarrow \mathbb{P}^N$ into projective space. Ampleness can be equivalently characterized by any of the following conditions:

- $L^{\otimes m}$ is very ample for $m \gg 0$,
- for every $x \in X$, there exists a section $s \in \Gamma(X, L)$ such that $X_s = \{s \neq 0\}$ is affine and contains $x$,
- for every coherent sheaf $F$, the tensor product $F \otimes L^{\otimes m}$ is base point free for $m \gg 0$, or
- for every coherent sheaf $F$ on $X$, the cohomology groups $\mathrm{H}^i(X, F \otimes L^{\otimes m}) = 0$ vanish for $i > 0$ and $m \gg 0$.

See [Har77, §II.7, III.5.3] or [SP, Tags 01PR and 0B5U].

**Proposition E.2.1** (Openness of Ampleness)**.** *Let $f\colon X \to Y$ be a proper, flat, and finitely presented morphism of schemes, and $L$ be a line bundle on $X$. If for some $y \in Y$, the restriction $L_y$ of $y$ to the fiber $X_y$ is ample (resp. very ample and $\mathrm{H}^i(X_y, L_y) = 0$ for $i > 0$), then there exists an open neighborhood $U \subset S$ of $s$ such that the restriction $L_U$ on $X_U$ is relatively ample (resp. relatively very ample) over $U$. In particular, for all $u \in U$, $L_u$ is ample (resp. very ample) on $X_u$.*

*Proof.* If $L_y$ is ample on $X_y$, then for $n \gg 0$, $L_y^{\otimes n}$ is very ample and $\mathrm{H}^i(X_y, L_y^{\otimes n}) = 0$ for $i > 0$. It therefore suffices to handle the very ample case. By Cohomology and Base Change (A.7.5), after replacing $Y$ with an open neighborhood of $y$, $f_*L$ is a vector bundle and the comparison map $f_*L \otimes \kappa(t) \to \mathrm{H}^i(X_t, L_t)$ is an isomorphism for $t \in Y$. By further replacing $Y$ with an affine open neighborhood, we can arrange that $\mathrm{H}^0(X, L)$ is freely generated by sections $s_0, \ldots, s_n$ that restrict to a basis in $\mathrm{H}^0(X_y, L_y)$. The vanishing locus $V := V(s_0, \ldots, s_n) \subset X$ is closed and disjoint from $X_y$. By replacing $Y$ with an affine open neighborhood of $s$ contained in $Y \setminus f(V)$, we may assume that the sections $s_i$ generate $L$ and that they define a morphism $g\colon X \to \mathbb{P}^n_Y$ over $Y$ restricting to a closed immersion $g_y\colon X_y \hookrightarrow \mathbb{P}^n_{\kappa(y)}$. By upper semi-continuity of fiber dimension, there is a closed locus $Z \subset \mathbb{P}^n_Y$ consisting of points $z$ such that $\dim g^{-1}(z) > 0$. Since $Z$ is disjoint from $\mathbb{P}^n_{\kappa(y)}$, we may shrink $Y$ further so that $g\colon X \to \mathbb{P}^n_Y$ is quasi-finite, and hence finite as $g$ is proper. The cokernel $\mathcal{O}_{\mathbb{P}^n_Y} \to g_*\mathcal{O}_X$ is coherent and its support is a closed subscheme of $\mathbb{P}^n$ disjoint from $\mathbb{P}^n_{\kappa(y)}$. By shrinking $Y$ further, we may arrange that $g\colon X \to \mathbb{P}^n_Y$ is a closed immersion and hence $L = g^*\mathcal{O}_{\mathbb{P}^n_Y}(1)$ is very ample.

See also [Laz04a, Thm. 1.2.17, Thm. 1.7.8], [EGA, $\mathrm{III}_1$.4.7.1, $\mathrm{IV}_3$.9.6.4], [KM98, Prop. 1.41] and [SP, Tags 0D3A and 0D3D]; the openness of ampleness holds without assuming flatness of $X \to Y$. □

We also recall that ampleness can be checked on finite covers.

