Stacks and Moduli

working draft

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February 17, 2025

Preface

These notes develop the foundations of moduli theory in algebraic geometry with the goal of providing self-contained proofs in characteristic 0 of the following theorems:

Theorem A. *The moduli stack $\overline{\mathcal{M}}_g$ of stable curves of genus $g \geq 2$ is a smooth, proper, and irreducible Deligne–Mumford stack of dimension $3g - 3$ which admits a projective coarse moduli space $\overline{M}_g$.*

Theorem B. *The moduli stack $\mathcal{B}un^{\mathrm{ss}}_{r,d}(C)$ of semistable vector bundles of rank r and degree d over a smooth, connected, and projective curve C of genus g is a smooth, universally closed, and irreducible algebraic stack of dimension $r^2(g-1)$ which admits a projective good moduli space $M^{\mathrm{ps}}_{r,d}(C)$.*

Along the way we build the foundations of algebraic spaces and stacks, which provide a convenient language to discuss and establish geometric properties of moduli spaces. Introducing these foundations requires developing several themes at the same time including:

- embracing the functorial perspective in algebraic geometry, where a scheme X is determined by its relation to all other schemes, i.e., by its representable functor $\mathrm{Sch} \to \mathrm{Sets}, T \mapsto \mathrm{Mor}(T, X)$;
- introducing stacks as a categorical gadget that encodes the symmetries of objects, e.g., we will think of $\mathcal{M}_g$ as the assignment of a scheme T to the *groupoid* of all families of smooth curves over T;
- replacing the Zariski topology on a scheme with the étale topology: we will introduce Grothendieck topologies as a generalization of topological spaces, and we will systematically use descent theory for étale morphisms; and
- relying on several advanced topics not typically seen in a first algebraic geometry course, e.g., properties of flat, étale, and smooth morphisms of schemes, algebraic groups and their actions, deformation theory, and the birational geometry of surfaces.

Choosing a linear order to present the foundations is no easy task. We mitigate this challenge by relegating much of the background to appendices, while keeping the main body of the notes focused on moduli theory with the above two theorems in mind.

Prerequisites. We assume that you are familiar with schemes, e.g., at the level of [Har77, Ch. 2–3] or [Vak17, Parts II-V], but not much more. On the other hand, you will see many important concepts in algebraic geometry *in action* in the context of moduli theory, e.g., the valuative criterion for properness and cohomology and base change. This provides the opportunity to fill in gaps in your background or to reinforce your understanding of key concepts in scheme theory. In fact, every section of [Har77] and [Vak17] will be relevant for at least some part of this text.

Sources. Moduli spaces have a rich history with its modern origins in Bernhard Riemann's work in the 19th century (see §0.1). Our approach to the development of moduli theory is due to Alexander Grothendieck, David Mumford, Michael Artin, János Kollàr, Johan de Jong, and many others. A primary source for many topics in this book is the *Stacks Project* [SP]. The moduli space of stable curves had its origins in [GIT] and [DM69] while the textbooks [HM98] and [ACG11] offer further details. Similarly, the origins of the moduli space of semistable vector bundles can be traced to [Mum63] and [Ses67], and the books [LP97] and [HL10] provide more extensive treatments.

Our approach. We choose a route to the summits of Mt. Mgbar (Theorem A) and Mt. BunC (Theorem B) with a gradual slope while minimizing technical pitches and crevasses. Our path offers frequent views of the summits ahead as well as the surrounding valleys and peaks. The appendices provide background training in the technical gear necessary for our climb, and we take a warm-up ascent of Mts. Hilbert and Quot (Theorems 1.1.2 and 1.1.3) to get practice with the equipment. While strongly advised against in mountain climbing, you can also head directly for the summits without any gear or training, waiting only until you fall into a crevasse before acquiring the background knowledge.

Motivated by the ethos that mathematical understanding is often best obtained by a thorough study of examples, this book illustrates concepts in moduli theory though three main examples: Hilbert and Quot schemes (as moduli spaces of objects with no automorphisms), the moduli stack $\overline{\mathcal{M}}_g$ of stable curves (with finite automorphisms), and the moduli stack $\mathcal{B}un^{\mathrm{ss}}_{r,d}(C)$ of semistable vector bundles (with infinite automorphisms) on a fixed curve. Certainly, before you hope to study moduli of higher dimensional varieties or moduli of Bridgeland semistable complexes on a surface, you ought to have a solid understanding of these examples.

This book is long (sorry!) and you should *not* read it linearly from start to finish. The hope is that each chapter and even each section can be read independently, and that the reader can skip around as desired. Throughout the text, we attempt to provide examples, counterexamples, and exercises for each new concept, always starting with the most elementary cases. The exercises come in different flavors, e.g., working out additional examples, proving further properties, or filling in technical details of a proof, but the purpose remains the same: to provide the reader a chance to practice and play around with the new mathematical objects.

While the approach in the *Stacks Project* is to prove theorems in the utmost level of generality, we often sacrifice the level of generality if it allows for a more intuitive or geometric proof. Most notably, while we work over any field $\Bbbk$ (or base scheme S) when possible, in both our proof of Stable Reduction (5.5.1) for curves and for the construction of the moduli space of semistable vector bundles, we assume that the field $\Bbbk$ is of characteristic 0. We sometimes add a noetherian hypothesis even if it is not strictly necessary, e.g., in the Valuative Criterion for Properness (3.8.7) for algebraic stacks, or we will establish a theorem only for Deligne–Mumford stacks rather than for algebraic stacks with finite inertia, e.g., in the Keel–Mori theorem (4.4.6).

Whenever possible, we give pointers to the *Stacks Project* and original sources containing either more general results or alternative arguments. We hope that this text can serve as a guide to the much more exhaustive development in the *Stacks Project*. With the tools and techniques developed in this book, the reader should be empowered to study geometric properties of these moduli spaces as in [HM98] or [HL10].

Acknowledgements. This book originated from an online graduate course *Math 582C: Introduction to Stacks and Moduli* taught by the author at the University of Washington in the Winter Quarter of 2021. The lecture notes have expanded over the years as I've had opportunities to teach similar graduate courses at the University of Washington and several graduate minicourses— University of Essex, Universidad Católica de Chile, Scuola Normale Superiore di Pisa, Universität Duisburg-Essen, L'Institut de Mathématiques de Bourgogne, University of Illinois at Chicago, Università degli Studi di Salerno, Johannes Gutenberg-Universität Mainz, Pohang University of Science and Technology, and Universidad de los Andes. I would like to thank the participants for their valuable feedback.

I would like to thank Ravi Vakil and Johan de Jong for introducing me to moduli spaces and stacks, and for their encouragement to write this textbook. This book would not have been possible without de Jong's contributions in the *Stacks Project* and the countless other arguments and examples that he has personally shared with me. I would also like to thank my collaborators—especially Maksym Fedorchuk, Jack Hall, Daniel Halpern-Leistner, Jochen Heinloth, Andrew Kresch, David Rydh, and David Smyth—for sharing their knowledge and insights. I thank Dhyan Aranha, Arend Bayer, Jim Bryan, Ben Church, Hannah Dell, Neeraj Deshmukh, Patricio Gallardo, Davide Gori, Vu Hien Giovanni Inchiostro, Adam Kapilow, Rega Karim, Myeonhu Kim, Aaron Landsman, Dion Leijnse, Jake Levinson, Max Lieblich, Benjamin Lim, Siddarth Matthur, Andrés Ibáñez Núñez, Brian Nungent, Dragos Oprea, Lachlan Potter, Denis Resin, Minseon Shin, Tuomas Tajakka, Alex Schefflin, Ian Selvaggi, Reinier Sorgdrager, Dario Weissmann, Maxime Wybouw, Xucheng Zhang, and many others for their comments and corrections.

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Chapter 0

Introduction and motivation

> *Besides being a form of cartography, the theory of moduli spaces has the wonderful feature of having many doors, many techniques by which this theory can be developed. Of course, there is traditional algebraic geometry, but there is also invariant theory, complex-analytic techniques such as Teichmüller theory, global topological techniques, and purely characteristic p methods such as counting objects over finite fields. This is another part of its charm.*
>
> David Mumford [Mum04, preface]

Moduli spaces arise as solutions to one of the most fundamental problems in mathematics:

> **Classification problem**: Can we classify the isomorphism classes of mathematical objects of a given type?

There are many types of objects that we may want to classify:

- subspaces $V \subseteq \mathbb{C}^n$ of dimension k;
- plane curves $C \subseteq \mathbb{P}^2$ of degree d;
- curves C of genus g together with a degree d morphism $C \to \mathbb{P}^1$;
- vector bundles on a fixed projective variety X; and
- representations of a group, e.g., an absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$, the fundamental group $\pi_1(\Sigma)$ of a topological surface of genus g, or the path algebra of quiver.

Our primary interest is in the following two examples, which will be used throughout this book to illustrate the concepts of moduli.

(1) smooth (or more generally stable) projective curves of genus g, and
(2) vector bundles (or more specifically semistable vector bundles) on a *fixed* smooth curve C.

> A **moduli space** is a *space* whose points are in *natural* bijection with isomorphism classes of the objects of a given type.

The keyword above is 'natural', and it is probably not clear to you what this could mean. Indeed, one of the main challenges in developing moduli theory is formulating precisely what this means. After all, any two complex manifolds or varieties of positive dimension are bijective as they both have the cardinality of the continuum. We do not want our moduli space to be a disjoint of a points, with one point for every isomorphism class. We need to specify a sense of proximity for objects. Our approach for formalizing the notion of a 'natural bijection' will be to introduce the notion of a *family of objects* and require that every family of objects over a space S corresponds uniquely to a map from S to the moduli space; see §0.3.

The structure of the 'space' depends on the context: if we are classifying topological objects, we might hope for the structure of a topological space, while if we are classifying analytic objects, we might hope for the structure of a manifold. In this book, we are mainly focused on classifying algebraic structures, and we desire a moduli space with the structure of an algebraic variety, and ideally a projective variety.

Ubiquity of moduli. Once you start viewing spaces through the lens of moduli, everything begins looking like a moduli space! This is true in a precise sense: every space M is the moduli space of its points. It is of course more interesting when there are equivalent descriptions with geometric meanings. Projective space $\mathbb{P}^1$ is the set of points in $\mathbb{P}^1$ (not so interesting), or lines in the plane passing through the origin (more interesting), or subschemes of $\mathbb{A}^2$ of length 2 supported at the origin (also interesting), or isomorphism classes of stable elliptic curves (very interesting). Fascinatingly, moduli spaces often afford equivalent viewpoints across different mathematical fields, as is the case in our primary two examples:

(1) $\mathcal{M}_g$ is the moduli of smooth projective algebraic curves of genus g, or the moduli of compact Riemann surfaces of genus g, or the moduli of complex structures on a fixed compact oriented topological surface Σ_g of genus g up to biholomorphisms, or the moduli of hyperbolic metrics on Σ_g up to isometries.

(2) $\mathcal{B}un^{\mathrm{ss}}_{r,d}(C)$ is the moduli of semistable algebraic vector bundle on a fixed curve C, or the moduli of holomorphic vector bundles on C with flat unitary connection, or the moduli of irreducible unitary representation of $\pi_1(C)$.

This leads to a rich interplay between algebraic, analytic, and topological approaches.

Discrete vs continuous moduli. Depending on the types of objects, the moduli space could be discrete or continuous, or a combination of the two.

- The moduli space of line bundles on $\mathbb{P}^1$ is the discrete set $\mathbb{Z}$: every line bundle on $\mathbb{P}^1$ is isomorphic to $\mathcal{O}(n)$ for a unique integer $n \in \mathbb{Z}$.
- The moduli space parameterizing quadric plane curves $C \subseteq \mathbb{P}^2$ is the connected space $\mathbb{P}^5$: a curve defined by $a_0x^2 + a_1xy + a_2xz + a_3y^2 + a_4yz + a_5z^2$ is uniquely determined by the point $[a_0, \dots, a_5] \in \mathbb{P}^5$, and as a plane curve varies continuously (i.e., by varying the coefficients a_i), the corresponding point in $\mathbb{P}^5$ does too.
- For smooth curves, the genus g is a discrete invariant. For fixed genus g, the moduli space M_g of smooth curves is a variety; it is in fact an *irreducible* quasi-projective variety, but we are now getting far ahead of ourselves. The moduli space of all smooth curves can be viewed as the disjoint union $\coprod_g M_g$.
- For vector bundles on a smooth curve C, the rank r and degree d are discrete invariants while the moduli space $M^{\mathrm{ss}}_{r,d}(C)$ of semistable bundles of rank r and degree d is an irreducible projective variety.

Why study moduli spaces? Properties of moduli spaces can inform us about the properties of the objects themselves. Many properties of objects are best formulated in terms of moduli spaces. For instance, to express the condition that a general genus 3 curve can be parameterized by an explicit coordinate system—namely a general genus 3 curve is canonically embedded into $\mathbb{P}^2$ as a plane quartic and thus parameterized by a point in the space $\mathbb{P}(\Gamma(\mathbb{P}^2, \mathcal{O}(4))) \cong \mathbb{P}^{14}$—we could say that the moduli space M_3 is *unirational*, i.e., there is a dominant rational map $\mathbb{P}^{14} \dashrightarrow M_3$.

Do moduli spaces exist? Before we can begin to study the geometric properties of a moduli space, we need to know that they exist. This is no easy task and is the main goal of this book. We develop the foundations of moduli theory in order to prove that there is a projective moduli space parameterizing stable curves of genus g (Theorem A) and a projective moduli space parameterizing semistable vector bundles of rank r and degree d on a fixed smooth curve (Theorem B).

It is an astonishing mathematical coincidence that moduli spaces of algebraic objects often exist as algebraic varieties. Their existence is the starting point of moduli theory. The beauty, allure, and elegance of moduli spaces is what first attracted me and countless others to the subject.

Trichotomy of moduli. A recurring theme in moduli is the influence of automorphism groups on both the properties of a moduli space and the techniques used to study its geometry. There is a trichotomy in moduli theory depending on the size of the automorphism groups: (1) no automorphisms, (2) finite automorphisms, and (3) infinite automorphisms. In (3), the moduli spaces are particularly well-behaved when the *closed* points—sometimes referred to as *polystable* objects—of the moduli stack have *reductive* automorphisms.

Automorphisms	None	Finite	Reductive at Closed Points
Type of space	Scheme/algebraic space	Deligne–Mumford stack	algebraic stack
Defining property	Zariski/étale locally an affine scheme	étale locally an affine scheme	smooth locally an affine scheme
Examples	$\mathbb{P}^n$, $\mathrm{Gr}(q,n)$, Hilb, Quot	$\mathcal{M}_g$	$\mathcal{B}un^{\mathrm{ss}}_{r,d}(C)$
Quotient stacks $[X/G]$	action is free	finite stabilizers	reductive stabilizers at closed orbits
Existence of moduli space	fine moduli space	coarse moduli space	good moduli space

Table 0.0.1: Trichotomy of moduli.

Roadmap of this chapter. We motivate the approach of this text by gradually adding more enriched structures to moduli sets of objects. We first introduce families of objects and the functorial worldview in §0.3, and then develop the groupoid perspective in §0.4. After motivating the étale topology in §0.5, we combine these perspectives by introducing moduli stacks in §0.6. We then sketch two main techniques to construct a projective moduli space in §0.7.

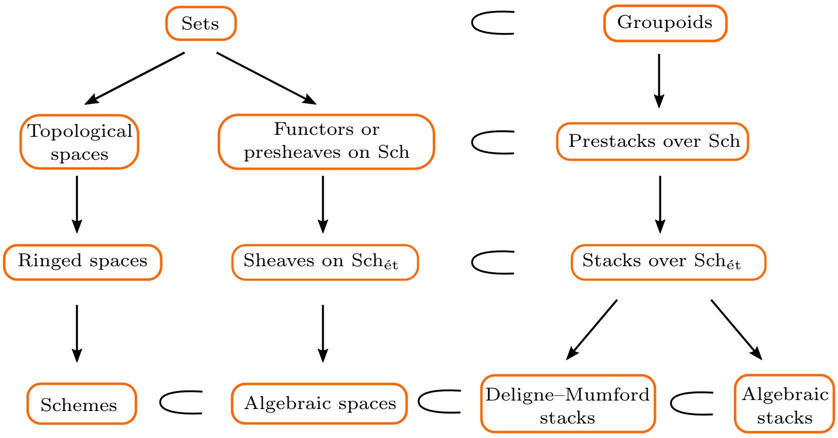


Figure 0.0.2: Schematic diagram of algebro-geometric enrichments of sets and groupoids.

0.1 A brief history of moduli

The spirit of Riemann will move future generations as it has moved us.

LARS AHLFORS [AHL53, P. 53]

The historical development of moduli theory provides a first glimpse of many themes in moduli.

0.1.1 Bernhard Riemann and the origins of M_g

Die $3p-3$ übrigen Verzweigungswerthe in jenen Systemen gleichverzweigter μ werthiger Functionen können daher beliebige Werthe annehmen; und es hängt also eine Klasse von Systemen gleichverzweigter $\overline{2p+1}$ fach zusammenhängender Functionen und die zu ihr gehörende Klasse algebraischer Gleichungen von $3p-3$ stetig veränderlichen Grössen ab, welche die Moduln dieser Klasse genannt werden sollen.

Translation: The remaining $3p-3$ branch points in these systems of similarly branching μ-valued functions can therefore be assigned any given values; and thus a class of systems of similarly branching functions with connectivity $2p+1$, and the corresponding class of algebraic equations, depends on $3p-3$ continuous variables, which we shall call the moduli of the class.

BERNHARD RIEMANN [RIE57, P. 33]

This is a remarkable sentence in a remarkable paper—Riemann both introduces the concept of 'moduli' and computes that the 'number of moduli' of M_g is $3g-3$. Riemann's idea went something like this: instead of considering abstract smooth curves, let us view curves as branched covers over $\mathbb{P}^1$ and consider the moduli space

$$\mathrm{Hur}_{d,g} = \left\{ [C \to \mathbb{P}^1] \;\middle|\; \begin{array}{l} \bullet\ C \text{ is a smooth curve of genus } g \\ \bullet\ C \to \mathbb{P}^1 \text{ is a simply branched covering of degree } d \end{array} \right\}. \tag{0.1.1}$$

Formally studied later by Hurwitz [Hur91], these moduli spaces—which are now referred to as *Hurwitz spaces*—play an essential role in the study of M_g and specifically in the proofs of irreducibility (see §5.7).

A *simply branched covering* is a finite map of smooth curves where the ramification indices are at most two and every fiber has at most one ramification point; see also Definition 5.7.2. By Riemann–Hurwitz (5.7.4), every simply branched covering $C \to \mathbb{P}^1$ is branched over $2g+2d-2$ distinct points of $\mathbb{P}^1$. This gives a commutative diagram

$$\begin{tikzcd} & \mathrm{Hur}_d(C) \arrow[r, hook] \arrow[dl] & \mathrm{Hur}_{d,g} \arrow[dl] \arrow[dr] & \\ {[C]} \arrow[r, hook] & M_g & & \mathrm{Sym}^{2d+2g-2}\,\mathbb{P}^1 \end{tikzcd} \tag{0.1.2}$$

where

- the map $\mathrm{Hur}_{d,g} \to \mathrm{Sym}^{2d+2g-2}\,\mathbb{P}^1$ sends a covering $[C \to \mathbb{P}^1]$ to the $2g+2d-2$ branched points; here $\mathrm{Sym}^N \mathbb{P}^1 = (\mathbb{P}^1)^N/S_N$ is the space classifying N *unordered* points,
- the map $\mathrm{Hur}_{d,g} \to M_g$ is defined by $[C \to \mathbb{P}^1] \mapsto [C]$, and
- $\mathrm{Hur}_d(C)$ is the preimage of $[C] \in M_g$ under $\mathrm{Hur}_{d,g} \to M_g$, i.e., $\mathrm{Hur}_d(C)$ classifies simply branched coverings $C \to \mathbb{P}^1$ where C is fixed.

If d is sufficiently large, then for every general collection of $2d+2g-2$ points of $\mathbb{P}^1$, there exists a genus g curve C and a simply branched covering $C \to \mathbb{P}^1$ branched over precisely these points, and moreover there are at most finitely many such maps. In other words, $\mathrm{Hur}_{d,g} \to \mathrm{Sym}^{2d+2g-2}\,\mathbb{P}^1$ has dense image and finite fibers; see §5.7.2 for precise details. Therefore,

$$\begin{aligned} \dim M_g &= \dim \mathrm{Hur}_{d,g} - \dim \mathrm{Hur}_d(C) \\ &= 2d + 2g - 2 - \dim \mathrm{Hur}_d(C). \end{aligned} \tag{0.1.3}$$

To compute the dimension of $\mathrm{Hur}_d(C)$, we observe that a simply branched covering $C \to \mathbb{P}^1$ is the data of a degree d line bundle L (i.e., an element of $\mathrm{Pic}^d(C)$) and two base point free sections such that the induced map to $\mathbb{P}^1$ is simply branched. Since a general choice of two sections defines a simply branched covering (Lemma 5.7.16), we can compute

$$\dim \mathrm{Hur}_d(C) = \dim \mathrm{Pic}^d(C) + 2h^0(C, L) - 1, \tag{0.1.4}$$

where we subtract one since scaling two sections defines the same map to $\mathbb{P}^1$. Riemann–Roch (5.1.5) tells us that $\mathrm{h}^0(C, L) = d + 1 - g$. On the other hand $\dim \mathrm{Pic}^d = \dim \mathrm{Pic}_0 = g$; this can be seen using the exponential sequence: $0 \to \mathbb{Z} \to \mathcal{O}_C \xrightarrow{\exp} \mathcal{O}_C^* \to 0$ yields a long exact sequence

$$\underbrace{\mathrm{H}^1(C,\mathbb{Z})}_{\mathbb{Z}^{2g}} \to \underbrace{\mathrm{H}^1(C,\mathcal{O}_C)}_{\mathbb{C}^g} \to \underbrace{\mathrm{H}^1(C,\mathcal{O}_C^*)}_{\mathrm{Pic}(C)} \xrightarrow{\deg} \underbrace{\mathrm{H}^2(C,\mathbb{Z})}_{\mathbb{Z}}, \tag{0.1.5}$$

and provides an identification $\operatorname{Pic}_0(C) \cong \mathbb{C}^g/\mathbb{Z}^{2g}$. Therefore, $\dim \operatorname{Pic}_0(C) = g$, and using (0.1.4), we compute that $\dim \operatorname{Hur}_d(C) = g + 2(d+1-g) - 1 = 2d - g + 1$. Plugging this into (0.1.3) yields

$$\dim M_g = (2d + 2g - 2) - (2d - g + 1) = 3g - 3.$$

Riemann in fact gave several other heuristic arguments computing the dimension of M_g. See [GH78, pp. 255-257] or [Mir95, pp. 211-215] for further discussion on the number of moduli of M_g, and see [AJP16] for historical background of Riemann's computations.

Riemann's moduli problem: Does M_g exist as a complex analytic space?

While Riemann's argument can be made completely rigorous with today's methods (as we do ourselves later in this text), there are foundational issues with Riemann's method—today we would say that Riemann computed the dimension of a 'local deformation space'. Most notably, M_g was not known to exist and it was not clear what type of space M_g was supposed to be. Despite this, Riemann had an instinctive grasp of its geometry—in fact in the same paper [Rie57], Riemann introduced the word 'Mannigfaltigkeit' (or 'manifold') to describe its geometry. Manifolds were not formally defined until much later in the 1940s following Teichmüller, Chern, and Weil.

0.1.2 Moduli of curves of low genus

Ritengo probabile che la varietà sia razionale o quanto meno che sia riferibile ad un'involuzione di gruppi di punti in uno spazio lineare...

Francesco Severi's conjecture that M_g is unirational for all g [Sev15, p. 881].

Genus 0. For $n \geq 3$, the moduli space $M_{0,n}$ of smooth genus 0 curves with n ordered distinct points can be described as

$$M_{0,n} = \left(\mathbb{P}^1 \smallsetminus \{0,1,\infty\}\right)^{n-3} \smallsetminus (\text{all diagonals}).$$

Indeed, given n ordered distinct points $p_1, \ldots, p_n$ on $\mathbb{P}^1$, there is a unique automorphism $g \in \operatorname{Aut}(\mathbb{P}^1) \cong \mathrm{PGL}_2$ taking (p_1, p_2, p_3) to $(0, 1, \infty)$. When $n = 4$, we obtain that a bijection $M_{0,4} = \mathbb{P}^1 \smallsetminus \{0, 1, \infty\}$ given by the classical cross-ratio of four points in $\mathbb{P}^1$ first discovered by Pappus of Alexandria [Ale86] in 300 AD; see also Example 7.2.8.

Genus 1. The moduli of elliptic curves was known to Dedekind and Klein. Every elliptic curve (E, p), i.e., a smooth genus 1 curve E with a marked point $p \in E$, can be described as a plane cubic in Weierstrass form

$$E_\lambda = V(y^2z - x(x-z)(x-\lambda z)) \subseteq \mathbb{P}^2$$

for some $\lambda \neq 0, 1$, where $p = [0:1:0] \in E_\lambda$. However, the choice of λ is not unique: the values λ, $1/\lambda$, $1-\lambda$, $1/(1-\lambda)$, $\lambda/(\lambda-1)$, and $(\lambda-1)/\lambda$ determine isomorphic elliptic curves. In other words, the map $\mathbb{A}^1 \smallsetminus \{0,1\} \to M_{1,1}$ given by $\lambda \mapsto [E_\lambda]$ is a 6-to-1 surjective map. The j-invariant on the other hand

$$j = 2^8 \frac{(\lambda^2 - \lambda + 1)^3}{\lambda^2(\lambda-1)^2}$$

uniquely determines the isomorphism class of the curve and thus gives a bijection $M_{1,1} \cong \mathbb{A}^1$. For a modern treatment, see [Har77, §4].

Genus 2. Every smooth genus 2 curve C is hyperelliptic and can be written as a double cover $y^2 = (x-a_1)\cdots(x-a_6)$ over $\mathbb{P}^1$. This is a consequence of the sheaf of differentials Ω_C being a base point free line bundle of degree 2 with 2 global sections; the induced map $C \to \mathbb{P}^1$ is ramified over 6 points by Riemann–Hurwitz (5.7.4). We obtain the description that

$$M_2 = \big(\Gamma(\mathbb{P}^1, \mathcal{O}(6)) \smallsetminus \Delta\big) / \operatorname{GL}_2,$$

where $\Delta \subseteq \Gamma(\mathbb{P}^1, \mathcal{O}(6))$ denotes the locus of binary sextics with a double root. After a projective change of coordinates on $\mathbb{P}^1$, we can arrange that the curve is ramified over 0, 1, ∞ and 3 other points $a_4, a_5, a_6 \in \mathbb{P}^1 \smallsetminus (0,1,\infty)$. In this way, we obtain a surjective map $(\mathbb{P}^1 \smallsetminus \{0,1,\infty\})^3 \smallsetminus \Delta \to M_2$.

Invariant theory of binary sextics (see [Cle70]) provides an even sharper description: the ring of invariant polynomials, i.e., polynomials in the coefficients of a binary sextic that are invariant under automorphisms of $\mathbb{P}^1$, is generated by invariants J_2, J_4, J_6, J_8, J_{10}, and J_{15}, whose degree is indicated by the subscript, with a single relation $J_{15}^2 = G(J_2, J_4, J_6, J_{10})$ for a polynomial G. The invariant J_{10} is the discriminant of a binary sextic, while J_{15} does not affect the scheme structure. This yields that M_2 is an open subset of weighted projective space

$$M_2 = \operatorname{Proj} \mathbb{C}[J_2, J_4, J_6, J_{10}] \smallsetminus \{J_{10} = 0\},$$

which implies that M_2 is an affine variety embedded into $\mathbb{A}^8$ via

$$\frac{J_2^5}{J_{10}}, \frac{J_2^3 J_4}{J_{10}}, \frac{J_2^2 J_6}{J_{10}}, \frac{J_2 J_4^2}{J_{10}}, \frac{J_2 J_6^3}{J_{10}^2}, \frac{J_4^5}{J_{10}^2}, \frac{J_4 J_6}{J_{10}}, \frac{J_6^5}{J_{10}^3}.$$

We can identify this coordinate ring with the invariant ring of the action of $\mathbb{Z}/5$ on $\mathbb{A}^3$ where a generator $\zeta \in \mathbb{Z}/5$ acts via $\zeta \cdot (x,y,z) = (\zeta x, \zeta^2 y, \zeta^3 z)$; the above functions are identified with the invariants x^5, $x^3 y$, $x^2 z$, xy^2, xz^3, y^5, yz, z^5. This yields the rather elegant global description

$$M_2 = \mathbb{A}^3 / (\mathbb{Z}/5).$$

This was studied classically by Bolza [Bol87] and more recently by Igusa [Igu60].

Genus 3. A non-hyperelliptic smooth genus 3 curve embeds as a quartic in $\mathbb{P}^2$ under the canonical embedding. Letting $\Delta \subseteq \Gamma(\mathbb{P}^2, \mathcal{O}(4))$ be the locus of singular quartics, we can describe the open locus in M_3 of non-hyperelliptic curves as the quotient $\big(\Gamma(\mathbb{P}^2, \mathcal{O}(4)) \smallsetminus \Delta\big) / \operatorname{GL}_3$—this is the first time that our description only describes a *general* curve. On the other hand, a hyperelliptic genus 3 curve is a double cover of $\mathbb{P}^1$ ramified over 8 points, and we obtain a *set-theoretic* decomposition

$$M_3 = \underbrace{\big(\Gamma(\mathbb{P}^2, \mathcal{O}(4)) \smallsetminus \Delta\big) / \operatorname{GL}_3}_{\dim = 6} \coprod \underbrace{\big(\Gamma(\mathbb{P}^1, \mathcal{O}(8)) \smallsetminus \Delta\big) / \operatorname{GL}_2}_{\dim = 5},$$

suggesting that the locus of hyperelliptic curves is a divisor in M_3.

Genus 4. A non-hyperelliptic smooth curve C of genus 4 embeds into $\mathbb{P}^3$ under its canonical embedding, and can be realized as the intersection $C = Q \cap S$ of a quadric surface Q and a cubic surface S. This gives a rational map $\Gamma(\mathbb{P}^3, \mathcal{O}(2)) \times \Gamma(\mathbb{P}^3, \mathcal{O}(3)) \dashrightarrow M_4$ whose image is the locus of non-hyperelliptic curves; as above the hyperelliptic locus can be parameterized by $\big(\Gamma(\mathbb{P}^1, \mathcal{O}(10)) \smallsetminus \Delta\big)/\operatorname{GL}_2$. Alternatively, a general non-hyperelliptic smooth genus 4 curve can be realized as the normalization of a plane quintic with precisely two nodes, or as a degree 3 cover of $\mathbb{P}^1$ branched over 12 points.

Genera 5–10. Classically, curves of low genus were described either as plane curves with prescribed singularities via the image of a map $C \to \mathbb{P}^2$, or as branched covers $C \to \mathbb{P}^1$. For a general genus g curve C, the smallest degree d such that C is realized as the normalization of a singular plane curve is $d = \lfloor \frac{2g+8}{3} \rfloor$. If the plane curve has at worst nodal singularities, then the number of nodes is $\delta := (d-1)(d-2)/2 - g$. Meanwhile, the minimum degree of a map $C \to \mathbb{P}^1$ is $\lfloor \frac{g+3}{2} \rfloor$. See also [Mum75a, p. 21].

g	$d =$ min degree of $\operatorname{im}(C \to \mathbb{P}^2)$	$\delta = \#$ of nodes	$\frac{(d+1)(d+2)}{2} - 3\delta$	min degree of $C \to \mathbb{P}^1$	# of branch pts
0	1	0	3	1	0
1	3	0	10	2	4
2	4	1	12	2	6
3	4	0	15	3	10
4	5	2	15	3	12
5	6	5	13	4	16
6	6	4	16	4	18
7	7	8	12	5	22
8	8	13	6	5	24
9	8	12	9	6	28
10	9	18	1	6	30
11	10	25	-9	7	34

Table 0.1.6: General curves of low genus.

In [Sev15] and [Sev21], Severi used such descriptions to show that M_g is unirational for $g \leq 10$. Like other mathematicians of his era, Severi did not precisely formulate what it meant for M_g to be a moduli space.

What goes wrong for $g \geq 11$? As the genus grows, it becomes more difficult to describe a general genus g curve. To give an indication of the challenges for $g \geq 11$, let us try to describe a genus g curve as a degree $d = \lfloor \frac{2g+8}{3} \rfloor$ planar curve with $\delta = (d-1)(d-2)/2 - g$ nodes at prescribed points $p_1, \ldots, p_\delta \in \mathbb{P}^2$. If the plane curve defined by $f \in \Gamma(\mathbb{P}^2, \mathcal{O}(d))$ has a node at each p_i, then the equations $f_x(p_i) = f_y(p_i) = f_z(p_i) = 0$ imposes 3δ linear equations on $\Gamma(\mathbb{P}^2, \mathcal{O}(d))$. For such nodal plane curves to exist, we would need

$$\dim \Gamma(\mathbb{P}^2, \mathcal{O}(d)) - 3\delta = \frac{(d+1)(d+2)}{2} - 3\Big(\frac{(d-1)(d-2)}{2} - g\Big) > 0.$$

As illustrated by Table 0.1.6, $g = 11$ is the first case where this does not hold!

Severi's conjecture. While Severi's conjecture that M_g is unirational for all g turned out to be false, it nevertheless motivated mathematicians for decades: "Whether more M_g's, $g \geq 11$, are unirational or not is a very interesting problem, but one which looks very hard too, especially if g is quite large" [Mum75a, p. 37]. In the 1980s, Eisenbud, Harris, and Mumford disproved this conjecture by showing that in some sense quite the opposite is true in large genus: M_g is of general type for $g \geq 24$ [HM82], [EH87].

Petri's description of canonical curves.

> *[Petri's approach] is unavoidably a bit messy, but just to be able to brag, I think it is a good idea to be able to say 'I have seen every curve once.'*
>
> David Mumford [Mum75a, p.17]

While most 19th and early 20th century mathematicians described curves as either plane singular curves or as covers of $\mathbb{P}^1$, Petri's explicit description [Pet23] of canonically embedded curves was an exception and is more reminiscent of modern approaches. Building on M. Noether's result [Noe80] that the canonical embedding $C \hookrightarrow \mathbb{P}^{g-1}$ of a non-hyperelliptic smooth curve C is projectively normal—that is, $\varphi\colon \operatorname{Sym}^* \Gamma(C, \Omega_C) \to \bigoplus_{d \geq 0} \Gamma(C, \Omega_C^{\otimes d})$ is surjective—and also building on work of Enriques [Enr19] and Babbage [Bab39], Petri showed that the homogeneous ideal $I = \ker \varphi$ is generated by quadrics unless C is a plane quintic ($g = 6$) or trigonal (i.e., a triple covering of $\mathbb{P}^1$) in which case I is generated in degree 2 and 3. Petri's analysis was remarkably constructive leading to explicit equations in $\mathbb{P}^{g-1}$ cutting out C along with explicit syzygies among the equations. Petri's work continues to inspire research in the theory of moduli and syzygies. We will not cover this perspective further in this text but we recommend [SD73], [Mum75a, pp. 17-21], [AS78], [Gre82], [Gre84], and [ACGH85, §III.3].

0.1.3 Analytic approaches and the Teichmüller space

> *It does not of necessity follow that, if the work delights you with its grace, the one who wrought it is worthy of your esteem.*
>
> Plutarch, The Parallel Lives

In the late 19th and early 20th century, Riemann surfaces were described as quotients of the upper half plane by a discrete subgroup of $\mathrm{PSL}_2(\mathbb{R})$; such subgroups are named *Fuchsian groups* after Fuchs [Fuc66]. Fricke and Klein classified Fuchsian groups using the theory of automorphic functions in their 1300 page volumes [FK92], [FK12]. They constructed what is now known as the Teichmüller space, showed that it is a contractible space, and even exhibited complex structures. Torelli showed that a Riemann surface can be constructed from its Jacobian [Tor13], and Siegel constructed the moduli space A_g of abelian varieties of dimension g as an analytic space [Sie35].

Oswald Teichmüller was the first to give a precise formulation of Riemann's moduli problem, to construct M_g as a complex analytic space, and to interpret $3g-3$ as its complex dimension [Tei40], [Tei44]. Teichmüller constructed the *Teichmüller space* T_g parameterizing complex structures on a topological surface Σ_g of genus

g up to homeomorphism. The space T_g is homeomorphic to a ball in $\mathbb{C}^{3g-3}$ and inherits an action of the *mapping class group* Γ_g of diffeomorphisms of Σ_g modulo the subgroup of diffeomorphisms isotopic to the identity. This action is properly discontinuous, and M_g is realized as the quotient T_g/Γ_g. Although largely forgotten for nearly 20 years, Teichmüller theory was later greatly expanded by Ahlfors, Bers, and Weil among others; see [Wei57], [Wei58], [AB60], [Ber60], and [Ahl61]. For modern expository treatments, see [Ber72], [Hub06], and [FM12].

Teichmüller also introduced the notion of families of Riemann surfaces and showed that the Teichmüller space satisfies a universal property. Grothendieck, in a series of ten lectures at Cartan's seminar [Gro61], developed a general theory of analytic moduli spaces in the language of categories and functors, reformulated Teichmüller theory in this setting, and showed that T_g represents a functor parameterizing families of Riemann surfaces. This set the stage for Grothendieck's later work on algebraic moduli: "One can hope that we shall be able one day to eliminate analysis completely from the theory of Teichmüller space, which should be purely geometric" [Gro61, Lecture I].

0.1.4 The origins of algebraic moduli theory

As for M_g there is virtually no doubt that it can be provided with the structure of an algebraic variety.

André Weil [Wei58, p. 383]

Boole, Cayley, Gordan, and Hilbert. The invariant theorists of the 19th century were interested in classifying homogeneous polynomials of degree d in n variables up to projective automorphisms, or in other words in the moduli space $\Gamma(\mathbb{P}^{n-1}, \mathcal{O}(d))/\operatorname{PGL}_n$. They attempted to describe this moduli space by exhibiting explicit invariant polynomials in the coefficients a_I of a polynomial $f = \sum_I a_I x^I$. The origins of invariant theory lie in work of George Boole [Boo41] and Arthur Cayley [Cay45], and were further developed by Paul Gordan and David Hilbert along with many others. Gordan exhibited explicit generators of the ring of invariants of binary forms ($n = 2$) [Gor68] and Hilbert later proved that the ring of invariants is finitely generated for any ring [Hil90], [Hil93]. We prove and discuss Hilbert's theorem in Corollary 6.5.8(3) and Remark 6.5.10.

Cayley constructed the moduli space of curves in $\mathbb{P}^3$ [Cay60], [Cay62], which is now referred to as the *Chow variety*. His idea was to associate to a degree d curve $C \subseteq \mathbb{P}^3$ the set of lines $L \subseteq \mathbb{P}^3$ meeting C non-trivially. This is a hypersurface of degree d in $\operatorname{Gr}(1, \mathbb{P}^3)$, and the Chow variety is the closure of all hypersurfaces in $\Gamma(\operatorname{Gr}(1, \mathbb{P}^3), \mathcal{O}(d))$ obtained this way. This construction was generalized later by Chow and van der Waerdan [CW37] to subvarieties of $\mathbb{P}^n$ of arbitrary dimension; see Section 1.4.5. Chow varieties will play a role in the construction of $\overline{M}_g$ in §5.8.

Weil. In André Weil's work on the Riemann hypothesis for curves over finite fields [Wei48], he needed to construct the Jacobian of a curve parameterizing degree 0 line bundles. At that point, varieties had only been considered as embedded in affine or projective space, and in his foundational work [Wei62], Weil enlarged the category to *abstract varieties*. This was enough to construct the Jacobian and give a proof—in fact his second proof—of the Riemann hypothesis for curves. Later Weil and Chow independently showed that the Jacobian was projective.

Baily. Walter Baily constructed the moduli space A_g of principally polarized abelian varieties as a quasi-projective variety [Bai60a], [Bai60b], showed that Satake's topological compactification [Sat56] is algebraic [Bai58], and together with Borel introduced what is now known as the *Baily–Borel compactification* [BB66]. Using the period map $M_g \to A_g$ associating a curve to its Jacobian and Torelli's theorem that this map is injective, Baily concluded that M_g has the structure of a quasi-projective variety. However, he did not prove that this provided a 'natural' structure of a variety nor that it had any uniqueness properties, i.e., that M_g is a coarse moduli space.

> *Thirdly, in order to call $\mathcal{E}$ the variety of moduli of Riemann surfaces of genus n, one should be able to state that it is unique and in some sense universal among normal parameter varieties of algebraic systems of curves of genus n. Namely, given any normal algebraic system of curves of genus n (by which we mean that the parameter variety is a normal variety) there should exist a natural map of the parameter variety of the nonsingular members of this system into $\mathcal{E}$.* — Baily [Bai60b, pp. 59-60]

Mumford credits Baily for the quasi-projectivity of M_g in [Mum75a, p. 98] just as Gieseker does in his commentary in [Mum04].

Grothendieck. After Alexander Grothendieck's formalization of analytic moduli theory in [Gro61], he applied his functorial approach to algebraic geometry in his 'FGA series' [FGAI]–[FGAVI]: he introduced the Hilbert, Quot, and Picard functors, showed that they were representable by projective schemes, developed descent theory, and introduced the notions of prestacks and stacks. Grothendieck of course later redeveloped the entire foundations of algebraic geometry by developing scheme theory. His profound influence on algebraic geometry and more broadly mathematics helped shape the future of moduli theory.

Although he did not publish on M_g, Grothendieck was nevertheless very much interested in the existence of M_g as a quasi-projective variety and its connectedness in any characteristic, as demonstrated from his written correspondence with Mumford and others in the early 1960s [Mum10, §II]. Grothendieck was aware that the presence of automorphisms obstructed the representability of the functor parameterizing smooth families of curves. He rigidified the moduli problem by also parameterizing a level n structure on a curve, i.e., a symplectic basis of $H_1(C, \mathbb{Z}/n\mathbb{Z})$. While he could show that the functor of smooth curves with level structure $n \geq 3$ was representable by a scheme, he struggled to show that it was quasi-projective. The idea was to construct M_g as a quotient of the rigidified moduli space by taking the quotient by the finite group acting on the choice of level n structure. The lack of quasi-projectivity impeded this approach as the quotient of a non-quasi-projective variety by a finite group need not exist as a variety.

Mumford. Motivated by Riemann's moduli space as well as by constructions of Chow varieties, Picard varieties, and the moduli of abelian varieties in the early 20th century, David Mumford made immense contributions to the foundations of moduli theory, and was the first to systematically study their geometry. By integrating Grothendieck's formalism of scheme theory with 19th century invariant theory, Mumford developed a theory of quotients in algebraic geometry now known as *Geometric Invariant Theory* (or *GIT*), which we develop in Chapter 7. Mumford applied his theory of GIT to construct both M_g and A_g. His theory was originally sketched in [Mum61] and fully worked out in the definitive text [GIT]. In fact, Mumford gave two constructions [GIT, Thms. 5.11 and 7.13] of the coarse moduli scheme M_g over $\operatorname{Spec}\mathbb{Z}$ and moreover that M_g is quasi-projective over $\operatorname{Spec}\mathbb{Z}[1/p]$ for every prime p.[1] The projectivity of $\overline{M}_g$ over $\operatorname{Spec}\mathbb{Z}$ was established later by other methods [Knu83b, Mum77, Gie82], which were more directly applicable to other moduli spaces.

Mumford also constructed a quasi-projective variety parameterizing *stable* vector bundles on a fixed smooth curve [Mum63], and Seshadri then showed that the moduli space of *semistable* vector bundles provides a projective compactification [Ses67].

In the seminal joint work [DM69], Deligne and Mumford introduced stable curves and the compactification $\overline{\mathcal{M}}_g$ of $\overline{\mathcal{M}}_g$, proved the Stable Reduction Theorem (5.5.1), and were the first to introduce the notion of an algebraic stack—now referred to as *Deligne–Mumford stacks*. Finally, they offered two proofs[2] of the connectedness of M_g in any characteristic; see §5.7. Unfortunately, Deligne and Mumford did not prove any of their statements on stacks: "The proofs of the results of this section will be given elsewhere" [DM69, p. 76]. The lack of rigorous foundations contributed to the formidable reputation of algebraic stacks over the following decades. It wasn't until the 2000s that algebraic stacks entered into mainstream algebraic geometry. In the 50+ years following Deligne and Mumford's paper, *most* of their statements now have full proofs in the literature. There are now excellent textbooks covering algebraic stacks [LMB00], [Ols16], and Johan de Jong's *Stacks Project* [SP] has provided an unquestionably solid foundation.

Artin. The theory of algebraic spaces and stacks was developed by Michael Artin. Similar to Weil's enlargement of affine and projective varieties to abstract varieties, enlarging the category of schemes to algebraic spaces allows us to construct the quotient of finite group actions or more generally any étale equivalence relation.[3] Knutson, a student of Artin, was the first to write down the theory of algebraic spaces [Knu71].

In the two papers [Art69a] and [Art69b], Artin proved two crucial results in moduli theory: Artin Approximation (B.5.18) and Artin Algebraization (C.6.8). In his groundbreaking paper [Art74b], Artin introduced a broader concept of algebraic stacks than Deligne and Mumford including stacks such as $\mathcal{B}un_{r,d}(C)$ with possibly infinite automorphism groups. He also provided a local deformation-theoretic characterization of algebraic stacks called Artin's Axioms for Algebraicity (C.7.1 and C.7.4), which can be applied to verify that a given moduli stack is algebraic; see §C.7.2.

For further historical background, we recommend [Mum75a], [Oor81], [Kle05], [JP13], [AJP16], and [Kol21].

[1] Interestingly, neither of Mumford's constructions actually uses GIT, or at least what is often considered as the 'standard GIT machinery' by verifying GIT stability using the Hilbert–Mumford Criterion. One of Mumford's constructions relies on the existence of the moduli space A_g of principally polarized abelian varieties, and the other on ad hoc method using covariants.

[2] While both proofs used the compactification $\overline{\mathcal{M}}_g$ and the Stable Reduction Theorem in a fundamental way, the use of algebraic stacks was not essential.

[3] Matsusaka also built a theory of $\mathbb{Q}$-*varieties* by considering certain quotients of equivalence relations [Mat64] but it was not as robust as the theory of algebraic spaces.

0.2 Moduli sets of curves, vector bundles, and triangles

To understand stacks, you really must first understand the moduli space of triangles.

ATTRIBUTED TO MICHAEL ARTIN

To define a moduli space as a set entails specifying two things:

(1) a class of certain types of objects, and

(2) an equivalence relation on objects.

Here is our first attempt at defining M_g:

Example 0.2.1 (Moduli set of smooth curves)**.** The objects of the *moduli set of smooth curves*, denoted as M_g, are smooth, connected, and projective curves of genus g over $\mathbb{C}$. Two curves are declared equivalent if they are isomorphic. There are many variants obtained by parameterizing additional structures or choosing different equivalence relations.

– We already saw the Hurwitz moduli set $\mathrm{Hur}_{d,g}$ in (0.1.1) parameterizing branched covers $C \to \mathbb{P}^1$ of degree d.

– The moduli set $M_{g,n}$ of n-pointed smooth genus g curves parameterizes the data of a smooth curve C together with n ordered distinct points $p_1, \ldots, p_n \in C$; two objects $(C, p_i) \sim (C', p_i')$ are equivalent if there is an isomorphism $\alpha \colon C \to C'$ with $\alpha(p_i) = p_i'$.

– The moduli set $M_g[n]$ of smooth genus g curves with level n structure parameterizes smooth, connected, and projective curves C of genus g over $\mathbb{C}$ together with a basis $(\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g)$ of $H_1(C, \mathbb{Z}/n\mathbb{Z})$ such that the intersection pairing is symplectic, while two objects $(C, \alpha_i, \beta_i) \sim (C', \alpha_i', \beta_i')$ are declared equivalent if there is an isomorphism $C \to C'$ taking α_i and β_i to α_i' and β_i'.

– For the moduli set whose objects are plane curves $C \subset \mathbb{P}^2$, there are several choices for equivalence relations $C \sim C'$: (a) C and C' are equal as subschemes, (b) C and C' are projectively equivalent (i.e., there is an automorphism of $\mathbb{P}^2$ taking C to C'), or (c) C and C' are abstractly isomorphic.

Example 0.2.2 (Moduli set of vector bundles on a curve)**.** The moduli set $\mathrm{Bun}_{r,d}(C)$ parameterizes vector bundles of rank r and degree d on a fixed smooth, connected, and projective curve C; the equivalence relation here is isomorphism. The special case of $r = 1$ yields the set $\mathrm{Pic}^d(C)$ parameterizing degree d line bundles on C. This is non-canonically identified with with the abelian variety $\mathrm{H}^1(C, \mathcal{O}_C)/\mathrm{H}^1(C, \mathbb{Z}) = \mathbb{C}^g/\mathbb{Z}^{2g}$ via the exponential exact sequence (0.1.5).

A recurring theme in moduli is the presentation of moduli spaces as quotients of group actions.

Example 0.2.3 (Moduli set of orbits)**.** Given a group action of a group G on a set X, we define the *moduli set of orbits* by taking the objects to be all elements $x \in X$ and by declaring x to be equivalent to x' if they have the same orbit $Gx = Gx'$. In other words, the moduli set of orbits is the quotient set X/G.

Some examples to keep in mind are the $\mathbb{Z}/2$-action on $\mathbb{A}^1$ via $(-1) \cdot x = -x$ and the usual scaling action of $\mathbb{G}_m$ on $\mathbb{A}^n$ via $t \cdot (x_1, \ldots, x_n) = (tx_1, \ldots, tx_n)$. The quotient set $(\mathbb{A}^n \smallsetminus 0)/\mathbb{G}_m$ is identified with $\mathbb{P}^{n-1}$. The quotient $\mathbb{A}^n/\mathbb{G}_m$ including the origin—and particularly the case of $\mathbb{A}^1/\mathbb{G}_m$—shows up repeatedly in this text. Another interesting example is the $\mathbb{G}_m$-action on $\mathbb{A}^2$ given by $t \cdot (x, y) = (tx, t^{-1}y)$.

0.2.1 Toy example: moduli of triangles

Before diving deeper into M_g and $\mathrm{Bun}_{r,d}(C)$, let us study the simple yet surprisingly fruitful example of the moduli of triangles. These moduli spaces are easy to visualize and are useful to illustrate various themes of stacks and moduli.

Example 0.2.4 (Labeled triangles)**.** A *labeled triangle* is a triangle in $\mathbb{R}^2$ where the vertices are labeled with '1', '2' and '3', and the distances of the edges are denoted as a, b, and c. We require that triangles have nonzero area or equivalently that their vertices are not collinear.

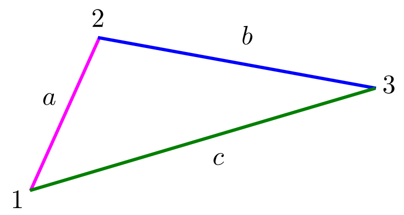


Figure 0.2.5: To keep track of the labeling, we color the edges.

We define the *moduli set of labeled triangles* M as the set of labeled triangles where two triangles are said to be equivalent if they are the same triangle in $\mathbb{R}^2$ with the same vertices and same labeling. By writing (x_1, y_1), (x_2, y_2), and (x_3, y_3) as the coordinates of the labeled vertices, we obtain a bijection

$$M \cong \left\{ (x_1, y_1, x_2, y_2, x_3, y_3) \mid \det \begin{pmatrix} x_2 - x_1 & x_3 - x_1 \\ y_2 - y_1 & y_3 - y_1 \end{pmatrix} \neq 0 \right\} \subseteq \mathbb{R}^6 \tag{0.2.6}$$

with the open subset of $\mathbb{R}^6$ whose complement is the codimension 1 closed subset defined by the condition that the vectors $(x_2, y_2) - (x_1, y_1)$ and $(x_3, y_3) - (x_1, y_1)$ are linearly dependent.

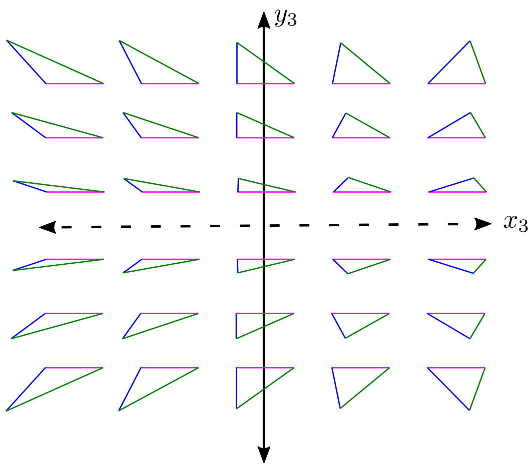


Figure 0.2.7: Picture of the slice of the moduli space M where $(x_1, y_1) = (0, 0)$ and $(x_2, y_2) = (1, 0)$. Triangles are described by their third vertex (x_3, y_3) with $y_3 \neq 0$. We have drawn representative triangles for a handful of points in the x_3y_3−plane.

Example 0.2.8 (Labeled triangles up to similarity)**.** We define the *moduli set of labeled triangles up to similarity*, denoted by M^{lab}, by taking the same class of objects as in the previous example—labeled triangles—but changing the equivalence relation to label-preserving similarity.

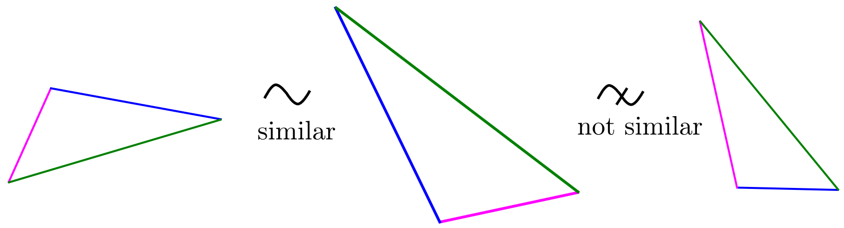


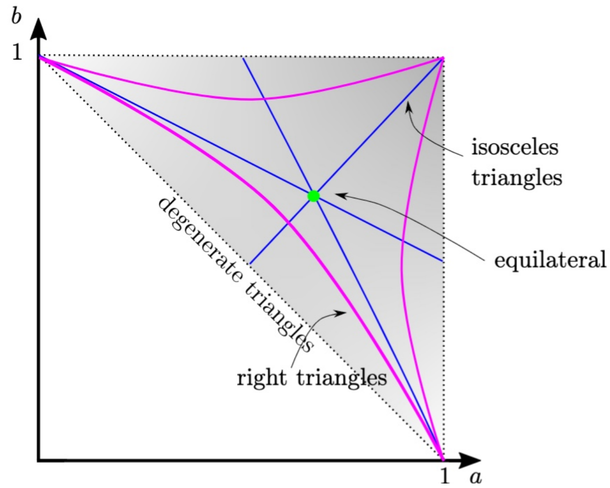
Figure 0.2.9: The two triangles on the left are similar, but the third is not.

Every labeled triangle is similar to a unique labeled triangle with perimeter $a+b+c=2$. We have the description

$$M^{\text{lab}} = \left\{ (a,b,c) \;\middle|\; \begin{array}{l} a+b+c=2 \\ 0<a<b+c \\ 0<b<a+c \\ 0<c<a+b \end{array} \right\}. \tag{0.2.10}$$

By setting $c = 2-a-b$, we may visualize M^{lab} as the analytic open subset of $\mathbb{R}^2$ defined by pairs (a,b) satisfying $0 < a, b < 1$ and $a+b > 1$.

Figure 0.2.11: M^{lab} is the shaded triangle. The magenta lines represent the right triangles defined by $a^2+b^2=c^2$, $a^2+c^2=b^2$, and $b^2+c^2=a^2$, the blue lines represent isosceles triangles defined by $a=b$, $b=c$, and $a=c$, and the green point is the unique equilateral triangle defined by $a=b=c$.



Example 0.2.12 (Unlabeled triangles up to similarity)**.** We now turn to the moduli of unlabeled triangles up to similarity, which reveals a new feature not seen in to the two previous examples: symmetry!

We define the *moduli set of unlabeled triangles up to similarity*, denoted by M^{unl}, where the objects are unlabeled triangles in $\mathbb{R}^2$ and the equivalence relation is similarity. We can describe an unlabeled triangle uniquely by the ordered tuple (a,b,c) of increasing side lengths:

$$M^{\text{unl}} = \left\{ (a,b,c) \;\middle|\; \begin{array}{l} 0<a\le b\le c<a+b \\ a+b+c=2 \end{array} \right\}. \tag{0.2.13}$$

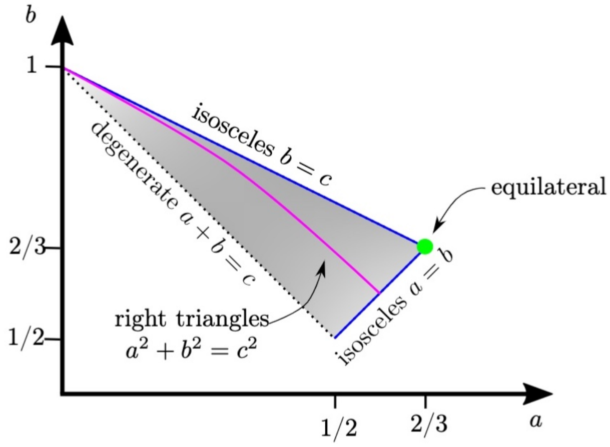


Figure 0.2.14: Picture of M^{unl}.

The isosceles triangles with $a = b$ or $b = c$ and the equilateral triangle with $a = b = c$ have symmetry groups of $\mathbb{Z}/2$ and S_3, respectively. This is unfortunately not encoded into our description M^{unl} above. Note that we can identify M^{unl} as the quotient M^{lab}/S_3 under the natural action of S_3 on the labelings, and that the stabilizers of isosceles and equilateral triangles are precisely their symmetry groups $\mathbb{Z}/2$ and S_3. The action of S_3 on the locus of triangles that are not isosceles nor equilateral is free.

0.3 The functorial worldview

Mathematical objects are determined by—and understood by—the network of relationships they enjoy with all the other objects of their species.

Barry Mazur [Maz08]

Defining a moduli functor requires specifying:

(1) families of objects,

(2) when two families of objects are equivalent, and

(3) how families pull back under morphisms.

In the algebraic category, this is packaged with a contravariant functor

$$F\colon \mathrm{Sch} \to \mathrm{Sets}, \qquad S \mapsto \{\text{families of objects over } S\}, \tag{0.3.1}$$

where for a map $f\colon S \to T$, $F(f)\colon F(T) \to F(S)$ gives the pullback map—sometimes written simply as f^*—from a family over T to a family over S. (To be a functor, pullback maps must commute with composition: $F(g \circ f) = F(f) \circ F(g)$ for maps $f\colon S \to T$ and $g\colon T \to U$.)

0.3.1 Family matters

Families allow us to provide a precise formulation of a moduli space M. A family of objects $\mathcal{C}$ over a space S defines a set-theoretic map

$$S \to M, \qquad s \mapsto [\mathcal{C}_s], \tag{0.3.2}$$

where the fiber $[\mathcal{C}_s] \in M$ is the pullback of $\mathcal{C}$ under the inclusion $\{s\} \hookrightarrow S$. In the topological (resp., algebraic) category, we desire that the map $S \to M$ is continuous (resp., algebraic). Ideally, there is a bijective correspondence between families over S and morphisms $S \to M$, or in other words that the space M *represents* the functor F of (0.3.1). If this happens, we call M a *fine moduli space*, but we will see shortly that this is often too much to hope for.

Defining moduli spaces via families has advantages:

- We can endow the moduli set M with enriched structures. To provide M a topology, we declare a subset $U \subseteq M$ to be *open* if for every family of objects $\mathcal{C}$ over S, the locus $\{s \in S \mid [\mathcal{C}_s] \in U\}$ is an open subset of S. A global function f on M can be defined as the data of compatible global functions $f_{\mathcal{X}}$ on S for every family $\mathcal{X}$ over S.

- When M is a fine moduli space, the identity map $\mathrm{id}\colon M \to M$ corresponds to a family of objects $\mathcal{U}$ over the moduli space M. This is the *universal family*: for any other family $\mathcal{C}$ over S, there is a unique morphism $S \to M$ (given set-theoretically by (0.3.2)) such that the universal family $\mathcal{U}$ pulls back to $\mathcal{C}$.

This is certainly a giant leap in abstraction! And it may seem that we just made life more difficult: rather than introducing a space by specifying its points, its topology, and possibly a complex or algebraic structure, we must specify an immense amount of categorical data. In practice, however, it is usually quite straightforward to define well-behaved notions of families. This change of perspective is enshrined in Yoneda's Lemma (0.3.12), which asserts that the functor of maps to a space uniquely determines the space.

Example 0.3.3 (Families of labeled triangles)**.** Revisiting the moduli of labeled triangles up to similarity introduced in Example 0.2.8, we define a *family of labeled triangles over a topological space* S as a tuple $(\mathcal{T}, \sigma_1, \sigma_2, \sigma_3)$ where $\mathcal{T} \to S$ is a fiber bundle with three sections $\sigma_i\colon S \to \mathcal{T}$ equipped with a continuous distance function $d\colon \mathcal{T} \times_S \mathcal{T} \to \mathbb{R}_{\geq 0}$. We require that for every point $s \in S$, the restriction $d_s\colon \mathcal{T}_s \times \mathcal{T}_s \to \mathbb{R}_{\geq 0}$ is a metric on the fiber $\mathcal{T}_s$ such that $\mathcal{T}_s$ isometric to a triangle with vertices $\sigma_i(s)$.

We say two families $(\mathcal{T}, (\sigma_i))$ and $(\mathcal{T}', (\sigma_i'))$ of labeled triangles over $S \in \mathrm{Top}$ are *similar* if there is a homeomorphism $f\colon \mathcal{T} \to \mathcal{T}'$ over S compatible with the sections (i.e., $f \circ \sigma_i = \sigma_i'$) such that for each $s \in S$, the induced map $\mathcal{T}_s \to \mathcal{T}_s'$ on fibers is a similarity of triangles, i.e., an isometry after rescaling. Given a family $\mathcal{T} \to S$ of labeled triangles and a continuous map $S' \to S$, the pullback family is defined as the fiber product $\mathcal{T} \times_S S'$ of sets together with the pullback sections $\sigma_i'\colon S' \to \mathcal{T}'$ and its inherited distance function.

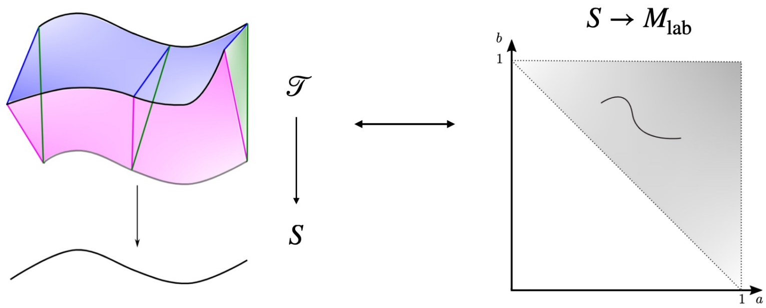


Figure 0.3.4: A family of labeled triangles over a curve corresponds to an arc in the moduli space.

We define the *moduli functor of labeled triangles* as

$$F_{M^{\text{lab}}}\colon \text{Top} \to \text{Sets}, \quad S \mapsto \{\text{families } (\mathcal{T} \to S, \sigma_i) \text{ of labeled triangles}\}/(\text{similarity}).$$

Recall from (0.2.10) that the assignment of a triangle to its side lengths yields a bijection between $F_{M^{\text{lab}}}$ and

$$M^{\text{lab}} = \left\{ (a,b,c) \;\middle|\; \begin{matrix} a+b+c=2 \\ 0<a<b+c \\ 0<b<a+c \\ 0<c<a+b \end{matrix} \right\}.$$

Since this extends to compatible isomorphisms $F_{M^{\text{lab}}}(S) \to \text{Mor}(S, M^{\text{lab}})$ for every topological space S, the topological space M^{lab} represents the functor $F_{M^{\text{lab}}}$. Consequently, there is a universal family $\mathcal{T}_{\text{univ}} \to M^{\text{lab}}$ with sections $\sigma_i\colon M^{\text{lab}} \to \mathcal{T}_{\text{univ}}$.

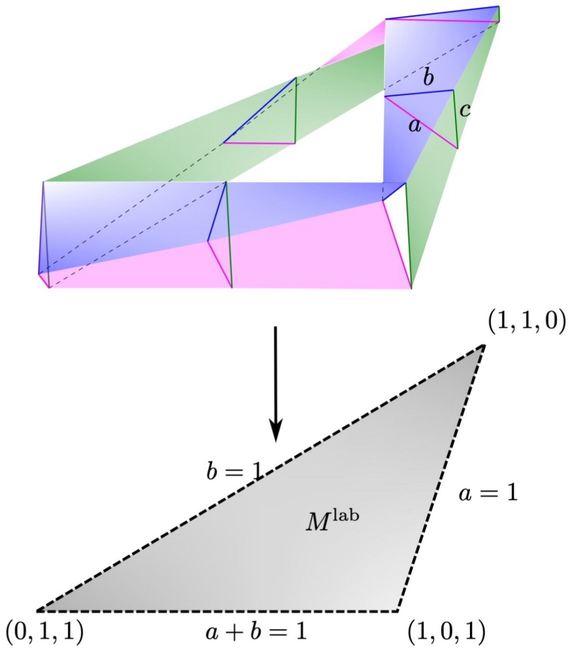


Figure 0.3.5: The universal family $U^{\text{lab}} \to M^{\text{lab}}$ of labeled triangles up to similarity.

Example 0.3.6 (Families of unlabeled triangles)**.** Revisiting Example 0.2.12, we define a *family of unlabeled triangles* as a fiber bundle $\mathcal{T} \to S$ equipped with a continuous distance function $d\colon \mathcal{T} \times_S \mathcal{T} \to \mathbb{R}_{\geq 0}$ that restricts to a metric on every fiber and such that every fiber is isometric to a triangle. Two families $\mathcal{T} \to S$ and $\mathcal{T}' \to S$ are *similar* if there is a homeomorphism $f\colon \mathcal{T} \to \mathcal{T}'$ over S compatible with the sections inducing similarities of triangles on fibers.

We define the functor

$$F_{M^{\mathrm{unl}}}\colon \mathrm{Top} \to \mathrm{Sets}, \quad S \mapsto \{\text{families of unlabeled triangles}\}/(\text{similarity})$$

but we can already see complications arising from the presence of symmetries of our objects—each equilateral triangle has symmetry group S_3 while the isosceles triangles have symmetry groups $\mathbb{Z}/2$. This functor is *not* representable by Proposition 0.3.29as there are non-trivial families of triangles $\mathcal{T}$ such that all fibers are similar triangles . For instance, we construct a non-trivial family of triangles over S^1 by gluing two trivial families via a symmetry of an equilateral triangle.

Figure 0.3.7: A trivial (left) and non-trivial (right) family of equilateral triangles. Image taken from a video produced by Jonathan Wise: see http://math.colorado.edu/~jonathan.wise/visual/moduli/index.html.

0.3.2 Moduli functors of curves, vector bundles, and orbits

Defining a moduli functor $F\colon \mathrm{Sch}/\mathbb{C} \to \mathrm{Sets}$ in the category of $\mathbb{C}$-schemes entails specifying for every $\mathbb{C}$-scheme S a set $F(S)$ of families of objects over S, and a pullback map $F(S) \to F(S')$ for every morphism $S' \to S$ of $\mathbb{C}$-schemes which are compatible under composition.

To gain intuition for a moduli functor, it is always useful to plug in special test schemes. For instance, by plugging in $S = \operatorname{Spec} \mathbb{C}$, we obtain the underlying moduli set $F(\operatorname{Spec} \mathbb{C})$ of objects. By plugging in $S = \mathbb{C}[\epsilon]/(\epsilon^2)$, we obtain a set of pairs consisting of a $\mathbb{C}$-point and a tangent vector, and plugging in a curve (or a DVR) gives families of objects over the curve.

Example 0.3.8 (Moduli functor of smooth curves)**.** A *family of smooth curves of genus g* is a smooth, proper morphism $\mathcal{C} \to S$ of schemes such that for every $s \in S$, the fiber $\mathcal{C}_s$ is a connected (smooth proper) curve of genus g.

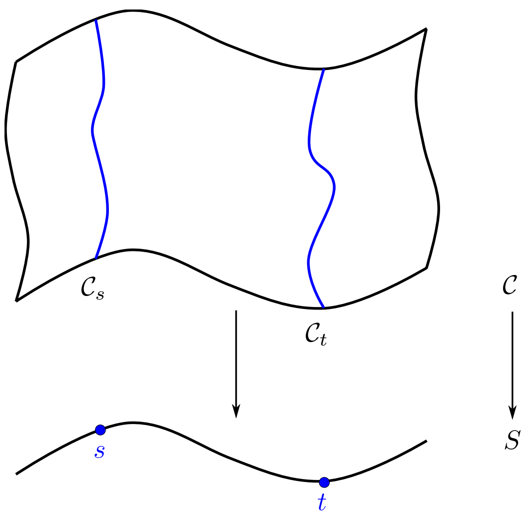


Figure 0.3.9: A family of curves over a curve S.

The *moduli functor of smooth curves of genus g* is

$$F_{M_g}\colon \mathrm{Sch}/\mathbb{C} \to \mathrm{Sets}, \quad S \mapsto \{\text{families of smooth curves } \mathcal{C} \to S \text{ of genus } g\} / \sim,$$

where two families $\mathcal{C} \to S$ and $\mathcal{C}' \to S$ are equivalent if there is an isomorphism $\mathcal{C} \to \mathcal{C}'$ over S. If $S' \to S$ is a map of $\mathbb{C}$-schemes and $\mathcal{C} \to S$ is a family of curves, the pullback is defined as the family $\mathcal{C} \times_S S' \to S'$.

Example 0.3.10 (Moduli functor of vector bundles on a curve)**.** Let C be a fixed smooth, connected, and projective curve over $\mathbb{C}$. A *family of vector bundles of rank r and degree d* over a $\mathbb{C}$-scheme S is a vector bundle $\mathcal{E}$ on $C \times S$ such that for every $s \in S$, the restriction $\mathcal{E}_s := \mathcal{E}|_{C_{\kappa(s)}}$ of $\mathcal{E}$ to $C_{\kappa(s)} := C \times_{\mathbb{C}} \kappa(s)$ has rank r and degree d. The *moduli functor of vector bundles on C of rank r and degree d* is

$$\mathrm{Sch}/\mathbb{C} \to \mathrm{Sets} \qquad S \mapsto \left\{ \begin{array}{l} \text{vector bundles } \mathcal{E} \text{ on } C \times S \\ \text{of rank } r \text{ and degree } d \end{array} \right\} / (\text{isomorphism}),$$

If $f\colon S' \to S$ is a map of $\mathbb{C}$-schemes and E is a vector bundle on $C \times S$, the pullback is defined as the vector bundle $(\mathrm{id} \times f)^*\mathcal{E}$ on $C \times S'$.

We will see in Section 0.3.5 that these two functors are not representable, and correspondingly that there is no fine moduli space.

Example 0.3.11 (Moduli functor of orbits)**.** Consider the action of an algebraic group G over $\mathbb{C}$ acting on a $\mathbb{C}$-scheme X. For every $\mathbb{C}$-scheme S, the abstract group $G(S)$ acts on the set $X(S)$—in fact, giving such actions functorial in S uniquely specifies the group action (Exercise B.1.10). We can consider the functor

$$\mathrm{Sch}/\mathbb{C} \to \mathrm{Sets} \qquad S \mapsto X(S)/G(S).$$

This is a very naive candidate for a moduli functor of a quotient, and very far from being representable even for free actions (see Exercise 0.3.37). We will modify this example in §0.6.5.

In some cases, you may know precisely which objects you want to parameterize, but it may not be straightforward to introduce a notion of families. Or there may be several candidate notions for a family of objects, which could translate to different scheme structures on the same topological space. This happens for instance for the moduli of higher dimensional varieties.

0.3.3 Yoneda's lemma and representable functors

The Yoneda lemma is the hardest trivial thing in mathematics.

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Following Grothendieck, we will study a scheme X by studying all maps to it! This psychological trick is the heart of the functorial approach: a geometric object is determined by its relationship to all other objects. This is made rigorous by the *Yoneda Lemma*: for an object X of a category $\mathcal{C}$, the contravariant functor

$$h_X\colon \mathcal{C} \to \text{Sets}, \qquad S \mapsto \operatorname{Mor}(S, X)$$

recovers the object X itself.

Lemma 0.3.12 (Yoneda Lemma)**.** *Let $\mathcal{C}$ be a category and X be an object. For every contravariant functor $G\colon \mathcal{C} \to \text{Sets}$, the map*

$$\operatorname{Mor}(h_X, G) \to G(X), \qquad \alpha \mapsto \alpha_X(\mathrm{id}_X)$$

is bijective and functorial with respect to both X and G, where the left-hand side denotes the set of natural transformations $h_X \to G$ and α_X denotes the map $h_X(X) = \operatorname{Mor}(X, X) \to G(X)$.

Exercise 0.3.13. (to be done at least once in your lifetime) Prove Yoneda's lemma.

This requires spelling out precisely what 'functorial with respect to both X and G' means.

Caution 0.3.14. Throughout this book, we will consistently abuse notation by conflating an element $g \in G(X)$ and the corresponding morphism $h_X \to G$, which we will often write simply as $X \to G$.

Definition 0.3.15 (Representable functors and fine moduli spaces)**.** We say that a functor $F\colon$ Sch $\to$ Sets is *representable by a scheme* if there exists a scheme X and an isomorphism of functors $F \xrightarrow{\sim} h_X$.

When F is a moduli functor representable by a scheme M, we say that M is a *fine moduli space.*

By the Yoneda Lemma (0.3.12), if a functor is representable, then it is representable by a *unique* scheme. One of our aims is to understand when a given moduli functor F has a fine moduli space, i.e., is representable by a scheme.

Example 0.3.16 (Projective space as a functor)**.** By [Har77, Thm. II.7.1], there is a functorial bijection

$$\operatorname{Mor}(S, \mathbb{P}^n_{\mathbb{Z}}) \cong \left\{ (L, s_0, \dots, s_n) \;\middle|\; \begin{array}{l} L \text{ is a line bundle on } S \text{ globally} \\ \text{generated by } s_0, \dots, s_n \in \Gamma(S, L) \end{array} \right\} / \sim,$$

where $(L, s_i) \sim (L', s'_i)$ if there exists an isomorphism $\alpha\colon L \to L'$ such that $s_i = \alpha^* s'_i$ for all i. In other words, the functor on the right is representable by the scheme $\mathbb{P}^n_{\mathbb{Z}}$. The condition that the sections s_i are globally generated translates to the condition that for every $x \in S$, at least one section $s_i(x) \in L \otimes \kappa(x)$ is nonzero, or equivalently to the surjectivity of $(s_0, \dots, s_n)\colon \mathcal{O}_S^{\oplus n+1} \to L$.

Example 0.3.17 (The Grassmannian functor)**.** As a set, the Grassmannian $\mathrm{Gr}(q,n)$ parameterizes q-dimensional *quotients* of n-dimensional space.[4] But what are families of q-dimensional quotients over a scheme S? A naive guess might be quotients $p\colon \mathcal{O}_S^n \twoheadrightarrow \mathcal{O}_S^k$ but this has no chance to be representable (see Exercise 0.3.37). The case of projective space suggests we define the *Grassmannian functor* as

$$\begin{aligned} \mathrm{Gr}(q,n)\colon\ &\mathrm{Sch} \to \mathrm{Sets} \\ &S \mapsto \left\{ [\mathcal{O}_S^{\oplus n} \twoheadrightarrow Q] \;\middle|\; Q \text{ is a vector bundle of rank } q \right\} / \sim, \end{aligned}$$

where $[p\colon \mathcal{O}_S^{\oplus n} \twoheadrightarrow Q] \sim [p'\colon \mathcal{O}_S^{\oplus n} \twoheadrightarrow Q']$ if there exists an isomorphism $\Psi\colon Q \xrightarrow{\sim} Q'$ such that

$$\begin{tikzcd} \mathcal{O}_S^{\oplus n} \arrow[r, "p"] \arrow[dr, "p'"'] & Q \arrow[d, "\Psi"] \\ & Q' \end{tikzcd}$$

commutes (i.e., $p' = \Psi \circ p$), or equivalently if $\ker(p) = \ker(p')$. Pullbacks are defined in the natural manner.

We will later show that $\mathrm{Gr}(q,n)$ is representable by a scheme projective over $\mathbb{Z}$ (Theorem 1.1.1). The proof of this result is a good illustration of the utility of the functorial approach and a warmup for the representability of Hilb and Quot (Theorems 1.1.2 and 1.1.3).

These exercises will give you some practice with the functorial approach.

Exercise 0.3.18 (Affine and Projective Space)**.** Let S be a scheme and E be a vector bundle on S.

(a) Show that the affine space $\mathbb{A}(E) := \mathcal{S}\mathrm{pec}_S(\mathrm{Sym}^* E)$ represents the functor assigning $f\colon T \to S$ to $\Gamma(T, f^*E^\vee)$, where $E^\vee := \mathscr{H}om_{\mathcal{O}_S}(E, \mathcal{O}_S)$. Note that a $\Bbbk$-point of $\mathbb{A}(E)$ is an element of the *dual* of $E \otimes_{\mathcal{O}_S} \Bbbk$.[5] Observe also the special case $\mathbb{A}(\mathcal{O}_S^{\oplus n}) \cong \mathbb{A}_S^n$.

(b) Show that the projectivization $\mathbb{P}(E) := \mathcal{P}\mathrm{roj}_S(\mathrm{Sym}^* E)$ of E represents the functor

$$\begin{aligned} &\mathrm{Sch}/S \to \mathrm{Sets} \\ &(T \xrightarrow{f} S) \mapsto \{\text{quotients } q\colon f^*E \twoheadrightarrow L \text{ where } L \text{ is a line bundle on } T\}/\sim \end{aligned}$$

where $[q\colon f^*E \twoheadrightarrow L] \sim [q'\colon f^*E \twoheadrightarrow L']$ if $\ker(q) = \ker(q')$ (or equivalently there is an isomorphism $\alpha\colon L \to L'$ with $q' = \alpha \circ q$). Note that E is naturally identified with the pushforward of $\mathcal{O}_{\mathbb{P}(E)}(1)$ along $\mathbb{P}(E) \to S$, and that when E is trivial, there is an identification $\mathbb{P}(\mathcal{O}_S^{n+1}) \cong \mathbb{P}^n$.

Exercise 0.3.19. Provide functorial descriptions of:

(a) $\mathbb{A}^n \smallsetminus 0$;

(b) $\operatorname{Spec} \Bbbk[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$;

(c) $\mathcal{S}\mathrm{pec}_S\, \mathcal{A}$ where $\mathcal{A}$ is a quasi-coherent sheaf of algebras on a scheme S; and

[4] Alternatively, the points could be considered as q-dimensional *subspaces* but in these notes, we will follow Grothendieck's convention using quotients.

[5] This is consistent with [Har77, Exc. 5.18, Def. p.162], [EGA, II.4.1.1], and [SP, Tag 01OB], but beware that some authors use the dual $E^\vee$ instead of E in defining $\mathbb{A}(E)$ and $\mathbb{P}(E)$.

(d) $\operatorname{Proj} R$ where R is a positively graded ring.

Exercise 0.3.20 (moderate, good practice)**.** Let X be a scheme and let F_1 and F_2 be vector bundles on X. Show that the functor

$$\underline{\mathscr{H}om}_{\mathcal{O}_X}(F_1, F_2) \colon \mathrm{Sch}/X \to \mathrm{Sets}, \quad (f \colon T \to X) \mapsto \operatorname{Hom}_{\mathcal{O}_T}(f^*F_1, f^*F_2)$$

is representable by $\mathbb{A}(\mathscr{H}om_{\mathcal{O}_X}(F_1, F_2)^\vee) \to X$.

Note the distinction between $\operatorname{Hom}_{\mathcal{O}_X}(F_1, F_2)$ *(group of* $\mathcal{O}_X$*-module homomorphisms),* $\mathscr{H}om_{\mathcal{O}_X}(F_1, F_2)$ *(sheaf of* $\mathcal{O}_X$*-module homomorphisms), and* $\underline{\mathscr{H}om}_{\mathcal{O}_X}(F_1, F_2)$ *(functor or scheme parameterizing* $\mathcal{O}_X$*-module homomorphisms).*

Exercise 0.3.21 (moderate)**.** Let X be a scheme over a field $\Bbbk$ and let F_1 and F_2 be vector bundles on X. Show that the functor

$$\underline{\operatorname{Ext}}^1_{\mathcal{O}_X}(F_1, F_2) \colon \mathrm{AffSch}/\Bbbk \to \mathrm{Sets}, \quad T \mapsto \operatorname{Ext}^1_{\mathcal{O}_{X \times T}}(p_1^*F_1, p_1^*F_2),$$

from the category of *affine* $\Bbbk$-schemes, is representable by $\mathbb{A}(\operatorname{Ext}^1_{\mathcal{O}_X}(F_1, F_2)^\vee)$. See also [LP97, §7.3] and Proposition 8.2.11.

Exercise 0.3.22 (moderate)**.** Let S be a noetherian scheme and let $E \to F$ be homomorphism of coherent sheaves on $\mathbb{P}^n_S$ with F flat over S. Show that the subfunctor of S (or more precisely of $h_S = \operatorname{Mor}(-, S)$) defined by

$$\mathrm{Sch} \to \mathrm{Sets}, \quad T \mapsto \{\text{morphisms } T \to S \text{ such that } E_T \to F_T \text{ is zero}\}$$

is representable by a closed subscheme of X.

Exercise 0.3.23 (Weil Restriction, hard)**.** If $S' \to S$ is a morphism of schemes, the *Weil restriction* of a morphism $X' \to S'$ is the functor

$$\operatorname{Res}_{S'/S}(X') \colon \mathrm{Sch}/S \to \mathrm{Sets}, \quad (T \to S) \mapsto X'(T \times_S S').$$

(1) If $\Bbbk/\Bbbk'$ is a field extension of degree d, show that $\operatorname{Res}_{\Bbbk'/\Bbbk}\mathbb{A}^1 \cong \mathbb{A}^d$.

(2) Show that $T := \operatorname{Res}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_{m,\mathbb{C}})$ is an algebraic group over $\mathbb{R}$, which is a non-split torus of rank 2, i.e., $T \times_{\mathbb{R}} \mathbb{C} \cong \mathbb{G}^2_{m,\mathbb{C}}$ but $T \ncong \mathbb{G}^2_{m,\mathbb{R}}$.

(3) (hard) Assume that $S' \to S$ is finite and flat, and that for every $s \in S$, every finite set of points of the fiber X'_s is contained in an affine, then $\operatorname{Res}_{S'/S}(X')$ is representable. See also [BLR90, Thm. 7.4].

0.3.4 Universal families

Definition 0.3.24. If $F \colon \mathrm{Sch} \to \mathrm{Sets}$ is a moduli functor representable by a scheme M via an isomorphism $\alpha \colon F \xrightarrow{\sim} h_M$ of functors, then the *universal family* of F is the object $u \in F(M)$ corresponding under α to the identity morphism $\mathrm{id}_M \in h_M(M) = \operatorname{Mor}(M, M)$.

Exercise 0.3.25 (easy but important)**.** Let $F \colon \mathrm{Sch} \to \mathrm{Sets}$ be a functor representable by a scheme M, and let $u \in F(M)$ be the universal family. Show that if $a \in F(T)$ is an object over a scheme T corresponding to a map $f_a \colon T \to M$, then the object a is the pullback of u under f_a, i.e., $a = F(f_a)(u)$.

Suspend your skepticism for a moment and suppose that there actually exists a scheme M_g representing the moduli functor of smooth curves of genus g (Example 0.3.8). Then corresponding to the identity map $M_g \to M_g$ is a family of genus g curves $U_g \to M_g$ satisfying the following universal property: for every smooth family of curves $\mathcal{C} \to S$ over a scheme S, there is a unique map $S \to M_g$ and cartesian diagram

$$\begin{tikzcd} \mathcal{C} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & U_g \arrow[d] \\ S \arrow[r] & M_g. \end{tikzcd}$$

The map $S \to M_g$ sends a point $s \in S$ to the curve $[\mathcal{C}_s] \in M_g$. While there does not exist a *scheme* M_g representing the moduli functor, there is an *algebraic stack* $\mathcal{M}_g$ parameterizing smooth curves which is equipped with a universal family; see §3.1.29.

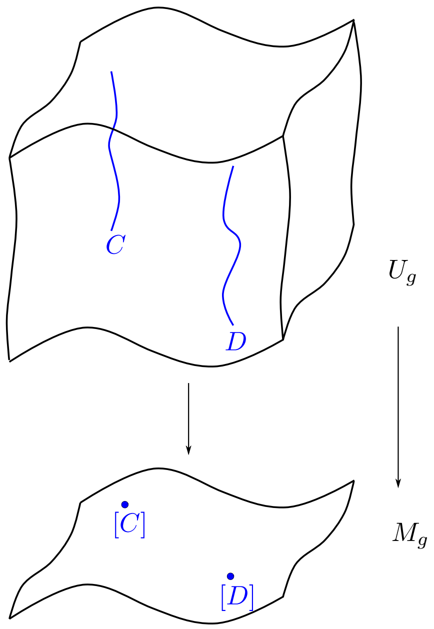

Figure 0.3.26: Visualization of a (non-existent) universal family over M_g.

Example 0.3.27. The universal family of the moduli functor of projective space (Example 0.3.16) is the line bundle $\mathcal{O}(1)$ on $\mathbb{P}^n$ together with the sections $x_0, \dots, x_n \in \Gamma(\mathbb{P}^n, \mathcal{O}(1))$.

Example 0.3.28 (Classifying spaces in algebraic topology)**.** Let G be a topological group and $\mathrm{Top}^{\mathrm{para}}$ be the category of paracompact topological spaces where morphisms are defined up to homotopy. It is a theorem in algebraic topology that the functor

$$\mathrm{Top}^{\mathrm{para}} \to \mathrm{Sets}, \qquad S \mapsto \{\text{principal } G\text{-bundles } P \to S\}/(\text{isomorphism}),$$

is representable by a topological space, which we denote by BG and call the *classifying space*. The universal family is usually denoted by $EG \to BG$. For example, the classifying space $B\mathbb{C}^*$ is the infinite dimensional manifold $\mathbb{CP}^\infty$. In algebraic geometry, however, the classifying stack $B\mathbb{G}_{m,\mathbb{C}}$ is an algebraic stack of dimension -1.

0.3.5 Examples of non-representable moduli functors

If $F\colon \mathrm{Sch}/\mathbb{C} \to \mathrm{Sets}$ is a moduli functor, then an object $E \in F(\mathbb{C})$ with a non-trivial automorphism can prevent the functor F from being representable. This is because we may glue trivial families using the automorphism to construct a *non-trivial* family $\mathcal{E}$ over a scheme S such that every fiber $\mathcal{E}_s$ (i.e., the pullback of $\mathcal{E}$ along $\operatorname{Spec}\mathbb{C} \to S$) is isomorphic to E.

Proposition 0.3.29. *Let $F\colon \mathrm{Sch}/\mathbb{C} \to \mathrm{Sets}$ be a moduli functor. If there is a family of objects $\mathcal{E} \in F(S)$ over a variety S such that*

- *(a) the fibers $\mathcal{E}_s$ are isomorphic for $s \in S(\mathbb{C})$, and*
- *(b) the family $\mathcal{E}$ is non-trivial, i.e., is not equal to the pullback of an object $E \in F(\mathbb{C})$ along the structure map $S \to \operatorname{Spec}\mathbb{C}$,*

then F is not representable.

Proof. Suppose by way of contradiction that F is representable by a scheme X. By condition (a), the restriction $E := \mathcal{E}_s$ is independent of $s \in S(\mathbb{C})$ and defines a unique point $x \in X(\mathbb{C})$. As S is reduced, the map $S \to X$ factors as $S \to \operatorname{Spec}\mathbb{C} \xrightarrow{x} X$. This implies that the family $\mathcal{E}$ is the pullback under the constant map $S \to \operatorname{Spec}\mathbb{C} \xrightarrow{x} X$, i.e., $\mathcal{E}$ is a trivial family, which contradicts condition (b). □

Exercise 0.3.30 (easy)**.** Show that the moduli functor $F\colon \mathrm{Sch}/\mathbb{C} \to \mathrm{Sets}$ assigning a scheme S to the set of isomorphism classes of vector bundles on S is not representable.

Example 0.3.31 (Moduli of elliptic curves)**.** An *elliptic curve* is a pair (E, p) where E is a smooth, connected, and projective curve E of genus 1 and $p \in E(\mathbb{C})$. A *family of elliptic curves* over a scheme S is a pair $(\mathcal{E} \to S, \sigma)$ where $\mathcal{E} \to S$ is smooth proper morphism with a section $\sigma\colon S \to \mathcal{E}$ such that for every $s \in S$, the fiber $(\mathcal{E}_s, \sigma(s))$ is an elliptic curve over the residue field $\kappa(s)$. The *moduli functor of elliptic curves* is

$$\begin{aligned} F_{M_{1,1}}\colon\ & \mathrm{Sch} \to \mathrm{Sets} \\ & S \mapsto \{\text{families } (\mathcal{E} \to S, \sigma) \text{ of elliptic curves }\}/\sim, \end{aligned}$$

where $(\mathcal{E} \to S, \sigma) \sim (\mathcal{E}' \to S, \sigma')$ if there is an S-isomorphism $\alpha\colon \mathcal{E} \to \mathcal{E}'$ compatible with the sections (i.e., $\sigma' = \alpha \circ \sigma$).