**Proposition E.2.2.** *Let $f\colon X \to Y$ be a finite morphism of noetherian schemes and $L$ be a line bundle on $Y$. If $L$ is ample, then so is $f^*L$. If $f$ is surjective, then the converse is true.*

*Proof.* See [Har77, Exer III.5.7]. □

**Remark E.2.3.** As an immediate consequence, we see that a line bundle $L$ on $X$ is ample if and only if its restriction $L_{(X_i)_{\mathrm{red}}}$ to the reduced subscheme of each irreducible component $X_i$ is ample.

### E.2.2 Nef line bundles

A line bundle $L$ on a proper scheme $X$ over a field $\Bbbk$ is *nef* if

$$\int_C c_1(L) \geq 0$$

for every irreducible curve. Here $\int_C c_1(L)$ is the same number as $C\dot{L}$ or $\deg L|_C$.

**Proposition E.2.4** (Openness of Nefness)**.** *Let $X$ be a proper and flat scheme over a DVR $R$ and $L$ be a line bundle on $X$. Let $0, \eta \in \operatorname{Spec} R$ be the closed and generic points. If $L|_{X_0}$ is nef, then so is $L|_{X_\eta}$.*

*Proof.* To be added. □

**Remark E.2.5.** For proper, flat, and surjective morphisms $X \to S$, it is shown in [Laz04a, Prop 1.4.14] that if $L|_{X_s}$ is ample for a point $s \in S$, then there exists a countable union $B \subset S$ of proper subschemes not containing $s$ such that $L|_{X_t}$ is nef for every $t \in S \setminus B$. It is unknown whether an open subset $s \in U \subset S$ with $L|_{X_t}$ nef for $t \in S$ exists.

**Proposition E.2.6.** *Let $f\colon X \to Y$ be a proper morphism of noetherian schemes, and let $L$ be a line bundle on $Y$. If $L$ is nef, then so is $f^*L$. If $f$ is surjective, then the converse is true.*

**Theorem E.2.7** (Kleiman's Theorem)**.** *If $L$ is a line bundle on a proper scheme $X$ over a field $\Bbbk$, then $L$ is nef if and only if for every irreducible subvariety $Z \subset X$ of dimension $k$,*

$$\int_Z c_1(L)^k \geq 0.$$

*Proof.* See [Laz04a, Thm. 1.4.9], [Kol96, Thm. 2.17], or the original source [Kle66]. □

**Remark E.2.8** (Ample and nef cones)**.** It's also worthwhile to keep in mind that ample and nef line bundles generate cones $\mathrm{Amp}(X), \mathrm{Nef}(X) \subset N^1(X)_{\mathbb{R}}$, called the *ample cone* and *nef cone*. As a consequence of Kleiman's theorem, one can show that for a projective variety, the nef cone is the closure of the ample cone and the ample cone is the interior of the nef cone; see [Laz04a, Thm. 1.4.23].

### E.2.3 Effective, base point free, and semiample line bundles

We have the following notions for a line bundle $L$ on $X$:

- $L$ is *effective* if $\Gamma(X, L) \neq 0$,
- $L$ is *base point free* (or *globally generated*) if for every $x \in X$, there exists $s \in \Gamma(X, L)$ with $s(x) \neq 0$, or equivalent the linear series $|L|$ defines a morphism $X \to \mathbb{P}^{\mathrm{h}^0(X,L)-1}$, and
- $L$ is *semiample* if for some $m > 0$, $L^{\otimes m}$ is base point free.