Exercise 0.3.32 (good practice)**.** Consider the family of elliptic curves defined over $\mathbb{A}^1 \smallsetminus 0$ (with coordinate t) by

$$\begin{tikzcd} \mathcal{E} := V(y^2z - x^3 + tz^3) \arrow[r, hook] \arrow[d] & (\mathbb{A}^1 \smallsetminus 0) \times \mathbb{P}^2 \\ \mathbb{A}^1 \smallsetminus 0 \end{tikzcd}$$

with section $\sigma\colon \mathbb{A}^1 \smallsetminus 0 \to \mathcal{E}$ given by $t \mapsto [0, 1, 0]$. Show that $(\mathcal{E} \to \mathbb{A}^1 \smallsetminus 0, \sigma)$ satisfies (a) and (b) in Proposition 0.3.29, and conclude that $F_{M_{1,1}}$ is not representable.

Example 0.3.33 (Moduli functor of smooth curves)**.** Let C be a curve with a non-trivial automorphism $\alpha \in \operatorname{Aut}(C)$, and let N be the nodal cubic curve, which we can think of as $\mathbb{P}^1$ with the points 0 and ∞ glued together. We can construct a family $\mathcal{C} \to N$ by taking the trivial family $\pi\colon C \times \mathbb{P}^1 \to \mathbb{P}^1$ and gluing the fiber $\pi^{-1}(0)$ with $\pi^{-1}(\infty)$ via the automorphism α. To show that the moduli functor

Sch/$\mathbb{C}$ $\to$ Sets of smooth curves is not representable, it suffices to show that $\mathcal{C} \to N$ is non-trivial.

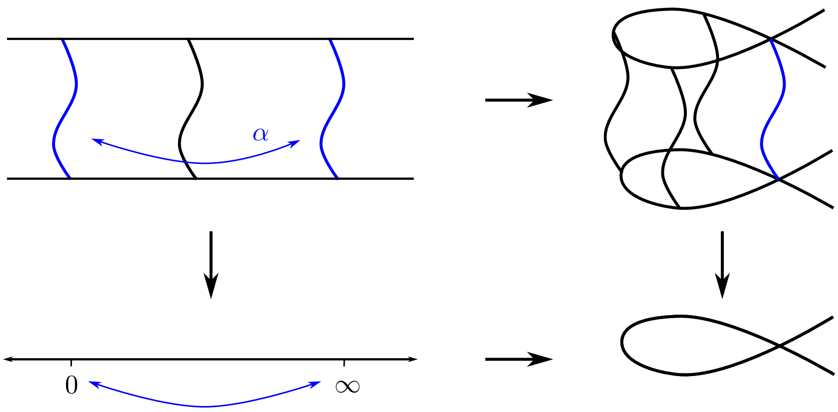

Figure 0.3.34: Family of curves over the nodal cubic obtaining by gluing the fibers over 0 and ∞ of the trivial family over $\mathbb{P}^1$ via α. (It would be more illustrative to draw a Möbius band as the family of curves over the nodal cubic.)

Exercise 0.3.35 (details)**.** Show that $\mathcal{C} \to N$ is a non-trivial family.

Exercise 0.3.36 (good practice)**.** Show that the moduli functor of vector bundles over a curve C is not representable.

0.3.6 Schemes are sheaves in the big Zariski topology

If $F\colon$ Sch $\to$ Sets is representable by a scheme X, then F is necessarily a *sheaf in the big Zariski topology*, that is, for every scheme S, the presheaf on the Zariski topology of S, defined by assigning to an open subset $U \subseteq S$ the set $F(U)$, is a sheaf on the Zariski topology of S. This is a restatement that morphisms into the fixed scheme X glue uniquely. The failure to be a sheaf therefore provides another obstruction to the representability of a given moduli functor F.

Exercise 0.3.37 (good practice)**.**

(a) Show that the following naive Grassmannian functor

$$F\colon \text{Sch} \to \text{Sets}, \quad S \mapsto \{\text{quotients } q\colon \mathcal{O}_S^n \twoheadrightarrow \mathcal{O}_S^k\}/\sim$$

is not representable.

(b) Under the usual scaling action of $\mathbb{G}_m$ on $\mathbb{A}^{n+1} \smallsetminus 0$, show that the functor $S \mapsto (\mathbb{A}^{n+1} \smallsetminus 0)(S)/\mathbb{G}_m(S)$ is not a sheaf.

The presence of non-trivial automorphisms often implies that a given moduli functor is not a sheaf in the big Zariski topology.

Example 0.3.38. Consider the moduli functor F_{M_g} of smooth curves from Example 0.3.8. Let $\{S_i\}$ be a Zariski open covering of a scheme S, and suppose that $\mathcal{C}_i \to S_i$ are families of smooth curves $\mathcal{C}_i \to S_i$ with isomorphisms $\alpha_{ij}\colon \mathcal{C}_i|_{S_{ij}} \xrightarrow{\sim} \mathcal{C}_j|_{S_{ij}}$ on the intersection $S_{ij} := S_i \cap S_j$. The requirement that F_{M_g} be a sheaf (when restricted to the Zariski topology on S) implies that the families $\mathcal{C}_i \to S_i$ glue uniquely to a family of curves $\mathcal{C} \to S$. However, we have not required the isomorphisms α_i to be compatible on the triple intersection (i.e., $\alpha_{ij}|_{S_{ijk}} \circ \alpha_{jk}|_{S_{ijk}} = \alpha_{ik}|_{S_{ijk}}$), which is necessary for gluing schemes [Har77, Exc. II.2.12]. For this reason, F_{M_g} fails to be a sheaf.

Exercise 0.3.39. Show that the moduli functors of smooth curves and elliptic curves are not sheaves by explicitly exhibiting a scheme S, an open cover $\{S_i\}$, and families of curves over S_i that do not glue to a family over S.

0.3.7 The yoga of functors

Contravariant functors $F\colon \mathrm{Sch} \to \mathrm{Sets}$ form a category $\mathrm{Fun}(\mathrm{Sch}, \mathrm{Sets})$ where morphisms are natural transformations. This category has fiber products: given morphisms $\alpha\colon F \to G$ and $\beta\colon G' \to G$, we define

$$\begin{aligned} F \times_G G' \colon\ & \mathrm{Sch} \to \mathrm{Sets} \\ & S \mapsto \{(a,b) \in F(S) \times G'(S) \mid \alpha_S(a) = \beta_S(b) \in G(S)\}\,. \end{aligned}$$

Exercise 0.3.40 (easy)**.** Show that $F \times_G G'$ satisfies the universal property for fiber products in $\mathrm{Fun}(\mathrm{Sch}, \mathrm{Sets})$.

Definition 0.3.41.

(1) We say that a morphism $F \to G$ of contravariant functors is *representable by schemes* if for every map $S \to G$ from a scheme S, the fiber product $F \times_G S$ is representable by a scheme.

(2) We say that a morphism $F \to G$ is an *open immersion* or that a subfunctor $F \subseteq G$ is *open* if for every morphism $S \to G$ from a scheme S, $F \times_G S$ is representable by an open subscheme of S.

(3) We say that a set of open subfunctors $\{F_i\}$ of F is a *Zariski open cover* if for every morphism $S \to F$ from a scheme S, $\{F_i \times_F S\}$ is a Zariski open cover of S (and in particular each F_i is an open subfunctor of F).

Each of these conditions can be checked on *affine* schemes.

These definitions give a recipe for checking that a given functor F is representable by a scheme: find a Zariski open cover $\{F_i\}$ where each F_i is representable.

Exercise 0.3.42 (good practice)**.**

(a) Let $F\colon \mathrm{Sch} \to \mathrm{Sets}$ be a functor which is a sheaf in the big Zariski topology, and let $\{F_i\}$ be a Zariski open cover of F. Show that if each F_i is representable by a scheme, then so is F.

(b) Show that a collection of open subfunctors $\{F_i\}$ of F is a Zariski open cover if and only if the map $\coprod_i F_i(\Bbbk) \to F(\Bbbk)$ is surjective for each algebraically closed field $\Bbbk$.

(c) Given morphisms of schemes $X \to Y$ and $Y' \to Y$, reprove the existence of the fiber product $X \times_Y Y'$ in the category of schemes by exhibiting a Zariski open cover $\{F_i\}$ of $X \times_Y Y'$ where each F_i is representable by an affine scheme.

Exercise 0.3.43 (Functorial definition of a scheme)**.** Show that a scheme can be equivalently defined as a contravariant functor $F\colon \mathrm{AffSch} \to \mathrm{Sets}$ on the category of affine schemes (or covariant functor on the category of rings) as follows. Let $\mathcal{C}$ be a full subcategory of the category $\mathrm{Fun}(\mathrm{AffSch}, \mathrm{Sets})$ of contravariant functors. Extending Definitions 0.3.15 and 0.3.41, we define a functor $F\colon \mathrm{AffSch} \to \mathrm{Sets}$ to be *representable in* $\mathcal{C}$ if there exists an object $X \in \mathcal{C}$ and a functorial equivalence

$F(S) = \mathrm{Mor}(S, X)$ for every $S \in \mathrm{AffSch}$. We say that a map $F \to G$ of functors from AffSch to Sets is *representable by open immersions in* $\mathcal{C}$ if for every morphism $\operatorname{Spec} B \to G$, the fiber product $F \times_G \operatorname{Spec} B$ is representable by an object $X \in \mathcal{C}$ which is an open subscheme of $\operatorname{Spec} B$. Finally, we say that a collection $\{F_i\}$ of subfunctors of F is a *Zariski open* $\mathcal{C}$*-cover* if each $F_i \to F$ is representable by open immersions in $\mathcal{C}$ and for each algebraically closed field $\Bbbk$, the map $\coprod_i F_i(\Bbbk) \to F(\Bbbk)$ is surjective.

(a) Letting $\mathcal{C} = \mathrm{AffSch}$, show that a scheme with affine diagonal can be equivalently defined as a functor $F\colon \mathrm{AffSch} \to \mathrm{Sets}$ such that there exists a Zariski open $\mathcal{C}$-cover $\{F_i\}$ of F with each F_i representable in $\mathcal{C}$.

(b) Letting $\mathcal{C}$ be the category of schemes with affine diagonal, show that a scheme can be equivalently defined as a functor $F\colon \mathrm{AffSch} \to \mathrm{Sets}$ such that there exists a Zariski open $\mathcal{C}$-cover $\{F_i\}$ with each F_i representable in $\mathcal{C}$.

(c) Alternatively, show that a scheme can be defined as a suitable contravariant functor on the category of *quasi-affine schemes*.

Replacing Zariski opens with étale morphisms in the above exercise leads to the definition of an algebraic space (Definition 3.1.2).

Exercise 0.3.44 (hard)**.** Let $X \to Y$ be a morphism of schemes each proper over a scheme S. If X is flat over S, show that the subfunctor $F \subseteq \mathrm{Mor}(-, S)$, parameterizing maps $T \to S$ of schemes such that $X_T \xrightarrow{\sim} Y_T$ is an isomorphism, is representable by an open subfunctor.

Hint: If $s \in S$ is a point such that $X_s \to Y_s$ is an isomorphism, use the Fibral Flatness Criterion (A.2.10) to show that $X \to Y$ is flat over s. Then reduce to the case when $X \to Y$ is finite étale.

0.4 Moduli groupoids

La conclusion pratique à laquelle je suis arrivé dès maintenant, c'est que chaque fois que en vertu de mes critères, une variété de modules (ou plutôt, un schéma de modules) pour la classification des variations (globales, ou infinitésimales) de certaines structures (variétés complètes non singulières, fibrés vectoriels, etc.) ne peut exister, malgré de bonnes hypothèses de platitude, propreté, et non singularité éventuellement, la raison en est seulement l'existence d'automorphismes de la structure qui empêche la technique de descente de marcher.

Alexander Grothendieck, letter to Serre, 1959 [CS01, p. 94]

We now change our perspective: rather than specifying *when* two objects are identified, we specify *how*!

One of the most desirable properties of a moduli space is the existence of a universal family (see §0.3.4), and the presence of automorphisms obstructs its existence (see §0.3.5). Encoding automorphisms into our descriptions will allow us to get around this problem. To define a moduli groupoid, we need to specify

(1) objects; and

(2) a set of equivalences (possibly empty) between any two objects.

Shortly we will combine the functorial worldview of the last section with this groupoid perspective to define moduli stacks.

0.4.1 Groupoids

A convenient mathematical structure to encode objects and their identifications is a *groupoid*.

Definition 0.4.1. A *groupoid* is a category $\mathcal{C}$ where every morphism is an isomorphism.

Two groupoids $\mathcal{C}_1$ and $\mathcal{C}_2$ are *equivalent* if there is an equivalence of categories $F\colon \mathcal{C}_1 \xrightarrow{\sim} \mathcal{C}_2$, i.e., there is a functor $G\colon \mathcal{C}_2 \to \mathcal{C}_1$ such that $F \circ G$ and $G \circ F$ are isomorphic to the identity functors, or equivalently F is fully faithful and essentially surjective.

Example 0.4.2 (Sets are groupoids)**.** If Σ is a set, the category $\mathcal{C}_\Sigma$, whose objects are elements of Σ and whose morphisms consist of only the identity morphisms, is a groupoid. We say that a groupoid $\mathcal{C}$ *is equivalent to a set* Σ if there is an equivalence of categories $\mathcal{C} \to \mathcal{C}_\Sigma$.

Example 0.4.3 (Classifying groupoid)**.** If G is a group, the *classifying groupoid* BG of G is defined as the category with one object $\star$ such that $\mathrm{Aut}(\star) = \mathrm{Mor}(\star, \star) = G$.

Example 0.4.4. The category FB of finite sets where morphisms are bijections is a groupoid. The isomorphism classes of FB are in bijection with $\mathbb{N}$ while $\mathrm{Aut}(\{1, \ldots, n\}) = S_n$ is the permutation group.

Example 0.4.5 (Projective space)**.** Projective space is identified with the moduli groupoid of lines $L \subset \mathbb{A}^{n+1}$ through the origin where the only morphisms are the identity maps. Alternatively, the objects are nonzero linear maps $x = (x_0, \ldots, x_n)\colon \mathbb{C} \to \mathbb{C}^{n+1}$ and there is a unique morphism $x \to x'$ if and only if $\mathrm{im}(x) = \mathrm{im}(x') \subseteq \mathbb{C}^{n+1}$ (i.e., there exists a $\lambda \in \mathbb{C}^*$ such that $x' = \lambda x$).

0.4.2 Moduli groupoid of orbits

Example 0.4.6 (Moduli groupoid of orbits)**.** Given an action of a group G on a set X, we define the *moduli groupoid of orbits* $[X/G]$[6] by taking the objects to be all elements $x \in X$ and by declaring $\mathrm{Mor}(x, x') = \{g \in G \mid x' = gx\}$.

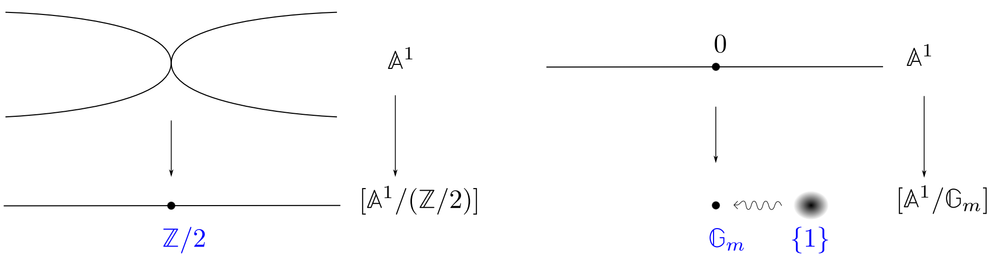

Figure 0.4.7: Pictures of the scaling actions of $\mathbb{Z}/2 = \{\pm 1\}$ and $\mathbb{G}_m$ on $\mathbb{A}^1$ over $\mathbb{C}$ with the automorphism groups listed in blue. Note that $[\mathbb{A}^1/\mathbb{G}_m]$ has two isomorphism classes of objects—0 and 1—corresponding to the two orbits—0 and $\mathbb{A}^1 \smallsetminus 0$—such that $0 \in \overline{\{1\}}$ if the set $\mathbb{A}^1/\mathbb{G}_m$ is endowed with the quotient topology.

[6] We use brackets to distinguish the groupoid quotient $[X/G]$ from the set quotient X/G. Later when G is an algebraic group and X is a scheme, $[X/G]$ will denote the quotient stack (which always exists) while X/G will denote the quotient space (if it exists).

Exercise 0.4.8 (easy)**.** Show that the moduli groupoid of orbits $[X/G]$ in Example 0.4.6 is equivalent to a set if and only if the action of G on X is free.

Example 0.4.9. Consider the category $\mathcal{C}$ with two objects x_1 and x_2 such that $\mathrm{Mor}(x_i, x_j) = \{\pm 1\}$ for $i, j = 1, 2$ where composition of morphisms is given by multiplication. Then $\mathcal{C}$ is equivalent $B(\mathbb{Z}/2)$.

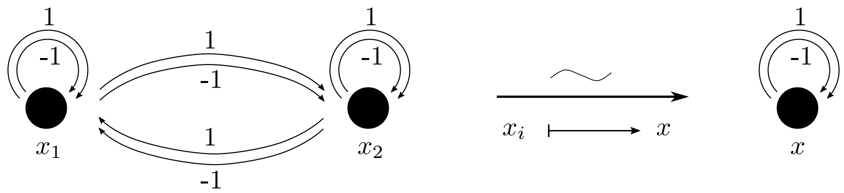

Figure 0.4.10: An equivalence of groupoids.

Exercise 0.4.11 (easy)**.** In Example 0.4.9, show that there is an equivalence of categories inducing a bijection on objects between $\mathcal{C}$ and either $[(\mathbb{Z}/2)/(\mathbb{Z}/4)]$ or $[(\mathbb{Z}/2)/(\mathbb{Z}/2 \times \mathbb{Z}/2)]$, where the action is given by surjections $\mathbb{Z}/4 \to \mathbb{Z}/2$ or $\mathbb{Z}/2 \times \mathbb{Z}/2 \to \mathbb{Z}/2$.

Example 0.4.12 (Projective space as a quotient)**.** The moduli groupoid of projective space (Example 0.4.5) can also be described as the moduli groupoid of orbits $[(\mathbb{A}^{n+1} \smallsetminus 0)/\mathbb{G}_m]$. We can also consider the quotient groupoid $[\mathbb{A}^{n+1}/\mathbb{G}_m]$, which is equivalent to the groupoid whose objects are (possibly zero) linear maps $x = (x_0, \ldots, x_n) \colon \mathbb{C} \to \mathbb{C}^{n+1}$ such that $\mathrm{Mor}(x, x') = \{t \in \mathbb{C}^* \mid x'_i = t x_i \text{ for all } i\}$. In this way, $\mathbb{P}^n$ is a subgroupoid of $[\mathbb{A}^{n+1}/\mathbb{G}_m]$.

Exercise 0.4.13 (easy, good practice)**.** If a group G acts on a set X and $x \in X$ is a point with stabilizer G_x, show that there is a fully faithful functor $BG_x \to [X/G]$. If the action is transitive, show that it is an equivalence.

A *morphisms of groupoids* $\mathcal{C}_1 \to \mathcal{C}_2$ is by definition a functor. The category $\mathrm{Mor}(\mathcal{C}_1, \mathcal{C}_2)$ has functors as objects and natural transformations as morphisms.

Exercise 0.4.14 (easy)**.** If $\mathcal{C}_1$ and $\mathcal{C}_2$ are groupoids, show that $\mathrm{Mor}(\mathcal{C}_1, \mathcal{C}_2)$ is a groupoid.

Exercise 0.4.15 (moderate, good practice)**.** If H and G are groups, show that there is an equivalence

$$\mathrm{Mor}(BH, BG) = \coprod_{\phi \in \mathrm{Hom}(H,G)/G} B\big(C^G(\mathrm{im}\,\phi)\big)$$

where $\mathrm{Hom}(H, G)/G$ denotes equivalence classes of homomorphisms $H \to G$ up to conjugation by G, and $C^G(\mathrm{im}\,\phi)$ denotes the centralizer of $\mathrm{im}\,\phi$ in G.

Exercise 0.4.16. Provide an example of group actions of H and G on sets X and Y and a map $[X/H] \to [Y/G]$ of groupoids that *does not* arise from a group homomorphism $\phi \colon H \to G$ and a ϕ-equivariant map $X \to Y$.

0.4.3 Examples of moduli groupoids

Example 0.4.17 (Moduli groupoid of smooth curves)**.** The objects are smooth, connected, and projective curves of genus g over $\mathbb{C}$ and for two curves C, C', the set of morphisms is defined as the set of isomorphisms

$$\mathrm{Mor}(C, C') = \{\text{isomorphisms } \alpha \colon C \xrightarrow{\sim} C'\}.$$

Example 0.4.18 (Moduli groupoid of vector bundles on a curve)**.** The objects are vector bundles E of rank r and degree d on a fixed curve C, and the morphisms are isomorphisms of vector bundles.

Example 0.4.19 (Moduli groupoid of unlabeled triangles)**.** Let us revisit the moduli M^{unl} of unlabeled triangles up to similarity from Example 0.2.12. Recall that we have already introduced families of unlabeled triangles and shown that this functor is not representable (Example 0.3.6).

We define the *moduli groupoid of unlabeled triangles up to similarity*, denoted by $\mathcal{M}^{\mathrm{unl}}$ (note the calligraphic font), where the objects are unlabeled triangles and the morphisms are similarities. For example, an isosceles triangle and an equilateral triangle have automorphism groups $\mathbb{Z}/2$ and S_3. We can draw essentially the same picture as Figure 0.2.14 except we record the automorphisms.

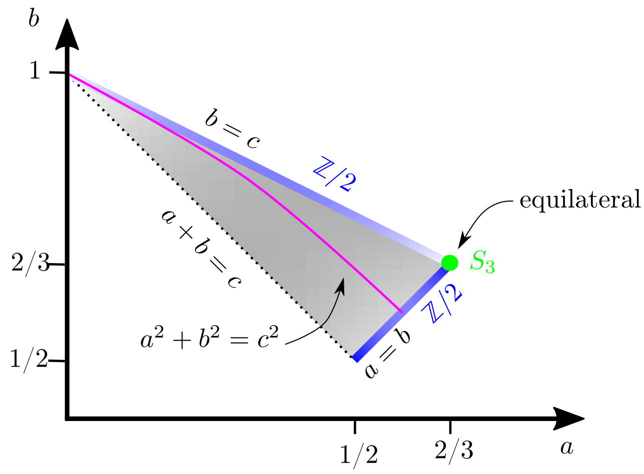

Figure 0.4.20: Picture of the moduli groupoid $\mathcal{M}^{\mathrm{unl}}$ with non-trivial automorphism groups labeled.

There is a functor

$$\mathcal{M}^{\mathrm{unl}} \to M^{\mathrm{unl}},$$

from the moduli groupoid to the moduli set, which is an equivalence on isomorphism classes of objects and collapses all morphisms to the identity. This is a first example of a *coarse moduli space*.

Exercise 0.4.21. Recalling the description of the moduli set M^{lab} of labeled triangles up to similarity from (0.2.6), show that there is a natural action of S_3 on the moduli set M^{lab} of labeled triangles up to similarity and that there is an identification $\mathcal{M}^{\mathrm{unl}} \cong [M^{\mathrm{lab}}/S_3]$ of groupoids.

Exercise 0.4.22. Define a moduli groupoid of *oriented triangles* and investigate its relation to the moduli groupoids of labeled/unlabeled triangles.

For a more detailed exposition of the moduli stack of triangles, see [Beh14].

0.5 Why the étale topology?

> *On peut dire qu'en passant de la topologie de Zariski à topologie étale, "on a fait ce qu'il fallait" pour obtenir "le bon"* H^1 *[...] pour un groupe de coefficients constant fini G.* C'est un fait remarquable, qui sera démontré dans la suite de ce séminaire, que cela suffit également pour trouver les "bons" $\mathrm{H}^i(X, G)$ pour tout groupe de coefficients de torsion (du moins si G est premier aux caractéristiques résiduelles de X).
>
> Alexander Grothendieck, [SGA4, VII.2.1]

Moduli stacks will be introduced in the next section by combining moduli functors with groupoids: one needs to specify families of objects over every scheme S along with identifications and pullbacks. For such data to define a *stack*, we will require that objects and their morphisms glue in the *étale topology*! Apparently Grothendieck coined the word 'étale' because étale morphisms reminded him of a calm sea at high tide under a full moon, just as Victor Hugo wrote in *Les travailleurs de la mer*: "La mer était étale, mais le reflux commençait à se faire sentir."

Why is the Zariski topology not sufficient for our purposes? The short answer is that there are not enough Zariski-open subsets and that étale morphisms serve as a good replacement of analytic open subsets.

0.5.1 What is an étale morphism anyway?

I have sometimes been baffled when a student is intimidated by étale morphisms, especially when she has already mastered conceptually more difficult notions of say properness and flatness. One factor could be the fact that the definition is buried in [Har77, Exercises III.10.3-6] and its importance is not highlighted there.

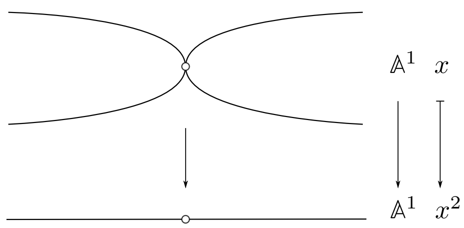

Figure 0.5.1: Picture of an étale double cover of $\mathbb{A}^1 \smallsetminus 0$.

The geometric picture to have in your mind is a covering space. There are several ways in which we can formulate an étale morphism $f\colon X \to Y$ of schemes of finite type over $\mathbb{C}$:

- f is smooth of relative dimension 0 (i.e., f is flat and all fibers are smooth of dimension 0);
- f is flat and unramified (i.e., for all $y \in Y(\mathbb{C})$, the scheme-theoretic fiber X_y is isomorphic to a disjoint union $\coprod_i \operatorname{Spec} \mathbb{C}$ of points);
- f is flat and $\Omega_{X/Y} = 0$;
- for all $x \in X(\mathbb{C})$, the induced map $\widehat{\mathcal{O}}_{Y,f(x)} \to \widehat{\mathcal{O}}_{X,x}$ on completions is an isomorphism; and

– assuming in addition that X and Y are smooth: for all $x \in X(\mathbb{C})$, the induced map $T_{X,x} \to T_{Y,f(x)}$ on tangent spaces is an isomorphism.

We say that f is *étale at* $x \in X$ if there is an open neighborhood U of x such that $f|_U$ is étale. See §A.3 for more background. These characterizations are all equivalent, but by no means this should be clear to you—some of the proofs are quite involved. Nevertheless, if you can take the equivalences on faith, it requires very little effort to not only internalize the concept but to master its use.

Exercise 0.5.2 (good practice)**.** Show that $f\colon \mathbb{A}^1 \to \mathbb{A}^1, x \mapsto x^2$ is étale over $\mathbb{A}^1 \setminus 0$ but is not étale at the origin. *Try to show this for as many of the above characterizations as you can.*

0.5.2 What can you see in the étale topology?

Working with the étale topology is like getting a pair of magnifying lenses for your algebraic geometry glasses allowing you to see finer details that you already could observe with your differential geometry glasses.

Example 0.5.3 (Reducibility of a node)**.** Consider the plane nodal cubic C defined by $y^2 = x^2(x-1)$ in the plane. While there is an analytic open neighborhood of the node $p = (0,0)$ which is reducible, there is no such Zariski-open neighborhood. However, taking a 'square root' of $x - 1$ yields a reducible étale neighborhood. More specifically, define $C' = \operatorname{Spec} \mathbb{C}[x,y,t]_t/(y^2 - x^3 + x^2, t^2 - x + 1)$ and consider

$$C' \to C, \qquad (x,y,t) \mapsto (x,y)$$

Since $y^2 - x^3 + x^2 = (y - xt)(y + xt)$, we see that C' is reducible. This is illustrated by the following picture, which is also featured in [Har77, Exc. III.10.6].

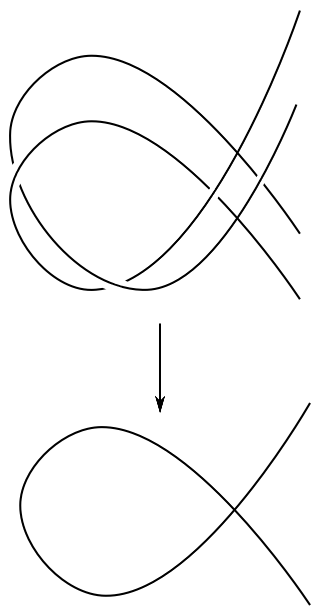

Figure 0.5.4: After an étale cover, the nodal cubic becomes reducible.

Example 0.5.5 (Étale cohomology)**.** Sheaf cohomology for the Zariski topology can be extended to the étale topology leading to the extremely robust theory of *étale* *cohomology*. For example, for a smooth projective curve C of genus g over $\mathbb{C}$, the étale cohomology $\mathrm{H}^1(C_{\text{ét}}, \mathbb{Z}/n)$ of the finite constant sheaf $\mathbb{Z}/n$ is isomorphic to $(\mathbb{Z}/n)^{2g}$ just like the ordinary cohomology groups, while the sheaf cohomology $\mathrm{H}^1(C, \mathbb{Z}/n)$ in the Zariski topology is 0. Finally, we would be remiss without mentioning the spectacular application of étale cohomology to prove the Weil conjectures.

Example 0.5.6 (Étale fundamental group)**.** Have you ever thought that there is a similarity between the bijection in Galois theory between intermediate field extensions and subgroups of the Galois group, and the bijection in algebraic topology between covering spaces and subgroups of the fundamental group? Well, you are in good company—Grothendieck also considered this and developed a beautiful theory of the *étale fundamental group* which packages Galois groups and fundamental groups in the same framework.

Example 0.5.7 (Quotients by free actions of finite groups)**.** If G is a finite group acting freely on a projective variety X, then there exists a quotient X/G as a projective variety. The essential reason for this is that every G-orbit (or in fact every finite set of points) is contained in an affine variety U, which is the complement of some hypersurface. Then the intersection $V = \bigcap_g gU$ of the G-translates is a G-invariant affine open containing Gx and $V/G = \operatorname{Spec} \Gamma(V, \mathcal{O}_V)^G$. These local quotients glue to form a global quotient X/G. See Corollary 4.2.13 and Exercise 4.2.14.

However, if X is not projective, the quotient does not necessarily exist as a scheme. As with most phenomena for smooth proper varieties that are not projective, a counterexample can be constructed by using Hironaka's examples of smooth, proper 3-folds; [Har77, App. B, Ex. 3.4.1]. There is a smooth, proper 3-fold with a free action by $G = \mathbb{Z}/2$ such that there is an orbit Gx not contained in any G-invariant affine open. This shows that X/G cannot exist as a scheme; indeed, if it did, then the image of x under the finite morphism $X \to X/G$ would be contained in some affine and its inverse would be an affine open subset of X containing Gx. See [Knu71, Ex. 1.3] or [Ols16, Ex. 5.3.2] for details.

Nevertheless, for a free action of a finite group G on a scheme X, then every point $x \in X$ has a G-equivariant étale neighborhood $U_x \to X$ where U_x is an affine scheme, and the quotients U_x/G can be glued in the *étale topology* to construct X/G as an *algebraic space* (Corollary 3.1.14). The upshot is that we can always take quotients of free actions by finite groups. This is a very desirable feature given the ubiquity of group actions in algebraic geometry, but it comes at the cost of enlarging our category from schemes to algebraic spaces.

Example 0.5.8 (Artin Approximation)**.** Artin Approximation (B.5.18) is a powerful and extremely deep result, due to Michael Artin, which implies that most properties which hold for the completion $\widehat{\mathcal{O}}_{X,x}$ of the local ring also in an étale neighborhood of x. For instance, since the completion of the local ring at a nodal singularity is reducible, Artin Approximation implies that there is a reducible étale neighborhood.

0.5.3 Working with the étale topology: descent theory

Another reason why the étale topology is so useful is that many properties of schemes and their morphisms can be checked on étale covers. Almost every property that can be checked on a Zariski-open cover $\{U_i\}$ of scheme X can also be checked on an étale cover $\{U_i \to X\}$; here each map $U_i \to X$ is étale and $\coprod_i U_i \to X$ is surjective. In fact, most properties can even be verified on a smooth or fppf cover. Descent theory is developed in §2.1 and is used to prove just about everything about algebraic spaces and stacks. Indeed, algebraic stacks (resp., Deligne–Mumford stacks) have by definition a smooth (resp., étale) cover by affine schemes. Similar to how many properties of schemes can be reduced to properties of affine schemes by taking a Zariski open affine cover, properties of algebraic spaces and stacks can also be established by reducing to affine schemes.

0.6 Moduli stacks

Was du ererbt von deinen Vätern hast, erwirb es, um es zu besitzen.

JOHANN WOLFGANG VON GOETHE

As promised, we now synthesize moduli functors with the groupoid perspective. To define a moduli stack, we need to specify:

(1) families of objects;

(2) how two families of objects are isomorphic; and

(3) how families pull back under morphisms.

Notice the difference from specifying a moduli functor is that rather than specifying *when* two families are isomorphic, we specify *how*. In other words, we need to specify an assignment

$$F\colon \mathrm{Sch} \to \mathrm{Groupoids}, \quad S \mapsto \mathrm{Fam}_S$$

taking a scheme S to a *groupoid* of families of objects over S. But what exactly do we mean by this? Groupoids form a '2-category' as they have objects (groupoids), morphisms (functors between groupoids), and 2-morphisms (natural transformations between functors). How can we precisely formulate such an assignment in down-to-earth terms? Well, we certainly need pullback functors $f^*\colon \mathrm{Fam}_T \to \mathrm{Fam}_S$ for each morphism $f\colon S \to T$. Given a composition $S \xrightarrow{f} T \xrightarrow{g} U$ of schemes, we should also have an isomorphism of functors (i.e., a 2-morphism) $\mu_{f,g}\colon (f^* \circ g^*) \xrightarrow{\sim} (g \circ f)^*$. Should the isomorphisms $\mu_{f,g}$ satisfy a compatibility condition under triples $S \xrightarrow{f} T \xrightarrow{g} U \xrightarrow{h} V$? Yes! This leads to the notion of a *pseudo-functor* but we will not spell it out here; we encourage the reader to work it out, or to look it up in [Vis05, Def. 3.10] or [SP, Tag 003N]. We take a slightly different approach using prestacks which is technically more convenient. It is nevertheless useful to think of a prestack as an assignment Sch $\to$ Groupoids.

0.6.1 Motivating the definition of a prestack

Instead of trying to define an assignment $S \mapsto \mathrm{Fam}_S$, we will build one massive category $\mathcal{X}$ encoding all of the groupoids Fam_S which will live over the category Sch of schemes. Loosely speaking, the objects of $\mathcal{X}$ will be a family a of objects over a scheme S, i.e., $a \in \mathrm{Fam}_S$, and a morphism $a \to b$ between a family a over S and a family b over T will be the data of a morphism $f\colon S \to T$ together with an isomorphism $a \xrightarrow{\sim} f^*b$ of a and the pullback family of b.

A *prestack* over Sch is a category $\mathcal{X}$ together with a functor $p\colon \mathcal{X} \to \mathrm{Sch}$, which we visualize as

$$\begin{tikzcd} \mathcal{X} \arrow[d, "p"] & a \arrow[r, "\alpha"] \arrow[d, mapsto] & b \arrow[d, mapsto] \\ \mathrm{Sch} & S \arrow[r, "f"] & T \end{tikzcd}$$

where the lower case letters a, b are objects in $\mathcal{X}$ and the upper case letters S, T are schemes. We say that *a is over S* and that *$\alpha\colon a \to b$ is over $f\colon S \to T$*. Moreover, we need to require that certain natural axioms hold for $p\colon \mathcal{X} \to \mathrm{Sch}$. Loosely speaking, we require the existence and uniqueness of pullbacks: given a map $S \to T$ and object $b \in \mathcal{X}$ over T, there should exist an arrow $a \xrightarrow{\alpha} b$ over f satisfying a suitable universal property; see Definition 2.4.1.

Given a scheme S, the *fiber category* $\mathcal{X}(S)$ is defined as the category of objects over S whose morphisms are over the identity. If $\mathcal{X}$ is built from the groupoids Fam_S as above, then $\mathcal{X}(S) = \mathrm{Fam}_S$.

Example 0.6.1 (Viewing a set-valued functor as a prestack)**.** A moduli functor $F\colon \mathrm{Sch} \to \mathrm{Sets}$ can be encoded as a moduli prestack as follows: we define the category $\mathcal{X}_F$ of pairs (S, a) where S is a scheme and $a \in F(S)$. A map $(S', a) \to (S, a)$ is a map $f\colon S' \to S$ such that $a' = f^*a$, where f^* is convenient shorthand for $F(f)\colon F(S) \to F(S')$. Observe that the fiber categories $\mathcal{X}_F(S)$ are equivalent (even equal) to the set $F(S)$.

Example 0.6.2 (Moduli prestack of smooth curves)**.** The *moduli prestack of smooth curves* is the category $\mathcal{M}_g$ of families of smooth curves $\mathcal{C} \to S$ together with the functor $p\colon \mathcal{M}_g \to \mathrm{Sch}$ defined by $(\mathcal{C} \to S) \mapsto S$. A morphism $(\mathcal{C}' \to S') \to (\mathcal{C} \to S)$ in $\mathcal{M}_g$ is the data of maps $\alpha\colon \mathcal{C}' \to \mathcal{C}$ and $f\colon S' \to S$ such that the diagram

$$\begin{tikzcd} \mathcal{C}' \arrow[r, "\alpha"] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{C} \arrow[d] \\ S' \arrow[r, "f"] & S \end{tikzcd}$$

is cartesian. Note that in the fiber category $\mathcal{M}_g(\mathbb{C})$, an object is a smooth curve C and the set of morphisms $C \to C$ is identified with the automorphism group $\mathrm{Aut}(C)$.

Example 0.6.3 (Moduli prestack of vector bundles)**.** The *moduli prestack of vector bundles* on a smooth curve C over $\mathbb{C}$ is the category $\mathcal{B}un_{r,d}(C)$ of pairs (E, S), where S is a $\mathbb{C}$-scheme and E is a vector bundle on $C_S = C \times_{\mathbb{C}} S$ such that for every $s \in S$, the restriction E_s to $C \times_{\mathbb{C}} \operatorname{Spec} \kappa(s)$ has rank r and degree d. The functor $p\colon \mathcal{B}un_{r,d}(C) \to \mathrm{Sch}/\mathbb{C}$ is defined by $(E, S) \mapsto S$. A map $(E', S') \to (E, S)$ consists of a map of schemes $f\colon S' \to S$ together with an isomorphism $(\mathrm{id} \times f)^*E \xrightarrow{\sim} E'$.

0.6.2 Motivating the definition of a stack

A stack is to a prestack as a sheaf is to a presheaf. The concept could not be more intuitive: we require that objects and morphisms glue uniquely.

Example 0.6.4 (Moduli stack of sheaves)**.** Define the category $\mathcal{X}$ over Sch of pairs (E, S) where E is a sheaf of abelian groups on a scheme S, and the functor $p\colon \mathcal{X} \to \mathrm{Sch}$ is given by $(E, S) \mapsto S$. A map $(E', S') \to (E, S)$ in $\mathcal{X}$ is a map of schemes $f\colon S' \to S$ together with an isomorphism $f^{-1}E \xrightarrow{\sim} E'$ (or alternatively a map $E \to f_*E'$ whose adjoint is an isomorphism).

You already know that morphisms of sheaves glue: let E and F be sheaves on schemes S and T and let $f\colon S \to T$ be a map. If $\{S_i\}$ is a Zariski open cover of S, then a map $f\colon S \to T$ is the same data as a collection of morphisms $f_i\colon S_i \to T$ such that $f_i|_{S_{ij}} = f_j|_{S_{ij}}$, where $S_{ij} = S_i \cap S_j$. Similarly, an isomorphism $f^{-1}E \xrightarrow{\sim} E'$ is the same data as isomorphisms $(f^{-1}E)|_{S_i} \to E'|_{S_i}$ that agree on the intersections S_{ij} [Har77, Exc. II.1.15]. Putting these together, a morphism $\alpha\colon (E, S) \to (F, T)$

is equivalent to morphisms $\alpha_i \colon (E|_{S_i}, S_i) \to (F, T)$ such that $\alpha_i|_{S_{ij}} = \alpha_j|_{S_{ij}}$. You also know how sheaves glue—it is more complicated than gluing morphisms as sheaves have automorphisms, and given two sheaves, we prefer to say that they are isomorphic rather than equal. If $\{S_i\}$ is a Zariski open cover of a scheme S, then giving a sheaf E on S is equivalent to giving a sheaf E_i on S_i and isomorphisms $\phi_{ij} \colon E_i|_{S_{ij}} \to E_j|_{S_{ij}}$ satisfying the cocycle condition $\phi_{ik} = \phi_{jk} \circ \phi_{ij}$ on the triple intersection $S_{ijk} = S_i \cap S_j \cap S_k$ [Har77, Exc. II.1.22].

In a similar way, we could have considered the stack of $\mathcal{O}$-modules, quasi-coherent sheaves, or vector bundles. Or for a scheme X, we could have stacks of sheaves, $\mathcal{O}$-modules, quasi-coherent sheaves, or vector bundles over X, where an object over a scheme S is a sheaf on $X \times S$.

The definition of a stack (Definition 2.5.1) simply axiomatizes these two natural gluing concepts.

0.6.3 Motivating the definition of an algebraic stack

For a stack to be a geometric object, we need to specify that it is locally like a scheme in a suitable sense. Without imposing such a condition would be like trying to study the geometry of an arbitrary ringed space $(X, \mathcal{O}_X)$ or a non-representable sheaf $F \colon \mathrm{Sch} \to \mathrm{Sets}$. If we wish to utilize our algebraic geometry toolkit (e.g., coherent sheaves, commutative algebra, cohomology, ...) to study stacks in a similar way that we study schemes, we must impose an algebraicity condition.

The conditions we impose are quite natural. In increasing generality, we define:

(1) A functor $X \colon \mathrm{Sch} \to \mathrm{Sets}$ is an *algebraic space* if it is a sheaf (i.e., objects glue uniquely in the étale topology), and there is an étale cover $\{U_i \to X\}$ where each U_i is an affine scheme.

(2) A stack $\mathcal{X} \to \mathrm{Sch}$ is *Deligne–Mumford* if there is an étale cover $\{U_i \to \mathcal{X}\}$ where each U_i is an affine scheme.

(3) A stack $\mathcal{X} \to \mathrm{Sch}$ is *algebraic* if there is a smooth cover $\{U_i \to \mathcal{X}\}$ where each U_i is an affine scheme.

Of course, we need to make precise the notions of étale and smooth covers. For a first approximation, when we say that $\{U_i \to X\}$ is an étale cover, we require that for every map $T \to X$ of functors where T is representable by a scheme, the fiber product of functors is representable by a scheme T_i, and moreover that $T_i \to T$ is étale and $\coprod T_i \to T$ is surjective. Note that in (1), if we replace 'étale' with 'Zariski', we would recover the notion of a scheme; see Exercise 0.3.42. It will take some time to develop the foundations to make this completely rigorous; precise definitions are postponed until §3.1.

Algebro-geometric space	Type of object	Obtained by gluing
Schemes	ringed space/ sheaf	affine schemes in the Zariski topology
Algebraic spaces	sheaf	affine schemes in the étale topology
Deligne–Mumford stacks	stack	affine schemes in the étale topology
Algebraic stacks	stack	affine schemes in the smooth topology

Table 0.6.5: Schemes, algebraic spaces, Deligne–Mumford stacks, and algebraic stacks are obtained by gluing affine schemes.

Why smooth covers? After all the fuss motivating the étale topology above, you might be surprised to see that an algebraic stack is *smooth locally* a scheme. For Deligne–Mumford stacks—which turn out to be precisely algebraic stacks with finite automorphism groups—étale covers are sufficient. But for algebraic stacks like $\mathcal{B}un_{r,d}(C)$ with infinite automorphism groups, we need smooth covers. For instance, we would like to be able to form the quotient $[\operatorname{Spec}\mathbb{C}/\mathbb{G}_m]$ (which we will call the *classifying stack* $B\mathbb{G}_m$) of the trivial action of $\mathbb{G}_m$ (or $\mathbb{C}^*$) on a point, and this will have no étale cover by a scheme.

0.6.4 Examples of moduli stacks

Given a stack encoding a moduli problem, constructing a smooth cover is a geometric problem inherent to the moduli problem. It can often be solved by rigidifying the moduli problem by parameterizing additional information. This concept is best absorbed in examples.

Example 0.6.6 (Moduli stack of elliptic curves)**.** An elliptic curve (E, p) is embedded into $\mathbb{P}^2$ via $\mathcal{O}_E(3p)$ such that E is defined by a Weierstrass equation

$$y^2 z = x(x - z)(x - \lambda z)$$

for some $\lambda \neq 0, 1$ [Har77, Prop. IV.4.6]. Setting $U = \mathbb{A}^1 \smallsetminus \{0, 1\}$ with coordinate λ, the family $\mathcal{E} \subseteq U \times \mathbb{P}^2$ of elliptic curves defined by this Weierstrass equation defines a map $U \to \mathcal{M}_{1,1}$ which is an étale cover.

Example 0.6.7 (Moduli stack of smooth curves)**.** For a smooth curve C of genus $g \geq 2$, the line bundle $\Omega_C^{\otimes 3}$ is very ample and defines an embedding $C \hookrightarrow \mathbb{P}(\Gamma(C, \Omega_C^{\otimes 3})) \cong \mathbb{P}^{5g-6}$. There is a Hilbert scheme $\operatorname{Hilb}^P(\mathbb{P}^{5g-6})$ (see Theorem 1.1.2) parameterizing closed subschemes of $\mathbb{P}^{5g-6}$ with the same Hilbert polynomial $P(z) = (6z - 1)(g - 1)$ as $C \subseteq \mathbb{P}^{5g-6}$, and there is a locally closed subscheme $H' \subseteq \operatorname{Hilb}^P(\mathbb{P}^{5g-6})$ parameterizing *smooth* subschemes such that $\Omega_C^{\otimes 3} \cong \mathcal{O}_C(1)$. The universal subscheme over H' defines a map $H' \to \mathcal{M}_g$ which is a smooth cover (see Theorem 3.1.17 for details) and thus $\mathcal{M}_g$ is an algebraic stack. We will show that it is Deligne–Mumford in Corollary 3.6.10.

Example 0.6.8 (Moduli stack of vector bundles)**.** For every vector bundle E of rank r and degree d on a smooth curve C with an ample line bundle $\mathcal{O}_C(1)$, for m sufficiently large, the twist $E(m)$ is globally generated and $P(m) = \mathrm{h}^0(C, E(m))$ defines the Hilbert polynomial of E. Therefore, for $m \gg 0$, we can view E as a quotient $\mathcal{O}_C(-m)^{P(m)} \twoheadrightarrow E$. There is a Quot scheme $\mathrm{Quot}^P(\mathcal{O}_C(-m)^{P(m)})$ (see Theorem 1.1.3) parameterizing quotients with Hilbert polynomial P. There is a locally closed subscheme $Q'_m \subseteq \mathrm{Quot}^P(\mathcal{O}_C(-m)^{P(m)})$ parameterizing vector bundle quotients $\pi\colon \mathcal{O}_C(-m)^{P(m)} \twoheadrightarrow E$ such that the induced map $\Gamma(\pi \otimes \mathcal{O}_C(m))\colon \mathbb{C}^{P(m)} \to \Gamma(C, E(m))$ is an isomorphism. The universal quotient over Q'_m defines a map $Q'_m \to \mathcal{B}un_{r,d}(C)$ which is smooth and the collection $\{Q'_m \to \mathcal{B}un_{r,d}(C)\}$ for $m \gg 0$ defines a smooth cover. This shows that an $\mathcal{B}un_{r,d}(C)$ is an algebraic stack; see Theorem 3.1.21 for details. It is not a Deligne–Mumford stack.

0.6.5 Quotient stacks

One of the most important examples of a stack is a quotient stack $[X/G]$ arising from an action of an algebraic group G on a scheme X. The geometry of $[X/G]$ could not be simpler: it is the G-equivariant geometry of X (see Table 0.6.15).

Similar to how toric varieties provide concrete examples of schemes, quotient stacks provide concrete examples of algebraic stacks that are useful to gain geometric intuition of general algebraic stacks, and at the same time provide a fertile testing ground for conjectural results. On the other hand, it turns out that many algebraic stacks are quotient stacks or are at least locally quotient stacks, and most properties that hold for quotient stacks can also be established for many algebraic stacks.

Quotient prestacks. Given an action of an algebraic group G on a scheme X, the *quotient prestack* $[X/G]^{\mathrm{pre}}$ is the prestack whose fiber category $[X/G]^{\mathrm{pre}}(S)$ over a scheme S is the quotient groupoid (or the moduli groupoid of orbits) $[X(S)/G(S)]$ as in Example 0.4.6. This will not satisfy the gluing axioms of a stack; even when the action is free, the quotient functor $\mathrm{Sch} \to \mathrm{Sets}$ defined by $S \mapsto X(S)/G(S)$ is not a sheaf (see Exercise 0.3.37). How can we make it into a stack? Well, instead of thinking of an object of $[X/G]^{\mathrm{pre}}$ over a scheme S as a morphism $f\colon S \to X$, let us think of it as a trivial G-bundle together with a map to X:

$$\begin{tikzcd} G \times S \arrow[r, "\widetilde{f}"] \arrow[d, "p_2"] & X, & (g,s) \arrow[r, mapsto] & g \cdot f(s) \\ S. \end{tikzcd}$$

Exercise 0.6.9 (easy)**.** Given two maps $f_1, f_2\colon S \to X$, show that an element of $\alpha \in G(S)$ satisfying $f_2 = \alpha \cdot f_1$ is the same data as an isomorphism of trivial G-bundles $G \times S \to G \times S$ compatible with the maps $\widetilde{f}_1$ and $\widetilde{f}_2$ to X.

From this perspective, it is even more clear that $[X/G]^{\mathrm{pre}}$ is not a stack even when X is a point: given a Zariski cover $\{S_i\}$ of a scheme S, trivial G-bundles $G \times S_i \to S_i$ together with isomorphisms over $S_i \cap S_j$ satisfying a cocycle condition will glue to a principal G-bundle $P \to S$ (Definition B.1.47), but it will *not* necessarily be trivial. This suggests that we should define an object of a quotient stack to be a principal G-bundle together with a G-equivariant map to X.

Quotient stacks. We define the *quotient stack* $[X/G]$ as the category over $\mathrm{Sch}/\mathbb{C}$

whose objects over a $\mathbb{C}$-scheme S are diagrams

$$\begin{tikzcd} P \arrow[r, "f"] \arrow[d] & X \\ S & \end{tikzcd}$$

where $P \to S$ is a principal G-bundle and $f\colon P \to X$ is a G-equivariant morphism. A morphism is the data of a commutative diagram

$$\begin{tikzcd} P' \arrow[r, "\varphi"] \arrow[d] \arrow[rr, bend left, "f'"] \arrow[dr, phantom, "\square"] & P \arrow[r, "f"] \arrow[d] & X \\ S' \arrow[r, "g"] & S & \end{tikzcd}$$

where the left square is cartesian. There is an object of $[X/G]$ over X given by the diagram

$$\begin{tikzcd} G \times X \arrow[r, "\sigma"] \arrow[d, "p_2"] & X \\ X, & \end{tikzcd}$$

where σ denotes the action map. By the 2-Yoneda Lemma (2.4.21), this defines a map $X \to [X/G]$. Even if the action of G on X is not free, the map $X \to [X/G]$ is a principal G-bundle. Let us pause to appreciate that:

The map $X \to [X/G]$ is a principal G-bundle even if the action of G on X is not free.

This is one of the great advantages of working with stacks. At the expense of enlarging our category from schemes to algebraic stacks, we are able to tautologically construct the quotient $[X/G]$ as a 'geometric space' with desirable properties.

Example 0.6.10 (Classifying stack)**.** We define the *classifying stack* of an algebraic group G as the category $BG := [\operatorname{Spec} \mathbb{C}/G]$ of principal G-bundles $P \to S$. The projection $\operatorname{Spec} \mathbb{C} \to BG$ is not only a principal G-bundle; it is the *universal principal G-bundle*. Given any other principal G-bundle $P \to S$, there is a unique map $S \to BG$ and a cartesian diagram

$$\begin{tikzcd} P \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \operatorname{Spec} \mathbb{C} \arrow[d] \\ S \arrow[r] & BG. \end{tikzcd}$$

Example 0.6.11 (Quotients by finite groups)**.** Quotients by free actions of finite groups exist as algebraic spaces! See Corollary 3.1.14.

Exercise 0.6.12. What is the universal family over the quotient stack $[X/G]$?

Moduli stacks can often be described as quotient stacks, and these descriptions can be leveraged to establish properties of the moduli stack.

Example 0.6.13 (Moduli stack of smooth curves as a quotient)**.** Reexamining Example 0.6.7, we see that the embedding of a smooth curve C via $|\Omega_C^{\otimes 3}|\colon C \hookrightarrow \mathbb{P}^{5g-6}$ depends on a choice of basis $\Gamma(C, \Omega_C^{\otimes 3}) \cong \mathbb{C}^{5g-5}$ and therefore is only unique up to a projective automorphism, i.e., an element of $\mathrm{PGL}_{5g-5} = \mathrm{Aut}(\mathbb{P}^{5g-6})$. The algebraic group PGL_{5g-5} acts on the subscheme $H' \subset \mathrm{Hilb}^P(\mathbb{P}^{5g-6})$ parameterizing smooth tricanonically embedded curves, and there is an isomorphism $\mathcal{M}_g \cong [H'/\mathrm{PGL}_{5g-6}]$.

Example 0.6.14 (Moduli stack of vector bundles as a quotient)**.** For the moduli stack of vector bundles (Example 0.6.8), the presentation of a vector bundle E as a quotient $\mathcal{O}_C(-m)^{P(m)} \twoheadrightarrow E$ depends on a choice of basis $\Gamma(C, E(m)) \cong \mathbb{C}^{P(m)}$. The algebraic group $\mathrm{GL}_{P(m)}$ acts on the scheme Q'_m and there is an identification

$$\mathcal{B}un_{r,d}(C) \cong \bigcup_{m \gg 0} [Q'_m/\mathrm{GL}_{P(m)}];$$

see Theorem 3.1.21.

Geometry of a quotient stack. While the definition of the quotient stack $[X/G]$ may appear abstract, its geometry is very familiar. The table below provides a dictionary between the geometry of a quotient stack $[X/G]$ and the G-equivariant geometry of X. The stack-theoretic concepts on the left-hand side will be introduced later.

Geometry of $[X/G]$	G-equivariant geometry of X
$\mathbb{C}$-point $\overline{x} \in [X/G]$	orbit Gx of $\mathbb{C}$-point $x \in X$ (with $\overline{x}$ the image of x under $X \to [X/G]$)
automorphism group $\mathrm{Aut}(\overline{x})$	stabilizer G_x
function $f \in \Gamma([X/G], \mathcal{O}_{[X/G]})$	G-equivariant function $f \in \Gamma(X, \mathcal{O}_X)^G$
map $[X/G] \to Y$ to a scheme Y	G-equivariant map $X \to Y$
line bundle	G-equivariant line bundle (or G-linearization)
quasi-coherent sheaf	G-equivariant quasi-coherent sheaf
tangent space $T_{[X/G],\overline{x}}$	normal space $T_{X,x}/T_{Gx,x}$ to the orbit
coarse moduli space $[X/G] \to Y$	geometric quotient $X \to Y$
good moduli space $[X/G] \to Y$	good GIT quotient $X \to Y$

Table 0.6.15: Dictionary between the geometry of $[X/G]$ and the G-equivariant geometry of X.

0.7 Constructing projective moduli spaces

The spaces $M_{g,\nu}$ are in my eyes (together with the group SL(2)*) the most beautiful and most fascinating objects that I have encountered in mathematics.*

GROTHENDIECK [GRO86, P. 129]

One of our incentives for introducing algebraic stacks is to ensure that a given moduli problem $\mathcal{M}$ is representable and equipped with a universal family. While many geometric questions can be studied (and arguably should be studied) on the moduli stack $\mathcal{M}$ itself, it is often very convenient to make a trade-off: by sacrificing the existence of a universal family, we can sometimes construct a more familiar geometric space, ideally a projective variety. This allows us to utilize the much larger toolkit of projective geometry (e.g., birational geometry, intersection theory, Hodge theory, ...) to study the moduli problem.

We highlight two approaches to construct projective moduli spaces:

(1) Geometric Invariant Theory (GIT), and

(2) Intrinsic construction of coarse/good moduli spaces.

There is a beautiful interplay between the intrinsic and extrinsic approaches. Ideas from GIT have inspired techniques in each of the six steps of the intrinsic approach below, and conversely the intrinsic approach sheds light back on GIT. GIT is also deeply intertwined with 19th century invariant theory, and determining the GIT semistable locus is an interesting and important problem on its own. It is valuable to keep both approaches in mind.

0.7.1 GIT approach

Outline of the GIT strategy

(A) Express the moduli stack $\mathcal{M}$ as a substack

$$\mathcal{M} \subseteq [X/G],$$

where G is reductive and $X \hookrightarrow \mathbb{P}(V)$ is a G-equivariant embedding into the projectivization of a G-representation V.

(B) Show that a point $x \in X$ is GIT semistable if and only if $x \in \mathcal{M}$, or in other words that $\mathcal{M} = [X^{\mathrm{ss}}/G]$.

For Step A, there are often natural ways to rigidify the moduli problem by parameterizing additional data. For a smooth curve C, a choice of basis of $\Gamma(C, \Omega_C^{\otimes 3})$ defines a tricanonical embedding $C \hookrightarrow \mathbb{P}^{5g-6}$ (Example 0.6.13), or for any $k \geq 3$, a choice of basis of $\Gamma(C, \Omega_C^{\otimes k})$ defines the kth pluricanonical embedding $C \hookrightarrow \mathbb{P}^{(2k-1)(g-1)-1}$. For a vector bundle E on a fixed smooth curve C, after choosing a sufficiently large integer m, a choice of basis of $\Gamma(C, E(m)) \cong \mathbb{C}^{P(m)}$ allows us to view E as a quotient $\mathcal{O}_C(-m)^P(m) \twoheadrightarrow E$. The rigidified moduli problem should have a compactification which is representable by a projective variety X—which is $\mathrm{Hilb}^P(\mathbb{P}^{5g-6})$ and $\mathrm{Quot}^P(\mathcal{O}_C(-m)^P(m))$ in our two examples—and the choice of additional data should be governed by an action of a group G. For the GIT approach to succeed, we need that G is reductive and that $\mathcal{M}$ is a substack of $[X/G]$. Finally, we need to choose a G-equivariant embedding $X \hookrightarrow \mathbb{P}(V)$ where V is a finite dimensional G-representation, or equivalently choose a G-linearization of an ample line bundle on X.

Step B is the hardest: we must show that $\mathcal{M}$ is precisely the open substack of $[X/G]$ of GIT semistable points. Using the Hilbert–Mumford Criterion (7.4.4), we can translate the problem to the following: a point $x \in X$ represents an object of the moduli problem $\mathcal{M}$ if and only if the *Hilbert–Mumford index* $\mu(x, \lambda) \geq 0$ for every one-parameter subgroup $\lambda\colon \mathbb{G}_m \to G$. This often reduces the goal to a tractable (but often still daunting) combinatorial problem.

The GIT quotient $M := X^{ss}/\!\!/G$ is necessarily projective. One beautiful feature of GIT is that even if the moduli stack $\mathcal{M}$ is not compact, the GIT strategy provides a compactification! If $\mathcal{M}$ has only finite automorphisms or equivalently there are no strictly semistable points, then $X^{ss} \to M$ is a geometric quotient and $\mathcal{M} = [X^{ss}/G] \to M$ is a coarse moduli space. In the presence of infinite automorphisms, $X^{ss} \to M$ is a good quotient and $\mathcal{M} \to M$ is a good moduli space.

The GIT approach is covered in detail in Chapter 7. We sketch the GIT construction of $\overline{M}_g$ in §5.8 and present a complete GIT construction of $\mathcal{B}un_{r,d}(C)$ in §??.

0.7.2 Intrinsic approach

<u>Six steps toward projective moduli</u>

① **Algebraicity:** Express the moduli stack $\mathcal{M}$ as a substack

$$\mathcal{M} \subseteq \mathcal{X}$$

of a larger moduli stack $\mathcal{X}$. Define an object $x \in \mathcal{X}$ to be *semistable* if it is in $\mathcal{M}$; this allows us to think of $\mathcal{M}$ as the semistable locus $\mathcal{X}^{ss}$. Show that $\mathcal{X}$ is an algebraic stack locally of finite type over $\mathbb{C}$.

② **Openness of semistability:** Show that semistability is an open condition, i.e., $\mathcal{M} = \mathcal{X}^{ss} \subseteq \mathcal{X}$ is an open substack.

③ **Boundedness of semistability:** Show that semistability is bounded, i.e., $\mathcal{M} = \mathcal{X}^{ss}$ is of finite type over $\mathbb{C}$.

④ **Semistable reduction:** Show that $\mathcal{M}$ satisfies the existence part of the valuative criterion for properness.

⑤ **Existence of a moduli space:** Show that there is a fine/coarse/good moduli space $\mathcal{M} \to M$ where M is a proper algebraic space.[7]

⑥ **Projectivity:** Show that a tautological line bundle on $\mathcal{M}$ descends to an ample line bundle on M, i.e., M is projective.

GIT magically solves all these steps at once! In Step A of the GIT approach, expressing the moduli stack $\mathcal{M}$ as a substack $[X/G]$ already implies 'boundedness.' Since GIT semistability is always an open condition, the identification in Step B of $\mathcal{M}$ with the semistable locus $[X^{ss}/G]$ gives 'openness of semistability' and thus 'algebraicity' of $\mathcal{M}$. Strikingly, GIT also implies each of the other steps: 'semistable reduction,' 'existence of a moduli space', and 'projectivity.'

Step 1 (Algebraicity). Many moduli stacks have natural enlargements. The stack $\mathcal{M}_g$ of smooth curves and the stack $\overline{\mathcal{M}}_g$ of stable curves are both contained in the stack of all curves or the stack of polarized curves. The stack of semistable vector bundles on a smooth curve is contained in the stack of all vector bundles or the even larger stack of all coherent sheaves. It is usually easier to first show that the enlargement $\mathcal{X}$ is an algebraic stack locally of finite type, and then use Steps 2 to conclude that $\mathcal{M}$ itself is algebraic.

[7]The calligraphic font $\mathcal{M}$ denotes the stack while the Roman font M denotes the space. This convention will be followed throughout the text.

To get started, we need to define the stacks $\mathcal{M}$ and its enlargement $\mathcal{X}$—this entails specifying families of objects along with pullbacks and identifications. To check that $\mathcal{X}$ is algebraic requires finding a smooth cover $U \to \mathcal{X}$ by a scheme. In many cases, we can even show that $\mathcal{X}$ is identified with a quotient stack $[U/G]$ in which case $U \to [U/G]$ provides a presentation. Alternatively, it is often possible to use Artin's Criteria (Theorem C.7.4) to establish algebraicity; this essentially amounts to verifying local properties of the moduli problem and in particular requires an understanding of the deformation and obstruction theory.

Step 2 (Openness of semistability). This translates to the following condition: for every family $\mathcal{E}$ of objects of $\mathcal{X}$ over a scheme S, the subset

$$\{s \in S \mid \mathcal{E}_s \text{ is semistable, i.e., } \mathcal{E}_s \in \mathcal{M}\}, \tag{0.7.1}$$

where $\mathcal{E}_s$ is the pullback of $\mathcal{E}$ along $\operatorname{Spec} \kappa(s) \to S$, is an open subset of S. This is precisely what it means for the inclusion $\mathcal{M} = \mathcal{X}^{\mathrm{ss}} \hookrightarrow \mathcal{X}$ to be representable by open immersions: for every map $S \to \mathcal{X}$ (corresponding to the family $\mathcal{E}$), the fiber product $\mathcal{M} \times_{\mathcal{X}} S$ (which is identified set-theoretically with (0.7.1)) is an open subscheme of S. This step ensures that $\mathcal{M}$ is also an algebraic stack locally of finite type over $\mathbb{C}$.

Step 3 (Boundedness of semistability). We say that an algebraic stack $\mathcal{M}$ over $\mathbb{C}$ is *bounded* if it is of finite type. Since Step 2 implies that $\mathcal{M}$ is *locally* of finite type over $\mathbb{C}$, boundedness translates into the quasi-compactness of $\mathcal{M}$. More concretely, boundedness is equivalent to the existence of a scheme Z of *finite type* over $\mathbb{C}$ and a family of objects $\mathcal{E}$ over Z such that every object E of $\mathcal{M}$ is isomorphic to $\mathcal{E}_z$ for some (not necessarily unique) $z \in Z$.

For example, $\mathcal{M}_g$ is bounded but the stack of all proper curves of genus g and the stack $\coprod_g \mathcal{M}_g$ of all smooth curves (of any genus) are not bounded. For vector bundles, we will show that the stack $\mathcal{B}un^{\mathrm{ss}}_{r,d}(C)$ of semistable vector bundles of fixed rank and degree is bounded. reference The stack of all vector bundles $\mathcal{B}un_{r,d}(C)$ of fixed rank and degree is not bounded, nor is the stack of semistable vector bundles of arbitrary rank and degree.

Step 4 (Semistable reduction). The existence part of the valuative criterion for properness is the assertion that for every DVR R (which you can think of as a local model of a smooth curve) with fraction field K (or punctured curve), every object $\mathcal{E}^\times$ over K extends to a family of objects $\mathcal{E}$ over R after possibly replacing R with an extension of DVRs. In other words, every diagram

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r, "\mathcal{E}^\times"] \arrow[d] & \mathcal{M} \\ \operatorname{Spec} R, \arrow[ur, dashed, "\mathcal{E}"'] & \end{tikzcd} \tag{0.7.2}$$

has an extension after replacing R with an extension. If the extension $\mathcal{E}$ over R is also unique, then we say that $\mathcal{M}$ satisfies the valuative criterion for properness, and this implies properness (Theorem 3.8.7) and in particular separatedness. Arguably the usefulness of valuative criteria in algebraic geometry is best witnessed in moduli theory.

The moduli stack of smooth curves is not compact and does not satisfy the existence part of the valuative criterion.

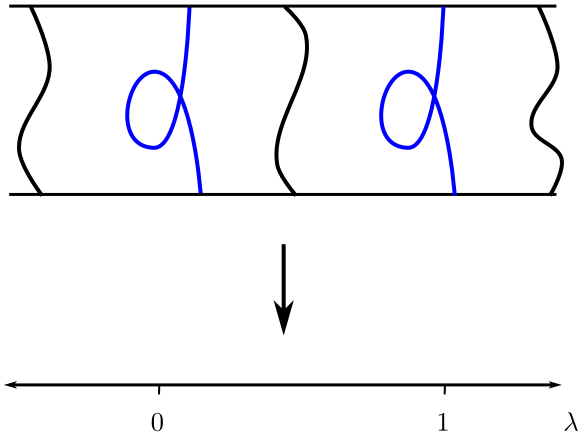

Figure 0.7.3: The family of elliptic curves $y^2z = x(x-z)(x-\lambda z)$ degenerates to the nodal cubic over $\lambda = 0, 1$.

Projective varieties are of course compact and satisfy the valuative criterion. If there's any hope to construct a projective moduli space, then the moduli stack better satisfy the existence part of the valuative criterion. Properness of $\overline{\mathcal{M}}_g$ was first proven by Deligne and Mumford in their influential paper [DM69]. We prove semistable reduction in characteristic 0 in §5.5.

For the moduli of vector bundles, semistable reduction was first proved by Mumford and Seshadri as a consequence of the GIT construction [Ses67]. An intrinsic geometric argument was later given by Langton [Lan75]. Note that unlike stable curves, the stack $\mathcal{B}un^{\mathrm{ss}}_{r,d}(C)$ is not separated as there may exist several non-isomorphic extensions of a vector bundle on C_K to C_R. Nevertheless, the moduli stack $\mathcal{B}un^{\mathrm{ss}}_{r,d}(C)$ satisfies a weaker notion of separatedness called *S-completeness.*

Step 5 (Existence of a moduli space). We would like to construct an algebraic space that is the best possible approximation of the moduli stack. This step depends on the automorphisms of the moduli problem:

- *No automorphisms*: in this case, the moduli stack $\mathcal{M}$ is already an algebraic space M. In other words, M is a *fine moduli space*: for a scheme S, there is a natural bijection between objects over a scheme S and maps $S \to M$.
- *Finite automorphisms*: we must show that $\mathcal{M}$ is separated or in other words that $\mathcal{M}$ satisfies the uniqueness part (in addition to the existence part) of the valuative criterion. The Keel–Mori theorem (Theorem 4.4.6) then establishes the existence of a *coarse moduli space* $\mathcal{M} \to M$ where M is a proper algebraic space. The map $\mathcal{M} \to M$ induces a bijection of $\mathbb{C}$-points and satisfies the universal property that any other map $\mathcal{M} \to Y$ to an algebraic space Y factors uniquely through M.
- *Reductive automorphisms*: in addition to show that $\mathcal{M}$ satisfies the existence part of the valuative criterion for properness, we must show that $\mathcal{M}$ satisfies two additional valuative criterion called Θ*-completeness* and *S-completeness*, which requires the existence of extensions of $\mathbb{G}_m$-equivariant families of objects over certain punctured surfaces with a $\mathbb{G}_m$-action; see §6.9.2. Given these properties, the Existence Theorem for Good Moduli Spaces (6.10.1) yields a *good moduli space* $\mathcal{M} \to M$ where M is a proper algebraic space. The map $\mathcal{M} \to M$ is no longer a bijection of $\mathbb{C}$-points as it identifies points whose

closures intersect in an analogous way to the orbit closure equivalence relation in GIT. But $\mathcal{M} \to M$ does induce a bijection between *closed* $\mathbb{C}$-points of $\mathcal{M}$ (sometimes called *polystable* objects) and the $\mathbb{C}$-points of M, and it also satisfies the universal property for maps to algebraic spaces.

Step 6 (Projectivity). This is usually the hardest step. It requires a solid understanding of the geometry of the moduli problem and sometimes relies on sophisticated techniques in birational geometry. Kollár introduced a strategy in [Kol90] to verify projectivity for moduli stacks of varieties and applied it to verify the projectivity of $\overline{M}_g$. We cover Kollár's method in §5.9. Faltings constructed projective moduli spaces of vector bundles in [Fal93] without using the theory of GIT, and we borrow several of his ideas in our construction in §**??**.

Chapter 1

Hilbert and Quot schemes

Wir müssen wissen. Wir werden wissen.

David Hilbert

We prove that the Grassmannian, Hilbert and Quot functors are representable by projective schemes. These results serve as the backbone of many results in moduli theory and more widely algebraic geometry. In particular, they are essential for establishing properties about the moduli stacks $\overline{\mathcal{M}}_g$ of stable curves and $\mathcal{B}un^{\mathrm{ss}}_{r,d}(C)$ of vector bundles over a curve C. While the reader could safely treat these results as black boxes (and we encourage some readers to do this), it is also worthwhile to dive into the details. We follow Mumford's simplification [Mum66a] of Grothendieck's original construction of Hilbert or Quot schemes [FGAIV].

1.1 The Grassmannian, Hilbert, and Quot functors

1.1.1 Statements of the main theorems

Theorem 1.1.1 (Projectivity of the Grassmannian)**.** *Let V be a vector bundle of rank n on a noetherian scheme S. For an integer $0 < q < n$, the functor*

$$\begin{aligned} \mathrm{Gr}(q, V)\colon\ &\mathrm{Sch} \to \mathrm{Sets} \\ &(T \to S) \mapsto \{\textit{vector bundle quotients } V_T \twoheadrightarrow Q \textit{ on } T \textit{ of rank } q\} \end{aligned}$$

is representable by a projective scheme over S.

When $S = \operatorname{Spec} \mathbb{Z}$ and $V = \mathcal{O}_{\operatorname{Spec}\mathbb{Z}}^{\oplus n}$, the functor $\mathrm{Gr}(q, V)$ is identified with the Grassmannian functor $\mathrm{Gr}(q, n)$ introduced in Example 0.3.17, and $\mathrm{Gr}(q, \mathcal{O}_S^{\oplus n}) \cong \mathrm{Gr}(q, n) \times_{\mathbb{Z}} S$.

Theorem 1.1.2 (Projectivity of the Hilbert Scheme)**.** *For every noetherian scheme S and every polynomial $P \in \mathbb{Q}[z]$, the functor*

$$\begin{aligned} &\mathrm{Hilb}^P(\mathbb{P}^n_S/S)\colon \mathrm{Sch}/S \to \mathrm{Sets} \\ &\qquad (T \to S) \mapsto \left\{ \begin{array}{l} \textit{closed subschemes } Z \subseteq \mathbb{P}^n_T \textit{ flat and finitely presented over } T \\ \textit{such that } Z_t \subseteq \mathbb{P}^n_{\kappa(t)} \textit{ has Hilbert polynomial } P \textit{ for all } t \in T \end{array} \right\} \end{aligned}$$

is representable by a projective scheme over S.

Observe that $\mathrm{Hilb}^P(\mathbb{P}^n_S/S) = \mathrm{Hilb}^P(\mathbb{P}^n_{\mathbb{Z}}/\mathbb{Z}) \times_{\mathbb{Z}} S$.

Theorem 1.1.3 (Projectivity of the Quot Scheme)**.** *Let S be a noetherian scheme and F be a coherent sheaf on $\mathbb{P}^n_S$ that can be written as a quotient of $\mathcal{O}_{\mathbb{P}^n_S}(-l)^{\oplus r}$ for some l and r. For every polynomial $P \in \mathbb{Q}[z]$, the functor*

$$\mathrm{Quot}^P(F/\mathbb{P}^n_S/S)\colon \mathrm{Sch}/S \to \mathrm{Sets}$$
$$(T \to S) \mapsto \left\{ \begin{array}{l} \textit{quasi-coherent and finitely presented quotients} \\ F_T \twoheadrightarrow Q \textit{ on } \mathbb{P}^n_T \textit{ such that } Q \textit{ is flat over } T \textit{ and} \\ Q_t := Q|_{\mathbb{P}^n_{\kappa(t)}} \textit{ has Hilbert polynomial } P \textit{ for all } t \in T \end{array} \right\}$$

is representable by a scheme projective over S.

In particular, $\mathrm{Hilb}^P(\mathbb{P}^n_S/S)$ and $\mathrm{Quot}^P(F/\mathbb{P}^n_S/S)$ are quasi-compact!

Caution 1.1.4. When the base scheme S is clear, we will often shorten our notation to $\mathrm{Hilb}^P(\mathbb{P}^n_S)$ and $\mathrm{Quot}^P(F/\mathbb{P}^n_S)$. We will also abuse notation by referring to $\mathrm{Gr}(q, V)$, $\mathrm{Hilb}^P(\mathbb{P}^n_S)$, and $\mathrm{Quot}^P(F/\mathbb{P}^n_S)$ as both the functor and the scheme representing it.

Remark 1.1.5.

(1) The Grassmannian and the Hilbert scheme are special cases of the Quot scheme: $\mathrm{Gr}(q, V) \cong \mathrm{Quot}^P(V/\mathbb{P}^0_S)$, where $P(z) = q$ is the constant polynomial, and $\mathrm{Hilb}^P(\mathbb{P}^n_S) = \mathrm{Quot}^P(\mathcal{O}_{\mathbb{P}^n_S}/\mathbb{P}^n_S)$.

(2) In the definition of the Grassmannian and Quot functor above, two quotients $p\colon F_T \twoheadrightarrow Q$ and $p'\colon F_T \twoheadrightarrow Q'$ are identified if $\ker(p) = \ker(p')$ as subsheaves of F_T, or equivalently if there exists an isomorphism $\alpha\colon Q \xrightarrow{\sim} Q'$ such that $p = p' \circ \alpha$. In the Hilbert functor, two subschemes of X_T are identified if they are equal as subschemes (or equivalently if their ideal sheaves are equal as subsheaves of $\mathcal{O}_{X_T}$).

(3) When T is noetherian, the conditions that Z be finitely presented over T and Q be of finite presentation in the definitions of $\mathrm{Hilb}^P(\mathbb{P}^n_S)$ and $\mathrm{Quot}^P(F/\mathbb{P}^n_S)$ are superfluous.

(4) If we do not fix the Hilbert polynomial P, the definitions above extend to functors $\mathrm{Hilb}(\mathbb{P}^n_S)$ and $\mathrm{Quot}(F/\mathbb{P}^n_S)$, which are representable by schemes *locally* of finite type. There are decompositions

$$\mathrm{Hilb}(\mathbb{P}^n_S) = \coprod_P \mathrm{Hilb}^P(\mathbb{P}^n_S) \quad \text{and} \quad \mathrm{Quot}(F/\mathbb{P}^n_S) = \coprod_P \mathrm{Quot}^P(F/\mathbb{P}^n_S);$$

this follows from the Semicontinuity Theorem (A.6.4) as the Hilbert polynomial of a sheaf Q on $\mathbb{P}^n_T$ flat over T is locally constant. Recall also that over a reduced base scheme T, the flatness of a quotient sheaf Q over T is equivalent to the local constancy of the Hilbert polynomial (Proposition A.2.4).

(5) If S is an affine scheme, every coherent sheaf F on $\mathbb{P}^n_S$ can be written as a quotient of of $\mathcal{O}_{\mathbb{P}^n_S}(-l)^{\oplus r}$ for some l and r.

(6) The representability and projectivity of the Hilbert and Quot scheme hold more generally. For instance, the main theorems can also be formulated and proved in the same way for the functors $\mathrm{Hilb}^P(X)$ and $\mathrm{Quot}^P(F/X)$, where $X \subseteq \mathbb{P}^n_S$ is a closed subscheme. They also hold more generally if X is a closed subscheme of $\mathbb{P}(E)$ for a vector bundle E on S, i.e. if $X \to S$ is *strongly projective*. Generalizations are discussed in §1.4.4, while variants such as the Chow scheme are discussed in §1.4.5.

Roadmap of this chapter. In §1.2, we reduce the projectivity of $\mathrm{Gr}(q, V)$ to the special case of $\mathrm{Gr}(q, n) = \mathrm{Gr}(q, \mathcal{O}_{\operatorname{Spec}\mathbb{Z}}^{\oplus n})$, which we show is representable by a projective scheme by using the functorial Plücker embedding $\mathrm{Gr}(q, n) \to \mathbb{P}(\bigwedge^q \mathcal{O}_{\operatorname{Spec}\mathbb{Z}}^{\oplus n})$: over a scheme S, a quotient $\mathcal{O}_S^{\oplus n} \to Q$ is mapped to the line bundle quotient $\bigwedge^q \mathcal{O}_S^{\oplus n} \to \bigwedge^q Q$. In §1.3, we introduce Castelnuovo–Mumford regularity and establish Mumford's result on Boundedness of Regularity (1.3.8). After reducing to the case of $F = \mathcal{O}_{\mathbb{P}^n}(-l)^{\oplus r}$, Cohomology and Base Change implies that for $d \gg 0$, the pushforward $\pi_* F(d)$ under $\pi \colon \mathbb{P}_S^n \to S$ is a vector bundle whose construction commutes with base change, and Boundedness of Regularity further implies that the morphism of functors

$$\begin{aligned} \mathrm{Quot}^P(F/\mathbb{P}_S^n) &\to \mathrm{Gr}(P(d), \pi_* F(d)) \\ [F_t \twoheadrightarrow Q] &\mapsto [\pi_{T,*}(F_T(d)) \to \pi_{T,*}(Q(d))]\,, \end{aligned} \tag{1.1.6}$$

defined on an S-scheme T, is well-defined. Note that for $\mathrm{Hilb}^P(\mathbb{P}_S^n)$ with $F = \mathcal{O}_{\mathbb{P}_S^n}$, we have that $\pi_*(\mathcal{O}_{\mathbb{P}_S^n}(d)) \cong \operatorname{Sym}^d \mathcal{O}_S^{\oplus n+1}$ so that (1.1.6) takes the form of $\mathrm{Hilb}^P(\mathbb{P}_S^n) \to \mathrm{Gr}(P(d), \operatorname{Sym}^d \mathcal{O}_S^{\oplus n+1})$. We then prove that (1.1.6) is representable by locally closed immersions (Proposition 1.4.1). Since $\mathrm{Gr}(q, \pi_* F(d))$ is representable by a projective scheme over S, this already establishes the representability and quasi-projectivity of $\mathrm{Quot}^P(F/\mathbb{P}_S^n)$. Finally, by checking the valuative criterion, we establish that $\mathrm{Quot}^P(F/\mathbb{P}_S^n)$ is proper over S (Proposition 1.4.2) which in turns implies the projectivity over S. This chapter follows the excellent expositions of [Mum66a, §14-15], [AK80, §2], [Kol96, §1], [Laz04a, §1.8], and [Nit05].

Historical comments. Grothendieck established the projectivity of the Hilbert and Quot scheme in [FGAIV, Thm. 3.2]. Our exposition largely follows Grothendieck's original strategy while incorporating Mumford's simplification to establish the boundedness (or equivalently the finite typeness) of the Hilbert and Quot schemes. Boundedness is one of the hardest parts of the proof, and almost every boundedness argument for a moduli space in algebraic geometry ultimately relies on the boundedness of Hilb or Quot. Grothendieck's approach for boundedness relied on Chow's boundedness result for the Chow scheme parameterizing reduced, pure-dimensional subschemes of fixed degree. In [Mum66a], Mumford introduced the *regularity* of a coherent sheaf—now called Castelnuovo–Mumford regularity—and proved that for sufficiently large integers m every subsheaf $F \subset \mathcal{O}_{\mathbb{P}^n}^{\oplus r}$ with fixed Hilbert polynomial is m-regular (Theorem 1.3.8). Mumford used this result to construct the Hilbert scheme of curves on a surface but his argument applies equally to construct the Quot scheme.

1.2 Projectivity of the Grassmannian

The art of doing mathematics consists in finding that special case which contains all the germs of generality.

David Hilbert

The Grassmannian provides a warmup to the functorial approach of constructing projective moduli spaces, and also plays an essential role in the representability and projectivity arguments of Hilb and Quot.

1.2.1 Representability by a scheme

We show that $\mathrm{Gr}(q,n) = \mathrm{Gr}(q, \mathcal{O}_{\operatorname{Spec}\mathbb{Z}}^{\oplus n})$ is representable by a scheme (Proposition 1.2.5) by exhibiting a Zariski open cover of $\mathrm{Gr}(q,n)$ by representable subfunctors; see Definition 0.3.41. Given a subset $I \subseteq \{1,\ldots,n\}$ of size q, define the subfunctor $\mathrm{Gr}_I \subseteq \mathrm{Gr}(q,n)$ as:

$$\begin{aligned} \mathrm{Gr}_I \colon \mathrm{Sch} &\to \mathrm{Sets} \\ T &\mapsto \left\{ \begin{array}{l} \text{quotients } [p\colon \mathcal{O}_T^{\oplus n} \twoheadrightarrow Q] \in \mathrm{Gr}(q,n)(T) \text{ such that the} \\ \text{composition } \mathcal{O}_T^{\oplus I} \xrightarrow{e_I} \mathcal{O}_T^{\oplus n} \overset{p}{\twoheadrightarrow} Q \text{ is an isomorphism} \end{array} \right\}, \end{aligned} \tag{1.2.1}$$

where e_I denotes the canonical inclusion.

Lemma 1.2.2. *For each subset $I \subseteq \{1,\ldots,n\}$ of size q, the functor Gr_I is representable by affine space $\mathbb{A}_{\mathbb{Z}}^{q\times(n-q)}$.*

Proof. We may assume that $I = \{1,\ldots,q\}$. We define a map of functors $\phi\colon \mathbb{A}^{q\times(n-q)} \to \mathrm{Gr}_I$, where over a scheme T, a $q \times (n-q)$ matrix $f = \big(f_{i,j}\big)_{1\le i\le q\,,\,1\le j\le n-q}$ of global functions on T is mapped to the quotient

$$\left(\begin{array}{cccc|ccc} 1 & & & & f_{1,1} & \cdots & f_{1,n-q} \\ & 1 & & & f_{2,1} & \cdots & f_{2,n-q} \\ & & \ddots & & \vdots & & \\ & & & 1 & f_{q,1} & \cdots & f_{q,n-q} \end{array}\right) : \mathcal{O}_T^{\oplus n} \to \mathcal{O}_T^{\oplus q}. \tag{1.2.3}$$

The injectivity of $\phi(T)\colon \mathbb{A}^{q\times(n-q)}(T) \to \mathrm{Gr}_I(T)$ follows from the fact that any two quotients written in the form of (1.2.3) which are equivalent in Gr_I are necessarily defined by the same equations. To see surjectivity, let $[p\colon \mathcal{O}_T^{\oplus n} \twoheadrightarrow Q] \in \mathrm{Gr}_I(T)$ where by definition $\mathcal{O}_T^{\oplus I} \xrightarrow{e_I} \mathcal{O}_T^{\oplus n} \overset{p}{\twoheadrightarrow} Q$ is an isomorphism. The tautological commutative diagram

$$\begin{tikzcd} \mathcal{O}_T^{\oplus n} \arrow[r, "p"] \arrow[dr, "(p\circ e_I)^{-1}\circ p"'] & Q \arrow[d, "(p\circ e_I)^{-1}"] \\ & \mathcal{O}_T^{\oplus I} \end{tikzcd}$$

shows that $[p\colon \mathcal{O}_T^{\oplus n} \twoheadrightarrow Q] = [(p\circ e_I)^{-1} \circ p\colon \mathcal{O}_T^{\oplus n} \twoheadrightarrow \mathcal{O}_T^{\oplus I}] \in \mathrm{Gr}(q,n)(T)$. Since the composition $\mathcal{O}_T^{\oplus I} \xrightarrow{e_I} \mathcal{O}_T^{\oplus n} \overset{(p\circ e_I)^{-1}}{\twoheadrightarrow} \mathcal{O}_T^{\oplus I}$ is the identity, the $q\times n$ matrix corresponding to $(p\circ e_I)^{-1}\circ p$ is necessarily of the form of (1.2.3) for functions $f_{i,j} \in \Gamma(T, \mathcal{O}_T)$. Therefore $\phi(T)(\{f_{i,j}\}) = [p\colon \mathcal{O}_T^{\oplus n} \twoheadrightarrow Q] \in \mathrm{Gr}(q,n)(T)$. $\square$

Lemma 1.2.4. *The set of subfunctors $\{\mathrm{Gr}_I\}$, where I ranges over all subsets of size q, is a Zariski open cover of $\mathrm{Gr}(q,n)$.*

Proof. For a fixed subset I, we first show that $\mathrm{Gr}_I \subseteq \mathrm{Gr}(q,n)$ is an open subfunctor. To this end, we consider a scheme T and a morphism $T \to \mathrm{Gr}(q,n)$ corresponding to a quotient $p\colon \mathcal{O}_T^{\oplus n} \to Q$. Let C denote the cokernel of the composition $p\circ e_I\colon \mathcal{O}_T^{\oplus I} \to Q$. Notice that if $C = 0$, then $p \circ e_I$ is an isomorphism. The fiber product

$$\begin{tikzcd} T_I \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & T \arrow[d, "{[\mathcal{O}_T^{\oplus n} \overset{p}{\twoheadrightarrow} Q]}"] \\ \mathrm{Gr}_I \arrow[r] & \mathrm{Gr}(q,n) \end{tikzcd}$$

of functors is representable by the open subscheme $U = T \setminus \mathrm{Supp}(C)$ (the reader is encouraged to verify this claim). Note that if T is not noetherian, then $\mathrm{Supp}(C) \subseteq T$ is still closed as C is a finite type quasi-coherent sheaf.

To check the surjectivity of $\amalg_I T_I \to T$, let $t \in T$ be a point. Since $p \otimes \kappa(t) \colon \kappa(t)^n \twoheadrightarrow Q \otimes \kappa(t)$ is a surjection of vector spaces, there is a nonzero $q \times q$ minor, given by a subset I, of the $q \times n$ matrix $p \otimes \kappa(t)$. This implies that $[p \otimes \kappa(t)] \in T_I(\kappa(t))$. $\square$

Lemmas 1.2.2 and 1.2.4 together imply:

Proposition 1.2.5. *The functor* $\mathrm{Gr}(q, n)$ *is representable by a scheme.* $\square$

Exercise 1.2.6 (easy)**.** Show that $\mathrm{Gr}(q, n)$ is a smooth scheme over $\mathbb{Z}$ of relative dimension $q(n - q)$.

Exercise 1.2.7 (good practice)**.** Use the valuative criterion of properness to show that $\mathrm{Gr}(q, n) \to \operatorname{Spec} \mathbb{Z}$ is proper.

1.2.2 Projectivity of the Grassmannian

We show that the Grassmannian scheme $\mathrm{Gr}(q, n)$ is projective by explicitly providing a projective embedding. The *Plücker embedding* is the map of functors

$$\begin{aligned} P \colon \mathrm{Gr}(q, n) &\to \mathbb{P}\left(\bigwedge^q \mathcal{O}_{\operatorname{Spec}\mathbb{Z}}^{\oplus n}\right) \\ [\mathcal{O}_T^{\oplus n} \twoheadrightarrow Q] &\mapsto \left[\bigwedge^q \mathcal{O}_T^{\oplus n} \to \bigwedge^q Q\right] \end{aligned}$$

defined above over a scheme T. As both sides are representable by schemes, the morphism P corresponds to a morphism of schemes via Yoneda's Lemma (0.3.12).

Proposition 1.2.8. *The morphism* $P \colon \mathrm{Gr}(q, n) \to \mathbb{P}\left(\bigwedge^q \mathcal{O}_{\operatorname{Spec}\mathbb{Z}}^{\oplus n}\right)$ *is a closed immersion. In particular,* $\mathrm{Gr}(q, n)$ *is a projective scheme over* $\mathbb{Z}$*.*

Proof. A subset $I \subseteq \{1, \ldots, n\}$ of size q corresponds to a coordinate x_I on $\mathbb{P}(\bigwedge^q \mathcal{O}_{\operatorname{Spec}\mathbb{Z}}^{\oplus n})$, and we set $\mathbb{P}(\bigwedge^q \mathcal{O}_{\operatorname{Spec}\mathbb{Z}}^{\oplus n})_I$ to be the open locus where $x_I \neq 0$. Note that $\mathbb{P}(\bigwedge^q \mathcal{O}_{\operatorname{Spec}\mathbb{Z}}^{\oplus n})_I \subseteq \mathbb{P}(\bigwedge^q \mathcal{O}_{\operatorname{Spec}\mathbb{Z}}^{\oplus n})$ is the subfunctor parameterizing line bundle quotients $\bigwedge^q \mathcal{O}_S^{\oplus n} \to L$ such that the composition $\mathcal{O}_S \xrightarrow{e_I} \bigwedge^q \mathcal{O}_S^{\oplus n} \to L$ (where the first map is the inclusion of the Ith term) is an isomorphism. It follows that there is a cartesian diagram of functors

$$\begin{tikzcd} \mathrm{Gr}_I \arrow[r, "P_I"] \arrow[d] \arrow[dr, phantom, "\square"] & \mathbb{P}(\bigwedge^q \mathcal{O}_{\operatorname{Spec}\mathbb{Z}}^{\oplus n})_I \arrow[d] \\ \mathrm{Gr}(q, n) \arrow[r, "P"] & \mathbb{P}(\bigwedge^q \mathcal{O}_{\operatorname{Spec}\mathbb{Z}}^{\oplus n}), \end{tikzcd}$$

where Gr_I is defined in (1.2.1). Since $\{\mathbb{P}(\bigwedge^q \mathcal{O}_{\operatorname{Spec}\mathbb{Z}}^{\oplus n})_I\}$ is a Zariski open cover of $\mathbb{P}(\bigwedge^q \mathcal{O}_{\operatorname{Spec}\mathbb{Z}}^{\oplus n})$, it suffices to show that each $P_I \colon \mathrm{Gr}(q, n)_I \to \mathbb{P}(\bigwedge^q \mathcal{O}_{\operatorname{Spec}\mathbb{Z}}^{\oplus n})_I$ is a closed immersion.

For simplicity, assume that $I = \{1, \ldots, q\}$. Under the isomorphisms $\mathrm{Gr}_I \cong \mathbb{A}_{\mathbb{Z}}^{q \times (n-q)}$ of Lemma 1.2.2 and $\mathbb{P}(\bigwedge^q \mathcal{O}_{\operatorname{Spec}\mathbb{Z}}^{\oplus n})_I \cong \mathbb{A}_{\mathbb{Z}}^{\binom{n}{q} - 1}$, the morphism P_I corresponds to the map

$$\mathbb{A}_{\mathbb{Z}}^{q \times (n-q)} \to \mathbb{A}_{\mathbb{Z}}^{\binom{n}{q} - 1}$$

assigning a $q \times (n-q)$ matrix $\{x_{i,j}\}$ to the element of $\mathbb{A}_{\mathbb{Z}}^{\binom{n}{q}-1}$ whose Jth coordinate, where $J \subset \{1, \dots, n\}$ is a subset of length q distinct from I, is the $\{1, \dots, q\} \times J$ minor of the $q \times n$ block matrix

$$\left(\begin{array}{cccc|ccc} 1 & & & & x_{1,1} & \cdots & x_{1,n-q} \\ & 1 & & & x_{2,1} & \cdots & x_{2,n-q} \\ & & \ddots & & \vdots & & \\ & & & 1 & x_{q,1} & \cdots & x_{q,n-q} \end{array}\right).$$

The coordinate $x_{i,j}$ on $\mathbb{A}_{\mathbb{Z}}^{q \times (n-q)}$ is the pullback of the coordinate corresponding to the subset $\{1, \cdots, \widehat{i}, \cdots, q, q+j\}$ (see Figure 1.5.5). This shows that the corresponding ring map is surjective thereby establishing that P_I is a closed immersion. $\square$

$$x_{i,j} = \det \left(\begin{array}{ccccc|ccccc} 1 & & & & & x_{1,1} & \cdots & x_{1,j} & \cdots & x_{1,n-q} \\ & \ddots & & & & \vdots & & \vdots & & \vdots \\ & & 1 & & & x_{i,1} & \cdots & x_{i,j} & \cdots & x_{i,n-q} \\ & & & \ddots & & \vdots & & \vdots & & \vdots \\ & & & & 1 & x_{q,1} & \cdots & x_{q,j} & \cdots & x_{q,n-q} \end{array}\right)$$

Figure 1.2.9: The minor obtained by removing the ith column and all columns $q+1, \dots, n$ other than $q+j$ is precisely $x_{i,j}$.

$$x_{i,j} = \det \left(\begin{array}{ccccc|ccccc} 1 & & & & & x_{1,1} & \cdots & x_{1,j} & \cdots & x_{1,n-q} \\ & \ddots & & & & \vdots & & \vdots & & \vdots \\ & & 1 & & & x_{i,1} & \cdots & x_{i,j} & \cdots & x_{i,n-q} \\ & & & \ddots & & \vdots & & \vdots & & \vdots \\ & & & & 1 & x_{q,1} & \cdots & x_{q,j} & \cdots & x_{q,n-q} \end{array}\right)$$

Exercise 1.2.10 (good practice)**.** For a field $\Bbbk$, define $\mathrm{Gr}(q,n)_{\Bbbk} := \mathrm{Gr}(q,n) \times_{\mathbb{Z}} \Bbbk$, and let $p \in \mathrm{Gr}(q,n)_{\Bbbk}$ be a point corresponding to a quotient $Q = \Bbbk^n/K$. Show that there is a natural bijection of the tangent space

$$T_p(\mathrm{Gr}(q,n)_{\Bbbk}) \xrightarrow{\sim} \mathrm{Hom}_{\Bbbk}(K, Q).$$

with the vector space of $\Bbbk$-linear maps $K \to Q$.

Exercise 1.2.11 (good practice)**.** Using the valuative criterion for properness for $\mathrm{Gr}(q,n)$ (Exercise 1.2.7), provide an alternative proof of projectivity.

Exercise 1.2.12 (hard)**.** Show that $\mathrm{Gr}(q,n)$ is identified with either the quotient U/GL_q where $U \subseteq \mathbb{A}^{q\times n}$ is the subset of $q \times n$ matrices of full rank, or the quotient GL_n/H, where

$$H = \left\{ \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \,\middle|\, A \in \mathrm{GL}_q, B \in \mathbb{A}^{q\times(n-q)}, C \in \mathrm{GL}_{n-q} \right\} \subseteq \mathrm{GL}_n .$$

Exercise 1.2.13 (hard)**.** Show that $\mathrm{Pic}(\mathrm{Gr}(q,n)) \cong \mathbb{Z}$ and is generated by the Plücker line bundle.

Exercise 1.2.14 (Flag varieties)**.** Fix a positive integer n and a decreasing sequence $n > q_1 > \ldots > q_s > 0$ of integers. Show that the functor $\mathrm{Sch} \to \mathrm{Sets}$, taking a scheme S to isomorphism classes of sequences of quotients

$$\mathcal{O}_S^{\oplus n} \twoheadrightarrow V_1 \twoheadrightarrow \cdots \twoheadrightarrow V_s,$$

where each V_i is a vector bundle on S of rank q_i, is representable by a projective scheme over $\mathbb{Z}$.

1.2.3 Relative version

We prove that the functor $\mathrm{Gr}(q,V)$ is representable and projective over S for any vector bundle V on a noetherian scheme S .

Proof of Theorem 1.1.1. If V is a vector bundle over S of rank n, there is the relative Plücker embedding

$$\begin{aligned} P\colon \mathrm{Gr}(q,V) &\to \mathbb{P}\left(\bigwedge^q V\right) \\ [V_T \twoheadrightarrow Q] &\mapsto \left[\bigwedge^q V_T \to \bigwedge^q Q\right] \end{aligned}$$

defined above over a S-scheme T, which is a morphism of functors over S. Since $\mathbb{P}(\bigwedge^q V)$ is projective over S, it suffices to show that this morphism is representable by closed immersions. This property can be checked Zariski-locally: if $U \subseteq S$ is an open subscheme where V is trivial, then the restriction of $\mathrm{Gr}(q,V) \to \mathbb{P}(\bigwedge^q V)$ to U is base change of the Plücker embedding $\mathrm{Gr}(q,\mathcal{O}_U^{\oplus n}) \to \mathbb{P}(\bigwedge^q \mathcal{O}_U^{\oplus n})$, which we've already verified to be a closed immersion (Proposition 1.2.8), by $U \to \mathrm{Spec}\,\mathbb{Z}$. □

Since the Grassmannian functor is representable, there is a universal family (see Definition 0.3.24), i.e., there is a *universal quotient* $\mathcal{O}_{\mathrm{Gr}(q,V)} \otimes_S V \to Q_{\mathrm{univ}}$, where $\mathcal{O}_{\mathrm{Gr}(q,V)} \otimes_S V$ is notation for the pullback of V under the structure morphism $\mathrm{Gr}(q,V) \to S$. The pullback of $\mathcal{O}(1)$ under the Plücker embedding is $\det(Q_{\mathrm{univ}})$, which we sometimes call the *Plücker line bundle*. Thus, we obtain:

Corollary 1.2.15. *The determinant* $\det(Q_{\mathrm{univ}})$ *of the universal quotient is a line bundle on* $\mathrm{Gr}(q,V)$ *which is relatively very ample over* S. □

1.3 Castelnuovo–Mumford regularity

> *Some of the deepest results in algebraic geometry concern the problem of giving criteria for the higher cohomology groups of a sheaf to be 0... We shall prove here (with the help of techniques developed and used by Nakai, Matsusaka and Kleiman) only a weak vanishing theorem, but one which is uniformly applicable to a large class of sheaves.*
>
> David Mumford [Mum66a, p. 99]

Serre's vanishing theorem—sometimes called the Cartan–Serre–Grothendieck theorem—states that if F is a coherent sheaf on a projective variety $(X, \mathcal{O}_X(1))$, then for $d \gg 0$

(1) $F(d)$ is base point free,

(2) $\mathrm{H}^i(X, F(d)) = 0$ for $i > 0$, and

(3) the multiplication map

$$\mathrm{H}^0(X, F(d)) \otimes \mathrm{H}^0(X, \mathcal{O}(k)) \to \mathrm{H}^0(X, F(d+k))$$

is surjective for all $k \geq 0$.

Castelnuovo–Mumford regularity provides a quantitative measure of the size of d necessary for the twist $F(d)$ to satisfy these three cohomological properties.

1.3.1 Definition and basic properties

Definition 1.3.1. Let F be a coherent sheaf on projective space $\mathbb{P}^n$ over a field $\Bbbk$. For an integer m, we say that F is *m-regular* if

$$\mathrm{H}^i(\mathbb{P}^n, F(m-i)) = 0$$

for all $i \geq 1$. The *regularity of F* is the smallest integer m such that $F(m)$ is m-regular.

It follows from the definition of regularity that if F is m-regular, then $F(d)$ is $(m-d)$-regular. We will show in Proposition 1.3.5 that if F is m-regular, it also d-regular for all $d \geq m$. While the requirement in the definition that the ith cohomology of the $(m-i)$th twist vanishes may appear mysterious at first, this definition is very convenient for induction arguments on the dimension, as evidenced by the following result.

Lemma 1.3.2. *Let F be an m-regular coherent sheaf on $\mathbb{P}^n$ over a field $\Bbbk$. If $H \subseteq \mathbb{P}^n$ is a hyperplane avoiding the associated points of F, then $F|_H$ is also m-regular.*

Proof. The hypotheses imply that over an affine open subscheme $U \subseteq \mathbb{P}^n$, the defining equation of H is a nonzerodivisor for the module $\Gamma(U, F)$. Thus $H \colon F(-1) \to F$ is injective, and for each integer $i > 0$, we have a short exact sequence

$$0 \to F(m-i-1) \to F(m-i) \to F|_H(m-i) \to 0,$$

which induces a long exact sequence on cohomology

$$\cdots \to \mathrm{H}^i(\mathbb{P}^n, F(m-i)) \to \mathrm{H}^i(H, F|_H(m-i)) \to \mathrm{H}^{i+1}(\mathbb{P}^n, F(m-i-1)) \to \cdots.$$

If F is m-regular, then $\mathrm{H}^i(\mathbb{P}^n, F(m-i)) = \mathrm{H}^{i+1}(\mathbb{P}^n, F(m-i-1)) = 0$. It follows that $\mathrm{H}^i(H, F|_H(m-i)) = 0$ for all $i > 0$, and thus $F|_H$ is also m-regular. □

Exercise 1.3.3 (easy, good practice)**.**

(a) Show that $\mathcal{O}(d)$ is $(-d)$-regular on $\mathbb{P}^n$.

(b) Show that the structure sheaf of a hypersurface $H \subseteq \mathbb{P}^n$ of degree d is $(d-1)$-regular.

(c) Show that the structure sheaf of a smooth curve $C \subseteq \mathbb{P}^n$ of genus g is $(2g-1)$-regular.

Exercise 1.3.4 (easy)**.** Let $0 \to K \to F \to Q \to 0$ be a short exact sequences of coherent sheaves on $\mathbb{P}^n$. If K is $(m+1)$-regular and F is m-regular, show that Q is m-regular.

Another advantage of regularity is the following result of Castelnuovo.

Proposition 1.3.5 (Properties of Regularity)**.** *Let F be an m-regular coherent sheaf on $\mathbb{P}^n$ over a field $\Bbbk$.*

(1) For $d \geq m$, F is d-regular.

(2) The multiplication map

$$\mathrm{H}^0(\mathbb{P}^n, F(d)) \otimes \mathrm{H}^0(\mathbb{P}^n, \mathcal{O}(k)) \to \mathrm{H}^0(\mathbb{P}^n, F(d+k))$$

is surjective if $d \geq m$ and $k \geq 0$.

(3) For $d \geq m$, $F(d)$ is globally generated and $\mathrm{H}^i(\mathbb{P}^n, F(d)) = 0$ for $i \geq 1$.

Proof. If $\Bbbk \to \Bbbk'$ is a field extension, then Flat Base Change (A.2.12) implies that $\mathrm{H}^i(\mathbb{P}^n_{\Bbbk}, F) \otimes_{\Bbbk} \Bbbk' = \mathrm{H}^i(\mathbb{P}^n_{\Bbbk'}, F \otimes_{\Bbbk} \Bbbk')$. As $\Bbbk \to \Bbbk'$ is faithfully flat, the assertions (1)–(3) can be checked after base change. We can thus assume that $\Bbbk$ is algebraically closed and, in particular, infinite. For (1) and (2), we will argue by induction on n with the base case of $n = 0$ being clear. If $n > 0$, since $\Bbbk$ is infinite, we may choose a hyperplane $H \subseteq \mathbb{P}^n$ avoiding the associated points of F. Since the restriction $F|_H$ is m-regular (Lemma 1.3.2) on $H \cong \mathbb{P}^{n-1}$, the inductive hypothesis implies that (1) and (2) hold for $F|_H$.

We prove (1) by also arguing via induction on the integer d. The base case $d = m$ holds by hypothesis. For $d > m$, the short exact sequence $0 \to F(d-i-1) \to F(d-i) \to F|_H(d-i) \to 0$ induces a long exact sequence on cohomology

$$\cdots \to \mathrm{H}^i(\mathbb{P}^n, F(d-i-1)) \to \mathrm{H}^i(\mathbb{P}^n, F(d-i)) \to \mathrm{H}^i(H, F|_H(d-i)) \to \cdots.$$

For $i > 0$, the first term vanishes by the induction hypothesis on d since F is $(d-1)$-regular. The third term above vanishes by the inductive hypothesis on n: $F|_H$ is m-regular by Lemma 1.3.2 and thus d-regular by the inductive hypothesis on n, hence $\mathrm{H}^i(H, F|_H(d-i)) = 0$. Thus, the second term vanishes and we have established (1).

To show (2), we use induction on k in addition to n. We denote the multiplication map by

$$\mu_{d,k} \colon \mathrm{H}^0(\mathbb{P}^n, F(d)) \otimes \mathrm{H}^0(\mathbb{P}^n, \mathcal{O}(k)) \to \mathrm{H}^0(\mathbb{P}^n, F(d+k)).$$

While the base case $k = 0$ is clear, the induction argument will require us to directly establish the case $k = 1$. To this end, we consider the commutative diagram

$$\begin{tikzcd}
& \mathrm{H}^0(\mathbb{P}^n, F(d)) \otimes \mathrm{H}^0(\mathbb{P}^n, \mathcal{O}(1)) \arrow[r, "\nu_d \otimes \mathrm{res}"] \arrow[d, "\mu_{d,1}"] & \mathrm{H}^0(H, F|_H(d)) \otimes \mathrm{H}^0(H, \mathcal{O}_H(1)) \arrow[d] \\
\mathrm{H}^0(\mathbb{P}^n, F(d)) \arrow[ur, "\mathrm{id} \otimes H"] \arrow[r, "\alpha"] & \mathrm{H}^0(\mathbb{P}^n, F(d+1)) \arrow[r, "\nu_{d+1}"] & \mathrm{H}^0(H, F|_H(d+1)).
\end{tikzcd} \tag{1.3.6}$$

The map α is given by multiplication by $H \in \mathrm{H}^0(\mathbb{P}^n, \mathcal{O}(1))$, and there is an inclusion $\operatorname{im}(\alpha) \subseteq \operatorname{im}(\mu_{d,1})$. Since $\mathrm{H}^1(\mathbb{P}^n, F(d)) = 0$ by (1), the restriction map $\nu_d \colon \mathrm{H}^0(\mathbb{P}^n, F(d)) \to \mathrm{H}^0(H, F|_H(d))$ is surjective. Likewise, since $\mathrm{H}^1(\mathbb{P}^n, \mathcal{O}) = 0$, $\mathrm{res} \colon \mathrm{H}^0(\mathbb{P}^n, \mathcal{O}(1)) \to \mathrm{H}^0(H, \mathcal{O}_H(1))$ is surjective, and so the top horizontal arrow is surjective. The inductive hypothesis applied to $H = \mathbb{P}^{n-1}$ implies that the right vertical arrow is surjective. Therefore, the composition $\nu_{d+1} \circ \mu_{d,1}$ is surjective and it follows that $\operatorname{im}(\mu_{d,1})$ surjects onto $\mathrm{H}^0(H, F|_H(d+1))$. By exactness of the bottom row, we have that

$$\mathrm{H}^0(\mathbb{P}^n, F(d+1)) = \operatorname{im}(\mu_{d,1}) + \ker(\nu_{d+1}) = \operatorname{im}(\mu_{d,1}) + \operatorname{im}(\alpha) = \operatorname{im}(\mu_{d,1}),$$

which shows that $\mu_{d,1}$ is surjective.

If $k > 1$, we consider the commutative square

$$\begin{tikzcd}
\mathrm{H}^0(\mathbb{P}^n, F(d)) \otimes \mathrm{H}^0(\mathbb{P}^n, \mathcal{O}(k-1)) \otimes \mathrm{H}^0(\mathbb{P}^n, \mathcal{O}(1)) \arrow[r] \arrow[d, "\mu_{d,k-1} \otimes \mathrm{id}"] & \mathrm{H}^0(\mathbb{P}^n, F(d)) \otimes \mathrm{H}^0(\mathbb{P}^n, \mathcal{O}(k)) \arrow[d, "\mu_{d,k}"] \\
\mathrm{H}^0(\mathbb{P}^n, F(d+k-1)) \otimes \mathrm{H}^0(\mathbb{P}^n, \mathcal{O}(1)) \arrow[r, "\mu_{d+k-1,1}"] & \mathrm{H}^0(\mathbb{P}^n, F(d+k)).
\end{tikzcd}$$

The left vertical map and bottom horizontal arrow are surjective by the inductive hypothesis applied to $k-1$ and $k = 1$, respectively. It follows that $\mu_{d,k}$ is surjective.

To show (3), we know that for $k \gg 0$, $F(d+k)$ is globally generated, i.e., $\gamma_{F(d+k)} \colon \mathrm{H}^0(\mathbb{P}^n, F(d+k)) \otimes \mathcal{O}_{\mathbb{P}^n} \to F(d+k)$ is surjective. Consider the commutative square

$$\begin{tikzcd}
\mathrm{H}^0(\mathbb{P}^n, F(d)) \otimes \mathrm{H}^0(\mathbb{P}^n, \mathcal{O}(k)) \otimes \mathcal{O}_{\mathbb{P}^n} \arrow[r, "\mu_{d,k} \otimes \mathrm{id}"] \arrow[d, "\gamma_{F(d)} \otimes \mathrm{id}"] & \mathrm{H}^0(\mathbb{P}^n, F(d+k)) \otimes \mathcal{O}_{\mathbb{P}^n} \arrow[d, "\gamma_{F(d+k)}"] \\
F(d) \otimes \left(\mathrm{H}^0(\mathbb{P}^n, \mathcal{O}(k)) \otimes \mathcal{O}_{\mathbb{P}^n}\right) \arrow[r, "\mathrm{id} \otimes \gamma_{\mathcal{O}(k)}"] & F(d) \otimes \mathcal{O}(k).
\end{tikzcd}$$

Since the top horizontal arrow is surjective by (2), the composition from the top left to the bottom right is surjective. Since the bottom horizontal map is defined by tensoring with $F(d)$, the map $\gamma_{F(d)}$ must be surjective. Finally, to see the vanishing of the higher cohomology of $F(d)$, observe that for each $i > 0$, the sheaf F is $(d+i)$-regular by (1) and thus $\mathrm{H}^i(\mathbb{P}^n, F(d)) = 0$. $\square$

One direct consequence of (1) is that if F is m-regular, then the restriction map

$$\nu_d \colon : \mathrm{H}^0(\mathbb{P}^n, F(d)) \to \mathrm{H}^0(H, F|_H(d))$$

is surjective for all $d \geq m$. Indeed, (1) implies that F is also d-regular and the surjectivity follows from the vanishing of $\mathrm{H}^1(\mathbb{P}^n, F(d-1))$. The following lemma—which will be used in the proof of Theorem 1.3.8—shows that we can still arrange for the surjectivity of ν_d under weaker hypotheses.

Lemma 1.3.7. *Let F be a coherent sheaf on $\mathbb{P}^n$ and H be a hyperplane avoiding the associated points of F. If $F|_H$ is m-regular and ν_d is surjective for some $d \geq m$, then ν_p is surjective for all $p \geq d$.*

Proof. By staring at diagram (1.3.6), we see that the top arrow $\nu_d \otimes \mathrm{res}$ is surjective (as both ν_d and res are surjective) and the vertical right multiplication morphism is surjective (by applying Proposition 1.3.5(2) to the m-regular sheaf $F|_H$). This implies that ν_{d+1} is surjective, and the statement follows by induction. $\square$

1.3.2 Regularity bounds

We prove a boundedness result for the regularity of subsheaves of the trivial vector bundle first established by Mumford in [Mum66a, p.101]. This is the basis of almost every boundedness result in algebraic geometry.[1]

Theorem 1.3.8 (Boundedness of Regularity). *For every pair of nonnegative integers r and n, and for every polynomial $P \in \mathbb{Q}[z]$, there exists an integer m_0 with the following property: for every field $\Bbbk$, every subsheaf $F \subset \mathcal{O}_{\mathbb{P}^n_{\Bbbk}}^{\oplus r}$ with Hilbert polynomial P is m_0-regular.*

Proof. Since being m-regular is insensitive to field extensions, we can assume that $\Bbbk$ is infinite. We will argue by induction on n. The base case of $n = 0$ holds as every sheaf F on $\mathbb{P}^0$ is m-regular for every integer m.

For $n \geq 1$ and a subsheaf $F \subseteq \mathcal{O}_{\mathbb{P}^n}^{\oplus r}$ with Hilbert polynomial P, we can choose a hyperplane $H \subseteq \mathbb{P}^n$ avoiding all associated points of $\mathcal{O}_{\mathbb{P}^n}^{\oplus r}/F$. This ensures that $\mathrm{Tor}_1^{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{O}_H, \mathcal{O}_{\mathbb{P}^n}^{\oplus r}/F) = 0$ and that the short exact sequence $0 \to F \to \mathcal{O}_{\mathbb{P}^n}^{\oplus r} \to \mathcal{O}_{\mathbb{P}^n}^{\oplus r}/F \to 0$ restricts to a short exact sequence

$$0 \to F|_H \to \mathcal{O}_H^{\oplus r} \to \mathcal{O}_H^{\oplus r}/F \to 0. \tag{1.3.9}$$

As $H \cong \mathbb{P}^{n-1}$, we may apply the inductive hypothesis to $F|_H \subseteq \mathcal{O}_H^{\oplus r}$. On the other hand, since $F \subseteq \mathcal{O}_{\mathbb{P}^n}^{\oplus r}$ is torsion free, we have a short exact sequence

$$0 \to F(-1) \xrightarrow{H} F \to F|_H \to 0, \tag{1.3.10}$$

and the Hilbert polynomial of $F|_H$ is $\chi(F|_H(d)) = \chi(F(d)) - \chi(F(d-1)) = P(d) - P(d-1)$. In particular, the Hilbert polynomial of $F|_H$ only depends on P, and the inductive hypothesis applied to $F|_H \subseteq \mathcal{O}_H^{\oplus r}$ gives an integer m_1 such that $F|_H$ is m_1-regular.

For $m \geq m_1 - 1$, since $\mathrm{H}^i(H, F|_H(m)) = 0$ for all $i \geq 1$, we have a long exact sequence

$$\begin{aligned} 0 \to \mathrm{H}^0(\mathbb{P}^n, F(m-1)) \to \mathrm{H}^0(\mathbb{P}^n, F(m)) \to \mathrm{H}^0(H, F|_H(m)) \to \\ \mathrm{H}^1(\mathbb{P}^n, F(m-1)) \to \mathrm{H}^1(\mathbb{P}^n, F(m)) \to 0 \end{aligned} \tag{1.3.11}$$

along with isomorphisms $\mathrm{H}^i(\mathbb{P}^n, F(m-1)) \xrightarrow{\sim} \mathrm{H}^i(\mathbb{P}^n, F(m))$ for $i \geq 2$. Since $\mathrm{H}^i(\mathbb{P}^n, F(d))$ vanishes for $d \gg 0$, we can conclude that $\mathrm{H}^i(\mathbb{P}^n, F(m-1)) = 0$ for $m \geq m_1 - 1$ and $i \geq 2$.

To handle H^1, we use the inequalities $\mathrm{h}^1(\mathbb{P}^n, F(m_1)) \geq \mathrm{h}^1(\mathbb{P}^n, F(m_1+1)) \geq \cdots$, which eventually stabilize to 0. We claim that in fact that the inequalities $\mathrm{h}^1(\mathbb{P}^n, F(m_1)) > \mathrm{h}^1(\mathbb{P}^n, F(m_1+1)) > \cdots$ are strict until they become 0. To see this, we observe that there is an equality $\mathrm{h}^1(\mathbb{P}^n, F(m-1)) = \mathrm{h}^1(\mathbb{P}^n, F(m))$ for $m \geq m_1$ if and only if $\nu_m \colon \mathrm{H}^0(\mathbb{P}^n, F(m)) \to \mathrm{H}^0(H, F|_H(m))$ is surjective. Suppose that $\nu_{m'}$ is surjective for some $m' \geq m$. Since $F|_H$ is m_1-regular, we may apply Lemma 1.3.7 to conclude that $\mu_{m''}$ is surjective for all $m'' \geq m'$. Thus $\mathrm{h}^1(\mathbb{P}^n, F(m''))$ is constant for $m'' \geq m'$, and therefore zero. This establishes the claim. Setting $m_2 = m_1 + 1 + \mathrm{h}^1(\mathbb{P}^n, F(m_1))$, we see that $\mathrm{h}^1(\mathbb{P}^n, F(m_2 - 1)) = 0$ and that F is m_2-regular.

[1] Even in [SP, Tag 0DPA] where Hilbert and Quot schemes are constructed using Artin's Axioms for Algebraicity (C.7.4), it is this result [SP, Tag 08AG] that is applied to establish quasi-compactness.

It remains to show that m_2 is bounded above by a constant m_0 independent of F. Since $F \subseteq \mathcal{O}_{\mathbb{P}^n}^{\oplus r}$, we have that $\mathrm{h}^0(\mathbb{P}^n, F(d)) \leq r\mathrm{h}^0(\mathbb{P}^n, \mathcal{O}(d)) = r\binom{n+d}{n}$ for any $d \geq 0$. Using the vanishing of $\mathrm{h}^i(\mathbb{P}^n, F(m_1))$ for $i \geq 2$, we have

$$\begin{aligned} \mathrm{h}^1(\mathbb{P}^n, F(m_1)) &= \mathrm{h}^0(\mathbb{P}^n, F(m_1)) - \chi(F(m_1)) \\ &\leq r\binom{n+m_1}{n} + P(m_1). \end{aligned}$$

Defining $m_0 := m_1 + 1 + r\binom{n+m_1}{n} + P(m_1) \geq m_2$ gives the desired constant such that every $F \subseteq \mathcal{O}_{\mathbb{P}^n}^{\oplus r}$ with Hilbert polynomial P is m_0-regular. □

Remark 1.3.12. The above proof establishes in fact a stronger statement. To formulate the result, we recall that every numerical polynomial $P \in \mathbb{Q}[z]$ (i.e., $P(d) \in \mathbb{Z}$ for integers $d \gg 0$) of degree n can be expressed uniquely as

$$P(d) = \sum_{i=0}^{n} a_i \binom{d}{i}$$

for $a_i \in \mathbb{Z}$; this follows from a straightforward inductive argument, c.f., [Har77, Prop. I.7.3]. For nonnegative integers r and n, there exists a polynomial

$$\Lambda_{r,n} \in \mathbb{Z}[x_0, \ldots, x_n] \tag{1.3.13}$$

with the following property: for every field $\Bbbk$, every subsheaf $F \subseteq \mathcal{O}_{\mathbb{P}^n_{\Bbbk}}^{\oplus r}$ with Hilbert polynomial $P(d) = \sum_i a_i \binom{d}{i}$ is m_0-regular for $m_0 = \Lambda_{r,n}(a_0, \ldots, a_n)$. See [Mum66a, p. 101].

Exercise 1.3.14. Let $P = d$ be the constant polynomial for a fixed integer d. What is the optimal m such that every ideal sheaf $I \subseteq \mathcal{O}_{\mathbb{P}^n}$ defining a dimension 0 subscheme of length d is m-regular?

Remark 1.3.15 (Optimal bounds). Although Mumford's result on Boundedness of Regularity (Theorem 1.3.8) provides an explicit bound and is sufficient for many applications including the construction of the Hilbert scheme, there is a more optimal bound established by Gotzmann: for a projective scheme $X \subseteq \mathbb{P}^N$ over a field $\Bbbk$ with Hilbert polynomial P, there are unique integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \geq 1$ such that P can be expressed as

$$P(d) = \binom{d+\lambda_1-1}{\lambda_1-1} + \binom{d+\lambda_2-2}{\lambda_2-1} + \cdots + \binom{d+\lambda_r-r}{\lambda_r-1},$$

and the ideal sheaf $\mathcal{I}_X$ of X is r-regular. See [Got78], [BH93, §4.3], [Gre89], and [Gre98, §3].

Exercise 1.3.16. Let $C \subseteq \mathbb{P}^n$ be a curve of degree d and genus g. Show that Gotzmann's bound implies that the ideal sheaf I_C of C is $(\binom{d}{2} + 1 - g)$-regular. Can you compare this to the bound given by the proof of Theorem 1.3.8, i.e., can you compute $\Lambda_{1,n}(1-g, d)$ from (1.3.13)?

Remark 1.3.17. It was shown in [GLP83] that the ideal sheaf I_C of an integral, non-degenerate curve $C \subseteq \mathbb{P}^N$ of degree d is $(d - N + 2)$-regular. It is conjectured more generally that the ideal sheaf of a smooth, non-degenerate projective subvariety $X \subseteq \mathbb{P}^N$ of dimension n and degree d is $(d - (N - n)) + 1)$-regular; see [GLP83] and [EG84].

Proposition 1.3.18 (Regularity in Families). *Let S be a noetherian scheme, $\pi\colon \mathbb{P}^n_S \to S$ be relative projective space, and Q be a coherent sheaf on $\mathbb{P}^n_S$ flat over S. Suppose that there is an integer $m > 0$ such that for every $s \in S$, the restriction Q_s to $\mathbb{P}^n_{\kappa(s)}$ is m-regular. Then for $d \geq m$,*

(1) $\pi_* Q(d)$ is a vector bundles whose construction commutes with base change by $f\colon T \to S$, i.e., $f^*\pi_*Q(d) \cong \pi_{T,*}Q_T(d)$,

(2) $R^i\pi_*Q(d) = 0$ for $i > 0$, and

(3) $\pi^*\pi_*Q(d) \to Q(d)$ is surjective.

Proof. Since $\mathrm{R}^i\pi_*Q(d) \otimes \kappa(s) \to \mathrm{H}^i(\mathbb{P}^n_\kappa(s), Q_s(d)) = 0$ is surjective for all $s \in S$, by Cohomology and Base Change (A.6.8) yields (1) and (2). For (3), let $C = \operatorname{coker}(\pi^*\pi_*Q(d) \to Q(d))$. For each $s \in S$, we have a cartesian diagram

$$\begin{tikzcd}
\mathbb{P}^n_{\kappa(s)} \arrow[r, "j"] \arrow[d, "\pi_s"] \arrow[dr, phantom, "\square"] & \mathbb{P}^n_S \arrow[d, "\pi"] \\
\operatorname{Spec}\kappa(s) \arrow[r, "i"] & S.
\end{tikzcd}$$

Using (1), $j^*\pi^*\pi_*Q(d) = \pi_s^* i^*\pi_*Q(d) = \pi_s^*\pi_{s,*}Q_s(d)$, the pullback of the adjunction map $\pi^*\pi_*Q(d) \to Q(d)$ under j corresponds to the adjunction map $\pi_s^*\pi_{s,*}Q_s(d) \to Q_s(d)$, which we know to be surjective by Properties of Regularity (1.3.5(3)). Thus $C \otimes \kappa(s) = \operatorname{coker}(\pi_s^*\pi_{s,*}Q_s(d) \to Q_s(d)) = 0$ for all $s \in S$, and $\pi^*\pi_*Q(d) \to Q(d)$ is surjective. $\square$

1.4 Projectivity of Hilb and Quot

Projective geometry is all geometry.

Arthur Cayley

We prove the Projectivity of Hilbert and Quot Schemes (1.1.2 and 1.1.3).

1.4.1 Special case when F is a direct sum of line bundles

Let S be a noetherian scheme and $\pi\colon \mathbb{P}^n_S \to S$ be relative projective space. We first consider the special case of a coherent sheaf

$$F := \mathcal{O}_{\mathbb{P}^n_S}(-l)^{\oplus r}$$

for integers l and $r \geq 1$. For each integer $d \geq 0$, we can try to define morphisms of functors

$$\begin{aligned}
\operatorname{Quot}^P(F/\mathbb{P}^n_S) &\to \operatorname{Gr}(P(d), \pi_*F(d)) \\
[F_T \twoheadrightarrow Q] &\mapsto [\pi_{T,*}F_T(d) \to \pi_{T,*}Q(d)],
\end{aligned}$$

where $[F_T \twoheadrightarrow Q]$ is an object over an S-scheme T.

Step 1: There exists an integer m_0 such that $\operatorname{Quot}^P(F/\mathbb{P}^n_S) \to \operatorname{Gr}(P(d), \pi_*F(d))$ is well-defined for $d \geq m_0$.

Proof. Since $F = \mathcal{O}_{\mathbb{P}^n_S}(-l)^{\oplus r}$, for $d \geq l$, we have that $\pi_*F(d) = \left(\operatorname{Sym}^{d-l}\Gamma(S, \mathcal{O}_S)\right)^{\oplus r}$ is a vector bundle whose construction commutes with base change by $T \to S$ and

$\mathrm{R}^i\pi_*F(d) = 0$ for all $i > 0$. For every field-valued point $s \in S(\Bbbk)$ and quotient $F_s = \mathcal{O}_{\mathbb{P}^n_{\Bbbk}}(-l)^{\oplus r} \twoheadrightarrow Q$ of Hilbert polynomial P with kernel $K_s = \ker(F_s \to Q)$, then $K(l)$ is a subsheaf of $\mathcal{O}_{\mathbb{P}^n_{\Bbbk}}^{\oplus r}$ whose Hilbert polynomial is determined by P and the Hilbert polynomial of $\mathcal{O}_{\mathbb{P}^n_{\Bbbk}}^{\oplus r}$. We may therefore apply Boundedness of Regularity (1.3.8) to choose $m_0 \geq l$ such that every such kernel K_s is m_0-regular. By possibly increasing m_0, we can arrange that F_s is also m_0-regular. Exercise 1.3.4 implies that each quotient Q is m_0-regular. By Regularity in Families (1.3.18(1)), for each $d \geq m_0$ and quotient sheaf $F_T \twoheadrightarrow Q$ flat over T with Hilbert polynomial P, $\pi_{T,*}Q(d)$ is a vector bundle on T of rank $P(d)$ whose construction commutes with base change. Since F_T and Q are flat over T, so is $K := \ker(F_T \to Q)$. Since every fiber K_s is m_0-regular, Regularity in Families (1.3.18(2)) implies that for $d \geq m_0$, $\mathrm{R}^1\pi_{T,*}K(d) = 0$. Therefore $\pi_{T,*}F_T(d) \to \pi_{T,*}Q(d)$ is surjective and (1.4.1) is well-defined. $\square$

Step 2: There exists an integer m_0 such that $\mathrm{Quot}^P(F/\mathbb{P}^n_S) \to \mathrm{Gr}(P(d), \pi_*F(d))$ is a monomorphism for $d \geq m_0$.

Proof. Let $F_T \twoheadrightarrow Q$ be a quotient sheaf flat over a scheme T with Hilbert polynomial P. The kernel $K = \ker(F_T \to Q)$ is also flat over T. For m_0 chosen as in Step 1, for every $s \in S$, the fibers K_s, $(F_T)_s$, and Q_s are m_0-regular. By Regularity of Families (1.3.18(2)), for $d \geq m_0$, $R^1\pi_{T,*}K(d) = 0$, which yields a commutative diagram of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \pi_T^*\pi_{T,*}K(d) \arrow[r] \arrow[d] \arrow[dr, "\alpha"] & \pi_T^*\pi_{T,*}F_T(d) \arrow[r] \arrow[d] & \pi_T^*\pi_{T,*}Q(d) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K(d) \arrow[r] & F_T(d) \arrow[r] & Q(d) \arrow[r] & 0
\end{tikzcd}$$

By Regularity of Families (1.3.18(3)), each vertical map is surjective. Thus $Q(d) = \mathrm{coker}(\alpha)$ is uniquely determined by $\pi_{T,*}F_T(d) \to \pi_{T,*}Q(d)$. $\square$

Step 3: There exists an integer m_0 such that $\mathrm{Quot}^P(F/\mathbb{P}^n_S) \to \mathrm{Gr}(P(d), \pi_*F(d))$ is a locally closed immersion for $d \geq m_0$.

Proof. Let $T \to \mathrm{Gr}(P(d), \pi_*F(d))$ be a map determined by a vector bundle quotient $(\pi_*F(d))_T = \pi_{T,*}F_T(d) \twoheadrightarrow V$, and let K be the kernel. Consider the fiber product

$$\begin{tikzcd}
Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & T \arrow[d, "{[\pi_{T,*}F_T(d) \twoheadrightarrow V]}"] \\
\mathrm{Quot}^P(F/\mathbb{P}^n_S) \arrow[r] & \mathrm{Gr}(P(d), \pi_*F(d)).
\end{tikzcd}$$

On $\mathbb{P}^n_T$, we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \pi_T^*K \arrow[r] \arrow[dr, "\alpha"] & \pi_T^*\pi_{T,*}F_T(d) \arrow[r] \arrow[d] & \pi_T^*V \arrow[r] \arrow[d] & 0 \\
 & & F_T(d) \arrow[r] & Q(d) \arrow[r] & 0,
\end{tikzcd}$$

of sheaves on $\mathbb{P}^n_T$, where Q is defined such that $Q(d) = \mathrm{coker}(\alpha)$. Since K is a vector bundle on T, $K = \pi_{T,*}\pi_T^*K$, and thus $\pi_{T,*}\alpha$ is identified with the inclusion $K \hookrightarrow$

$\pi_{T,*}F_T(d)$. Applying $\pi_{T,*}$ to the bottom sequence and using that $\mathrm{R}^1\pi_{T,*}(\pi_T^*K) = 0$, we obtain a short exact sequence $0 \to K = \pi_{T,*}\pi_T^*K \to \pi_{T,*}F_T(d) \to \pi_{T,*}Q(d) \to 0$. This gives the identification $\pi_{T,*}Q(d) = V$.

If Q is flat over T with Hilbert polynomial P, then $[F_T \twoheadrightarrow Q] \colon T \to \mathrm{Quot}^P(F/\mathbb{P}^n_S)$ provides a lift of the given map $[\pi_{T,*}F_T(d) \twoheadrightarrow V] \colon T \to \mathrm{Gr}(P(d), \pi_*F(d))$, or, in other words, $Y \xrightarrow{\sim} T$ is an isomorphism. This analysis shows that Y is identified with the subfunctor of T defined by

$$\begin{aligned} &\mathrm{Sch}/T \to \mathrm{Sets}, \\ &(T' \to T) \mapsto \begin{cases} \{*\} & \text{if } Q_{T'} \text{ is flat over } T' \text{ with Hilbert polynomial } P \\ \emptyset & \text{otherwise.} \end{cases} \end{aligned}$$

Existence of Flattening Stratifications[2] (A.2.16) implies that Y is representable by a locally closed subscheme of T. $\square$

We summarize the conclusion as:

Proposition 1.4.1. *Let S be a noetherian scheme and $\pi \colon \mathbb{P}^n \to S$ be relative projective space. If $F := \mathcal{O}_{\mathbb{P}^n_S}(-l)^{\oplus r}$ and $P \in \mathbb{Q}[z]$, there exists an integer m_0 such that for all $d \geq m_0$,*

$$\mathrm{Quot}^P(F/\mathbb{P}^n_S) \to \mathrm{Gr}(P(d), \pi_*F(d))$$

is a locally closed immersion. In particular, $\mathrm{Quot}^P(F/\mathbb{P}^n_S)$ is representable by a quasi-projective scheme over S. $\square$

1.4.2 Valuative criteria for Hilb and Quot

To establish that Quot is projective, it will be sufficient to know that it is proper.

Proposition 1.4.2. *For every noetherian scheme S, coherent sheaf F on $\mathbb{P}^n_S$, and polynomial $P \in \mathbb{Q}[z]$, the functor $\mathrm{Quot}^P(F/\mathbb{P}^n_S)$ satisfies the valuative criterion for properness over S, i.e., for every DVR R over S with fraction field K, every flat coherent quotient $F_K \to Q$ on $\mathbb{P}^n_K$ with Hilbert polynomial P extends uniquely to a flat coherent quotient $F_R \to \widetilde{Q}$ on $\mathbb{P}^n_R$ with Hilbert polynomial P.*

Remark 1.4.3. In other words, the proposition implies that for every commutative diagram

$$\begin{tikzcd} \mathrm{Spec}\, K \arrow[r, "Q"] \arrow[d] & \mathrm{Quot}^P_{X/S}(F) \arrow[d] \\ \mathrm{Spec}\, R \arrow[r] \arrow[ur, dashed, "\widetilde{Q}"] & S, \end{tikzcd}$$

has a unique lift. See §3.8 for a discussion of the valuative criterion for functors and stacks.

Proof. If we write $j \colon X_K \hookrightarrow X_R$ as the open immersion, we define $\widetilde{Q}$ as the image of the composition $F_R \to j_*F_K \to j_*Q$, where the first map is given by the adjunction $F_R \to j_*j^*F_R = j_*F_K$. Since $\widetilde{Q}$ is a subsheaf of j_*Q, it is torsion free over R and thus flat (as R is a DVR). Finally, since Q if flat over R and $\mathrm{Spec}\, R$ is connected, its Hilbert polynomial is constant. $\square$

[2]The Existence of Flattening Stratifications is arguably the deepest component of the proof of the Projectivity of the Quot Scheme (1.1.3).

Remark 1.4.4. For $\mathrm{Hilb}^P_{X/S}$, the argument translates into the following: the unique extension of a closed subscheme $Z \subseteq X_K$ is the scheme-theoretic image $\widetilde{Z} = \mathrm{im}(Z \to X_K \hookrightarrow X_R)$. The scheme $\widetilde{Z}$ is flat over R as all associated points live over the generic point of $\operatorname{Spec} R$.

1.4.3 Projectivity

The following completes the proofs of Theorems 1.1.2 and 1.1.3.

Theorem 1.4.5. *Let S be a noetherian scheme and $\pi\colon \mathbb{P}^n \to S$ be relative projective space. If $F := \mathcal{O}_{\mathbb{P}^n_S}(-l)^{\oplus r}$ and $P \in \mathbb{Q}[z]$, there exists an integer m_0 such that for all $d \geq m_0$,*

$$\mathrm{Quot}^P(F/\mathbb{P}^n_S) \to \mathrm{Gr}(P(d), \pi_* F(d))$$

is a closed immersion. In particular, $\mathrm{Quot}^P(F/\mathbb{P}^n_S)$ is representable by a projective scheme over S. $\square$

Proof. Let S be a noetherian scheme, $\pi\colon \mathbb{P}^n_S \to S$ relative projective space, and F be any coherent sheaf on $\mathbb{P}^n_S$. Choosing integers l and r together with a surjection $G := \mathcal{O}_{\mathbb{P}^n_S}(-l)^{\oplus r} \twoheadrightarrow F$, we obtain a morphism of functors

$$\begin{aligned} \mathrm{Quot}^P(F/\mathbb{P}^n_S) &\to \mathrm{Quot}^P(G/\mathbb{P}^n_S) \\ [F_T \to Q] &\mapsto [G_T \twoheadrightarrow F_T \twoheadrightarrow Q], \end{aligned}$$

defined over an S-scheme T. The functor $\mathrm{Quot}^P(G/\mathbb{P}^n_S)$ is representable by a quasi-projective scheme over S by Proposition 1.4.1 and is proper over S by Proposition 1.4.2, hence it is projective.

To prove that $\mathrm{Quot}^P(F/\mathbb{P}^n_S)$ is projective over S, it suffices to prove that the above morphism is representable by closed immersions. This boils down to the claim that for an S-scheme T and quotient $G_T \twoheadrightarrow Q$ with Q flat over T, there is a closed subscheme $Z \subseteq T$ such that a morphism $T' \to T$ factors through Z if and only if $G_{T'} \to Q_{T'}$ factors through $F_{T'}$. Defining $K = \ker(G_T \to F_T)$ and considering the diagram

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r] \arrow[dr] & G_T \arrow[r] \arrow[d, two heads] & F_T \arrow[r] & 0 \\ & & Q, & & \end{tikzcd}$$

then $G_{T'} \to Q_{T'}$ factors through $F_{T'}$ if and only if $K_{T'} \to Q_{T'}$ is zero. By Exercise 0.3.22, the subfunctor of T, parameterizing maps $T' \to T$ such that $K_{T'} \to Q_{T'}$ is zero, is representable by a closed subscheme. $\square$

Taking $G = \mathcal{O}_{\mathbb{P}^n_S}(-l)^{\oplus r} \twoheadrightarrow F$ to be a surjection as in the proof above, for $d \gg 0$, we have a composition of closed immersions

$$\mathrm{Quot}^P(F/\mathbb{P}^n_S) \hookrightarrow \mathrm{Quot}^P(G/\mathbb{P}^n_S) \hookrightarrow \mathrm{Gr}(P(d), \pi_* G(d)) \hookrightarrow \mathbb{P}(\textstyle\bigwedge^{P(d)} \pi_* G(d)).$$

Letting $F_{\mathrm{Quot}^P(F/\mathbb{P}^n_S)} \to \mathcal{Q}_{\mathrm{univ}}$ denote the universal quotient on $\mathbb{P}^n_S \times_S \mathrm{Quot}^P(F/\mathbb{P}^n_S)$, the pullback of $\mathcal{O}(1)$ under the above composition is identified with $\det p_{2,*}(\mathcal{Q}_{\mathrm{univ}}(d))$. We summarize this as follows:

Corollary 1.4.6. *For $d \gg 0$, the line bundle $\det\big(p_{2,*}(\mathcal{Q}_{\mathrm{univ}}(d))\big)$ on $\mathrm{Quot}^P(F/\mathbb{P}^n_S)$ is relatively very ample over S.* $\square$

1.4.4 Generalizations

The first generalization we consider is the *Flag Scheme*.

Exercise 1.4.7 (Flag Schemes)**.** Let S be a noetherian scheme and F be a coherent sheaf on $\mathbb{P}^n_S$ that can be written as a quotient of $\mathcal{O}_{\mathbb{P}^n_S}(-l)^{\oplus r}$ for some l and r. For every sequence $P_1, \dots, P_k \in \mathbb{Q}[z]$ of polynomials, show that the functor

$$\mathrm{Flag}^{P_1,\dots,P_k}(F/\mathbb{P}^n_S/S) \colon \mathrm{Sch}/S \to \mathrm{Sets}$$

$$(T \to S) \mapsto \left\{ \begin{array}{l} \text{filtrations } 0 = F_0 \subseteq \cdots \subseteq F_k = F_T \text{ of quasi-} \\ \text{coherent sheaves on } \mathbb{P}^n_T \text{ such that each quotient } F^i/F^{i-1} \\ \text{is a finitely presented } \mathcal{O}_{\mathbb{P}^n_T}\text{-module and flat over } T \\ \text{with Hilbert polynomial } P_i \end{array} \right\}$$

is representable by a scheme projective over S.

Hint: Construct the Flag scheme as an iteration of relative Quot schemes, i.e., consider $\mathrm{Flag} := \mathrm{Flag}^{P_2,\dots,P_k}(F/\mathbb{P}^n_S/S)$ *and the universal filtration* $0 = \mathcal{F}_{\mathrm{univ},0} \subseteq \mathcal{F}_{univ,2} \subseteq \cdots \subseteq p_1^*\mathcal{F}$ *on* $\mathbb{P}^n_S \times_S \mathrm{Flag}$*, and show that* $\mathrm{Flag}^{P_1,\dots,P_k}(F/\mathbb{P}^n_S/S) \cong \mathrm{Quot}^{P_1}(\mathcal{F}_{\mathrm{univ},2}/\mathrm{Flag}/S)$*.*

The existence and projectivity of the Quot scheme $\mathrm{Quot}^P(F/\mathbb{P}^n_S)$ was proven in Theorem 1.1.3 for coherent sheaves F on $\mathbb{P}^n_S$ that that can be written as a quotient of $\mathcal{O}_{\mathbb{P}^n_S}(-l)^{\oplus r}$. We now consider more general setups. Recall that there are three distinct notions of projectivity for a morphism $X \to S$, listed below in increasing generality:

- $X \to S$ factors as a closed immersion $X \hookrightarrow \mathbb{P}^n_S$ and $\mathbb{P}^n_S \to S$; this is sometimes referred to as *H-projective* as this is the definition in [Har77, II.4],
- $X \to S$ factors as a closed immersion $X \hookrightarrow \mathbb{P}(E)$ and $\mathbb{P}(E) \to S$, where E is a vector bundle over S; this is called *strongly projective*, and
- $X \to S$ factors as a closed immersion $X \hookrightarrow \mathbb{P}(E)$ and $\mathbb{P}(E) \to S$, where E is a finite type, quasi-coherent sheaf on S (i.e., a coherent sheaf when S is noetherian); this is the definition in [EGA, §II.5], [SP, Tag 01W8].

When S has the resolution property, i.e., every coherent sheaf is the quotient of a vector bundle, then a projective morphism is strongly projective. This holds for every smooth or quasi-projective scheme. The definitions of the Hilbert functor $\mathrm{Hilb}^P(X/S)$ and Quot functor $\mathrm{Quot}^P(F/X/S)$ extend to any projective morphism $X \to S$ and finite type, quasi-coherent sheaf F on X.

Exercise 1.4.8. Let $X \to S$ be a projective morphism of schemes and F be a coherent sheaf on X.

(a) (easy) Show that $\mathrm{Quot}^P(F/X/S)$ is H-projective if $X \to S$ is H-projective and F is a quotient of $\mathcal{O}_X(-l)^{\oplus r}$.

(b) (moderate) Show that $\mathrm{Quot}^P(F/X/S)$ is strongly projective if $f \colon X \to S$ is strongly projective and F is a quotient of $f^*V \otimes \mathcal{O}_X(-l)$ where V is a vector bundle on S.

(c) (moderate) Show that $\mathrm{Quot}^P(F/X/S)$ is a proper scheme over S if $X \to S$ is projective.

(d) (hard) Show that $\mathrm{Quot}^P(F/X/S)$ is projective if $X \to S$ is projective and F is flat over S.

(e) (open) Is $\mathrm{Quot}^P(F/X/S)$ always projective if $X \to S$ is projective?

See also [AK80, Thm. 2.6] and [Nit05, Thm. 5.2].

Exercise 1.4.9 (hard)**.** Suppose that $X \to S$ is *strongly quasi-projective* morphism of noetherian scheme, i.e., there is a locally closed immersion $X \hookrightarrow \mathbb{P}(V)$ where V is a vector bundle on S. If F is a coherent sheaf on X and $P \in \mathbb{Q}[z]$ is a polynomial, we can modify the Quot functor as follows:

$$\mathrm{Quot}^P(F/X/S)\colon \mathrm{Sch}/S \to \mathrm{Sets}$$
$$(T \to S) \mapsto \left\{ \begin{array}{l} \text{finitely presented, quasi-coherent quotients} \\ F_T \to Q \text{ on } X_T \text{ which are flat and have} \\ \textit{proper support} \text{ over } T \text{ such that } Q_t \text{ on } X_t \\ \text{has Hilbert polynomial } P \text{ for all } t \in T \end{array} \right\}.$$

The Hilbert functor $\mathrm{Hilb}^P(X/S)$ is defined analogously by only parameterizing closed subschemes with *proper support* over the base. Show that $\mathrm{Hilb}^P_{X/S}$ and $\mathrm{Quot}^P_{X/S}(F)$ are representable by strongly quasi-projective schemes over S. See also [FGAIV, §4], [AK80, Thm. 2.6], or [Nit05, §6].

Remark 1.4.10. If $X \to S$ is merely a separated morphism of noetherian schemes, then one can define functors $\mathrm{Hilb}(X/S)$ and $\mathrm{Quot}(F/X/S)$ as above after dropping the condition on the Hilbert polynomial P. Artin's Axioms for Algebraicity (C.7.4) can be applied to show that these functors are representable by algebraic spaces separated and locally of finite type over S; see [Art69b, Thm. 6.1] and [SP, Tag 09TQ], or Theorem C.7.7 for a special case. Examples of Hironaka produce smooth proper (but not projective) threefolds X over $\mathbb{C}$ such that $\mathrm{Hilb}(X/S)$ is not a scheme.

1.4.5 Chow varieties and other variants

The Chow variety is a variant of the Hilbert scheme that is easier to construct but doesn't afford as nice functorial properties. The Chow variety $\mathrm{Chow}_{r,d}(\mathbb{P}^n_{\Bbbk})$ parameterizes effective cycles on $\mathbb{P}^n_{\Bbbk}$ of dimension r and degree d, and can be constructed by using the Chow form Chow_α of an effective cycle α. For an effective cycle $\alpha = \sum_i a_i[X_i]$, one can define $\mathrm{Chow}_\alpha = \prod_i \mathrm{Chow}_{X_i}^{a_i}$, so it suffices to define the Chow form for an integral closed subscheme $X \subseteq \mathbb{P}^n_{\Bbbk}$ of dimension r and degree d. Letting $\mathbb{P}^{n,\vee}$ denote the dual projective space parameterizing hyperplanes $H \in \Gamma(\mathbb{P}^n, \mathcal{O}(1))$, define the locus

$$\{(H_0, \ldots, H_r) \mid X \cap H_0 \cap \cdots\cdots \cap H_r \neq \emptyset\} \subseteq (\mathbb{P}^{n,\vee})^{r+1}. \tag{1.4.11}$$

As this is a divisor, there is a polynomial in the variables $u_{i,j}$ with $0 \le i \le r$ and $0 \le j \le n$

$$\mathrm{Chow}_X(u_{ij}) \in \mathrm{Sym}^d(\Bbbk^{n+1})^{\otimes(r+1)},$$

which is homogenous of degree d in $u_{i,0}, \ldots, u_{i,n}$ for each i, such that $\mathrm{Chow}_X(H_0, \ldots, H_r) = 0$ if and only if $X \cap H_0 \cap \cdots\cdots \cap H_r \neq \emptyset$.

The *Chow variety* $\mathrm{Chow}_{r,d}(\mathbb{P}^n)$ is the closure of the set of Chow forms of all effective cycles on $\mathbb{P}^n$ of dimension r and degree d. The main existence theorem in characteristic 0 asserts that

- $\mathrm{Chow}_{r,d}(\mathbb{P}^n)$ is projective and seminormal (i.e. every finite bijective morphism $Y \to \mathrm{Chow}_{r,d}(\mathbb{P}^n)$ inducing isomorphisms on residue fields is an isomorphism),

– $\mathrm{Chow}_{r,d}(\mathbb{P}^n)$ represents a functor on the category of *seminormal* $\Bbbk$-schemes: for a seminormal $\Bbbk$ scheme S, $\mathrm{Mor}(S, \mathrm{Chow}_{r,d}(\mathbb{P}^n))$ are identified with well-defined families of effective algebraic cycles on $\mathbb{P}^n_S$ of dimension r and degree d, and
– for a polynomial $P(t) = d \cdot \frac{t^r}{r!} + (\text{lower terms})$, there is a *Hilbert–Chow morphism*

$$\mathrm{Hilb}^P(\mathbb{P}^n)^{\mathrm{sn}} \to \mathrm{Chow}_{r,d}(\mathbb{P}^n),$$

from the seminormalization of $\mathrm{Hilb}^P(\mathbb{P}^n)$, taking a closed subscheme $Z \subset \mathbb{P}^n$ to the cycle $a_i[Z_i]$ were the Z_i are the reduced scheme structures of the ith irreducible components and a_i is its multiplicity of Z_i at its generic point.

See [CW37], [HP47, §X.6-8], and [Sam55], or the modern treatments in [GIT, §5.4] and [Kol96, I.3.21, I.6.1]. In our sketch of the GIT construction of $\overline{M}_g$, we will utilize the Chow variety for curves in $\mathbb{P}^n$ (i.e., the $r = 1$ case), in which case the statements are slightly easier to prove.

Example 1.4.12 (Additional variants)**.** There are further variants and generalizations. For instance, Vistoli constructs a Hilbert stack parameterizing finite and unramified morphisms to a separated scheme X [Vis91]. Alexeev and Knutson's moduli of branch varieties parameterizes finite morphisms from a geometrically reduced proper scheme to a separated scheme X [AK10].

Exercise 1.4.13 (Schemes of morphisms)**.** For strongly projective morphisms $X \to S$ and $Y \to S$ of noetherian schemes, consider the functor

$$\begin{aligned} \underline{\mathrm{Mor}}_S(X, Y) \colon \mathrm{Sch}/S &\to \mathrm{Sets} \\ (T \to S) &\mapsto \mathrm{Mor}_T(X_T, Y_T) \end{aligned}$$

assigning an S-scheme T to the set of T-morphisms $X_T \to Y_T$. By using a suitable closed subscheme of $\mathrm{Hilb}^P_{X \times_S Y/X}$ parameterizing graphs $X \subseteq X \times_S Y$ of morphisms $X \to Y$, show that $\underline{\mathrm{Mor}}_S(X, Y)$ is representable by a projective scheme over S.

1.5 An invitation to the geometry of Hilbert schemes

Murphy's Law for Hilbert Schemes: There is no geometric possibility so horrible that it cannot be found generically on some component of the Hilbert scheme.

Joe Harris and Ian Morrison [HM98, p.18]

Hilbert schemes are some of the most well-studied moduli spaces, with perhaps only $\overline{M}_g$ and $\mathcal{B}un^{\mathrm{ps}}_{r,d}(C)$ having received greater attention over the last 50 years. As such, we will not attempt a systematic exposition, but merely offer a few interesting examples and features.

1.5.1 First examples

In this section, we work over an algebraically closed field $\Bbbk$. The Hilbert polynomial $P(z) = \sum_{i=0}^d a_i z^i$ of a projective scheme $X \subset \mathbb{P}^n$ encodes invariants of X. For instance, $\dim X$ is the degree d of P and $\deg X$ is the normalized leading coefficient $d! a_d$. Riemann–Roch implies that $P(z) = \deg(C)z + (1 - g)$ for a smooth curve

$C \subseteq \mathbb{P}^n$ and $P(z) = \frac{1}{2}(zH \cdot (zH - K)) + (1 - p_a)$ for a smooth surface $S \subset \mathbb{P}^n$, where H is a hyperplane divisor, K is the canonical divisor, and $p_a = 1 - \chi(\mathcal{O}_S)$ is the arithmetic genus. In arbitrary dimension, Hirzebruch–Riemann–Roch states that $P(z) = \int_X \operatorname{ch}(\mathcal{O}_X(z))\operatorname{td}(X)$, where $\operatorname{ch}(\mathcal{O}_X(z))$ is the Chern character and $\operatorname{td}(X)$ the Todd class.

Exercise 1.5.1 (Hypersurfaces and linear subspaces)**.**

(a) A hypersurface $H \subseteq \mathbb{P}^n$ of degree d has Hilbert polynomial

$$P(z) = \chi(\mathcal{O}_{\mathbb{P}^n}(z)) - \chi(\mathcal{O}_{\mathbb{P}^n}(z-d)) = \binom{n+z}{n} - \binom{n+z-d}{n}.$$

Show that $\operatorname{Hilb}^P(\mathbb{P}^n) \cong \mathbb{P}(\Gamma(\mathbb{P}^n, \mathcal{O}(d)))$.

(b) A linear subspace $L \subseteq \mathbb{P}^n$ of dimension k has Hilbert polynomial $P(z) = \binom{z+k}{k}$ and $\operatorname{Hilb}^P(\mathbb{P}^n) = \operatorname{Gr}(k+1, n+1)$.

Example 1.5.2 (Hilbert scheme of points on a curve)**.** If C is a smooth projective curve, then the Hilbert scheme $\operatorname{Hilb}^n(C)$ of n points (viewing n as the constant polynomial) is a smooth irreducible projective variety isomorphic to the symmetric product

$$\operatorname{Sym}^n C := \underbrace{C \times \cdots \times C}_{n} / S_n,$$

where S_n acts by permuting the factors. The quotient exists as a projective variety since $C \times \cdots \times C$ is projective; see Exercise 4.2.14.

Example 1.5.3 (Hilbert scheme of points on a surface)**.** If S is a smooth irreducible projective surface, then the Hilbert scheme of n points $\operatorname{Hilb}^n(S)$ is a smooth irreducible projective variety [Fog68]. See also [Nak99a] and [Mac07, §4]. There is a birational morphism

$$\operatorname{Hilb}^n(S) \to \operatorname{Sym}^n(S) := \underbrace{S \times \cdots \times S}_{n} / S_n,$$

of projective varieties. The symmetric product $\operatorname{Sym}^n(S)$ is not smooth for $n > 1$ and this provides a resolution of singularities. For an unordered collection of (possibly non-distinct) points $(p_1, \dots, p_n) \in \operatorname{Sym}^n(S)$, the fiber consists of all possible scheme structures on $\{p_1, \dots, p_n\}$ of length n.

The case of $n = 2$ is the first interesting case, as $\operatorname{Hilb}^1(S) = S$. For any point $p \in S$, there are many non-reduced scheme structures of length 2 supported at p. They are parameterized by their "tangent direction": given $[a : b] \in \mathbb{P}^1$, $\Bbbk[x,y]/(x^2, xy, y^2, ay - bx)$ defines a length 2 non-reduced subscheme. In this case, $\operatorname{Hilb}^2(S) \to \operatorname{Sym}^2(S)$ is the blowup of the diagonal $S \hookrightarrow \operatorname{Sym}^2(S)$ given by $p \mapsto (p,p)$. For $n > 2$, the map $\operatorname{Hilb}^n(S) \to \operatorname{Sym}^n(S)$ is a blowup along some ideal sheaf [Hai98] but the description of the ideal sheaf is more complicated. When X is of arbitrary dimension, $\operatorname{Hilb}^n(X)$ is smooth at (reduced) closed subschemes $Z \subseteq X$ consisting of n distinct smooth points of X. If X is reduced, there is an open subscheme of $\operatorname{Hilb}^n(X)$ dimension $n \dim(X)$ parameterizing n distinct smooth points. Another result of Fogarty is that $\operatorname{Hilb}^n(X)$ is connected as long as X is connected [Fog68]. Moreover, for every projective scheme X, there is an irreducible component $\operatorname{Hilb}^n(X)$, called the "good component," that can be identified with the blowup of $\operatorname{Sym}^n(X)$ along an ideal sheaf [ES14].

Example 1.5.4 (Twisted cubics)**.** The Hilbert scheme $\mathrm{Hilb}^{3z+1}(\mathbb{P}^3)$ consists of the union of two smooth rational irreducible components H and H' of dimensions 12 and 15 intersecting transversely along a smooth rational subvariety of dimension 11 [PS85]. The locus H is the closure of the locus H_0 consisting of *twisted cubics*, i.e., rational smooth curves in $\mathbb{P}^3$ of degree 3. Each twisted cubic can be represented by a map $\mathbb{P}^1 \to \mathbb{P}^3$ given by the line bundle $\mathcal{O}_{\mathbb{P}^1}(3)$ and a choice of basis of $\Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(3))$, and this representation is unique up to automorphisms of $\mathbb{P}^1$. All such curves are projectively equivalent, i.e., differ by an automorphism of $\mathbb{P}^3$, so we see that H_0 is identified with the homogeneous space $\mathrm{Aut}(\mathbb{P}^3)/\mathrm{Aut}(\mathbb{P}^1) = \mathrm{PGL}_4/\mathrm{PGL}_2$, which is smooth and irreducible of dimension 12. The locus H_0 is not proper as it includes families such as $\mathbb{P}^1 \hookrightarrow \mathbb{P}^3$ given by $[x,y] \mapsto [x^3, x^2y, xy^2, ty^3]$ parameterized by $t \in \mathbb{A}^1$ whose limit is a singular curve C_0 supported on a nodal cubic in $V(w) = \mathbb{P}^2$ (where w is the 4th coordinate) but with an embedded point at the node; see [Har77, Ex. 9.8.4].

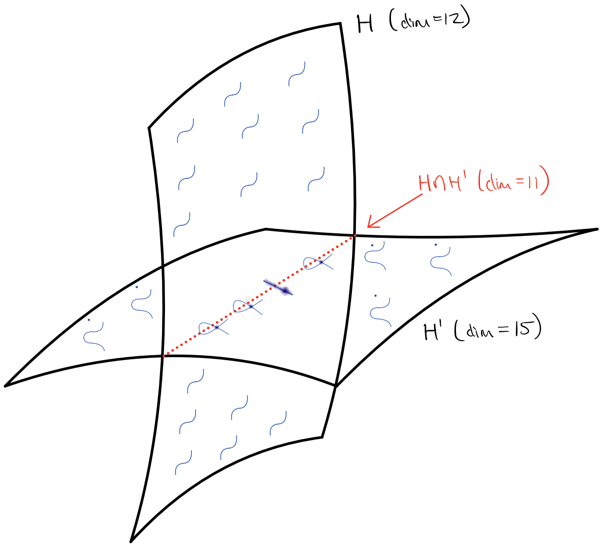

Figure 1.5.5: $\mathrm{Hilb}^{3z+1}(\mathbb{P}^3)$

The locus H' is the closure of the locus H'_0 consisting of subschemes $C \amalg \{p\}$ where C is a smooth cubic curve contained in a hyperplane H and $p \in \mathbb{P}^3 \smallsetminus C$. To count the dimension, observe that the choice of hyperplane $H \in \mathbb{P}(\mathrm{H}^0(\mathbb{P}^3, \mathcal{O}(1)))$ is given by 3 parameters, the choice of plane cubic $C \in \mathbb{P}(\mathrm{H}^0(H, \mathcal{O}_H(3)))$ is given by 9 parameters, and the point $p \in \mathbb{P}^3 \smallsetminus C$ is given by 3 parameters. It follows that the locus H'_0 is smooth and irreducible of dimension 15. The locus H'_0 is not proper and its closure contains the degenerations where the point p lies on the curve.

The intersection $H \cap H'$ consists of plane singular cubic curves with an embedded point at the singular point. This locus contains curves such as C_0 above but it also contains even more degenerate curves such as a triple line with an embedded point. Every curve $C \in H \cap H'$ is in fact projectively equivalent to the curve defined by $V(xz, yz, z^2, q(x,y,w))$ where $q(x,y,w)$ is a homogeneous cubic polynomial with a singular point at $(0,0,1)$. This depends on 11 parameters.

1.5.2 Geometric properties

Exercise 1.5.6 (Local properties)**.** Let X be a projective scheme over a field $\Bbbk$ and F be a coherent sheaf on X.

(a) Let $p \in \operatorname{Quot}^P(F/X)$ be the point corresponding to a quotient $Q = F/K$. Show that $T_p \operatorname{Quot}^P(F/X) \cong \operatorname{Hom}_{\mathcal{O}_X}(K, Q)$. This generalizes Exercise 1.2.10 computing the tangent space of the Grassmannian.

(b) Conclude that if $p \in \operatorname{Hilb}^P(X)$ is a point corresponding to a closed subscheme $Z \subseteq X$ defined by a sheaf of ideals I, then $T_p \operatorname{Hilb}^P(X) \cong \mathrm{H}^0(Z, N_{Z/X})$ where $N_{Z/X}$ is the normal sheaf $\operatorname{Hom}_{\mathcal{O}_Z}(I_Z/I_Z^2, \mathcal{O}_Z)$. (This recovers Proposition C.1.3.)

Non-emptiness. The Hilbert scheme $\operatorname{Hilb}^P(\mathbb{P}^n)$ is non-empty if and only if the Hilbert polynomial P can be written as as

$$P(z) = \binom{z+\lambda_1-1}{\lambda_1-1} + \binom{z+\lambda_2-2}{\lambda_2-1} + \cdots + \binom{z+\lambda_r-r}{\lambda_r-1}, \tag{1.5.7}$$

integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \geq 1$. This is a result of Hartshorne [Har66b, Cor. 5.7].

Connectedness. Hartshorne's Connectedness Theorem asserts that the Hilbert scheme $\operatorname{Hilb}^P(\mathbb{P}^n)$ is connected for every Hilbert polynomial P [Har66b]. The proof strategy is to exhibit a degeneration from any closed subscheme $Z \subseteq \mathbb{P}^n$ to a subscheme $V(I) \subseteq \mathbb{P}^n$ defined by a monomial ideal. This reduces the theorem to the combinatorial question of connecting any two monomial ideals by a family over $\mathbb{A}^1$. This turns out to be a purely deformation and combinatorial question, or as Hartshorne writes: "It also appears that the Hilbert scheme is never actually needed in the proof."

Murphy's Law. The first pathology was exhibited by Mumford: there is an irreducible component of $\operatorname{Hilb}^{14z-23}(\mathbb{P}^3)$ which is generically non-reduced [Mum62]. Ellia—Hirschowitz—Mezzetti show that the number of irreducible components in $\operatorname{Hilb}^{az+b}(\mathbb{P}^3)$ is not bounded by a polynomial in a, b [EHM92]. Murphy's Law was made precise by Vakil [Vak06]: for every scheme X finite type over $\mathbb{Z}$ and point $x \in X$, there exists a point $[Z \subseteq \mathbb{P}^n] \in \operatorname{Hilb}^P(\mathbb{P}^n)$ of some Hilbert scheme such that (X, x) and $(\operatorname{Hilb}^P(\mathbb{P}^n), [Z \subseteq \mathbb{P}^n])$ are smooth locally isomorphic, i.e., their complete local rings become isomorphic after appending power series rings. In fact, one can take $[Z \subseteq \mathbb{P}^n]$ to be a smooth curve! Many other moduli spaces satisfy Murphy's Law: Kontsevich's moduli space of maps, moduli of canonically polarized smooth surfaces, moduli of curves with linear systems, and the moduli space of stable sheaves.

Smoothness. Despite Murphy's Law, Hilbert schemes are surprisingly often smooth. We have seen before that the Hilbert scheme of points on a smooth surface is smooth. A recent theorem of Skjelnes–Smith [SS20] gives necessary and sufficient conditions for the Hilbert scheme $\operatorname{Hilb}^P(\mathbb{P}^n)$ to be smooth in terms of the partition $\lambda = (\lambda_1, \ldots, \lambda_r)$ defining P as in (1.5.7).

Chapter 2

Sites, sheaves, and stacks

If there is one thing in mathematics that fascinates me more than anything else (and doubtless always has), it is neither 'number' nor 'size,' but always form.

Alexander Grothendieck

This chapter introduces the core categorical constructions—sites, sheaves, and stacks—necessary to define algebraic spaces and stacks. Grothendieck introduced stacks in [FGAI, §A.1] and [SGA1, §6] as a way to package objects, e.g., quasi-coherent sheaves, satisfying descent.

2.1 Descent theory

It is hard to overstate the importance of descent in moduli theory. The central idea is strikingly simple, and a special case is already familiar to you: quasi-coherent sheaves, morphisms of schemes, and schemes themselves can be constructed locally on a Zariski cover, and moreover most of their properties can be checked Zariski locally. Indeed, a central technique in the development of scheme theory is to reduce to the case of affine schemes, and then apply results from commutative algebra. Descent theory implies that each of these objects (quasi-coherent sheaves, morphisms of schemes, and schemes) can be constructed not only Zariski locally but étale locally (and even fppf or fpqc locally), and moreover that their properties can be verified locally. This allows us to prove statements about algebraic stacks by reducing to the case of schemes (or even affine schemes).

Descent theory was originally developed by Grothendieck in [FGAI], [SGA1, §6], and [EGA, §IV] as a grand generalization of the theory of Galois descent initiated by Weil [Wei56]. Applications of descent theory extend far beyond moduli theory. For instance, since field extensions are faithfully flat, one can reduce properties of a scheme over a field $\Bbbk$ to the case of an algebraically closed field. Similarly, since the map $A \to \widehat{A}$ of a noetherian local ring to its completion is faithfully flat, properties of A can be reduced to properties of its completion. As there are already wonderful expositions on descent theory such as [BLR90, §6], [Vis05], and [SP, Tag 0238], our treatment will sometimes be short on details.

2.1.1 Descending quasi-coherent sheaves

The following key algebraic fact is the basis for descent for quasi-coherent sheaves.

Proposition 2.1.1. *If $\phi\colon A \to B$ is a faithfully flat ring map, then*

$$0 \longrightarrow A \xrightarrow{\phi} B \underset{b\mapsto 1\otimes b}{\overset{b\mapsto b\otimes 1}{\rightrightarrows}} B \otimes_A B$$

is exact (i.e., an equalizer sequence). More generally, if M is an A-module,

$$0 \longrightarrow M \xrightarrow{m\mapsto m\otimes 1} M \otimes_A B \underset{m\otimes b\mapsto m\otimes 1\otimes b}{\overset{m\otimes b\mapsto m\otimes b\otimes 1}{\rightrightarrows}} M \otimes_A B \otimes_A B \tag{2.1.2}$$

is exact.

Proof. Note that $A \to B$ and $M \to M \otimes_A B$ are necessarily injective by Faithfully Flat Equivalences (A.2.19). Since $A \to B$ is faithfully flat, the sequence (2.1.2) is exact if and only if

$$0 \longrightarrow M \otimes_A B \xrightarrow{m\otimes b'\mapsto m\otimes 1\otimes b'} M \otimes_A B \otimes_A B \underset{m\otimes b\otimes b'\mapsto m\otimes 1\otimes b\otimes b'}{\overset{m\otimes b\otimes b'\mapsto m\otimes b\otimes 1\otimes b'}{\rightrightarrows}} M \otimes_A B \otimes_A B \otimes_A B$$

is exact. The above sequence can be rewritten as

$$0 \longrightarrow M \otimes_A B \xrightarrow{x\mapsto x\otimes 1} (M \otimes_A B) \otimes_B (B \otimes_A B) \underset{x\otimes y\mapsto x\otimes 1\otimes y}{\overset{x\otimes y\mapsto x\otimes y\otimes 1}{\rightrightarrows}} (M \otimes_A B) \otimes_B (B \otimes_A B) \otimes_B (B \otimes_A B)$$

which is precisely sequence (2.1.2) applied to the ring map $B \to B \otimes_A B$, defined by $b \mapsto 1 \otimes b$, and the B-module $M \otimes_A B$. Since $B \to B \otimes_A B$ has a left inverse given by $b \otimes b' \mapsto bb'$, we are reduced to proving the proposition when $\phi\colon A \to B$ has a left inverse $s\colon B \to A$ with $s \circ \phi = \mathrm{id}_A$. Let $x = \sum_i m_i \otimes b_i \in M \otimes_A B$ such that

$$\sum_i m_i \otimes b_i \otimes 1 = \sum_i m_i \otimes 1 \otimes b_i \in M \otimes_A B \otimes_A B.$$

Applying $\mathrm{id}_M \otimes \mathrm{id}_B \otimes s\colon M \otimes_A B \otimes_A B \to M \otimes_A B \otimes_A A \cong M \otimes_A B$ to this identity shows that $x = \sum_i m_i \otimes \phi(s(b_i)) = \sum_i \phi(s(b_i))m_i \otimes 1$ is in the image of $M \to M \otimes_A B$. □

Exercise 2.1.3. Denoting $(B/A)^{\otimes n}$ as the n-fold tensor product $B \otimes_A \cdots \otimes_A B$, show that the exact sequence (2.1.2) extends to a long exact sequence

$$0 \to M \xrightarrow{d} M \otimes_A (B/A)^{\otimes 1} \xrightarrow{d} M \otimes_A (B/A)^{\otimes 2} \xrightarrow{d} \cdots$$

with differentials

$$\begin{aligned} d\colon M \otimes_A (B/A)^{\otimes n} &\to M \otimes_A (B/A)^{\otimes (n+1)} \\ m \otimes b_1 \otimes \cdots \otimes b_n &\mapsto \sum_{i=0}^{n+1} (-1)^i m \otimes b_1 \otimes \cdots \otimes b_{i-1} \otimes 1 \otimes b_{i+1} \otimes \cdots \otimes b_n. \end{aligned}$$

Recall from Definition A.2.20 that a morphism of schemes $f\colon S' \to S$ is *fpqc* if f is faithfully flat and every quasi-compact open subset of S is the image of quasi-compact open subset of S'. Fpqc morphisms include faithfully flat and quasi-compact morphisms, but the definition is broader as it includes for instance fppf morphisms (i.e. faithfully flat morphisms locally of finite presentation).

It is always instructive to keep in the mind the special case of an étale morphism $f\colon S' = \amalg_i U_i \to S$ induced from a Zariski cover $\{U_i\}$ of S. In this case, the fiber product $S' \times_S S'$ is $\amalg_{i,j} U_i \cap U_j$ and the higher fiber products afford similar descriptions.

Proposition 2.1.4 (Fpqc Descent for Quasi-Coherent Sheaves)**.** *Let $f\colon S' \to S$ be an fpqc morphism of schemes.*

(1) Let F and G be quasi-coherent $\mathcal{O}_S$-modules. Let $p_1, p_2\colon S' \times_S S' \rightrightarrows S'$ be the two projections and $q\colon S' \times_S S' \to S$ be the composition $f \circ p_i$. Then the sequence

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{O}_S}(F,G) \xrightarrow{f^*} \mathrm{Hom}_{\mathcal{O}_{S'}}(f^*F, f^*G) \underset{p_2^*}{\overset{p_1^*}{\rightrightarrows}} \mathrm{Hom}_{\mathcal{O}_{S'\times_S S'}}(q^*F, q^*G)$$

is exact.

(2) Let H be a quasi-coherent $\mathcal{O}_{S'}$-module and $\alpha\colon p_1^*H \to p_2^*H$ be an isomorphism of $\mathcal{O}_{S'\times_S S'}$-modules satisfying the cocycle condition $p_{23}^*\alpha \circ p_{12}^*\alpha = p_{13}^*\alpha$ on $S' \times_S S' \times_S S'$. Then there exists a quasi-coherent $\mathcal{O}_S$-module G and an isomorphism $\phi\colon H \to f^*G$ such that $p_1^*\phi = p_2^*\phi \circ \alpha$ on $S' \times_S S'$. The data (G, ϕ) is unique up to unique isomorphism.

The following diagram may help to internalize (2):

$$\begin{tikzcd}
p_{23}^*\alpha \circ p_{12}^*\alpha = p_{13}^*\alpha \arrow[d, dash] & p_1^*H \xrightarrow{\alpha} p_2^*H \arrow[d, dash] & H \arrow[d, dash] & G \arrow[d, dash, dashed] \\
S' \times_S S' \times_S S' \arrow[r, shift left=2, "p_{12}"] \arrow[r, "p_{13}" description] \arrow[r, shift right=2, "p_{23}"'] & S' \times_S S' \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & S' \arrow[r, "f"] & S.
\end{tikzcd}$$

The cocycle condition $p_{23}^*\alpha \circ p_{12}^*\alpha = p_{13}^*\alpha$ and the equality $p_1^*\phi = p_2^*\phi \circ \alpha$ should be understood as the commutativity of

$$\begin{tikzcd}
 & p_{12}^*p_1^*H \arrow[r, "p_{12}^*\alpha"] \arrow[dl, equal] & p_{12}^*p_2^*H \arrow[dr, equal] & \\
p_{13}^*p_1^*H \arrow[dr, "p_{13}^*\alpha"'] & & & p_{23}^*p_1^*H \arrow[dl, "p_{23}^*\alpha"] \\
 & p_{13}^*p_2^*H \arrow[r, equal] & p_{23}^*p_2^*H &
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
p_1^*H \arrow[r, "p_1^*\phi"] \arrow[d, "\alpha"] & p_1^*f^*G \arrow[d, equal] \\
p_2^*H \arrow[r, "p_2^*\phi"] & p_2^*f^*G.
\end{tikzcd}$$

Proposition 2.1.4 can be reformulated as an equivalence of categories

$$\mathrm{QCoh}(S) \xrightarrow{\sim} \mathrm{QCoh}(S' \to S), \qquad G \mapsto (f^*G, \mathrm{can}), \tag{2.1.5}$$

where $\mathrm{QCoh}(S' \to S)$ is the *category of descent data for* $S' \to S$, whose objects are pairs (H, α) consisting of a quasi-coherent $\mathcal{O}_{S'}$-module H and an isomorphism $\alpha\colon p_1^*H \to p_2^*H$ satisfying the cocycle condition, and a morphism $(H', \alpha') \to (H, \alpha)$ is a morphism $\beta\colon H' \to H$ such that $\alpha \circ p_1^*\beta = p_2^*\beta \circ \alpha'$. Note that if $H = f^*G$ for $G \in \mathrm{QCoh}(S)$, then there a canonical isomorphism $\mathrm{can}\colon p_1^*H \xrightarrow{\sim} p_2^*H$ since the compositions $f \circ p_1$ and $f \circ p_2$ are equal. In yet other language, this result asserts that every descent data (H, α) for $S' \to S$ is *effective*, i.e., in the essential image of (2.1.5). Finally, we will see shortly that this is equivalent to the statement that the prestack $\underline{\mathrm{QCoh}}$ parameterizing quasi-coherent sheaves is a stack in the fpqc topology (Example 2.5.9).

Proof. If $S' = \operatorname{Spec} A'$ and $S = \operatorname{Spec} A$ are affine, write $F = \widetilde{M}$ and $G = \widetilde{N}$. Proposition 2.1.1 implies that $0 \to N \to N \otimes_A A' \rightrightarrows N \otimes_A A' \otimes_A A'$ is exact. Part (1) follows from applying $\operatorname{Hom}_A(M, -)$ and using tensor-hom adjunction, e.g., $\operatorname{Hom}_A(M, N \otimes_A A') = \operatorname{Hom}_{A'}(M \otimes_A A', N \otimes_A A')$. For (2), writing $H = \widetilde{M'}$, we define the A-module M as the equalizer

$$0 \longrightarrow M \longrightarrow M' \underset{m \mapsto \alpha(m \otimes 1)}{\overset{m \mapsto m \otimes 1}{\rightrightarrows}} M' \otimes_A A'$$

Tensoring this sequence of A-modules with A' expresses $M \otimes_A A'$ as the equalizer of $M' \otimes_A A' \rightrightarrows M' \otimes_A A' \otimes_A A'$. On the other hand, the key algebra result of Proposition 2.1.1 shows that M' is identified with the same equalizer. This gives an isomorphism $\phi \colon M' \to M \otimes_A A'$ of A'-modules and one checks that $p_1^* \phi = p_2^* \phi \circ \alpha$.

The general case is Zariski local on S so we may assume that S is affine. Since f is fpqc, S is the image of a quasi-compact open subset $U' \subset S'$. By choosing a finite affine cover $\{U'_i\}$ of U', we can reduce to the case of a faithfully flat map $\amalg_i U'_i \to S$ of affine cases (details left to the reader), where we have already verified the result. It is, in fact, a general result (see Exercises 2.3.7 and 2.5.5) that to verify properties (1)–(2) for an fpqc morphism $S' \to S$, it suffices to verify them for maps $\amalg_i U_i \to S$ induced by a Zariski cover $\{U_i\}$ of S (which we already know) and for a faithfully flat map $S' \to S$ of affine schemes (which we just verified). See also [FGAI, Thm. 1, p. 315], [BLR90, Thm. 6.4], [Vis05, Thm. 4.23], and [SP, Tag 023T]. ☐

Remark 2.1.6. It turns out that descent for modules holds for a class of ring maps $A \to B$ larger than just faithfully flat maps. It holds for universally injective maps (see Definition A.2.22), and remarkably the converse is true! More precisely, $A \to B$ is universally injective if and only if the functor to the category of descent data

$$\begin{aligned} \mathrm{Mod}_A &\to \left\{ (N, \alpha) \;\middle|\; \begin{array}{l} N \in \mathrm{Mod}_B,\ \alpha \colon N \otimes_{B, p_1} (B \otimes_A B) \xrightarrow{\sim} N \otimes_{B, p_2} (B \otimes_A B) \\ \text{satisfying the cocycle condition } p_{23}^* \alpha \circ p_{12}^* \alpha = p_{13}^* \alpha \end{array} \right\}. \\ M &\mapsto (M \otimes_A B, \mathrm{can}) \end{aligned}$$

is an equivalence of categories. See [Mes00] or [SP, Tag 08XA].

2.1.2 Descending morphisms

Proposition 2.1.7 (Fpqc Descent for Morphisms)**.** *Let Y be a scheme and $f \colon S' \to S$ be an fpqc morphism of schemes. If $g \colon S' \to Y$ is a morphism such that $g \circ p_1 = g \circ p_2$, then there exists a unique morphism $h \colon S \to Y$ filling in the commutative diagram*

$$\begin{tikzcd} S' \times_S S' \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & S' \arrow[r, "f"] \arrow[dr, "g"'] & S \arrow[d, dashed, "h"] \\ & & Y. \end{tikzcd}$$

In other words, an fpqc morphism $f\colon S' \to S$ is an *effective epimorphism* in the category of schemes, i.e., for every scheme Y, the sequence

$$\mathrm{Mor}(S, Y) \to \mathrm{Mor}(S', Y) \rightrightarrows \mathrm{Mor}(S' \times_S S', Y), \tag{2.1.8}$$

is exact; being only an epimorphism translates to the injectivity of the first map. As we will see shortly, this also translates into the functor $\mathrm{Mor}(-, Y)$ being a sheaf in the fpqc topology (see Proposition 2.3.8).

Proof. The affine case is straightforward. Writing $S' = \operatorname{Spec} A'$, $S = \operatorname{Spec} A$, and $Y = \operatorname{Spec} R$, then Proposition 2.1.1 yields that $A \to A' \rightrightarrows A' \otimes_A A'$ is exact, and applying $\mathrm{Hom}(R, -)$ shows that $\mathrm{Hom}(R, A) \to \mathrm{Hom}(R, A') \rightrightarrows \mathrm{Hom}(R, A' \otimes_A A')$ is also exact, which translates to the exactness of (2.1.8) under the duality between affine schemes and rings. To reduce to the affine case, observe that question is local on S so we may assume that S is affine. As $S' \to S$ is fpqc, there is a quasi-compact open subset $U' \subseteq S'$ surjecting onto S. After choosing a finite affine covering $\{U'_i\}$ of U', we can replace S' with the affine scheme $\amalg_i U'_i$ (details left to the reader).

Reducing to the case that Y is affine is a little harder. We first observe that

$$|S' \times_S S'| \rightrightarrows |S'| \to |S| \tag{2.1.9}$$

is a coequalizer diagram of sets. Indeed, we already know that $|S'| \to |S|$ is surjective, and that if $s'_1, s'_2 \in |S'|$ have the same image in S, then there exists a point $q \in |S' \times_S S'|$ with $p_1(q) = s'_1$ and $p_2(q) = s'_2$. Thus there exists a map $|h|\colon |S| \to |Y|$ of sets such that $|g| = |h| \circ |f|$. Letting $U \subseteq Y$ be an open affine subset, then $V := |h|^{-1}(U)$ is a subset of S such that $f^{-1}(V) = g^{-1}(U)$ is an open subset of S'. Since $S' \to S$ is submersive (i.e., S has the quotient topology) by Exercise A.4.9, it follows that $V \subseteq S$ is open. Since $f^{-1}(V) \to V$ is also fpqc, we may assume that Y is affine. See also [FGAI, Thm. 2, p. 317], [BLR90, Thm. 6(a)], [Vis05, Thm. 2.55], and [SP, Tag 023Q]. □

The following generalization also holds.