A semiample line bundle $L$ is necessarily nef; indeed if for some $m > 0$, $L^{\otimes m}$ defines a morphism $f\colon X \to \mathbb{P}^N$ with $f^*\mathcal{O}(1) \cong L^{\otimes m}$, then the projection formula implies that $\int_C c_1(L^{\otimes m}) = \int_{f(C)} c_1(\mathcal{O}(1)) \geq 0$. We thus have the implications

$$\text{base point free} \implies \text{semiample} \implies \text{nef}.$$

### E.2.4 Big line bundles

A line bundle $L$ on a normal variety $X$ is *big* if for some $m > 0$, the rational map $\phi_m\colon X \overset{|L^{\otimes m}|}{\dashrightarrow} \mathbb{P}^N$ is birational onto its image for some $m > 0$. For a possibly non-normal variety $X$, we say a line bundle $L$ is big if its pullback to the normalization is big.

**Proposition E.2.9** (Kodaira's Lemma)**.** *Let $X$ be a projective variety and $L$ be a big line bundle. If $E$ is an effective line bundle, then for $m$ sufficiently divisible, $L^{\otimes m} \otimes E^\vee$ is effective.*

*Proof.* See [Laz04a, Prop. 2.2.6]. □

**Proposition E.2.10** (Equivalences of Bigness). *We have the following equivalences for a line bundle $L = \mathcal{O}_X(D)$ on an irreducible variety:*

$$\begin{aligned}
L \text{ is big} \iff\ & \dim \operatorname{im} \phi_m = \dim X \text{ for } m \text{ sufficiently large} \\
\iff\ & \text{there exists a constant } C \text{ such that } \mathrm{h}^0(X, L^{\otimes m}) \geq C \cdot m^{\dim X} \\
& \text{for } m \text{ sufficiently large} \\
\iff\ & \text{for every ample divisor } A \text{ on } X\text{, there exists a positive} \\
& \text{integer } m > 0 \text{ and an effective divisor } N \text{ on } X \text{ such that} \\
& mD = A + N \text{ (linear equivalence).} \\
\iff\ & \text{there exists an ample divisor } A \text{ on } X\text{, a positive integer} \\
& m > 0\text{, and an effective divisor } N \text{ on } X \text{ such that } mD \equiv \\
& A + N \text{ (numerical equivalence).}
\end{aligned}$$

*Proof.* See [Laz04a, §2.2] for details; the last three equivalences follow from Kodaira's lemma. □

As a consequence of Proposition E.2.10, we see that up to scaling (i.e. taking positive tensor powers), a big line bundle is the same as the sum of an ample and effective line bundle. In particular, the sum of a big and effective line bundle is also big. To summarize,

$$\text{big} \xLeftrightarrow{\text{up to scaling}} \text{ample} + \text{effective}$$

$$\text{big} + \text{effective} \implies \text{big}.$$

**Proposition E.2.11.** *Let $f\colon X \to Y$ be a generically quasi-finite and proper morphism of varieties such that $f_*\mathcal{O}_X = \mathcal{O}_Y$ (e.g. $f$ is a proper birational morphism of normal varieties). For a line bundle $L$ on $Y$, $L$ is big if and only if $f^*L$ is big.*

*Proof.* The projection formula

$$f_* f^* L^{\otimes m} \cong L^{\otimes m} \otimes f_* \mathcal{O}_X \cong L^{\otimes m}$$

implies that $\Gamma(Y, f^*L^{\otimes m}) = \Gamma(X, L^{\otimes m})$. Since $X$ and $Y$ have the same dimension, the result follows from the above equivalences of bigness. □

**Theorem E.2.12** (Asymptotic Riemann-Roch). *Let $X$ be a proper scheme over a field $\Bbbk$ of dimension $n$, and let $L$ be a nef line bundle on $X$. Then the Euler characteristic $\chi(X, L^{\otimes m})$ is a polynomial of degree $\leq n$ in $m$*

$$\mathrm{h}^0(X, L^{\otimes m}) = \frac{(c_1(L)^n)}{n!} m^n + O(m^{n-1}).$$

**Remark E.2.13.** See [Laz04a, Cor. 1.4.41] for a proof in the projective case and [Kol96, Thm. VI.2.15] in general.

This immediately yields the following useful characterization of bigness for nef line bundles.