Corollary 2.1.10. *Let $f\colon S' \to S$ be an fpqc morphism of schemes.*

(1) If $S \to T$ is a morphism of schemes and Y is an T-scheme, then

$$\mathrm{Mor}_T(S, Y) \to \mathrm{Mor}_T(S', Y) \rightrightarrows \mathrm{Mor}_T(S' \times_S S', Y),$$

is exact.

(2) If X and Y are schemes over S, then

$$\mathrm{Mor}_S(X, Y) \to \mathrm{Mor}_{S'}(X_{S'}, Y_{S'}) \rightrightarrows \mathrm{Mor}_{S''}(X_{S''}, Y_{S''}),$$

is exact where $S'' = S' \times_S S'$.

Proof. For (1), it follows from Proposition 2.1.7 that $\mathrm{Mor}_T(S, Y) \to \mathrm{Mor}_T(S', Y)$ is injective and that if $g\colon S' \to Y$ is an T-morphism such that $p_1 \circ g = p_2 \circ g$, there exists a map $h\colon S \to Y$ of schemes with $g = h \circ f$. Letting $p_S\colon S \to T$, $p_{S'}\colon S' \to T$, and $p_Y\colon Y \to T$ denote the structure morphisms, observe that $p_Y \circ h$ and p_S are elements of $\mathrm{Mor}(S, T)$ mapping to $p_Y \circ g = p_{S'} \in \mathrm{Mor}(S', T)$. By Proposition 2.1.7, the inclusion $\mathrm{Mor}(S, T) \to \mathrm{Mor}(S', T)$ is injective and we conclude that $h\colon S \to Y$ is a T-morphism. Part (2) follows from applying (1) to the fpqc morphism $f\colon X_{S'} \to X$ over $T = S$. □

2.1.3 Descending schemes

Proposition 2.1.11 (Fpqc Descent for Open/Closed Subschemes)**.** *Let $f\colon S' \to S$ be an fpqc morphism of schemes. If $Z' \subseteq S'$ is a closed (resp., open) subscheme such that $p_1^{-1}(Z') = p_2^{-1}(Z')$ as subschemes of $S' \times_S S'$, then there exists a closed (resp., open) subscheme $Z \subseteq S$ such that $Z' = f^{-1}(Z)$.*

Proof. If $Z' \hookrightarrow S'$ is a closed immersion defined by an ideal sheaf $I_{Z'} \subseteq \mathcal{O}_{S'}$, then Fpqc Descent for Quasi-Coherent Sheaves (2.1.4) implies that $I_{Z'}$ descends to a quasi-coherent sheaf I_Z on S and the inclusion $I_{Z'} \hookrightarrow \mathcal{O}_{S'}$ descends to an inclusion $I_Z \hookrightarrow \mathcal{O}_S$. It follows that Z' descends to closed subscheme $Z \subseteq S$ defined by I_Z. The case of an open immersion is handled by passing to the reduced complement. □

For the following results, it will be convenient to denote f^*X as the base change of $X \to S$ by a morphism $f\colon S' \to S$.

Proposition 2.1.12 (Fpqc Descent for Affine/Quasi-affine Schemes)**.** *Let $f\colon S' \to S$ be an fpqc morphism of schemes. If $X' \to S'$ is an affine (resp., quasi-affine) morphism and $\alpha\colon {p_1}^*(X') \xrightarrow{\sim} p_2^*(X')$ is an isomorphism over $S' \times_S S'$ satisfying $p_{23}^*\alpha \circ p_{12}^*\alpha = p_{13}^*\alpha$, then there exists an affine (resp., quasi-affine) morphism $X \to S$ of schemes and an isomorphism $\phi\colon X' \to f^*(X)$ over S' such that $p_1^*\phi = p_2^*\phi \circ \alpha$.*

In other words, there exists dotted arrows completing the diagram

$$\begin{tikzcd}
p_{23}^*\alpha \circ p_{12}^*\alpha = p_{13}^*\alpha \arrow[d, dash] & p_1^*X' \overset{\alpha}{\xrightarrow{\sim}} p_2^*X' \arrow[d] & X' \arrow[r, dashed] \arrow[d] & X \arrow[d, dashed] \\
S' \times_S S' \times_S S' \arrow[r, shift left=2, "p_{12}"] \arrow[r, "p_{13}" description] \arrow[r, shift right=2, "p_{23}"'] & S' \times_S S' \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & S' \arrow[r, "f"] & S.
\end{tikzcd}$$

Proof. If $X' \to S'$ is affine, we can write $X' = \mathcal{S}\mathrm{pec}_X\, \mathcal{A}'$ for a quasi-coherent sheaf $\mathcal{A}'$ of $\mathcal{O}_{S'}$-algebras. Fpqc Descent for Quasi-Coherent Sheaves (2.1.4) allows us to first descend $\mathcal{A}'$ to a quasi-coherent sheaf $\mathcal{A}$ on X, and then to descend the multiplication map $\mathcal{A}' \otimes_{\mathcal{O}_{S'}} \mathcal{A}' \to \mathcal{A}'$ to a map $\mathcal{A} \otimes_{\mathcal{O}_S} \mathcal{A} \to \mathcal{A}$, which by descent will necessarily satisfy the axioms making $\mathcal{A}$ into a quasi-coherent $\mathcal{O}_S$-algebra. It follows that $X' \to S'$ descends to the affine morphism $X := \mathcal{S}\mathrm{pec}_S\, \mathcal{A} \to S$. The case of quasi-affine morphisms is handled by using the canonical factorization $X' \hookrightarrow \mathcal{S}\mathrm{pec}_{S'}\, g'_*\mathcal{O}_{X'} \to S'$ into an open immersion followed by an affine morphism, and then combining the affine case above with Fpqc Descent for Open Subschemes (2.1.11). See also [BLR90, Thm. 6.6] and [Vis05, Thm. 4.33]. □

Remark 2.1.13 (Cocycle condition)*.* The cocycle condition is necessary for a scheme to descend. Shimura showed that for a genus 2 curve C' over $\mathbb{C}$ defined by $y^2 = x^6 + ax^5 + bx^4 + x^3 - \overline{b}x^2 + \overline{a}x - 1$ for a general $a, b \in \mathbb{C}$ is isomorphic to its complex conjugate but does not descend to a real curve [Shi72]. In other words, under the cover $S' = \operatorname{Spec}\mathbb{C} \to \operatorname{Spec}\mathbb{R}$, there is an isomorphism $\alpha\colon p_1^*C' \xrightarrow{\sim} p_2^*C'$ but $C' \to S'$ does not descend to a curve $C \to S$.

Theorem 2.1.14 (Fppf Descent for Locally Quasi-finite and Separated Schemes)**.** *Let $f\colon S' \to S$ be an fppf morphism of schemes. If $X' \to S'$ is a locally quasi-finite and separated morphism of schemes and $\alpha\colon p_1^*(X') \xrightarrow{\sim} p_2^*(X')$ is an isomorphism over $S' \times_S S'$ satisfying $p_{23}^*\alpha \circ p_{12}^*\alpha = p_{13}^*\alpha$, then there exists a locally quasi-finite and separated morphism $X \to S$ of schemes and an isomorphism $\phi\colon X' \to f^*(X)$ over S' such that $p_1^*\phi = p_2^*\phi \circ \alpha$.*

This will imply that the prestack SepLQFin over Sch parameterizing locally quasi-finite and separated morphisms $X \to S$ is a stack in the fppf topology.

Proof. Our strategy is to reduce to Fpqc Descent for Quasi-affine Morphisms (2.1.12). For each quasi-compact open subset $U' \subseteq X'$, the composition $U' \hookrightarrow X' \to S$ is quasi-affine by Zariski's Main Theorem (A.7.3). Since $S' \to S$ is fppf, so is $p_2 \colon S' \times_S S' \to S'$, and hence the image $V' := p_2(\alpha(p_1^* U'))$ is an open subset of X'. The image V' is also quasi-compact and contains U', and moreover α restricts to an isomorphism $p_1^*(V') \xrightarrow{\sim} p_2^*(V')$ satisfying the cocycle condition. As the map $V' \to S'$ is quasi-affine, Fpqc Descent for Quasi-affine Morphisms implies that $V' \to S'$ descends to a quasi-affine morphism $V \to S$. Covering X' with quasi-compact open subsets U'_i, the subsets $V'_i := p_2(\alpha(p_1^* U'_i))$ also cover X', and since each V'_i descends to a scheme V_i quasi-affine over S, we may glue the schemes V_i to a scheme X over S which pulls back to X'. That $X \to S$ is locally quasi-finite and separated follows from a straightforward descent argument; see Proposition 2.1.26. See also [GR71, Lem. 5.7.2] and [SP, Tag 02W7]. □

Remark 2.1.15 (Historical comment). If $X' \to S'$ is quasi-compact, then it is quasi-affine in which case the argument for effective descent is straightforward (see Proposition 2.1.12). In a 1963 letter to Mumford [Mum10, p. 681], Grothendieck indicates that he could prove effective descent for locally quasi-finite and separated morphisms. This result appeared in unpublished work of Murre—later published in [Mur95, Prop. 1]—under a locally noetherian hypothesis and in [$\mathrm{SGA3}_{\mathrm{II}}$, Lem. X.5.4] under a locally of finite presentation hypothesis. In full generality, this theorem was proven by Raynaud and Gruson, but unfortunately was disguised in the proof of [GR71, Lem. 5.7.2]. That this theorem was not widely known at the time may have been a contributing reason to why algebraic spaces and stacks were first defined in [Knu71] and [Art74b] with a quasi-compact hypothesis on the diagonal.

We will often apply the above result in the form of Proposition 2.3.17 to show that a given sheaf in the big étale or fppf topology is representable by a scheme.

Example 2.1.16 (Non-effective descent)**.** An arbitrary morphism $X' \to S'$ of schemes with descent data along an fppf (or even étale) morphism $S' \to S$ may not descend to a morphism $X \to S$ of *schemes*. Hironaka's example of a proper three-fold that is not projective can be adapted to give an example of a projective morphism $X' \to S'$ that does not descend; see [SP, Tag 08KE]. Additionally, Raynaud constructed a normal noetherian local ring A of dimension 2, an étale cover $S' \to S = \operatorname{Spec} A$, and a family $\mathcal{C}' \to S'$ of smooth genus 1 curves that does not descend to a family $\mathcal{C} \to S$ [Ray70, XIII 3.2]. There is also an example of a DVR R, an étale cover $S' \to S = \operatorname{Spec} R$, and a family $\mathcal{C}' \to S'$ of nodal genus 1 curves with smooth generic fiber that does not descend to a family $\mathcal{C} \to S$ [BLR90, § 6.7], and an example of a projective surface S, an étale cover $S' \to S$, and a family $\mathcal{C}' \to S'$ of nodal genus 0 curves with smooth generic fiber that does not descend to a family $\mathcal{C} \to S$ [Ful10, Ex. 2.3].

On the other hand, an map $X' \to S'$ of schemes with descent data along an fppf cover $S' \to S$ always descends to a morphism $X \to S$ of *algebraic spaces* (see Theorem 3.4.13 for the étale case, and Corollary 6.3.6 in general). In fact, the prestack $\underline{\mathrm{AlgSp}}$, whose objects over a scheme S are morphisms $X \to S$ of algebraic spaces, is a stack in the fppf topology (see Exercise 4.5.15).

Effective descent does, however, hold in some other settings. For instance, we will see in the next result that it holds for principal G-bundles. It also holds for pairs $(X' \to S', \mathcal{L}')$, where $X' \to S'$ is a quasi-compact morphism of schemes and $\mathcal{L}'$

is a line bundle on X' relatively ample over S'; see [BLR90, Thm. 6.7] and [Vis05, Thm. 4.38].

Proposition 2.1.17 (Fpqc Descent for Principal G-bundles)**.** *Let $G \to T$ be an fppf affine group scheme, and let $f: S' \to S$ be an fpqc morphism of schemes over T. If $P' \to S'$ is a principal G-bundle and $\alpha: p_1^* P' \xrightarrow{\sim} p_2^* P'$ is an isomorphism of principal G-bundles over $S' \times_S S'$ satisfying $p_{12}^*\alpha \circ p_{23}^*\alpha = p_{13}^*\alpha$, then there exists a principal G-bundle $P \to S$ and an isomorphism $\phi: P' \to f^*P$ of principal G-bundles such that $p_1^*\phi = p_2^*\phi \circ \alpha$.*

Proof. Since $G \to T$ is affine, the principal G-bundle $P' \to S'$ is affine. By Fpqc Descent for Affine Schemes (Proposition 2.1.12), there is an affine morphism $P \to S$ and an isomorphism $\phi: P \to f^*Q$ of schemes with $p_1^*\phi = p_2^*\phi \circ \alpha$. By Fpqc Descent for Morphisms (2.1.7), we can descend the action $G \times_S P' \to P'$ to an action $G \times_S P \to P$ giving P the structure of a principal G-bundle and making $\phi: P' \to f^*P$ a G-equivariant isomorphism. □

2.1.4 Descending properties

Proposition 2.1.18 (Fpqc Local Properties of Quasi-Coherent Sheaves)**.** *Let $f: S' \to S$ be an fpqc morphism of schemes.*

(1) A homomorphism $F \to G$ of quasi-coherent $\mathcal{O}_S$-module is an isomorphism (resp., injective, surjective) if and only if $f^*F \to f^*G$ is.

(2) A quasi-coherent $\mathcal{O}_S$-module F is of finite type (resp., of finite presentation, flat, a vector bundle, a line bundle) if and only if f^*G is. If S and S' are noetherian, then the same holds for coherence.

(3) A quasi-coherent $\mathcal{O}_X$-module F on an S-scheme X is flat over S if and only if the pullback of F to $X \times_S S'$ is flat over S'.

In other words, each of these properties is *fpqc local on S*.

Proof. Part (1) reduces to the algebra statement: if $A \to A'$ is a faithfully flat ring map, an A-module map $M \to N$ is an isomorphism (resp., injection, surjection) if and only if $M \otimes_A A' \to N \otimes_A A'$ is. This follows directly from the faithful exactness of $- \otimes_A A'$. Part (2) reduces to: if $A \to A'$ is faithfully flat, an A-module M is finitely generated (resp., finitely presented, flat, locally free of rank r) if and only if $M \otimes_A A'$ is. The ($\Rightarrow$) implications are clear. Conversely, if $M \otimes_A A'$ is finitely generated, then let $x_1', \ldots, x_m' \in M \otimes_A A'$ be generators and write $x_i' = \sum_j x_{ij} \otimes a_{ij}'$ with $x_{ij} \in M$ and $a_{ij}' \in A'$. Letting n be the number of the x_{ij}, the map $(x_{ij}): A^{\oplus n} \to M$ is surjective since it becomes surjective after base changing by the faithfully flat map $A \to A'$. Repeating this argument to the kernel, we see that the property of being finite presentation descends. For flatness, suppose that $M \otimes_A A'$ is flat. By the faithful flatness of $A \to A'$, the exactness of $M \otimes_A -$ is equivalent to the exactness of $(M \otimes_A A') \otimes_{A'} (A' \otimes_A -)$, which follows from the flatness of $A \to A'$ and the flatness of the A'-module $M \otimes_A A'$. As being locally free of finite rank is equivalent to being finitely presented and flat, the final statement also follows. Part (3) reduces to: if $A \to A'$ is faithfully flat and $A \to B$ is a ring map, a B-module N is flat over A if and only if $N \otimes_A A'$ is flat over A'. This is special case of (2). See also [EGA, IV_2.2.5] and [SP, Tag 05AY]. □

The following, perhaps surprising, fact that regularity descends under faithful flatness will come in handy.

Lemma 2.1.19. *If $A \to B$ is a flat local ring map of noetherian local rings and B is regular, then so is A.*

Proof. Recall that a noetherian local ring R is regular if and only if every finitely generated R-module M has a finite resolution by free modules of finite rank. Moreover, if R is regular of dimension d and

$$0 \to K \to R^{\oplus k_{d-1}} \to \cdots \to R^{\oplus k_0} \to M \to 0 \tag{2.1.20}$$

is an exact sequence of R-modules, then K is free; see [Eis95, Thm. 19.12] and [SP, Tag 00OC]. If M is a finitely generated A-module, choose an exact sequence (2.1.20). Since B is regular, $K \otimes_A B$ is free. Since being locally free is an Fpqc Local Property of Quasi-Coherent Sheaves (2.1.18(2)) and R is local, K is free. Therefore A is regular. See also [EGA, IV_0.17.3.3] and [SP, Tag 00OF]. □

Proposition 2.1.21 (Fpqc Descent for Properties of Schemes)**.** *Let $X \to Y$ be an fpqc morphism of schemes. If X is quasi-compact (resp., locally noetherian, noetherian, integral, reduced, normal, regular), then so is Y.*

Proof. First, note that quasi-compactness descends under any surjective map. The other parts reduce to the algebraic statement: if $A \to B$ is a faithfully flat map of rings and B is noetherian (resp., a domain, reduced, normal, or regular), then so is A. As $A \to B$ is faithfully flat, $A \to B$ is injective and $I = IB \cap A$ for every ideal $I \subseteq A$. By the injectivity of $A \to B$, the 'domain' and 'reduced' cases are clear. For noetherianness, if $I_1 \subseteq I_2 \subseteq \cdots$ is an ascending chain of ideals, then since $I_1B \subseteq I_2B \subseteq \cdots$ terminates, so does $I_1 = I_1B \cap A \subseteq I_2 = I_2B \cap A \subseteq \cdots$. If B is a normal domain and a/b is integral over A where $a, b \in A$, then $a/b \in B$. This implies that $a \in (b) \subseteq B$, hence $a\colon B \to B/bB$ is the zero map. As this map is the base change of $a\colon A \to A/bA$, faithful flatness implies that $a\colon A \to A/bA$ is the zero map, hence $a \in (b) \subseteq A$ and $a/b \in A$. The regularity statement follows from Lemma 2.1.19. See also [SP, Tags 033D, 034B, and 06QL]. □

Proposition 2.1.22 (Fpqc Descent for Properties on the Source)**.** *Let $X' \to X$ be an fpqc morphism of schemes. If $X \to Y$ is a morphism of schemes such that $X' \to X \to Y$ is smooth (resp., étale), then $X \to Y$ is smooth (resp., étale).*

Proof. The smooth case follows from Lemma 2.1.19. The étale case follows from the smooth case with the observation that for $y \in Y$, the map $X'_y \to X_y$ on fibers is surjective, and hence if $\dim X'_y = 0$, then $\dim X_y = 0$. See also [EGA, IV_4.17.7.7] and [SP, Tag 05B5]. □

Note that smoothness and étaleness are, however, not fpqc (or fppf) local properties, i.e., properties that hold if *and only if* they hold after an fpqc (or fppf) cover. For instance, there are non-smooth (but necessarily flat) schemes of finite type over a field.

Proposition 2.1.23 (Fppf/Smooth Local Properties of Schemes)**.**

(1) If $X \to Y$ is an fppf morphism of schemes, then X is locally noetherian if and only if Y is.

(2) If $X \to Y$ be a surjective smooth morphism of schemes, then X is reduced (resp., normal, regular) if and only if Y is.

Proof. The ($\Rightarrow$) implications follows from Proposition 2.1.21. For (1), if Y is locally noetherian, so is X by Hilbert's Basis Theorem. Part (2) reduces to the algebra statement that if $A \to B$ is a smooth ring map and A is reduced (resp., normal, regular), then so is B, which we leave to the reader. See also [SP, Tag 034D]. $\square$

Remark 2.1.24. The property of being a domain is not étale local, e.g., there is a reducible étale neighborhood of the nodal cubic (see Example 0.5.3). Reducedness, normality, and regularity are not fppf local: there are non-reduced schemes of finite type over a field. On the other hand, if $X \to Y$ is a flat morphism of noetherian schemes such that Y is normal and every fiber X_y is normal, then X is normal; see [EGA, IV_2.6.5.4] or [SP, Tag 0C22].

Remark 2.1.25. If A is a noetherian local ring, the map $A \to \widehat{A}$ to its completion is faithfully flat. If the completion $\widehat{A}$ is reduced (resp., normal, regular), Fpqc Descent for Properties of Schemes (2.1.21) implies that the same holds for A. While the converse holds for regularity, it does not hold in general for reducedness and normality. However, if A is essentially of finite type over a field (or more generally excellent), then A is reduced (resp., normal) if and only if $\widehat{A}$ is, and moreover in this case the normalization commutes with completion. See [SP, Tags 07NZ and 0C23].

Proposition 2.1.26 (Fpqc Local Properties on the Target)**.** *Let $S' \to S$ be an fpqc morphism of schemes and $\mathcal{P}$ be one of the following properties of a morphism of schemes: surjective, quasi-compact, quasi-separated, isomorphism, open immersion, closed immersion, monomorphism, affine, quasi-affine, quasi-compact locally closed immersion, locally of finite type, locally of finite presentation, separated, proper, universally closed, universally open, universally submersive, finite, locally quasi-finite, quasi-finite, flat, fppf, smooth, étale, unramified, or syntomic. Then $X \to S$ has $\mathcal{P}$ if and only if $X \times_S S' \to S'$ does.*

In other words, each of these properties is *fpqc local on the target*.

Proof. To setup the notation, consider the cartesian diagram

$$\begin{tikzcd}
X' \times_X X' \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] \arrow[d] \arrow[dr, phantom, "\square"] & X' \arrow[r, "g'"] \arrow[d, "f'"] \arrow[dr, phantom, "\square"] & X \arrow[d, "f"] \\
S' \times_S S' \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & S' \arrow[r, "g"] & S.
\end{tikzcd}$$

We already know that each property is stable under base change. It is not hard to see that the properties 'surjective', 'quasi-compact', and 'quasi-separated' descend.

For 'isomorphism', let $h' \colon S' \to X'$ be an inverse of f'. As inverses are unique, $p_1^* h' = p_2^* h'$ as morphisms $S' \times_S S' \to X' \times_X X'$. By Fpqc Descent for Morphisms (2.1.10(2)) applied to $X' \to X$, the S'-morphism h' descends to a S-morphism $h \colon S \to X$. This yields that $f \circ h = \mathrm{id}_Y$. To see that $h \circ f = \mathrm{id}_X$, observe that the two morphisms $\mathrm{id}_X, h \circ f \colon X \to X$ become equal after precomposing with $X' \to X$, and thus Fpqc Descent for Morphisms (2.1.7) implies that $h \circ f = \mathrm{id}_X$. For 'open immersion', if f' is an open immersion, then $f'(X') = g^{-1}(f(X))$ is open. As g is universally submersive (Exercise A.4.9), $f(X)$ is open and we have reduced to show that $X \to f(X)$ is an isomorphism, which follows from the previous case. The case of 'closed immersion'

and 'quasi-compact locally closed immersion' follow For 'affine' and 'quasi-affine', we can assume that f is quasi-compact and quasi-separated. We appeal to the canonical factorization $f \colon X \to \mathcal{S}\mathrm{pec}_Y\, f_* \mathcal{O}_X \to S$ which commutes with flat base change by $S' \to S$, and use that f is affine (resp., quasi-affine) if and only if $X \to \mathcal{S}\mathrm{pec}_Y\, f_*\mathcal{O}_X$ is an isomorphism (resp., open immersion).

The 'locally of finite type' (resp., 'locally of finite presentation') case reduces to: if $A \to A'$ is faithfully flat, then a ring map $A \to B$ is of finite type (resp., finite presentation) if and only if $A' \to A' \otimes_B B'$ is. If $A' \to A' \otimes_A B$ is of finite type, there are A'-algebra generators $b_1', \ldots, b_n'$ which we can write as $b_i' = \sum_j a_{ij}' \otimes b_{ij}$ with $a_{ij}' \in A'$ and $b_{ij} \in B$. If $\widetilde{B} \subseteq B$ denotes the A-subalgebra generated by the b_{ij}, then since $\widetilde{B} \otimes_A A' = B \otimes_A A'$, the faithful flatness of $A \to A'$ implies that $\widetilde{B} = B$. If $A' \to A' \otimes_A B$ is of finite presentation, then we have just seen that $A \to B$ is of finite type and we can write $B = A[x_1, \ldots, x_n]/I$. Since $A \to A'$ is flat, $B \otimes_A A' = A'[x_1, \ldots, x_n]/I'$, where $I' = I \otimes_{A[x_1,\ldots,x_n]} A'[x_1, \ldots, x_n]$. Since I' is a finitely generated ideal, Fpqc Descent for Properties of Quasi-Coherent Sheaves (2.1.18(2)) implies that I is also finitely generated.

The 'flat' case is easy and follows from Proposition 2.1.18(2). Since the property 'smooth' is equivalent to flat, locally of finite presentation, and smoothness of every fiber, this case reduces to the algebra fact: a finite type algebra A over an algebraically closed field K is regular if and only if $A \otimes_K L$ is regular for every algebraically closed field extension L/K. The remaining cases are left to the reader. See also [EGA, IV$_2$.2.6–7 and IV$_4$.17.7.4] and [SP, Tag 02YJ]. □

Proposition 2.1.27 (Fppf/Smooth/Étale Local Properties on the Source)**.**

(1) If $X' \to X$ is an fppf morphism of schemes, a morphism $X \to Y$ of schemes is locally of finite presentation, (resp., locally of finite type, surjective, flat, fppf) if and only if $X' \to X \to Y$ is.

(2) If $X' \to X$ is a surjective smooth morphism of schemes, a morphism $X \to Y$ of schemes is smooth if and only if $X' \to X \to Y$ is.

(3) If $X' \to X$ is a surjective étale morphism of schemes, a morphism $X \to Y$ of schemes is étale, (resp., locally quasi-finite, unramified) if and only if $X' \to X \to Y$ is.

In other words, each property is *fppf/smooth/étale local on the source.*

Proof. Part (1) reduces to: if $A \to A'$ is a faithfully flat and finitely presented map of R-algebras, then $R \to A$ is of finite type (resp., finite presentation) if and only if $R \to A'$ is. Using Limit Methods (B.3.2 and B.3.3), there exists a finite type R-algebra A_0, a faithfully flat and finitely presented map $A_0 \to A_0'$ of R-algebras, and a commutative diagram

$$\begin{tikzcd} R \arrow[r] \arrow[dr] & A_0 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & A_0' \arrow[d] \\ & A \arrow[r] & A' \end{tikzcd}$$

such that $A' \cong A \otimes_{A_0} A_0'$. If $R \to A'$ is of finite type, after possibly enlarging A_0', we may arrange that $A_0' \to A'$ is surjective. As $A \to A'$ is faithfully flat, this implies that $A_0 \to A$ is surjective. Hence $R \to A$ is also of finite type. We leave the finite presentation case to the reader. For (2), smoothness descends in fact under fpqc morphisms by Proposition 2.1.22. Conversely, smooth morphisms are stable under composition. For (3), it easy to see that locally quasi-finiteness and unramifiedness descend, and since étaleness is equivalent to smoothness and unramifiedness, étaleness also descends. See also [EGA, IV$_4$.17.7.5] and [SP, Tags 036M, 036T, and 036V]. □

2.2 Grothendieck topologies and sites

The introduction of the digit 0 or the group concept was general nonsense too, and mathematics was more or less stagnating for thousands of years because nobody was around to take such childish steps...

Alexander Grothendieck, letter to Ronald Brown

To utilize the descent properties of the preceding section, it will be convenient to generalize the concept of a topological space so that we can view étale/fppf/fpqc morphisms as 'opens', and so that we can formulate the axioms for sheaves and stacks in these generalized topologies. Grothendieck topologies were introduced in [SGA4, Def. II.1.3]. Our exposition follows [Art62], [Vis05], [Ols16, §2], and [SP, Tag 00UZ].

2.2.1 Definitions and first examples

Definition 2.2.1 (Sites)**.** A *Grothendieck topology* on a category $\mathcal{S}$ consists of the following data: for each object $X \in \mathcal{S}$, there is a set $\mathrm{Cov}(X)$ consisting of *coverings* of X, i.e., collections of morphisms $\{X_i \to X\}_{i \in I}$ in $\mathcal{S}$. We require that:

(1) (identity) If $X' \to X$ is an isomorphism, then $(X' \to X) \in \mathrm{Cov}(X)$.

(2) (restriction) If $\{X_i \to X\}_{i\in I} \in \mathrm{Cov}(X)$ and $Y \to X$ is a morphism, then the fiber products $X_i \times_X Y$ exist in $\mathcal{S}$ and the collection $\{X_i \times_X Y \to Y\}_{i\in I} \in \mathrm{Cov}(Y)$.

(3) (composition) If $\{X_i \to X\}_{i\in I} \in \mathrm{Cov}(X)$ and $\{X_{ij} \to X_i\}_{j \in J_i} \in \mathrm{Cov}(X_i)$ for each $i \in I$, then $\{X_{ij} \to X_i \to X\}_{i\in I, j\in J_i} \in \mathrm{Cov}(X)$.

A *site* is a category $\mathcal{S}$ with a Grothendieck topology.

Caution 2.2.2. The definition requires that $\mathrm{Cov}(X)$ is a *set*, but this is not true in many cases of interest, such as the big étale site. We will ignore set-theoretic issues in this book, but they are usually easy (but annoying) to address by working with a suitable subcategory containing all morphisms of interest which defines the same category of sheaves; see [SP, Tag 00VI].

Example 2.2.3 (Topological spaces)**.** If X is a topological space, let $\mathrm{Op}(X)$ denote the category of open sets $U \subseteq X$, where there is a unique morphism $U \to V$ if and only if $U \subseteq V$. We say that a covering of U (i.e., an element of $\mathrm{Cov}(U)$) is a collection of open subsets $\{U_i\}_{i\in I}$ such that $U = \bigcup_{i\in I} U_i$. This defines a Grothendieck topology on $\mathrm{Op}(X)$. In particular, if X is a scheme, the Zariski topology on X defines a site X_{Zar}, called the *small Zariski site on X*.

By replacing Zariski open immersions with étale morphisms, we obtain the small étale site.

Example 2.2.4 (Small étale site)**.** If X is a scheme, the *small étale site on* X is the category $X_{\text{ét}}$ of étale morphisms $U \to X$ such that a morphism $(U \to X) \to (V \to X)$ is simply an X-morphism $U \to V$ (which is necessarily étale). In other words, $X_{\text{ét}}$ is the full subcategory of Sch/X consisting of schemes étale over X. A covering of an object $(U \to X) \in X_{\text{ét}}$ is a collection of étale morphisms $\{U_i \to U\}$ such that $\coprod_i U_i \to U$ is surjective. Later we will introduce the small étale site $\mathcal{X}_{\text{ét}}$ of an algebraic space or Deligne–Mumford stack (Definition 4.1.1), which we use to define sheaves on $\mathcal{X}$.

2.2.2 Big sites

Example 2.2.5 (Big étale site)**.** The *big étale site* is the category Sch where a covering of a scheme U is a collection of étale morphisms $\{U_i \to U\}$ in Sch such that $\coprod_i U_i \to U$ is surjective. We denote this site as $\mathrm{Sch}_{\text{ét}}$.

The big étale site $\mathrm{Sch}_{\text{ét}}$ is the most important site in this text. It is used to define the most central notions in this book: an algebraic space is a sheaf on $\mathrm{Sch}_{\text{ét}}$ that is étale locally a scheme (Definition 3.1.2) while an algebraic stack is a stack over $\mathrm{Sch}_{\text{ét}}$ that is smooth locally a scheme (Definition 3.1.6). There are various analogous sites.

Example 2.2.6 (Big topological site)**.** Let Top be the category of topological spaces. A covering of $U \in \mathrm{Top}$ is a collection of open subspaces $\{U_i \hookrightarrow U\}_{i\in I}$ such that $U = \bigcup_{i\in I} U_i$.

Example 2.2.7 (Big Zariski site)**.** Replacing étale morphisms in Example 2.2.5 with open immersions defines the *big Zariski site* $\mathrm{Sch}_{\mathrm{Zar}}$.

Example 2.2.8 (Big fppf site)**.** A morphism of schemes is *fppf* if it is surjective, flat, and locally of finite presentation (see Definition A.2.20). The *big fppf site* $\mathrm{Sch}_{\mathrm{fppf}}$ is the category Sch of schemes where a covering $\{U_i \to U\}$ is a collection of morphisms such that $\coprod_i U_i \to U$ is fppf.

The big fppf site $\mathrm{Sch}_{\mathrm{fppf}}$ is also important in moduli theory. In [SP], algebraic spaces and stacks are defined using $\mathrm{Sch}_{\mathrm{fppf}}$ rather than $\mathrm{Sch}_{\text{ét}}$, but these two sites nevertheless define equivalent notions [SP, Tag 076U]. In this text, we use the big fppf site in §6.3 to discuss gerbes and quotients stacks by non-smooth groups schemes.

Example 2.2.9 (Big fpqc site)**.** A morphism of schemes is *fpqc* if it is surjective, flat, and every quasi-compact subset of the target is the image of a quasi-compact subset (see Definition A.2.20). The *big fpqc site* $\mathrm{Sch}_{\mathrm{fppf}}$ is the category Sch where a covering $\{U_i \to U\}$ is a collection of morphisms such that $\coprod_i U_i \to U$ is fpqc.

Caution 2.2.10. There are serious set-theoretic issues in defining an fpqc site, arising from the presence of too many fpqc covers—given any nonzero ring R, there does not exist a set of fpqc coverings of $\operatorname{Spec} R$ which can refine every fpqc covering; see [SP, Tags 0BBK and 03NV]. If one defines the big fpqc site ignoring set-theoretic issues (as we just did!), there are presheaves that do not have a sheafification [Wat75, Thm. 5.5]. Fortunately, we have no need for fpqc sites in this text. On the other hand, the notion of a sheaf or stack in the fpqc topology is well-defined, and this allows us to formulate general statements (but we usually only invoke the étale case).

Example 2.2.11 (Lisse-étale site)**.** On a scheme X, the *lisse-étale site* $X_{\text{lis-ét}}$ is the category of schemes smooth over X where morphisms in $X_{\text{lis-ét}}$ are (not necessarily smooth) morphisms of schemes over X. A covering $\{U_i \to U\}$ of an X-scheme U is a collection of X-morphisms such that $\coprod_i U_i \to U$ is surjective and étale.

We introduce the lisse-étale site of an algebraic stack in Definition 6.1.1 in order to define quasi-coherent sheaves.

Example 2.2.12 (Restricted categories and sites)**.** If $\mathcal{S}$ is a category and $S \in \mathcal{S}$, define the *restricted category* (sometimes referred to as the *localized category*) as the category $\mathcal{S}/S$ whose objects are maps $T \to S$ in $\mathcal{S}$. A morphism $(T' \to S) \to (T \to S)$ is a map $T' \to T$ over S. If $\mathcal{S}$ is a site, $\mathcal{S}/S$ is also a site where a covering of $T \to S$

in $\mathcal{S}/S$ is a covering $\{T_i \to T\}$ in $\mathcal{S}$. Applying this construction to a scheme S yields the relative versions of the big Zariski, étale, fppf, and fpqc sites $(\mathrm{Sch}/S)_{\mathrm{Zar}}$, $(\mathrm{Sch}/S)_{\text{ét}}$, $(\mathrm{Sch}/S)_{\mathrm{fppf}}$, and $(\mathrm{Sch}/S)_{\mathrm{fpqc}}$.

Example 2.2.13 (Grothendieck topologies on the category of affine schemes)**.** In the literature, authors sometimes use the big sites $\mathrm{AffSch}_{\mathrm{Zar}}$, $\mathrm{AffSch}_{\text{ét}}$, and $\mathrm{AffSch}_{\mathrm{fppf}}$ on the category of affine schemes. These define the same categories of sheaves as the corresponding big sites on Sch.

2.3 Presheaves and sheaves

According to A. Grothendieck one really does not need a space to do geometry, all one needs is a category of sheaves on this would-be space.

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Recall that if X is a topological space, a presheaf of sets on X is simply a contravariant functor $F\colon \mathrm{Op}(X) \to \mathrm{Sets}$ on the category $\mathrm{Op}(X)$ of open sets. The sheaf axiom translates succinctly into the condition that for each covering $U = \bigcup_i U_i$, the sequence

$$F(U) \to \prod_i F(U_i) \rightrightarrows \prod_{i,j} F(U_i \cap U_j)$$

is exact (i.e., is an equalizer diagram), where the two maps $F(U_i) \rightrightarrows F(U_i \cap U_j)$ are induced by restricting along the two inclusions $U_i \cap U_j \subset U_i$ and $U_i \cap U_j \subseteq U_j$. Also note that the intersections $U_i \cap U_j$ can also be viewed as fiber products $U_i \times_X U_j$.

The definition of presheaves and sheaves on a general site mirrors the topological case. Using descent, we prove some first examples of sheaves on the big étale site.

2.3.1 Definitions

Definition 2.3.1 (Presheaves)**.** A *presheaf* on a category $\mathcal{S}$ is a contravariant functor $\mathcal{S} \to \mathrm{Sets}$.

Remark 2.3.2. If $F\colon \mathcal{S} \to \mathrm{Sets}$ is a presheaf and $f\colon S \to T$ is a map in $\mathcal{S}$, then the pullback $F(f)(b)$ of an element $b \in F(T)$ is sometimes denoted as f^*b or $b|_S$.

Example 2.3.3 (Representable presheaves)**.** If $\mathcal{S}$ is a category and $S \in \mathcal{S}$, then $\mathrm{Mor}_{\mathcal{S}}(-, S)\colon \mathcal{S} \to \mathrm{Sets}$ defines a presheaf.

Definition 2.3.4 (Sheaves)**.** A *sheaf* on a site $\mathcal{S}$ is a presheaf $F\colon \mathcal{S} \to \mathrm{Sets}$ such that for every object $S \in \mathcal{S}$ and covering $\{S_i \to S\} \in \mathrm{Cov}(S)$, the sequence

$$F(S) \to \prod_i F(S_i) \rightrightarrows \prod_{i,j} F(S_i \times_S S_j) \tag{2.3.5}$$

is exact, where the two maps $F(S_i) \rightrightarrows F(S_i \times_S S_j)$ are induced by the two maps $S_i \times_S S_j \to S_i$ and $S_i \times_S S_j \to S_i$.

A *morphism* of presheaves or sheaves is by definition a natural transformation.

Remark 2.3.6. The exactness of (2.3.5) means that it is an equalizer diagram: $F(S)$ is identified with the subset of $\prod_i F(S_i)$ consisting of elements whose images under the two maps $F(S_i) \rightrightarrows F(S_i \times_S S_j)$ are equal.

Exercise 2.3.7. Let F be a presheaf on Sch. Show that the following are equivalent:

(1) F is a sheaf on $\mathrm{Sch}_{\text{ét}}$ (resp., $\mathrm{Sch}_{\text{fppf}}$, $\mathrm{Sch}_{\text{fpqc}}$),

(2) F sends coproducts to products (i.e., $F(\coprod_i U_i) = \prod_i F(U_i)$ for schemes U_i) and for every surjective étale (resp., fppf, faithfully flat) morphism $S' \to S$ of schemes, the sequence $F(S) \to F(S') \rightrightarrows F(S' \times_S S')$ is exact.

(3) F is a sheaf in the big Zariski topology $\mathrm{Sch}_{\text{Zar}}$ and for every surjective étale (resp., fppf, faithfully flat) morphism $S' \to S$ of *affine* schemes, the sequence $F(S) \to F(S') \rightrightarrows F(S' \times_S S')$ is exact.

Hint: Given a covering $\{S_i \to S\}$, consider the map $\coprod_i S_i \to S$.

Proposition 2.3.8 (Schemes are Sheaves)**.** *If $X \to S$ is a morphism of schemes, then* $\mathrm{Mor}_S(-, X)\colon \mathrm{Sch}/S \to \mathrm{Sets}$ *is a sheaf on* $(\mathrm{Sch}/S)_{\text{fpqc}}$ *and therefore also a sheaf on* $(\mathrm{Sch}/S)_{\text{ét}}$ *and* $(\mathrm{Sch}/S)_{\text{fppf}}$*.*

Proof. As $\mathrm{Mor}_S(-, X)$ is a sheaf in the big Zariski topology, it suffices by Exercise 2.3.7 to show that if $T' \to T$ is a faithfully flat morphism of affine schemes over S, then the sequence

$$\mathrm{Mor}_S(T, X) \to \mathrm{Mor}_S(T', X) \rightrightarrows \mathrm{Mor}_S(T' \times_T T', X)$$

is exact, which is precisely Fpqc Descent for Morphisms (Corollary 2.1.10). $\square$

Exercise 2.3.9.

(a) If F and G are sheaves on a site $\mathcal{S}$, show that the presheaf $\mathrm{Mor}_{\mathcal{S}}(F, G)$, defined by $S \mapsto \mathrm{Mor}_{\mathrm{Sets}}(F(S), G(S))$, is a sheaf on $\mathcal{S}$.

(b) Conclude that if X and Y are schemes over S, the functor $\underline{\mathrm{Mor}}_S(X, Y)\colon \mathrm{Sch}/S \to \mathrm{Sets}$, assigning an S-scheme T to $\mathrm{Mor}_T(X_T, Y_T)$, is a sheaf in the fpqc topology. (This gives another proof of Corollary 2.1.10(2).)

Exercise 2.3.10 (Gluing sheaves)**.** Let $\mathcal{S}$ be a site and $(X_i \to X)$ be a covering in $\mathcal{S}$. If F_i are sheaves on the restricted sites $\mathcal{S}/X_i$ and $\alpha_{ij}\colon F_i|_{X_{ij}} \to F_j|_{X_{ij}}$ are isomorphisms of sheaves on $\mathcal{S}/X_{ij}$ (where $X_{ij} := X_i \times_X X_j$) satisfying the cocycle condition $\alpha_{jk} \circ \alpha_{ij} = \alpha_{ik}$ on $\mathcal{S}/X_{ijk}$ (where $X_{ijk} = X_i \times_X X_j \times_X X_k$), show that there exists a unique sheaf F on $\mathcal{S}$ and isomorphisms $\phi_i\colon F|_{X_i} \to F_i$ satisfying $\phi_j|_{X_{ij}} = \alpha_{ij} \circ \phi_i|_{X_{ij}}$.

Exercise 2.3.11. Show that a surjective smooth (resp., fppf, fpqc) morphism of schemes is an epimorphism of sheaves on $\mathrm{Sch}_{\text{ét}}$ (resp., $\mathrm{Sch}_{\text{fppf}}$, $\mathrm{Sch}_{\text{fpqc}}$).

2.3.2 Fiber products

By Yoneda's Lemma (0.3.12), if $X \in \mathcal{S}$ is an object of a category $\mathcal{S}$ and F is a presheaf on $\mathcal{S}$, a morphism $\alpha\colon X \to F$ (which we interpret as a morphism of presheaves $\mathrm{Mor}(-, X) \to F$) corresponds to an element in $F(X)$, which by abuse of notation we also denote by α.

Definition 2.3.12. Given morphisms $\alpha\colon F \to G$ and $\beta\colon G' \to G$ of presheaves on a category $\mathcal{S}$, the *fiber product of α and β* is the presheaf $F \times_G G'$ whose set of sections over $S \in \mathcal{S}$ is $F(S) \times_{G(S)} G'(S)$, i.e.,

$$\begin{aligned} F \times_G G' \colon \mathcal{S} &\to \mathrm{Sets} \\ S &\mapsto \{(a, b) \in F(S) \times G'(S) \mid \alpha_S(a) = \beta_S(b)\}\,. \end{aligned} \tag{2.3.13}$$

Exercise 2.3.14.

(a) Show that (2.3.13) is a fiber product $F \times_G G'$ in $\mathrm{Pre}(\mathcal{S})$. (This is a generalization of Exercise 0.3.40 but the same proof should work.)

(b) Show that if F, G, and G' are sheaves on a site $\mathcal{S}$, then so is $F \times_G G'$. In particular, (2.3.13) is also a fiber product $F \times_G G'$ in $\mathrm{Sh}(\mathcal{S})$.

2.3.3 Sheafification

Theorem 2.3.15 (Sheafification)**.** *Let $\mathcal{S}$ be a site. The forgetful functor $\mathrm{Sh}(\mathcal{S}) \to \mathrm{Pre}(\mathcal{S})$ admits a left adjoint $F \mapsto F^{\mathrm{sh}}$, called the sheafification.*

Proof. A presheaf F on $\mathcal{S}$ is called *separated* if for every covering $\{S_i \to S\}$ of an object S, the map $F(S) \to \prod_i F(S_i)$ is injective (i.e., if sections glue, they glue uniquely). Let $\mathrm{Pre}(\mathcal{S})$ and $\mathrm{Sh}(\mathcal{S})$ be the categories of presheaves and sheaves, and let $\mathrm{Pre}^{\mathrm{sep}}(\mathcal{S}) \subset \mathrm{Pre}(\mathcal{S})$ be the full subcategory of separated presheaves. We will construct left adjoints

$$\begin{tikzcd} \mathrm{Sh}(\mathcal{S}) \arrow[r, hook] & \mathrm{Pre}^{\mathrm{sep}}(\mathcal{S}) \arrow[l, bend right, "\mathrm{sh}_2"'] \arrow[r, hook] & \mathrm{Pre}(\mathcal{S}). \arrow[l, bend right, "\mathrm{sh}_1"'] \end{tikzcd}$$

For $F \in \mathrm{Pre}(\mathcal{S})$, we define $\mathrm{sh}_1(F)$ by $S \mapsto F(S)/\sim$ where $a \sim b$ if there exists a covering $\{S_i \to S\}$ such that $a|_{S_i} = b|_{S_i}$ for all i. For $F \in \mathrm{Pre}^{\mathrm{sep}}(\mathcal{S})$, we define $\mathrm{sh}_2(F)$ by

$$S \mapsto \left\{ (\{S_i \to S\}, \{a_i\}) \;\middle|\; \begin{array}{l} \{S_i \to S\} \in \mathrm{Cov}(S) \text{ and } a_i \in F(S_i) \\ \text{such that } a_i|_{S_{ij}} = a_j|_{S_{ij}} \text{ for all } i, j \end{array} \right\} / \sim$$

where $(\{S_i \to S\}, \{a_i\}) \sim (\{S'_j \to S\}, \{a'_j\})$ if $a_i|_{S_i \times_S S'_j} = a'_j|_{S_i \times_S S'_j}$ for all i, j. The details are left to the reader. $\square$

Remark 2.3.16 (Topos). A *topos* is a category equivalent to the category of sheaves on a site. Two different sites may have equivalent categories of sheaves, and the topos is a more fundamental invariant. While topoi are undoubtedly important in moduli theory, they will not play a role in these notes.

2.3.4 A criterion for a sheaf to be a scheme

The following is a reinterpretation of Fppf Descent for Schemes (2.1.11, 2.1.12, and 2.1.14).

Proposition 2.3.17 (Descent Criterion for an Fppf Sheaf to be a Scheme)**.** *Let $\mathcal{P}$ be one of the following properties of morphisms of schemes: open immersion, closed immersion, affine, quasi-affine, or locally quasi-finite and separated. Let $X \to Y$ be a surjective smooth (resp., fppf) morphism of schemes. Let F be a sheaf on $(\mathrm{Sch}/Y)_{\text{ét}}$ (resp., $(\mathrm{Sch}/Y)_{\mathrm{fppf}}$). Consider the fiber product*

$$\begin{tikzcd} F_X \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X \arrow[d] \\ F \arrow[r] & Y \end{tikzcd}$$

of sheaves. If F_X is a scheme and $F_X \to X$ has $\mathcal{P}$, then F is a scheme and $F \to Y$ has $\mathcal{P}$.

Proof. As F_X is the pullback of F, there is a canonical isomorphism $\alpha\colon p_1^* F_X \to p_2^* F_X$ on $X \times_Y X$ satisfying the cocycle condition $p_{23}^*\alpha \circ p_{12}^*\alpha = p_{13}^*\alpha$. By Fppf Descent for Schemes (2.1.11, 2.1.12, and 2.1.14), there exists a morphism of schemes $W \to Y$ satisfying $\mathcal{P}$ that pulls back to $F_X \to X$. The sheaf F is identified with the sheafification of the presheaf on Sch/Y defined by

$$F^{\mathrm{pre}}\colon (T \to Y) \mapsto \mathrm{Eq}\left(F_X(T \to Y) \rightrightarrows (F_X \times_X F_X)(T \to Y)\right).$$

By Schemes are Sheaves (2.3.8), W is a sheaf and there is a morphism $F^{\mathrm{pre}} \to W$ of presheaves. By the universal property of sheafification, there is a morphism $\alpha\colon F \to W$ of sheaves which pulls back under $X \to Y$ to an isomorphism $\alpha_X\colon F_X \to W_X$. It is not hard to see that this implies that α is an isomorphism. For instance, since $\mathrm{Hom}(W, F)$ is a sheaf (Exercise 2.3.9) and the inverse β_X of α_X defines a section over $X \to Y$ whose two pullbacks to $X \times_Y X$ agree, the inverse β_X descends to a section β of $\mathrm{Hom}(W, F)$ over $\mathrm{id}\colon Y \to Y$. Since $\mathrm{Hom}(F, F)$ and $\mathrm{Hom}(W, W)$ are sheaves, it follows that $\beta \circ \alpha = \mathrm{id}_F$ and $\alpha \circ \beta = \mathrm{id}_W$. See also [SP, Tag 02W5]. $\square$

2.4 Prestacks

> *The Red Queen shook her head. "You may call it 'nonsense' if you like," she said, "but I've heard nonsense, compared with which that would be as sensible as a dictionary!"*
>
> LEWIS CARROLL, *THROUGH THE LOOKING-GLASS*

Prestacks and stacks were introduced by Grothendieck in [FGAI, §A.1] and [SGA1, §6] to express the categorical structure of objects satisfying fpqc descent. The language of prestacks was further developed by Giraud [Gir64] and [Gir71]. Motivation for prestacks was provided in §0.6.1: in an effort to keep track of automorphisms, we were naively led to consider a 'functor'

$$F\colon \mathcal{S} \to \text{Groupoids}\,.$$

While this is a good way to think about prestacks, it is more convenient to define a prestack by packaging the groupoids $F(S)$ for $S \in \mathcal{S}$ into one massive category $\mathcal{X}$ over $\mathcal{S}$ parameterizing pairs (a, S) where $S \in \mathcal{S}$ and $a \in F(S)$.

2.4.1 Definition of a prestack

Let $\mathcal{S}$ be a category and $p\colon \mathcal{X} \to \mathcal{S}$ be a functor of categories. We visualize this data as

$$\begin{tikzcd} \mathcal{X} \arrow[d, mapsto, "p"] & a \arrow[r, "\alpha"] \arrow[d, mapsto] & b \arrow[d, mapsto] \\ \mathcal{S} & S \arrow[r, "f"] & T \end{tikzcd}$$

where the lower case letters a, b are objects of $\mathcal{X}$ and the upper case letters S, T are objects of $\mathcal{S}$. We say that a is over S and that a morphism $\alpha\colon a \to b$ is over $f\colon S \to T$.

Definition 2.4.1 (Prestacks)**.** A functor $p\colon \mathcal{X} \to \mathcal{S}$ is a *prestack over a category* $\mathcal{S}$ if

(1) (pullbacks exist) for every diagram

$$\begin{tikzcd} a \arrow[r, dashed] \arrow[d, mapsto, dashed] & b \arrow[d, mapsto] \\ S \arrow[r] & T \end{tikzcd}$$

of solid arrows, there exists a morphism $a \to b$ over $S \to T$; and

(2) (universal property for pullbacks) for every diagram

$$\begin{tikzcd} a \arrow[r, dashed] \arrow[d, mapsto] \arrow[rr, bend left] & b \arrow[r] \arrow[d, mapsto] & c \arrow[d, mapsto] \\ R \arrow[r] & S \arrow[r] & T \end{tikzcd}$$

of solid arrows, there exists a unique arrow $a \to b$ over $R \to S$ filling in the diagram.

Caution 2.4.2. When defining and discussing prestacks, we often write $\mathcal{X}$ instead of $\mathcal{X} \to \mathcal{S}$, but when necessary, we denote the projection by $p_{\mathcal{X}} \colon \mathcal{X} \to \mathcal{S}$. We do not usually spell out the definition of the functor $\mathcal{X} \to \mathcal{S}$ as it should be clear to the reader. Moreover, when defining a prestack $\mathcal{X}$, we often only define the objects and morphisms in $\mathcal{X}$ and leave the composition law to the reader.

Remark 2.4.3. Axiom (2) above implies that the pullback in Axiom (1) is unique up to unique isomorphism. We often write f^*b or simply $b|_S$ to indicate a *choice* of a pullback.

Definition 2.4.4 (Fiber categories)**.** If $\mathcal{X}$ is a prestack over $\mathcal{S}$, the *fiber category* $\mathcal{X}(S)$ over $S \in \mathcal{S}$ is the category of objects in $\mathcal{X}$ over S with morphisms over id_S.

Exercise 2.4.5. Show that the fiber category $\mathcal{X}(S)$ is a groupoid.

Caution 2.4.6. Our terminology is not standard. Prestacks are usually referred to as *categories fibered in groupoids*. In the literature (c.f., [FGI+05, §4], [Ols16, §4.6]), a prestack is sometimes defined as a category fibered in groupoids together with Axiom 2.5.1(1) of a prestack.

It is also standard to call a morphism $b \to c$ in $\mathcal{X}$ *cartesian* if it satisfies the universal property in Axiom 2.5.1(2) and $p \colon \mathcal{X} \to S$ a *fibered category* if for every diagram as in Axiom 2.5.1(1), there exists a cartesian morphism $a \to b$ over $S \to T$. With this terminology, a prestack (as we have defined it) is a fibered category where every arrow is cartesian, or equivalently where every fiber category $\mathcal{X}(S)$ is a groupoid.

2.4.2 Examples

Example 2.4.7 (Presheaves as prestacks)**.** If $F \colon \mathcal{S} \to \mathrm{Sets}$ is a presheaf, we can construct a prestack $\mathcal{X}_F$ as the category of pairs (a, S) where $S \in \mathcal{S}$ and $a \in F(S)$. A morphism $(a', S') \to (a, S)$ in $\mathcal{X}_F$ is a map $f \colon S' \to S$ such that $a' = f^*a$, where f^* is convenient shorthand for $F(f) \colon F(S) \to F(S')$. The projection $\mathcal{X}_F \to \mathcal{S}$ is defined by $(a, S) \mapsto S$. Observe that the fiber categories $\mathcal{X}_F(S)$ are equivalent (even equal) to the set $F(S)$. We will often abuse notation by conflating F and $\mathcal{X}_F$.

Example 2.4.8 (Representable prestacks)**.** If S is an object of category $\mathcal{S}$, then the restricted category $\mathcal{S}/S$ of objects over S defines a prestack over $\mathcal{S}$, where the projection morphism $\mathcal{S}/S \to \mathcal{S}$ is given by $(T \to S) \mapsto T$. This is the same prestack obtained by applying the previous example to the representable presheaf $\mathrm{Mor}(-, S) \colon \mathcal{S} \to \mathrm{Sets}$.

Example 2.4.9 (Schemes as prestacks)**.** A scheme X defines the prestack Sch/X of schemes over X as in the previous example. The projection $\mathrm{Sch}/X \to \mathrm{Sch}$ is given by $(T \to X) \mapsto T$. We often abuse notation by referring to Sch/X simply as X.

Example 2.4.10 (Prestack of smooth curves)**.** We define the prestack $\mathcal{M}$ over Sch as the category of families of smooth curves $\mathcal{C} \to S$, i.e., smooth and proper morphisms $\mathcal{C} \to S$ of schemes such that every geometric fiber is a connected curve. A map $(\mathcal{C}' \to S') \to (\mathcal{C} \to S)$ is the data of maps $\alpha \colon \mathcal{C}' \to \mathcal{C}$ and $f \colon S' \to S$ such that the diagram

$$\begin{tikzcd} \mathcal{C}' \arrow[r, "\alpha"] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{C} \arrow[d] \\ S' \arrow[r, "f"] & S \end{tikzcd}$$

is cartesian. The projection $\mathcal{M} \to \mathrm{Sch}$ is given by $(\mathcal{C} \to S) \mapsto S$.

The prestack $\mathcal{M}_g$ is defined as the full subcategory of $\mathcal{M}$ consisting of families of smooth curves $\mathcal{C} \to S$ where every geometric fiber has genus g. Note that the fiber category $\mathcal{M}_g(\Bbbk)$ over an algebraically closed field $\Bbbk$ is the groupoid of smooth, connected, and projective curves C over $\Bbbk$ of genus g such that $\mathrm{Mor}_{\mathcal{M}_g(\Bbbk)}(C, C') = \mathrm{Isom}_{\mathrm{Sch}/\Bbbk}(C, C')$.

Exercise 2.4.11 (easy but important)**.** Verify that $\mathcal{M}$ and $\mathcal{M}_g$ are prestacks.

Example 2.4.12 (Prestack of coherent sheaves and vector bundles)**.** Let X be a scheme over a field $\Bbbk$. We define the prestack $\underline{\mathrm{QCoh}}(X)$ over $\mathrm{Sch}/\Bbbk$ as the category of pairs (E, S) where S is a scheme over $\Bbbk$ and E is a quasi-coherent sheaf on $X_S = X \times_{\Bbbk} S$ flat over S. A morphism $(E', S') \to (E, S)$ consists of a map of schemes $f \colon S' \to S$ together with an isomorphism $f^*E \to E'$ of $\mathcal{O}_{X_{S'}}$-modules.[1] The projection $\underline{\mathrm{QCoh}}(X) \to \mathrm{Sch}/\Bbbk$ is defined by $(E, S) \mapsto S$.

The substack $\underline{\mathrm{Coh}}(X) \subseteq \underline{\mathrm{QCoh}}(X)$ is the full subcategory consisting of pairs (E, S) where E is a finitely presented, quasi-coherent sheaf on X_S (or equivalently a coherent sheaf when X_S is noetherian). Similarly, $\mathcal{B}un(X) \subseteq \underline{\mathrm{Coh}}(X)$ is the full subcategory where E is a vector bundle on X_S (i.e., locally free quasi-coherent sheaf of finite rank).

Exercise 2.4.13. Verify that $\underline{\mathrm{QCoh}}(X)$, $\underline{\mathrm{Coh}}(X)$, and $\mathcal{B}un(X)$ are prestacks.

[1] This definition of a morphism is not completely precise as the pullback f^*E is not canonical. Recall that f^*E is defined as $f^{-1}E \otimes_{f^{-1}\mathcal{O}_S} \mathcal{O}_{S'}$, and while it exists and is unique up to unique isomorphism, a choice of pullback f^*E involves a choice of a limit in the definition of $f^{-1}E$, choices of tensor products, and a choice of sheafification. Instead, one can define a morphism $(E', S') \to (E, S)$ as an equivalence class of triples (f, F', α) where $f \colon S' \to S$ is a map of schemes, F' is a *choice* of a pullback of E, and an isomorphism $\alpha \colon F' \to E'$, where $(f, F', \alpha) \sim (g, G', \beta)$ if $f = g$ and the canonical isomorphism $\gamma \colon F' \xrightarrow{\sim} G'$ satisfies $\alpha = \beta \circ \gamma$. Alternatively, since the pushforward f_*E' is canonical, a morphism $(E', S') \to (E, S)$ can be defined as a map $f \colon S' \to S$ and a morphism $E \to f_*E'$ of $\mathcal{O}_{X_S}$-modules whose adjoint is an isomorphism.

2.4.3 Classifying stacks and quotient stacks

Classifying and quotient stacks were motivated in §0.6.5. Their definitions involve the notion of a principal G-bundle. For a smooth affine group scheme $G \to S$[2], a *principal G-bundle over an S-scheme T* is a morphism $P \to T$ of schemes with an action of G on P via $\sigma \colon G \times_S P \to P$ such that $P \to T$ is a G-invariant smooth morphism and

$$(\sigma, p_2) \colon G \times_S P \to P \times_T P, \qquad (g, p) \mapsto (gp, p)$$

is an isomorphism (see Definition B.1.47); in other words, G acts freely and transitively on P with quotient T. Equivalently, $P \to T$ is a principal G-bundle if there is étale cover $T' \to T$ such that $P \times_T T'$ is G-equivariantly isomorphic to the trivial principal G-bundle $G \times_S T'$ (Proposition B.1.49). See §B.1.7 for further background and many examples.

Definition 2.4.14 (Classifying stacks)**.** The *classifying stack BG* of a smooth affine group scheme $G \to S$ is the category over Sch/S whose objects are principal G-bundles $P \to T$ and a morphism $(P' \to T') \to (P \to T)$ is the data of a G-equivariant morphism $P' \to P$ such that

$$\begin{tikzcd} P' \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & P \arrow[d] \\ T' \arrow[r] & T \end{tikzcd}$$

is cartesian.

Definition 2.4.15 (Quotient prestacks and stacks)**.** Let $G \to S$ be a smooth affine group scheme acting on a scheme U over S. The *quotient prestack* $[U/G]^{\mathrm{pre}}$ of an action of a smooth affine group scheme $G \to S$ on an S-scheme U is the category over Sch/S consisting of pairs (T, u) where T is an S-scheme and $u \in U(T)$. A morphism $(T', u') \to (T, u)$ is the data of a map $f \colon T' \to T$ of S-schemes and an element $g \in G(T')$ such that $f^*u = g \cdot u'$. Note that the fiber category $[U/G]^{\mathrm{pre}}(T)$ is identified with the quotient groupoid $[U(T)/G(T)]$ from Example 0.4.6.

The *quotient stack* $[U/G]$ is the category over Sch/S consisting of diagrams

$$\begin{tikzcd} P \arrow[r] \arrow[d] & U \\ T & \end{tikzcd}$$

where $P \to T$ is a principal G-bundle and $P \to U$ is a G-equivariant morphism of S-schemes. A morphism $(T' \leftarrow P' \to U) \to (T \leftarrow P \to U)$ consists of a morphism $T' \to T$ and a G-equivariant morphism $P' \to P$ of schemes such that the diagram

$$\begin{tikzcd} P' \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] \arrow[rr, bend left] & P \arrow[r] \arrow[d] & U \\ T' \arrow[r] & T & \end{tikzcd}$$

is commutative and the left square is cartesian.

[2]In §6.3.2, we will define principal G-bundles, classifying stacks, and quotient stacks more generally for fppf group schemes.

Exercise 2.4.16 (easy)**.** Verify that $[U/G]^{\mathrm{pre}}$ and $[U/G]$ are prestacks over Sch/S.

We show shortly that $[U/G]$ and $BG = [S/G]$ are stacks over $(\mathrm{Sch}/S)_{\text{ét}}$ (Proposition 2.5.13), which justifies our terminology of a 'quotient stack' and 'classifying stack'. We also show that $[U/G]$ is identified as the stackification of $[U/G]^{\mathrm{pre}}$ (Exercise 2.5.21), and later show that $[U/G]$ is algebraic (Theorem 3.1.10).

2.4.4 Morphisms of prestacks

Definition 2.4.17.

(1) A *morphism of prestacks* $f\colon \mathcal{X} \to \mathcal{Y}$ is a functor $f\colon \mathcal{X} \to \mathcal{Y}$ such that the diagram

$$\begin{tikzcd} \mathcal{X} \arrow[rr, "f"] \arrow[dr, "p_{\mathcal{X}}"'] & & \mathcal{Y} \arrow[dl, "p_{\mathcal{Y}}"] \\ & \mathcal{S} & \end{tikzcd}$$

strictly commutes, i.e., for every object $a \in \mathrm{Ob}(\mathcal{X})$, there is an *equality* $p_{\mathcal{X}}(a) = p_{\mathcal{Y}}(f(a))$ of objects in $\mathcal{S}$, and for every morphism $\alpha\colon a \to b$, $p_{\mathcal{X}}(\alpha) = p_{\mathcal{Y}}(f(\alpha))$.

(2) If $f, g\colon \mathcal{X} \to \mathcal{Y}$ are morphisms of prestacks, a *2-isomorphism* (or *2-morphism*) $\alpha\colon f \to g$ is a natural transformation $\alpha\colon f \to g$ such that for every object $a \in \mathcal{X}$, the morphism $\alpha_a\colon f(a) \to g(a)$ in $\mathcal{Y}$ (which is an isomorphism) is over the identity in $\mathcal{S}$. We often describe the 2-isomorphism α schematically as

$$\begin{tikzcd} \mathcal{X} \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathcal{Y}. \end{tikzcd}$$

(3) We define the category $\mathrm{Mor}(\mathcal{X}, \mathcal{Y})$ whose objects are morphisms of prestacks and whose morphisms are 2-isomorphisms.

(4) A *2-commutative diagram* (which we often call simply a *commutative diagram*) is a diagram

$$\begin{tikzcd} \mathcal{X}' \arrow[r, "f'"] \arrow[d, "g'"'] \arrow[dr, phantom, "\swarrow\alpha"] & \mathcal{Y}' \arrow[d, "g"] \\ \mathcal{X} \arrow[r, "f"] & \mathcal{Y} \end{tikzcd}$$

together with a 2-isomorphism $\alpha\colon g \circ f' \xrightarrow{\sim} f \circ g'$.

(5) A morphism $f\colon \mathcal{X} \to \mathcal{Y}$ of prestacks is a *monomorphism* (resp., *epimorphism*) if f is fully faithful (resp., essentially surjective), and f is an *isomorphism* if there exists a morphism $g\colon \mathcal{Y} \to \mathcal{X}$ of prestacks and 2-isomorphisms $g \circ f \xrightarrow{\sim} \mathrm{id}_{\mathcal{X}}$ and $f \circ g \xrightarrow{\sim} \mathrm{id}_{\mathcal{Y}}$.

Exercise 2.4.18 (easy)**.** Show that every 2-isomorphism is an isomorphism of functors, or in other words that $\mathrm{Mor}(\mathcal{X}, \mathcal{Y})$ is a groupoid.

Exercise 2.4.19 (details)**.** Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a morphism of prestacks over a category $\mathcal{S}$.

(a) Show that f is a monomorphism if and only if $f_S\colon \mathcal{X}(S) \to \mathcal{Y}(S)$ is fully faithful for every $S \in \mathcal{S}$.

(b) Show that f is an isomorphism if and only if f is fully faithful and essentially surjective.

A prestack $\mathcal{X}$ is *equivalent to a presheaf* if there is a presheaf F and an isomorphism between $\mathcal{X}$ and the prestack $\mathcal{X}_F$ corresponding to F (see Example 2.4.7).

Exercise 2.4.20 (good practice)**.** Show that G acts freely on U (i.e., the action map $(\sigma, p_2)\colon G \times_S U \to U \times_S U$ is a monomorphism) if and only if $[U/G]^{\mathrm{pre}}$ (resp., $[U/G]$) is equivalent to a presheaf. We denote these presheaves by $(U/G)^{\mathrm{pre}}$ and U/G.

2.4.5 The 2-Yoneda Lemma

The Yoneda Lemma (0.3.12) states that for a presheaf $F\colon \mathcal{S} \to \mathrm{Sets}$ on a category $\mathcal{S}$ and an object $S \in \mathcal{S}$, there is a bijection $\mathrm{Mor}(S, F) \xrightarrow{\sim} F(S)$. In particular, there is a fully faithful embedding $\mathcal{S} \to \mathrm{Pre}(\mathcal{S})$, from $\mathcal{S}$ into the category of presheaves on $\mathcal{S}$, given by $S \mapsto \mathrm{Mor}(-, S)$. We will need an analog of Yoneda's lemma for prestacks. Recall from Example 2.4.8 that an object $S \in \mathcal{S}$ can be viewed as a prestack over $\mathcal{S}$, which we also denote by S, whose objects over $T \in \mathcal{S}$ are morphisms $T \to S$ and a morphism $(T \to S) \to (T' \to S)$ is an S-morphism $T \to T'$.

Lemma 2.4.21 (The 2-Yoneda Lemma)**.** *Let $\mathcal{X}$ be a prestack over a category $\mathcal{S}$ and $S \in \mathcal{S}$. The functor*

$$\mathrm{Mor}(S, \mathcal{X}) \to \mathcal{X}(S), \qquad f \mapsto f_S(\mathrm{id}_S)$$

is an equivalence of categories.

Proof. We will construct a quasi-inverse $\Psi\colon \mathcal{X}(S) \to \mathrm{Mor}(S, \mathcal{X})$ as follows.

On objects: For $a \in \mathcal{X}(S)$, we define $\Psi(a)\colon S \to \mathcal{X}$ as the morphism of prestacks sending an object $(f\colon T \to S)$ (of the prestack corresponding to S) over T to a *choice* of pullback $f^*a \in \mathcal{X}(T)$ and a morphism $(f'\colon T' \to S) \to (f\colon T \to S)$, given by an S-morphism $g\colon T' \to T$, to the morphism $f'^*a \to f^*a$ uniquely filling in the diagram

$$\begin{tikzcd}
f'^*a \arrow[r, dashed] \arrow[d] \arrow[rr, bend left] & f^*a \arrow[r] \arrow[d, mapsto] & a \arrow[d, mapsto] \\
T' \arrow[r, "g"] & T \arrow[r, "f"] & S,
\end{tikzcd}$$

using Axiom (2) of a prestack.

On morphisms: If $\alpha\colon a' \to a$ is a morphism in $\mathcal{X}(S)$, then $\Psi(\alpha)\colon \Psi(a') \to \Psi(a)$ is defined as the morphism of functors which maps a morphism $f\colon T \to S$ (i.e., an object in S over T) to the unique morphism $f^*a' \to f^*a$ filling in the diagram

$$\begin{tikzcd}
f^*a' \arrow[r, dashed] \arrow[d] & f^*a \arrow[d] \\
a' \arrow[r, "\alpha"] & a
\end{tikzcd}
\quad \text{over} \quad
\begin{tikzcd}
T \arrow[d, "f"] \\
S
\end{tikzcd}$$

using again Axiom (2) of a prestack.

We leave the verification that Ψ is a quasi-inverse to the reader. $\square$

Caution 2.4.22. We will use the 2-Yoneda Lemma, often without mention, throughout these notes in passing between morphisms $S \to \mathcal{X}$ and objects of $\mathcal{X}$ over S.

Remark 2.4.23. If $f, g\colon S \to \mathcal{X}$ are morphisms from a scheme S, corresponding via the 2-Yoneda Lemma to objects $a, b \in \mathcal{X}(S)$, then a 2-isomorphism $\alpha\colon f \xrightarrow{\sim} g$ corresponds to an isomorphism $a \xrightarrow{\sim} b$ in $\mathcal{X}(S)$.

Remark 2.4.24 (Universal families). When $F\colon \mathrm{Sch} \to \mathrm{Sets}$ is the functor $h_M = \mathrm{Mor}(-, M)$ representable by a scheme M, the (usual) Yoneda Lemma (0.3.12) gives a bijection $\mathrm{Mor}(h_M, F) \cong F(M)$ and the object $U \in F(M)$ corresponding to the identity map is a universal family (see §0.3.4). The Yoneda 2-Lemma does not immediately apply to give a universal family as the category $\mathcal{X}(\mathcal{X})$ of objects of $\mathcal{X}$ over $\mathcal{X}$ needs to be defined. Later in §3.1.7, we prove a Generalized 2-Yoneda Lemma (3.1.25) for algebraic stacks and use it to define universal families.

Remark 2.4.25 (Cleavages and splittings). In the proof of the Yoneda 2-Lemma, we made a *choice of pullbacks* $f^*\colon \mathcal{X}(S) \to \mathcal{X}(T)$ for every morphism $T \to S$ as in [SP, Tag 02XN]. This is often called a *cleavage* in the literature [SGA1, VI.7], [Vis05, Def. 3.9]. A choice of pullbacks defines a *pseudo-functor* $\mathcal{S} \to \mathrm{Groupoids}$. If for a composition $T \xrightarrow{f} S \xrightarrow{g} U$, the functors $(g \circ f)^*$ and $f^* \circ g^*$ are *equal*, then it is often said that this choice of pullbacks defines a *splitting* or that $\mathcal{X}$ is *split*. In this case, the induced pseudo-functor $\mathcal{S} \to \mathrm{Groupoids}$ is a strict functor. Every prestack (or more generally fibered category) is equivalent in the 2-categorical sense to a split prestack [SP, Tag 004A]. Do not worry; we will not make use of this terminology.

Example 2.4.26 (Quotient stack presentations)**.** Consider the prestack $[U/G]$ in Definition 2.4.15 arising from a group action $\sigma\colon G \times_S U \to U$. The object of $[U/G]$ over U given by the diagram

$$\begin{tikzcd} G \times_S U \arrow[r, "\sigma"] \arrow[d, "p_2"] & U \\ U & \end{tikzcd}$$

corresponds via the 2-Yoneda Lemma (2.4.21) to a morphism $U \to [U/G]$. Later we will verify that $U \to [U/G]$ is surjective, smooth, and representable, or in other words a *smooth presentation*. This will verify that $[U/G]$ is an algebraic stack (Theorem 3.1.10).

Exercise 2.4.27 (good practice)**.**

(a) Show that there is a morphism $p\colon U \to [U/G]^{\mathrm{pre}}$ and a 2-commutative diagram

$$\begin{tikzcd} G \times_S U \arrow[r, "\sigma"] \arrow[d, "p_2"] \arrow[dr, phantom, "\not\Downarrow\alpha"] & U \arrow[d, "p"] \\ U \arrow[r, "p"] & {[U/G]^{\mathrm{pre}}}. \end{tikzcd}$$

(b) Show that $U \to [U/G]^{\mathrm{pre}}$ is a categorical quotient among prestacks, i.e., for every 2-commutative diagram

$$\begin{tikzcd} G \times_S U \arrow[r, "\sigma"] \arrow[d, "p_2"'] \arrow[dr, phantom, "\not\Downarrow\alpha"] & U \arrow[d, "p"] \arrow[ddr, bend left, "\varphi"] & \\ U \arrow[r, "p"] \arrow[drr, bend right, "\varphi"'] & {[U/G]^{\mathrm{pre}}} \arrow[dr, phantom, "\not\Downarrow\tau"] & \\ & & \mathcal{Z} \end{tikzcd}$$

of prestacks, there exists a morphism $\chi\colon [U/G]^{\mathrm{pre}} \to \mathcal{Z}$ and a 2-isomorphism $\beta\colon \varphi \xrightarrow{\sim} \chi \circ p$ which is compatible with α and τ (i.e., the two natural transformations $\varphi \circ \sigma \xrightarrow{\beta \circ \sigma} \chi \circ p \circ \sigma \xrightarrow{\chi \circ \alpha} \chi \circ p \circ p_2$ and $\varphi \circ \sigma \xrightarrow{\tau} \varphi \circ p_2 \xrightarrow{\beta \circ p_2} \chi \circ p \circ p_2$ agree. Show that χ is unique up to unique 2-isomorphism.

2.4.6 Warmup: fiber products of groupoids

Fiber products of prestacks brings out many of the 2-categorical subtleties. As understanding fiber products is essential to working with algebraic stacks, we recommend the reader spend time working out many examples. It is instructive to begin with fiber products of groupoids.

Construction 2.4.28. Let $f\colon \mathcal{C} \to \mathcal{D}$ and $g\colon \mathcal{D}' \to \mathcal{D}$ be functors of groupoids. Define the groupoid $\mathcal{C} \times_{\mathcal{D}} \mathcal{D}'$ as the category of triples (c, d', γ) where $c \in \mathcal{C}$ and $d' \in \mathcal{D}'$ are objects, and $\gamma\colon f(c) \xrightarrow{\sim} g(d')$ is an isomorphism in $\mathcal{D}$. A morphism $(c_1, d_1', \gamma_1) \to (c_2, d_2', \gamma_2)$ is the data of morphisms $\chi\colon c_1 \xrightarrow{\sim} c_2$ and $\lambda\colon d_1' \xrightarrow{\sim} d_2'$ such that

$$\begin{tikzcd}
f(c_1) \arrow[r, "f(\chi)"] \arrow[d, "\gamma_1"] & f(c_2) \arrow[d, "\gamma_2"] \\
g(d_1') \arrow[r, "g(\lambda)"] & g(d_2')
\end{tikzcd}$$

commutes.

Exercise 2.4.29 (details)**.** Formulate a university property for fiber products of groupoids and show that $\mathcal{C} \times_{\mathcal{D}} \mathcal{D}'$ satisfies it.

The following foreshadows important cartesian diagrams involving quotient stacks.

Exercise 2.4.30 (important, good practice)**.** Let G be a group acting on a set U via $\sigma\colon G \times U \to U$. Let $[U/G]$ denote the quotient groupoid as defined in Exercise 0.4.8: objects are elements $u \in U$ and a morphism $u \to u'$ is an element $g \in G$ with $u' = gu$. Let $p\colon U \to [U/G]$ denote the projection.

(a) Let P be a set with a free action of G and quotient $T = P/G$. If $f\colon P \to U$ is a G-equivariant map, show that there is a cartesian diagram

$$\begin{tikzcd}
P \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, "\square"] & U \arrow[d, "p"] \\
T \arrow[r] & {[U/G]},
\end{tikzcd}$$

i.e., that P is equivalent to the fiber product.

(b) Show that there are cartesian diagrams

$$\begin{tikzcd}
G \times U \arrow[r, "\sigma"] \arrow[d, "p_2"] \arrow[dr, phantom, "\square"] & U \arrow[d, "p"] \\
U \arrow[r, "p"] & {[U/G]}
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
G \times U \arrow[r, "(\sigma, p_2)"] \arrow[d] \arrow[dr, phantom, "\square"] & U \times U \arrow[d, "p \times p"] \\
{[U/G]} \arrow[r, "\Delta"] & {[U/G] \times [U/G]}.
\end{tikzcd}$$

Exercise 2.4.31 (good practice)**.** Recall from Example 0.4.3 that the classifying groupoid BG of a group G is the category with one object $*$ with $\mathrm{Mor}(*, *) = G$.

(a) Let $\phi\colon H \to G$ be a homomorphism of groups. Show that there is an induced morphism $BH \to BG$ of groupoids and that $BH \times_{BG} \mathrm{pt} \cong [G/H]$, where pt denotes the groupoid with one object and one morphism.

(b) If $K \trianglelefteq G$ is a normal subgroup with quotient $Q = G/K$, show that there is a cartesian diagram

$$\begin{tikzcd} Q \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & BK \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathrm{pt} \arrow[d] \\ \mathrm{pt} \arrow[r] & BG \arrow[r] & BQ. \end{tikzcd}$$

(c) Let G be a group acting on a set U and let $[U/G]$ be the groupoid quotient. If $u \in U$ is a point with stabilizer G_u and orbit Gus, show that there is a morphism $BG_u \to [U/G]$ of groupoids and a cartesian diagram

$$\begin{tikzcd} Gu \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & U \arrow[d, "p"] \\ BG_u \arrow[r] & {[U/G]}. \end{tikzcd}$$

Exercise 2.4.32 (important)**.** Show that a groupoid $\mathcal{C}$ is equivalent to a set if and only if $\mathcal{C} \to \mathcal{C} \times \mathcal{C}$ is fully faithful.

2.4.7 Fiber products of prestacks

The fiber product of morphisms $\mathcal{X} \to \mathcal{Y}$ and $\mathcal{Y}' \to \mathcal{Y}$ of prestacks over a category $\mathcal{S}$ is the prestack $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}'$ whose fiber category over $S \in \mathcal{S}$ is the fiber product $\mathcal{X}(S) \times_{\mathcal{Y}(S)} \mathcal{Y}'(S)$ of groupoids.

Construction 2.4.33. Let $f\colon \mathcal{X} \to \mathcal{Y}$ and $g\colon \mathcal{Y}' \to \mathcal{Y}$ be morphisms of prestacks over a category $\mathcal{S}$. Define the prestack $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}'$ over $\mathcal{S}$ as the category of triples (x, y', γ) where $x \in \mathcal{X}$ and $y' \in \mathcal{Y}'$ are objects over the *same* object $S := p_{\mathcal{X}}(x) = p_{\mathcal{Y}'}(y') \in \mathcal{S}$, and $\gamma\colon f(x) \xrightarrow{\sim} g(y')$ is an isomorphism in $\mathcal{Y}(S)$. A morphism $(x_1, y_1', \gamma_1) \to (x_2, y_2', \gamma_2)$ consists of a triple (h, χ, λ) where $h\colon p_{\mathcal{X}}(x_1) = p_{\mathcal{Y}'}(y_1') \to p_{\mathcal{Y}'}(y_2') = p_{\mathcal{X}}(x_2)$ is a morphism in $\mathcal{S}$, and $\chi\colon x_1 \to x_2$ and $\lambda\colon y_1' \to y_2'$ are morphisms in $\mathcal{X}$ and $\mathcal{Y}'$ over h such that

$$\begin{tikzcd} f(x_1) \arrow[r, "f(\chi)"] \arrow[d, "\gamma_1"] & f(x_2) \arrow[d, "\gamma_2"] \\ g(y_1') \arrow[r, "g(\lambda)"] & g(y_2') \end{tikzcd}$$

commutes.

Letting $p_1\colon \mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}' \to \mathcal{X}$ and $p_2\colon \mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}' \to \mathcal{Y}'$ denote the projections $(x, y', \gamma) \mapsto x$ and $(x, y', \gamma) \mapsto y'$, define the 2-isomorphism $\alpha\colon f \circ p_1 \xrightarrow{\sim} g \circ p_2$, which is defined on an object $(x, y', \gamma) \in \mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}'$ by setting $\alpha_{(x,y',\gamma)} := \gamma\colon f(x) \to g(y')$. This yields a 2-commutative diagram

$$\begin{tikzcd} \mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}' \arrow[r, "p_2"] \arrow[d, "p_1"] \arrow[dr, phantom, "\alpha \nearrow"] & \mathcal{Y}' \arrow[d, "g"] \\ \mathcal{X} \arrow[r, "f"] & \mathcal{Y}. \end{tikzcd} \tag{2.4.34}$$

Theorem 2.4.35. *The prestack $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}'$ together with the morphisms p_1 and p_2 and the 2-isomorphism α as in (2.4.34) satisfy the following universal property: for every 2-commutative diagram*

$$\begin{tikzcd}
 & & \mathcal{Y}' \arrow[dr, "g"] & \\
\mathcal{T} \arrow[urr, bend left, "q_2"] \arrow[drr, bend right, "q_1"'] \arrow[r, phantom, "\tau\Uparrow"] & \mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}' \arrow[ur, "p_2"] \arrow[dr, "p_1"'] \arrow[rr, phantom, "\alpha\Uparrow"] & & \mathcal{Y} \\
 & & \mathcal{X} \arrow[ur, "f"'] &
\end{tikzcd}$$

with 2-isomorphism $\tau\colon f \circ q_1 \xrightarrow{\sim} g \circ q_2$, there exists a morphism $h\colon \mathcal{T} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}'$ and 2-isomorphisms $\beta\colon q_1 \to p_1 \circ h$ and $\rho\colon q_2 \to p_2 \circ h$ yielding a 2-commutative diagram

$$\begin{tikzcd}
 & & \mathcal{Y}' \arrow[dr, "g"] & \\
\mathcal{T} \arrow[urr, bend left, "q_2"] \arrow[drr, bend right, "q_1"'] \arrow[r, "h"] \arrow[urr, phantom, "\Downarrow\rho"] \arrow[drr, phantom, "\Uparrow\beta"] & \mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}' \arrow[ur, "p_2"] \arrow[dr, "p_1"'] \arrow[rr, phantom, "\alpha\Uparrow"] & & \mathcal{Y} \\
 & & \mathcal{X} \arrow[ur, "f"'] &
\end{tikzcd}$$

such that

$$\begin{tikzcd}
f \circ q_1 \arrow[r, "f(\beta)"] \arrow[d, "\tau"] & f \circ p_1 \circ h \arrow[d, "\alpha \circ h"] \\
g \circ q_2 \arrow[r, "g(\rho)"] & g \circ p_2 \circ h
\end{tikzcd}$$

commutes. The data (h, β, ρ) is unique up to unique isomorphism.

Proof. We define $h\colon \mathcal{T} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}'$ on objects by $t \mapsto \big(q_1(t), q_2(t), \tau_t\colon f(q_1(t)) \xrightarrow{\sim} g(q_2(t))\big)$ and on morphisms as $(\Psi\colon t \to t') \mapsto (p_{\mathcal{T}}(\Psi), q_1(\Psi), q_2(\Psi))$. There are *equalities* of functors $q_1 = p_1 \circ h$ and $q_2 = p_2 \circ h$ so we can define β and ρ as the identity natural transformations. The remaining details are left to the reader. $\square$

Definition 2.4.36. We say that a 2-commutative diagram

$$\begin{tikzcd}
\mathcal{X}' \arrow[r] \arrow[d] \arrow[dr, phantom, "\alpha\Nearrow"] & \mathcal{Y}' \arrow[d] \\
\mathcal{X} \arrow[r] & \mathcal{Y}.
\end{tikzcd}$$

is *cartesian* if it satisfies the universal property of Theorem 2.4.35. We often write a cartesian diagram of prestacks as

$$\begin{tikzcd}
\mathcal{X}' \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{Y}' \arrow[d] \\
\mathcal{X} \arrow[r] & \mathcal{Y},
\end{tikzcd}$$

where the existence of the 2-isomorphism α is implicit.

When $\mathcal{X}$ and $\mathcal{Y}$ are prestacks over Sch, then $\mathcal{X} \times_{\mathrm{Sch}} \mathcal{Y} = \mathcal{X} \times_{\operatorname{Spec} \mathbb{Z}} \mathcal{Y}$, which we sometimes abbreviate as $\mathcal{X} \times \mathcal{Y}$. Likewise, when working over a field $\Bbbk$, $\mathcal{X} \times_{\mathrm{Sch}/\Bbbk} \mathcal{Y} = \mathcal{X} \times_{\operatorname{Spec} \Bbbk} \mathcal{Y}$ is abbreviated as $\mathcal{X} \times_{\Bbbk} \mathcal{Y}$ or, if there is no possible confusion, as $\mathcal{X} \times \mathcal{Y}$.

2.4.8 Examples

The following exercise is essential for verifying the algebraicity and establishing properties of quotient stacks (e.g., see Theorem 3.1.10).

Exercise 2.4.37 (important, good practice)**.** Let $G \to S$ be a smooth affine group scheme acting on a scheme U over S via $\sigma \colon G \times_S U \to U$, and let $[U/G]$ be the quotient stack (Definition 2.4.15).

(a) Let $T \to [U/G]$ be a morphism corresponding via the 2-Yoneda Lemma (2.4.21) to a principal G-bundle $P \to T$ and a G-equivariant map $f \colon P \to U$. Show that there is a cartesian diagram

$$\begin{tikzcd} P \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, "\square"] & U \arrow[d] \\ T \arrow[r] & {[U/G]}. \end{tikzcd}$$

(We will later see that $[U/G]$ is an algebraic stack and that $U \to [U/G]$ is principal G-bundle (Theorem 3.1.10). This will allow us to identify the principal G-bundle $U \to [U/G]$ together with the identity map $U \to U$ as the universal family over $[U/G]$, corresponding via the 2-Yoneda Lemma to the identity map $[U/G] \to [U/G]$.)

(b) Show that there are cartesian diagrams

$$\begin{tikzcd} G \times_S U \arrow[r, "\sigma"] \arrow[d, "p_2"] \arrow[dr, phantom, "\square"] & U \arrow[d, "p"] \\ U \arrow[r, "p"] & {[U/G]} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} G \times_S U \arrow[r, "{(\sigma, p_2)}"] \arrow[d] \arrow[dr, phantom, "\square"] & U \times_S U \arrow[d, "p \times p"] \\ {[U/G]} \arrow[r, "\Delta"] & {[U/G] \times_S [U/G]}. \end{tikzcd}$$

The diagram in the next exercise is utilized extensively, just as it is in the case of schemes. It will be used to define stabilizers and the inertia stack in §3.2.2.

Exercise 2.4.38 (important)**.** Let $\mathcal{X}$ be a prestack over a category $\mathcal{S}$. Let $S, T \in \mathcal{S}$ be objects which we can view as prestacks over $\mathcal{S}$ via Example 2.4.9. Show that for all morphisms $a \colon S \to \mathcal{X}$ and $b \colon T \to \mathcal{X}$, there is a cartesian diagram

$$\begin{tikzcd} S \times_{\mathcal{X}} T \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & S \times T \arrow[d, "a \times b"] \\ \mathcal{X} \arrow[r, "\Delta"] & \mathcal{X} \times \mathcal{X}, \end{tikzcd}$$

where the fiber products $S \times T$ and $\mathcal{X} \times \mathcal{X}$ are taken over $\mathcal{S}$.

Exercise 2.4.39 (Isom presheaves, important)**.** Let $\mathcal{X}$ be a prestack over a category $\mathcal{S}$. For $S \in \mathcal{S}$, recall from Example 2.2.12 that the restricted category $\mathcal{S}/S$ denotes the category whose objects are morphisms $T \to S$ in $\mathcal{S}$ and whose morphisms are S-morphisms.

(a) Show that for objects a and b of $\mathcal{X}$ over S that the functor

$$\begin{aligned} \underline{\mathrm{Isom}}_{\mathcal{X}(S)}(a,b)\colon \mathcal{S}/S &\to \mathrm{Sets} \\ (T \xrightarrow{f} S) &\mapsto \mathrm{Mor}_{\mathcal{X}(T)}(f^*a, f^*b), \end{aligned}$$

where f^*a and f^*b are choices of pullbacks, defines a presheaf on $\mathcal{S}/S$.

(b) Show that there is a cartesian diagram

$$\begin{tikzcd} \underline{\mathrm{Isom}}_{\mathcal{X}(S)}(a,b) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & S \arrow[d, "(a,b)"] \\ \mathcal{X} \arrow[r, "\Delta"] & \mathcal{X} \times \mathcal{X}. \end{tikzcd}$$

(c) Show that the presheaf $\underline{\mathrm{Aut}}_{\mathcal{X}(T)}(a) = \underline{\mathrm{Isom}}_{\mathcal{X}(T)}(a,a)$ is naturally a presheaf in groups.

(d) Show that $\mathcal{X}$ is equivalent to a presheaf if and only if the diagonal $\mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is fully faithful.

Exercise 2.4.40 (good practice)**.**

(a) If $H \to G$ is a morphism of smooth affine group schemes over a scheme S, define a morphism of prestacks $BH \to BG$ over Sch/S by using Definition B.1.60 to construct a principal G-bundle from a principal H-bundle.

(b) Show that $BH \times_{BG} S \cong [G/H]$.

(c) If $1 \to K \to G \to Q \to 1$ is an exact sequence of smooth affine algebraic groups over a field $\Bbbk$, show that there is a cartesian diagram

$$\begin{tikzcd} Q \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & BK \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \operatorname{Spec} \Bbbk \arrow[d] \\ \operatorname{Spec} \Bbbk \arrow[r] & BG \arrow[r] & BQ. \end{tikzcd}$$

Exercise 2.4.41 (good practice)**.** Let G and H be smooth affine group schemes over a scheme S.

(a) Show that $B(G \times_S H) \cong BG \times_S BH$.

(b) If X and Y are S-schemes with a actions by G and H, show that $[(X \times_S Y)/(G \times_S H)] \cong [X/G] \times_S [Y/H]$.

(c) Conclude that $[\mathbb{A}^n/\mathbb{G}_m^n] \cong \underbrace{[\mathbb{A}^1/\mathbb{G}_m] \times \cdots \times [\mathbb{A}^1/\mathbb{G}_m]}_{n \text{ times}}$ over $\operatorname{Spec} \mathbb{Z}$.

Exercise 2.4.42 (important, good practice)**.** Analogous to the prestack $\mathcal{M}_g$ of smooth curves (Example 2.4.10), let $\mathcal{M}_{g,1}$ be the prestack, where an object over a scheme S is a family of smooth curves $\mathcal{C} \to S$ and a section $\sigma\colon S \to \mathcal{C}$. Let $\mathcal{M}_{g,1} \to \mathcal{M}_g$ be the morphism of prestacks forgetting the section. Show that if $S \to \mathcal{M}_g$ is a morphism corresponding to a family of curves $\mathcal{C} \to S$, there is a cartesian diagram

$$\begin{tikzcd} \mathcal{C} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{M}_{g,1} \arrow[d] \\ S \arrow[r] & \mathcal{M}_g. \end{tikzcd}$$

In other words, $\mathcal{M}_{g,1} \to \mathcal{M}_g$ is the universal family, as defined later in Definition 3.1.27.

Exercise 2.4.43 (good practice)**.** Let H and G be smooth affine group schemes over a scheme S. Let $\underline{\mathrm{Hom}}(H, G)$ be the sheaf on $(\mathrm{Sch}/S)_{\text{ét}}$ whose sections over an S-scheme T are homomorphisms $H_T \to G_T$, and let $\underline{\mathrm{Mor}}(BH, BG)$ be the prestack over $\mathcal{S}$ whose objects over $T \in \mathcal{S}$ are morphisms $B(H_T) \to B(G_T)$. Show that

$$\underline{\mathrm{Mor}}(BH, BG) \cong [\underline{\mathrm{Hom}}(H, G)/G],$$

where G acts via conjugation.

2.5 Stacks

An absence of proof is a challenge; an absence of definition is deadly.

Pierre Deligne

A stack over a site $\mathcal{S}$ is a prestack $\mathcal{X}$ where the objects and morphisms glue uniquely in the Grothendieck topology of $\mathcal{S}$.

2.5.1 The definition

Given a covering $\{S_i \to S\}$ in a site, we will use the convention that S_{ij} denotes $S_i \times_S S_j$ and S_{ijk} denotes $S_i \times_S S_j \times_S S_k$.

Definition 2.5.1 (Stacks)**.** A prestack $\mathcal{X}$ over a site $\mathcal{S}$ is a *stack* if the following conditions hold for all coverings $\{S_i \to S\}$ of an object $S \in \mathcal{S}$:

(1) (morphisms glue) For objects a and b in $\mathcal{X}$ over S and morphisms $\phi_i \colon a|_{S_i} \to b$ such that $\phi_i|_{S_{ij}} = \phi_j|_{S_{ij}}$ as displayed in the diagram

$$\begin{tikzcd}
& a|_{S_i} \arrow[dr] \arrow[drr, bend left, "\phi_i"] & & \\
a|_{S_{ij}} \arrow[ur] \arrow[dr] & & a \arrow[r, dashed] & b \\
& a|_{S_j} \arrow[ur] \arrow[urr, bend right, "\phi_j"'] & &
\end{tikzcd}
\quad \text{over} \quad
\begin{tikzcd}
& S_i \arrow[dr] & \\
S_{ij} \arrow[ur] \arrow[dr] & & S, \\
& S_j \arrow[ur] &
\end{tikzcd}$$

there exists a unique morphism $\phi \colon a \to b$ over id_S with $\phi|_{S_i} = \phi_i$.

(2) (objects glue) For objects a_i over S_i and isomorphisms $\alpha_{ij} \colon a_i|_{S_{ij}} \to a_j|_{S_{ij}}$, as displayed in the diagram

$$\begin{tikzcd}
& a_i \arrow[dr] & \\
a_i|_{S_{ij}} \arrow[ur] \arrow[d, "\alpha_{ij}"] & & a \\
a_j|_{S_{ij}} \arrow[dr] & & \\
& a_j \arrow[uur] &
\end{tikzcd}
\quad \text{over} \quad
\begin{tikzcd}
& S_i \arrow[dr] & \\
S_{ij} \arrow[ur] \arrow[dr] & & S \\
& S_j \arrow[ur] &
\end{tikzcd}$$

satisfying the cocycle condition $\alpha_{jk}|_{S_{ijk}} \circ \alpha_{ij}|_{S_{ijk}} = \alpha_{ik}|_{S_{ijk}}$ on S_{ijk}, there exists an object a over S and isomorphisms $\phi_i \colon a|_{S_i} \to a_i$ over id_{S_i} such that $\phi_j|_{S_{ij}} = \alpha_{ij} \circ \phi_i|_{S_{ij}}$ on S_{ij}.

A *morphism* of stacks is a morphism of prestacks.

Remark 2.5.2. If the covering consists of a single map $S' \to S$ and $a' \in \mathcal{X}$ is an object over S', the cocycle condition for an isomorphism $\alpha\colon p_1^* a' \xrightarrow{\sim} p_2^* a'$ over $S' \times_S S'$ translates to the commutativity of

$$\begin{tikzcd}[column sep=small]
 & p_{12}^* p_1^* a \arrow[r, "p_{12}^*\alpha"] & p_{12}^* p_2^* a \arrow[dr, equal] & \\
p_{13}^* p_1^* a \arrow[ur, equal] \arrow[dr, "p_{13}^*\alpha"'] & & & p_{23}^* p_1^* a \arrow[dl, "p_{23}^*\alpha"] \\
 & p_{13}^* p_2^* a \arrow[r, equal] & p_{23}^* p_2^* a &
\end{tikzcd}$$

over $S' \times_S S' \times_S S'$. Axiom (2) requires the existence of an object $a \in \mathcal{X}(S)$ and an isomorphism $\phi\colon a' \xrightarrow{\sim} a|_{S'}$ satisfying $p_1^*\phi = p_2^*\phi \circ \alpha$.

Remark 2.5.3. Analogous to the sheaf axiom of a presheaf $F\colon \mathcal{S} \to \mathrm{Sets}$ requiring that $F(S) \to \prod_i F(S_i) \rightrightarrows \prod_{i,j} F(S_i \times_S S_j)$ is exact for coverings $\{S_i \to S\}$, the stack axioms can be interpreted as the 'exactness' of

$$\mathcal{X}(S) \longrightarrow \textstyle\prod_i \mathcal{X}(S_i) \rightrightarrows \prod_{i,j} \mathcal{X}(S_i \times_S S_j) \Rrightarrow \prod_{i,j,k} \mathcal{X}(S_i \times_S S_j \times_S S_k).$$

Exercise 2.5.4. Show that Axiom (1) is equivalent to the condition that for all objects a and b of $\mathcal{X}$ over $S \in \mathcal{S}$, the Isom presheaf $\underline{\mathrm{Isom}}_{\mathcal{X}(S)}(a, b)$ (see Exercise 2.4.39) is a sheaf on $\mathcal{S}/S$.

Exercise 2.5.5. Generalizing Exercise 2.3.7 from sheaves to stacks, show that a prestack $\mathcal{X}$ over Sch is a stack over $\mathrm{Sch}_{\text{ét}}$ (resp., $\mathrm{Sch}_{\text{fppf}}$, $\mathrm{Sch}_{\text{fpqc}}$) if and only if $\mathcal{X}$ is a stack over $\mathrm{Sch}_{\text{Zar}}$ and Axioms (1) and (2) hold for a surjective étale (resp., fppf, fpqc) morphism $\operatorname{Spec} A' \to \operatorname{Spec} A$ of affine schemes.

Exercise 2.5.6 (Fiber product of stacks)**.** Show that if $\mathcal{X} \to \mathcal{Y}$ and $\mathcal{Y}' \to \mathcal{Y}$ are morphisms of stacks over a site $\mathcal{S}$, then $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}'$ is also a stack over $\mathcal{S}$.

2.5.2 First examples of stacks

Example 2.5.7 (Sheaves and schemes as stacks)**.** A presheaf F on a site $\mathcal{S}$ defines a prestack $\mathcal{X}_F$ over $\mathcal{S}$ whose objects are pairs (a, S) where $S \in \mathcal{S}$ and $a \in F(S)$ (see Example 2.4.7), and F is a sheaf if and only if $\mathcal{X}_F$ is a stack. We often abuse notation by writing F also as the stack $\mathcal{X}_F$.

Since a scheme X is a sheaf on $\mathrm{Sch}_{\text{ét}}$ (Proposition 2.3.8), the prestack Sch/X—often denoted simply as X—whose objects over a scheme S are morphisms $S \to X$, is a stack over $\mathrm{Sch}_{\text{ét}}$.

Example 2.5.8 (Stacks of sheaves)**.** Let $\underline{\mathrm{Sheaves}}$ be the prestack over Sch whose objects are pairs (T, F) where T is a scheme and F is a sheaf on the Zariski topology of T. A morphism $(T', F') \to (T, F)$ is pair (f, α) where $f\colon T' \to T$ is a map of schemes and $\alpha\colon f^{-1}F \to F'$ is an isomorphism of sheaves. The projection $\underline{\mathrm{Sheaves}} \to \mathrm{Sch}$ is defined by $(T, F) \mapsto T$. Because sheaves and their morphisms glue in the Zariski topology [Har77, Exc. II.1.15 and 22], $\mathcal{X}$ is a stack over the big Zariski site $\mathrm{Sch}_{\text{Zar}}$.

Example 2.5.9 (Stack of quasi-coherent sheaves)**.** Define the $\underline{\mathrm{QCoh}}$, $\underline{\mathrm{Coh}}$, and $\mathcal{B}un$ as the category over Sch consisting of pairs (T, F) where F is a quasi-coherent sheaf (resp., finitely presented, quasi-coherent sheaf, vector bundle) on a scheme T and a morphism $(T, F) \to (T', F')$ is a map $f : T \to T'$ and an isomorphism $\alpha : f^*F' \to F$. To see that $\underline{\mathrm{QCoh}}$ is a stack over $\mathrm{Sch}_{\mathrm{fpqc}}$, by Exercise 2.5.5, it suffices to verify Axioms (1) and (2) with respect to an fpqc map $T' \to T$, and these translate literally to the two parts of Fpqc Descent of Quasi-Coherent Sheaves (2.1.4(2)). In particular, $\underline{\mathrm{QCoh}}$ is a stack over $\mathrm{Sch}_{\text{ét}}$. By Fpqc Descent of Properties of Quasi-Coherent Sheaves (2.1.18), $\underline{\mathrm{Coh}}$ and $\mathcal{B}un$ are also stacks over $\mathrm{Sch}_{\mathrm{fpqc}}$ and $\mathrm{Sch}_{\text{ét}}$.

Exercise 2.5.10 (not important)**.** Show that the prestack $\underline{\mathrm{Mod}}$ parameterizing pairs (T, F) where T is a scheme and F is a sheaf of $\mathcal{O}_T$-modules is *not* a stack over $\mathrm{Sch}_{\text{ét}}$.

Exercise 2.5.11 (not easy but also not so important)**.** Define the prestack of sheaves over any site and apply Exercises 2.3.9 and 2.3.10 to conclude that it is a stack.

Example 2.5.12 (Stack of schemes)**.** Define $\underline{\mathrm{Schemes}}$ as the prestack over Sch consisting of morphisms $X \to S$ of schemes where a morphism $(X' \to S') \to (X \to S)$ consists of morphisms $X' \to X$ and $S' \to S$ that forms a cartesian diagram. The projection $\underline{\mathrm{Schemes}} \to \mathrm{Sch}$ takes $X \to S$ to S. Since schemes glue in the Zariski topology [Har77, Exc. II.2.12], $\underline{\mathrm{Schemes}}$ is a stack over $\mathrm{Sch}_{\mathrm{Zar}}$. However, $\underline{\mathrm{Schemes}}$ is *not* a stack over $\mathrm{Sch}_{\text{ét}}$; see Example 2.1.16. Schemes can be glued to algebraic spaces in the étale topology and there is a stack of algebraic spaces over $\mathrm{Sch}_{\text{ét}}$; see Exercise 4.5.15.

On the other hand, the subcategories ClSubSch (resp., OpenSubSch, Aff, QAff, SepLQFin) parameterizing morphisms $X \to S$ which are closed immersions (resp., open immersions, affine, quasi-affine, locally quasi-finite and separated) are stacks over $\mathrm{Sch}_{\text{ét}}$ by Fppf Descent for Schemes (2.1.11, 2.1.12, and 2.1.14).

2.5.3 Classifying stacks and quotient stacks

Let $G \to S$ be a smooth affine group scheme acting on an S-scheme U, and let $[U/G]$ be the category over Sch/S defined in Definition 2.4.15: an object over an S-scheme T is a diagram

$$\begin{tikzcd} P \arrow[r, "f"] \arrow[d] & U \\ T \end{tikzcd}$$

where $P \to T$ is a principal G-bundle and $f : P \to U$ is a G-equivariant morphism of schemes.

Proposition 2.5.13. *If $G \to S$ be a smooth affine group scheme acting on an S-scheme U, then $[U/G]$ is a stack over $\mathrm{Sch}_{\text{ét}}$. In particular, the classifying stack $BG = [S/G]$ is a stack over $\mathrm{Sch}_{\text{ét}}$.*

Proof. We will show in fact that $[U/G]$ is a stack over $\mathrm{Sch}_{\mathrm{fpqc}}$. Since schemes and their morphisms glue in the Zariski topology, it suffices by Exercise 2.5.5 to verify Axioms (1) and (2) with respect to an fpqc map $T' \to T$. For Axiom (1), let $(P' \to T, f' : P \to U)$ and $(P \to T, f : P \to U)$ be objects over an S-scheme T, $T' \to T$ be an fpqc map, and $\phi' : P'_{T'} \to P$ be a G-equivariant morphism inducing an isomorphism $P'_{T'} \xrightarrow{\sim} P_{T'}$ and compatible with f and f'. By Fpqc Descent for Morphisms (2.1.7), there exists a unique morphism $P' \to P$ compatible with f' and f. By Fpqc Local Properties on the Target (2.1.26), $P' \to P$ is an isomorphism. For Axiom (2), let $(P' \to T', f' \colon P' \to U)$ be an object over T' and $\alpha \colon p_1^* P' \to p_2^* P'$ be an isomorphism commuting with f' and satisfying the cocycle condition $p_{23}^* \alpha \circ p_{12}^* \alpha = p_{13}^* \alpha$ on $T' \times_T T' \times_T T'$. The existence of a principal G-bundle $P \to T$ pulling back to $P' \to T$ follows from Fpqc Descent of Principal G-bundles (2.1.17). The existence of a G-equivariant morphism $P \to U$ follows from Fpqc Descent for Morphisms (2.1.7). $\square$

2.5.4 Moduli stack of curves

Let $\mathcal{M}_g$ denote the prestack of families of smooth curves $\mathcal{C} \to S$ of genus g as defined in Example 2.4.10.

Proposition 2.5.14 (Moduli stack of smooth curves)**.** *If $g \geq 2$, then $\mathcal{M}_g$ is a stack over* $\mathrm{Sch}_{\text{ét}}$.

Proof. We check that $\mathcal{M}_g$ is a stack over $\mathrm{Sch}_{\text{fpqc}}$. As smooth curves and their morphisms glue in the Zariski topology, it suffices by Exercise 2.5.5 to verify Axioms (1) and (2) with respect to an fpqc map $S' \to S$. Axiom (1) translates to: for families of smooth curves $\mathcal{C} \to S$ and $\mathcal{D} \to S$ of genus g, every commutative diagram

$$\begin{tikzcd}
\mathcal{C}_{S' \times_S S'} \arrow[r, shift left] \arrow[r, shift right] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{C}_{S'} \arrow[r] \arrow[d] \arrow[rr, bend left, "f'"] \arrow[dr, phantom, "\square"] & \mathcal{C} \arrow[r, dashed, "f"] \arrow[d] & \mathcal{D} \arrow[dl] \\
S' \times_S S' \arrow[r, shift left] \arrow[r, shift right] & S' \arrow[r] & S &
\end{tikzcd}$$

of solid arrows can be uniquely filled in. The existence and uniqueness of f follow from Fpqc Descent for Morphisms (2.1.7). Since the base change $\mathcal{C}_{S'} \to \mathcal{D}_{S'}$ is an isomorphism (because of how we've defined morphisms in $\mathcal{M}_g$), the morphism $f \colon \mathcal{C} \to \mathcal{D}$ is also an isomorphism by étale descent (Proposition 2.1.26).

Axiom (2) will require some geometry: we must show that given a diagram

$$\begin{tikzcd}
p_1^*\mathcal{C}' \arrow[r, "\alpha"] \arrow[dr] \arrow[rr, bend left] & p_2^*\mathcal{C}' \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{C}' \arrow[r, dashed] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{C} \arrow[d, dashed] \\
 & S' \times_S S' \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & S' \arrow[r] & S,
\end{tikzcd}$$

where $\mathcal{C}' \to S'$ is a family of smooth curves and $\alpha \colon p_1^* \mathcal{C}' \to p_2^* \mathcal{C}'$ is an isomorphism satisfying the cocycle condition $p_{23}^* \alpha \circ p_{12}^* \alpha = p_{13}^* \alpha$ on $S' \times_S S' \times_S S'$, there is family of smooth curves $\mathcal{C} \to S$ and an isomorphism $\phi \colon \mathcal{C}|_{S'} \to \mathcal{C}'$ such that $p_1^* \phi = p_2^* \phi \circ \alpha$. We will apply Properties of Families of Smooth Curves (5.1.16): $\Omega_{\mathcal{C}'/S'}^{\otimes 3}$ is relatively very ample on S (as $g \geq 2$) and $E' := \pi_*(\Omega_{\mathcal{C}'/S'}^{\otimes 3})$ is a vector bundle on S' of rank $5(g-1)$ whose construction commutes with base change. This implies that $\Omega_{\mathcal{C}'/S'}^{\otimes 3}$ defines a closed immersion $\mathcal{C}' \hookrightarrow \mathbb{P}(E')$ over S'. The isomorphism α induces an isomorphism $\beta \colon p_1^* E' \to p_2^* E'$ satisfying the cocycle condition $p_{23}^* \beta \circ p_{12}^* \beta = p_{13}^* \beta$ on $S' \times_S S' \times_S S'$. Fpqc Descent for Quasi-Coherent Sheaves (2.1.4) yields a quasi-coherent sheaf E on S and isomorphisms $\psi \colon E' \to E|_{S'}$ such that $p_1^* \psi = p_2^* \psi \circ \beta$. It follows from Fpqc Descent of Properties of Quasi-Coherent Sheaves (2.1.18) that E is a vector bundle. Pictorially, we have

$$
\begin{tikzcd}
 & & & \mathbb{P}(E') \arrow[r] \arrow[ddl] & \mathbb{P}(E) \arrow[ddl] \\
p_1^*\mathcal{C}' \arrow[r, "\alpha"] \arrow[rr, bend left] \arrow[d] & p_2^*\mathcal{C}' \arrow[r] \arrow[dl] & \mathcal{C}' \arrow[ur, hook] \arrow[r] \arrow[d] & \mathcal{C} \arrow[ur, hook, dashed] \arrow[d, dashed] & \\
S' \times_S S' \arrow[rr, shift left, "p_1"] \arrow[rr, shift right, "p_2"'] & & S' \arrow[r] & S, &
\end{tikzcd}
$$

Under the identifications of $\mathbb{P}(p_1^*E')$ and $\mathbb{P}(p_2^*E')$ with $\mathbb{P}(E) \times_S (S' \times_S S')$, the preimages $p_1^*\mathcal{C}'$ and $p_2^*\mathcal{C}'$ are equal. By Fpqc Descent for Closed Subschemes (2.1.11), there is a closed subscheme $\mathcal{C} \subseteq \mathbb{P}(E)$ pulling back to $\mathcal{C}'$. Smoothness and properness are Fpqc Local Properties on the Target (2.1.26), and thus $\mathcal{C} \to S$ is smooth and proper. Every geometric fiber of $\mathcal{C} \to S$ is identified with a geometric fiber of $\mathcal{C}' \to S'$ and is thus a connected genus g curve. $\square$

Exercise 2.5.15 (Moduli of genus 0 and elliptic curves, good practice)**.**

(a) Use the correspondence between families of genus 0 curves and principal PGL_2-torsors (Exercise B.1.67) to show that the prestack $\mathcal{M}_0$ is a stack on $\mathrm{Sch}_{\text{ét}}$ isomorphic to $B\,\mathrm{PGL}_2$ over $\operatorname{Spec}\mathbb{Z}$.

(b) A *family of elliptic curves* over a scheme S is a pair $(\mathcal{E} \to S, \sigma)$ where $\mathcal{E} \to S$ is smooth proper morphism with a section $\sigma\colon S \to \mathcal{E}$ such that for every $s \in S$, the fiber $(\mathcal{E}_s, \sigma(s))$ is an elliptic curve over the residue field $\kappa(s)$. Show that the moduli stack $\mathcal{M}_{1,1}$, whose objects are families of elliptic curves, is a stack on $\mathrm{Sch}_{\text{ét}}$.

Remark 2.5.16 (Moduli of genus 1 curves). The prestack $\mathcal{M}_1$, whose objects are smooth and proper morphisms $\mathcal{C} \to S$ of *schemes* whose geometric fibers are connected genus 1 curves, is *not* a stack over $\mathrm{Sch}_{\text{ét}}$. Unlike $\mathcal{M}_g$ with $g \geq 2$ and $\mathcal{M}_{1,1}$, there is no natural line bundle defining an embedding into projective space that we can use as above to verify Axiom (2). Raynaud constructed an étale cover $S' \to S$ and a family $\mathcal{C}' \to S'$ of smooth genus 1 curves which does not descend to a family $\mathcal{C} \to S$ [Ray70, XIII 3.2]. However, similar to Example 2.5.12, if we redefine $\mathcal{M}_1$ as the category of smooth and proper morphisms $\mathcal{C} \to S$ from an *algebraic space* such that every geometric fiber is a connected curve of genus 1, then $\mathcal{M}_1$ is a stack.

2.5.5 Moduli stack of coherent sheaves and vector bundles

If X is a scheme over a field $\Bbbk$, the prestacks $\underline{\mathrm{QCoh}}(X)$, $\underline{\mathrm{Coh}}(X)$, and $\mathcal{B}un(X)$ introduced in Example 2.4.12 parameterizes pairs (T, F) where T is a scheme and F is a quasi-coherent, coherent sheaf (more precisely, a finitely presented quasi-coherent sheaf), or a vector bundle on $X_T = X \times_{\Bbbk} T$. Note that the prestacks $\underline{\mathrm{QCoh}}$, $\underline{\mathrm{Coh}}$, and $\mathcal{B}un$ of Example 2.5.9 parameterizes pairs (T, F) where T is a scheme and F is a quasi-coherent sheaf on T.

Proposition 2.5.17. *The prestacks* $\underline{\mathrm{QCoh}}(X)$*,* $\underline{\mathrm{Coh}}(X)$*, and* $\mathcal{B}un(X)$ *are stacks over* $(\mathrm{Sch}/\Bbbk)_{\text{ét}}$.

Proof. Just as in the proof of Example 2.5.9, the statement follows directly from Fpqc Descent for Quasi-Coherent Sheaves (2.1.4) and Fpqc Descent of Properties of Quasi-Coherent Sheaves (2.1.18). □

2.5.6 Stackification

Recall from Sheafification (2.3.15) that for any a presheaf F on a site $\mathcal{S}$, there is a map $F \to F^{\mathrm{sh}}$ which is a left adjoint to the inclusion, i.e., $\mathrm{Mor}(F^{\mathrm{sh}}, G) \to \mathrm{Mor}(F, G)$ is bijective for every sheaf G on $\mathcal{S}$. Similarly, there is a stackification $\mathcal{X} \to \mathcal{X}^{\mathrm{st}}$ of a prestack $\mathcal{X}$ over $\mathcal{S}$.

Theorem 2.5.18 (Stackification)**.** *If $\mathcal{X}$ is a prestack over a site $\mathcal{S}$, there exists a stack $\mathcal{X}^{\mathrm{st}}$, which we call the* stackification*, and a morphism $\mathcal{X} \to \mathcal{X}^{\mathrm{st}}$ of prestacks such that for every stack $\mathcal{Y}$ over $\mathcal{S}$, the induced functor*

$$\mathrm{Mor}(\mathcal{X}^{\mathrm{st}}, \mathcal{Y}) \to \mathrm{Mor}(\mathcal{X}, \mathcal{Y}) \tag{2.5.19}$$

is an equivalence of categories.

Proof. As in the construction of the sheafification in Theorem 2.3.15, we construct the stackification in stages. Most details are left to the reader.

First, given a prestack $\mathcal{X}$, we can construct a prestack $\mathcal{X}^{\mathrm{st}_1}$ satisfying Axiom (1) and a morphism $\mathcal{X} \to \mathcal{X}^{\mathrm{st}_1}$ of prestacks such that

$$\mathrm{Mor}(\mathcal{X}^{\mathrm{st}_1}, \mathcal{Y}) \to \mathrm{Mor}(\mathcal{X}, \mathcal{Y})$$

is an equivalence for all prestacks $\mathcal{Y}$ satisfying Axiom (1). The objects of $\mathcal{X}^{\mathrm{st}_1}$ are the same as $\mathcal{X}$, and for objects $a, b \in \mathcal{X}$ over $S, T \in \mathcal{S}$, the set of morphisms $a \to b$ in $\mathcal{X}^{\mathrm{st}_1}$ over a given morphism $f \colon S \to T$ is the global sections $\Gamma(S, \underline{\mathrm{Isom}}_{\mathcal{X}(S)}(a, f^*b)^{\mathrm{sh}})$ of the sheafification of the Isom presheaf introduced in Exercise 2.4.39.

Second, given a prestack $\mathcal{X}$ satisfying Axiom (1), we construct a stack $\mathcal{X}$ and a morphism $\mathcal{X} \to \mathcal{X}^{\mathrm{st}_2}$ of prestacks such that (2.5.19) is an equivalence for all stacks $\mathcal{Y}$. An object of $\mathcal{X}^{\mathrm{st}_2}$ over $S \in \mathcal{S}$ is given by a triple consisting of a covering $\{S_i \to S\}$, objects a_i of $\mathcal{X}$ over S_i, and isomorphisms $\alpha_{ij} \colon a_i|_{S_{ij}} \to a_j|_{S_{ij}}$ satisfying the cocycle condition $\alpha_{jk}|_{S_{ijk}} \circ \alpha_{ij}|_{S_{ijk}} = \alpha_{ik}|_{S_{ijk}}$ on S_{ijk}. Morphisms

$$\big(\{S_i \to S\}, \{a_i\}, \{\alpha_{ij}\}\big) \to \big(\{T_\mu \to T\}, \{b_\mu\}, \{\beta_{\mu\nu}\}\big)$$

in $\mathcal{X}^{\mathrm{st}_2}$ over $S \to T$ are defined as follows: for a choice of pullbacks $a_i|_{S_i \times_S T_\mu}$ and $b_\mu|_{S_i \times_S T_\mu}$ with respect to the induced cover $\{S_i \times_S T_\mu \to S\}_{i,\mu}$, a morphism is the data of maps $\Psi_{i\mu} \colon a_i|_{S_i \times_S T_\mu} \to b_\mu|_{S_i \times_S T_\mu}$ for all i, μ which are compatible with α_{ij} and $\beta_{\mu\nu}$ (i.e., $\Psi_{j\nu} \circ \alpha_{ij} = \beta_{\mu\nu} \circ \Psi_{i\mu}$ on $S_{ij} \times_S T_{\mu\nu}$). □

Exercise 2.5.20 (details)**.** Show that stackification commutes with fiber products: if $\mathcal{X} \to \mathcal{Y}$ and $\mathcal{Z} \to \mathcal{Y}$ are morphisms of prestacks, then $(\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z})^{\mathrm{st}} \cong \mathcal{X}^{\mathrm{st}} \times_{\mathcal{Y}^{\mathrm{st}}} \mathcal{Z}^{\mathrm{st}}$.

Exercise 2.5.21 (good practice)**.** Let $G \to S$ be a smooth affine group scheme acting on a scheme U over S. Recall from Definition 2.4.15 that the quotient prestack $[U/G]^{\mathrm{pre}}$ and quotient stack $[U/G]$ denote the prestacks over Sch/S classifying trivial principal G-bundles (resp., principal G-bundles) $P \to T$ and G-equivariant maps $P \to U$.

(a) Show that $[U/G]^{\mathrm{pre}}$ satisfies Axiom (1) of a stack over $(\mathrm{Sch}/S)_{\text{ét}}$.

(b) Show that the $[U/G]$ is isomorphic to the stackification of $[U/G]^{\mathrm{pre}}$ over $(\mathrm{Sch}/S)_{\text{ét}}$, and that $[U/G]^{\mathrm{pre}} \to [U/G]$ is fully faithful.

Exercise 2.5.22. Extending Exercise 2.4.27, show that $U \to [U/G]$ is a categorical quotient among stacks.

Chapter 3

Algebraic spaces and stacks

The notion of stacks came up in the sixties. But to swallow schemes was already enough for one generation of mathematicians.

Gerd Faltings [Sto95, p. 45]

3.1 Definitions of algebraic spaces and stacks

The most abstract definition, once you are familiar with it, is not abstract anymore. It's like a beautiful mountain that you see very well, because the air is very clear and there is light that lets you see all the details.

Claire Voisin

What are algebraic spaces, Deligne–Mumford stacks, and algebraic stacks? After giving their definitions, we will verify the algebraicity of quotient stacks $[U/G]$, the moduli stack of curves $\mathcal{M}_g$, and the moduli stack of vector bundles $\mathcal{B}un_{r,d}(C)$.

3.1.1 Algebraic spaces

Definition 3.1.1 (Morphisms representable by schemes)**.** A morphism $\mathcal{X} \to \mathcal{Y}$ of prestacks (or presheaves) over Sch is *representable by schemes* (or sometimes called *schematic*) if for every morphism $T \to \mathcal{Y}$ from a scheme, the fiber product $\mathcal{X} \times_{\mathcal{Y}} T$ is a scheme.

If $\mathcal{P}$ is a property of morphisms of schemes stable under base change (e.g., surjective or étale), a morphism $\mathcal{X} \to \mathcal{Y}$ of prestacks representable by schemes *has property* $\mathcal{P}$ if for every morphism $T \to \mathcal{Y}$ from a scheme, the morphism $\mathcal{X} \times_{\mathcal{Y}} T \to T$ of schemes has property $\mathcal{P}$.

Definition 3.1.2 (Algebraic spaces)**.** An *algebraic space* is a sheaf X on $\mathrm{Sch}_{\text{ét}}$ such that there exists a scheme U and a surjective étale morphism $U \to X$ representable by schemes.

The map $U \to X$ is called an *étale presentation.* Morphisms of algebraic spaces are by definition morphisms of sheaves. Every scheme is an algebraic space.

3.1.2 Deligne–Mumford stacks

Definition 3.1.3 (Representable morphisms and their properties)**.** A morphism $\mathcal{X} \to \mathcal{Y}$ of prestacks (or presheaves) over Sch is *representable* if for every morphism $T \to \mathcal{Y}$ from a scheme T, the fiber product $\mathcal{X} \times_{\mathcal{Y}} T$ is an algebraic space.

Let $\mathcal{P}$ be a property of morphisms of schemes stable under base change and étale-local on the source (i.e., if $X' \to X$ is a surjective étale morphism, then a morphism $X \to Y$ of schemes has $\mathcal{P}$ if and only if $X' \to X \to Y$ has $\mathcal{P}$); examples include the properties of being surjective, étale, or smooth. We say that a representable morphism $\mathcal{X} \to \mathcal{Y}$ of prestacks *has property* $\mathcal{P}$ if for every morphism $T \to \mathcal{Y}$ from a scheme and étale presentation $U \to \mathcal{X} \times_{\mathcal{Y}} T$ by a scheme, the composition $U \to \mathcal{X} \times_{\mathcal{Y}} T \to T$ has property $\mathcal{P}$.

Definition 3.1.4 (Deligne–Mumford stacks)**.** A *Deligne–Mumford stack* is a stack $\mathcal{X}$ over $\mathrm{Sch}_{\text{ét}}$ such that there exists a scheme U and a surjective, étale, and representable morphism $U \to \mathcal{X}$.

The morphism $U \to \mathcal{X}$ is called an *étale presentation*. Morphisms of Deligne–Mumford stacks are by definition morphisms of stacks. Every algebraic space is a Deligne–Mumford stack via Example 2.4.7. We show later that a Deligne–Mumford stack (or even an algebraic stack) that is a sheaf is an algebraic space (Theorems 3.6.6 and 4.5.10).

Remark 3.1.5. While the essential difference between an algebraic space and a Deligne–Mumford stack is that one is a sheaf while the other is a stack, there is also the technical difference that an étale presentation of an algebraic space is *representable by schemes* while an étale presentation of a Deligne–Mumford stack is only required to be *representable*. If the diagonal of a Deligne–Mumford stack is separated and quasi-compact, then it is representable by schemes and every presentation $U \to \mathcal{X}$ is representable by schemes (Corollary 4.5.8). While requiring that the diagonal be separated and quasi-compact is a very mild condition and satisfied in almost every moduli problem of interest, you should be aware that there are Deligne–Mumford stacks whose diagonal is not quasi-compact, not separated, or not representable by schemes, e.g., BG for an étale group algebraic space that is not quasi-compact, not separated, or not a scheme; see Examples 3.9.35 and 3.9.37.

3.1.3 Algebraic stacks

Definition 3.1.6 (Algebraic stacks)**.** An *algebraic stack* is a stack $\mathcal{X}$ over $\mathrm{Sch}_{\text{ét}}$ such that there exists a scheme U and a surjective, smooth, and representable morphism $U \to \mathcal{X}$.

The morphism $U \to \mathcal{X}$ is called a *smooth presentation*. Morphisms of algebraic stacks are by definition morphisms of prestacks. Every scheme, algebraic space, or Deligne–Mumford stack is also an algebraic stack.

Caution 3.1.7. In the literature, most authors add a representability condition on the diagonal. However, as we show in Theorem 3.2.1, the existence of a smooth presentation implies the representability of the diagonal: the diagonal of an algebraic space is representable by schemes and that the diagonal of an algebraic stack is representable.

Definition 3.1.8 (Open and closed substacks)**.** A substack $\mathcal{T} \subseteq \mathcal{X}$ of a stack over $\mathrm{Sch}_{\text{ét}}$ is called an *open substack* (resp., *closed substack*) if the inclusion $\mathcal{T} \to \mathcal{X}$ is representable by schemes and an open immersion (resp., closed immersion).

Exercise 3.1.9 (Fiber products)**.** Show that fiber products exist for algebraic spaces, Deligne–Mumford stacks, and algebraic stacks.

3.1.4 Algebraicity of quotient stacks

Theorem 3.1.10 (Algebraicity of Quotient Stacks)**.** *If $G \to S$ is a smooth affine group scheme acting on an algebraic space $U \to S$, the quotient stack $[U/G]$ is an algebraic stack over S such that $U \to [U/G]$ is a principal G-bundle and in particular surjective, smooth, and affine. In particular, the classifying stack $BG = [S/G]$ is algebraic.*

Remark 3.1.11*.* An eagle-eyed reader may have noticed that we only defined $[U/G]$ when U is a scheme. It is not hard to extend the definition. An *action* of a smooth affine group scheme $G \to S$ on an algebraic space $U \to S$ is a morphism $\sigma\colon G \times_S U \to U$ satisfying the same axioms as in Definition B.1.9, and we define the quotient stack $[U/G]$ as the stackification of the prestack $[U/G]^{\mathrm{pre}}$, whose fiber category over an S-scheme T is the quotient groupoid $[U(T)/G(T)]$, as in Definition 2.4.15. Objects of $[U/G]$ over an S-scheme T are principal G-bundles $P \to T$ and G-equivariant morphisms $P \to U$. Since morphisms to algebraic spaces glue uniquely in the étale topology, the argument of Proposition 2.5.13 extends to show that $[U/G]$ is a stack. Finally, we note that saying that $U \to [U/G]$ is a principal G-bundle means that it is representable by schemes and every base change by a map $T \to [U/G]$ from scheme is a principal G-bundle.

Proof. We will use the natural projection $U \to [U/G]$ corresponding via the 2-Yoneda Lemma (2.4.21) to the trivial principal G-bundle $p_2\colon G \times U \to U$ and the G-equivariant map $\sigma\colon G \times U \to U$ given by multiplication. To show that $U \to [U/G]$ is a principal G-bundle, let $T \to [U/G]$ be a morphism from an S-scheme classified by a principal G-bundle $P \to T$ and a G-equivariant map $P \to U$. By Exercise 2.4.37, there is a cartesian diagram

$$\begin{tikzcd} P \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & U \arrow[d] \\ T \arrow[r] & {[U/G]}. \end{tikzcd}$$

Since every base change is a principal G-bundle, so is $U \to [U/G]$. The map $U \to [U/G]$ is a smooth presentation if U is a scheme. In general, letting $U' \to U$ be an étale presentation with U' a scheme, the composition $U' \to U \to [U/G]$ provides a smooth presentation. $\square$

Example 3.1.12 ($B\mathbb{G}_m$, $B\mathrm{GL}_n$, and $B\,\mathrm{PGL}_n$)**.** Since principal $\mathbb{G}_m$-bundles correspond to line bundles (Exercise B.1.51), the classifying stack $B\mathbb{G}_m$ is equivalent to the category of pairs (S, V) consisting of a scheme S and a line bundle V on S. Similarly, $B\mathrm{GL}_n$ is the stack of pairs (S, V) where V is a vector bundle of rank n on a scheme S (Exercise B.1.56). The classifying stack $B\,\mathrm{PGL}_n$ can be described equivalently using either principal PGL_n-bundles, Brauer–Severi schemes, or Azumaya algebras over S (see Exercises B.1.66 and B.1.68).

Exercise 3.1.13 ($BO(q)$)**.** Let $\Bbbk$ be a field of $\mathrm{char}(\Bbbk) \neq 2$. For a non-degenerate quadratic form q on an n-dimensional vector space V, the orthogonal group $O(q)$ is the subgroup of $\mathrm{GL}(V)$ containing matrices preserving q. If q and q' are non-degenerate quadratic forms, show that $BO(q) \cong BO(q')$ even though $O(q)$ and $O(q')$ may be non-isomorphic.

We obtain the desirable consequence that the category of algebraic spaces is closed under taking quotients of free actions of finite groups. Recall from Example 0.5.7 that the quotient of a free action by a finite group on a scheme need not exist as a scheme.

Corollary 3.1.14. *Let G be an finite abstract group viewed as a group scheme over a scheme S. If G acts freely on an algebraic space U over S, then the quotient sheaf U/G is an algebraic space.*

Proof. Since the action is free, the quotient stack $[U/G]$ is equivalent to a sheaf, which we denote by U/G (see Exercise 2.4.20). Theorem 3.1.10 implies that U/G is an algebraic stack and that $U \to U/G$ is a principal G-bundle so in particular finite, étale, surjective and representable by schemes. Taking $U' \to U$ to be an étale presentation by a scheme, the composition $U' \to U \to U/G$ yields an étale presentation of U/G. □

Remark 3.1.15. In addition, it shows that the category of algebraic spaces itself is closed under taking quotients by free actions of finite groups so that we do not need to enlarge our category even further.

Exercise 3.1.16 (easy)**.** Let $G \to S$ be a smooth affine group scheme acting on S-schemes X and Y. Show that a G-equivariant morphism $X \to Y$ induces a morphism $[X/G] \to [Y/G]$ of algebraic stacks, and conversely that $[X/G] \to [Y/G]$ is induced by a G-equivariant morphism if and only if $[X/G] \to [Y/G]$ is a morphism over BG.

3.1.5 Algebraicity of $\mathcal{M}_g$

Why is $\mathcal{M}_g$ algebraic? Here is one reason: every smooth, connected, and projective curve C is tricanonically embedded $C \hookrightarrow \mathbb{P}^{5g-6}$ by the very ample line bundle $\Omega_C^{\otimes 3}$ and the locally closed subscheme $H' \subseteq \mathrm{Hilb}^P(\mathbb{P}^{5g-6})$ parameterizing smooth families of tricanonically embedded curves provides a smooth presentation $H' \to \mathcal{M}_g$. The technical details however are a bit involved. The Algebraicity of $\mathcal{B}un_{r,d}(C)$ (3.1.21) will be easier.

Theorem 3.1.17 (Algebraicity of $\mathcal{M}_g$)**.** *If $g \geq 2$, then $\mathcal{M}_g$ is an algebraic stack over* $\operatorname{Spec} \mathbb{Z}$. *Moreover, $\mathcal{M}_g \cong [H'/\operatorname{PGL}_{5g-5}]$ where H' is a locally closed subscheme of a projective Hilbert scheme.*

Proof. As in the proof that $\mathcal{M}_g$ is a stack (Proposition 2.5.14), we will use Properties of Families of Smooth Curves (5.1.16): for a family of smooth curves $p\colon \mathcal{D} \to S$, $\Omega_{\mathcal{D}/S}^{\otimes 3}$ is relatively very ample on S and $p_*(\Omega_{\mathcal{D}/S}^{\otimes 3})$ is a vector bundle of rank $5(g-1)$. It follows that $\Omega_{\mathcal{D}/S}^{\otimes 3}$ defines a closed immersion $\mathcal{D} \hookrightarrow \mathbb{P}(p_*(\Omega_{\mathcal{D}/S}^{\otimes 3}))$ over S. By Riemann–Roch (5.1.2), the Hilbert polynomial of a fiber $\mathcal{D}_s \hookrightarrow \mathbb{P}_{\kappa(s)}^{5g-6}$ is given by

$$P(n) := \chi(\mathcal{O}_{\mathcal{D}_s}(n)) = \deg(\Omega_{\mathcal{D}_s}^{\otimes 3n}) + 1 - g = (6n-1)(g-1).$$

and we define

$$H := \mathrm{Hilb}^P(\mathbb{P}_{\mathbb{Z}}^{5g-6})$$

as the (projective) Hilbert scheme parameterizing closed subschemes of $\mathbb{P}^{5g-6}$ with Hilbert polynomial P (Theorem 1.1.2). Let $\mathcal{C} \hookrightarrow \mathbb{P}^{5g-6} \times H$ be the universal closed subscheme and let $\pi\colon \mathcal{C} \to H$ be the projection. We claim that there is a locally closed subscheme $H' \subset H$ such that the family $\mathcal{C}_{H'} \to H'$ satisfies

(a) for every $h \in H'$, $\mathcal{C}_h \to \operatorname{Spec} \kappa(h)$ is smooth and geometrically connected,

(b) the natural map $p_{2,*}\mathcal{O}_{\mathbb{P}^{5g-6} \times H'}(1) \to p_{2,*}\mathcal{O}_{\mathcal{C}_{H'}}(1)$ induced by the closed immersion $\mathcal{C}_{H'} \hookrightarrow \mathbb{P}^{5g-6} \times H'$ is an isomorphism (or equivalently $\mathrm{H}^0(\mathbb{P}^{5g-6}_{\kappa(h)}, \mathcal{O}(1)) \to \mathrm{H}^0(\mathcal{C}_h, \mathcal{O}_{\mathcal{C}_h}(1))$ is an isomorphism for all $h \in H'$), and

(c) the line bundles $\Omega^{\otimes 3}_{\mathcal{C}_{H'}/H'}$ and $\mathcal{O}_{\mathcal{C}_{H'}}(1)$ differ by a pullback of a line bundle from H'.

Moreover, if $T \to H$ is a morphism of schemes such that (a)–(c) hold for the family $\mathcal{C}_T \to T$, then $T \to H$ factors through H'. In particular, $H' \subseteq H$ is unique. Note that (a)–(c) imply that $\mathcal{C}_h \hookrightarrow \mathbb{P}^{5g-6}_{\kappa(h)}$ is embedded by the complete linear series $\Omega^{\otimes 3}_{\mathcal{C}_h/\kappa(h)}$ for $h \in H$.

Since the condition that a fiber of a proper morphism (of finite presentation) is smooth is an open condition on the target (Corollary A.3.9), the condition on H that $\mathcal{C}_h$ is smooth is open. Consider the Stein factorization [Har77, Cor. 11.5] $\mathcal{C} \to \widetilde{H} = \operatorname{Spec}_H \pi_* \mathcal{O}_{\mathcal{C}} \to H$, where $\mathcal{C} \to \widetilde{H}$ has geometrically connected fibers and $\widetilde{H} \to H$ is finite. Since the kernel and cokernel of $\mathcal{O}_H \to \pi_*\mathcal{O}_{\mathcal{C}}$ have closed support, $\widetilde{H} \to H$ is an isomorphism over an open subscheme of H, which is precisely where the fibers of $\mathcal{C} \to H$ are geometrically connected. In summary, the set of $h \in H$ satisfying (a) is an open subscheme of H, which we will denote by H_1. To arrange (b), observe that Cohomology and Base Change (A.6.8) implies that both $p_{2,*}\mathcal{O}_{\mathbb{P}^{5g-6} \times H_1}(1)$ and $p_{2,*}\mathcal{O}_{\mathcal{C}_{H_1}}(1)$ are vector bundles whose constructions commute with base change, and Riemann–Roch (5.1.2) further implies they have the same rank. Therefore, $\alpha \colon p_{2,*}\mathcal{O}_{\mathbb{P}^{5g-6} \times H_1}(1) \to p_{2,*}\mathcal{O}_{\mathcal{C}_{H_1}}(1)$ is an isomorphism if and only if it is surjective. Setting $H_2 := H_1 \setminus \operatorname{Supp}(\operatorname{coker}(\alpha))$, we conclude that (a)–(b) hold over H_2. It is also clear that any map $T \to H$ over which (a)–(b) hold factors through H_2. To arrange property (c), as $\mathcal{C}_{H_2} \to H_2$ has geometrically integral fibers and the relative canonical sheaf $\Omega := \Omega_{\mathcal{C}_{H_2}/H_2}$ is a line bundle, we may use Proposition A.6.17 to conclude that there exists a closed subscheme $H_3 \hookrightarrow H_2$ such that a morphism $T \to H_2$ factors through H_3 if and only if $\Omega^{\otimes 3}|_{\mathcal{C}_T}$ and $\mathcal{O}_{\mathcal{C}}(1)|_{\mathcal{C}_T}$ differ by the pullback of a line bundle on T. The subscheme $H' := H_3$ satisfies (a)–(c) along with the universal property.

The group scheme $\mathrm{PGL}_{5g-5} = \underline{\mathrm{Aut}}(\mathbb{P}^{5g-6}_{\mathbb{Z}})$ over $\mathbb{Z}$ acts naturally on $H = \mathrm{Hilb}^P(\mathbb{P}^{5g-6}_{\mathbb{Z}})$: if $g \in \mathrm{PGL}_{5g-5}$ and $[\mathcal{D} \subseteq \mathbb{P}^{5g-6}_S] \in H(S)$, then $g \cdot [\mathcal{D} \subseteq \mathbb{P}^{5g-6}_S] = [g(\mathcal{D}) \subseteq \mathbb{P}^{5g-6}_S]$. The closed subscheme $H' \subseteq H$ is PGL_{5g-5}-invariant. We claim that $\mathcal{M}_g \cong [H'/\mathrm{PGL}_{5g-5}]$. This claim finishes the theorem by the Algebraicity of Quotient Stacks (3.1.10). To see the claim, consider the morphism $H' \to \mathcal{M}_g$ defined by the restriction $\mathcal{C}_{H'} \to H'$ of the universal family of the Hilbert scheme. This morphism forgets the embedding, i.e., assigns a closed subscheme $\mathcal{D} \subseteq \mathbb{P}^{5g-6}_S$ to the family $\mathcal{D} \to S$. This morphism is PGL_{5g-5}-invariant and descends to a morphism $[H'/\mathrm{PGL}_{5g-5}]^{\mathrm{pre}} \to \mathcal{M}_g$ of prestacks. We first show that this map is fully faithful: let $S \to H'$ be a map corresponding to a closed subscheme

$$\begin{tikzcd} \mathcal{D} \arrow[r, hook] \arrow[dr, "q"'] & \mathbb{P}^{5g-6} \times S \arrow[d, "p_2"] \\ & S. \end{tikzcd}$$

We will exploit the equivalences

$$\begin{aligned}
\mathrm{H}^0(\mathbb{P}^{5g-6}_{\mathbb{Z}}, \mathcal{O}(1)) \otimes \mathcal{O}_S &\cong p_{2,*}\mathcal{O}_{\mathbb{P}^{5g-6}\times S}(1) & \\
&\cong q_*\mathcal{O}_{\mathcal{D}}(1) & \text{(property (c))} \\
&\cong q_*(\Omega^{\otimes 3}_{\mathcal{D}/S} \otimes q^*M) & \text{(property (b) for } M \in \operatorname{Pic}(S)) \\
&\cong q_*(\Omega^{\otimes 3}_{\mathcal{D}/S}) \otimes M & \text{(projection formula).}
\end{aligned}$$

An automorphism of $\mathcal{D} \to S$ induces a unique automorphism of $\Omega^{\otimes 3}_{\mathcal{D}/S}$ and thus an unique automorphism of $q_*(\Omega^{\otimes 3}_{\mathcal{D}/S}) \otimes M$, which in turn induces a unique automorphism of $\mathbb{P}^{5g-6} \times S$ preserving $\mathcal{D}$. This shows that the natural map $\operatorname{Stab}^{\mathrm{PGL}_{5g-5}}([\mathcal{D} \subseteq \mathbb{P}^{5g-6}_S]) \xrightarrow{\sim} \operatorname{Aut}(\mathcal{D} \to S)$ is bijective.

Since $\mathcal{M}_g$ is a stack (Proposition 2.5.14), the universal property of stackification yields a morphism $[H'/\mathrm{PGL}_{5g-5}] \to \mathcal{M}_g$; this map is fully faithful since $[H'/\mathrm{PGL}_{5g-5}]^{\mathrm{pre}} \to [H'/\mathrm{PGL}_{5g-5}]$ is fully faithful (Exercise 2.5.21). It remains to check that $[H'/\mathrm{PGL}_{5g-5}] \to \mathcal{M}_g$ is essentially surjective. For this, it suffices to check that if $q\colon \mathcal{D} \to S$ is a family of smooth curves, there exists an étale cover $\{S_i \to S\}$ such that each $\mathcal{D}_{S_i}$ is in the image of $H'(S_i) \to \mathcal{M}_g(S_i)$. Since $\Omega^{\otimes 3}_{\mathcal{D}/S}$ defines a closed immersion $\mathcal{D} \hookrightarrow \mathbb{P}(q_*(\Omega^{\otimes 3}_{\mathcal{D}/S}))$ over S and $q_*(\Omega^{\otimes 3}_{\mathcal{D}/S})$ is locally free of rank $5g-5$, we may simply take $\{S_i\}$ to be a Zariski open cover (and thus étale cover) such that each $(q_*(\Omega_{\mathcal{D}/S})^{\otimes 3})|_{S_i}$ is free. We conclude that $\mathcal{M}_g \cong [H'/\mathrm{PGL}_{5g-5}]$ and Theorem 3.1.10 therefore implies that $\mathcal{M}_g$ is algebraic. □

Remark 3.1.18. The entire stack $\mathcal{M}$ of smooth curves (Example 2.4.10) is also algebraic since $\mathcal{M} = \coprod_g \mathcal{M}_g$.

Exercise 3.1.19 (Moduli of elliptic curves)**.** Recall from Exercise 2.5.15 that $\mathcal{M}_{1,1}$ denotes the stack over $\mathrm{Sch}_{\text{ét}}$ parameterizing families of elliptic curves.

(a) Show that $\mathcal{M}_{1,1}$ is an algebraic stack over $\mathbb{Z}$.

(b) Use the Weierstrass form $y^2 = x^3 + ax + b$ (see [Sil09, §3.1]) to show that if we invert the primes 2 and 3, there is an isomorphism

$$\mathcal{M}_{1,1} \times_{\mathbb{Z}} \mathbb{Z}[1/6] \cong [(\mathbb{A}^2 \smallsetminus V(\Delta))/\mathbb{G}_m],$$

where the action is given by $t \cdot (a, b) = (t^4 a, t^6 b)$ and Δ is the discriminant $4a^3 + 27b^2$. See also [MS72, §3, Thm. 1].

(c) Define a *stable elliptic curve* over a field $\Bbbk$ as a pair (E, p) where E is an irreducible projective curve over $\Bbbk$ of arithmetic genus 1 with at worst nodal singularities and $p \in E(\Bbbk)$ is a smooth point. Over a scheme S, a *family of stable elliptic curves over* S is a proper, flat, and finitely presented morphism $\mathcal{E} \to S$ together with a section $\sigma\colon S \to \mathcal{E}$ such that every fiber is a stable elliptic curve. Denoting $\overline{\mathcal{M}}_{1,1}$ as the stack over Sch classifying stable elliptic curves, show that

$$\overline{\mathcal{M}}_{1,1} \times_{\mathbb{Z}} \mathbb{Z}[1/6] \cong [(\mathbb{A}^2 \smallsetminus 0)/\mathbb{G}_m]$$

with the same action as above.

Exercise 3.1.20. An *n-pointed family of genus* 0 *smooth curves* is a smooth, proper morphism $X \to S$ of schemes with n sections $\sigma_1, \ldots, \sigma_n\colon S \to X$ such that for every $s\colon \operatorname{Spec} \Bbbk \to S$, X_s is a genus 0 curve with and $\sigma_1(s), \ldots, \sigma_n(s) \in X_s$ are distinct. In Exercise 2.5.15, we have identified $\mathcal{M}_{0,0}$ with the classifying stack $B\,\mathrm{PGL}_2$.

(a) Show that the prestack $\mathcal{M}_{0,n}$ parameterizing n-pointed families of genus 0 curves is a stack over $\mathrm{Sch}_{\text{ét}}$.
(b) Show that $\mathcal{M}_{0,1} \cong BU_2$, where $U_2 \subseteq \mathrm{PGL}_2$ is the two-dimensional subgroup of upper triangular matrices.
(c) Show that $\mathcal{M}_{0,2} \cong B\mathbb{G}_m$.
(d) Show that $\mathcal{M}_{0,3} \cong \operatorname{Spec} \mathbb{Z}$.
(e) Show that for $n > 3$, $\mathcal{M}_{0,n} \cong (\mathbb{P}^1 \smallsetminus \{0, 1, \infty\})^{n-3} \smallsetminus \Delta$, where Δ is the closed subscheme where at least two of the $n-3$ points are equal.

3.1.6 Algebraicity of $\mathcal{B}un(C)$

In Proposition 2.5.17, we showed that $\underline{\mathrm{QCoh}}(X)$, $\underline{\mathrm{Coh}}(X)$, and $\mathcal{B}un(X)$ are stacks over $(\mathrm{Sch}/\Bbbk)_{\text{ét}}$ for a scheme X over a field $\Bbbk$. We now specialize to the case of a curve C, even though the following result holds in far greater generality (see Exercise 3.1.23). We define

$$\underline{\mathrm{Coh}}_{r,d}(C) \subseteq \underline{\mathrm{Coh}}(C) \quad \text{and} \quad \mathcal{B}un_{r,d}(C) \subset \mathcal{B}un(C)$$

as the full subcategories parameterizing pairs (E, S) such that for every geometric point $\operatorname{Spec} K \to S$, E_K is a coherent sheaf on C_K of rank r and degree d.

Theorem 3.1.21 (Algebraicity of $\mathcal{B}un(C)$)**.** *If C is a smooth, connected, and projective curve over an algebraically closed field $\Bbbk$, then the stacks $\mathcal{B}un(C)$ and $\underline{\mathrm{Coh}}(C)$ are algebraic, and $\mathcal{B}un(C) \subseteq \underline{\mathrm{Coh}}(C)$ is an open substack. For integers $r \geq 0$ and d, $\mathcal{B}un_{r,d}(C)$ and $\underline{\mathrm{Coh}}_{r,d}(C)$ are algebraic stacks, and $\mathcal{B}un_{r,d}(C) \subseteq \mathcal{B}un(C)$ and $\underline{\mathrm{Coh}}_{r,d}(C) \subseteq \underline{\mathrm{Coh}}(C)$ are open and closed substacks.*

Remark 3.1.22. The theorem yields decompositions

$$\underline{\mathrm{Coh}}(C) = \coprod_{r,d} \underline{\mathrm{Coh}}_{r,d}(C) \quad \text{and} \quad \mathcal{B}un(C) = \coprod_{r,d} \mathcal{B}un_{r,d}(C).$$

While $\mathcal{B}un_{r,d}(C)$ and $\underline{\mathrm{Coh}}_{r,d}(C)$ are not quasi-compact (Definition 3.3.21), the proof below shows that every quasi-compact open substack of $\mathcal{B}un_{r,d}(C)$ or $\underline{\mathrm{Coh}}_{r,d}(C)$ is a quotient stack of a quasi-projective scheme (in fact, an open subscheme of a Quot scheme) by GL_N.

Proof. We first note that $\underline{\mathrm{Coh}}_{r,d}(C)$ and $\mathcal{B}un_{r,d}(C)$ are stacks over $(\mathrm{Sch}/\Bbbk)_{\text{ét}}$ since they are defined as the full subcategories of the stacks $\underline{\mathrm{Coh}}(C)$ and $\mathcal{B}un(C)$ by a condition on the geometric fiber. To see that $\mathcal{B}un(C) \subseteq \underline{\mathrm{Coh}}(C)$ is an open substack, let $S \to \underline{\mathrm{Coh}}(C)$ be a morphism classified by a finitely presented, quasi-coherent sheaf E on $C \times S$ flat over S. If $V \subseteq C \times S$ is the open locus where E is a vector bundle, then $S \smallsetminus p_2(C \times S \smallsetminus V)$ is open and identified with the fiber product $S \times_{\underline{\mathrm{Coh}}(C)} \mathcal{B}un(C)$. By Riemann–Roch for Coherent Sheaves (8.1.2), the Hilbert polynomial of a coherent sheaf E on C of rank r and degree d is

$$P(n) := \chi(E(n)) = \deg(E(n)) + \operatorname{rk}(E(n))(1-g) = d + rn + r(1-g).$$

Since the Hilbert polynomial is locally constant in flat families (Proposition A.2.4), the inclusions $\mathcal{B}un_{r,d}(C) \hookrightarrow \mathcal{B}un(C)$ and $\underline{\mathrm{Coh}}_{r,d}(C) \hookrightarrow \underline{\mathrm{Coh}}(C)$ are open and closed substacks. It therefore suffices to show that $\underline{\mathrm{Coh}}_{r,d}(C)$ is algebraic.

If E is a coherent sheaf C of rank r and degree d, then Serre vanishing implies that $E(N)$ is globally generated and $\mathrm{H}^1(C, E(N)) = 0$ for $N \gg 0$. This yields a surjection $\Gamma(C, E(N)) \otimes_{\Bbbk} \mathcal{O}_C \twoheadrightarrow E(N)$ inducing an isomorphism on global sections. For each integer N, let

$$\mathcal{U}_N \subseteq \underline{\mathrm{Coh}}_{r,d}(C)$$

be the substack parameterizing pairs $(S, \mathcal{E}) \in \underline{\mathrm{Coh}}_{r,d}(C)$ such that for every $s \in S$, $\mathcal{E}_s(N) := \mathcal{E}(N)|_{C \times \operatorname{Spec} \kappa(s)}$ is globally generated on $C_s := C \times_{\Bbbk} \kappa(s)$ and $\mathrm{H}^1(C_s, \mathcal{E}_s(N)) = 0$. Note that since $P(N) = \mathrm{h}^0(C_s, \mathcal{E}_s(N)) - \mathrm{h}^1(C_s, \mathcal{E}_s(N))$, these conditions imply that $\mathrm{h}^0(C, \mathcal{E}_s(N)) = P(N)$ and that the surjection $\Gamma(C, \mathcal{E}(N)) \otimes \mathcal{O}_C \twoheadrightarrow \mathcal{E}(N)$ induces an isomorphism on global sections.

We claim that $\mathcal{U}_N \subseteq \underline{\mathrm{Coh}}_{r,d}(C)$ is an open substack. To verify the claim, let S be a scheme and $\mathcal{E}$ be a finitely presented, quasi-coherent sheaf on $C \times S$ flat over S with Hilbert polynomial P, and consider the diagram

$$\begin{tikzcd} & C \times S \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ C & & S. \end{tikzcd}$$

A simple application of Cohomology and Base Change (see Proposition A.6.11) implies that the locus $S' \subseteq S$ of points $s \in S$ such that $\mathrm{H}^1(C, \mathcal{E}_s(N)) = 0$ is open, and moreover that $\big(\mathrm{R}^1 p_{2,*}\mathcal{E}(N)\big)|_{S'} = 0$ and $\big(p_{2,*}\mathcal{E}(N)\big)|_{S'}$ is a vector bundle of rank $P(N)$ whose construction commutes base change. The sheaf $\mathcal{F} := \operatorname{coker}\big(p_2^* p_{2,*}\mathcal{E}(N) \to \mathcal{E}(N)\big)$ has closed support and the open subset $S'' := S' \smallsetminus p_2(\operatorname{Supp}(\mathcal{F})) \subseteq S$ is the locus of points $s \in S$ such that $\mathcal{E}_s(N)$ is globally generated and $\mathrm{H}^1(C, \mathcal{E}_s(N)) = 0$. If $S \to \underline{\mathrm{Coh}}_{r,d}(C)$ is the map classifying $\mathcal{E}$, the base change $\mathcal{U}_N \times_{\underline{\mathrm{Coh}}_{r,d}(C)} S$ is identified with S'', and the claim is established.

For each N, consider the Quot scheme

$$Q_N := \operatorname{Quot}^P(\mathcal{O}_C(-N)^{\oplus P(N)}/C/\Bbbk)$$

parameterizing quotients $\mathcal{O}_C(-N)^{\oplus P(N)} \twoheadrightarrow F$ with Hilbert polynomial P (Theorem 1.1.3). A similar argument as above shows that there is an open subscheme $Q'_N \subseteq Q_N$ parameterizing quotients $q \colon \mathcal{O}_C(-N)^{\oplus P(N)} \twoheadrightarrow F$ such that $\mathrm{H}^0(q(N)) \colon \mathrm{H}^0(C, \mathcal{O}_C)^{\oplus P(N)} \to \mathrm{H}^0(C, F(N))$ is surjective and $\mathrm{H}^1(C, F(N)) = 0$. The Quot scheme Q_N inherits a natural action from $\mathrm{GL}_{P(N)}$: given $g \in \mathrm{GL}_{P(N)}$ and $[q \colon \mathcal{O}_C(-N)^{\oplus P(N)} \twoheadrightarrow F] \in Q_N$,

$$g \cdot q := [\mathcal{O}_C(-N)^{\oplus P(N)} \xrightarrow{g^{-1}} \mathcal{O}_C(-N)^{\oplus P(N)} \xrightarrow{q} F] \in Q_N.$$

The locus $Q'_N \subseteq Q_N$ is clearly $\mathrm{GL}_{P(N)}$-invariant. The morphism

$$Q'_N \to \mathcal{U}_N, \qquad [\mathcal{O}_C(-N)^{\oplus P(N)} \twoheadrightarrow F] \mapsto F,$$

is $\mathrm{GL}_{P(N)}$-invariant (i.e., $\mathrm{GL}_{P(N)}$-equivariant with respect to the trivial action on $\mathcal{U}_N$) and thus descends to a morphism $\Psi^{\mathrm{pre}} \colon [Q'_N/\mathrm{GL}_{P(N)}]^{\mathrm{pre}} \to \mathcal{U}_N$ of prestacks. To see that Ψ^{pre} is fully faithful, consider a quotient $[q \colon \mathcal{O}_C(-N)^{\oplus P(N)} \twoheadrightarrow F] \in Q_N(\Bbbk)$ and the induced map $\operatorname{Stab}^{\mathrm{GL}_{P(N)}}(q) \to \operatorname{Aut}(F)$. An automorphism $\alpha \in \operatorname{Aut}(F)$ induces an automorphism of $\mathrm{H}^0(C, F(N)) \cong \Bbbk^{P(N)}$ fixing the kernel of $\mathrm{H}^0(q(N))$. This defines an inverse to $\operatorname{Stab}(q) \to \operatorname{Aut}(F)$, and it is not hard to extend this construction to families of quotients.

Since $\underline{\mathrm{Coh}}_{r,d}(C)$ is a stack (Proposition 2.5.17), so is $\mathcal{U}_N$, and there is an induced morphism $\Psi \colon [Q'_N/\mathrm{GL}_{P(N)}] \to \mathcal{U}_N$ of stacks which is fully faithful (by

Exercise 2.5.21) and essentially surjective (by construction). We conclude that $\mathcal{U}_N = [Q'_N / \mathrm{GL}_{P(m)}]$ and that

$$\underline{\mathrm{Coh}}_{r,d}(C) = \bigcup_N [Q'_N / \mathrm{GL}_{P(N)}].$$

The algebraicity of quotient stacks (Theorem 3.1.10) implies the algebraicity of $\underline{\mathrm{Coh}}_{r,d}(C)$. □

Exercise 3.1.23.

(1) (easy) Modify the above argument to show that $\mathcal{B}un(X)$ and $\underline{\mathrm{Coh}}(X)$ are algebraic stacks if X is a projective scheme over a field $\Bbbk$ of arbitrary dimension.

(2) (moderate) More generally, show that if $X \to S$ is a strongly projective (see §1.4.4) morphism of noetherian schemes, then the stack $\underline{\mathrm{Coh}}(X/S)$, whose objects over an S-scheme T are finitely presented, quasi-coherent sheaves on X_T flat over T, is an algebraic stack. (See also [LMB00, Thm. 4.6.2.1].)

See also Theorem C.7.7 for a proof using Artin's Axioms.

Exercise 3.1.24 (hard)**.** Let X be a projective scheme over a field $\Bbbk$ and G be an affine algebraic group over $\Bbbk$. Define a stack $\mathcal{B}un_G(X)$ of principal G-bundles over X, and prove that it is algebraic.

Hint: Choose a faithful embedding $G \hookrightarrow \mathrm{GL}_n$ and show that inducing principal bundles defines a representable morphism $\mathcal{B}un_G(X) \to \mathcal{B}un_{\mathrm{GL}_n}(X)$ to the stack of vector bundles on X.

3.1.7 Universal families

Lemma 3.1.25 (Generalized 2-Yoneda Lemma)**.** *Let $\mathcal{X}$ be a stack over $\mathrm{Sch}_{\text{ét}}$. If $\mathcal{T}$ is an algebraic stack and $U \to \mathcal{T}$ is a smooth presentation, define the category $\mathcal{X}(\mathcal{T})$ as the equalizer*

$$\mathcal{X}(\mathcal{T}) := \mathrm{Eq}\left(\mathcal{X}(U) \rightrightarrows \mathcal{X}(U \times_{\mathcal{T}} U) \Rrightarrow \mathcal{X}(U \times_{\mathcal{T}} U \times_{\mathcal{T}} U) \right),$$

i.e., an object is the data of a pair (a, α) where $a \in \mathcal{X}(U)$ and $\alpha\colon p_1^* a \xrightarrow{\sim} p_2^* a$ is an isomorphism satisfying the cocycle condition $p_{23}^*\alpha \circ p_{12}^*\alpha = p_{13}^*\alpha$, while a morphism $(a,\alpha) \to (a',\alpha')$ is the data of a morphism $\beta\colon a \to a'$ satisfying $p_2^*\beta \circ \alpha = \alpha' \circ p_1^*\beta$. There is a natural equivalence of categories

$$\mathrm{Mor}(\mathcal{T}, \mathcal{X}) \to \mathcal{X}(\mathcal{T}). \tag{3.1.26}$$

In particular, $\mathcal{X}(\mathcal{T})$ is independent of the presentation.

Proof. Let $q\colon U \to \mathcal{T}$ denote the smooth presentation, and consider the 2-commutative diagram

$$\begin{tikzcd} U \times_{\mathcal{T}} U \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\swArrow_{\gamma}"] & U \arrow[d, "q"] \\ U \arrow[r, "q"] & \mathcal{T}. \end{tikzcd}$$

Given a morphism $f\colon \mathcal{T} \to \mathcal{X}$, let $a \in \mathcal{X}(U)$ be the object corresponding via the 2-Yoneda Lemma (2.4.21) to the composition $f \circ q\colon U \to \mathcal{T} \to \mathcal{X}$. Composing the 2-isomorphism $\gamma\colon q \circ p_2 \to q \circ p_1$ with f induces a 2-isomorphism $\alpha\colon p_1^* a \xrightarrow{\sim} p_2^* a$. The map (3.1.26) is defined by taking the morphism f to the pair $(a, \alpha) \in \mathcal{X}(\mathcal{T})$.

To show that (3.1.26) is an equivalence, we first assume that it holds for algebraic spaces, and so it holds for U, $U \times_{\mathcal{T}} U$, and $U \times_{\mathcal{T}} U \times_{\mathcal{T}} U$. To construct an inverse $\Psi\colon \mathcal{X}(\mathcal{T}) \to \mathrm{Mor}(\mathcal{T}, \mathcal{X})$ of (3.1.26), let $(a, \alpha) \in \mathcal{X}(\mathcal{T})$ and $f_a\colon U \to \mathcal{X}$ be the map corresponding to a. To define a morphism $\Psi(a, \alpha)\colon \mathcal{T} \to \mathcal{X}$ of stacks, let $b \in \mathcal{T}$ be an object over a scheme S classified by a map $f_b\colon S \to \mathcal{T}$, and consider the commutative diagram

$$\begin{tikzcd} S_2 \arrow[r, hook] \arrow[drr] & S_1 \arrow[r] & S_0 \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & U \arrow[r, "f_a"] \arrow[d, "q"] & \mathcal{X} \\ & & S \arrow[r, "f_b"] & \mathcal{T}, & \end{tikzcd}$$

where $S_0 = S \times_{\mathcal{T}} U$ is an algebraic space, $S_1 \to S_0$ is an étale presentation, and $S_2 \to S$ is an etale surjection of schemes factoring through S_1, which exists because smooth morphisms have sections étale locally (Corollary A.3.5). The composition in the top row defines an object c_2 of $\mathcal{X}$ over S_2. The isomorphism $\alpha\colon p_1^* a \xrightarrow{\sim} p_2^* a$ over $U \times_{\mathcal{T}} U$ induces an isomorphism $\beta\colon p_1^* c_2 \xrightarrow{\sim} p_2^* c_2$ over $S_2 \times_S S_2$. Since α satisfies the cocycle condition and (3.1.26) is an equivalence for $U \times_{\mathcal{T}} U \times_{\mathcal{T}} U$, β also satisfies the cocycle condition, and thus there is an object c of $\mathcal{X}$ over S pulling back to c_2. We set $\Psi(a, \alpha)(b) = c \in \mathcal{X}(S)$. The details of verifying that is an inverse are left to the reader. Finally, the same argument applies to show that (3.1.26) is an equivalence when $\mathcal{T}$ is an algebraic space by using an étale presentation $U \to \mathcal{T}$. $\square$

Definition 3.1.27 (Universal family)**.** If $\mathcal{X}$ is an algebraic stack, the *universal family over* $\mathcal{X}$ is the object $u \in \mathcal{X}(\mathcal{X})$ (unique up to unique isomorphism) corresponding to the identity morphism $\mathrm{id}_{\mathcal{X}}\colon \mathcal{X} \to \mathcal{X}$ under the equivalence (3.1.26).

Exercise 3.1.28 (details)**.** Let $\mathcal{X}$ be an algebraic stack and $u \in \mathcal{X}(\mathcal{X})$ be the universal family. If $g\colon \mathcal{S} \to \mathcal{T}$ is a morphism of algebraic stacks, show that there is a natural pullback functor $g^*\colon \mathcal{X}(\mathcal{T}) \to \mathcal{X}(\mathcal{S})$, and that if $f_a\colon T \to \mathcal{X}$ is a map from scheme classified by an object $a \in \mathcal{X}(T)$, then a is isomorphic to the pullback $f_a^* u$.

In practice, for an algebraic stack $\mathcal{X}$ arising from a moduli problem, the geometric significance of objects of $\mathcal{X}(S)$ is usually clear.

Example 3.1.29 (Universal family of $\mathcal{M}_g$)**.** A *family of smooth curves* $\mathcal{C} \to \mathcal{T}$ over an algebraic stack $\mathcal{T}$ is a morphism of algebraic stacks which is representable by schemes, proper, flat, and finitely presented such that for every geometric point $\mathrm{Spec}\,\Bbbk \to \mathcal{T}$, the fiber $\mathcal{C}_s := \mathcal{C} \times_S \Bbbk$ is a smooth, connected, and projective curve of genus g. Descent theory provides an identification of $\mathcal{M}_g(\mathcal{T})$ with the category of family of smooth curves over $\mathcal{T}$. Let $\mathcal{U}_g \to \mathcal{M}_g$ be the family of curves corresponding to the universal family $u \in \mathcal{M}_g(\mathcal{M}_g)$. By Exercise 2.4.42, the universal family is identified with the map $\mathcal{M}_{g,1} \to \mathcal{M}_g$ forgetting a section, where $\mathcal{M}_{g,1}$ is the stack of smooth 1-pointed curves. For every morphism $S \to \mathcal{M}_g$ corresponding to a family $\mathcal{C} \to S$ of smooth curves, there is a cartesian diagram

$$\begin{tikzcd} \mathcal{C} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{U}_g = \mathcal{M}_{g,1} \arrow[d] \\ S \arrow[r] & \mathcal{M}_g. \end{tikzcd}$$

See Figure 0.3.26 for a visualization of $\mathcal{U}_g \to \mathcal{M}_g$.

Example 3.1.30 (Universal family of $\mathcal{B}un_{r,d}(C)$)**.** Later we will define vector bundles on an algebraic stack using the theory of quasi-coherent sheaves (see §6.1). By decent theory, $\mathcal{B}un_{r,d}(C)(\mathcal{T})$ is identified with the groupoid of vector bundles on $C \times \mathcal{T}$ of rank r and degree d. The universal family is a vector bundle $\mathcal{E}_{\mathrm{univ}}$ on $C \times \mathcal{B}un_{r,d}(C)$ with the property that if S is a scheme S and E is a vector bundle on $C \times S$ classified by a map $f\colon S \to \mathcal{B}un_{r,d}(C)$, then $E \cong (\mathrm{id} \times f)^* \mathcal{E}_{\mathrm{univ}}$.

3.1.8 Desideratum

We will develop the foundations of algebraic spaces and stacks in the forthcoming chapters but we first share some of the highlights.

The importance of the diagonal. When overhearing others discussing algebraic stacks, you may have wondered what is all the fuss about the diagonal. Well, I will tell you—the diagonal encodes the stackiness!

First and foremost, the diagonal $\mathcal{X} \to \mathcal{X} \times \mathcal{X}$ of an algebraic stack is representable and the diagonal $X \to X \times X$ of an algebraic space is representable by schemes (Theorem 3.2.1). The *automorphism group* or *stabilizer* of a field-valued point $x\colon \operatorname{Spec} \Bbbk \to \mathcal{X}$ is defined as the sheaf

$$G_x := \underline{\mathrm{Aut}}_{\mathcal{X}(\Bbbk)}(x) = \underline{\mathrm{Isom}}_{\mathcal{X}(\Bbbk)}(x, x)$$

and is identified with the fiber product $\mathcal{X} \times_{\mathcal{X} \times \mathcal{X}} \operatorname{Spec} \Bbbk$ by Exercise 2.4.39. By Representability of the Diagonal (3.2.1), the stabilizer G_x is representable by a group algebraic space over $\Bbbk$. See §3.2.2 for a further discussion of stabilizers.

For schemes (resp., separated schemes), the diagonal is an immersion (resp., closed immersion). For algebraic spaces, the diagonal need not be an immersion, and for algebraic stacks, the diagonal need not even be a monomorphism as the fiber over $(x, x)\colon \operatorname{Spec} \Bbbk \to \mathcal{X} \times \mathcal{X}$ (that is, the stabilizer G_x), may be non-trivial. Theorems 3.6.4 and 3.6.6 characterize algebraic spaces and Deligne–Mumford stacks in terms of the diagonal, as expressed by the following table:

Type of space	Property of the diagonal	Property of stabilizers
algebraic space	monomorphism	trivial
Deligne–Mumford stack	unramified	reduced finite groups[1]
algebraic stack	arbitrary	arbitrary

Table 3.1.31: Characterization of algebraic spaces and Deligne–Mumford stacks. See Table 0.0.1, Figure 0.0.2, and Table 0.6.5 for further diagrammatic explanations of the trichotomy of moduli.

This characterization will allow us to conclude that $\mathcal{M}_g$ is Deligne–Mumford (Corollary 3.6.10). We show later in Corollaries 3.6.9 and 4.5.11 that a morphism $f\colon \mathcal{X} \to \mathcal{Y}$ of algebraic stacks is representable if and only if the induced map on stabilizer groups $G_x \to G_{f(x)}$ is injective for every field-valued point $x \in \mathcal{X}(\Bbbk)$.

Properties of algebraic spaces.

[1] If the diagonal is not quasi-compact, the stabilizers will only be discrete and reduced.

- (Algebraicity of Quotients by Equivalence Relations) If $R \rightrightarrows U$ is an étale equivalence relation of schemes, the quotient sheaf U/R is an algebraic space (Theorem 3.4.13). This is extended to smooth equivalence relations of algebraic spaces in Corollary 4.5.12 and fppf equivalence relations in Corollary 6.3.6. In particular, the quotient of a free action by an algebraic group on an algebraic space exists as an algebraic space.
- (Zariski's Main Theorem) Generalizing Zariski's Main Theorem for Schemes (A.7.3), if $X \to Y$ is a quasi-finite and separated morphism of noetherian algebraic spaces, then there exists a factorization $X \hookrightarrow \widetilde{X} \to Y$ where $X \hookrightarrow \widetilde{X}$ is an open immersion and $\widetilde{X} \to Y$ is finite (Theorem 4.5.9). In particular, $X \to Y$ is quasi-affine. Zariski's Main Theorem also holds for representable morphisms of Deligne–Mumford stacks, and is extended to representable morphisms of algebraic stacks in Theorem 6.1.18.
- (Affine Criteria) By Serre's Criterion for Affineness (4.5.16), an algebraic space X is an affine scheme if and only if $\Gamma(X, -)$ is exact on the category of quasi-coherent sheaves. By Chevalley's Criterion for Affineness (4.5.20), if $X \to Y$ is a finite surjection of noetherian algebraic spaces and X is affine, then Y is also affine.
- (Algebraic spaces vs schemes) If X is a quasi-separated algebraic space, there exists a dense open subspace $U \subseteq X$ which is a scheme (Theorem 4.5.1). A quasi-separated group algebraic space locally of finite type over a field is a scheme (Theorem 4.5.28); in particular, the stabilizer of a field-valued point of an algebraic stack with quasi-separated diagonal is a group scheme. A separated one dimensional algebraic spaces is a scheme (Theorem 4.5.32).

Properties of Deligne–Mumford stacks.

- (Algebraicity of Quotients by Groupoids) If $R \rightrightarrows U$ is an étale groupoid of schemes, the quotient stack $[U/R]$ is a Deligne–Mumford stack (Theorem 3.4.13).
- (Local Structure of Deligne–Mumford Stacks) If $\mathcal{X}$ is a separated Deligne–Mumford stack and $x \in \mathcal{X}$ is a finite type point with stabilizer G_x, there exists an affine étale neighborhood $[\operatorname{Spec} A/G_x] \to \mathcal{X}$ of x (Theorem 4.3.1).
- (Keel–Mori Theorem) If $\mathcal{X}$ is an Deligne–Mumford stack separated and of finite type over a noetherian scheme S, there exists a coarse moduli space $\mathcal{X} \to X$ where X is an algebraic space separated over S (Theorem 4.4.6).
- (Le Lemme de Gabber) If $\mathcal{X}$ is a Deligne–Mumford stack (e.g., algebraic space) separated and of finite type over a noetherian scheme S, there exists a scheme Z and a finite surjection $Z \to \mathcal{X}$ (Theorem 4.6.1).

Properties of algebraic stacks.

- (Algebraicity of Quotients by Groupoids) If $R \rightrightarrows U$ is a smooth groupoid of schemes, the quotient stack $[U/R]$ is an algebraic stack (Theorem 3.4.13). This is extended to fppf groupoids of algebraic spaces in Corollary 6.3.6.
- (Residual Gerbes and Minimal Presentations) If $\mathcal{X}$ is a noetherian algebraic stack and $x \in |\mathcal{X}|$ is a finite type point, then the residual gerbe $\mathcal{G}_x$ exists and $\mathcal{G}_x \hookrightarrow \mathcal{X}$ is a locally closed immersion (Proposition 3.5.16), and if in addition the stabilizer G_x is smooth, there is a smooth presentation $(U, u) \to (\mathcal{X}, x)$ of relative dimension $\dim G_x$ such that $\mathcal{G}_x \times_{\mathcal{X}} U \cong \operatorname{Spec} \kappa(u)$ (Theorem 3.6.1). Later we establish the existence of residual gerbes for any point $x \in |\mathcal{X}|$ (Proposition 6.4.26).

- (Infinitesimal Lifting Criteria) We establish infinitesimal lifting criteria characterizing morphisms that are smooth, étale, unramified, or that have unramified diagonal (Theorem 3.7.1).
- (Valuative Criteria) We provide valuative criteria characterizing morphisms that are proper, universally closed, separated, or that have separated diagonal (Theorem 3.8.7).
- (Local Structure of Algebraic Stacks) If $\mathcal{X}$ is an algebraic stack of finite type over an algebraically closed field $\Bbbk$ with affine diagonal, every point $x \in \mathcal{X}(\Bbbk)$ with linearly reductive stabilizer G_x has an affine étale neighborhood $[\mathrm{Spec}(A)/G_x] \to \mathcal{X}$ of x (Theorem 6.7.1).
- (Existence of Good Moduli Spaces) Let $\mathcal{X}$ be an algebraic stack, of finite type over an algebraically closed field $\Bbbk$ of characteristic 0, with affine diagonal. If $\mathcal{X}$ is S-complete and Θ-complete, then there exists a good moduli space $\mathcal{X} \to X$ where X is a separated algebraic space of finite type over $\Bbbk$ (Theorem 6.10.1).

Historical comments

Deligne–Mumford and algebraic stacks were first introduced in [DM69] and [Art74b]—and in both cases referred to as *algebraic stacks*—with conventions slightly different than ours. Namely, [DM69, Def. 4.6] assumed that algebraic stacks had an étale presentation $U \to \mathcal{X}$ which is representable by schemes (which as they point out in a footnote gives the "right" definition when the diagonal is separated and quasi-compact). On the other hand, [Art74b, Def. 5.1] assumed that algebraic stacks are locally of finite type over an excellent Dedekind domain. The term *Artin stack*—which we refrain from using—is sometimes reserved for stacks that satisfy Artin's Axioms (C.7.1 or C.7.4) as first established in [Art74b, Thm. 5.3].

We follow the conventions of [LMB00], [Ols16], and [SP] with the following exceptions:

- [LMB00] imposes a separated and quasi-compact hypothesis on the diagonal, but we do not as in [Ols16] and [SP].
- We define algebraic spaces and stacks over $\mathrm{Sch}_{\text{ét}}$, while [LMB00] works over the big étale site $\mathrm{AffSch}_{\text{ét}}$ of affine schemes and [SP] works over $\mathrm{Sch}_{\mathrm{fppf}}$. These gives equivalent notions of algebraic stacks (c.f., [SP, Tag 076U]).
- What we call *representable* (resp., *representable by schemes*) for a morphism of stacks is called *representable by algebraic spaces* (resp., *representable*) in [SP, Tags 02ZW and 02Y7].

3.2 Representability of the diagonal

> *The diagonal gives a three-dimensional feeling to dancers that cannot be achieved when they are only front and back. On the diagonal, more movement is automatically visible.*
>
> Lucinda Childs

We prove the diagonals of algebraic spaces and stacks are necessarily representable. The diagonal is used to define stabilizers and the inertia stack.

3.2.1 Representability

Theorem 3.2.1 (Representability of the Diagonal)**.**

(1) The diagonal of an algebraic space is representable by schemes.

(2) The diagonal of an algebraic stack is representable.

Proof. Let X be an algebraic space and $U \to X$ be an étale presentation. Define the scheme $R := U \times_X U$. If $T \to X \times X$ is a morphism from a scheme, we need to show that the sheaf $Q_T = X \times_{X \times X} T$ is, in fact, a scheme. Since $U \to X$ is étale, surjective, and representable by schemes, so is $U \times U \to X \times X$. The base change of $T \to X \times X$ by $U \times U \to X \times X$ is a scheme T' which is surjective étale over T. In the cartesian cube

$$\begin{tikzcd}
& Q_{T'} \arrow[rr] \arrow[dl] \arrow[dd] & & T' \arrow[dl] \arrow[dd] \\
R \arrow[rr, crossing over] \arrow[dd] & & U \times U \arrow[dd, crossing over] & \\
& Q_T \arrow[rr] \arrow[dl] & & T \arrow[dl] \\
X \arrow[rr] & & X \times X, &
\end{tikzcd} \tag{3.2.2}$$

Q_T is a sheaf on $\mathrm{Sch}_{\text{ét}}$ while $Q_{T'}$ is a scheme. Since $R \to U \times U$ is a locally quasi-finite and separated morphism of schemes, so is $Q_{T'} \to T'$. (If X has quasi-compact diagonal, then by Zariski's main theorem $R \to U \times U$ is quasi-affine and thus so is $Q_{T'} \to T'$.) Since Q_T is a sheaf in the étale topology that pulls back to a scheme $Q_{T'}$ locally quasi-finite and separated over T', we may apply the Descent Criterion for an Fppf Sheaf to be a Scheme (2.3.17) to conclude that Q_T is a scheme.

If $\mathcal{X}$ is an algebraic stack and $U \to \mathcal{X}$ is a smooth presentation, we may imitate the above argument. The fiber product $R := U \times_{\mathcal{X}} U$ is an algebraic space. If $T \to \mathcal{X} \times \mathcal{X}$ is a morphism from a scheme, its base change along $U \times U \to \mathcal{X} \times \mathcal{X}$ yields an algebraic space T_1 which is surjective smooth over T. Choose an étale presentation $T_2 \to T_1$. Then $T_2 \to T$ is a surjective smooth morphism of schemes which has a section after an étale cover $T' \to T$ (Proposition A.3.4). The composition $T' \to T_2 \to T_1 \to U \times U$ provides a lift of $T \to \mathcal{X} \times \mathcal{X}$. We obtain a diagram similar to (3.2.2) but where the left and right squares are not necessarily cartesian. Since Q_T is identified with $\underline{\mathrm{Isom}}_{\mathcal{X}(T)}(a, a)$ where $a \colon T \to \mathcal{X}$ (see Exercise 2.4.39), Axiom (1) for $\mathcal{X}$ being a stack implies that Q_T is a sheaf. The morphism $Q_{T'} \to Q_T$ is étale, surjective, and representable by schemes (as $T' \to T$ is). Choosing an étale presentation $V \to Q_{T'}$ of the algebraic space $Q_{T'}$, the composition $V \to Q_{T'} \to Q_T$ yields an étale presentation showing that Q_T is an algebraic space. □

Corollary 3.2.3.

(1) If the diagonal of a stack $\mathcal{X}$ is representable (resp., representable by a scheme), then every morphism $U \to \mathcal{X}$ from a scheme is representable (resp., representable by a scheme).

(2) Every morphism from a scheme to an algebraic stack (resp., algebraic spaces) is representable (resp., representable by schemes).

Proof. The first part follows from the cartesian diagram of Exercise 2.4.38

$$\begin{tikzcd}
T_1 \times_{\mathcal{X}} T_2 \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & T_1 \times T_2 \arrow[d] \\
\mathcal{X} \arrow[r] & \mathcal{X} \times \mathcal{X}.
\end{tikzcd}$$

associated to any two maps $T_1 \to \mathcal{X}$ and $T_2 \to \mathcal{X}$ from schemes to an algebraic stack. The second part follows directly from the first part and the Representability of the Diagonal (Theorem 3.2.1). $\square$

Exercise 3.2.4.

(a) If $\mathcal{X} \to \mathcal{Y}$ is a representable morphism of algebraic stacks (e.g., a morphism of algebraic spaces), show that $\mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ is representable by schemes.

(b) If $\mathcal{X} \to \mathcal{Y}$ is a morphism of algebraic stacks, show that $\mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ is representable.

3.2.2 Stabilizer groups and the inertia stack

Now that we know the diagonal is representable, we can discuss its properties. Crucially, the diagonal encodes the automorphism or stabilizer groups. stabilizer groups.