**Corollary E.2.14.** *On a proper scheme of dimension $n$, a nef line bundle $L$ is big if and only if $c_1(L)^n > 0$.*

**Remark E.2.15** (Big and pseudo-effective cones)**.** Big and effective divisors generate cones $\operatorname{Big}(X), \operatorname{Eff}(X) \subset N^1(X)_{\mathbb{R}}$, called the *big cone* and *effective cone*. The closure $\overline{\operatorname{Eff}}(X)$ of $\operatorname{Eff}(X)$ is called the *pseudo-effective cone*. The big cone $\operatorname{Big}(X)$ is the interior of $\overline{\operatorname{Eff}}(X)$, and $\overline{\operatorname{Eff}}(X) = \overline{\operatorname{Big}(X)}$ [Laz04a, Thm. 2.2.6].

In particular, we have the implication:

$$\text{big} + \text{nef} \Rightarrow \text{big}.$$

### E.2.5 Ampleness criteria

We review general techniques here to show that a line bundle $L$ on a proper scheme $X$ is ample. Perhaps the first strategy to keep in mind is that if $L$ is semiample and strictly nef, then $L$ is ample.

**Lemma E.2.16.** *Let $X$ be a proper scheme. If $L$ is a semiample line bundle and $\int_T c_1(L) = \deg L|_C > 0$ for all curves $T$, then $L$ is ample.*

*Proof.* For some $m > 0$, $L^{\otimes m}$ defines a morphism $f\colon X \to \mathbb{P}^N$ which does not contract any curves. It follows that $f\colon X \to \mathbb{P}^N$ is a proper and quasi-finite morphism of schemes, thus finite. Therefore, $L^{\otimes m} = f^*\mathcal{O}(1)$ is ample. $\square$

See also Lemma 5.8.1 for a similar property of algebraic spaces and Deligne–Mumford stacks.

**Remark E.2.17.** The semiampleness condition can be very challenging to verify in practice. However, there are powerful base point free theorems in birational geometry stemming either from vanishing theorems or analytic methods that can reduce semiampleness to bigness and nefness. For instance, Kawamata's base point free theorem states that if $(X, \Delta)$ is a proper klt pair with $\Delta$ effective and $D$ is a nef Cartier divisor such that $aD - K_X - \Delta$ is nef and big for some $a > 0$, then $D$ is semiample [KM98, Thm. 3.3]. One can contrast this result with the Abundance Conjecture that states that if $(X, \Delta)$ is a proper log canonical pair with $\Delta$ effective, then the nefness of $K_X + \Delta$ implies semiampleness [KM98, Conj. 3.12].

Alternatively, it is a classical result of Zariski and Wilson that if $X$ is a normal projective variety and $D$ is a nef and big divisor, then $D$ is semiample if and only if its graded section ring $\bigoplus_n \Gamma(X, \mathcal{O}_X(nD))$ is finitely generated; see [Laz04a, Thm. 2.3.15]. While [BCHM10] can sometimes be applied to verify the finite generation, this result already presumes the projectivity of $X$; nevertheless, this can be applied for instance to show that a given birational model of $X$ is projective.

In positive characteristic, Keel's theorem provides another technique: on a projective variety $X$, a nef line bundle $L$ is semiample if and only if the restriction of $L$ to the *exceptional locus* $E$ is semiample, where the exceptional locus $E$ is defined as the union of irreducible subvarieties the $Z \subset X$ satisfying $L^{\dim Z} \cdot Z = 0$ [Kee99].

### E.2.6 Numerical criteria for ampleness

The Nakai–Moishezon Criterion[1] for ampleness provides a convenient method to establish projectivity. We state the criteria for proper schemes, but this is extended to proper algebraic spaces in Theorem 5.8.4.

**Theorem E.2.18** (Nakai–Moishezon Criterion)**.** *If $X$ is a proper scheme, a line bundle $L$ is ample if and only if for all irreducible closed subvarieties $Z \subset X$, $c_1(L)^{\dim Z} \cdot Z > 0$*

**Remark E.2.19.** Using Corollary E.2.14, the Nakai–Moishezon Criterion translates to:

$$L \text{ is ample} \iff L|_Z \text{ is big for all irreducible closed subvarieties } Z \subset X.$$

[1] This is also known as the Nakai Criterion or the Nakai–Moishezon–Kleiman Criterion. See [Laz04a, §1.2.B] for a historical account and further references.

*Proof.* Let $n = \dim X$. First, if $L$ is very ample, then for some $m > 0$, $L^{\otimes m}$ is very ample and $m^n c_1(L)^{\dim Z} \cdot Z = c_1(L^{\otimes m})^{\dim Z} \cdot Z > 0$ as its the degree of $Z$ under the projective embedding defined by $L^{\otimes m}$. To show the converse, we follow the proofs of [Laz04a, Thm. 1.2.23], [Kol96, Thm VI.2.18], and [Har77, Thm. V.1.10] (surface case). Since we already know that $L$ is nef, it suffices to show that $L$ is semiample (Lemma E.2.16).

First, by Proposition E.2.2, we may assume that $X$ is a normal variety, and we write $L = \mathcal{O}_X(D)$ for a divisor $D$. Since $D$ is big on $X$, some positive multiple $mD$ is effective; replacing $D$ by $mD$, there exists a nonzero section $s \in \mathrm{H}^0(X, \mathcal{O}_X(D))$. In particular, $\mathcal{O}_X(D)$ is base point free away from the support of $D$. We aim to show that for $m \gg 0$, $\mathcal{O}_X(mD)$ is also base point free on $D$.

By induction on $n = \dim X$, we can assume that $\mathcal{O}_X(D)|_D$ is ample; the base case for the induction is $n = 1$, where a line bundle is ample if and only if it has positive degree. Consider the exact sequence

$$0 \to \mathcal{O}_X((m-1)D) \to \mathcal{O}_X(mD) \to \mathcal{O}_D(mD) \to 0.$$

For $m \gg 0$, $\mathcal{O}_D(mD)$ is base point free and $\mathrm{H}^1(X, \mathcal{O}_D(mD) = 0$. It follows that $\mathrm{H}^1(X, \mathcal{O}_X((m-1)D)) \twoheadrightarrow \mathrm{H}^1(X, \mathcal{O}_X(mD))$ is surjective, but since each vector space is finitely generated, we see that these surjections eventually become isomorphisms for $m \gg 0$. Thus, for $m \gg 0$, $\mathrm{H}^0(X, \mathcal{O}_X(mD)) \to \mathrm{H}^0(D, \mathcal{O}_D(mD))$ is surjective and $\mathcal{O}_X(mD)$ is base point free on $D$. $\square$

We use this criterion to establish Kollar's Ampleness Criteria (Theorem 5.8.5), which we in turn apply to establish the projectivity of $\overline{M}_g$. The following two additional numerical criteria for ampleness will not be used in these notes but are included to offer a more complete treatment.

**Theorem E.2.20** (Kleiman's Criterion)**.** *If $X$ is a projective scheme, a divisor $D$ is ample if and only if for all $C \in \overline{\mathrm{NE}(X)}$, $D \cdot C > 0$.*

**Remark E.2.21.** See [?], [Kol96, Thm. VI.2.19], and [Laz04a, Thm. 1.4.23]. Note that it is not enough to check that $D \cdot C$ for only irreducible curves $C \subset X$; one must check it on curve classes in the closure $\overline{\mathrm{NE}(X)}$ of the effective cone of curves. See [Har66a, p.50-56] for a counterexample due to Mumford.

Kleiman's Criterion also holds for $\mathbb{Q}$-factorial (e.g. smooth) proper schemes but is unknown in general for proper schemes or algebraic spaces.