Definition 3.2.5 (Stabilizers)**.** If $\mathcal{X}$ is an algebraic stack and $x\colon \operatorname{Spec} \Bbbk \to \mathcal{X}$ is a field-valued point, the *stabilizer of* x (or *automorphism group of* x) is defined as the group algebraic space $G_x := \underline{\operatorname{Aut}}_{\mathcal{X}(\Bbbk)}(x)$.

The stabilizer G_x is identified with the fiber product

$$\begin{tikzcd}
G_x = \underline{\operatorname{Aut}}_{\mathcal{X}(\Bbbk)}(x) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \operatorname{Spec} \Bbbk \arrow[d, "{(x,x)}"] \\
\mathcal{X} \arrow[r, "\Delta"] & \mathcal{X} \times \mathcal{X}
\end{tikzcd}$$

by Exercise 2.4.39, and is representable by an algebraic space over $\Bbbk$ by Representability of the Diagonal (3.2.1). The stabilizer G_x is a *group algebraic space*, i.e., an algebraic space G_x with multiplication, inverse, and identity morphisms satisfying the commutativity conditions of Definition B.1.1 (or equivalently a group object in the category of algebraic spaces). In fact, G_x is actually a group scheme locally of finite type as long as the diagonal of $\mathcal{X}$ is quasi-separated (Corollary 4.5.31).

Remark 3.2.6. Let G be a group scheme over a field $\Bbbk$ acting on a $\Bbbk$-scheme U via $\sigma\colon G \times U \to U$, and let $u \in U(\Bbbk)$. The stabilizer of the image of u in $[U/G]$ is the usual stabilizer group scheme, i.e., the fiber product of $(\sigma, p_2)\colon G \times U \to U \times U$ along $(u, u)\colon \operatorname{Spec} \Bbbk \to U \times U$.

Exercise 3.2.7.

(a) Show that the stabilizer of a field-valued point of a fiber product of algebraic stacks is the fiber product of stabilizers, i.e., for $x' \in (\mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}')(\Bbbk)$, then $G_{x'} \cong G_x \times_{G_y} G_{y'}$, where x, y and y' are the images of x'.

(b) Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks and $x \in \mathcal{X}(\Bbbk)$ be a field-valued point. Show that the fiber of the diagonal $\mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ over the point $(x, x, \mathrm{id}) \in (\mathcal{X} \times_{\mathcal{Y}} \mathcal{X})(\Bbbk)$ is identified with $\ker(G_x \to G_y)$. What is the fiber of the diagonal over an arbitrary field-valued point of $\mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$?

Exercise 3.2.8. Let $\mathcal{X}$ be a Deligne–Mumford stack .

(a) For a field-valued point $x \in \mathcal{X}(\Bbbk)$, show that G_x is a separated étale group scheme over $\Bbbk$. If in addition G_x is quasi-compact[2], show that G_x is a finite étale group scheme over $\Bbbk$.

Hint: First show that G_x is an etale group algebraic space over $\Bbbk$ which becomes a scheme after the base change by a finite field extension $\Bbbk \to \Bbbk'$. Apply Proposition B.1.8 to conclude that $G_x \times_{\Bbbk} \Bbbk'$ is separated. Then apply the Descent Criterion for an Fppf Sheaf to be a Scheme (2.3.17).

(b) Show that the diagonal of $\mathcal{X}$ is unramified.[3]

We will see later that these properties characterize Deligne–Mumford stacks; see Theorem 3.6.4.

Remark 3.2.9. If $\mathcal{X}$ is a quasi-separated Deligne–Mumford stack, the above exercise implies that the stabilizer G_x of $x \in \mathcal{X}(\Bbbk)$ is a finite étale group scheme. When $\Bbbk$ is algebraically closed, then G_x corresponds to the finite group scheme corresponding to the abstract finite group $\mathrm{Aut}_{\mathcal{X}(\Bbbk)}(\Bbbk)$.

The stabilizer group G_x varies as $x \in \mathcal{X}$ varies, and the stabilzer groups naturally form a family. We have already seen this: if $a\colon T \to \mathcal{X}$ is an object, then $\mathrm{Isom}_{\mathcal{X}(T)}(a) \to S$ is a group algebraic space such that the fiber over a point $s \in S$ is the stabilizer of the restriction $a|_{\operatorname{Spec} \kappa(s)}$ of a to $\operatorname{Spec} \kappa(s)$. Applying this to the identity map $\mathrm{id}_{\mathcal{X}}\colon \mathcal{X} \to \mathcal{X}$ produces the inertia stack.

Definition 3.2.10 (Inertia stack)**.** The *inertia stack* of an algebraic stack $\mathcal{X}$ is the fiber product

$$\begin{tikzcd} I_{\mathcal{X}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{X} \arrow[d] \\ \mathcal{X} \arrow[r] & \mathcal{X} \times \mathcal{X}. \end{tikzcd}$$

In the relative setting of a morphism $\mathcal{X} \to \mathcal{Y}$ of algebraic stacks, the *relative inertia stack* is $I_{\mathcal{X}/\mathcal{Y}} := \mathcal{X} \times_{\mathcal{X} \times_{\mathcal{Y}} \mathcal{X}} \mathcal{X}$.

The inertia stack $I_{\mathcal{X}}$ is equivalent to the category of pairs (x, α), where $x \in \mathcal{X}$ is an object over $S \in \mathrm{Sch}$ and $\alpha \in \mathrm{Aut}_{\mathcal{X}(S)}(x)$ (see Exercise 3.2.15(a)). The fiber of $I_{\mathcal{X}} \to \mathcal{X}$ over a field-valued point $x \in \mathcal{X}(\Bbbk)$ is precisely the stabilizer G_x. We can therefore think of $I_{\mathcal{X}}$ as a group scheme (or really group algebraic space) over $\mathcal{X}$ incorporating all of the stabilizers of $\mathcal{X}$. If we let $(\mathrm{Sch}/\mathcal{X})_{\text{ét}}$ be the big étale site of schemes over $\mathcal{X}$, then $I_{\mathcal{X}}$ is a sheaf of groups on $(\mathrm{Sch}/\mathcal{X})_{\text{ét}}$ where $I_{\mathcal{X}}(a) = \mathrm{Aut}_S(a)$ for $a \in \mathcal{X}(S)$.

If $\alpha\colon a' \to a$ is a morphism over $S' \to S$, there is a natural pullback functor $\alpha^*\colon \mathrm{Aut}_S(a) \to \mathrm{Aut}_{S'}(a')$ defined as follows: for $\beta \in \mathrm{Aut}_S(a)$, the image $\alpha^*(\beta)$ is the unique dotted arrow (provided by Axiom (2) of a prestack (Definition 2.4.1)) making the diagram

$$\begin{tikzcd} a' \arrow[r, dashed, "\alpha^*(\beta)"'] \arrow[rr, bend left, "\beta\circ\alpha"] & a' \arrow[r, "\alpha"'] & a \end{tikzcd} \tag{3.2.11}$$

commute. Note that if $\alpha\colon a' \to a$ is an isomorphism, then $\alpha^*(\beta) = \alpha^{-1} \circ \beta \circ \alpha$ is conjugation by α.

[2]This is implied by the quasi-separatedness of $\mathcal{X}$; see Definition 3.3.11.

[3]Since unramified morphisms are schemes are stable under base change and étale local on the source, the notion is well-defined for representable morphism of algebraic stacks.

Exercise 3.2.12. Let $G \to S$ be a group scheme acting on a scheme $U \to S$, and let $\mathcal{X} = [U/G]$ be the quotient stack. Show that there is a cartesian diagram

$$\begin{tikzcd} S_U \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & U \arrow[d] \\ I_{\mathcal{X}} \arrow[r] & \mathcal{X} \end{tikzcd}$$

where $S_U \to U$ is the stabilizer group scheme, i.e., the fiber product of the action map $G \times U \to U \times U$ and the diagonal $U \to U \times U$.

Example 3.2.13. The inertia stack of the classifying stack $B\mathbb{G}_m$ is $I_{B\mathbb{G}_m} \cong \mathbb{G}_m \times B\mathbb{G}_m$. Similarly, if we let $\mathbb{G}_m$ act on $\mathbb{G}_m \times \mathbb{A}^1$ via the product of the trivial and the scaling action and we let $V(x(t-1)) \subseteq \mathbb{G}_m \times \mathbb{A}^1$ be the $\mathbb{G}_m$-invariant closed subscheme, then $I_{[\mathbb{A}^1/\mathbb{G}_m]} \cong [V(x(t-1))/\mathbb{G}_m]$.

Exercise 3.2.14.

(a) If G is a smooth affine algebraic group over a field $\Bbbk$, show that the inertia stack of BG is the quotient $[G/G]$ where G acts on itself via conjugation. In particular, if G is abelian then $I_{BG} \cong G \times BG$.

(b) More generally, show that if G acts on a $\Bbbk$-scheme U, show that $I_{[U/G]} \cong [(G \times U)/G]$ where the action is given by $g \cdot (h, u) = (ghg^{-1}, gu)$.

(c) Let G be a group scheme over $\Bbbk$ corresponding to a finite abstract group. If G acts on a $\Bbbk$-scheme U, then

$$I_{[U/G]} = \coprod_{g \in \mathrm{Conj}(G)} [U^g/C_g],$$

where $\mathrm{Conj}(G)$ is the set of conjugacy classes of elements of G, C_g is the centralizer of g, and $U^g := \{x \in U \mid gx = x\}$ is the closed locus fixed by g.

(d) Explicitly compute the inertia stack for the quotient stack $[\mathbb{A}^3/S_3]$ of the permutation action.

Exercise 3.2.15. Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks.

(a) Show that the relative inertia stack $I_{\mathcal{X}/\mathcal{Y}}$ is equivalent to the category of pairs (x, α) where $x \in \mathcal{X}$ and $\alpha\colon x \xrightarrow{\sim} x$ is an isomorphism such that $f(\alpha) = \mathrm{id}_{f(x)}$, and show that the identity section $\mathcal{X} \to I_{\mathcal{X}/\mathcal{Y}}$ takes an object x to (x, id_x).

(b) Show that there are morphisms $I_{\mathcal{X}/\mathcal{Y}} \to I_{\mathcal{X}} \to I_{\mathcal{Y}} \times_{\mathcal{Y}} \mathcal{X}$ of algebraic stacks over $\mathcal{X}$ which corresponds over a field-valued point $x \in \mathcal{X}(\Bbbk)$ to a left exact sequence $1 \to K_x \to G_x \to G_{f(x)}$ of group algebraic spaces.

(c) Show that there is a cartesian diagram

$$\begin{tikzcd} I_{\mathcal{X}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & I_{\mathcal{Y}} \times_{\mathcal{Y}} \mathcal{X} \arrow[d] \\ \mathcal{X} \arrow[r] & \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}. \end{tikzcd}$$

Hint for (c): An object of $I_{\mathcal{Y}} \times_{\mathcal{Y}} \mathcal{X}$ over a scheme S is a quadruple (y, α, x, β) where $y \in \mathcal{Y}(S)$, $\alpha\colon y \xrightarrow{\sim} y$, $x \in \mathcal{X}(S)$, and $\beta\colon y \xrightarrow{\sim} f(x)$. On the other hand, an object of $\mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ over S is a triple (x_1, x_2, γ) where $x_1, x_2 \in \mathcal{X}(S)$ and $\gamma\colon f(x_1) \xrightarrow{\sim} f(x_2)$. Define $I_{\mathcal{Y}} \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ on fiber categories by $(y, \alpha, x, \beta) \mapsto (x, x, \beta \circ \alpha \circ \beta^{-1})$. Construct a map $I_{\mathcal{X}}(S)$ to the fiber product of $\mathcal{X}(S)$ and $(I_{\mathcal{Y}} \times_{\mathcal{Y}} \mathcal{X})(S)$ over $(\mathcal{X} \times_{\mathcal{Y}} \mathcal{X})(S)$, and show that it is an equivalence.

3.3 First properties

'Of course, here I'm working with the moduli stack rather than with the moduli space. For those of you who aren't familiar with stacks, don't worry: basically, all it means is that I'm allowed to pretend that the moduli space is smooth and that there's a universal family over it.'

Who hasn't heard these words, or their equivalent, spoken in a talk? And who hasn't fantasized about grabbing the speaker by the lapels and shaking him until he says what — exactly — he means by them?

Joe Harris and Ian Morrison [HM98, p.140]

We define our first properties of morphisms of algebraic stacks (e.g., closed immersion, affine, locally of finite type, smooth, quasi-separated, quasi-finite, and étale) and some first properties of algebraic stacks (e.g., locally noetherian, quasi-compact, connected, reduced, regular, and noetherian). We also introduce the underlying topological space $|\mathcal{X}|$ of an algebraic stack.

3.3.1 Properties of morphisms

Recall that a morphism of prestacks $\mathcal{X} \to \mathcal{Y}$ over $\mathrm{Sch}_{\text{ét}}$ is representable by schemes (resp., representable) if for every morphism $T \to \mathcal{Y}$ from a scheme, the base change $\mathcal{X} \times_{\mathcal{Y}} T$ is a scheme (resp., algebraic space); see Definitions 3.1.1 and 3.1.3. Both notions are clearly stable under base change. Morphisms representable by schemes are also clearly stable under composition, and the following lemma shows the same for representable morphisms.

Lemma 3.3.1.

(1) If $\mathcal{X} \to Y$ is a representable morphism of prestacks over $\mathrm{Sch}_{\text{ét}}$ *and Y is an algebraic space, then $\mathcal{X}$ is an algebraic space.*

(2) The composition of representable morphisms is representable.

Proof. For the first statement, if $V \to Y$ is an étale presentation by a scheme V, the base change $\mathcal{X}_V$ is an algebraic space. Since the diagonal $\mathcal{X} \to \mathcal{X} \times_Y \mathcal{X}$ base changes under $V \to Y$ to a monomorphism $\mathcal{X}_V \to \mathcal{X}_V \times_V \mathcal{X}_V$, étale descent implies that $\mathcal{X} \to \mathcal{X} \times_Y \mathcal{X}$ is also a monomorphism. Hence, $\mathcal{X}$ is equivalent to a sheaf (Exercise 2.4.39). Moreover, the base change $\mathcal{X}_V \to \mathcal{X}$ is a morphism of algebraic spaces which is étale, surjective and representable by schemes. Letting $U \to X_V$ be an étale presentation, then the composition $U \to \mathcal{X}_V \to \mathcal{X}$ is étale, surjective, and representable by schemes, and thus $\mathcal{X}$ is an algebraic space. The second statement follows from the first. □

Definition 3.3.2. Let $\mathcal{P}$ be a property of morphisms of schemes.

(1) We say that $\mathcal{P}$ is *étale local on the source* if for every étale surjection $X' \to X$ of schemes, a morphism $X \to Y$ satisfies $\mathcal{P}$ if and only if $X' \to X \to Y$ does, and that $\mathcal{P}$ is *étale local on the target* if for every étale surjection $Y' \to Y$ of schemes, a morphism $X \to Y$ satisfies $\mathcal{P}$ if and only if $X \times_Y Y' \to Y$ does. The notions of being *smooth (resp., fppf, fpqc) local on the source/target* are defined similarly.

(2) If $\mathcal{P}$ is étale local (resp., smooth local) on the source and target and is stable under composition and base change, a morphism $\mathcal{X} \to \mathcal{Y}$ of Deligne–Mumford

stacks (resp., algebraic stacks) *has property* $\mathcal{P}$ if for all étale (resp., smooth) presentations $V \to \mathcal{Y}$ and $U \to \mathcal{X} \times_{\mathcal{Y}} V$ yielding a diagram

$$\begin{tikzcd} U \arrow[r] & \mathcal{X} \times_{\mathcal{Y}} V \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & V \arrow[d] \\ & \mathcal{X} \arrow[r] & \mathcal{Y}, \end{tikzcd}$$

the composition $U \to V$ has $\mathcal{P}$. It is enough to check this for specific presentations $V \to \mathcal{Y}$ and $U \to \mathcal{X} \times_{\mathcal{Y}} V$.

(3) A morphism $\mathcal{X} \to \mathcal{Y}$ of algebraic stacks representable by schemes *has property* $\mathcal{P}$ if for every morphism $T \to \mathcal{Y}$ from a scheme, the base change $\mathcal{X} \times_{\mathcal{Y}} T \to T$ has $\mathcal{P}$. In particular, a morphism $\mathcal{X} \to \mathcal{Y}$ of algebraic stacks is an *isomorphism*, *open immersion*, *closed immersion*, *locally closed immersion*, *affine*, or *quasi-affine* if it is representable by schemes and has the corresponding property.

The properties of flatness, smoothness (resp., smoothness of relative dimension n), surjectivity, locally of finite presentation, and locally of finite type are smooth local on the source and target. Using (2), these properties extend to morphisms of algebraic stacks. Likewise, étaleness and unramifiedness are étale local on the source and target, and thus extend to morphisms of Deligne–Mumford stacks. These properties are stable under composition and base change.

Example 3.3.3. If $G \to S$ is a smooth affine group scheme acting on an algebraic space $U \to S$, then $[U/G] \to S$ is flat (resp., smooth, surjective, locally of finite presentation, locally of finite type) if and only if $U \to S$ is. In particular, using the quotient stack presentations in the proofs of Theorems 3.1.17 and 3.1.21, we conclude that $\mathcal{M}_g$ is locally of finite type over $\mathbb{Z}$ and that both $\underline{\mathrm{Coh}}_{r,d}(C)$ and $\mathcal{B}un_{r,d}(C)$ are locally of finite type over $\Bbbk$.

Exercise 3.3.4 (easy)**.** Let $G = \mathbb{Z}/2$ act on $\mathbb{A}^2$ over a field $\Bbbk$ via $(-1) \cdot (x, y) = (-x, -y)$. Show that $[\mathbb{A}^2/G]$ is smooth but that the scheme quotient $\mathbb{A}^2/G := \operatorname{Spec} \Bbbk[x, y]^G$ is singular.

Exercise 3.3.5 (easy)**.** Show that the diagonal of a morphism $\mathcal{X} \to \mathcal{Y}$ of algebraic stacks is locally of finite type.

We now show that certain morphisms of algebraic stacks are smooth local on the target. They are even fppf local, but this is postponed until Exercise 6.3.5.

Proposition 3.3.6. *Let $\mathcal{P}$ be one of the following properties of morphisms of algebraic stacks: representable, isomorphism, open immersion, closed immersion, affine, or quasi-affine. Consider a cartesian diagram*

$$\begin{tikzcd} \mathcal{X}' \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{Y}' \arrow[d] \\ \mathcal{X} \arrow[r] & \mathcal{Y} \end{tikzcd}$$

of algebraic stacks where $\mathcal{Y}' \to \mathcal{Y}$ is smooth and surjective. Then $\mathcal{X} \to \mathcal{Y}$ has $\mathcal{P}$ if and only if $\mathcal{X}' \to \mathcal{Y}'$ has $\mathcal{P}$.

Proof. We will show the ($\Leftarrow$) implications as the other directions are clear. For representability, we may assume that $\mathcal{Y}$ and $\mathcal{Y}'$ are schemes. It suffices to show that the every automorphism $\alpha\colon a \to a$ of an object $a \in \mathcal{X}$ over a scheme T is trivial. The base change T' of $a\colon T \to \mathcal{X}$ by $\mathcal{X}' \to \mathcal{X}$ is identified with $T \times_{\mathcal{Y}} \mathcal{Y}'$, and thus is a scheme smooth over T. Since smooth morphisms étale locally have sections (Corollary A.3.5), there is an étale cover $g\colon \widetilde{T} \to T$ that factors through T'. The automorphism α defines a section of $\underline{\mathrm{Aut}}_T(a)$ over T. Since $\underline{\mathrm{Aut}}_T(a)$ is a sheaf on $(\mathrm{Sch}/T)_{\text{ét}}$ and $g^*\alpha = \mathrm{id}$, we have that $\alpha = \mathrm{id}$.

For the other properties, we already know that $\mathcal{X} \to \mathcal{Y}$ is representable, and it thus suffices to assume that $\mathcal{Y}$, $\mathcal{Y}'$ and $\mathcal{X}'$ are schemes and that $\mathcal{X}$ is an algebraic space. Fortunately, we can apply Descent Criterion for an Fppf Sheaf to be a Scheme (Proposition 2.3.17) to conclude that $\mathcal{X}$ is a scheme. The result therefore follows as these properties of morphisms of schemes are Fpqc Local Properties on the Target (2.1.26) □

3.3.2 Properties of algebraic spaces and stacks

Definition 3.3.7 (Properties of algebraic spaces and stacks)**.** Let $\mathcal{P}$ be a property of schemes which is étale (resp., smooth) local, i.e., if $X \to Y$ is an étale (resp., smooth) surjection of schemes, then X has $\mathcal{P}$ if and only if Y has $\mathcal{P}$. We say that a Deligne–Mumford stack (resp., algebraic stack) $\mathcal{X}$ *has property* $\mathcal{P}$ if for an étale (resp., smooth) presentation (equivalently for all presentations) $U \to \mathcal{X}$, the scheme U has $\mathcal{P}$.

The properties of being locally noetherian, reduced, or regular are smooth local (Proposition 2.1.21).

Example 3.3.8. Let $G \to S$ be a smooth affine group scheme acting on a scheme U over S. Then $[U/G]$ is locally noetherian, reduced, or regular if and only if U is.

Definition 3.3.9 (Substacks)**.** If $\mathcal{X}$ is an algebraic stack, a substack $\mathcal{Z} \subseteq \mathcal{X}$ is *closed* (resp., *open*, *locally closed*) if the induced morphism $\mathcal{Z} \to \mathcal{X}$ is a closed immersion (resp., open immersion, locally closed immersion).

Exercise 3.3.10. For an action of a smooth affine group scheme $G \to S$ on a scheme U over S, show that there is an equivalence between closed (resp., open) substacks of $[U/G]$ and G-invariant closed (resp., open) subschemes of U.

3.3.3 Separation properties

Separation properties for algebraic stacks are defined in terms of the diagonal.

Definition 3.3.11.

(1) A morphism of algebraic stacks $\mathcal{X} \to \mathcal{Y}$ has *affine diagonal* (resp., *quasi-affine diagonal, separated diagonal*) if the diagonal $\mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ is affine (resp., quasi-affine, separated). An algebraic stack $\mathcal{X}$ has *affine diagonal* (resp., *quasi-affine diagonal, separated diagonal*) if $\mathcal{X} \to \operatorname{Spec} \mathbb{Z}$ does.

(2) A morphism of algebraic stacks $\mathcal{X} \to \mathcal{Y}$ is *quasi-separated* if the diagonal $\mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ and second diagonal $\mathcal{X} \to \mathcal{X} \times_{\mathcal{X} \times_{\mathcal{Y}} \mathcal{X}} \mathcal{X}$ are quasi-compact. An algebraic stack $\mathcal{X}$ is *quasi-separated* if it is quasi-separated over $\operatorname{Spec} \mathbb{Z}$.

(3) A morphism of algebraic stacks $\mathcal{X} \to \mathcal{Y}$ is of *finite presentation* if it locally of finite presentation, quasi-compact, and quasi-separated.

(4) A representable morphism $\mathcal{X} \to \mathcal{Y}$ of algebraic stacks is *separated* if the morphism $\mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$, which is representable by schemes (Exercise 3.2.4), is proper. Separated morphisms are defined in general in Definition 3.8.1.

Conditions on the diagonal translate to conditions on the Isom sheaves since the base change of $\mathcal{X} \to \mathcal{X} \times \mathcal{X}$ by a morphism $(a, b) \colon S \to \mathcal{X} \times \mathcal{X}$ from a scheme S is identified with $\underline{\mathrm{Isom}}_{\mathcal{X}(S)}(a, b)$ (see Exercise 2.4.39), which is an algebraic space by Representability of the Diagonal (Theorem 3.2.1(2)). In particular, $\mathcal{X}$ has affine diagonal if and only if $\underline{\mathrm{Isom}}_{\mathcal{X}(S)}(a, b)$ is affine for every scheme S and $a, b \in \mathcal{X}(S)$. Every algebraic stack with affine or quasi-affine diagonal is necessarily quasi-separated.

Remark 3.3.12. For morphisms of schemes, the definition of separatedness above agrees with the usual notion as the diagonal of a morphism of schemes is a closed immersion if and only if it is proper.

Remark 3.3.13. A quasi-separated Deligne–Mumford stack has *finite* and reduced stabilizer groups (see Exercise 3.2.8).

Lemma 3.3.14. *Let S be an affine scheme and $G \to S$ be a smooth affine group scheme acting on an algebraic space U over S. If U has affine diagonal (resp., quasi-affine diagonal), then so does $[U/G]$.*

Proof. Recall that we established that $[U/G]$ is an algebraic stack in Theorem 3.1.10. Representability of the Diagonal (Theorem 3.2.1(2)) implies that $[U/G] \to [U/G] \times_S [U/G]$ is a representable morphism. Applying smooth descent (Proposition 3.3.6) to the cartesian diagram

$$\begin{tikzcd}
G \times_S U \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & U \times_S U \arrow[d] \\
{[U/G]} \arrow[r] & {[U/G] \times_S [U/G]}
\end{tikzcd}$$

of Exercise 2.4.37, it suffices to show that $G \times_S U \to U \times_S U$ is affine (resp., quasi-affine). Since G is affine, so is the composition $G \times_S U \to U \times_S U \xrightarrow{p_2} U$. On the other hand, $p_2 \colon U \times_S U \to U$ has affine diagonal (resp., quasi-affine diagonal) since U does. The cancellation property for morphisms implies that $G \times_S U \to U \times_S U$ is affine (resp., quasi-affine). $\square$

The condition of having affine diagonal is satisfied by most moduli problems (except for example $\mathcal{M}_1$).

Example 3.3.15. The moduli stacks $\mathcal{M}_g$ and $\mathcal{B}un_{r,d}(C)$ have affine diagonal and are thus quasi-separated. The statement for $\mathcal{M}_g$ follows from the above lemma and the quotient presentation $\mathcal{M}_g = [H' / \mathrm{PGL}_{5g-5}]$ in Theorem 3.1.17 as H' is locally closed subscheme of a projective Hilbert scheme. We will show later that $\mathcal{M}_g$ is separated (Proposition 5.5.20).

Similarly in Theorem 3.1.21, we expressed every quasi-compact open substack of $\mathcal{B}un_{r,d}(C)$ as a quotient stack $[Q' / \mathrm{GL}_N]$ where Q' is an open subscheme of a projective Quot scheme. To see that $\mathcal{B}un_{r,d}(C)$ has affine diagonal, it suffices to show that the base change of the along a morphism $\operatorname{Spec} A \to \mathcal{B}un_{r,d}(C) \times \mathcal{B}un_{r,d}(C)$ is affine. But such a morphism factors through $\mathcal{U} \times \mathcal{U}$ for some quasi-compact open substack $\mathcal{U} \subseteq \mathcal{B}un_{r,d}(C)$, and we know that $\mathcal{U}$ has affine diagonal.

Example 3.3.16. The non-separated union $\mathbb{A}^\infty \bigcup_{\mathbb{A}^\infty \smallsetminus 0} \mathbb{A}^\infty$ is a prototypical example of a non-quasi-separated scheme. For algebraic spaces and stacks, there are additional pathologies coming from actions of non-quasi-compact group schemes. For instance, $[\mathbb{A}^1 / \underline{\mathbb{Z}}]$ is a non-quasi-separated algebraic space (see Example 3.9.35), while $B\underline{\mathbb{Z}}$ is a non-quasi-separated algebraic stack (see Example 3.9.35).

3.3.4 The topological space of a stack

We can associate a topological space $|\mathcal{X}|$ to every algebraic stack $\mathcal{X}$.

Definition 3.3.17 (Topological space of an algebraic stack)**.** If $\mathcal{X}$ is an algebraic stack, we define *the topological space of* $\mathcal{X}$ as the set $|\mathcal{X}|$ consisting of field-valued morphisms $x\colon \operatorname{Spec} K \to \mathcal{X}$, where morphisms $x_1\colon \operatorname{Spec} K_1 \to \mathcal{X}$ and $x_2\colon \operatorname{Spec} K_2 \to \mathcal{X}$ are identified in $|\mathcal{X}|$ if there exists field extensions $K_1 \to K_3$ and $K_2 \to K_3$ such that $x_1|_{\operatorname{Spec} K_3}$ and $x_2|_{\operatorname{Spec} K_3}$ are isomorphic in $\mathcal{X}(K_3)$. A subset $U \subseteq |\mathcal{X}|$ is open if there exists an open substack $\mathcal{U} \subseteq \mathcal{X}$ such that $U = |\mathcal{U}| \subseteq |\mathcal{X}|$.

A morphism of algebraic stacks $\mathcal{X} \to \mathcal{Y}$ induces a continuous map $|\mathcal{X}| \to |\mathcal{Y}|$.

Exercise 3.3.18. Show that if $\mathcal{X}$ is an algebraic stack and $U \subseteq |\mathcal{X}|$ is an open subset, then there exists a reduced closed substack $\mathcal{Z} \hookrightarrow \mathcal{X}$ such that $|\mathcal{Z}| = |\mathcal{X}| \setminus U$.

Example 3.3.19. The topological space of the quotient stack $|[\mathbb{A}^1/\mathbb{G}_m]|$ over a field $\Bbbk$, where $\mathbb{G}_m$ acts via the standard scaling action on $\mathbb{A}^1$, consists of two points with representatives $x_0\colon \operatorname{Spec} k \xrightarrow{0} \mathbb{A}^1 \to [\mathbb{A}^1/\mathbb{G}_m]$ and $x_1\colon \operatorname{Spec} k \xrightarrow{1} \mathbb{A}^1 \to [\mathbb{A}^1/\mathbb{G}_m]$. In particular, the inclusion of the generic point $\operatorname{Spec} \Bbbk(x) \to \mathbb{A}^1 \to [\mathbb{A}^1/\mathbb{G}_m]$ is equivalent to x_1.

Given $x \in |\mathcal{X}|$, the stabilizer group G_{x_1} depends on the choice of representative $x_1\colon \operatorname{Spec} \Bbbk \to \mathcal{X}$ of x but its dimension—which we denote by $\dim G_x$—is independent of this choice. Similarly, the properties of being smooth, unramified, affine, finite, and reduced are also independent of this choice.

Exercise 3.3.20 (easy)**.** Let $x \in |\mathcal{X}|$ be a point of an algebraic stack with two representatives $x_1\colon \operatorname{Spec} \Bbbk_1 \to \mathcal{X}$ and $x_2\colon \operatorname{Spec} \Bbbk_2 \to \mathcal{X}$.

(1) Show that the stabilizer group G_{x_1} is smooth (resp., étale, unramified, affine, finite) if and only if G_{x_2} is.

(2) Show that $\dim G_{x_1} = \dim G_{x_2}$.

(3) If $\mathcal{X}$ is Deligne–Mumford and both $\Bbbk_1$ and $\Bbbk_2$ are algebraically closed, show that the abstract discrete groups corresponding to G_{x_1} and G_{x_2} are isomorphic.

As a consequence of the above exercise, it makes sense to say that $x \in |\mathcal{X}|$ has *smooth* (resp., *étale*, *unramified*, *affine*, *finite*) *stabilizer*. For a Deligne–Mumford stack $\mathcal{X}$, we define the *geometric stabilizer of* x as the discrete group $G = G_{\overline{x}}(\Bbbk)$ where $\overline{x}\colon \operatorname{Spec} \Bbbk \to \mathcal{X}$ is a geometric point representing x.

We can now define topological properties of algebraic stacks and their morphisms.

Definition 3.3.21. We say that an algebraic stack $\mathcal{X}$ is *quasi-compact*, *connected*, or *irreducible* if $|\mathcal{X}|$ is, and we say that $\mathcal{X}$ is *noetherian* if it is locally noetherian (Definition 3.3.7), quasi-separated (Definition 3.3.11(2)), and quasi-compact.

Exercise 3.3.22 (easy)**.** Show that an algebraic stack $\mathcal{X}$ is quasi-compact if and only if there exists a smooth presentation $\operatorname{Spec} A \to \mathcal{X}$ from an affine scheme, and that a quasi-separated algebraic stack $\mathcal{X}$ is noetherian if and only if there exists a smooth presentation $\operatorname{Spec} A \to \mathcal{X}$ where A is a noetherian ring.

Example 3.3.23. The moduli stack $\mathcal{M}_g$ is noetherian and in particular quasi-compact. This follows from the above exercise using the quotient presentation $\mathcal{M}_g = [H'/\operatorname{PGL}_{5g-5}]$ from Theorem 3.1.17. While $\mathcal{B}un_{r,d}(C)$ is quasi-separated and locally noetherian, it is not quasi-compact, and hence not noetherian.

Exercise 3.3.24 (easy)**.**

(a) Show that a morphism $\mathcal{X} \to \mathcal{Y}$ of algebraic stacks is surjective if and only if $|\mathcal{X}| \to |\mathcal{Y}|$ is surjective.

(b) Show that if $\mathcal{X} \to \mathcal{Y}$ and $\mathcal{Y}' \to \mathcal{Y}$ are morphisms of algebraic stacks, then $|\mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}'| \to |\mathcal{X}| \times_{|\mathcal{Y}|} |\mathcal{Y}'|$ is surjective.

Exercise 3.3.25 (easy)**.** If $\mathcal{X}$ is a quasi-compact and locally noetherian algebraic stack, show that $|\mathcal{X}|$ is a noetherian topological space.

Exercise 3.3.26 (moderate)**.** If $U \to \mathcal{X}$ is a smooth presentation of an algebraic stack, show that $|U| \to |\mathcal{X}|$ is submersive, i.e., is surjective and $|\mathcal{X}|$ has the quotient topology.

Hint: If $V \subseteq |\mathcal{X}|$ is a subset whose preimage U' in U is open, define $\mathcal{V} \subseteq \mathcal{X}$ as the substack of objects $T \to \mathcal{X}$ such that $U'_T = U_T$.

Exercise 3.3.27 (easy)**.** *Universally open* morphisms of algebraic stacks can be defined using Definition 3.3.2(1) since they are smooth local on the source and target. For instance, faithfully flat morphisms locally of finite presentation are universally open. If $f \colon \mathcal{X} \to \mathcal{Y}$ is a universally open morphism of algebraic stacks, show that $f(|\mathcal{X}|) \subseteq |\mathcal{Y}|$ is open. Conclude that for every morphism $\mathcal{Y}' \to \mathcal{Y}$ of algebraic stacks, the map $|\mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}'| \to |\mathcal{Y}'|$ is open.

Definition 3.3.28. A morphism $\mathcal{X} \to \mathcal{Y}$ of algebraic stacks is *quasi-compact* if for every morphism $\operatorname{Spec} B \to \mathcal{Y}$, the fiber product $\mathcal{X} \times_{\mathcal{Y}} \operatorname{Spec} B$ is quasi-compact. We say that $\mathcal{X} \to \mathcal{Y}$ is *of finite type* if $\mathcal{X} \to \mathcal{Y}$ is locally of finite type and quasi-compact.

Example 3.3.29. The moduli stack $\mathcal{M}_g$ is finite type over $\mathbb{Z}$. On the other hand, $\mathcal{B}un_{r,d}(C)$ is locally of finite type over $\Bbbk$ but not of finite type.

Remark 3.3.30. A quasi-compact morphism $\mathcal{X} \to \mathcal{Y}$ induces a quasi-compact morphism $|\mathcal{X}| \to |\mathcal{Y}|$ on topological spaces. The converse is true if $\mathcal{Y}$ is quasi-separated but not in general, e.g., $\operatorname{Spec} \Bbbk \to B\mathbb{Z}$ (see Example 3.9.35).

Exercise 3.3.31 (challenging)**.**

(a) Let $f \colon \mathcal{X} \to \mathcal{Y}$ be a quasi-compact morphism of algebraic stacks. For $y \in |\mathcal{Y}|$, show that $y \in \overline{f(|\mathcal{X}|)}$ if and only if there exists $x \in |\mathcal{X}|$ and a specialization $f(x) \rightsquigarrow y$.

(b) Show an open morphism $f \colon \mathcal{X} \to \mathcal{Y}$ of algebraic stacks (i.e., $|\mathcal{X}| \to |\mathcal{Y}|$ is open) satisfies the following lifting property: if $x \in |\mathcal{X}|$ is a point, then every specialization $y' \rightsquigarrow f(x)$ lifts to a specialization $x' \rightsquigarrow x$. Show that the converse is true for finitely presented morphisms.

(c) If $\mathcal{X}$ is a quasi-separated algebraic stack, show that $|\mathcal{X}|$ is a sober topological space, i.e., every irreducible closed subset has a unique generic point.

(d) Generalize Chevalley's criterion to algebraic stacks: if $f \colon \mathcal{X} \to \mathcal{Y}$ is a finitely presented morphism of algebraic stacks, then the image $f(|\mathcal{X}|) \subseteq |\mathcal{Y}|$ is constructible.

See also [LMB00, §5.7-5.9] and [SP, Tags 0DQQ and 0GVY].

Exercise 3.3.32 (Generic Flatness)**.** Extend Generic Flatness (A.2.13) from morphisms of schemes to algebraic stacks: if $\mathcal{X} \to \mathcal{Y}$ is a finite type morphism of algebraic stacks with $\mathcal{Y}$ reduced, then there exists a dense open substack $\mathcal{U} \subseteq \mathcal{Y}$ such that the base change $X_{\mathcal{U}} \to \mathcal{U}$ is flat and of finite presentation.

Exercise 3.3.33 (technical)**.** Extend the characterization of locally of finite presentation morphisms given in Proposition A.1.1 to algebraic stacks: a morphism $f\colon \mathcal{X} \to \mathcal{Y}$ of algebraic stacks is locally of finite presentation if and only if for every directed system $\{\operatorname{Spec} A_\lambda\}_{\lambda \in I}$ of affine schemes over $\mathcal{Y}$, the natural map

$$\operatorname{colim}_\lambda \operatorname{Mor}_{\mathcal{Y}}(\operatorname{Spec} A_\lambda, \mathcal{X}) \to \operatorname{Mor}_{\mathcal{Y}}(\operatorname{Spec}(\operatorname{colim}_\lambda A_\lambda), \mathcal{X})$$

is an equivalence of categories.

3.3.5 Quasi-finite morphisms

A morphism of schemes is *locally quasi-finite* if it is locally of finite type and every fiber is discrete, i.e., has the discrete topology. Since this property is étale local on the source and target, we can extend this property to morphisms of *algebraic spaces* using Definition 3.3.2.

Definition 3.3.34.

(1) A representable morphism $\mathcal{X} \to \mathcal{Y}$ of algebraic stacks is *locally quasi-finite* if for every morphism $T \to \mathcal{Y}$ from a scheme, the algebraic space $\mathcal{X} \times_{\mathcal{Y}} T$ is locally quasi-finite over T.

(2) A morphism $\mathcal{X} \to \mathcal{Y}$ of algebraic stacks is *locally quasi-finite* if is locally of finite type, the diagonal $\mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ is locally quasi-finite, and for every morphism $\operatorname{Spec} \Bbbk \to \mathcal{Y}$ from a field, the topological space $|\mathcal{X} \times_{\mathcal{Y}} \operatorname{Spec} \Bbbk|$ is discrete.

(3) A morphism $\mathcal{X} \to \mathcal{Y}$ of algebraic stacks is *quasi-finite* if it is locally quasi-finite and quasi-compact.

To understand condition (2), recall that the diagonal $\mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ is always a representable morphism (Exercise 3.2.4). The diagonal is quasi-finite (resp., locally quasi-finite) if and only if for every field-valued point $x \in \mathcal{X}(\Bbbk)$ with image $y \in \mathcal{Y}(\Bbbk)$, the kernel $\ker(G_x \to G_y)$ of the induced map of stabilizer groups is finite (resp., discrete); see Exercise 3.2.7. In particular, if $\mathcal{Y}$ is a scheme, the diagonal is quasi-finite if and only if all stabilizers of $\mathcal{Y}$ are finite. For instance, if G is a finite group scheme over a field $\Bbbk$ (e.g., μ_p), then $BG \to \operatorname{Spec} \Bbbk$ is quasi-finite. On the other hand, $B\mathbb{G}_m \to \operatorname{Spec} \Bbbk$ is not quasi-finite despite that $|B\mathbb{G}_m|$ is a single point.

Exercise 3.3.35 (easy)**.** Show that a finite type morphism $f\colon \mathcal{X} \to \mathcal{Y}$ of algebraic stacks is *quasi-finite* if and only if $|\mathcal{X}| \to |\mathcal{Y}|$ has finite fibers and for every field-valued point $x \in \mathcal{X}(\Bbbk)$, the map $\operatorname{Aut}_{\mathcal{X}(\Bbbk)}(x) \to \operatorname{Aut}_{\mathcal{Y}(\Bbbk)}(f(x))$ has finite cokernel.

We will later establish that every representable, quasi-finite, and separated morphism is quasi-affine (Proposition 4.5.5).

3.3.6 Étale and unramified morphisms

We will require that étale and unramified morphisms are relatively Deligne–Mumford.

Definition 3.3.36. A morphism of stacks $\mathcal{X} \to \mathcal{Y}$ over $\mathrm{Sch}_{\text{ét}}$ is *relatively Deligne–Mumford* if for every morphism $T \to \mathcal{Y}$ from a scheme, the fiber product $\mathcal{X} \times_{\mathcal{Y}} T$ is a Deligne–Mumford stack.

We will see in Corollary 3.6.5 that relatively Deligne–Mumford morphisms are characterized by the unramifiedness of the diagonal. A morphism $\mathcal{X} \to \mathcal{Y}$ satisfying the weaker condition that the diagonal $\mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ is locally quasi-finite is called *quasi-DM* [SP, Tag 04YW].

For morphisms of schemes, étaleness and unramifiedness are étale local on the source and smooth local on the target. These notions thus extend to relatively Deligne–Mumford morphisms.

Definition 3.3.37. A morphism $\mathcal{X} \to \mathcal{Y}$ of algebraic stacks is *étale* (resp., *unramified*) if it is relatively Deligne–Mumford and for every smooth presentation $V \to \mathcal{Y}$ and étale presentation $U \to \mathcal{X} \times_{\mathcal{Y}} V$, the induced morphism $U \to V$ of schemes is étale (resp., unramified).

A morphism is étale if and only if it is smooth and unramified, and a morphism is unramified if and only if the diagonal is étale. These follows from the analogous facts for morphisms of schemes (Theorems A.3.2 and A.3.3), noting that for a morphism of schemes, the diagonal is étale if and only if it is an open immersion.

While étale morphisms are smooth and locally quasi-finite, the converse is not true, e.g., over a field $\Bbbk$ of characteristic p, the map $B\boldsymbol{\mu}_p \to \operatorname{Spec} \Bbbk$ is smooth and quasi-finite but is not étale as $B\boldsymbol{\mu}_p$ is not Deligne–Mumford (see Exercise 6.3.12). Similarly, étale morphisms are smooth of relative dimension 0, but again the converse doesn't hold, e.g., $B\boldsymbol{\mu}_p \to \operatorname{Spec} \Bbbk$ in characteristic p or $[\mathbb{A}^1/\mathbb{G}_m] \to \operatorname{Spec} \Bbbk$ in any characteristic.

Exercise 3.3.38 (technical)**.** For a morphism $f\colon \mathcal{X} \to \mathcal{Y}$ of algebraic stacks, show that the following are equivalent:

(1) the diagonal $\Delta_f\colon \mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ is unramified (resp., separated, quasi-separated),

(2) the relative inertia $I_{\mathcal{X}/\mathcal{Y}} \to \mathcal{X}$ is unramified (resp., separated, quasi-separated), and

(3) the double diagonal (or equivalently the identity section of the relative inertia) $\mathcal{X} \to I_{\mathcal{X}/\mathcal{Y}} = \mathcal{X} \times_{\mathcal{X} \times_{\mathcal{Y}} \mathcal{X}} \mathcal{X}$ is an open immersion (resp., closed immersion, quasi-compact).

Hint: Extend the characterization of unramified (resp., separated, quasi-separated) group schemes of Proposition B.1.8(4) to group algebraic spaces.

3.4 Equivalence relations and groupoids

Different people at different times will use the different techniques for different purposes.

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We show that quotients of equivalence relations and groupoids can be constructed as algebraic stacks and spaces (Theorem 3.4.13). Moreover, any presentation of an algebraic space or stack induces a equivalence relation or groupoid of schemes. In addition to giving a useful way to construct algebraic spaces and stacks, equivalence relation and groupoids provide an alternative way to think about algebraic spaces and stacks.

3.4.1 Definitions

Definition 3.4.1. An *étale (resp., smooth) groupoid of schemes* is a pair of schemes U and R together with étale (resp., smooth) morphisms $s\colon R \to U$ called the *source* and $t\colon R \to U$ called the *target*, and a *composition* morphism $c\colon R \times_{s,U,t} R \to R$ satisfying:

(1) (associativity) the following diagram commutes

$$\begin{tikzcd}
R \times_{s,U,t} R \times_{s,U,t} R \arrow[r, "c\times\mathrm{id}"] \arrow[d, "\mathrm{id}\times c"] & R \times_{s,U,t} R \arrow[d, "c"] \\
R \times_{s,U,t} R \arrow[r, "c"] & R,
\end{tikzcd}$$

(2) (identity) there exists a morphism $e\colon U \to R$ (called the *identity*) such that the following diagrams commute

$$\begin{tikzcd}
 & U \arrow[dl, "\mathrm{id}"'] \arrow[d, "e"] \arrow[dr, "\mathrm{id}"] & \\
U & R \arrow[l, "s"] \arrow[r, "t"'] & U
\end{tikzcd}
\qquad
\begin{tikzcd}
R \arrow[r, "(e\circ s,\mathrm{id})"] \arrow[dr, "\mathrm{id}"'] & R \times_{s,U,t} R \arrow[d, "c"] & R \arrow[l, "(e\circ t,\mathrm{id})"'] \arrow[dl, "\mathrm{id}"] \\
 & R, &
\end{tikzcd}$$

(3) (inverse) there exists a morphism $i\colon R \to R$ (called the *inverse*) such that the following diagrams commute

$$\begin{tikzcd}
R \arrow[r, "i"] \arrow[dr, "s"'] & R \arrow[r, "i"] \arrow[d, "t"] & R \arrow[dl, "s"] \\
 & U &
\end{tikzcd}
\quad
\begin{tikzcd}
R \arrow[r, "s"] \arrow[d, "{(i,\mathrm{id})}"] & U \arrow[d, "e"] \\
R \times_{s,U,t} R \arrow[r, "c"] & R
\end{tikzcd}
\quad
\begin{tikzcd}
R \arrow[r, "t"] \arrow[d, "{(\mathrm{id},i)}"] & U \arrow[d, "e"] \\
R \times_{s,U,t} R \arrow[r, "c"] & R.
\end{tikzcd}$$

We often abuse notation by denoting this data simply as $s,t\colon R \rightrightarrows U$.

If $(s,t)\colon R \to U \times U$ is a monomorphism, then we say that $s,t\colon R \rightrightarrows U$ is an *étale (resp., smooth) equivalence relation.*

If U and R are algebraic spaces, and the source, target, and composition are morphisms of algebraic spaces, we obtain the notion of an *étale (resp., smooth) groupoid of algebraic spaces* and similarly an *étale (resp., smooth) equivalence relation of algebraic spaces.*

Groupoids of schemes were introduced in [$\mathrm{SGA3}_{\mathrm{I}}$, Exp. V].

Remark 3.4.2 (relations)*.* We can view R as a *scheme of relations on* U: a point $r \in R$ specifies a relation on the points $s(r), t(r) \in U$, which we sometimes write as $r\colon s(r) \to t(r)$. For every scheme T, the morphisms $R(T) \rightrightarrows U(T)$ define a groupoid of sets, i.e., there is composition morphism $R(T) \times_{s,U(T),t} R(T) \to R(T)$ satisfying axioms analogous to (1)–(3). We can think of an element $r \in R(T)$ as specifying a relation $r\colon u \to v$ between elements $u, v \in U(T)$. The composition morphism composes relations $r\colon u \to v$ and $r'\colon v \to w$ to the relation $r' \circ r\colon u \to w$, while the identity morphism takes $u \in U(T)$ to $\mathrm{id}\colon u \to u$ and the inverse morphism takes $r\colon u \to v$ to $r^{-1}\colon v \to u$. When $R \rightrightarrows U$ is an equivalence relation, the morphism $R(T) \to U(T) \times U(T)$ is injective, and there is at most one relation between any two elements of $U(T)$.

The prototypical examples of groupoids arise from group actions.

Example 3.4.3 (Group actions)**.** If $G \to S$ is an étale (resp., smooth) group scheme with multiplication $m\colon G \times_S G \to G$ acting on a scheme U over S via multiplication $\sigma\colon G \times U \to U$, then

$$p_2, \sigma\colon G \times_S U \rightrightarrows U$$

is an étale (resp., smooth) groupoid of schemes. The inverse $G \times_S U \to G \times_S U$ is given by $(g, u) \mapsto (g^{-1}, gu)$ and the composition is

$$(G \times_S U) \times_{p_2, U, \sigma} (G \times_S U) \to G \times_S U, \quad \big((g', u'), (g, u) \mid u' = gu\big) \mapsto (g'g, u).$$

A T-valued point (g, u) of $G \times_S U$ as the relation $u \to gu$.

The special case when $U = S$ defines a groupoid $s, t \colon G \rightrightarrows S$ with both arrows equal to the structure morphism $G \to S$.

Example 3.4.4. For an interesting example of an étale equivalence relation, consider the action of $\mathbb{Z}/2 = \{\pm 1\}$ on $\mathbb{A}^1$ with multiplication $\sigma \colon \mathbb{Z}/2 \times \mathbb{A}^1 \to \mathbb{A}^1$ defined by $-1 \cdot x = -x$. Removing the non-identity element of the stabilizer of the origin, we obtain a scheme $R = (\mathbb{Z}/2 \times \mathbb{A}^1) \smallsetminus \{(-1, 0)\}$ and an equivalence relation $p_2, \sigma \colon R \rightrightarrows \mathbb{A}^1$. We will show later that the quotient $X = \mathbb{A}^1/R$ is an algebraic space that is not a scheme (Example 3.9.2).

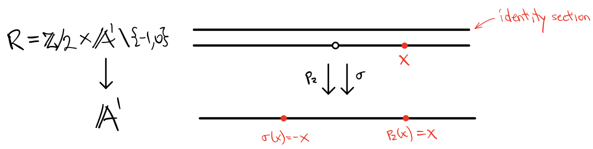


Figure 3.4.4: The quotient $X = \mathbb{A}^1/R$ looks like $\mathbb{A}^1_{\mathbb{R}}$ except that the origin has residue field $\mathbb{C}$.

Example 3.4.5 (Groupoids induced from presentations)**.** Let $\mathcal{X}$ be a Deligne–Mumford stack (resp., algebraic stack) and $p \colon U \to \mathcal{X}$ be an étale (resp., smooth) presentation which we assume is not only representable but representable by schemes.

Define the algebraic space $R := U \times_{\mathcal{X}} U$ (which is a scheme if the diagonal of $\mathcal{X}$ is representable by schemes). By the definition of fiber products, a T-valued point of R is a triple (u_1, u_2, α) where $\alpha \colon p(u_1) \to p(u_2)$ is an isomorphism in $\mathcal{X}(T)$, and this is viewed as the relation $\alpha \colon u_1 \to u_2$. If we let the source morphism $s = p_1 \colon R \to U$ be the first projection and the target morphism $t = p_2 \colon R \to U$ be the second projection, the composition morphism can be defined as $R \times_{s,U,t} R \to U \times_{\mathcal{X}} U$ taking $(\alpha_2 \colon u_2 \to u_3, \alpha_1 \colon u_1 \to u_2)$ to $\alpha_2 \circ \alpha_1 \colon u_1 \to u_3$. This gives the structure of an étale (resp., smooth) groupoid $R \rightrightarrows U$ of algebraic spaces. Observe that if $\mathcal{X}$ is an algebraic space, then the diagonal $\mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is a monomorphism. Hence, $(s, t) \colon R \to U \times U$ is a monomorphism, and $R \rightrightarrows U$ is an étale equivalence relation of schemes.

Choosing different presentations yields different groupoids which are equivalent under a notion called *Morita equivalence*. While this is an important notion, we will not use it in this text.

If $U \to [U/G]$ is a presentation of a quotient stack, then the induced groupoid is precisely the groupoid $p_2, \sigma \colon G \times_S U \rightrightarrows U$ discussed in the previous example. Indeed, there is a natural identification $U \times_{[U/G]} U \xrightarrow{\sim} G \times_S U$ defined by $u_1 \times_g u_2 \mapsto (g, u_2)$, where $u_1 \times_g u_2$ is shorthand notation for the triple $(u_1 = gu_2, u_2, g)$ defining an element of the fiber product. Similarly $U \times_{[U/G]} U \times_{[U/G]} U \xrightarrow{\sim} G \times_S G \times_S U$ is given by $u_1 \times_{g_1} u_2 \times_{g_2} u_3 \mapsto (g_1, g_2, u_3)$. The various projection maps in the groupoid can be identified with the maps arising from the group action:

$$\begin{tikzcd}
U \times_{[U/G]} U \times_{[U/G]} U \arrow[r, "\sim"] \arrow[d, shift right=4, "p_{12}"'] \arrow[d, "p_{13}"] \arrow[d, shift left=4, "p_{23}"] & G \times_S G \times_S U \arrow[d, shift right=4, "\mathrm{id}\times\sigma"'] \arrow[d, "m\times\mathrm{id}"] \arrow[d, shift left=4, "p_{23}"] \\
U \times_{[U/G]} U \arrow[r, "\sim"] \arrow[d, shift right, "p_1"'] \arrow[d, shift left, "p_2"] & G \times_S U \arrow[d, shift right, "p_2"'] \arrow[d, shift left, "\sigma"] \\
U \arrow[r, equal] \arrow[d] & U \\
{[U/G].} &
\end{tikzcd}$$

Exercise 3.4.6. Show that there are identifications

$$(U/[U/G])^n := \underbrace{U \times_{[U/G]} \cdots \times_{[U/G]} U}_{n \text{ times}} \xrightarrow{\sim} G^{n-1} \times_S U,$$
$$u_1 \times_{g_1} \cdots \times_{g_{n-1}} u_n \mapsto (g_1, \ldots, g_{n-1}, u_n),$$

where $u_i = g_i u_{i+1}$, such that the projection map $(U/[U/G])^{n+1} \to (U/[U/G])^n$ forgetting the kth term is identified with that map

$$G^n \times U \to G^{n-1} \times U,$$
$$(g_1, \ldots, g_n, u_{n+1}) \mapsto \begin{cases} (g_2, \ldots, g_n, u_{n+1}) & \text{if } k = 1 \\ (g_1, \ldots, g_{k-2}, g_{k-1}g_k, g_{k+1}, \ldots, g_n, u_{n+1}) & \text{if } k = 2, \ldots, n \\ (g_1, \ldots, g_n, g_n u_{n+1}) & \text{if } k = n+1. \end{cases}$$

Definition 3.4.7 (Orbits and stabilizers)**.** Let $R \rightrightarrows U$ be a smooth groupoid of algebraic spaces, and let $x\colon \operatorname{Spec} \Bbbk \to U$ be a field-valued point. The *stabilizer* G_x *of* x (or the *automorphism group* $\underline{\mathrm{Aut}}(x)$ *of* x) is defined as the fiber product of $(s,t)\colon R \to U \times U$ by $(x,x)\colon \operatorname{Spec} \Bbbk \to U \times U$. The *orbit* $O(x)$ is defined as the *set* $s(t^{-1}(x)) \subseteq U$.

Remark 3.4.8. Observe that if the groupoid $R \rightrightarrows U$ arises from a smooth presentation $U \to \mathcal{X}$ of an algebraic stack, then the stabilizer of $x \in U(\Bbbk)$ is identified with the stabilizer of its image in $\mathcal{X}(\Bbbk)$. On the other hand, the orbit $O(x)$ is the set of points $y \in U$ such that there exists a relation $x \xrightarrow{\sim} y$ in R, or equivalently the set of points $y \in U$ whose image in $\mathcal{X}$ is isomorphic to the image of x.

Exercise 3.4.9 (not important)**.** In Definition 3.4.1, show that the identity and inverse morphism are uniquely determined.

3.4.2 Algebraicity of quotients by groupoids

Definition 3.4.10 (Quotient stack of a smooth groupoid)**.** Let $s,t\colon R \rightrightarrows U$ be a smooth groupoid of algebraic spaces. Define $[U/R]^{\mathrm{pre}}$ as the prestack whose objects are morphisms $T \to U$ from a scheme T, where a morphism $(a\colon S \to U) \to (b\colon T \to U)$ is the data of a morphism of schemes $f\colon S \to T$ and an element $r \in R(S)$ such that $s(r) = a$ and $t(r) = f \circ b$. Define $[U/R]$ to be the stackification of $[U/R]^{\mathrm{pre}}$ in the big étale topology $\mathrm{Sch}_{\text{ét}}$. If in addition $R \rightrightarrows U$ is an equivalence relation, then $[U/R]$ is isomorphic to a sheaf (Exercise 3.4.12) and we denote it as U/R.

The fiber category $[U/R]^{\mathrm{pre}}(T)$ is the groupoid whose objects are $U(T)$ and morphisms are $R(T)$. The identity morphism $\mathrm{id}\colon U \to U$ defines a map $U \to [U/R]^{\mathrm{pre}}$ and therefore a map $p\colon U \to [U/R]$.

Exercise 3.4.11 (important)**.** Extend Exercise 2.4.37 to show that if $s, t\colon R \rightrightarrows U$ is a smooth groupoid of algebraic spaces, the following diagrams are cartesian:

$$\begin{tikzcd} R \arrow[r, "s"] \arrow[d, "t"] \arrow[dr, phantom, "\square"] & U \arrow[d, "p"] \\ U \arrow[r, "p"] & {[U/R]} \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} R \arrow[r, "{(s,t)}"] \arrow[d] \arrow[dr, phantom, "\square"] & U \times U \arrow[d, "p \times p"] \\ {[U/R]} \arrow[r, "\Delta"] & {[U/R] \times [U/R].} \end{tikzcd}$$

Exercise 3.4.12. Let $R \rightrightarrows U$ be a smooth groupoid of algebraic spaces. Show that $[U/R]$ is equivalent to a sheaf if and only if $R \rightrightarrows U$ is an equivalence relation.

Theorem 3.4.13 (Algebraicity of Quotients by Groupoids)**.**

(1) If $R \rightrightarrows U$ is an étale (resp., smooth) groupoid of algebraic spaces. Then $[U/R]$ is a Deligne–Mumford stack (resp., algebraic stack) and $U \to [U/R]$ is an étale (resp., smooth) presentation.

(2) If $R \rightrightarrows U$ be an étale equivalence relation of schemes, then U/R is an algebraic space and $U \to U/R$ is an étale presentation.

Remark 3.4.14. In Corollary 4.5.12, we show that in fact the quotient U/R of an étale equivalence relation of *algebraic spaces* is an algebraic space, establishing that one doesn't obtain new algebro-geometric objects by considering sheaves which are étale locally algebraic spaces. This result is delayed until §4.5 as it takes more work to show that the diagonal of U/R is representable by *schemes*. More generally, if $R \rightrightarrows U$ is an fppf groupoid (resp., fppf equivalence relation) of algebraic spaces, then $[U/R]$ is an algebraic stack (resp., U/R is an algebraic space); see Corollary 6.3.6.

Proof. For (1), we will show that $U \to \mathcal{X} := [U/R]$ is surjective, smooth, and representable. Let $T \to \mathcal{X}$ be a morphism from a scheme T. It follows from the definition of $[U/R]$ as the stackification of $[U/R]^{\mathrm{pre}}$ that there exists an étale cover $T' \to T$ and a commutative diagram

$$\begin{tikzcd} T' \arrow[r] \arrow[d] & U \arrow[d] \\ T \arrow[r] & \mathcal{X}. \end{tikzcd}$$

In the commutative cube

$$\begin{tikzcd}[row sep=small, column sep=small] & U_{T'} \arrow[rr] \arrow[dl] \arrow[dd] & & T' \arrow[dl] \arrow[dd] \\ R \arrow[rr, crossing over] \arrow[dd] & & U & \\ & U_T \arrow[rr] \arrow[dl] & & T \arrow[dl] \\ U \arrow[rr] & & \mathcal{X} \arrow[from=uu, crossing over] & \end{tikzcd} \tag{3.4.15}$$

of prestacks, the front, back, top, and bottom squares are cartesian. We need to show that the sheaf U_T is an algebraic space, but this isn't hard: $U_{T'}$ is a scheme and $U_{T'} \to U_T$ is a surjective étale morphism representable by schemes (as it is the base change of $T' \to T$). Since $R \to U$ is surjective and étale (resp., smooth), so is $U_{T'} \to T'$, and therefore by étale descent, $U_T \to T$ is also.

For (2), we already know that U/R is a sheaf and that $U \to U/R$ is surjective, étale, and representable. We need to show that $U \to U/R$ is representable by *schemes*. It suffices to show that the diagonal of X is representable by schemes; see Corollary 3.2.3(1). Let $T \to X \times X$ be a morphism from a scheme and define Q_T and $Q_{T'}$ using the cartesian cube

$$\begin{tikzcd}
& Q_{T'} \arrow[rr] \arrow[dl] \arrow[dd] & & T' \arrow[dl] \arrow[dd] \\
R \arrow[rr, crossing over] \arrow[dd] & & U \times U & \\
& Q_T \arrow[rr] \arrow[dl] & & T \arrow[dl] \\
X \arrow[rr] & & X \times X. \arrow[from=uu, crossing over] &
\end{tikzcd} \tag{3.4.16}$$

Since $R \to U \times U$ is locally quasi-finite and separated, so is $Q_{T'} \to T'$. The Descent Criterion for an Fppf Sheaf to be a Scheme (Proposition 2.3.17) implies that Q_T is a scheme. $\square$

Remark 3.4.17 (Quotient stacks revisited). As a consequence, wee see that the hypothesis in Algebraicity of Quotient Stacks (3.1.10) that hypothesis that the group scheme $G \to S$ is affine is not necessary for the quotient stack $[X/G]$ or classifying stack BG to be algebraic.

Exercise 3.4.18 (important)**.** Show that if $\mathcal{X}$ is an algebraic stack (resp., algebraic space) and $U \to \mathcal{X}$ is a smooth presentation, then $\mathcal{X}$ is isomorphic to the quotient stack $[U/R]$ (resp., quotient sheaf U/R) of the étale groupoid (resp., equivalence relation) $R \rightrightarrows U$ where $R = U \times_{\mathcal{X}} U$.

3.4.3 Inducing and slicing presentations

We provide here two useful techniques to build new presentations from given ones.

First, consider a quotient stack $\mathcal{X} = [X/H]$ where H is a smooth affine algebraic group acting on a scheme X over $\Bbbk$. If G is another group that contains $H \subseteq G$ as a subgroup, then H acts freely on $G \times X$ via $h \cdot (g, x) = (gh^{-1}, hx)$, and we can define the quotient algebraic space

$$G \times^H X := (G \times X)/H,$$

which is sometimes referred to as the *Borel construction*. So far, we only that this is a sheaf representable by an algebraic stack (Theorem 3.4.13(1)), but it will follow from Corollary 3.6.8 that it is, in fact, an algebraic space. (If H is a finite abstract group, then Theorem 3.4.13(2) already implies that it is an algebraic space.) In any case, there is an action of G on $G \times^H X$ via $g \cdot (g', x) = (gg', x)$.

Exercise 3.4.19. Show that $[X/H] \cong [(G \times^H X)/G]$.

The quotient presentation $[G \times^H X/G]$ *is called the* induced presentation*.*

The second technique we introduce is called *slicing a groupoid*. Let $U \to \mathcal{X}$ be a smooth presentation of an algebraic stack with the corresponding groupoid $s, t\colon R = U \times_{\mathcal{X}} U \rightrightarrows U$. If $g\colon U' \to U$ is a morphism, we define *the restriction of* $R \rightrightarrows U$ *along* $U' \to U$ to be the groupoid $R|_{U'} \rightrightarrows U'$ defined by the fiber product

$$\begin{tikzcd}
R|_{U'} \arrow[r, "{(t',s')}"] \arrow[d] \arrow[dr, phantom, "\square"] & U' \times U' \arrow[d] \\
R \arrow[r, "{(t,s)}"] & U \times U
\end{tikzcd}$$

Exercise 3.4.20.

(a) Show that $R|_{U'}$ fits into a cartesian diagram

$$\begin{tikzcd}
R|_{U'} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & R \times_{s,U} U' \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & U' \arrow[d, "g"] \\
U' \times_{U,t} R \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & R \arrow[r, "s"] \arrow[d, "t"] \arrow[dr, phantom, "\square"] & U \arrow[d] \\
U' \arrow[r, "g"] & U \arrow[r] & {[U/R]}
\end{tikzcd}$$

Assume in addition that $U' \times_{U,t} R \to R \xrightarrow{s} U$ is étale (resp., smooth).

(2) Show that $R|_{U'} \rightrightarrows U'$ is an étale (resp., smooth) groupoid.

(3) Show that there is an open immersion $[U'/R|_{U'}] \to [U/R]$.

(4) Show that $[U'/R|_{U'}] \to [U/R]$ is an isomorphism if and only if for every point $u \in U$, there exists a point $u' \in U$ and a relation $u \to g(u')$ in R.

Exercise 3.4.21 (Moduli of elliptic curves revisited)**.** Consider the moduli stack $\mathcal{M}_{1,1} \cong [(\mathbb{A}^2 \smallsetminus V(\Delta))/\mathbb{G}_m]$ of elliptic curves from Exercise 3.1.19, where $\Bbbk$ is a field with $\mathrm{char}(\Bbbk) \neq 2, 3$ and $\Delta = 4a^3 + 27b^2$ is the discriminant (with a and b the coordinates on $\mathbb{A}^2$). Slice along the closed immersion $V(\Delta - 1) \hookrightarrow \mathbb{A}^2 \smallsetminus V(\Delta)$ to show that $\mathcal{M}_{1,1} \cong [V(\Delta - 1)/\mu_{12}]$ is a quotient of an affine scheme by a finite group.

3.5 Dimension, tangent spaces, and residual gerbes

A point is that which has no part.

EUCLID

After extending the notions of dimension and tangent spaces to algebraic stacks, we show that every finite type point has a *residual gerbe*, which serves as a replacement for the notion of the residue field of a point of a scheme.

3.5.1 Dimension

Recall that the dimension $\dim X$ of a scheme X is the Krull dimension of the underlying topological space while the dimension $\dim_x X$ at a point $x \in X$ is the minimum dimension of open subsets containing x. This is equal to $\dim \mathcal{O}_{X,x}$ when X is of finite type over a field and $x \in X$ is a closed point. We now extend these definitions to algebraic spaces and stacks.

Definition 3.5.1 (Dimension)**.**

(1) Let X be a noetherian algebraic space and $x \in |X|$. We define the *dimension of X at x* to be

$$\dim_x X = \dim_u U \in \mathbb{Z}_{\geq 0} \cup \infty$$

where $U \to X$ is an étale presentation and $u \in U$ is a preimage of x.

(2) Let $\mathcal{X}$ be a noetherian algebraic stack and $x \in |\mathcal{X}|$. If $U \to \mathcal{X}$ is a smooth presentation and $s, t \colon R \rightrightarrows U$ is the induced smooth groupoid, and $u \in U$ is a preimage of $x \in |\mathcal{X}|$. We define the *dimension of $\mathcal{X}$ at x* to be

$$\dim_x \mathcal{X} = \dim_u U - \dim_{e(u)} R_u \in \mathbb{Z} \cup \infty,$$

where R_u is the fiber of $s \colon R \to U$ over u and $e \colon U \to R$ denotes the identity morphism in the groupoid.

(3) If $\mathcal{X}$ is a noetherian algebraic space or stack, we define the *dimension of $\mathcal{X}$* to be

$$\dim \mathcal{X} = \sup_{x \in |\mathcal{X}|} \dim_x \mathcal{X} \in \mathbb{Z} \cup \infty.$$

Proposition 3.5.2. *The definition of $\dim_x \mathcal{X}$ is independent of the choices of presentation $U \to \mathcal{X}$ and the preimage u of x.*

Proof. The dimension of an algebraic space at a point is well-defined as étale morphisms have relative dimension 0. If $U \to \mathcal{X}$ is a smooth presentation (with U a scheme) and $u \in U$ is a preimage of x with residue field $\kappa(u)$, then the fiber R_u is identified with the fiber product

$$\begin{tikzcd} R_u \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & R \arrow[r, "t"] \arrow[d, "s"] \arrow[dr, phantom, "\square"] & U \arrow[d] \\ \operatorname{Spec} \kappa(u) \arrow[r] & U \arrow[r] & \mathcal{X}, \end{tikzcd}$$

and is a smooth algebraic space over $\kappa(u)$. Let $U' \to \mathcal{X}$ be a second presentation inducing a smooth groupoid $R' \rightrightarrows U'$ and $u' \in U'$ be a preimage of x. Since we can form the smooth presentation $U \times_{\mathcal{X}} U'$ of $\mathcal{X}$ dominating both U and U', we may assume that there is a smooth morphism $U' \to U$ of schemes, taking u' to u. By the dimension formula for smooth morphisms,

$$\dim_{u'} U' = \dim_u U + \dim_{u'} U'_u; \tag{3.5.3}$$

see Proposition A.3.10. To relate $\dim_{u'} U'_u$ to $\dim_{e(u)} R_u$ and $\dim_{e'(u')} R'_{u'}$, we form the cartesian diagram

$$\begin{tikzcd}
R'_{u'} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & U''_u \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & U'' \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & U' \arrow[d] \\
R_u \times_{\kappa(u)} \kappa(u') \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & R_u \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & R \arrow[r, "t"] \arrow[d, "s"] \arrow[dr, phantom, "\square"] & U \arrow[d] \\
\operatorname{Spec} \kappa(u') \arrow[r] & \operatorname{Spec} \kappa(u) \arrow[r] & U \arrow[r] & \mathcal{X}.
\end{tikzcd}$$

We will apply again the dimension formula for smooth morphisms to $U''_u \to R_u$, but this time using its generalization (Exercise 3.5.5(a)) to algebraic space. Using that the fiber over $e(u) \in |R_u|$ is identified with U'_u and letting $u'' \in U''_u$ is a preimage of $e(u) \in |R_u|$ and $u' \in U'$, we have that

$$\dim_{u''} U''_u = \dim_{e(u)} R_u + \dim_{u'} U'_u. \tag{3.5.4}$$

Since dimension is preserved under field extensions (Exercise 3.5.5(b)), $\dim_{e'(u')} R'_{u'} = \dim_{u''} U''_u$. Combining (3.5.3) and (3.5.4) gives the desired result. □

Exercise 3.5.5 (details)**.**

(a) Show that if $X \to Y$ is a smooth morphism of noetherian algebraic spaces and $x \in |X|$ is a point with image $y \in |Y|$, then

$$\dim_x X = \dim_y Y + \dim_x X_y.$$

This generalizes Proposition A.3.10 to algebraic spaces.

(b) If X be a noetherian algebraic stack over a field $\Bbbk$, $\Bbbk \to \Bbbk'$ is a field extension, and $x' \in |\mathcal{X}_{\Bbbk'}|$ is a point with image $x \in |\mathcal{X}|$, show that $\dim_{x'} \mathcal{X}_{\Bbbk'} = \dim_x \mathcal{X}$.

Example 3.5.6. If G is a smooth affine algebraic group over a field $\Bbbk$ acting on a pure dimensional scheme U of finite type over $\Bbbk$, then

$$\dim\,[U/G] = \dim U - \dim G.$$

In particular, the dimension of the classifying stack BG is

$$\dim BG = -\dim G,$$

which can be negative! Similarly, $\dim\,[\mathbb{A}^1_{\Bbbk}/\mathbb{G}_m] = 0$.

3.5.2 Tangent spaces

The *dual numbers* is the ring $\Bbbk[\epsilon] := \Bbbk[\epsilon]/\epsilon^2$ defined over a field $\Bbbk$.

Definition 3.5.7. For an algebraic stack $\mathcal{X}$ and a point $x\colon \operatorname{Spec} \Bbbk \to \mathcal{X}$, we define the *Zariski tangent space* (or simply the *tangent space* of $\mathcal{X}$ at x) as the set

$$T_{\mathcal{X},x} := \left\{ \text{2-commutative diagrams} \quad \begin{tikzcd} \operatorname{Spec} \Bbbk \arrow[d, hook] \arrow[dr, "x"] & \\ \operatorname{Spec} \Bbbk[\epsilon] \arrow[r, "\tau"'] \arrow[ur, phantom, "\alpha \Leftarrow"] & \mathcal{X} \end{tikzcd} \right\} \Big/ \sim$$

In other words, $T_{\mathcal{X},x}$ is the set of pairs (τ, α) with $\tau\colon \operatorname{Spec} \Bbbk[\epsilon] \to \mathcal{X}$ and $\alpha\colon x \xrightarrow{\sim} \tau|_{\Bbbk}$, where two pairs are equivalent $(\tau, \alpha) \sim (\tau', \alpha')$ if there is an isomorphism $\beta\colon \tau \xrightarrow{\sim} \tau'$ in $\mathcal{X}(\Bbbk[\epsilon])$ compatible with α and α', i.e., $\alpha' = \beta|_{\operatorname{Spec} \Bbbk} \circ \alpha$.

Example 3.5.8. Consider a smooth, connected, and projective curve $[C] \in \mathcal{M}_g(\Bbbk)$ defined over $\Bbbk$ of genus $g \geq 2$. Deformation Theory (C.1.11) implies that $T_{\mathcal{M}_g,[C]} = H^1(C, T_C)$. Since $\deg T_C < 0$, $\mathrm{H}^0(C, T_C) = 0$ and Riemann–Roch implies

$$\dim T_{\mathcal{M}_g,[C]} = \dim \mathrm{H}^1(C, T_C) = -\chi(T_C) = -(\deg T_C + (1-g)) = 3g-3.$$

Example 3.5.9. Let C be a smooth, connected, and projective curve over $\Bbbk$ and $E \in \mathcal{B}un_{r,d}(C)(\Bbbk)$ be a vector bundle on C of rank r and degree d. Deformation Theory (C.1.18) implies that $T_{\mathcal{B}un_{r,d}(C),[E]} = \mathrm{Ext}^1_{\mathcal{O}_C}(E, E) = H^1(C, E \otimes E^\vee)$. By Riemann–Roch (8.1.2), $\chi(E \otimes E^\vee) = r^2(1-g)$. Since $\dim \mathrm{Aut}(E) = \dim_\Bbbk \mathrm{Hom}_{\mathcal{O}_C}(E, E) = \mathrm{H}^0(C, E \otimes E^\vee)$, we compute that

$$\dim T_{\mathcal{B}un_{r,d}(C),[E]} = \dim \mathrm{Ext}^1_{\mathcal{O}_C}(F, F) = \dim \mathrm{Aut}(F) + r^2(g-1).$$

Proposition 3.5.10. *If $\mathcal{X}$ is an algebraic stack with affine diagonal and $x \in \mathcal{X}(\Bbbk)$, then $T_{\mathcal{X},x}$ is naturally a $\Bbbk$-vector space.*

Proof. Scalar multiplication of $c \in \Bbbk$ on $(\tau, \alpha) \in T_{\mathcal{X},x}$ is defined as the composition $\operatorname{Spec} \Bbbk[\epsilon] \to \operatorname{Spec} \Bbbk[\epsilon] \xrightarrow{\tau} \mathcal{X}$, where the first map is defined by $\epsilon \mapsto c\epsilon$ and with the same 2-isomorphism α. To define addition, we will show that there is an equivalence of categories

$$\mathcal{X}(\Bbbk[\epsilon_1] \times_\Bbbk \Bbbk[\epsilon_2]) \to \mathcal{X}(\Bbbk[\epsilon_1]) \times_{\mathcal{X}(\Bbbk)} \mathcal{X}(\Bbbk[\epsilon_2]),$$

or, in other words, that

$$\begin{tikzcd}
\operatorname{Spec} \Bbbk \arrow[r, hook] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner"] & \operatorname{Spec} \Bbbk[\epsilon_1] \arrow[d, hook] \\
\operatorname{Spec} \Bbbk[\epsilon_2] \arrow[r, hook] & \operatorname{Spec}(\Bbbk[\epsilon_1] \times_\Bbbk \Bbbk[\epsilon_2])
\end{tikzcd}$$

is a pushout among algebraic stacks with affine diagonal. (This is related to the homogeneity conditions in Schlessinger's Criteria (C.4.6) and Artin's Axioms for Algebraicity (C.7.4); see also §B.4 for a discussion of pushouts of schemes.) Once this is established, we define addition of (τ_1, α_1) and (τ_2, α_2) by the composition $\operatorname{Spec} \Bbbk[\epsilon] \to \operatorname{Spec}(\Bbbk[\epsilon_1] \times_\Bbbk \Bbbk[\epsilon_2]) \to \mathcal{X}$, where the first map is defined sending both $(\epsilon_1, 0)$ and $(0, \epsilon_2)$ to ϵ.

To establish the equivalence, choose a smooth morphism $(U, u) \to (\mathcal{X}, x)$ from an affine scheme U. Since $\mathcal{X}$ has affine diagonal, $U \to \mathcal{X}$ is an affine morphism. Let $\operatorname{Spec} A_0 = \operatorname{Spec} \Bbbk \times_\mathcal{X} U$, $\operatorname{Spec} A_1 = \operatorname{Spec} \Bbbk[\epsilon_1] \times_\mathcal{X} U$ and $\operatorname{Spec} A_2 = \operatorname{Spec} \Bbbk[\epsilon_2] \times_\mathcal{X} U$. Since $\operatorname{Spec}(A_1 \times_A A_2)$ is the pushout of $\operatorname{Spec} A_0 \hookrightarrow \operatorname{Spec} A_1$ and $\operatorname{Spec} A_0 \hookrightarrow \operatorname{Spec} A_2$ in the category of affine schemes, there are unique morphisms $\operatorname{Spec}(A_1 \times_A A_2) \to \operatorname{Spec}(\Bbbk[\epsilon_1] \times_\Bbbk \Bbbk[\epsilon_2])$ and $\operatorname{Spec}(A_1 \times_A A_2) \to U$ completing the diagram

$$\begin{tikzcd}
 & \operatorname{Spec} A_0 \arrow[rr] \arrow[dl] \arrow[dd] & & \operatorname{Spec} A_2 \arrow[dl] \arrow[dd] & \\
\operatorname{Spec} \Bbbk \arrow[rr] \arrow[dd] & & \operatorname{Spec} \Bbbk[\epsilon_1] \arrow[dd] \arrow[dddr, bend left, "\tau_1"] & & \\
 & \operatorname{Spec} A_1 \arrow[rr] \arrow[dl] & & \operatorname{Spec}(A_1 \times_A A_2) \arrow[dl] \arrow[dr] & \\
\operatorname{Spec} \Bbbk[\epsilon_2] \arrow[rr] \arrow[drrr, bend right, "\tau_2"] & & \operatorname{Spec}(\Bbbk[\epsilon_1] \times_\Bbbk \Bbbk[\epsilon_2]) \arrow[dr, dashed] & & U \arrow[dl] \\
 & & & \mathcal{X} &
\end{tikzcd}$$

By the Flatness Criterion over Artinian Rings (A.2.3), we see that the map $\operatorname{Spec}(A_1 \times_A A_2) \to \operatorname{Spec}(\Bbbk[\epsilon_1] \times_{\Bbbk} \Bbbk[\epsilon_2])$ is faithfully flat. Repeating this argument on $U \times_{\mathcal{X}} U$ shows that $\operatorname{Spec}(A_1 \times_A A_2) \to U$ descends uniquely to a map $U \to \mathcal{X}$. These details and the verification that scalar multiplication and addition give $T_{\mathcal{X},x}$ the structure of a vector space are left to the reader. □

Exercise 3.5.11 (hard)**.**

(a) Show that $T_{\mathcal{X},x}$ is naturally a representation of G_x which is given set-theoretically by: $g \cdot (\tau, \alpha) = (\tau, g \circ \alpha)$ for $g \in G_x$ and $(\tau, \alpha) \in T_{\mathcal{X},x}$.

(b) Show that the affine diagonal in Proposition 3.5.10 is superfluous.

3.5.3 Residual gerbes

Attached to every point $x \in X$ of a scheme is a residue field $\kappa(x)$ and a monomorphism $\operatorname{Spec} \kappa(x) \to X$ with image x. The residual gerbe will provide us with an analogous property for algebraic stacks. Note that non-trivial stabilizers prevent field-valued points from being monomorphisms, e.g., $\operatorname{Spec} \Bbbk \to BG$ is not a monomorphism for a non-trivial algebraic group.

Definition 3.5.12. Let $\mathcal{X}$ be an algebraic stack and $x \in |\mathcal{X}|$ be a point. We say that the *residual gerbe at x exists* if there is a reduced, locally noetherian algebraic stack $\mathcal{G}_x$ and a monomorphism $\mathcal{G}_x \hookrightarrow \mathcal{X}$ such that $|\mathcal{G}_x|$ is a point mapping to x. The algebraic stack $\mathcal{G}_x$ is called the *residual gerbe at x*.

We define gerbes later in Definition 6.4.6 and show that $\mathcal{G}_x$ is a gerbe over a field $\kappa(x)$, called the *residue field* (Proposition 6.4.26). While not apparent from the definition, it is not hard to see that residual gerbes are in fact unique (see Proposition 3.5.16). These facts justify our terminology of calling $\mathcal{G}_x$ *the* residual *gerbe*. We will discuss the existence of residual gerbes more generally in §6.4.4, but in the meantime, we will be content with verifying the existence of residual gerbes at *finite type points*.

Definition 3.5.13. A point $x \in |\mathcal{X}|$ in an algebraic stack is *of finite type* if there exists a representative $\operatorname{Spec} \Bbbk \to \mathcal{X}$ of x that is locally of finite type.[4]

Remark 3.5.14. If X is a noetherian scheme, a point $x \in X$ is of finite type if and only if $x \in X$ is locally closed. More generally, a morphism $\operatorname{Spec} \Bbbk \to X$ from a field with image x is of finite type if and only if the image $x \in X$ is locally closed and $\kappa(x)/\Bbbk$ is a finite extension. Indeed, to see the non-trivial implication ($\Rightarrow$), we replace X with $\overline{\{x\}}$, and since $\operatorname{Spec} \Bbbk \to X$ is of finite type with dense image, Generic Flatness (A.2.13) implies that $\operatorname{Spec} \Bbbk \to X$ is fppf and thus its image is open.

An example of a finite type point of a scheme that is not closed is the generic point of a DVR. On the other hand, if X is a scheme of finite type over a field $\Bbbk$, then every finite type point is a closed point. The analogous fact is *not* true for algebraic stacks of finite type over $\Bbbk$, e.g., $1 \colon \operatorname{Spec} \Bbbk \to [\mathbb{A}^1/\mathbb{G}_m]$ is an open finite type point.

Exercise 3.5.15. Let $\mathcal{X}$ be an algebraic stack.

[4] If $\mathcal{X}$ has quasi-compact diagonal, e.g., $\mathcal{X}$ is quasi-separated (Definition 3.3.11), then every field-valued point $\operatorname{Spec} \Bbbk \to \mathcal{X}$ is automatically quasi-compact, and thus the locally of finite typeness of $\operatorname{Spec} \Bbbk \to \mathcal{X}$ is equivalent to finite typeness.

(a) Show that a point $x \in |\mathcal{X}|$ is of finite type if and only if there exists a scheme U, a closed point $u \in U$, and a smooth morphism $(U, u) \to (\mathcal{X}, x)$.

(b) Show that any algebraic stack (resp., quasi-compact algebraic stack) has a finite type point (resp., closed point).

Proposition 3.5.16 (Existence of Residual Gerbes)**.** *If $\mathcal{X}$ is a noetherian algebraic stack and $x \in |\mathcal{X}|$ is a finite type point, then there exists a unique residual gerbe $\mathcal{G}_x$ at x. Moreover, $\mathcal{G}_x$ satisfies the following:*

(1) The algebraic stack $\mathcal{G}_x$ is regular and the morphism $\mathcal{G}_x \hookrightarrow \mathcal{X}$ is a locally closed immersion.

(2) If in addition $\mathcal{X}$ is of finite type over a field $\Bbbk$ and $x \in \mathcal{X}(\Bbbk)$ has a smooth affine stabilizer G_x, then $\mathcal{G}_x \cong BG_x$.

(3) If in addition $\mathcal{X}$ is a noetherian algebraic space, then $\mathcal{G}_x \cong \operatorname{Spec} \kappa(x)$ for a field $\kappa(x)$, called the residue field of x*.*

Proof. After replacing $\mathcal{X}$ with $\overline{\{x\}}$, we may assume that $\mathcal{X}$ is reduced and $x \in |\mathcal{X}|$ is dense. Let $\operatorname{Spec} \Bbbk \to \mathcal{X}$ be a finite type representative of x. By Generic Flatness (3.3.32), $\operatorname{Spec} \Bbbk \to \mathcal{X}$ is flat. Therefore, the image $\{x\} \subseteq |\mathcal{X}|$ of $\operatorname{Spec} \Bbbk \to \mathcal{X}$ is open (Exercise 3.3.27). The corresponding open substack $\mathcal{G}_x \subseteq \mathcal{X}$ satisfies the properties of being a residual gerbe. Since $\operatorname{Spec} \Bbbk \to \mathcal{G}_x$ is fppf and the property of being regular descends under fppf morphisms (Proposition 2.1.21), $\mathcal{G}_x$ is regular.

For the uniqueness, suppose that $\mathcal{G}$ and $\mathcal{G}'$ are reduced, locally noetherian algebraic stacks with monomorphisms $\mathcal{G} \hookrightarrow \mathcal{X}$ and $\mathcal{G}' \hookrightarrow \mathcal{X}$ such that $|\mathcal{G}|$ and $|\mathcal{G}'|$ are singletons mapping to x. Then $\mathcal{G}'' := \mathcal{G} \times_{\mathcal{X}} \mathcal{G}'$ is a nonempty algebraic stack with monomorphisms $\mathcal{G}'' \to \mathcal{G}$ and $\mathcal{G}'' \to \mathcal{G}'$. Let $\operatorname{Spec} \Bbbk \to \mathcal{G}$ be a finite type morphism from a field, which exists by Exercise 3.5.15 and is fppf by Generic Flatness (3.3.32). The base change $\mathcal{G}'' \times_{\mathcal{G}} \operatorname{Spec} \Bbbk$ is a nonempty noetherian algebraic space equipped with a monomorphism to $\operatorname{Spec} \Bbbk$, which by Exercise 3.5.17 implies that $\mathcal{G}'' \times_{\mathcal{G}} \operatorname{Spec} \Bbbk \cong \operatorname{Spec} \Bbbk$. By fppf descent, $\mathcal{G}'' \to \mathcal{G}$ is also an isomorphism. Similarly, $\mathcal{G}'' \to \mathcal{G}'$ is an isomorphism.

For (2), there is a monomorphism of prestacks $BG_x^{\mathrm{pre}} \to \mathcal{X}$: for a $\Bbbk$-scheme T, there is a unique object of BG_x^{pre} over T, and this object gets mapped to the composition $T \to \operatorname{Spec} \Bbbk \xrightarrow{x} \mathcal{X}$. Similarly, a morphism over $T' \to T$ corresponds to a map $T' \to G_x$, and this gets mapped to the corresponding morphism in $\mathcal{X}$. Under stackification, this induces a monomorphism $BG_x \to \mathcal{X}$, and we see that BG_x satisfies the properties of a residual gerbe. Finally, (3) follows from Exercise 3.5.17. □

Exercise 3.5.17 (details)**.** Let X be a reduced noetherian algebraic space such that $|X|$ is a point. Show that $X \cong \operatorname{Spec} \Bbbk$ for a field $\Bbbk$. (This will also follow from later facts, e.g., Theorem 4.5.1.)

Exercise 3.5.18 (technical)**.** Let $\mathcal{X}$ be a (possibly non-noetherian) algebraic stack and $x \in \mathcal{X}$ be a finite type point such that the stabilizer is unramified (i.e., the stabilizer group scheme of any representative is unramified). Show that the residual gerbe exists and is unique. See also [SP, Tag 06G3].

This corollary shows that the orbit $O(u) \subseteq U$ of a finite type point in a smooth groupoid is locally closed.

Corollary 3.5.19. *Let $x \in |\mathcal{X}|$ be a finite type point of a noetherian algebraic stack $\mathcal{X}$. If $(U, u) \to (\mathcal{X}, x)$ is a smooth morphism from a scheme U with $u \in U$ a finite* *type point, then there is a cartesian diagram*

$$\begin{tikzcd} O(u) \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & U \arrow[d] \\ \mathcal{G}_x \arrow[r, hook] & \mathcal{X} \end{tikzcd} \tag{3.5.20}$$

where $O(u)$ is identified set-theoretically with the orbit $s(t^{-1}(u))$ of the induced groupoid $s, t\colon R := U \times_{\mathcal{X}} U \rightrightarrows U$. $\square$

Remark 3.5.21. If $\mathcal{X} = [U/G]$ is the quotient stack of a smooth affine algebraic group over a field $\Bbbk$ acting on a noetherian $\Bbbk$-scheme U and $u \in U(\Bbbk)$, there is a cartesian diagram

$$\begin{tikzcd} Gu \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & U \arrow[d] \\ BG_x \arrow[r, hook] & {[U/G],} \end{tikzcd}$$

and we recover the familiar fact that orbit $Gu \hookrightarrow U$ is locally closed (Proposition B.1.17(5)).

Exercise 3.5.22. Let $\mathcal{X}$ be a noetherian algebraic stack and $x \in |\mathcal{X}|$ be a finite type point with smooth stabilizer. Let $\overline{x}\colon \operatorname{Spec} \Bbbk \to \mathcal{X}$ be a representative of x. Show that $\dim \mathcal{G}_x = -\dim G_{\overline{x}}$.

3.6 Characterization of Deligne–Mumford stacks

After proving the existence of minimal presentations around finite type points with smooth stabilizers, we prove characterizations of Deligne–Mumford stacks and algebraic spaces. We use this conclude that $\mathcal{M}_g$ is Deligne–Mumford.

3.6.1 Existence of minimal presentations

Theorem 3.6.1 (Existence of Minimal Presentations)**.** *Let $\mathcal{X}$ be a noetherian algebraic stack and let $x \in |\mathcal{X}|$ be a finite type point with smooth stabilizer G_x. Then there exists a scheme U, a closed point $u \in U$, and a smooth morphism $(U, u) \to (\mathcal{X}, x)$ of relative dimension $\dim G_x$ from a scheme U such that the diagram*

$$\begin{tikzcd} \operatorname{Spec} \kappa(u) \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & U \arrow[d] \\ \mathcal{G}_x \arrow[r, hook] & \mathcal{X} \end{tikzcd}$$

is cartesian. In particular, if G_x is finite and reduced, there is an étale morphism $(U, u) \to (\mathcal{X}, x)$ from a scheme.

Proof. Let $(U, u) \to (\mathcal{X}, x)$ be a smooth morphism of relative dimension n from a scheme U such that $u \in U$ is a finite type point. By Proposition 3.5.16, the residual gerbe $\mathcal{G}_x$ at x exists, the inclusion $\mathcal{G}_x \hookrightarrow \mathcal{X}$ is locally closed, and $\mathcal{G}_x$ is regular of dimension $-\dim G_x$ (Exercise 3.5.22). We obtain a cartesian diagram

$$\begin{tikzcd} O(u) \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & U \arrow[d] \\ \mathcal{G}_x \arrow[r, hook] & \mathcal{X}, \end{tikzcd}$$

where $O(u)$ is a regular scheme of dimension $c := n - \dim G_x$. Let $f_1, \ldots, f_c \in \mathcal{O}_{O(u),u}$ be a regular sequence generating the maximal ideal at u. After replacing U with an open affine neighborhood of u, we may assume that each f_i is a global function on U. The closed subscheme $W := V(f_1, \ldots, f_c)$ intersects $O(u)$ transversely at U, i.e., $W \cap O(u) = \operatorname{Spec} \kappa(u)$ scheme-theoretically. We claim that inductively applying the Slicing Criterion for Flatness (A.2.9) implies that the composition $W \hookrightarrow U \to \mathcal{X}$ is flat at u. Indeed, by smooth descent, it suffices to verify that $W \times_{\mathcal{X}} U \hookrightarrow U \times_{\mathcal{X}} U \xrightarrow{p_2} U$ is flat at a preimage u' of u under $p_1 \colon W \times_{\mathcal{X}} U \to W$. Since the induced map $\operatorname{Spec} \kappa(u) \to \mathcal{G}_x$ is flat, so is the base change $U \times_{\mathcal{X}} \operatorname{Spec} \kappa(u) \to O(u)$, and therefore the pullbacks of $f_1, \ldots, f_c$ define a regular sequence in the local ring of $U \times_{\mathcal{X}} \operatorname{Spec} \kappa(u)$ at u'. Inductively applying the Slicing Criterion for Flatness gives the claim.

Since G_x is smooth, so is $\operatorname{Spec} \kappa(u) \to \mathcal{G}_x$. For flat morphisms, smoothness is a property that can be checked on fibers and thus (again arguing via smooth descent) $W \to \mathcal{X}$ is smooth at u. The statement follows after replacing W with an open neighborhood of u. □

Remark 3.6.2. A smooth presentation $p \colon U \to \mathcal{X}$ is called a *miniversal at* $u \in U(\Bbbk)$ if $T_{U,u} \to T_{\mathcal{X},p(u)}$ is an isomorphism of $\Bbbk$-vector spaces. We will see that the above presentations are miniversal in Proposition 3.7.6.

If the stabilizer G_x is not smooth, there are two candidates for 'minimal presentations.' There still exists a miniversal presentation $(U, u) \to (\mathcal{X}, x)$, but its relative dimension is equal to the dimension of the Lie algebra of G_x (rather than $\dim G_x$), and the fiber product $\mathcal{G}_x \times_{\mathcal{X}} U$ may thus be positive dimensional. For example, $B\boldsymbol{\mu}_p$ is an algebraic stack in characteristic p (Proposition 6.3.10) and it can be realized as the quotient of $\mathbb{G}_m$ acting on itself via $t \cdot x = t^p x$; here $\mathbb{G}_m \to B\boldsymbol{\mu}_p$ is a miniversal presentation. On the other hand, the above argument extends to show that there exists an fppf (but not always smooth) morphism $(U, u) \to (\mathcal{X}, x)$ such that $\mathcal{G}_x \times_{\mathcal{X}} U \cong \operatorname{Spec} \kappa(u)$. If in addition $\mathcal{X}$ has quasi-finite diagonal, $U \to \mathcal{X}$ is an fppf and quasi-finite presentation. See also [SP, Tag 06MC].

Exercise 3.6.3 (technical)**.** If $\mathcal{X}$ is a (possibly non-noetherian) algebraic stack and $x \in \mathcal{X}$ is a finite type point with discrete and unramified stabilizer G_x, show that there is an étale morphism $(U, u) \to (\mathcal{X}, x)$ from a scheme U where $u \in U$ is a closed point.

Hint: Replicate the above argument using Exercise 3.5.18.

3.6.2 Equivalent characterizations

Theorem 3.6.4 (Characterization of Deligne–Mumford Stacks)**.** *Let $\mathcal{X}$ be an algebraic stack. The following are equivalent:*

(1) the stack $\mathcal{X}$ is a Deligne–Mumford;

(2) the diagonal $\mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is unramified; and

(3) every point of $\mathcal{X}$ has a discrete and reduced stabilizer group.

If $\mathcal{X}$ has quasi-compact diagonal (e.g., $\mathcal{X}$ is quasi-separated), then (3) is equivalent to requiring that every stabilizer is *finite* and reduced.

Proof. The equivalence (2) $\Leftrightarrow$ (3) is essentially the definition of unramified: since the diagonal $\mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is always locally of finite type (Exercise 3.3.5), it is unramified if and only if every geometric fiber (which is either empty or isomorphic to a stabilizer) is discrete and reduced. It is not hard to see that a Deligne–Mumford stack has unramified diagonal (Exercise 3.2.8). For the converse, Existence of Minimal Presentations (Theorem 3.6.1 and Exercise 3.6.3) shows that for every finite type point $x \in \mathcal{X}$, there is an étale morphism $U \to \mathcal{X}$ from a scheme whose image contains x. Thus $\mathcal{X}$ is Deligne–Mumford. See also [LMB00, Thm 8.1] and [SP, Tag 06N3]. $\square$

Corollary 3.6.5. *A morphism $\mathcal{X} \to \mathcal{Y}$ of algebraic stacks is relatively Deligne–Mumford (Definition 3.3.36) if and only if $\mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ is unramified.*

Theorem 3.6.6 (Characterization of Algebraic Spaces)**.** *Let $\mathcal{X}$ be an algebraic stack whose diagonal is representable by schemes. The following are equivalent:*

(1) the stack $\mathcal{X}$ is an algebraic space,

(2) the diagonal $\mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is a monomorphism, and

(3) every point of $\mathcal{X}$ has a trivial stabilizer.

Remark 3.6.7. We will remove the pesky hypothesis that $\Delta_{\mathcal{X}}$ is representable by schemes in Theorem 4.5.10.

Proof. Condition (2) is equivalent to the condition that $\mathcal{X}$ is a sheaf (Exercise 2.4.39(d)). The implication (1) $\Rightarrow$ (2) follows from the definition of an algebraic space. For the converse, if $\mathcal{X}$ is a sheaf, then Existence of Minimal Presentations (3.6.1) implies that there exists a surjective, étale, and representable morphism $U \to \mathcal{X}$ from a scheme. Since $\Delta_{\mathcal{X}}$ is representable by schemes, so is $U \to \mathcal{X}$. The equivalence (2) $\Leftrightarrow$ (3) follows from the fact that a group scheme of finite type is trivial if and only if every fiber is trivial (Proposition B.1.8). $\square$

Corollary 3.6.8. *Let $G \to S$ be a smooth affine group scheme acting on an algebraic space U over S. Then*

(1) $[U/G]$ is Deligne–Mumford if and only if every point of U has a discrete and reduced stabilizer group, or equivalently if and only if the action map $G \times U \to U \times U$ is unramified.

(2) $[U/G]$ is an algebraic space if and only if every point of U has a trivial stabilizer group, or equivalently if and only if the action map $G \times U \to U \times U$ is a monomorphism. $\square$

Corollary 3.6.9 (Characterization of Representable Morphisms)**.** *Let $\mathcal{X} \to \mathcal{Y}$ be a morphism of noetherian algebraic stacks whose diagonal is representable by schemes. Then $\mathcal{X} \to \mathcal{Y}$ is representable if and only if for every geometric point $x \in \mathcal{X}(\Bbbk)$, the map $G_x \to G_{f(x)}$ on automorphism groups is injective.* $\square$

Corollary 3.6.10. *For $g \geq 2$, $\mathcal{M}_g$ is a Deligne–Mumford stack of finite type over $\mathbb{Z}$ with affine diagonal.*

Proof. It only remains to show that $\mathcal{M}_g$ is Deligne–Mumford, and by the Characterization of Deligne–Mumford Stacks (3.6.4), it suffices to show that every smooth, connected, and projective curve C over a field $\Bbbk$ has a discrete and reduced automorphism group scheme $\mathrm{Aut}(C)$. In other words, we need to show that the dimension of the Lie algebra $T_{\mathrm{Aut}(C),e}$ is zero. Consider the diagrams

$$\begin{tikzcd} \operatorname{Spec}\Bbbk \arrow[r, "e"] \arrow[d, hook] & \mathrm{Aut}(C) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{M}_g \arrow[d] \\ \operatorname{Spec}\Bbbk[\epsilon] \arrow[r] \arrow[ur, "\tau"] & \operatorname{Spec}\Bbbk \arrow[r, "({[C]},{[C]})"] & \mathcal{M}_g \times \mathcal{M}_g \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \operatorname{Spec}\Bbbk \arrow[r, "{[C]}"] \arrow[d, hook] & \mathcal{M}_g \arrow[d] \\ \operatorname{Spec}\Bbbk[\epsilon] \arrow[r, "p"'] \arrow[ur, bend left] \arrow[ur, bend right] \arrow[ur, phantom, "\Downarrow\tau"] & \operatorname{Spec}\Bbbk. \end{tikzcd}$$

A lifting $\tau\colon \operatorname{Spec}\Bbbk[\epsilon] \to \mathrm{Aut}(C)$ of the left diagram, i.e., a tangent vector $\tau \in T_{\mathrm{Aut}(C),e}$, translates on the right diagram to an automorphism τ of the trivial first order deformation $[C] \circ p$. By Deformation Theory (C.2.4), the automorphism group of the trivial first order deformation is identified with $\mathrm{H}^0(C, T_C)$, but this vector space is zero since the degree of $T_C = \Omega_C^\vee$ is $2 - 2g < 0$. $\square$

3.7 Smoothness and the Infinitesimal Lifting Criteron

Truth is ever to be found in simplicity, and not in the multiplicity and confusion of things.

Isaac Newton

The Infinitesimal Lifting Criteria (A.3.1(3), A.3.2(5), and A.3.3(4)) for schemes provides functorial characterizations of morphisms that are smooth, étale, or unramified. We provide below a generalization to algebraic stacks (Theorem 3.7.1). As an application, we show that the moduli stacks $\mathcal{M}_g$ of smooth curves and $\underline{\mathrm{Coh}}_{r,d}(C)$ of coherent sheaves on a fixed smooth curve C are smooth (Propositions 3.7.7 and 3.7.9).

3.7.1 Infinitesimal Lifting Criteria

Recall that smoothness is well-defined for morphisms of algebraic stacks since smoothness is a smooth local property on the source and target. On the other hand, étaleness and unramifiedness are smooth local on the target, but only étale-local on the source, and thus well-defined for morphisms $\mathcal{X} \to \mathcal{Y}$ that are *relatively Deligne–Mumford*, i.e., if every base change $\mathcal{X} \times_{\mathcal{Y}} T$ by a map $T \to \mathcal{Y}$ from a scheme is Deligne–Mumford; see Definition 3.3.36. A morphism $\mathcal{X} \to \mathcal{Y}$ of algebraic stacks is étale (resp., unramified) if it is relatively Deligne–Mumford and every base change $\mathcal{X} \times_{\mathcal{Y}} T \to T$ is étale (resp., unramified); see Definition 3.3.37. We also recall that $\mathcal{X} \to \mathcal{Y}$ is relatively Deligne–Mumford if and only if the diagonal $\mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ is unramified (Corollary 3.6.5).

Theorem 3.7.1 (Infinitesimal Lifting Criteria for Smoothness/Étaleness/Unramifedness)**.** *Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a locally of finite type morphism of locally noetherian algebraic stacks with quasi-compact and separated diagonals. Consider a 2-commutative dia-*

gram

$$
\begin{tikzcd}
\operatorname{Spec} A_0 \arrow[r] \arrow[d, hook] & \mathcal{X} \arrow[d, "f"] \\
\operatorname{Spec} A \arrow[r] \arrow[ur, dashed] & \mathcal{Y},
\end{tikzcd}
\tag{3.7.2}
$$

of solid arrows where $A \to A_0$ is a surjection of artinian local rings with residue field $\Bbbk$ such that $\ker(A \to A_0) \cong \Bbbk$. *Then*

(1) f is smooth if and only if there exists a lifting of every diagram (3.7.2);

(2) f is étale if and only if there exists a lifting, which is unique up to unique isomorphism, of every diagram (3.7.2),

(3) f is unramified if and only if every two liftings of a diagram (3.7.2) are uniquely isomorphic; and

(4) f has unramified diagonal if and only if every automorphism of a lifting of a diagram (3.7.2) is trivial.

Remark 3.7.3. To be explicit, a *lifting* of a 2-commutative diagram

$$
\begin{tikzcd}
S \arrow[r, "x"] \arrow[d, "g"'] \arrow[dr, phantom, "\not\Downarrow_{\alpha}"] & \mathcal{X} \arrow[d, "f"] \\
T \arrow[r, "y"'] & \mathcal{Y},
\end{tikzcd}
\tag{3.7.4}
$$

is a triple $(\widetilde{x}, \beta, \gamma)$ where $\widetilde{x}\colon T \to \mathcal{X}$ is a map and $\beta\colon x \xrightarrow{\sim} \widetilde{x} \circ g$ and $\gamma\colon f \circ \widetilde{x} \xrightarrow{\sim} y$ are 2-isomorphisms such that

$$
\begin{tikzcd}
 & f \circ x \arrow[dl, "f(\beta)"'] \arrow[dr, "\alpha"] & \\
f \circ \widetilde{x} \circ g \arrow[rr, "g^*\gamma"] & & y \circ g
\end{tikzcd}
$$

commutes. We may view a lifting as a 2-commutative diagram

$$
\begin{tikzcd}
S \arrow[r, "x"] \arrow[d, "g"'] & \mathcal{X} \arrow[d, "f"] \\
T \arrow[r, "y"'] \arrow[ur, "\widetilde{x}"] & \mathcal{X}.
\end{tikzcd}
$$

A morphism $(\widetilde{x}, \beta, \gamma) \to (\widetilde{x}', \beta', \gamma')$ of liftings is a 2-isomorphism $\Theta\colon \widetilde{x} \xrightarrow{\sim} \widetilde{x}'$ such that $\beta = g^*\Theta \circ \beta'$ and $\gamma = \gamma' \circ f(\Theta)$.

We can also interpret liftings using the map

$$\Psi\colon \mathcal{X}(T) \to \mathcal{X}(S) \times_{\mathcal{Y}(S)} \mathcal{Y}(T)$$

of groupoids. A diagram (3.7.4) defines an object $(x, y, \alpha) \in \mathcal{X}(S) \times_{\mathcal{Y}(S)} \mathcal{Y}(T)$ and the category of liftings is the fiber category over this object. The criteria above states that $f\colon \mathcal{X} \to \mathcal{Y}$ is smooth (resp., is étale, is unramified, has unramified diagonal) if and only if $\mathcal{X}(A) \to \mathcal{X}(A_0) \times_{\mathcal{Y}(A_0)} \mathcal{Y}(A)$ is essentially surjective (resp., an equivalence, fully faithful, faithful).

Proof. We handle the criterion for smoothness leaving the remaining cases to the reader. Suppose that $f\colon \mathcal{X} \to \mathcal{Y}$ is smooth and that we are given a diagram (3.7.2). By replacing $\mathcal{Y}$ with $\operatorname{Spec} A$ and $\mathcal{X}$ with $\mathcal{X} \times_{\mathcal{Y}} \operatorname{Spec} A$, we may assume that $\mathcal{Y} = \operatorname{Spec} A$ is affine, and we need to show that a section over $\operatorname{Spec} A_0$

$$\begin{tikzcd}
 & \mathcal{X} \arrow[d] \\
\operatorname{Spec} A_0 \arrow[ur] \arrow[r, hook] & \operatorname{Spec} A \arrow[u, dashed, bend right]
\end{tikzcd}$$

extends to a section over $\operatorname{Spec} A$. If $\mathcal{X}$ is a scheme, the existence of a lifting is provided by the Infinitesimal Lifting Criterion for Smoothness (A.3.1) for schemes. If $\mathcal{X}$ is an algebraic space, we may apply Proposition 4.3.4 (proven later but independently) to construct a smooth presentation $U \to \mathcal{X}$ from a scheme and a lifting $\operatorname{Spec} \Bbbk \to U$ of $\operatorname{Spec} \Bbbk \hookrightarrow \operatorname{Spec} A_0 \to \mathcal{X}$. This gives a diagram

$$\begin{tikzcd}
 & & U \arrow[d] \\
 & & \mathcal{X} \arrow[d] \\
\operatorname{Spec} \Bbbk \arrow[uurr] \arrow[r, hook] & \operatorname{Spec} A_0 \arrow[ur] \arrow[uur, dashed] \arrow[r, hook] & \operatorname{Spec} A. \arrow[uu, dashed, bend right]
\end{tikzcd}$$

Since $U \to \mathcal{X}$ is representable by schemes and smooth, applying the Infinitesimal Lifting Criterion (A.3.1) inductively to the square-zero extensions $A_0/\mathfrak{m}^{k+1} \to A_0/\mathfrak{m}^k$ gives a lifting $\operatorname{Spec} A_0 \to U$ of $\operatorname{Spec} A_0 \to \mathcal{X}$. Since $U \to \operatorname{Spec} A$ is a smooth morphism of schemes, another application of the Infinitesimal Lifting Criterion gives a lifting $\operatorname{Spec} A \to U$ of $\operatorname{Spec} A_0 \to U$, and the composition $\operatorname{Spec} A \to U \to \mathcal{X}$ gives the desired extension. Finally, if $\mathcal{X}$ is an algebraic stack, we can repeat the above argument by applying Proposition 4.3.4 to construct a smooth presentation $U \to \mathcal{X}$ with a lifting $\operatorname{Spec} \Bbbk \to U$, where we use the representability of $U \to \mathcal{X}$ and the algebraic space case to construct the lifting $\operatorname{Spec} A_0 \to U$.

Conversely, suppose that every diagram (3.7.2) for $f\colon \mathcal{X} \to \mathcal{Y}$ has a lifting. Choose smooth presentations $V \to \mathcal{Y}$ and $U \to \mathcal{X}_V$ giving a commutative diagram

$$\begin{tikzcd}
U \arrow[r] \arrow[dr] & \mathcal{X}_V \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{X} \arrow[d, "f"] \\
 & V \arrow[r] & \mathcal{Y}.
\end{tikzcd}$$

Since $U \to \mathcal{X}_V$ is smooth, by the implication already proven, every diagram (3.7.2) for $U \to \mathcal{X}_V$ has a lifting. Since every diagram (3.7.2) for $\mathcal{X}_V \to V$ also has a lifting (as it is the base change of $\mathcal{X} \to \mathcal{Y}$), so does the composition $U \to V$. As $U \to V$ is a morphism of schemes, the Infinitesimal Lifting Criterion (A.3.1) implies that it is smooth. Since smoothness is a smooth local property on the source and target, $\mathcal{X} \to \mathcal{Y}$ is smooth. See also [LMB00, Prop. 4.15] and [SP, Tag 0DP0]. $\square$

Exercise 3.7.5. Prove the remaining parts of the Infinitesimal Lifting Criteria for étaleness, unramifiedness, and unramified diagonal.

Hint: For the case of unramified diagonal, relate an automorphism of a lifting for the morphism $f\colon \mathcal{X} \to \mathcal{Y}$ to a lifting of the diagonal $\Delta_f\colon \mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$. Reduce the case *of étaleness to unramifiedness. For unramifiedness, use Corollary 3.6.5 to reduce to showing that a relatively Deligne–Mumford morphism $f\colon \mathcal{X} \to \mathcal{Y}$ is unramified if and only if every two liftings of a diagram (3.7.2) are isomorphic, and establish this using similar methods to the proof of the smoothness criterion.*

As a first application, we see that the presentation produced by the Existence of Minimal Presentations (Theorem 3.6.1) is miniversal, i.e., induces an isomorphism on tangent spaces at the chosen preimage, and we can also express the dimension of a smooth algebraic stack in terms of its tangent space and stabilizer.

Proposition 3.7.6. *Let $\mathcal{X}$ be a noetherian algebraic stack and $x \in |\mathcal{X}|$ be a finite type point with smooth stabilizer. Let $f\colon (U,u) \to (\mathcal{X},x)$ be a smooth morphism from a scheme such that $\mathcal{G}_x \times_{\mathcal{X}} U \cong \operatorname{Spec} \kappa(u)$. Then $U \to \mathcal{X}$ is miniversal at u, i.e., $T_{U,u} \to T_{\mathcal{X},f(u)}$ is an isomorphism of $\kappa(u)$-vector spaces. In particular, if $\mathcal{X}$ is smooth over a field $\Bbbk$ and $x \in \mathcal{X}(\mathbb{F})$ is a point with smooth stabilizer over a finite extension $\mathbb{F}/\Bbbk$, then*

$$\dim_x \mathcal{X} = \dim T_{\mathcal{X},x} - \dim G_x.$$

Proof. Surjectivity of $T_{U,u} \to T_{\mathcal{X},f(u)}$ follows from applying the Infinitesimal Lifting Criterion (3.7.1) to the smooth morphism $U \to \mathcal{X}$. Injectivity follows from the fact that

$$\begin{tikzcd}
\operatorname{Spec} \kappa(u) \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & U \arrow[d] \\
\mathcal{G}_x \arrow[r, hook] & \mathcal{X}
\end{tikzcd}$$

is cartesian. Indeed, if $\tau\colon \operatorname{Spec} \kappa(u)[\epsilon] \to U$ is an element of $T_{U,u}$ mapping to $0 \in T_{\mathcal{X},f(u)}$, then the composition $\operatorname{Spec} \kappa(u)[\epsilon] \to U \to \mathcal{X}$ factors through the residual gerbe $\mathcal{G}_x$ and therefore also factors through $\operatorname{Spec} \kappa(u)$. We conclude that $\tau = 0$. For the final statement, Existence of Minimal Presentations (3.6.1) produces a smooth morphism $f\colon (U,u) \to (\mathcal{X},x)$ of relative dimension $\dim G_x$ such that $\mathcal{G}_x \times_{\mathcal{X}} U \cong \operatorname{Spec} \kappa(u)$. Therefore, $\dim_x \mathcal{X} = \dim_u U - \dim G_x$, and since U is smooth at u, $\dim_u U = \dim T_{U,u} = \dim T_{\mathcal{X},x}$. $\square$

3.7.2 Smoothness of moduli stacks

Combining the Infinitesimal Lifting Criterion for Smoothness (3.7.1) with deformation theory allows both for the verification of smoothness of a moduli stack and the computation of its dimension.

Proposition 3.7.7. *For $g \geq 2$, the Deligne–Mumford stack $\mathcal{M}_g$ is smooth over* $\operatorname{Spec} \mathbb{Z}$ *of relative dimension* $3g-3$.

Proof. Let $\operatorname{Spec} \Bbbk \to \mathcal{M}_g$ be a morphism from a field $\Bbbk$ corresponding to smooth, connected, and projective curve C over $\Bbbk$. Consider a diagram

$$\begin{tikzcd}
\operatorname{Spec} \Bbbk \arrow[r] \arrow[rr, bend left, "{[C]}"] & \operatorname{Spec} A_0 \arrow[r] \arrow[d, hook] & \mathcal{M}_g \arrow[d] \\
& \operatorname{Spec} A \arrow[r] \arrow[ur, dashed] & \operatorname{Spec} \mathbb{Z},
\end{tikzcd} \tag{3.7.8}$$

where $A \to A_0$ is surjection of artinian local rings with residue field $\Bbbk$ such that $\Bbbk = \ker(A \to A_0)$. The map $\operatorname{Spec} A_0 \to \mathcal{M}_g$ corresponds to a family of curves $\mathcal{C}_0 \to \operatorname{Spec} A_0$, and a lifting in (3.7.8) translates to an extension

$$\begin{tikzcd} C \arrow[r, hook] \arrow[d] & \mathcal{C}_0 \arrow[r, hook, dashed] \arrow[d] & \mathcal{C} \arrow[d, dashed] \\ \operatorname{Spec} \Bbbk \arrow[r, hook] & \operatorname{Spec} A_0 \arrow[r, hook] & \operatorname{Spec} A \end{tikzcd}$$

of solid arrows. By Deformation Theory (C.2.4), there is a cohomology class $\mathrm{ob}_C \in \mathrm{H}^2(C, T_C)$ such that the above diagram has an extension if and only if $\mathrm{ob}_C = 0$. Since C is a curve, $\mathrm{H}^2(C, T_C) = 0$; thus, $\mathrm{ob}_C = 0$ and there is a lifting. Finally, Deformation Theory (C.2.4) also gives the identification $T_{\mathcal{M}_g,[C]} = \mathrm{H}^1(C, T_C)$, which has dimension $3g - 3$ by a Riemann–Roch computation (see Example 3.5.8). Since $\dim \operatorname{Aut}(C) = 0$, we conclude using Proposition 3.7.6 that $\dim_{[C]}(\mathcal{M}_g \times_{\mathbb{Z}} \Bbbk) = 3g - 3$. $\square$

Proposition 3.7.9. *For a smooth, connected, and projective curve C over an algebraically closed field $\Bbbk$, the algebraic stack $\underline{\mathrm{Coh}}_{r,d}(C)$ is smooth over $\operatorname{Spec} \Bbbk$ of dimension $r^2(g-1)$.*

Proof. By Exercise 3.7.10, $\underline{\mathrm{Coh}}_{r,d}(C)$ is nonempty. Let $[F] \in \underline{\mathrm{Coh}}_{r,d}(C)(\mathbb{F})$ be a coherent sheaf on $C_{\mathbb{F}} = C \times_{\Bbbk} \mathbb{F}$ of rank r and degree d. Let $A \to A_0$ be a surjection of artinian local rings with residue field $\mathbb{F}$ such that $\mathbb{F} = \ker(A \to A_0)$. We need to check that every coherent sheaf $\mathcal{F}_0$ on C_{A_0} that restricts to F extends to a coherent sheaf $\mathcal{F}$ on C_A. By Deformation Theory (C.2.11), there is an element $\mathrm{ob}_F \in \operatorname{Ext}^2_{\mathcal{O}_C}(F, F)$ such that $\mathrm{ob}_F = 0$ if and only if there exists an extension. Since C is a smooth curve, $\operatorname{Ext}^2_{\mathcal{O}_C}(F, F) = \mathrm{H}^2(C_{\mathbb{F}}, F \otimes F^\vee) = 0$. Deformation theory also provides an identification $T_{\underline{\mathrm{Coh}}_{r,d}(C),[F]} = \operatorname{Ext}^1_{\mathcal{O}_C}(F, F)$ and a Riemann–Roch calculation yields that $\dim \operatorname{Ext}^1_{\mathcal{O}_C}(F, F) = \dim \operatorname{Aut}(F) + r^2(g-1)$ (see Example 3.5.9). Therefore, $\dim_{[F]} \underline{\mathrm{Coh}}_{r,d}(C) = \dim \operatorname{Ext}^1_{\mathcal{O}_C}(F, F) - \dim \operatorname{Aut}(F) = r^2(g-1)$. $\square$

Exercise 3.7.10. For every line bundle L and rank $r > 0$, show that there exists a vector bundle E of rank r with $\det E \cong L$.

3.8 Properness and the Valuative Criterion

Working with noncompact spaces is like trying to keep change with holes in your pockets.

ANGELO VISTOLI

After defining universally closed, separated, and proper morphisms of algebraic stacks, we prove valuative criteria (Theorem 3.8.7) providing a generalization of the usual valuative criteria for schemes (Theorem A.4.5). These valuative criteria are essential in moduli theory. They are applied in this book to show that $\overline{\mathcal{M}}_g$ is proper (Theorem 5.5.23) and $\mathcal{B}un^{\mathrm{ss}}_{r,d}(C)$ is universally closed.

3.8.1 Definitions

With some care, we define separatedness and properness for morphisms of algebraic stacks. Recall from Definition 3.3.11 that a representable morphism $\mathcal{X} \to \mathcal{Y}$ of algebraic stacks is *separated* if the diagonal $\mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ (which is representable by schemes) is proper.

Definition 3.8.1.

(1) A morphism $\mathcal{X} \to \mathcal{Y}$ of algebraic stacks is *universally closed* if for every morphism $\mathcal{Y}' \to \mathcal{Y}$ of algebraic stacks, the morphism $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}' \to \mathcal{Y}'$ induces a closed map $|\mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}'| \to |\mathcal{Y}'|$.

(2) A representable morphism $\mathcal{X} \to \mathcal{Y}$ of algebraic stacks is *proper* if it is universally closed, separated, and of finite type.

(3) A morphism $\mathcal{X} \to \mathcal{Y}$ of algebraic stacks is *separated* if the representable morphism $\mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ is proper.

(4) A morphism $\mathcal{X} \to \mathcal{Y}$ of algebraic stacks is *proper* if it is universally closed, separated, and of finite type.

For a morphism of schemes, properness is equivalent to the diagonal being a closed immersion. This is also true for algebraic spaces as proper monomorphisms of algebraic spaces are closed immersions (Corollary 4.5.14). This fails for morphisms of algebraic stacks as the diagonal need not be a monomorphism. Recall that the stabilizer G_x of a field-valued point $x\colon \operatorname{Spec} \Bbbk \to \mathcal{X}$ is given by the cartesian diagram

$$\begin{tikzcd} G_x \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \operatorname{Spec} \Bbbk \arrow[d, "{(x,x)}"] \\ \mathcal{X} \arrow[r] & \mathcal{X} \times \mathcal{X}. \end{tikzcd}$$

If $\mathcal{X}$ is a separated algebraic stack over a scheme S, then G_x is a proper group algebraic space over $\Bbbk$, and even a group *scheme* by Theorem 4.5.28. If $\mathcal{X}$ is separated and has affine diagonal, then G_x is proper and affine, thus finite.

Example 3.8.2. If G is an abstract finite group, then $BG \to \operatorname{Spec} \mathbb{Z}$ is proper. If $G \to S$ is a proper group scheme (e.g., an abelian scheme), then $B\mathbb{G}_m \to S$ proper. On the other hand, if $G \to S$ is a non-finite affine group scheme (e.g., $\mathbb{G}_m \to \operatorname{Spec} \mathbb{Z}$), then $BG \to S$ is universally closed but not separated.

Exercise 3.8.3 (easy)**.** Show that $\mathcal{M}_{(0,0)}$, $\mathcal{M}_{(0,1)}$, $\mathcal{M}_{(0,2)}$, and $\mathcal{M}_{(1,0)}$ are *not* separated.

Example 3.8.4 ($\mathcal{B}un_{r,d}(C)$ is not separated)**.** Since $\mathcal{B}un_{r,d}(C)$ has affine diagonal (Example 3.3.15) and infinite automorphism groups, $\mathcal{B}un_{r,d}(C)$ is *not* separated over the base field $\Bbbk$.

Remark 3.8.5. As universal closedness is a smooth local property on the target, $\mathcal{X} \to \mathcal{Y}$ is universally closed if and only if $\mathcal{X} \times_{\mathcal{Y}} T \to T$ is closed for all maps $T \to \mathcal{Y}$ from schemes, or equivalently $\mathcal{X} \times_{\mathcal{Y}} V \to V$ is universally closed for a smooth presentation $V \to \mathcal{Y}$. In the proof of Theorem 3.8.7, we will show that when $\mathcal{Y}$ is noetherian, it suffices to consider *finite type* base changes $T \to \mathcal{Y}$ from schemes.

Exercise 3.8.6. An action of an algebraic group G over a field $\Bbbk$ on an algebraic space U is called *proper* if the action map

$$\Psi\colon G \times U \to U \times U, \quad (g,u) \mapsto (gu, u)$$

is proper.

(a) Show that the action of G on U is proper if and only if $[U/G]$ is separated.

(b) For $u \in U(\Bbbk)$, let $\Psi_u \colon G \to U$ be the map defined by $g \mapsto gu$ (viewing Ψ as a morphism over U via the projections on the second components, then Ψ_u is the fiber of Ψ over u). Show that the following are equivalent:

   (i) $\Psi_u \colon G \to U$ is proper,

   (ii) $u\colon \operatorname{Spec} \Bbbk \to [U/G]$ is proper,

   (iii) $Gu \subseteq U$ is closed and G_u is proper.

Hint for (b): To show that (i) or (ii) implies (iii), replace U with the reduced orbit $(Gu)_{\mathrm{red}}$, use Generic Flatness (3.3.32) to show that the morphisms Φ_u or u are faithfully flat, and then use fppf descent. See also [GIT, Lem. 0.3].

3.8.2 Valuative Criteria

For moduli problems, the valuative criterion for properness translates to the geometric question of whether an object over the fraction field of a DVR R (which we think of as a punctured curve) extends uniquely to a family over the DVR (or curve). In the formulation below, we will use the notions of liftings of 2-commutative diagrams and their morphisms as defined formally in Remark 3.7.3.

Theorem 3.8.7 (Valuative Criteria for Properness/Universal Closedness/Separatedness)**.** *Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a quasi-compact and quasi-separated morphism of locally noetherian algebraic stacks. Consider a 2-commutative diagram*

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] \arrow[dr, phantom, "\swarrow_{\alpha}"] & \mathcal{X} \arrow[d, "f"] \\ \operatorname{Spec} R \arrow[r] & \mathcal{Y} \end{tikzcd} \tag{3.8.8}$$

where R is a DVR with fraction field K. Then

(1) *f is universally closed if and only if for every diagram (3.8.8), there exists an extension $R \to R'$ of DVRs with $K \to K' = \operatorname{Frac}(R)$ of finite transcendence degree and a lifting*

$$\begin{tikzcd} \operatorname{Spec} K' \arrow[r] \arrow[d] & \operatorname{Spec} K \arrow[r] \arrow[d] & \mathcal{X} \arrow[d, "f"] \\ \operatorname{Spec} R' \arrow[r] \arrow[urr, dashed] & \operatorname{Spec} R \arrow[r] & \mathcal{Y}; \end{tikzcd} \tag{3.8.9}$$

(2) *f is proper if and only if f is finite type and for every diagram (3.8.8), there exists an extension $R \to R'$ of DVRs with the map $K \to K'$ on fraction fields having finite transcendence degree and a lifting as in (3.8.9), which is unique up to unique isomorphism;*

(3) *f is separated if and only if every two liftings of a diagram (3.8.8) are uniquely isomorphic; and*

(4) *f has separated diagonal if and only if every automorphism of a lifting of a diagram (3.8.8) is trivial.*

Before proving this theorem in the next subsection, we provide some examples.

Exercise 3.8.10 (good practice)**.** Use the valuative criterion to show the following:

(a) If G is an abstract finite group, then $BG \to \operatorname{Spec} \mathbb{Z}$ is proper.

(b) The map $B\mathbb{G}_m \to \operatorname{Spec} \mathbb{Z}$ is universally closed but not separated.

Exercise 3.8.11 (hard)**.** Show that the stack $\overline{\mathcal{M}}_{1,1}$ of stable elliptic curves introduced in Exercise 3.1.19(c) is proper over $\operatorname{Spec} \mathbb{Z}$.

We will prove later that $\overline{\mathcal{M}}_{g,n}$ is proper as long as $g \geq 2$, or $g = 1$ and $n \geq 1$, or $g = 0$ and $n \geq 3$ (Theorem 5.5.23).

Example 3.8.12 (Properness of $\overline{\mathcal{M}}_{g,n}$)**.** The moduli stack $\overline{\mathcal{M}}_{g,n}$ is proper over $\operatorname{Spec} \mathbb{Z}$ (Theorem 5.5.23) as long as $2g - 2 + n > 0$ (i.e., we exclude $(g, n) = (0, 0)$, $(0, 1)$, $(0, 2)$, and $(1, 0)$). We prove this in §5.5.1 in characteristic 0 by verifying the valuative criterion.

Exercise 3.8.13 (good practice)**.** For a smooth, geometrically connected, and projective curve C over a field $\Bbbk$, show that the algebraic stacks $\underline{\mathrm{Coh}}_{r,d}(C)$ and $\mathcal{B}un_{r,d}(C)$ satisfy the valuative criterion for universally closedness. Observe that in this case the valuative criterion holds without requiring an extension of DVRs.

Remark 3.8.14. Note that the above exercise does *not* imply that $\mathcal{B}un_{r,d}(C)$ is universally closed. Indeed, the Valuative Criterion for Universal Closedness requires quasi-compactness and $\mathcal{B}un_{r,d}(C)$ is *not* quasi-compact (see Exercise 8.2.3). In fact, a universally closed morphism of schemes is necessarily quasi-compact [SP, Tag 04XU], and presumably the same holds for morphisms of algebraic stacks. The stack of vector bundles over a curve and the stack of all curves are not universally closed (see Exercises 3.8.15 and 5.4.18).

Exercise 3.8.15.

(a) Show that $\mathcal{B}un_{r,d}(C)$ is *not* universally closed

(b) For an integral proper scheme X over $\Bbbk$ of arbitrary dimension, show that the stack $\underline{\mathrm{Coh}}^{\mathrm{tf}}$ of torsion free sheaves on X satisfies the valuative criterion for universally closedness.

Are base changes necessary? For the Valuative Criterion for Properness (A.4.5) for a morphism $X \to Y$ of schemes, it is not necessary to allow extensions of the DVR, i.e., there exists a unique lift

$$\begin{tikzcd}
\operatorname{Spec} K \arrow[r] \arrow[d] & X \arrow[d] \\
\operatorname{Spec} R \arrow[r] \arrow[ur, dashed] & Y.
\end{tikzcd}$$

The same holds for morphisms of algebraic spaces [SP, Tag 0A40]. For general morphisms of algebraic stacks, it is necessary to allow extensions.

Example 3.8.16. Consider the Deligne–Mumford stack $\mathcal{X} = B\boldsymbol{\mu}_n$ over a field $\Bbbk$ whose characteristic is prime to n. Let $R = \Bbbk[x]_{(x)}$ with fraction field $K = \Bbbk(x)$. Then $\operatorname{Spec} K \to \operatorname{Spec} K$, given by $x \mapsto x^n$, is a non-trivial principal $\boldsymbol{\mu}_n$-torsor. The classifying map $\operatorname{Spec} K \to B\boldsymbol{\mu}_n$ does not extend to a map $\operatorname{Spec} R \to B\boldsymbol{\mu}_n$ (as all $\boldsymbol{\mu}_n$-torsors over $\operatorname{Spec} R$ are trivial). Note however that the principal $\boldsymbol{\mu}_n$-bundle $\operatorname{Spec} K \to \operatorname{Spec} K$ becomes trivial after base changing by the field extension $K(x^{1/n})$ of K, and therefore the composition $\operatorname{Spec} K(x^{1/n}) \to B\boldsymbol{\mu}_n$ extends to the map $\operatorname{Spec} R[x^{1/n}] \to B\boldsymbol{\mu}_n$ classifying the trivial bundle.

Example 3.8.17. Let $X = \operatorname{Spec} R$ be the spectrum of a DVR and $\mathcal{X}$ be the nth root stack $X(\sqrt[n]{\mathcal{O}_X/\pi})$ (Example 3.9.22). The morphism $\mathcal{X} \to X$ is an isomorphism over the generic point, but the section $\operatorname{Spec} K \to \mathcal{X}$ does not extend to a global section $X \to \mathcal{X}$.

If G is a special algebraic group over a field (i.e., every principal G-bundle is Zariski-locally trivial) such as SL_n or GL_n, then BG satisfies the valuative criterion for universal closedness without a base change: any map $\operatorname{Spec} K \to BG$ corresponds to the trivial principal G-bundle and thus extends to a map $\operatorname{Spec} R \to BG$. On the other hand, base changes are necessary for $B\mathrm{PGL}_n$.

Exercise 3.8.18 (hard)**.** Show that there is a principal PGL_n-bundle over the fraction field of a DVR that does not extend to the DVR.

For the valuative criterion of properness for $\overline{\mathcal{M}}_g$, extensions of the DVR are necessary (see Example 5.5.11). On the other hand, the valuative criterion for universal closedness for $\mathcal{B}un_{r,d}(C)$ holds without extensions (Exercise 3.8.13).

3.8.3 Proof of the Valuative Criteria

We modify the proof of the valuative criterion for schemes given in §A.4. The starting point is a lifting criterion for closed morphisms generalizing Lemma A.4.1.

Lemma 3.8.19. *Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a quasi-compact morphism algebraic stacks. Then f is closed if and only for every point $x \in |\mathcal{X}|$, every specialization $f(x) \rightsquigarrow y_0$ lifts to a specialization $x \rightsquigarrow x_0$.*

Proof. If f is closed, then $f(\overline{\{x\}})$ is closed and equal to $\overline{\{f(x)\}}$. Every point $y_0 \in \overline{\{f(x)\}} = f(\overline{\{x\}})$ is therefore the image of a specialization of x. For the converse direction, after replacing $\mathcal{X}$ with any closed substack, it suffices to show that $f(|\mathcal{X}|)$ is closed. By Exercise 3.3.31(a), it suffices show that $f(|\mathcal{X}|)$ is closed under specialization. □

Specializations are induced by maps from DVRs, just as in the case of schemes (Proposition A.4.4).

Proposition 3.8.20. *If $f\colon \mathcal{X} \to \mathcal{Y}$ is a finite type morphism of noetherian algebraic stacks, $x \in |\mathcal{X}|$, and $f(x) \rightsquigarrow y_0$ is a specialization, then there exists a diagram*

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & \mathcal{X} \arrow[d, "f"] & x \arrow[d, mapsto] \\ \operatorname{Spec} R \arrow[r] & \mathcal{Y} & f(x) \arrow[r, rightsquigarrow] & y_0, \end{tikzcd}$$

where R is a DVR with fraction field K, the image of $\operatorname{Spec} K \to \mathcal{X}$ is x, and $\operatorname{Spec} R \to \mathcal{Y}$ realizes the specialization $f(x) \rightsquigarrow y_0$. In particular, every specialization $x \rightsquigarrow x_0$ in a noetherian algebraic stack is realized by a map $\operatorname{Spec} R \to \mathcal{X}$ from a DVR.

Proof. Let $V \to \mathcal{Y}$ be a smooth presentation and $v_0 \in V$ be a preimage of y_0. Since $V \to \mathcal{Y}$ is smooth, it is an open morphism (Exercise 3.3.27), and thus there exists a specialization $v \rightsquigarrow v_0$ over $f(x) \rightsquigarrow y_0$ (Exercise 3.3.31(b)). Let $x' \in |\mathcal{X}_V|$ be a preimage of $v \in V$ and $x \in |\mathcal{X}|$. Let $U \to \mathcal{X}_V$ be a smooth presentation and $u \in U$ be a preimage of x'. Applying Proposition A.4.4 to the morphism $U \to V$ of schemes and $u \mapsto v \rightsquigarrow v_0$ gives the desired diagram. □

Proof of Theorem 3.8.7. The quasi-separatedness of $f\colon \mathcal{X} \to \mathcal{Y}$ means that the diagonal $\Delta_f\colon \mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ and the double diagonal $\Delta_{\Delta_f}\colon \mathcal{X} \to \mathcal{X} \times_{\mathcal{X}\times_{\mathcal{Y}}\mathcal{X}} \mathcal{X}$ are quasi-compact. The quasi-compactness of f, Δ_f, and Δ_{Δ_f} are needed in the proof for the implication that the valuative criterion implies universally closedness, separatedness, and separated diagonal, respectively.

We first establish the criterion for universal closedness. Suppose that the valuative criterion holds and that $f\colon \mathcal{X} \to \mathcal{Y}$ is not universally closed. Since universal closedness is a smooth local property on the target, we may assume that $\mathcal{Y} = Y$ is a noetherian scheme and that there is a map $T \to Y$ of schemes such that $f_T\colon \mathcal{X}_T \to T$ is not closed. We will reduce to the case that $T \to Y$ is a *finite type* morphism. By Lemma 3.8.19, there exists $z \in |\mathcal{X}_T|$ and a specialization $f_T(z) \rightsquigarrow t_0$ which doesn't lift to a specialization $z \rightsquigarrow z_0$. This implies that $\mathcal{Z} = \overline{\{z\}} \subseteq \mathcal{X}_T$ has trivial intersection with the fiber $(\mathcal{X}_T)_{t_0}$. If $p\colon X \to \mathcal{X}$ is a smooth presentation, then the preimage Z of $\mathcal{Z}$ under $X_T \to \mathcal{X}_T$ does not meet the fiber $(X_T)_{t_0}$. Applying Lemma A.4.8 to $X_T \to T$ shows that after replacing T with an open neighborhood of t_0, there exists a factorization $T \xrightarrow{g} T' \to Y$ and a closed subscheme $Z' \subseteq X_{T'}$ such that $T' \to Y$ is of finite type, $Z \cap (X_{T'})_{g(t_0)} = \emptyset$, and $\operatorname{im}(Z \hookrightarrow X_T \to X_{T'}) \subseteq Z'$. Letting $z' \in |\mathcal{X}_{T'}|$ be the image of $z \in |\mathcal{X}_T|$, we have that z' maps to $g(f_T(z)) \in T'$ and that there is a specialization $g(f_T(z)) \rightsquigarrow g(t_0)$ which does not lift to a specialization of z'. By Lemma 3.8.19, $\mathcal{X}_{T'} \to T'$ is not closed.

If $T \to Y$ is a finite type morphism, the base change $\mathcal{X}_T \to T$ is a finite type morphism of noetherian algebraic stacks which also satisfies the valuative criterion for universal closedness. It therefore suffices to show that $f\colon \mathcal{X} \to Y$ is closed. By Lemma 3.8.19, we need to show that given a point $x \in |\mathcal{X}|$, every specialization $f(x) \rightsquigarrow y_0$ lifts to a specialization $x \rightsquigarrow x_0$. By Proposition 3.8.20, there exists a diagram

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & \mathcal{X} \arrow[d, "f"] & x \arrow[d, mapsto] \\ \operatorname{Spec} R \arrow[r] & \mathcal{Y} & f(x) \arrow[r, rightsquigarrow] & y_0. \end{tikzcd} \tag{3.8.21}$$

The valuative criterion implies that after replacing R with an extension there exists a lift $\operatorname{Spec} R \to \mathcal{X}$, which in turn yields a specialization $x \rightsquigarrow x_0$ lifting $f(x) \rightsquigarrow y_0$.

Conversely, assume that $f\colon \mathcal{X} \to \mathcal{Y}$ is universally closed and that we are given a diagram as in (3.8.21). By replacing $\mathcal{Y}$ with $\operatorname{Spec} R$ and $\mathcal{X}$ with $\mathcal{X} \times_{\mathcal{Y}} \operatorname{Spec} R$, we may assume that $\mathcal{Y} = \operatorname{Spec} R$ and that we have a diagram

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[dr] & \mathcal{X} \arrow[d, "f"] \\ & \operatorname{Spec} R. \end{tikzcd}$$

By replacing $\mathcal{X}$ with $\overline{\{x\}}$, we may assume that $\mathcal{X}$ is integral with generic point x. Since $\mathcal{X} \to \operatorname{Spec} R$ is closed, there exists a specialization $x \rightsquigarrow x_0$ mapping to the specialization of the generic point to the closed point in $\operatorname{Spec} R$. Since f is quasi-separated, $\operatorname{Spec} K \to \mathcal{X}$ is quasi-compact. Applying Proposition 3.8.20 to

$\operatorname{Spec} K \to \mathcal{X}$ yields a DVR R' with fraction field K' and a commutative diagram

$$\begin{tikzcd} \operatorname{Spec} K' \arrow[r] \arrow[d] & \operatorname{Spec} K \arrow[d] \arrow[dr] & \\ \operatorname{Spec} R' \arrow[r] & \mathcal{X} \arrow[r, "f"] & \operatorname{Spec} R \end{tikzcd}$$

such that $\operatorname{Spec} R' \to \mathcal{X}$ realizes the specialization $x \rightsquigarrow x_0$. As $\operatorname{Spec} R' \to \operatorname{Spec} R$ is surjective,$R \to R'$ is an extension of DVRs and $\operatorname{Spec} R' \to \mathcal{X}$ provides a lift of (3.8.21).

The criterion for separated diagonal is left for the reader (Exercise 3.8.22). Assuming that f has separated diagonal, the valuative criterion for the separatedness of $f\colon \mathcal{X} \to \mathcal{Y}$ translates to the valuative criterion for the universal closedness of the diagonal $\Delta_f\colon \mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$. Thus, the valuative criteria for properness and separatedness follow from the valuative criterion for universal closedness. See also [LMB00, Thm. 7.10], [Fal03, §4], and [SP, Tags 0CLQ, 0CLS, 0CLV, and 0CLY]. $\square$

Exercise 3.8.22. Verify the valuative criterion for separated diagonal.

Try to use a similar argument as for the Infinitesimal Lifting Criterion for Unramified Diagonal (Exercise 3.7.5), using instead that Δ_f is separated if and only if Δ_{Δ_f} is a closed immersion (Exercise 3.3.38), and noting that since Δ_{Δ_f} is a quasi-compact monomorphism representable by schemes, the Valuative Criterion (A.4.5) for schemes applies to verify that it is a closed immersion.

3.9 Further examples

The extreme possibilities are the most illuminating.

David Mumford

This section provides examples of algebraic spaces (§3.9.1), Deligne–Mumford stacks (§3.9.2), and algebraic stacks (§3.9.3), as well as some extremely non-separated examples (§3.9.4).

3.9.1 Examples of algebraic spaces

Example 3.9.1. As discussed in Example 0.5.7, there exists a smooth proper complex 3-fold U with a free action of $\mathbb{Z}/2 = \{\pm 1\}$ such that there is an orbit $\{u, -1 \cdot u\}$ not contained in an affine open subscheme. The quotient sheaf $X = U/(\mathbb{Z}/2)$ is an algebraic space (Corollary 3.1.14), but is not a scheme. If X were a scheme, there would be an affine open around the image of u and the preimage of this open under the finite morphism $U \to X$ would be an affine open containing the orbit.

Example 3.9.2 (Bug-eyed cover)**.** Consider the étale equivalence relation $(p_2, \sigma)\colon R = (\mathbb{Z}/2 \times \mathbb{A}^1) \smallsetminus \{(-1, 0)\} \rightrightarrows \mathbb{A}^1$ defined in Example 3.4.4 obtained by restricting the $\mathbb{Z}/2$-action on $\mathbb{A}^1$ where $(-1) \cdot x = -x$. Let $X = \mathbb{A}^1/R$ be the algebraic space quotient.

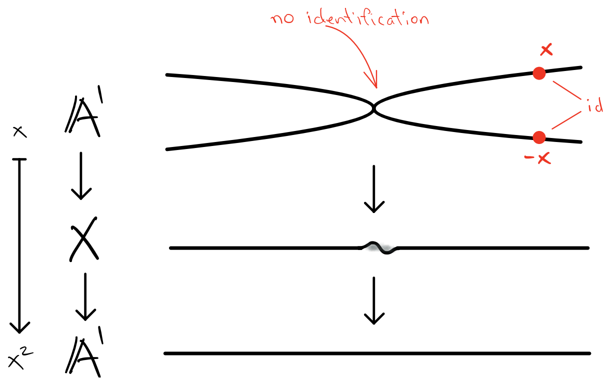


Figure 3.9.3: $X \to \mathbb{A}^1$ is a bug-eyed cover.

We claim that the algebraic space quotient $X = \mathbb{A}^1/R$ is not a scheme. To achieve this, we will show that the diagonal $X \to X \times X$ is not a locally closed immersion. This follows from the cartesian diagram

$$\begin{tikzcd}
(\mathbb{A}^1 \smallsetminus 0) \amalg \{0\} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & R \arrow[r] \arrow[d, "(p_2,\sigma)"] \arrow[dr, phantom, "\square"] & X \arrow[d, "\Delta"] \\
\mathbb{A}^1 \arrow[r, "t \mapsto (t,-t)"] & \mathbb{A}^1 \times \mathbb{A}^1 \arrow[r] & X \times X
\end{tikzcd}$$

as the vertical arrow on the left is transparently not a locally closed immersion.

The algebraic space X is smooth of dimension one and looks eerily similar to the affine line. This example first appeared in [Art74a, p. 311]. The terminology of a bug-eyed cover was coined in [Kol92, §4.2.2], where more general covers were considered: if $Y \to Z$ is a generically étale morphism of schemes with étale locus $Y^{\text{ét}}$, then $Z^{\text{bug}} = Y/R \to Z$ is a *bug-eyed cover*, where $R = Y \amalg (Y^{\text{ét}} \times_Z Y^{\text{ét}} \smallsetminus \Delta^{\text{ét}})$, where $\Delta^{\text{ét}}$ is the diagonal of $Y^{\text{ét}} \to Z$.

Exercise 3.9.4 (Alternative descriptions of the bug-eyed cover)**.** Let X be the bug-eyed cover from the previous example.

(a) Show that X is isomorphic to the quotient $U/(\mathbb{Z}/2)$ of the free $\mathbb{Z}/2$-action on the non-separated affine line $U = \mathbb{A}^1 \bigcup_{\mathbb{A}^1 \smallsetminus 0} \mathbb{A}^1$ over $\Bbbk$ by swapping the origins and by $(-1) \cdot x = -x$ for $x \neq 0$.

(b) Show that there is a universal homeomorphism $X \to \mathbb{A}^1$ which is an isomorphism away from the origin but ramified over the origin.

(c) Show that every map to a *scheme* $X \to Z$ factors through $X \to \mathbb{A}^1$. (In other words, $\mathbb{A}^1$ is the categorical quotient of U by $\mathbb{Z}/2$ in the category of *schemes* but not in the category of algebraic spaces.)

(d) (hard) Consider the SL_2 action on $V_d = \mathrm{Sym}^d \Bbbk^2$, the space of homogeneous polynomials in x and y of degree d. Let $W \subseteq V_1 \times V_4$ be the reduced, locally closed subscheme defined as the set of pairs (L, F) such that $L \neq 0$ and F is the square of a homogeneous quadratic form of discriminant 1. Show that the induced SL_2-action on W is free and that $X \cong W/\mathrm{SL}_2$.

While the descriptions of X as $\mathbb{A}^1/R$ and $U/(\mathbb{Z}/2)$ may seem pathological, this exercise shows that X also arises as a quotient of a quasi-affine variety by SL_2*. See also [GIT, Ex. 0.4] and [Kol97, Ex. 2.18].*

(e) (hard) Let $\pi\colon \mathcal{Y} \to \mathbb{A}^1$ be a proper flat family of surfaces which is smooth over $\mathbb{A}^1 \smallsetminus 0$. Assume also that the central fiber $\mathcal{Y}_0$ has an isolated nodal singularity at a point p given locally by $z^2 + xy = t$, where t is a local coordinate of $\mathbb{A}^1$. Show that the functor

$$\begin{aligned} F\colon (\mathrm{Sch}/\mathbb{A}^1) &\to \mathrm{Sets} \\ (T \to \mathbb{A}^1) &\mapsto \{\text{simultaneous resolutions } \widetilde{\mathcal{Y}} \to \mathcal{Y} \times_{\mathbb{A}^1} T\}, \end{aligned}$$

where a *simultaneous resolution* is a proper morphism which is a minimal resolution on every geometric fiber, is representable by the bug-eyed cover X. See also [Art74a, p. 331].

Exercise 3.9.5. Let $V = \operatorname{Spec} \Bbbk[x,y]/(xy)$ and $U = V \bigcup_{V \smallsetminus 0} V$ be the non-separated union. Let $\mathbb{Z}/2$ act on U exchanging the branches and swapping the origins, and let $X = U/(\mathbb{Z}/2)$ be the quotient algebraic space.

Figure 3.9.6: The affine line with a nodal singularity at the origin.

(a) Show that the diagonal of X is not a locally closed immersion, and conclude that X is not a scheme.

(b) Show that X has dimension one with a singularity at the origin, and that there is a universal homeomorphism $X \to \mathbb{A}^1$ which is an isomorphism away from the origin.

(c) Generalize this example by constructing an algebraic space, which looks like the affine line with an A_k singularity (i.e., $y^2 = x^k + 1$) at the origin.

(d) Construct an affine line with the (non-lci) singularity $\Bbbk[\![x,y,z]\!]/(xy, yz, xz)$ (three coordinate axis in $\mathbb{A}^3$) at the origin.

Exercise 3.9.7. Let $\mathbb{Z}/2 = \{\pm 1\}$ act on $\mathbb{A}^1_{\mathbb{C}}$ via conjugation over $\operatorname{Spec} \mathbb{R}$. Let $R = (\mathbb{Z}/2 \times \mathbb{A}^1_{\mathbb{C}}) \smallsetminus \{(-1,0)\} \rightrightarrows U$ be the induced étale equivalence relation. Show that the algebraic space $X = \mathbb{A}^1_{\mathbb{C}}/R$ is not a scheme. Observe also that, unlike the bug-eyed cover and the previous example, the diagonal of X is a locally closed immersion. (The condition that the diagonal be a locally closed immersion is sometimes referred to as *locally separated*, which is a different notion that being *Zariski locally separated*.)

Hint: Identify X with the quotient $U/(\mathbb{Z}/2)$ of the non-separated affine line $U = \mathbb{A}^1_{\mathbb{C}} \bigcup_{\mathbb{A}^1_{\mathbb{C}} \smallsetminus 0} \mathbb{A}^1_{\mathbb{C}}$ by the $\mathbb{Z}/2$-action that acts via conjugation and swaps the origins. If there were an affine neighborhood of the origin, show that the preimage would be an affine neighborhood containing both neighborhoods.

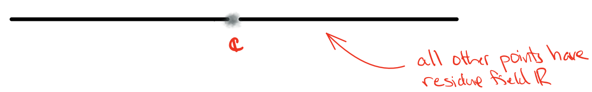


Figure 3.9.8: The quotient $X = \mathbb{A}^1/R$ looks like $\mathbb{A}^1_{\mathbb{R}}$ except that the origin has residue field $\mathbb{C}$.

Remark 3.9.9. Typical constructions in algebraic geometry often only exist in the category of algebraic spaces. The quotient of a free action of a finite group on a scheme exists as a scheme if every orbit is contained in an affine (Exercise 4.2.14), but not in general (Example 3.9.1). Similarly, the pushout $X \amalg_{X_0} Y_0$ of a closed immersion $X_0 \hookrightarrow X$ and finite (or even affine) morphism $X_0 \to Y_0$ always exists as an algebraic space [Art70, Thm. 6.1], but is not always a scheme, unless for instance the preimage of any point in Y_0 in X_0 is contained in an affine (Theorem B.4.1).

While Castelnuovo proved that curves with negative self-intersection in smooth surfaces can be contracted (Theorem B.2.6), contractions in higher dimension do not always exist as scheme. Artin [Art70, Thm. 3.1] (see also [Maz75]) provided conditions for when they exist as algebraic spaces. Likewise, simultaneous resolutions of singularities of families of surfaces (as in Exercise 3.9.4(e)) do not always exist as schemes, but do exist as algebraic spaces [Art74a, Thm. 1].

3.9.2 Examples of Deligne–Mumford stacks

Example 3.9.10 (Classifying stacks)**.** If G is an finite abstract group scheme over a field $\Bbbk$, then the *classifying stack BG of G* is the stack defined as the category of pairs (T, P) where T is a scheme and $P \to T$ is a G-torsor (Definition 2.4.15). Then BG is a smooth and proper Deligne–Mumford stack over $\Bbbk$ of dimension 0.

Example 3.9.11. Suppose $\operatorname{char}(\Bbbk) \neq 2$. Let $\mathbb{Z}/2$ act on $\mathbb{A}^2$ via $-1 \cdot (x, y) = (-x, -y)$. Show that $[\mathbb{A}^2/(\mathbb{Z}/2)]$ is a smooth Deligne–Mumford stack over a field $\Bbbk$ and that there is a proper and bijective morphism $[\mathbb{A}^2/(\mathbb{Z}/2)] \to Y$ where Y is the singular variety $\operatorname{Spec} \Bbbk[x^2, xy, y^2]$ defined by the $\mathbb{Z}/2$-invariants of $\Gamma(\mathbb{A}^2, \mathcal{O}_{\mathbb{A}^2})$.

Example 3.9.12 (Quasi-finite but non-finite inertia)**.** Let $G = \mathbb{A}^1 \amalg (\mathbb{A}^1 \smallsetminus 0) \to \mathbb{A}^1$ be the group scheme where every fiber is $\mathbb{Z}/2$ except over the origin where it is trivial. Then $BG \to \mathbb{A}^1$ is a Deligne–Mumford stack with quasi-finite but not finite diagonal, and hence is not separated. Similarly, the Deligne–Mumford locus of the moduli stack $[\operatorname{Sym}^4 \mathbb{P}^1/\operatorname{PGL}_2]$ of four unordered points in $\mathbb{P}^1$ is not separated (see Example 4.4.17).

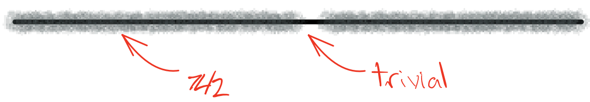


Figure 3.9.13: The stabilizer group jumps down over the origin.

Example 3.9.14 (Weighted projective stacks)**.** For a tuple of positive integers $(d_0, \ldots, d_n)$, let $\mathbb{G}_m$ act on $\mathbb{A}^{n+1}$ via $t \cdot (x_0, \ldots, x_n) = (t^{d_0} x_0, \ldots, t^{d_n} x_n)$. We define the *weighted projective stack* as

$$\mathcal{P}(d_0, \ldots, d_n) = [(\mathbb{A}^{n+1} \smallsetminus 0)/\mathbb{G}_m].$$

If the d_i are all 1, then we recover projective space $\mathbb{P}^n$; otherwise, $\mathcal{P}(d_0, \ldots, d_n)$ is not an algebraic space. For example, over $\mathbb{Z}[1/6]$ the stack of stable elliptic curves $\overline{\mathcal{M}}_{1,1}$ is isomorphic to $\mathcal{P}(4, 6)$ by Exercise 3.1.19(c).

More generally, if R is a finitely generated positively graded $\Bbbk$-algebra, we can define *stacky proj* as $\mathcal{P}\mathrm{roj}\, R = [(\operatorname{Spec}(R) \smallsetminus 0)/\mathbb{G}_m]$, where $\mathbb{G}_m$ acts such that the weight of x_i is the same as its degree.

Exercise 3.9.15.

(a) If $\Bbbk$ is a field of characteristic p, show that $\mathcal{P}(d_0, \ldots, d_n)$ (or more precisely $\mathcal{P}(d_0, \ldots, d_n)) \times_{\mathbb{Z}} \Bbbk)$ is a Deligne–Mumford stack over $\Bbbk$ if and only if p doesn't divide each d_i.

(b) Classify all the points of $\mathcal{P}(3, 3, 4, 6)$ that have non-trivial stabilizers.

(c) We say that an algebraic stack $\mathcal{X}$ has *generically trivial stabilizer* if there exists a dense open substack $U \subset \mathcal{X}$ which is an algebraic space. Provide conditions for when $\mathcal{P}(d_0, \ldots, d_n)$ has generically trivial stabilizer.

(d) Show that there is a bijective morphism $\mathcal{P}(d_0, \ldots, d_n)$ to weighted projective space $\operatorname{Proj} \Bbbk[x_0, \ldots, x_n]$, where x_i has degree d_i. (This is an example of a coarse moduli space.)

Example 3.9.16 (Stacky curves)**.** A *stacky curve* is a one dimensional Deligne—Mumford stack of finite type over a field $\Bbbk$. For positive integers m and n, the weighted projective stack $\mathcal{P}(m, n)$ is an example of a smooth and proper stacky curve as long as the characteristic is relatively prime to m and n.

Exercise 3.9.17.

(a) Show that a smooth stacky curve has abelian stabilizers.

(b) Generalize this to nodal stacky curves.

Example 3.9.18 (Football curves)**.** For integers m and n, the *football curve* $\mathcal{F}(m, n)$ is the proper stacky curve obtained by gluing $[\mathbb{A}^1/\mu_m]$ and $[\mathbb{A}^1/\mu_n]$ along $[(\mathbb{A}^1 \smallsetminus 0)/\mu_m] \cong \mathbb{A}^1 \smallsetminus 0 \cong [(\mathbb{A}^1 \smallsetminus 0)/\mu_n]$. The topological space $|\mathcal{F}(m, n)|$ is identified with $|\mathbb{P}^1|$, and $\mathcal{F}(m, n) \to \mathbb{P}^1$ is an example of a coarse moduli space.

Exercise 3.9.19. Show that $\mathcal{F}(m, n) \cong \mathcal{P}(m, n)$ if and only if m and n are relatively prime.

Remark 3.9.20 (Uniformization of stacky curves)*.* In [BN06], it is shown that every smooth and separated stacky curve over $\mathbb{C}$ has a universal cover which is a simply connected, smooth, and separated stacky curve with generically trivial stabilizers and is isomorphic to $\mathbb{H}$, $\mathbb{C}$, or $\mathcal{P}(m, n)$ for $\gcd(m, n) = 1$.

Root gerbes and root stacks are important examples that were first introduced in [Cad07, §2].

Example 3.9.21 (Root gerbes)**.** Let X be a scheme and L be a line bundle. This data determines a morphism $[L] \colon X \to B\mathbb{G}_m$. For a positive integer r, let $r \colon B\mathbb{G}_m \to B\mathbb{G}_m$ be the morphism induced from the rth power map $r \colon \mathbb{G}_m \to \mathbb{G}_m$, where $t \mapsto t^r$; alternatively $r \colon B\mathbb{G}_m \to B\mathbb{G}_m$ is defined functorially on objects by the assignment $M \mapsto M^{\otimes r}$ on line bundles. Define the *rth root gerbe* $X(\sqrt[r]{L})$ *of X and L* (sometimes denoted as $\sqrt[r]{L/X}$) as the fiber product

$$\begin{tikzcd}
X(\sqrt[r]{L}) \arrow[r] \arrow[d, "\pi"] \arrow[dr, phantom, "\square"] & B\mathbb{G}_m \arrow[d, "r"] \\
X \arrow[r, "{[L]}"] & B\mathbb{G}_m.
\end{tikzcd}$$

Gerbes are discussed later in §6.4.3. The map $X(\sqrt[r]{L}) \to X$ is an example of a *banded $\boldsymbol{\mu}_r$-gerbe* (Exercise 6.4.25) and, in particular, is a coarse moduli space.

Example 3.9.22 (Root stacks)**.** Let X be a scheme, L be a line bundle, and $s \in \Gamma(X, L)$ be a section. This data determines a morphism $[L, s] \colon X \to [\mathbb{A}^1/\mathbb{G}_m]$

(see Example 3.9.25 below). Let $r\colon [\mathbb{A}^1/\mathbb{G}_m] \to [\mathbb{A}^1/\mathbb{G}_m]$ be the morphism induced from the rth power map $r\colon \mathbb{A}^1 \to \mathbb{A}^1$, given by $x \mapsto x^r$, which is equivariant under the rth power map $r\colon \mathbb{G}_m \to \mathbb{G}_m$; alternatively $r\colon [\mathbb{A}^1/\mathbb{G}_m] \to [\mathbb{A}^1/\mathbb{G}_m]$ is defined functorially by $(M,t) \mapsto (M^{\otimes r}, t^{\otimes r})$ on line bundles and sections. For a positive integer r, define the *rth root stack* $X(\sqrt[r]{L,s})$ *of X and L along s* (sometimes denoted as $\sqrt[r]{(L,s)/X}$) as the fiber product

$$\begin{tikzcd}
X(\sqrt[r]{L,s}) \arrow[r] \arrow[d, "\pi"] \arrow[dr, phantom, "\square"] & {[\mathbb{A}^1/\mathbb{G}_m]} \arrow[d, "r"] \\
X \arrow[r, "{[L,s]}"] & {[\mathbb{A}^1/\mathbb{G}_m]}.
\end{tikzcd}$$

The structure map $\pi\colon X(\sqrt[r]{L,s}) \to X$ is an example of a coarse moduli space.

Caution: if $s = 0$ is the zero section, then $X(\sqrt[r]{L,0})$ is not isomorphic to the root gerbe $X(\sqrt[r]{L})$, even though they have the same reduced structures and same coarse moduli space.

Exercise 3.9.23. Let S be a scheme and r be an integer invertible in $\Gamma(S, \mathcal{O}_S)$. (This hypothesis ensures that $\boldsymbol{\mu}_{r,S} \to S$ is an étale group scheme; it will be removed in Exercise 6.4.25.)

(a) Show that $X(\sqrt[r]{L})$ and $X(\sqrt[r]{L,s})$ are Deligne–Mumford stacks.

(b) Show that $X(\sqrt[r]{L})$ has the equivalent description as the category of tuples $(f\colon T \to X, M, \alpha)$ where $f\colon T \to X$ is a morphism from a scheme, M is a line bundle on T and $\alpha\colon M^{\otimes r} \xrightarrow{\sim} f^*L$ is an isomorphism. In particular, there is a line bundle $L^{1/r}$ on $X(\sqrt[r]{L})$ and an isomorphism $(L^{1/r})^{\otimes r} \xrightarrow{\sim} \pi^*L$.

(c) Show that $X(\sqrt[r]{L,s})$ has the equivalent description as the category of triples $(T \xrightarrow{f} X, M, \alpha, t)$ where $f\colon T \to X$ is a morphism from a scheme, M is a line bundle on T, $\alpha\colon M^{\otimes r} \to f^*L$ is an isomorphism, and $t \in \Gamma(T, M)$ is a section such that $\alpha(t^{\otimes r}) = f^*s$. In particular, there is a line bundle $L^{1/r}$ on $X(\sqrt[r]{L,s})$ with a section $s^{1/r}$ together with an isomorphism $(L^{1/r})^{\otimes r} \xrightarrow{\sim} \pi^*L$ identifying $(s^{1/r})^{\otimes r}$ with π^*s.

(d) If $X = \operatorname{Spec} A$ is an affine scheme over S and $L = \mathcal{O}_X$ is trivial, show that

$$X(\sqrt[r]{L}) \cong [X/\boldsymbol{\mu}_r] \quad \text{and} \quad X(\sqrt[r]{L,s}) \cong [\operatorname{Spec}\big(A[x]/(x^r - s)\big)/\boldsymbol{\mu}_r]$$

where $\boldsymbol{\mu}_r$ acts trivially on X and acts on $\operatorname{Spec}\big(A[x]/(x^r - s)\big)$ via $t \cdot x = tx$.

(e) Show that the fiber of $X(\sqrt[r]{L}) \to X$ at a point $x \in X$ is isomorphic to $B\boldsymbol{\mu}_{r,\kappa(x)}$.

(f) Show that $X(\sqrt[r]{L,s}) \to X$ is an isomorphism over $X_s = \{s \neq 0\}$ and that it restricts to an infinitesimal extension of the root gerbe $V(s)(\sqrt[r]{L|_{V(s)}})$ over $V(s)$.

3.9.3 Examples of algebraic stacks

Example 3.9.24. The classifying stack $B\mathrm{GL}_n$ over $\operatorname{Spec}\mathbb{Z}$ classifies vector bundles of rank n. When $n = 1$, $B\mathbb{G}_m = B\mathrm{GL}_1$ classifies line bundles. The stack $B\mathrm{GL}_n$ is a universally closed and smooth algebraic stack over $\operatorname{Spec}\mathbb{Z}$ of relative dimension $-n^2$ with affine diagonal. However, $B\mathrm{GL}_n$ is not separated nor Deligne–Mumford. The structure morphism $B\mathrm{GL}_n \to \operatorname{Spec}\mathbb{Z}$ is an example of a coarse moduli space and trivial GL_n-banded gerbe.

Example 3.9.25. If $\mathbb{G}_m$ acts on $\mathbb{A}^1$ over $\operatorname{Spec} \mathbb{Z}$ via scaling, the objects of the quotient stack $[\mathbb{A}^1/\mathbb{G}_m]$ over a scheme T are pairs (L, s), where L is a line bundle on T and $s \in \Gamma(T, L)$. The stack $[\mathbb{A}^1/\mathbb{G}_m]$ is an algebraic stack universally closed and smooth over $\operatorname{Spec} \mathbb{Z}$ of relative dimension 0 with affine diagonal. The stack $[\mathbb{A}^1/\mathbb{G}_m]$ is, however, *not* separated nor Deligne–Mumford. Over a field $\Bbbk$, $[\mathbb{A}^1/\mathbb{G}_m]$ has two points—one open and one closed—corresponding to the two $\mathbb{G}_m$-orbits (see Figure 0.4.7). There is an open immersion and closed immersion

$$1\colon \operatorname{Spec} \Bbbk \xhookrightarrow{\text{op}} [\mathbb{A}^1/\mathbb{G}_m] \xhookleftarrow{\text{cl}} B\mathbb{G}_m : 0$$

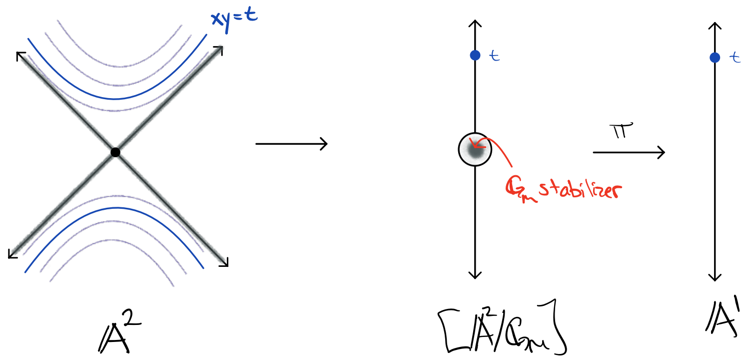
The morphism $[\mathbb{A}^1/\mathbb{G}_m] \to \operatorname{Spec} \Bbbk$ identifies the two orbits and is an example of a good moduli space.

Exercise 3.9.26. Show that $[\mathbb{P}^1/\mathbb{G}_m] \cong [(\mathbb{A}^1 \bigcup_{\mathbb{A}^1 \smallsetminus 0} \mathbb{A}^1)/\mathbb{G}_m]$.

Example 3.9.27. Analogous to $[\mathbb{A}^1/\mathbb{G}_m]$, the quotient stack $[\mathbb{P}^1/\mathbb{G}_a]$ of the action $t \cdot [x : y] = [x + ty : y]$ has two points with one open and one closed

$$0\colon \operatorname{Spec} \Bbbk \xhookrightarrow{\text{op}} [\mathbb{P}^1/\mathbb{G}_a] \xhookleftarrow{\text{cl}} B\mathbb{G}_a : \infty.$$

Example 3.9.28. Working over a field $\Bbbk$, let $\mathbb{G}_m$ act on $\mathbb{A}^2$ via $t \cdot (x, y) = (tx, t^{-1}y)$. The quotient stack $[\mathbb{A}^2/\mathbb{G}_m]$ is a smooth algebraic stack of dimension 1. An object of $[\mathbb{A}^2/\mathbb{G}_m]$ over a scheme T is a triple (L, s, t) where L is a line bundle on T, $s \in \Gamma(T, L)$ and $t \in \Gamma(T, L^{-1})$. The complement $[\mathbb{A}^2/\mathbb{G}_m] \smallsetminus 0$ of the origin is isomorphic to the non-separated affine line. There is a morphism $\pi\colon [\mathbb{A}^2/\mathbb{G}_m] \to \mathbb{A}^1$ defined by $(x, y) \mapsto xy$, which is an isomorphism over $\mathbb{A}^1 \smallsetminus 0$ and identifies the three orbits defined by $xy = 0$; this is an example of a good moduli space.



Example 3.9.30 (Toric stacks)**.** A fan Σ on a lattice $L = \mathbb{Z}^n$ defines a toric variety $X(\Sigma)$, i.e., a normal separated variety with an action of $\mathbb{G}_m^n$ such that there is a dense orbit with trivial stabilizer; see [Ful93]. Meanwhile, a *stacky fan* is a pair (Σ, β) where Σ is a fan on a lattice L and $\beta\colon L \to N$ is a homomorphism of lattices. As L and N are lattices (i.e., finitely generated free abelian groups), the $\mathbb{Z}$-linear duals define tori $T_L := D(L^\vee)$ and $T_N := D(N^\vee)$ (Example B.1.6) where T_L is a torus for the toric variety $X(\Sigma)$. The map β induces a homomorphism $T_\beta\colon T_L \to T_N$, naturally identifying β with the induced map on lattices of 1-parameter subgroups. We can then define $G_\beta = \ker(T_\beta)$ and the *toric stack*

$$X(\Sigma, \beta) := [X(\Sigma)/G_\beta].$$

Similar to the role of toric varieties, toric stacks provide a rich source of combinatorial examples that can serve as a testing ground for general conjectures. See [FMN10] and [GS17].

Example 3.9.31 (Picard schemes and stacks)**.** Let X be a proper integral scheme over an algebraically closed field $\Bbbk$, the *Picard functor of* X and *Picard stack of* X are defined as the sheaf $\underline{\mathrm{Pic}}(X)$ and stack $\underline{\mathcal{P}\mathrm{ic}}\,(X)$ on $\mathrm{Sch}_{\text{ét}}$ by

$$\begin{aligned}\underline{\mathrm{Pic}}(X) &= \text{sheafification of } T \mapsto \mathrm{Pic}(X_T)\\ \underline{\mathcal{P}\mathrm{ic}}\,(X)(T) &= \{\text{groupoid of line bundles } L \text{ on } X_T\}\end{aligned}$$

A morphism $(T, L) \to (T', L')$ in $\underline{\mathcal{P}\mathrm{ic}}\,(X)$ is the data of a morphism $f\colon T \to T'$ of schemes and an isomorphism $\alpha\colon L \to f^*L'$. The sheaf $\underline{\mathrm{Pic}}(X)$ is representable by a projective scheme and the tensor product of line bundles provides it with the structure of a group scheme, hence an abelian variety. Moreover, $\underline{\mathcal{P}\mathrm{ic}}\,(X)$ is a smooth algebraic stack over $\Bbbk$. The map $\underline{\mathcal{P}\mathrm{ic}}\,(X) \to \underline{\mathrm{Pic}}(X)$ forgetting automorphism groups is an example of a banded $\mathbb{G}_m$-gerbe. See §6.4.7.

Example 3.9.32 (Vector bundle stacks)**.** Let $K^\bullet\colon K^0 \xrightarrow{d} K^1$ be a two-term complex of vector bundles on a scheme X. The *vector bundle stack* $\mathfrak{C}(K^\bullet)$, sometimes called a *cone stack* and written as $[K^1/K^0]$ or $[h^0/h^1]$, is defined as the quotient stack over $(\mathrm{Sch}/X)^{\text{ét}}$

$$\mathfrak{C}(K^\bullet) = [\mathbb{A}(K^{1,\vee})/\mathbb{A}(K^{0,\vee})],$$

where $\mathbb{A}(K^{0,\vee})$ is viewed as the additive group scheme and the action $\sigma\colon \mathbb{A}(K^{0,\vee}) \times \mathbb{A}(K^{1,\vee}) \to \mathbb{A}(K^{1,\vee})$ is defined by $\sigma(x_0, x_1) = x_1 + d(x_0)$. When $X = \operatorname{Spec}\Bbbk$ with $\Bbbk$ a field, then $\mathfrak{C}(K^\bullet)(\Bbbk)$ is the groupoid whose objects are $\mathbb{A}(K^{1,\vee})(\Bbbk) = K^1$ and where a morphism $v \to v'$ is an element $\alpha \in \mathbb{A}(K^{0,\vee})(\Bbbk) = K^0$ such that $d(\alpha) = v - v'$. Moreover, $\mathfrak{C}(K^\bullet)$ is identified with the quotient $[\mathbb{A}(\mathrm{H}^1(K^\bullet)^\vee)/\mathbb{A}(\mathrm{H}^0(K^\bullet)^\vee)]$ of the *trivial action*, and in particular the groupoid $\mathfrak{C}(K^\bullet)(\Bbbk)$ is equivalent to $[\mathrm{H}^1(K^\bullet)/\mathrm{H}^0(K^\bullet)]$.

For general X, the fiber of $\mathfrak{C}(K^\bullet) \to X$ over a geometric point $\operatorname{Spec}\Bbbk \to X$ is $\mathfrak{C}(K^\bullet \otimes_{\mathcal{O}_X} \Bbbk) \cong [\mathbb{A}(\mathrm{H}^1(K^\bullet \otimes_{\mathcal{O}_X} \Bbbk)^\vee)/\mathbb{A}(\mathrm{H}^0(K^\bullet \otimes_{\mathcal{O}_X} \Bbbk)^\vee)]$. If $g\colon T \to X$ is a morphism from an affine scheme, then since isomorphism classes of principal $\mathbb{A}(K^{0,\vee})$-bundles over T are in bijection with $\mathrm{H}^1(T, K^{0,\vee}) = 0$, every principal bundle is trivial. Hence,

$$\mathfrak{C}(K^\bullet)(T) = [\mathbb{A}(K^{1,\vee})(T)/\mathbb{A}(K^{0,\vee})(T)] \cong [\mathrm{H}^0(T, g^*K^1)/\mathrm{H}^0(T, g^*K^0)],$$

where we have used the bijections

$$\begin{aligned}\mathbb{A}(K^{i,\vee})(T \xrightarrow{g} S) &= \mathrm{Mor}_S(T, \mathbb{A}(K^{i,\vee}))\\ &= \mathrm{Hom}_{\mathcal{O}_S-\mathrm{alg}}(\mathrm{Sym}^* K^{i,\vee}, g_*\mathcal{O}_T)\\ &= \mathrm{Hom}_{\mathcal{O}_S}(K^{i,\vee}, g_*\mathcal{O}_T)\\ &= \mathrm{H}^0(S, K^i \otimes g_*\mathcal{O}_T)\\ &= \mathrm{H}^0(S, g_*g^*K^i) && \text{(projection formula)}\\ &= \mathrm{H}^0(T, g^*K^i)\end{aligned} \tag{3.9.33}$$

By construction, we have that $\mathfrak{C}(K^\bullet) \to X$ is a smooth morphism of relative dimension $\mathrm{h}^1(K^\bullet \otimes_{\mathcal{O}_X} \Bbbk) - \mathrm{h}^0(K^\bullet \otimes_{\mathcal{O}_X} \Bbbk)$ for any $\Bbbk$-point $\operatorname{Spec}\Bbbk \to X$.

Exercise 3.9.34. Show that a quasi-isomorphism $K^\bullet \to L^\bullet$ of two-term complexes on a scheme X induces an isomorphism $\mathfrak{C}(K^\bullet) \xrightarrow{\sim} \mathfrak{C}(L^\bullet)$ of vector bundle stacks.

The exercise above shows that the vector bundle stack $\mathfrak{C}(K^\bullet)$ is well-defined for a complex $K^\bullet \in D^b(X)$ with amplitude in $[0,1]$. We will use this construction in Proposition 8.2.11 to describe the stack of extensions $\mathfrak{C}(Rp_{2,*}\mathscr{H}om_{\mathcal{O}_{C\times S}}(F_1,F_2)) \to S$ of vector bundles F_1,F_2 on $C\times S$, where C is a smooth curve. For a syntomic morphism $X\to Y$, $L_{X/Y}$ is perfect complex with amplitude in $[-1,0]$, one can consider the vector bundle stack $\mathfrak{C}(L_{X/Y}^\vee)$, which plays a role in defining the *intrinsic normal cone* in [BF97].

3.9.4 Extremely non-separated examples

The following are often counterexamples for properties that hold for algebraic stacks with separated and quasi-compact diagonal, but fail in general. For instance, if the diagonal of a Deligne–Mumford stack is quasi-compact and separated diagonal, then it is quasi-affine (Corollary 4.5.8) and in particular representable by schemes, and the examples below illustrate that both the quasi-compactness and separated hypotheses are necessary.

Example 3.9.35 (Deligne–Mumford stacks with non-quasi-compact diagonals)**.** Let $\underline{\mathbb{Z}}$ denote the constant group scheme over $\operatorname{Spec}\mathbb{Z}$ associated to the abstract discrete group $\mathbb{Z}$. The classifying stack $B\underline{\mathbb{Z}}$ is a smooth algebraic stack of dimension 0 whose diagonal is not quasi-compact. In particular, $B\underline{\mathbb{Z}}$ is not quasi-separated.

Let $\underline{\mathbb{Z}}$ act on $\mathbb{A}^1$ over a characteristic zero field $\Bbbk$ via $n\cdot x = x+n$ for $x\in\mathbb{A}^1$ and $n\in\underline{\mathbb{Z}}$. The quotient $G := \mathbb{A}^1/\underline{\mathbb{Z}}$ is a quasi-compact, locally noetherian group algebraic space with non-quasi-compact diagonal that is not a scheme. Any quasi-separated group algebraic space locally of finite type over a field $\Bbbk$ is a scheme (Theorem 4.5.28), and this example shows that the quasi-separatedness hypothesis is necessary.

The classifying stack BG is a Deligne–Mumford stack whose diagonal Δ_{BG} is quasi-compact but whose second diagonal $\Delta_{\Delta_{BG}}$ is *not* quasi-compact; in particular BG is not quasi-separated. Moreover, the diagonal of BG is not representable by schemes.

Similarly, one can consider the algebraic space quotient $\mathbb{A}^1_{\mathbb{C}}/\underline{\mathbb{Z}}^2$ where $(a,b)\cdot x = x+a+ib$. While the analytic quotient $\mathbb{C}/\mathbb{Z}^2$ of this action is an elliptic curve over $\mathbb{C}$, the algebraic space quotient is not quasi-separated and not a scheme.

Exercise 3.9.36. Let $G=\mathbb{A}^1/\underline{\mathbb{Z}}$.

(a) Show that G is not a scheme.

(b) Show that the generic point $\operatorname{Spec}\Bbbk(x)\to\mathbb{A}^1\to X$ is fixed under the $\underline{\mathbb{Z}}$-action, and that the composition $\operatorname{Spec}\Bbbk(x)\to G$ does not factor through a monomorphism $\operatorname{Spec} L\to G$ for a field L, i.e., the generic point of G does not have a residue field.

Example 3.9.37 (Deligne–Mumford stacks with non-separated diagonal)**.** The non-separated affine line $Q := \mathbb{A}^1\bigcup_{\mathbb{A}^1\smallsetminus 0}\mathbb{A}^1$ is a group scheme over $\mathbb{A}^1$, where every fiber is trivial except over the origin. The classifying stack BQ is a Deligne–Mumford stack whose diagonal is non-separated.

Here is a construction to produce similar examples. Let $G\to S$ be a finite étale group scheme. If $H\subseteq G$ is a subgroup scheme over S, then G/H is separated if and only if $H\subset G$ is closed. If H is normal in G, then $G/H\to S$ is a group algebraic space. For instance, taking $G=\mathbb{Z}/2\times\mathbb{A}^1\to\mathbb{A}^1$ and the subgroup $H=G\smallsetminus\{-1,0\}$, the quotient $Q=G/H$ is the non-separated affine line ver $\mathbb{A}^1$.

Likewise, consider $\mu_{3,\mathbb{Q}} = \operatorname{Spec}\mathbb{Q} \amalg G'$ over $\mathbb{Q}$ where $G' = \operatorname{Spec}\mathbb{Q}[x]/(x^2+x+1)$ is the non-identity component , and let $G = \mu_{3,\mathbb{Q}} \times \mathbb{A}^1_{\mathbb{Q}}$. Then $Q = (\mu_{3,\mathbb{Q}} \times \mathbb{A}^1_{\mathbb{Q}})/H$ is a non-separated étale group algebraic space over $\mathbb{A}^1_{\mathbb{Q}}$ which is not a scheme. It is not a scheme for several reasons to Exercise 3.9.7: $Q \to \mathbb{A}^1$ is a form of the $\mathbb{A}^1_{\mathbb{Q}} \bigcup_{\mathbb{A}^1_{\mathbb{Q}} \smallsetminus 0} \mathbb{A}^1_{\mathbb{Q}} \bigcup_{\mathbb{A}^1_{\mathbb{Q}} \smallsetminus 0} \mathbb{A}^1_{\mathbb{Q}}$ of the group scheme which is trivial except over the origin where it is $\mathbb{Z}/2$. If the non-identity point in Q_0 contained an affine open, its preimage under $\mathbb{A}^1_{\mathbb{Q}[x]/(x^2+x+1)} \to \mathbb{A}^1_{\mathbb{Q}}$ would be an affine open containing two of the origins, a contradiction.

In this case, BQ is Deligne–Mumford stack which is quasi-compact and quasi-separated (i.e., BG, the first diagonal Δ_{BG}, and second diagonal $\Delta_{\Delta_{BG}}$ are quasi-compact), but whose diagonal is not separated and not representable by schemes.

While the vast majority of algebraic stacks that appear in moduli theory have separated diagonal, there are a few exceptions, e.g., the stack $\mathcal{L}\mathrm{og}^{\mathrm{alg}}$ of log structures [Ols03].

Chapter 4

Geometry of Deligne–Mumford stacks

Geometry is the cilantro of math.