**Theorem E.2.22** (Sesahdri's criterion)**.** *If $X$ is a proper scheme, a line bundle $L$ is ample if and only if there exists an $\epsilon > 0$ such that for every point $x \in X$ and every irreducible curve $C \subset X$, $c_1(L) \cdot C > \epsilon \,\mathrm{mult}_x(C)$, where $\mathrm{mult}_x(C)$ denotes the multiplicity of $C$ at $x$.*

**Remark E.2.23.** See [Laz04a, Thm. 1.4.13] or [Kol96, Thm. 2.18] for a proof. This criterion also holds for proper algebraic spaces; see [Cor93].

### E.2.7 Nef vector bundles

In Kollár's Criterion (Theorem 5.8.5), nefness of vector bundles plays an essential role:

**Definition E.2.24.** A vector bundle $E$ on a scheme $X$ is called *nef* (or *semipositive*) if for every map $f\colon C \to X$ from a proper curve, every quotient line bundle of $f^*E \twoheadrightarrow L$ has nonnegative degree.

We note that when $E$ is a line bundle, then this is clearly equivalent to the usual notion of nefness: for all proper curves $C \subset X$, $\deg L|_C \geq 0$.

**Proposition E.2.25.** *Let $E$ be a vector bundle on a proper scheme $X$. Then the following are equivalent:*

$$\begin{aligned} E \text{ is nef} &\iff \text{for every map } f\colon C \to X \text{ from a proper curve, every quotient vector bundle of } f^*E \twoheadrightarrow W \text{ has nonnegative degree;} \\ &\iff \mathcal{O}_{\mathbb{P}(E)}(1) \text{ is nef on } \mathbb{P}(E) \to X. \end{aligned}$$

**Remark E.2.26.** See [Har66a] or [Laz04b, Ch. 6] for details. There is a similar notion of an ample vector bundle (which we won't need in these notes) where one defines a vector bundle $E$ to be ample if $\mathcal{O}_{\mathbb{P}(E)}(1)$ is ample on $\mathbb{P}(E)$. This notion also has some nice equivalences. If $X$ is an irreducible projective variety and $E$ is base point free, then $E$ is ample if and only if for every map $f\colon C \to X$ from a proper curve, every quotient line bundle of $f^*E \twoheadrightarrow L$ is non-trivial. There are also cohomological characterizations of ampleness for vector bundles in the same spirit as their line bundle counterparts. Moreover, nefness of $E$ can then be characterized as requiring that for every map $f\colon C \to X$ from a proper curve and for every ample line bundle $H$ on $C$, the vector bundle $H \otimes f^*E$ is ample.

**Proposition E.2.27.**

(1) *Quotients and extensions of nef vector bundles are nef.*

(2) *If $E$ is nef, then so is $\bigwedge^k E$ and $\operatorname{Sym}^k E$ for $k \geq 0$.*

*Proof.* Part (1) follows from the definition of nefness. Part (2): to be added. □

As a consequence of Proposition E.2.4 and the first equivalence of Proposition E.2.25, we obtain that nefness is open in a proper flat family over a DVR:

**Proposition E.2.28** (Openness of Nefness)**.** *Let $X$ be a proper and flat scheme over a DVR $R$ and $E$ be a vector bundle on $X$. Let $0, \eta \in \operatorname{Spec} R$ be the closed and generic points. If $E|_{X_0}$ is nef, then so is $E|_{X_\eta}$.*

## E.3 Vanishing theorems

Kollár's argument for the projectivity of $\overline{M}_g$ makes use of the following vanishing theorem in positive characteristic due to Ekedahl [Eke88]. The characteristic zero version is due to Bombieri [Bom73].

**Theorem E.3.1** (Bombieri–Ekedahl vanishing)**.** *Let $S$ be a smooth projective surface over $\Bbbk$ which is minimal and of general type. If $\operatorname{char}(\Bbbk) \neq 2$, then $\mathrm{H}^1(S, K_S^{\otimes -n}) = 0$ for all $n \geq 1$. If $\operatorname{char}(\Bbbk) = 2$, then $\mathrm{h}^1(S, K_S^{\otimes -n}) \leq 1$ for all $n \geq 2$.*

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