Jordan Ellenberg

4.1 Quasi-coherent sheaves and cohomology

4.1.1 Sheaves

The small étale site of a Deligne–Mumford stack can be defined analogously to the small étale site of a scheme (Example 2.2.4).

Definition 4.1.1. The *small étale site* of a Deligne–Mumford stack $\mathcal{X}$ is the category $\mathcal{X}_{\text{ét}}$ of schemes étale over $\mathcal{X}$. A covering of an $\mathcal{X}$-scheme U is a collection of étale morphisms $\{U_i \to U\}$ over $\mathcal{X}$ such that $\coprod_i U_i \to U$ is surjective.

We can therefore discuss sheaves of abelian groups on $\mathcal{X}_{\text{ét}}$ and their morphisms. We denote $\text{Ab}(\mathcal{X}_{\text{ét}})$ as the category of abelian sheaves on $\mathcal{X}_{\text{ét}}$. For an abelian sheaf F on $\mathcal{X}_{\text{ét}}$, the sections over an étale $\mathcal{X}$-scheme U are denoted by $F(U \to \mathcal{X})$.

Example 4.1.2 (Structure sheaf)**.** The structure sheaf $\mathcal{O}_{\mathcal{X}}$ on a Deligne–Mumford stack is defined by $\mathcal{O}_{\mathcal{X}}(U \to \mathcal{X}) = \Gamma(U, \mathcal{O}_U)$ on an étale $\mathcal{X}$-scheme U.

Example 4.1.3 (Differentials)**.** If $\mathcal{X}$ is a Deligne–Mumford stack over a scheme S, the relative sheaf of differentials $\Omega_{\mathcal{X}/S}$ is defined by $\Omega_{\mathcal{X}/S}(U \to \mathcal{X}) = \Gamma(U, \Omega_{U/S})$.

Example 4.1.4 (Hodge bundle)**.** Define the sheaf $\mathcal{H}$ on $\mathcal{M}_g$ (for $g \geq 2$) as follows: for an étale morphism $U \to \mathcal{M}_g$ classified by a family $\mathcal{C} \to U$ of smooth curves, we set $\mathcal{H}(U \to \mathcal{M}_g) = \Gamma(\mathcal{C}, \Omega_{\mathcal{C}/U})$. We will see shortly that $\mathcal{H}$ is a coherent $\mathcal{O}_{\mathcal{M}_g}$-module and, in fact, a vector bundle of rank g.

Exercise 4.1.5 (easy)**.** If $\Bbbk$ is an algebraically closed field, show that the functor $\text{Ab}((\operatorname{Spec} \Bbbk)_{\text{ét}}) \to \text{Ab}$, defined by $F \mapsto F(\operatorname{Spec} \Bbbk)$, is an equivalence of categories.

While sections of a sheaf F on $\mathcal{X}_{\text{ét}}$ are only defined on étale $\mathcal{X}$-schemes, we can extend the definition to a Deligne–Mumford stack $\mathcal{U}$ étale over $\mathcal{X}$ as follows: choose étale presentations $U \to \mathcal{U}$ and $R \to U \times_{\mathcal{U}} U$ by schemes and define

$$F(\mathcal{U} \to \mathcal{X}) := \text{Eq}(F(U \to \mathcal{X}) \rightrightarrows F(R \to \mathcal{X})).$$

One checks that this is independent of the choice of presentation. In particular, it makes sense to discuss global sections

$$\Gamma(\mathcal{X}, F) := F(\mathcal{X} \xrightarrow{\mathrm{id}} \mathcal{X}).$$

Exercise 4.1.6 (easy)**.** If F is an abelian sheaf on a Deligne–Mumford stack $\mathcal{X}$, show that $\Gamma(\mathcal{X}, F) = \mathrm{Hom}_{\mathrm{Ab}(\mathcal{X}_{\text{ét}})}(\underline{\mathbb{Z}}, F)$ where $\underline{\mathbb{Z}}$ is the constant sheaf.

Given a morphism $f\colon \mathcal{X} \to \mathcal{Y}$ of Deligne–Mumford stacks, there are functors

$$\begin{tikzcd} \mathrm{Ab}(\mathcal{X}_{\text{ét}}) \arrow[r, bend left, "f_*"] & \mathrm{Ab}(\mathcal{Y}_{\text{ét}}) \arrow[l, bend left, "f^{-1}"] \end{tikzcd}$$

where the *pushforward* $f_*F(V \to \mathcal{Y}) := F(\mathcal{X} \times_{\mathcal{Y}} V \to \mathcal{X})$ and the *inverse image* $f^{-1}G$ is the sheafification of the presheaf

$$(U \to \mathcal{X}) \mapsto \mathrm{colim}_{(V \to \mathcal{Y}, U \to \mathcal{X} \times_{\mathcal{Y}} V)}\, G(V \to \mathcal{Y}),$$

where the colimit is over pairs of étale morphisms $V \to \mathcal{Y}$ and $U \to \mathcal{X} \times_{\mathcal{Y}} V$, i.e., commutative diagrams

$$\begin{tikzcd} U \arrow[r] \arrow[d, "\text{ét}"] & V \arrow[d, "\text{ét}"] \\ \mathcal{X} \arrow[r] & \mathcal{Y}. \end{tikzcd}$$

Exercise 4.1.7 (easy)**.** Show that if $f\colon \mathcal{X} \to \mathcal{Y}$ is an étale morphism of Deligne–Mumford stacks and G is a sheaf on $\mathcal{Y}_{\text{ét}}$, then $f^{-1}G = G|_{\mathcal{X}}$ is the restriction of G to $\mathcal{X}_{\text{ét}}$, i.e., the sheaf $(U \to \mathcal{X}) \mapsto G(U \to \mathcal{X} \to \mathcal{Y})$. In particular, show that $f^{-1}\mathcal{O}_{\mathcal{Y}} = \mathcal{O}_{\mathcal{X}}$.

Exercise 4.1.8 (easy)**.** Let X be a scheme and $x \in X$.

(a) Denote $i_x\colon \operatorname{Spec}\kappa(x) \to X$, show that the global sections of $i_x^{-1}\mathcal{O}_X$ is the henselization $\mathcal{O}_{X,x}^{\mathrm{h}}$ of $\mathcal{O}_{X,x}$.

(b) Denoting $i_{\overline{x}}\colon \operatorname{Spec}\overline{\kappa(x)} \to X$, show that $i_{\overline{x}}^{-1}$ is identified with the strict henselization $\mathcal{O}_{X,x}^{\mathrm{sh}}$ of $\mathcal{O}_{X,x}$.

See §B.5.3 for background on (strict) henselizations.

Exercise 4.1.9. Let $\Bbbk$ be a field, $\Bbbk^{\mathrm{sep}}$ be a separable closure, and $G := \mathrm{Gal}(\Bbbk^{\mathrm{sep}}/\Bbbk)$ be its Galois group with the profinite topology.

(a) Show that a covariant functor

$$F\colon \{\text{étale } \Bbbk\text{-algebras}\} \to \mathrm{Ab}$$

defines a sheaf of abelian groups on $(\operatorname{Spec}\Bbbk)_{\text{ét}}$ if and only if $F(\prod_i A_i) = \bigoplus F(A_i)$ for every finite set of étale $\Bbbk$-algebras $\{A_i\}$ and $F(K) = F(L)^{\mathrm{Gal}(L/K)}$ for every finite Galois extension L/K of finite separable field extensions of $\Bbbk$.

(b) Show that the category of sheaves on $(\operatorname{Spec}\Bbbk)_{\text{ét}}$ is equivalent to the category of discrete G-modules, i.e., abelian groups with a continuous G-action.

Hint: Given F, define a discrete G-module by $M_F := \mathrm{colim}_{\Bbbk'/\Bbbk} F(\Bbbk')$, where the colimit is over finite separable field extensions. Conversely, given M, show that the functor on finite field extensions $\Bbbk' \mapsto M^{\mathrm{Gal}(\Bbbk^{\mathrm{sep}}/\Bbbk')}$ extends to a sheaf F_M on $(\operatorname{Spec}\Bbbk)_{\text{ét}}$. Show that $M \mapsto M_F$ and $M \mapsto F_M$ are inverse functors.

Exercise 4.1.10. Let $\mathcal{X}$ be a Deligne–Mumford stack and $x\colon \operatorname{Spec} \Bbbk \to \mathcal{X}$ be a geometric point (i.e., $\Bbbk$ is an algebraically closed field). Define the *stalk* of an abelian sheaf F on $\mathcal{X}_{\text{ét}}$ at x as $F_x := x^{-1}F$.

(a) Show that $\text{Ab}(\mathcal{X}_{\text{ét}}) \to \text{Ab}$, defined by $F \mapsto F_x$, is an exact functor.

(b) Show that a sequence of abelian sheaves on $\mathcal{X}_{\text{ét}}$ is exact if and only if the induced sequence of stalks is exact for every geometric point.

(c) Show that f^{-1} is exact for a morphism $f\colon \mathcal{X} \to \mathcal{Y}$ of Deligne–Mumford stacks.

Exercise 4.1.11 (details)**.** Show that f^{-1} is left adjoint to f_*.

Exercise 4.1.12 (technical)**.** If $\mathcal{X}$ is a Deligne–Mumford stack, define instead the site $\mathcal{X}_{\text{ét}'}$ as the category of *algebraic spaces* étale over $\mathcal{X}$ where coverings are étale coverings. Show that the categories of sheaves on $\mathcal{X}_{\text{ét}}$ and $\mathcal{X}_{\text{ét}'}$ are equivalent.

4.1.2 $\mathcal{O}_{\mathcal{X}}$-modules

The structure sheaf $\mathcal{O}_{\mathcal{X}}$ is a ring object in $\text{Ab}(\mathcal{X}_{\text{ét}})$, and $\mathcal{O}_{\mathcal{X}}$-modules are defined as module objects over $\mathcal{O}_{\mathcal{X}}$.

Definition 4.1.13. If $\mathcal{X}$ is a Deligne–Mumford stack, a *sheaf of $\mathcal{O}_{\mathcal{X}}$-modules* (or simply an *$\mathcal{O}_{\mathcal{X}}$-module*) is a sheaf F on $\mathcal{X}_{\text{ét}}$ which is a module object for $\mathcal{O}_{\mathcal{X}}$ in the category of sheaves, i.e., for every étale $\mathcal{X}$-scheme U, $F(U \to \mathcal{X})$ is an module over $\mathcal{O}_{\mathcal{X}}(U \to \mathcal{X}) = \Gamma(U, \mathcal{O}_U)$ and the module structure is compatible with respect to restriction along morphisms $V \to U$ of étale $\mathcal{X}$-schemes.

We denote $\text{Mod}(\mathcal{O}_{\mathcal{X}})$ as the category of $\mathcal{O}_{\mathcal{X}}$-modules. Given two $\mathcal{O}_{\mathcal{X}}$-modules F and G, we can define the *tensor product* $F \otimes G := F \otimes_{\mathcal{O}_{\mathcal{X}}} G$ as the sheafification of the $\mathcal{O}_{\mathcal{X}}$-module given by $(U \to \mathcal{X}) \mapsto F(U \to \mathcal{X}) \otimes_{\mathcal{O}_{\mathcal{X}}(U \to \mathcal{X})} G(U \to \mathcal{X})$. The *Hom sheaf* $\mathscr{H}om_{\mathcal{O}_{\mathcal{X}}}(F, G)$ has sections $\operatorname{Hom}_{\mathcal{O}_U}(F|_U, G|_U)$ over an étale morphism $f\colon U \to \mathcal{X}$ from scheme, where $F|_U = f^{-1}F$ denotes the restriction of F to $U_{\text{ét}}$.

Exercise 4.1.14 (easy)**.** If F is an $\mathcal{O}_{\mathcal{X}}$-module, show that $\Gamma(\mathcal{X}, F) = \operatorname{Hom}_{\mathcal{O}_{\mathcal{X}}}(\mathcal{O}_{\mathcal{X}}, F)$.

Given a morphism $f\colon \mathcal{X} \to \mathcal{Y}$ of Deligne–Mumford stacks, there are functors

$$\text{Mod}(\mathcal{O}_{\mathcal{X}}) \underset{f^*}{\overset{f_*}{\rightleftarrows}} \text{Mod}(\mathcal{O}_{\mathcal{Y}})$$

where for an $\mathcal{O}_{\mathcal{X}}$-module F, f_*F is the pushforward as sheaves and is naturally an $\mathcal{O}_{\mathcal{Y}}$-module. For an $\mathcal{O}_{\mathcal{Y}}$-module G, since there is a morphism $f^{-1}\mathcal{O}_{\mathcal{Y}} \to \mathcal{O}_{\mathcal{X}}$ of sheaves of rings in $\mathcal{X}_{\text{ét}}$ and $f^{-1}G$ is a $f^{-1}\mathcal{O}_{\mathcal{Y}}$-module, it makes sense to define the *pullback* as the $\mathcal{O}_{\mathcal{X}}$-module

$$f^*G := f^{-1}G \otimes_{f^{-1}\mathcal{O}_{\mathcal{Y}}} \mathcal{O}_{\mathcal{X}}.$$

If f is étale, then $f^*G = f^{-1}G = G|_{\mathcal{X}}$ since $f^{-1}\mathcal{O}_{\mathcal{Y}} = \mathcal{O}_{\mathcal{X}}$ (by Exercise 4.1.7)

Exercise 4.1.15 (details)**.** Show that f^* is left adjoint to f_*.

Exercise 4.1.16 (details)**.** Show that $\text{Mod}(\mathcal{O}_{\mathcal{X}})$ is an abelian category.

4.1.3 Quasi-coherent sheaves

Let F be an $\mathcal{O}_{\mathcal{X}}$-module on a Deligne–Mumford stack $\mathcal{X}$. For an étale $\mathcal{X}$-scheme U, we have the restriction $F|_{U_{\text{ét}}}$ to the small étale site $U_{\text{ét}}$ of U and the further restriction $F|_{U_{\text{Zar}}}$ restricted to the Zariski topology of U. Note that when X is a scheme, $\mathcal{O}_X$ could refer to the structure sheaf either in $X_{\text{ét}}$ or X_{Zar}, and we write either $\mathcal{O}_{X_{\text{Zar}}}$ or $\mathcal{O}_{X_{\text{ét}}}$ if there is any possibility of confusion.

Definition 4.1.17. Let $\mathcal{X}$ be a Deligne–Mumford stack. An $\mathcal{O}_{\mathcal{X}}$-module F is *quasi-coherent* if

(1) for every étale $\mathcal{X}$-scheme U, the restriction $F|_{U_{\text{Zar}}}$ is a quasi-coherent $\mathcal{O}_{U_{\text{Zar}}}$-module, and

(2) for every étale morphism $f\colon U \to V$ of étale $\mathcal{X}$-schemes, the natural morphism $f^*(F|_{V_{\text{Zar}}}) \to F|_{U_{\text{Zar}}}$ is an isomorphism.

A quasi-coherent $\mathcal{O}_{\mathcal{X}}$-module F on $\mathcal{X}$ is a *vector bundle* (resp., *vector bundle of rank* r, *line bundle*) if $F|_{U_{\text{Zar}}}$ is for every étale morphism $U \to \mathcal{X}$ from a scheme. If in addition $\mathcal{X}$ is locally noetherian, we say F is *coherent* if $F|_{U_{\text{Zar}}}$ is coherent for every étale morphism $U \to \mathcal{X}$.

We denote by $\operatorname{QCoh}(\mathcal{X})$ and $\operatorname{Coh}(\mathcal{X})$ (in the noetherian setting) the categories of quasi-coherent and coherent sheaves. In a moment, we will see that this agrees with the usual definition when $\mathcal{X}$ is a scheme, but first we give some examples:

Example 4.1.18. The structure sheaf $\mathcal{O}_{\mathcal{X}}$ is quasi-coherent. It is a line bundle, which is coherent when $\mathcal{X}$ is locally noetherian.

Example 4.1.19. For a Deligne–Mumford stack $\mathcal{X}$ over a scheme S, the relative sheaf of differentials $\Omega_{\mathcal{X}/S}$ of Example 4.1.3 is quasi-coherent since for an étale morphisms $f\colon U \to V$ of étale $\mathcal{X}$-schemes, $f^*\Omega_{V/\mathcal{S}} \to \Omega_{U/S}$ is an isomorphism. It is a vector bundle when $\mathcal{X} \to S$ is smooth.

Example 4.1.20. For $\mathcal{M}_g$ with $g \geq 2$, the Hodge bundle $\mathcal{H}$ of Example 4.1.4 is a vector bundle of rank g. This follows from Proposition 5.1.16(2): for a smooth family $\pi\colon \mathcal{C} \to V$ of genus g curves corresponding to a $\mathcal{M}_g$-scheme V, the construction of $\pi_*\Omega_{\mathcal{C}/V}$ commutes with the base change along a map $f\colon U \to V$, i.e., $f^*(\pi_*\Omega_{\mathcal{C}/V}) \xrightarrow{\sim} \pi_{U,*}\Omega_{\mathcal{C}_U/U}$ (which implies that $\mathcal{H}$ is quasi-coherent), and moreover $\pi_*\Omega_{\mathcal{C}/V}$ is a vector bundle on V of rank g (which implies that $\mathcal{H}$ is a vector bundle of rank g).

Example 4.1.21. If G is a finite abstract group viewed as a group scheme over a field $\Bbbk$, a quasi-coherent sheaf on BG corresponds to a representation V of G. If G acts on an affine $\Bbbk$-scheme $\operatorname{Spec} A$, a quasi-coherent sheaf on $[\operatorname{Spec} A/G]$ is the data of an A-module M equipped with a group homomorphism $G \to \operatorname{End}_A(M)$. These descriptions follow from Exercise 4.1.24(b).

Exercise 4.1.22 (Equivalences I)**.** Let X be a scheme.

(a) Show that the following are equivalent for an an $\mathcal{O}_{X_{\text{ét}}}$-module F:

(i) F is quasi-coherent as in Definition 4.1.17.

(ii) There is a quasi-coherent $\mathcal{O}_{X_{\text{Zar}}}$-module F' and an isomorphism $F \cong \alpha^* F'$, where

$$\alpha^*\colon \operatorname{Mod}(\mathcal{O}_{X_{\text{Zar}}}) \to \operatorname{Mod}(\mathcal{O}_{X_{\text{ét}}})$$

is the functor taking an $\mathcal{O}_{X_{Zar}}$-module F' to the $\mathcal{O}_{X_{\text{ét}}}$-module $\alpha^* F'$ defined by $\alpha^* F'(U \xrightarrow{g} X) = \Gamma(U, g^* F')$.

(iii) F is quasi-coherent as a module object in the ringed site $(X_{\text{ét}}, \mathcal{O}_{X_{\text{ét}}})$ using the general definition in a ringed site $(\mathcal{S}, \mathcal{O})$ [SP, Tag 03DL]: an $\mathcal{O}$-module F is *quasi-coherent* if for every object $U \in \mathcal{S}$, there is a covering $\{U_i \to U\}$ such that the restriction $F|_{U_i}$ to the restricted site $\mathcal{S}/U_i$ has a free presentation

$$\mathcal{O}_{U_i}^{\oplus J} \to \mathcal{O}_{U_i}^{\oplus I} \to F|_{U_i} \to 0.$$

Hint: To show that (iii) implies the other conditions, use Fpqc Descent for Quasi-Coherent Sheaves (2.1.4) to construct a quasi-coherent sheaf.

(b) Show that α^* restricts to an equivalence of categories $\mathrm{QCoh}(X_{\mathrm{Zar}}) \to \mathrm{QCoh}(X_{\text{ét}})$.

Hint: Show that restriction $F|_{X_{\mathrm{Zar}}}$ of an $\mathcal{O}_{X_{\text{ét}}}$-module to the Zariski topology defines a left adjoint α_* to α^*. Use adjunction and the identification $\alpha_*\alpha^*F \cong F$ for a quasi-coherent $\mathcal{O}_{X_{\mathrm{Zar}}}$-module F to show that α^* is fully faithful on quasi-coherent sheaves.

Exercise 4.1.23. Provide an example of a scheme X and a non-quasi-coherent $\mathcal{O}_{X_{\text{ét}}}$-module F such that $F|_{X_{\mathrm{Zar}}}$ is quasi-coherent.

This shows why condition (2) is necessary in Definition 4.1.17.

Exercise 4.1.24 (Equivalences II)**.** Let $\mathcal{X}$ be a Deligne–Mumford stack.

(a) Show that the following are equivalent for an $\mathcal{O}_{\mathcal{X}_{\text{ét}}}$-module F:

(i) F is quasi-coherent as in Definition 4.1.17.

(ii) There is an étale presentation $U \to \mathcal{X}$ by a scheme such that $F|_{U_{\text{ét}}}$ is quasi-coherent.

(iii) F is quasi-coherent as a sheaf of modules in the ringed site $(\mathcal{X}_{\text{ét}}, \mathcal{O}_{\mathcal{X}_{\text{ét}}})$.

(b) Let $U \to \mathcal{X}$ be an étale presentation and set $R = U \times_{\mathcal{X}} U$. Show that the category of quasi-coherent sheaves on $\mathcal{X}$ is equivalent to the category of pairs (G, α) where G is a quasi-coherent sheaf on U and $\alpha\colon p_1^*G \xrightarrow{\sim} p_2^*G$ is an isomorphism on $R := U \times_{\mathcal{X}} U$ satisfying the cocycle condition $p_{23}^*\alpha \circ p_{12}^*\alpha = p_{13}^*\alpha$.

(c) Show that a quasi-coherent sheaf on $\mathcal{X}$ is equivalent to the data of a quasi-coherent sheaf F_U on every scheme U over $\mathcal{X}$ and compatible isomorphisms $f^*F_V \to F_U$ for every map $f\colon U \to V$ of schemes over $\mathcal{X}$.

Exercise 4.1.24(c) provides a convenient and functorial way to think about quasi-coherent sheaves on moduli stacks. For instance, the Hodge bundle on $\mathcal{M}_g$ can be viewed as the data of the sheaf $\pi_*\Omega_{\mathcal{C}/S}$ for every smooth family of curves $\pi\colon \mathcal{C} \to S$.

Exercise 4.1.25 (Equivalences III)**.** Given a Deligne–Mumford stack $\mathcal{X}$, define the big étale site $(\mathrm{Sch}/\mathcal{X})_{\text{ét}}$, big fppf site $(\mathrm{Sch}/\mathcal{X})_{\mathrm{fppf}}$, and lisse-étale site $\mathcal{X}_{\text{lis-ét}}$ analogously to the definitions on a scheme as in §2.2.2. In each site, there is structure sheaf $\mathcal{O}$ and we say that an $\mathcal{O}$-module is *quasi-coherent* if it is quasi-coherent as a sheaf of modules in the ringed site. Show that there are equivalences

$$\mathrm{QCoh}(\mathcal{X}_{\text{ét}}) \xrightarrow{\sim} \mathrm{QCoh}(\mathcal{X}_{\text{lis-ét}}) \xrightarrow{\sim} \mathrm{QCoh}((\mathrm{Sch}/\mathcal{X})_{\text{ét}}) \xrightarrow{\sim} \mathrm{QCoh}((\mathrm{Sch}/\mathcal{X})_{\mathrm{fppf}}).$$

defined analogously to the equivalence $\mathrm{QCoh}(X_{\mathrm{Zar}}) \xrightarrow{\sim} \mathrm{QCoh}(X_{\text{ét}})$ in Exercise 4.1.22 over scheme X.

4.1.4 Pushforwards and pullbacks

Exercise 4.1.26 (Pushforward–Pullback Adjunction)**.** Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a morphism of Deligne–Mumford stacks.

(a) Show that f^* preserves quasi-coherence.

(b) If f is quasi-compact and quasi-separated, show that f_* preserves quasi-coherence and conclude that

$$\begin{tikzcd} \mathrm{QCoh}(\mathcal{X}) \arrow[r, bend left, "f_*"] & \mathrm{QCoh}(\mathcal{Y}) \arrow[l, bend left, "f^*"] \end{tikzcd}$$

are adjoint functors (with f_* the right adjoint).

Exercise 4.1.27 (good practice)**.** Let G be a finite group and $\Bbbk$ be a field.

(a) Consider a G-representation V viewed also as a quasi-coherent sheaf on BG using Example 4.1.21. Under the composition $\operatorname{Spec}\Bbbk \xrightarrow{p} BG \xrightarrow{\pi} \operatorname{Spec}\Bbbk$, show that $\pi_*V = V^G$, where V^G is the subspace of G-invariants, and that $p^*V = V$ forgetting the G-action. Moreover, for a $\Bbbk$-vector space W, show that $\pi^*W = W$ with the trivial G-action and that $p_*W = W \otimes p_*\Bbbk$, where $p_*\Bbbk$ is the regular representation $\Gamma(G, \mathcal{O}_G)$.

(b) Given an action of G an an affine $\Bbbk$-scheme $\operatorname{Spec} A$, consider the diagram

$$\begin{tikzcd} \operatorname{Spec} A \arrow[r, "p"] & {[\operatorname{Spec} A/G]} \arrow[r, "\pi"] \arrow[d, "q"] & \operatorname{Spec} A^G \\ & BG & \end{tikzcd}$$

and recall from Example 4.1.21 that a quasi-coherent sheaf on $[\operatorname{Spec} A/G]$ is an A-module M with a group homomorphism $G \to \operatorname{End}_A(M)$. Provide explicit descriptions of the functors $p_*, p^*, \pi_*, \pi^*, q_*$ and q^* on quasi-coherent sheaves.

Exercise 4.1.28 (hard)**.** Let $\mathcal{X}$ be a noetherian Deligne–Mumford stack. Prove the following two statements:

(a) Every quasi-coherent sheaf on $\mathcal{X}$ is a directed colimit of its coherent subsheaves.

(b) If $\mathcal{U} \subseteq \mathcal{X}$ is an open substack, then every coherent sheaf on $\mathcal{U}$ extends to a coherent sheaf on $\mathcal{X}$.

This exercise extends [Har77, Exc. II.5.15] from schemes to Deligne–Mumford stacks; see also [LMB00, Prop. 15.4], [Ols16, Prop. 7.1.11], and [SP, Tag 0GRE].

4.1.5 Quasi-coherent constructions

A *quasi-coherent* $\mathcal{O}_{\mathcal{X}}$*-algebra* on a Deligne–Mumford stack is a quasi-coherent $\mathcal{O}_{\mathcal{X}}$-module with the compatible structure of a ring object in $\mathrm{Ab}(\mathcal{X}_{\text{ét}})$. We define the *relative spectrum* $\mathcal{S}\mathrm{pec}_{\mathcal{X}}\,\mathcal{A}$ as the stack whose objects over a scheme S consists of a morphism $f\colon S \to \mathcal{X}$ and a morphism $f^*\mathcal{A} \to \mathcal{O}_S$ of $\mathcal{O}_S$-algebras.

Exercise 4.1.29 (easy, good practice)**.** Show that $\mathcal{S}\mathrm{pec}_{\mathcal{X}}\,\mathcal{A}$ is an algebraic stack affine over $\mathcal{X}$.

Example 4.1.30 (Reduction)**.** Let $\mathcal{X}$ be a Deligne-Mumford stack and let $\mathcal{O}_{\mathcal{X}}^{\mathrm{red}}$ be the sheaf of $\mathcal{O}_{\mathcal{X}}$-algebras defined by $\mathcal{O}_{\mathcal{X}}^{\mathrm{red}}(U \to \mathcal{X}) = \Gamma(U, \mathcal{O}_U)_{\mathrm{red}}$ for an étale $\mathcal{X}$-scheme U. Then $\mathcal{O}_{\mathcal{X}}^{\mathrm{red}}$ is a quasi-coherent $\mathcal{O}_{\mathcal{X}}$-algebra and $\mathcal{X}_{\mathrm{red}} := \mathcal{S}\mathrm{pec}_{\mathcal{X}}\, \mathcal{O}_{\mathcal{X}}^{\mathrm{red}}$ is a reduced algebraic stack equipped with a closed immersion $\mathcal{X}_{\mathrm{red}} \hookrightarrow \mathcal{X}$ such that $|\mathcal{X}_{\mathrm{red}}| = |\mathcal{X}|$. We call $\mathcal{X}_{\mathrm{red}}$ the *reduction of* $\mathcal{X}$.

Example 4.1.31 (Normalization)**.** Let $\mathcal{X}$ be an integral Deligne-Mumford stack and let $\mathcal{A}$ be the sheaf of $\mathcal{O}_{\mathcal{X}}$-algebras defined by setting $\mathcal{A}(U \to \mathcal{X})$ to be the normalization of $\Gamma(U, \mathcal{O}_U)$. Since normalization commutes with étale extensions (Proposition A.7.4), $\mathcal{A}$ is a quasi-coherent $\mathcal{O}_{\mathcal{X}}$-algebra. The *normalization of* $\mathcal{X}$ is defined as $\widetilde{\mathcal{X}} := \mathcal{S}\mathrm{pec}_{\mathcal{X}}\, \mathcal{A}$.

Exercise 4.1.32 (good practice)**.** Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a quasi-compact and quasi-separated morphism of Deligne–Mumford stacks.

(a) Show that there is factorization $f\colon \mathcal{X} \to \mathcal{S}\mathrm{pec}\, f_*\mathcal{O}_{\mathcal{X}} \to \mathcal{Y}$.

(b) Show that f is affine if and only if $\mathcal{X} \to \mathcal{S}\mathrm{pec}\, f_*\mathcal{O}_{\mathcal{X}}$ is an isomorphism.

(c) Show that f is quasi-affine if and only if $\mathcal{X} \to \mathcal{S}\mathrm{pec}\, f_*\mathcal{O}_{\mathcal{X}}$ is an open immersion.

Exercise 4.1.33 (details)**.** Use Exercise 4.1.28 to show that every quasi-coherent sheaf of algebras on a noetherian Deligne–Mumford stack is a directed colimit of finite type subalgebras.

4.1.6 Picard groups

We denote by $\mathrm{Pic}(\mathcal{X})$ the set of isomorphism classes of line bundles on a Deligne–Mumford stack $\mathcal{X}$. It is an abelian group under tensor product.

Exercise 4.1.34.

(a) Show that $\mathrm{Pic}(B\mu_n) = \mathbb{Z}/n\mathbb{Z}$ over a field whose characteristic is relatively prime to n.

(b) More generally, if G is finite group whose order is prime to the characteristic of a field $\Bbbk$, show that $\mathrm{Pic}(BG) \cong \mathbb{X}_*(G)$, where $\mathbb{X}_*(G)$ denotes the group of characters of G, i.e., group homomorphisms $G \to \mathbb{G}_m$.

The hypothesis on the characteristic is only to ensure that BG is Deligne–Mumford stack.

Exercise 4.1.35 (hard)**.** Show that $\mathrm{Pic}(\mathcal{M}_{1,1}) = \mathbb{Z}/12$ over a field with $\mathrm{char}(\Bbbk) \neq 2, 3$ and is generated by the Hodge bundle (see Example 4.1.4).

This was the main conclusion of Mumford's influential paper [Mum65], where the result was formulated without the language of Deligne–Mumford stacks (which had not yet been introduced). This exercise will be easier once we have more technology, namely quasi-coherent sheaves; see Exercise 6.1.22.

4.1.7 Cohomology

We develop a cohomology theory for abelian sheaves on Deligne–Mumford stacks. Despite utilizing the cohomology of quasi-coherent sheaves on schemes throughout these notes, we have surprisingly little need for cohomology on algebraic spaces and Deligne–Mumford stacks, and many of the results here are included only for completeness.

The existence of enough injective objects is shown analogously to the case of schemes [Har77, Prop. 2.2].

Lemma 4.1.36. *If $\mathcal{X}$ is a Deligne–Mumford stack, the categories* $\mathrm{Ab}(\mathcal{X}_{\text{ét}})$ *and* $\mathrm{Mod}(\mathcal{O}_{\mathcal{X}})$ *have enough injectives. If in addition $\mathcal{X}$ is quasi-separated, then* $\mathrm{QCoh}(\mathcal{X})$ *has enough injectives.*

Proof. Recall that a functor $R\colon \mathcal{A} \to \mathcal{B}$ between abelian categories with an exact left adjoint L preserves injectives since for an injective $I \in \mathcal{A}$, $\mathrm{Hom}_{\mathcal{B}}(-, R(I)) = \mathrm{Hom}_{\mathcal{A}}(L(-), I)$ is exact. We claim that for every sheaf of rings Λ on $\mathcal{X}_{\text{ét}}$, the category $\mathrm{Mod}(\Lambda)$ of Λ-modules has enough injectives. Taking Λ to be the constant sheaf $\underline{\mathbb{Z}}$ (resp., the structure sheaf $\mathcal{O}_{\mathcal{X}}$) establishes that $\mathrm{Ab}(\mathcal{X}_{\text{ét}})$ (resp., $\mathrm{Mod}(\mathcal{O}_{\mathcal{X}})$) has enough injectives. Let F be a Λ-module and let $U \to \mathcal{X}$ be an étale presentation. For each geometric point $\mathrm{Spec}\, \Bbbk \to U$, consider the composition $j_u\colon \mathrm{Spec}\, \Bbbk \to U \to \mathcal{X}$. Since an étale sheaf on $\mathrm{Spec}\, \Bbbk$ corresponds to an ordinary group (Exercise 4.1.5), the preimage $F_u := j_u^{-1}F$ corresponds to an ordinary module over the ring $\Lambda_u := j_u^{-1}\Lambda$. Choose an inclusion $F_u \hookrightarrow I_u$ into an injective Λ_u-module. Adjunction gives a map $F \to j_{u,*}I_u$, and since j_u^{-1} is exact (Exercise 4.1.10), $j_{u,*}I_u \in \mathrm{Ab}(\mathcal{X}_{\text{ét}})$ is injective. By taking the product, we obtain an injection $F \hookrightarrow \prod_{u \in U} j_{u,*}I_u$ into an injective Λ-module.

For the final statement, let $F \in \mathrm{QCoh}(\mathcal{X})$ and let $p\colon U = \coprod_i \mathrm{Spec}\, A_i \to \mathcal{X}$ be an étale presentation. Choose an injection $p^*F \hookrightarrow I$ into an injective quasi-coherent $\mathcal{O}_U$-module. The composition $F \hookrightarrow p_*p^*F \hookrightarrow p_*I$ is injective and since p^* is exact, p_*I is injective. □

Remark 4.1.37. The above argument extends to any ringed site with enough points (see [Ols16, Thm. 2.3.2]), and the result holds even in any ringed site [SP, Tag 01DP]. The category of quasi-coherent sheaves on a Deligne–Mumford stack $\mathcal{X}$ is a Grothendieck abelian category [SP, Tag 0781] and any such category has enough injectives [Gro57b], [SP, Tag 079H].

Definition 4.1.38 (Cohomology)**.** Let $\mathcal{X}$ be a Deligne–Mumford stack and F a sheaf of abelian groups on $\mathcal{X}_{\text{ét}}$. The *cohomology group* $\mathrm{H}^p(\mathcal{X}_{\text{ét}}, F)$ is defined as the pth right derived functor of the global sections functor $\Gamma\colon \mathrm{Ab}(\mathcal{X}_{\text{ét}}) \to \mathrm{Ab}$.

Given a morphism $f\colon \mathcal{X} \to \mathcal{Y}$ of Deligne–Mumford stacks, the *higher direct image* $\mathrm{R}^p f_* F$ is defined as the pth right derived functor of $f_*\colon \mathrm{Ab}(\mathcal{X}_{\text{ét}}) \to \mathrm{Ab}(\mathcal{Y}_{\text{ét}})$.

In order to show that the cohomology of a quasi-coherent sheaf on a scheme can be computed using either the usual Zariski topology or the small étale site (Proposition 4.1.43), we will use Čech cohomology. Just as with schemes, Čech cohomology provides an extremely effective tool to compute cohomology groups. Čech cohomology in the étale topology is defined similarly to the case of the Zariski topology [Har77, III.4] replacing intersections $U_{i_0} \cap \cdots \cap U_{i_n}$ with fiber products $U_{i_0} \times_{\mathcal{X}} \cdots \times_{\mathcal{X}} U_{i_n}$ and considering all (possibly non-distinct) indices $i_0, \ldots, i_n$ in any order.

Definition 4.1.39 (Čech cohomology)**.** Let F be an abelian sheaf on a Deligne–Mumford stack $\mathcal{X}$. If $\mathcal{U} = \{U_i \to \mathcal{X}\}_{i \in I}$ is an étale covering, the *Čech complex of F with respect to $\mathcal{U}$* is $\check{C}^\bullet(\mathcal{U}, F)$ where

$$\check{C}^n(\mathcal{U}, F) = \prod_{(i_0, \ldots, i_n) \in I^{n+1}} F(U_{i_0} \times_{\mathcal{X}} \cdots \times_{\mathcal{X}} U_{i_n})$$

with differential

$$d^n\colon \check{C}^n(\mathcal{U}, F) \to \check{C}^{n+1}(\mathcal{U}, F), \qquad (s_{i_0, \ldots, i_n}) \mapsto \left(\sum_{k=0}^{n+1} (-1)^k \widehat{p}_k^* s_{i_0, \ldots, \widehat{i_k}, \ldots, i_{n+1}} \right)_{(i_0, \ldots, i_{n+1})}$$

where $\widehat{p}_k \colon U_{i_0} \times_{\mathcal{X}} \cdots \times_{\mathcal{X}} U_{i_{n+1}} \to U_{i_0} \times_{\mathcal{X}} \cdots \times_{\mathcal{X}} \widehat{U_{i_k}} \times_{\mathcal{X}} \cdots \times_{\mathcal{X}} U_{i_{n+1}}$ is the map forgetting the kth component (with indexing starting at 0). The *Čech cohomology of F with respect to $\mathcal{U}$* is

$$\check{\mathrm{H}}^p(\mathcal{U}, F) := \mathrm{H}^p(\check{\mathcal{C}}^\bullet(\mathcal{U}, F)).$$

The following is a standard result in Čech cohomology whose proof is analogous to topological spaces. It is often referred to as Cartan's criterion.

Lemma 4.1.40. *Let $\mathcal{X}$ be a Deligne–Mumford stack and let F be an abelian sheaf on $\mathcal{X}_{\text{ét}}$. Suppose that Ob' is a subset of the set of schemes étale over $\mathcal{X}$ and that Cov' is a subset of all the coverings in $\mathcal{X}_{\text{ét}}$ such that for each $\{U_i \to U)\} \in \mathrm{Cov}'$, we have that U, each U_i, and each fiber product $U_{i_1} \times_U \cdots \times_U U_{i_n}$ is in Ob'. Assume that every covering of an étale $\mathcal{X}$-scheme U has a refinement in Cov'. If for every covering $\mathcal{U} \in \mathrm{Cov}'$, $\check{\mathrm{H}}^p(\mathcal{U}, F) = 0$ for $p > 0$, then $\mathrm{H}^p(U, F) = 0$ for every $U \in \mathrm{Ob}'$.*

Proof. This holds in any site; see [God58, II.5.9.2], [Mil80, Prop. III.2.12], [SP, Tag 03F9], or [Ols16, Prop. 2.3.15]. □

Cohomology of an étale sheaf can be computed as the Čech cohomology with respect to a covering with vanishing cohomology, just as in the case of ordinary topological spaces [Har77, Exc. III.4.11].

Lemma 4.1.41. *Let F be an abelian sheaf on $\mathcal{X}_{\text{ét}}$ and $(U_i \to \mathcal{X})_{i \in I}$ an étale covering. If $\mathrm{H}^p(U_{i_0} \times_U \cdots \times_U U_{i_n}, F) = 0$ for all $p > 0$, $n \geq 0$ and $i_0, \ldots, i_n \in I$, then $\check{\mathrm{H}}^p(\mathcal{U}, F) = \mathrm{H}^p(\mathcal{X}_{\text{ét}}, F)$.*

Proof. This also holds in any site and follows from Čech-to-derived functor spectral sequence; see [SP, Tag 03F7]. □

Theorem 4.1.42. *For a quasi-coherent sheaf F on an affine scheme X, $\mathrm{H}^p(X_{\text{ét}}, F) = 0$ for all $p > 0$.*

Proof. Let $X = \operatorname{Spec} A$, $F = \widetilde{M}$ be a quasi-coherent $\mathcal{O}_{X_{\text{Zar}}}$-module, which we can also view as a quasi-coherent $\mathcal{O}_{X_{\text{ét}}}$-module (Exercise 4.1.22). The set of étale coverings of the form $\mathcal{U} = \{\operatorname{Spec} B \to \operatorname{Spec} A\}$ is sufficient to refine any other covering. For the covering $\mathcal{U}$, faithfully flat descent (Exercise 2.1.3) gives a long exact sequence

$$0 \to M \to M \otimes_A B \to M \otimes_A B \otimes_A B \to M \otimes_A B \otimes_A B \otimes_A B \to \cdots,$$

which is identified with the Čech complex $\check{\mathcal{C}}^\bullet(\mathcal{U}, F)$. This shows that $\check{\mathrm{H}}^p(\mathcal{U}, F) = 0$ for $p > 0$ and thus Lemma 4.1.40 implies that $\mathrm{H}^p(X_{\text{ét}}, F) = 0$. □

Proposition 4.1.43. *If X is a scheme with affine diagonal and F is a quasi-coherent sheaf, then $\mathrm{H}^p(X, F) = \mathrm{H}^p(X_{\text{ét}}, F)$ for all p.*

Proof. Let $\mathcal{U} := \{U_i \hookrightarrow X\}$ be a Zariski covering of X by affine scheme. Since X has affine diagonal, each fiber product $U_{i_0} \times_X \cdots \times_X U_{i_n}$ is also affine. By Theorem 4.1.42, $\mathrm{H}^p(U_{i_0} \times_U \cdots \times_U U_{i_n}, F) = 0$, and therefore we may apply Lemma 4.1.41 to conclude that $\mathrm{H}^p(X, F) = \check{\mathrm{H}}^p(\mathcal{U}, F) = \mathrm{H}^p(X_{\text{ét}}, F)$. The same result holds more generally without the affine diagonal hypothesis and in other sites, e.g., the big fppf site $(\mathrm{Sch}/X)_{\text{fppf}}$; see [Mil80, Prop. 3.7] and [SP, Tag 03DW]. □

We now compare the cohomologies computed in $\mathrm{Ab}(\mathcal{X}_{\text{ét}})$, $\mathrm{Mod}(\mathcal{O}_{\mathcal{X}_{\text{ét}}})$, and $\mathrm{QCoh}(\mathcal{X})$.

Proposition 4.1.44. *Let $\mathcal{X}$ be a Deligne–Mumford stack.*

(1) *If F is an $\mathcal{O}_{\mathcal{X}}$-module, then the cohomology $\mathrm{H}^p(\mathcal{X}_{\text{ét}}, F)$ of F as an abelian sheaf agrees with the pth right derived functor of $\Gamma\colon \operatorname{Mod}(\mathcal{O}_{\mathcal{X}}) \to \mathrm{Ab}$.*

(2) *If $\mathcal{X}$ is quasi-compact with affine diagonal and F is a quasi-coherent sheaf on $\mathcal{X}$, then the cohomology $\mathrm{H}^p(\mathcal{X}_{\text{ét}}, F)$ of F as an abelian sheaf agrees with the pth right derived functor of $\Gamma\colon \operatorname{QCoh}(\mathcal{X}) \to \mathrm{Ab}$.*

The same holds for the higher direct images $\mathrm{R}^p f_* F$ of a morphism $f\colon \mathcal{X} \to \mathcal{Y}$ of Deligne–Mumford stacks.

Proof. For (1), we need to show that an injective object in $\operatorname{Mod}(\mathcal{O}_{\mathcal{X}})$ is acyclic in $\mathrm{Ab}(\mathcal{X}_{\text{ét}})$. This uses a standard technique in Čech cohomology. We will use the notation that if $\mathcal{U} = \{U_i \to \mathcal{X}\}_{i\in I}$ is an étale covering and $j = (j_0, \ldots, j_n) \in I^{n+1}$, then $U_j := U_{j_0} \times_{\mathcal{X}} \cdots \times_{\mathcal{X}} U_{j_n}$ with structure morphism $q_j\colon U_j \to \mathcal{X}$. There is a chain complex $\underline{\mathbb{Z}}_{\mathcal{U},\bullet}$ of presheaves on $\mathcal{X}$ defined by

$$\underline{\mathbb{Z}}_{\mathcal{U},n} := \bigoplus_{j\in I^{n+1}} q_{j,!}\underline{\mathbb{Z}},$$

where $\underline{\mathbb{Z}}$ denotes the constant presheaf and $q_{j,!}\underline{\mathbb{Z}}(V \to \mathcal{X}) := \bigoplus_{\operatorname{Mor}_{\mathcal{X}}(V,U_i)} \mathbb{Z}$, and where the differentials are the alternating sums of the natural maps. This complex of presheaves is exact in positive degrees and has the property that for every presheaf F

$$\check{C}(\mathcal{U}, F) = \operatorname{Mor}_{\operatorname{PAb}(\mathcal{X}_{\text{ét}})}(\underline{\mathbb{Z}}_{\mathcal{U},\bullet}, F) = \operatorname{Mor}_{\operatorname{PMod}(\mathcal{O}_{\mathcal{X}})}(\underline{\mathbb{Z}}_{\mathcal{U},\bullet} \otimes_{\underline{\mathbb{Z}}} \mathcal{O}_{\mathcal{X}}, F),$$

where morphisms are computed in the categories $\operatorname{PAb}(\mathcal{X}_{\text{ét}})$ and $\operatorname{PMod}(\mathcal{O}_{\mathcal{X}})$ of presheaves. If $F \in \operatorname{Mod}(\mathcal{O}_{\mathcal{X}})$ is injective, then it is also injective as a presheaf of $\mathcal{O}_{\mathcal{X}}$-modules. It follows that $\check{C}(\mathcal{U}, F)$ is exact in positive degrees and thus $\check{\mathrm{H}}^p(\mathcal{U}, F) = 0$ for $p > 0$. Therefore, Lemma 4.1.40 implies that $\mathrm{H}^p(\mathcal{X}_{\text{ét}}, F) = 0$. This result holds more generally in any ringed site; see [Ols16, Cor. 2.3.16] or [SP, Tag 03FD].

For (2), let $F \in \operatorname{QCoh}(\mathcal{X})$ be an injective object. Let $q\colon U = \operatorname{Spec} A \to \mathcal{X}$ be an étale presentation and choose an injection $q^*F \hookrightarrow G$ into an injective object $G \in \operatorname{QCoh}(U)$. Then pushforward q_*G is injective (as the right adjoint q^* is exact), and we have an inclusion $F \hookrightarrow q_*q^*F \hookrightarrow q_*G$ of injectives which necessarily splits. It thus suffices to show that q_*G is acyclic in $\mathrm{Ab}(\mathcal{X}_{\text{ét}})$. Since $\mathcal{X}$ has affine diagonal, $p\colon U \to \mathcal{X}$ is an affine morphism. By Flat Base Change (A.2.12) and étale descent, q_* is exact on the category of quasi-coherent sheaves. By Theorem 4.1.42, $\mathrm{H}^p(\mathcal{X}_{\text{ét}}, q_*G) = \mathrm{H}^p(U_{\text{ét}}, G) = 0$. $\square$

Exercise 4.1.45 (Flat Base Change)**.** Consider a cartesian diagram

$$\begin{tikzcd} \mathcal{X}' \arrow[r, "g'"] \arrow[d, "f'"'] \arrow[dr, phantom, "\square"] & \mathcal{X} \arrow[d, "f"] \\ \mathcal{Y}' \arrow[r, "g"] & \mathcal{Y} \end{tikzcd}$$

of Deligne–Mumford stacks, and let F be a quasi-coherent sheaf on $\mathcal{X}$. If $g\colon \mathcal{Y}' \to \mathcal{Y}$ is flat and $f\colon \mathcal{X} \to \mathcal{Y}$ is quasi-compact and quasi-separated, the natural adjunction map

$$g^*\mathrm{R}^p f_* F \to \mathrm{R}^p f'_* g'^* F$$

is an isomorphism for all $p \geq 0$.

Exercise 4.1.46. If $\mathcal{X}$ is a Deligne–Mumford stack and F_i is a directed system of abelian sheaves in $\mathcal{X}_{\text{ét}}$, show that $\operatorname{colim}_i \mathrm{H}^p(\mathcal{X}_{\text{ét}}, F_i) \to \mathrm{H}^p(\mathcal{X}_{\text{ét}}, \operatorname{colim}_i F_i)$ is an isomorphism.

Example 4.1.47 (Group cohomology)**.** Let G be a finite abstract group viewed as a group scheme over a field $\Bbbk$, and let V be a G-representation which we can also view as a quasi-coherent sheaf on BG. The *group cohomology* $\mathrm{H}^p(G, V)$ is defined as the pth right derived functor of

$$\operatorname{Rep}(G) \to \operatorname{Vect}_{\Bbbk}, \qquad V \mapsto V^G.$$

Since $\operatorname{Rep}(G) \cong \operatorname{QCoh}(BG)$, $\Gamma(BG, V) = V^G$, and $\mathrm{H}^p((BG)_{\text{ét}}, V)$ can be computed in $\operatorname{QCoh}(BG)$ (Proposition 4.1.44(2)), we have the identification

$$\mathrm{H}^p(G, V) \cong \mathrm{H}^p((BG)_{\text{ét}}, V).$$

The Čech complex $\check{C}^\bullet(\mathcal{U}, V)$ of V with respect to the étale cover $\mathcal{U} = \{\operatorname{Spec}\Bbbk \to BG\}$ has terms

$$\check{C}^n(\mathcal{U}, V) = \Gamma(G, \mathcal{O}_G)^{\otimes n} \otimes V$$

and differentials

$$\begin{aligned} d^n \colon \Gamma(G, \mathcal{O}_G)^{\otimes n} \otimes V \to& \Gamma(G, \mathcal{O}_G)^{\otimes (n+1)} \otimes V \\ f \otimes v \mapsto& \sum_{k=0}^{n} (-1)^k \mu_k^*(f) \otimes v + (-1)^{n+1} f \otimes \sigma(v) \end{aligned} \tag{4.1.48}$$

where $\sigma \colon V \to \Gamma(G, \mathcal{O}_G) \otimes V$ is the coaction and $\mu_k \colon G^{n+1} \to G^n$ is defined by sending $(g_1, \ldots, g_{n+1})$ to $(g_2, \ldots, g_{n+1})$ for $k = 0$ and to $(g_1, \ldots, g_{k-1}, g_k g_{k+1}, g_{k+2}, \ldots, g_{n+1})$ for $k = 1, \ldots, n$; see Exercise 3.4.6. In low degrees, we have $d^0(v) = v - \sigma(v)$ and $d^1(f, v) = 1 \otimes f \otimes v - \mu^*(f_1) \otimes v + f_1 \otimes \sigma(v)$ where $\mu = \mu_1$ is group multiplication $G \times G \to G$. This is called the *standard resolution* or *bar resolution*, and provides an effective way to compute group cohomology. Since G is finite, this complex can also be interpreted using the identifications $\Gamma(G^n, \mathcal{O}_{G^n}) \otimes V \cong \operatorname{Map}(G^n, V)$ with set-theoretic maps. See also [Wei94, §6] and [Bro94, §1].

The following example illustrates that coherent sheaf cohomology on a Deligne–Mumford stack can be nonzero in arbitrary high degrees. This doesn't happen in characteristic 0 or for *tame* Deligne–Mumford stacks (see Corollary 4.4.23).

Exercise 4.1.49. Let $\Bbbk$ be a field of characteristic $p > 0$ and let $\mathbb{Z}/p\mathbb{Z}$ be the group scheme over $\operatorname{Spec}\Bbbk$ corresponding to the abstract cyclic group. Show that $\mathrm{H}^i(B(\mathbb{Z}/p\mathbb{Z}), \mathcal{O}_{B(\mathbb{Z}/p\mathbb{Z})}) = \Bbbk$ for each $i \geq 0$.

Exercise 4.1.50 (hard)**.** Generalize Grothendieck's Vanishing Theorem [Har77, Thm. III.2.7] to algebraic spaces: if X is a noetherian algebraic space and F is a quasi-coherent sheaf, then $\mathrm{H}^i(X, F) = 0$ for $i > \dim X$. See also [SP, Tag 0A4Q].

4.2 Quotients by finite groups

> *Symmetry, as wide or narrow as you may define its meaning, is one idea by which man through the ages has tried to comprehend and create order, beauty, and perfection.*
>
> Hermann Weyl

Quotient stacks $[\operatorname{Spec} A/G]$ of affine schemes by finite abstract groups form a particularly nice class of Deligne–Mumford stacks. Their geometry is the G-equivariant geometry of $\operatorname{Spec} A$ (see Table 0.6.15). In this section, we show that the natural map $[\operatorname{Spec} A/G] \to \operatorname{Spec} A^G$ is a coarse moduli space (Theorem 4.2.3). In the next section, we will show that every Deligne–Mumford stack is étale locally isomorphic to a quotient stack of the form $[\operatorname{Spec} A/G]$ (Theorem 4.3.1). Combined these two theorems will be applied to establish the existence of a coarse moduli spaces, i.e., the Keel–Mori Theorem (4.4.6).

4.2.1 Coarse moduli spaces and geometric quotients

Definition 4.2.1 (Coarse moduli spaces)**.** A morphism $\pi\colon \mathcal{X} \to X$ from an algebraic stack to an algebraic space is a *coarse moduli space* if

(1) for every algebraically closed field $\Bbbk$, the induced map $\mathcal{X}(\Bbbk)/{\sim} \to X(\Bbbk)$, from the set of isomorphism classes of objects of $\mathcal{X}$ over $\Bbbk$, is bijective, and

(2) π is universal for maps to algebraic spaces, i.e., every map $\mathcal{X} \to Y$ to an algebraic space factors uniquely as

$$\begin{tikzcd} \mathcal{X} \arrow[d, "\pi"'] \arrow[dr] & \\ X \arrow[r, dashed] & Y. \end{tikzcd}$$

Definition 4.2.2 (Geometric quotients)**.** If G is an fppf group scheme over a scheme S (e.g., a finite group scheme associated to a finite abstract group) acting on an algebraic space U over S, a G-invariant morphism $U \to X$ over S to an algebraic space is a *geometric quotient* if $[U/G] \to X$ is a coarse moduli space, i.e.,

(1) for every algebraically closed field $\Bbbk$, the map $U \to X$ induces a bijection $U(\Bbbk)/G \xrightarrow{\sim} X(\Bbbk)$, and

(2) $U \to X$ is universal for G-invariant maps to algebraic spaces, i.e., for every G-invariant map $U \to Y$ to an algebraic space, there is a unique map $X \to Y$ such that $U \to Y$ factors as $U \to X \to Y$.

If $\pi\colon U \to X$ is a geometric quotient, we often write $X = U/G$. A geometric quotient $U \to U/G$ factors as

$$\begin{tikzcd} U \arrow[r] \arrow[dr, "\substack{\text{geometric}\\ \text{quotient}}"'] & {[U/G]} \arrow[d, "\substack{\text{coarse moduli}\\ \text{space}}"] \\ & U/G \end{tikzcd}$$

In the case that G acts freely on U (i.e., the action map $G \times U \to U \times U$ is a monomorphism), then we have already defined the algebraic space quotient U/G (Corollary 3.1.14) and this is identified with the geometric quotient (Corollary 4.2.13).

If a finite abstract group G acts on an affine scheme $\operatorname{Spec} A$, then G also acts on the ring A. We define the *invariant ring* as

$$A^G = \{f \in A \mid g \cdot f = f \text{ for all } g \in G\}.$$

The main result of this section is that $\operatorname{Spec} A \to \operatorname{Spec} A^G$ is a geometric quotient.

Theorem 4.2.3. *If G is a finite abstract group acting on an affine scheme* $\operatorname{Spec} A$, *then* $\pi \colon [\operatorname{Spec} A/G] \to \operatorname{Spec} A^G$ *is a coarse moduli space. Moreover,*

(1) π *is a universal homeomorphism;*

(2) the base change of π *along a flat morphism* $X' \to \operatorname{Spec} A^G$ *of algebraic spaces is a coarse moduli space;*

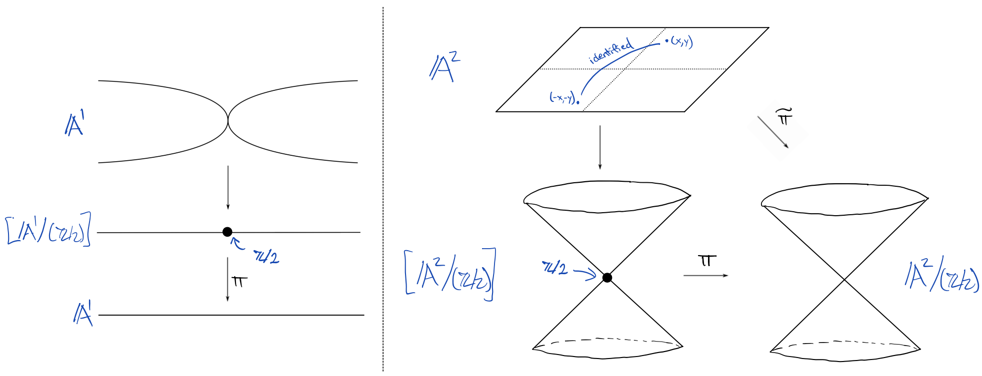
(3) the natural map $\mathcal{O}_X \to \pi_* \mathcal{O}_{\mathcal{X}}$ *is an isomorphism; and*

(4) if A *is finitely generated over a noetherian ring* R, *then* A^G *is finitely generated over* R *and* π *is proper.*

Quotients of schemes by finite groups is covered in many places in the literature, e.g., [Bou89, §V.9], [Ser88, §III.12], and [Mum70a, p. 66]. Our proof below follows standard arguments for the existence of the quotient scheme U/G, but it requires additional effort to prove that it is universal in the category of *algebraic spaces*.

Example 4.2.4. Assume $\operatorname{char}(\Bbbk) \neq 2$. If $G = \mathbb{Z}/2$ acts on $\mathbb{A}^1 = \operatorname{Spec} \Bbbk[x]$ via $-1 \cdot x = -x$, then $\Bbbk[x]^G = \Bbbk[x^2]$. The geometric quotient is the map $\mathbb{A}^1 = \operatorname{Spec} \Bbbk[x] \to \operatorname{Spec} \Bbbk[x^2] = \mathbb{A}^1$ sending t to t^2. The map $[\mathbb{A}^1/G] \to \operatorname{Spec} \Bbbk[x^2]$ is the coarse moduli space.

Similarly, if $G = \mathbb{Z}/2$ acts on $\mathbb{A}^2 = \operatorname{Spec} \Bbbk[x,y]$ via $-1 \cdot (x,y) = (-x,-y)$, then $\Bbbk[x,y]^G = \Bbbk[x^2, xy, y^2]$ and the geometric quotient is $\mathbb{A}^2 \to \mathbb{A}^2/G = \operatorname{Spec} \Bbbk[x^2, xy, y^2]$. By setting $A = x^2, B = xy$ and $C = y^2$, the invariant ring can be identified with $\Bbbk[A,B,C]/(B^2 - AC)$ so the quotient $\mathbb{A}^2/G$ is a cone over a conic and in particular singular.



4.2.2 Properties of invariant rings

Lemma 4.2.6. *If G is a finite abstract group acting on a R-algebra A via R-algebra automorphisms, then $A^G \to A$ is integral. If R is noetherian and A is finitely generated over R, then $A^G \to A$ is finite and A^G is finitely generated over R.*

Proof. To see that $A^G \to A$ is integral, observe that for every element $a \in A$, the product $\prod_{g \in G}(x - ga) \in A^G[x]$ is polynomial with invariant coefficients which has a as a root. If R is noetherian and $R \to A$ is of finite type, then $A^G \to A$ is also of finite type. Because $A^G \to A$ is integral and of finite type, it is finite (c.f., [AM69, Cor. 5.2]). Since R is noetherian, we may conclude by the Artin–Tate Lemma (c.f., [AM69, Prop. 7.8]) that $R \to A^G$ is of finite type. $\square$

The invariant ring is compatible with flat base change.

Lemma 4.2.7. *Let G be a finite abstract group acting on an affine scheme $\operatorname{Spec} A$. If $A^G \to B$ is a flat ring homomorphism, then G acts on the affine scheme $\operatorname{Spec}(B \otimes_{A^G} A)$ and $B = (B \otimes_{A^G} A)^G$.*

Proof. By definition, the invariant ring is the equalizer

$$0 \to A^G \to A \overset{p_1}{\underset{p_2}{\rightrightarrows}} \prod_{g \in G} A$$

where $p_1(f) = (f)_{g \in G}$ and $p_2(f) = (gf)_{g \in G}$. Since $A^G \to B$ is flat, we have that

$$0 \to B \to A \otimes_{A^G} B \overset{p_1}{\underset{p_2}{\rightrightarrows}} \prod_{g \in G} A \otimes_{A^G} B$$

is also exact, and we conclude that $B = (B \otimes_{A^G} A)^G$. $\square$

Exercise 4.2.8. Let $A^G \to B$ be a ring homomorphism and consider the commutative diagram

$$\begin{tikzcd}
 & \operatorname{Spec} B \otimes_{A^G} A \arrow[r] \arrow[dl] \arrow[d] \arrow[dr, phantom, "\square"] & \operatorname{Spec} A \arrow[d] \\
\operatorname{Spec}(B \otimes_{A^G} A)^G \arrow[r] & \operatorname{Spec} B \arrow[r] & \operatorname{Spec} A^G.
\end{tikzcd}$$

(a) Show that $\operatorname{Spec}(B \otimes_{A^G} A)^G \to \operatorname{Spec} B$ is an integral homeomorphism.

(b) If $|G|$ is invertible in A, show that $B \to (B \otimes_{A^G} A)^G$ is an isomorphism.

(c) Provide an example where $B \to (B \otimes_{A^G} A)^G$ is not an isomorphism.

Hint: In characteristic p, consider the action of $\mathbb{Z}/p$ on $A = \Bbbk[x, y]$ where a generator acts via $(x, y) \mapsto (x + y, y)$ and take $B = A^G/(y)$.

4.2.3 Properties of coarse moduli spaces

Lemma 4.2.9. *Let $\pi \colon \mathcal{X} \to X$ be a coarse moduli space such that for every étale morphism $X' \to X$ from an affine scheme, the base change $\mathcal{X} \times_X X' \to X'$ is a coarse moduli space. Then the natural map $\mathcal{O}_X \to \pi_* \mathcal{O}_{\mathcal{X}}$ is an isomorphism.*

Proof. As π is universal for maps to algebraic spaces, we have that $\operatorname{Map}(X, \mathbb{A}^1) \to \operatorname{Map}(\mathcal{X}, \mathbb{A}^1)$ is bijective or in other words $\Gamma(X, \mathcal{O}_X) \cong \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$. For every étale map $X' \to X$, the base change $\mathcal{X}' = \mathcal{X} \times_X X' \to X'$ is also a coarse moduli space and thus $\Gamma(X', \mathcal{O}_{X'}) \cong \Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})$. This shows that $\mathcal{O}_X \to \pi_* \mathcal{O}_{\mathcal{X}}$ is isomorphism. $\square$

The property of being a coarse moduli space is étale local.

Lemma 4.2.10. *Let $\pi \colon \mathcal{X} \to X$ be a morphism to an algebraic space. Suppose that there is an étale covering $\{X_i \to X\}$ such that $\mathcal{X} \times_X X_i \to X_i$ is a coarse moduli space for each i. Then $\pi \colon \mathcal{X} \to X$ is a coarse moduli space.* $\square$

Proof. Axiom (1) of a coarse moduli space is a condition on geometric fibers and can thus be checked étale locally, while Axiom (2) follows from the fact that algebraic spaces are sheaves in the étale topology. $\square$

4.2.4 $\operatorname{Spec} A \to \operatorname{Spec} A^G$ is a geometric quotient

Proposition 4.2.11. *If G is a finite abstract group acting on an affine scheme $\operatorname{Spec} A$, then $\operatorname{Spec} A \to \operatorname{Spec} A^G$ is a geometric quotient.*

Proof. Consider the commutative diagram

$$\begin{tikzcd}
U = \operatorname{Spec} A \arrow[d] \arrow[dr, "\widetilde{\pi}"] & \\
\mathcal{X} = [U/G] \arrow[r, "\pi"] & X = \operatorname{Spec} A^G.
\end{tikzcd}$$

Since $\widetilde{\pi}$ is integral and dominant, it is surjective. To see that $\widetilde{\pi}$ is injective on G-orbits of geometric points, let $\Bbbk$ be an algebraically closed field and suppose that $x, x' \in U(\Bbbk)$ are points such that under the induced G-action on $U \times_X \Bbbk$, the G-orbits of $x, x' \in U \times_X \Bbbk$ are distinct. As the orbits are disjoint and there exists a function $f \in A \otimes_{A^G} \Bbbk$ with $f|_{Gx} = 0$ and $f|_{Gx'} = 1$. Then $\widetilde{f} = \prod_{g \in G} gf \in (A \otimes_{A^G} \Bbbk)^G$ is a G-invariant function with $\widetilde{f}(x) = 0$ and $\widetilde{f}(x') = 1$, and this implies that $\widetilde{\pi}(x) \neq \widetilde{\pi}(x') \in X(\Bbbk)$.

It remains to show that the natural map

$$\operatorname{Map}(X, Y) \to \operatorname{Map}(\mathcal{X}, Y) \tag{4.2.12}$$

is bijective for every algebraic space Y. We note that this is immediate when Y is affine as $\Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) = \Gamma(X, \mathcal{O}_X)$. (The case when Y is a scheme can be reduced to this case without much effort: if $g \colon \mathcal{X} \to Y$ is a map, an affine covering Y_i of Y induces an open covering $X_i = X \smallsetminus \pi(\mathcal{X} \smallsetminus g^{-1}(Y_i))$ of X, and g restricts to a map $\pi^{-1}(X_i) \to Y_i$ which factors uniquely through X_i since $\pi_* \mathcal{O}_{\mathcal{X}} = \mathcal{O}_X$; see also [GIT, §0.6].)

For the injectivity of (4.2.12), let $h_1, h_2 \colon X \to Y$ be two maps such that $h_1 \circ \pi = h_2 \circ \pi$. Let $E \to X$ be the equalizer of h_1 and h_2, i.e., the pullback of the diagonal $Y \to Y \times Y$ along $(h_1, h_2) \colon X \to Y \times Y$. The equalizer $E \to X$ is a monomorphism and locally of finite type. By construction $\pi \colon \mathcal{X} \to X$ factors through $E \to X$ and since π is universally closed and schematically dominant (i.e., $\mathcal{O}_X \to \pi_* \mathcal{O}_{\mathcal{X}}$ is injective), so is $E \to X$. As every universally closed and locally of finite type monomorphism is a closed immersion (see Remark A.7.6), we conclude that $E \to X$ is an isomorphism.

For the surjectivity of (4.2.12), let $g \colon \mathcal{X} \to Y$ be a morphism. We will begin with the easy reductions. By Lemmas 4.2.7 and 4.2.10, the question is étale-local on X and, in particular, we can assume that X is affine. Moreover, since $\mathcal{X}$ is quasi-compact, we may assume that Y is quasi-compact as $g \colon \mathcal{X} \to Y$ factors through a quasi-compact open algebraic subspace of Y. Let $Y' \to Y$ be an étale presentation from an affine scheme and let $\mathcal{X}' := \mathcal{X} \times_Y Y'$.

We claim that after replacing X with an étale cover $V \to X$ and $\mathcal{X}$ with the base change $\mathcal{X} \times_X V$, there is a section $s \colon \mathcal{X} \to \mathcal{X}'$ of $\mathcal{X}' \to \mathcal{X}$ in the commutative diagram

$$\begin{tikzcd}
\mathcal{X}' \arrow[r] \arrow[d, "g'"'] \arrow[dr, phantom, "\square"] & \mathcal{X} \arrow[l, bend right, "s"'] \arrow[r, "\pi"] \arrow[d, "g"] & X \arrow[dl, dashed] \\
Y' \arrow[r] & Y. &
\end{tikzcd}$$

The surjectivity of (4.2.12) follows from this claim: since X and Y' are affine, the equality $\Gamma(X, \mathcal{O}_X) = \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ implies that $\mathcal{X} \xrightarrow{s} \mathcal{X}' \xrightarrow{g'} Y'$ factors through $\pi\colon \mathcal{X} \to X$ via a morphism $X \to Y'$. The composition $X \to Y' \to Y$ yields the desired dotted arrow above.

To establish the claim, we will use limit methods to further reduce to the case that $X = \operatorname{Spec} A^G$ is the spectrum of a strictly henselian local ring. For a closed point u of $U := \operatorname{Spec} A$ over $x \in |\mathcal{X}|$, the strict henselization $X^{\mathrm{sh}} := \mathcal{O}^{\mathrm{sh}}_{X,\pi(x)}$ is the limit $\lim_\lambda X_\lambda$ over all affine étale neighborhoods $X_\lambda \to X$ of $\pi(x)$. The base change $U^{\mathrm{sh}} := U \times_X X^{\mathrm{sh}}$ is the limit of the affine schemes $U_\lambda := U \times_X X_\lambda$. We also set $\mathcal{X}^{\mathrm{sh}} := \mathcal{X} \times_X X^{\mathrm{sh}} = [U^{\mathrm{sh}}/G]$ and $\mathcal{X}_\lambda := \mathcal{X} \times_X X_\lambda = [U_\lambda/G]$. Since $\mathcal{X}' \to \mathcal{X}$ is locally of finite presentation, the natural map

$$\operatorname{colim}_\lambda \operatorname{Mor}_{\mathcal{X}}(\mathcal{X}_\lambda, \mathcal{X}') \to \operatorname{Mor}_{\mathcal{X}}(\mathcal{X}^{\mathrm{sh}}, \mathcal{X}')$$

is an equivalence; this follows from Exercise 3.3.33 (which asserts the equality when the first terms $\mathcal{X}_\lambda$ and $\mathcal{X}^{\mathrm{sh}}$ are affine) by expressing $\operatorname{Mor}_{\mathcal{X}}(\mathcal{X}^{\mathrm{sh}}, \mathcal{X}')$ as the equalizer of $\operatorname{Mor}_{\mathcal{X}}(U^{\mathrm{sh}}, \mathcal{X}') \rightrightarrows \operatorname{Mor}_{\mathcal{X}}(G \times U^{\mathrm{sh}}, \mathcal{X}')$ and similarly for the left-hand side. A section of $\mathcal{X}' \times_{\mathcal{X}} \mathcal{X}^{\mathrm{sh}} \to \mathcal{X}^{\mathrm{sh}}$ is determined by a map $\mathcal{X}^{\mathrm{sh}} \to \mathcal{X}'$ over $\mathcal{X}$. This map extends to a morphism $\mathcal{X}_\lambda \to \mathcal{X}'$ for some λ, giving us the desired section.

Let κ be the residue field of A^G. As $A^G \to A$ is finite, $A = A_1 \times \cdots \times A_r$ is a product of strictly henselian local rings, each finite over A^G (Proposition B.5.10). If $u \in \operatorname{Spec} A_1 \subseteq \operatorname{Spec} A$ is a closed point, then $\operatorname{Spec} A_1$ is G_u-invariant and the orbit Gu is in bijection with the r connected components of $\operatorname{Spec} A$. There is an isomorphism $\mathcal{X} \cong [\operatorname{Spec} A_1/G_u]$; this can be verified directly by for instance slicing the groupoid $G \times \operatorname{Spec} A \rightrightarrows \operatorname{Spec} A$ by $\operatorname{Spec} A_1 \hookrightarrow \operatorname{Spec} A$ (as in Exercise 3.4.20). We may thus replace $\mathcal{X} = [\operatorname{Spec} A/G]$ with $[\operatorname{Spec} A_1/G_u]$, and we can assume that there is a unique closed point $u \in \operatorname{Spec} A$ which is set-theoretically fixed by G. As $Y' \to Y$ is representable by schemes, we can write $\mathcal{X}' = [U'/G]$ for a scheme U'. Let $u' \in U'$ be a preimage of $u \in \operatorname{Spec} A$. As A is strictly henselian and the G-equivariant morphism $U' \to U$ is the base change of the étale morphism $Y' \to Y$, we see that $\kappa(u') = \kappa(u)$ and $G_{u'} = G_u = G$, and moreover the stabilizers act trivially on the residue fields. Again using that A is strictly henselian, there is a unique section $s\colon \operatorname{Spec} A \to U'$ with $s(u) = u'$ (Proposition B.5.9). This section is G-invariant because for every $g \in G$, both $s \circ g$ and $g \circ s$ are sections of $U' \to \operatorname{Spec} A \xrightarrow{g^{-1}} \operatorname{Spec} A$ with $u' \mapsto u$, and therefore $s \circ g = g \circ s$ by the uniqueness of the sections. It follows that $s\colon \operatorname{Spec} AS \to U$ descends to a section $\mathcal{X} = [\operatorname{Spec} A/G] \to [U'/G] = \mathcal{X}'$ of $\mathcal{X}' \to \mathcal{X}$. This finishes the proof that $\operatorname{Spec} A \to \operatorname{Spec} A^G$ is a geometric quotient. □

We now conclude not only that $[\operatorname{Spec} A/G] \to \operatorname{Spec} A^G$ is a coarse moduli space, but that it enjoys properties (1)–(4) of Theorem 4.2.3.

Proof of Theorem 4.2.3. Proposition 4.2.11 implies that $\pi\colon [\operatorname{Spec} A/G] \to \operatorname{Spec} A^G$ is a coarse moduli space. To show property (1), i.e., that π is a universal homeomorphism, observe that since π is bijective and closed, its set-theoretic inverse is continuous, and thus π is a homeomorphism. The base change of π along a morphism $\operatorname{Spec} B \to \operatorname{Spec} A^G$ factors as $[\operatorname{Spec}(B \otimes_{A^G} A)/G] \to \operatorname{Spec}(B \otimes_{A^G} A)^G \to \operatorname{Spec} B$ where the first map is a homeomorphism by the above argument and the second is a homeomorphism by Exercise 4.2.8. Property (2), i.e., the base change of π by a flat morphism $X' \to \operatorname{Spec} A^G$ is also a coarse moduli space, follows from Lemma 4.2.7 after reducing to the case that X' is affine using Lemma 4.2.10. Property (3) that $\mathcal{O}_X \to \pi_* \mathcal{O}_{\mathcal{X}}$ is an isomorphism follows direct from (2) by Lemma 4.2.9.

Finally, Lemma 4.2.6 implies (4): if R is noetherian and $R \to A$ is of finite type, then $R \to A^G$ is of finite type. In particular, $A^G \to A$ is of finite type and $\pi\colon [\operatorname{Spec} A/G] \to \operatorname{Spec} A^G$ is separated, universally closed, and of finite type, i.e., proper. $\square$

Corollary 4.2.13. *Let G be a finite abstract group acting freely on an affine scheme $U = \operatorname{Spec} A$, then the algebraic space quotient U/G is isomorphic to* $\operatorname{Spec} A^G$.

Proof. Both the algebraic space quotient and the geometric quotient are universal for maps to algebraic spaces. $\square$

Exercise 4.2.14. Let G be a finite abstract group acting on a scheme U.

(a) Show that the geometric quotient U/G is a scheme if and only if every G-orbit is contained in an affine open subscheme.

(b) If U is affine (resp., quasi-affine, quasi-projective, projective) scheme over a noetherian ring R, show that U/G is affine (resp., quasi-affine, quasi-projective, projective) scheme over R.

Exercise 4.2.15. Suppose that G is a finite abstract group acting on an affine scheme $\operatorname{Spec} A$ of finite type over a noetherian ring R. If $x \in \operatorname{Spec} A$ is a closed point, show that there is an isomorphism

$$\widehat{A}^{G_x} \cong \widehat{A^G}$$

between the G_x-invariants of the completion at $\operatorname{Spec} A$ at x and the completion of $\operatorname{Spec} A^G$ at the image of x.

The following exercise generalizes Theorem 4.2.3 from quotients of finite groups to quotients of finite flat groupoids.

Exercise 4.2.16 (hard)**.** Let $s, t\colon R \rightrightarrows U$ be a finite flat groupoid of affine schemes, and define $A^R \subseteq A$ as the subring of R-invariants, i.e., the subring of elements $a \in A$ such that $s^*a = t^*a \in \Gamma(R, \mathcal{O}_R)$. Show that $[U/R] \to X := \operatorname{Spec} A^R$ is a coarse moduli space with properties (1)–(4) of Theorem 3.4.13.

Hint: To show that $A^R \to A$ is integral, for an element $a \in A$, rather than using the product $\prod_{g \in G}(x - ga) \in A^G[x]$ as in the proof of Lemma 4.2.6, use the characteristic polynomial *of $\delta_0(a) \in B$ as a locally free module over A via δ_1 where $\delta_0, \delta_1\colon A \rightrightarrows B = \Gamma(U, \mathcal{O}_U)$ are the two finite flat ring maps. See also [SGA3$_I$, Thm. V.4.1], [KM97, Prop. 5.1], [Con05b, Thm. 3.1], [Ols16, Thm. 6.2.2], and [SP, Tag 0DUR].*

4.3 The local structure of Deligne–Mumford stacks

If only I had the theorems! Then I should find the proofs easily enough.

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We show that a Deligne–Mumford stack $\mathcal{X}$ near a point x is étale locally the quotient stack $[\operatorname{Spec} A/G_x]$ of an affine scheme by the stabilizer group scheme. Conceptually, this tells us that just as schemes (resp., algebraic spaces) are obtained by gluing affine schemes in the Zariski topology (resp., étale topology), Deligne–Mumford stacks are obtained by gluing quotient stacks $[\operatorname{Spec} A/G]$ in the étale topology.[1] This has the practical application of allowing one to reduce many properties of Deligne–Mumford stacks to quotient stacks $[\operatorname{Spec} A/G]$. We will exploit the Local Structure Theorem to construct a coarse moduli space (Theorem 4.4.6).

4.3.1 The Local Structure Theorem

The *geometric stabilizer* of a point $x \in |\mathcal{X}|$ of a Deligne–Mumford stack $\mathcal{X}$ is the abstract group defined as the stabilizer of any geometric point $\operatorname{Spec} \Bbbk \to \mathcal{X}$ with image x.

Theorem 4.3.1 (Local Structure Theorem of Deligne–Mumford Stacks)**.** *Let $\mathcal{X}$ be a quasi-separated Deligne–Mumford stack and $x \in |\mathcal{X}|$ be a finite type point with geometric stabilizer G_x. There exists an étale morphism*

$$f\colon ([\operatorname{Spec} A/G_x], w) \to (\mathcal{X}, x)$$

where $w \in [\operatorname{Spec} A/G_x]$ such that f induces an isomorphism of geometric stabilizer groups at w. Moreover, if $\mathcal{X}$ has separated (resp., affine) diagonal, then it can be arranged that f is representable and separated (resp., affine).

Proof. We first handle the case when $\mathcal{X}$ has affine diagonal. Choose a geometric point $\operatorname{Spec} \Bbbk \to \mathcal{X}$ of representing x, and let d be the cardinality of G_x. Viewing G_x as a group scheme over $\operatorname{Spec} \Bbbk$, let $BG_x \to \mathcal{X}$ be the induced map. Let $(U, u) \to (\mathcal{X}, x)$ be an étale representable morphism from an affine scheme. Since $\mathcal{X}$ has affine diagonal, $U \to \mathcal{X}$ is affine. Define the quasi-affine scheme

$$\mathrm{SEC}_d := \underbrace{U \times_{\mathcal{X}} \cdots \times_{\mathcal{X}} U}_{d \text{ times}} \setminus \Delta,$$

where Δ is the union of all pairwise diagonals A map $S \to \mathrm{SEC}_d$ from a scheme is classified by a morphism $S \to \mathcal{X}$ and d sections $s_1, \ldots, s_d$ of $U_S := U \times_{\mathcal{X}} S \to S$ which are disjoint (i.e., the intersection of s_i and s_j is empty for $i \neq j$). There is an action of S_d on SEC_d given by permuting the sections and we define the quotient stack

$$\mathrm{ET}_d := [\mathrm{SEC}_d / S_d].$$

By the correspondence between principal S_d-bundles and finite étale covers of degree d (Exercise B.1.52), an object of ET_d over a scheme S corresponds to a diagram

$$\begin{tikzcd} Z \arrow[r, hook] \arrow[dr] & U_S \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & U \arrow[d] \\ & S \arrow[r] & \mathcal{X} \end{tikzcd}$$

where $Z \hookrightarrow U_S$ is a closed subscheme and $Z \to S$ is finite étale of degree d. There is an induced étale morphism $\mathrm{ET}_d \to \mathcal{X}$ and a commutative diagram

$$\begin{tikzcd} \mathrm{SEC}_d \arrow[rr] \arrow[d, hook] & & \mathrm{ET}_d \arrow[dr] & \\ U \times_{\mathcal{X}} \cdots \times_{\mathcal{X}} U \arrow[rr] & & U \arrow[r] & \mathcal{X} \end{tikzcd}$$

[1] Of course, Deligne–Mumford stacks are also étale locally schemes but the étale neighborhoods $([\operatorname{Spec} A/G_x], w) \to (\mathcal{X}, x)$ produced by Theorem 4.3.1 preserve the stabilizer group at w.

To construct a preimage $w \in \mathrm{ET}_d(\Bbbk)$ of x, we observe that the fiber product $BG_x \times_{\mathcal{X}} U$ is a finite disjoint union of $\operatorname{Spec}\Bbbk$'s and we may choose one of them. This leads to a commutative

$$\begin{tikzcd}
\operatorname{Spec}\Bbbk \arrow[r, hook] \arrow[dr] & \coprod \operatorname{Spec}\Bbbk \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & U \arrow[d] \\
& BG_x \arrow[r] & \mathcal{X}
\end{tikzcd}$$

with $\operatorname{Spec}\Bbbk \to BG_x$ finite étale of degree d, and this defines a lift $BG_x \to \mathrm{ET}_d$ of $BG_x \to \mathcal{X}$. The composition $w\colon \operatorname{Spec}\Bbbk \to BG_x \to \mathrm{ET}_d$ is a preimage of x. We claim the map $\mathrm{ET}_d \to \mathcal{X}$ induces an isomorphism of geometric stabilizers at w. To see this, it suffices to assume that $\mathcal{X} = BG_x$ and $U = \operatorname{Spec}\Bbbk$. In this case, there is an isomorphism $U \times_{\mathcal{X}} \cdots \times_{\mathcal{X}} U \cong G_x^{d-1}$, which we can further identify with the quotient G_x^d/G_x of the diagonal action. Then $\mathrm{SEC}_d \subseteq U \times_{\mathcal{X}} \cdots \times_{\mathcal{X}} U$ is identified with the quotient $(G_x^d \smallsetminus \Delta)/G_x$. The permutation S_d-action is transitive with stabilizer isomorphic to G_x. In fact, SEC_d is S_d-equivariantly isomorphic to the quotient S_d/G of the regular representation $G \subseteq S_d$, and thus $\mathrm{ET}_d \cong BG_x$.

Since ET_d is separated and $\mathcal{X}$ has separated diagonal, the relative inertia stack $I_{\mathrm{ET}_d/\mathcal{X}} \to \mathrm{ET}_d$ is finite. As $\mathrm{ET}_d \to \mathcal{X}$ induces an isomorphism of stabilizers at w, $I_{\mathrm{ET}_d/\mathcal{X}} \to \mathrm{ET}_d$ is an isomorphism over an open substack $[W/S_d] \subseteq \mathrm{ET}_d$ around w, where $W \subseteq \mathrm{SEC}_d$ is a quasi-affine scheme. It follows that $[W/S_d] \to \mathcal{X}$ is a separated, étale, and representable morphism inducing an isomorphism on stabilizer groups at w. By quotienting out by $G_x \subseteq S_d$ instead, $[W/G_x] \to \mathcal{X}$ is also a separated, étale, and representable morphism inducing an isomorphism of stabilizer groups at w. Letting $W' \subseteq W$ be an affine open subscheme containing w, we may replace W with the G_x-invariant *affine* open subscheme $\bigcap_{g \in G_x} g \cdot W'$. It remains to show that $[W/G_x] \to \mathcal{X}$ is affine. Since $\mathcal{X}$ has affine diagonal, the morphism $W \to \mathcal{X}$ from the affine scheme W is affine. The fiber product

$$\begin{tikzcd}
{[W/G_x] \times_{\mathcal{X}} W} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & W \arrow[d] \\
{[W/G_x]} \arrow[r] & \mathcal{X}
\end{tikzcd}$$

is affine over $[W/G_x]$ and thus isomorphic to a quotient stack $[\operatorname{Spec} B/G_x]$. On the other hand, since $[W/G_x] \to \mathcal{X}$ is representable, the quotient stack $[\operatorname{Spec} B/G_x]$ is an algebraic space and the action of G_x on $\operatorname{Spec} B$ is free. By Corollary 4.2.13, $[\operatorname{Spec} B/G_x]$ is isomorphic to the affine scheme $\operatorname{Spec} B^{G_x}$. By étale descent $[W/G_x] \to \mathcal{X}$ is affine.

If $\mathcal{X}$ has separated diagonal, then the above argument produces a separated scheme SEC_d with an action of G_x and an separated, étale, and representable morphism $([\mathrm{SEC}_d/G_x], w) \to (\mathcal{X}, x)$ inducing an isomorphism at w. By applying the argument again to the *separated* Deligne–Mumford stack $[\mathrm{SEC}_d/G_x]$ gives the desired local quotient presentation. If $\mathcal{X}$ is only quasi-separated, the argument produces a (not necessarily representable) morphism $([\mathrm{SEC}_d/G_x], w) \to (\mathcal{X}, x)$ inducing an isomorphism at w where SEC_d is a quasi-separated algebraic space. As SEC_d has separated diagonal (as the diagonal is a monomorphism), $[\mathrm{SEC}_d/S_d]$ is a Deligne–Mumford stack with separated diagonal, and we may apply the statement in the case of separated diagonal to obtain a local quotient presentation by the quotient of an affine scheme. See also [LMB00, Thm. 6.2]. $\square$

Exercise 4.3.2. Let $\mathcal{X}$ be a Deligne–Mumford stack. Show that $\mathcal{X}$ is isomorphic to a quotient stack $[U/G]$, where U is an affine scheme (resp., scheme, algebraic space) and G is a finite abstract group, if and only if there exists a finite étale morphism $V \to \mathcal{X}$ from an affine scheme (resp., scheme, algebraic space).

Hint: If $V \to \mathcal{X}$ is a finite étale cover of degree d, consider the associated principal S_d-torsor $\underbrace{V \times_{\mathcal{X}} \cdots \times_{\mathcal{X}} V}_{d \text{ times}} \smallsetminus \Delta \to \mathcal{X}$ as in Exercise B.1.52.

Proposition 4.3.3. *If $R \rightrightarrows U$ is a finite étale equivalence relation of affine schemes, then the algebraic space quotient U/R is an affine scheme.*

Proof. By Exercise 4.3.2, the algebraic space U/R is isomorphism to V/G for the free action of a finite group G on an affine scheme $V = \operatorname{Spec} B$. Theorem 4.2.3 shows that $V/G \to \operatorname{Spec} B^G$ is universal for maps to algebraic spaces and thus an isomorphism. Alternatively, this follows from Exercise 4.2.16: if $U = \operatorname{Spec} A$, then $U/R \to \operatorname{Spec} A^R$ is universal for maps to algebraic spaces and thus an isomorphism. $\square$

4.3.2 Lifting rational points to presentations

With a similar technique to the proof of the Local Structure Theorem (4.3.1), we can prove the existence of presentations with a lift of a given field-valued point.

Proposition 4.3.4. *If $\mathcal{X}$ is an algebraic stack with separated diagonal (resp., Deligne–Mumford stack with separated and quasi-compact diagonal) and $x \in \mathcal{X}(\Bbbk)$ is a field-valued point, then there exists a smooth (resp., étale) morphism $U \to \mathcal{X}$ from an affine scheme and a point $u \in U(\Bbbk)$ over x.*

Proof. Let $U \to \mathcal{X}$ be a smooth morphism from an affine scheme such that x is contained in its image. Consider the fiber product

$$\begin{tikzcd} U_x \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & U \arrow[d] \\ \operatorname{Spec} \Bbbk \arrow[r, "x"] & \mathcal{X}. \end{tikzcd}$$

Since $\mathcal{X}$ has separated diagonal, the morphism $U \to \mathcal{X}$ is separated. As U_x is finite type and quasi-separated over $\operatorname{Spec} \Bbbk$, it is a noetherian algebraic space. Choosing a closed point $u \in U_x$, by the Existence of Residual Gerbes (3.5.16), there is a closed immersion $\operatorname{Spec} \kappa(u) \to U_x$ from the residue field. The field extension $\Bbbk \to \kappa(u)$ is finite and separable, and we let d be its degree.

Following the notation of the proof of Theorem 4.3.1, we consider the open subspace $\mathrm{SEC}_d \subseteq U \times_{\mathcal{X}} \cdots \times_{\mathcal{X}} U$ parameterizing d disjoint sections of $U \to \mathcal{X}$, which is a separated algebraic space. Define the separated Deligne–Mumford stack $\mathrm{ET}_d = [\mathrm{SEC}_d / S_d]$ and consider the induced morphism $\mathrm{ET}_d \to \mathcal{X}$. As $\operatorname{Spec} \kappa(u) \to \operatorname{Spec} \Bbbk$ is finite étale of degree d, the closed immersion $\operatorname{Spec} \kappa(u) \hookrightarrow U_x$ defines a $\Bbbk$-point v of ET_d. This gives a commutative diagram

$$\begin{tikzcd} & & \operatorname{Spec} \Bbbk \arrow[dl, "v"'] \arrow[d, "x"] \\ \mathrm{SEC}_d \arrow[r] & \mathrm{ET}_d \arrow[r] & \mathcal{X}. \end{tikzcd}$$

The point $v \in \mathrm{ET}_d(\Bbbk)$ does not necessarily lift to SEC_d, but we can use the following trick to choose a different quotient presentation of ET_d where v does lift.

Namely, choose a faithful representation $S_d \subseteq \mathrm{GL}_n$ and write $\mathrm{ET}_d \cong [V/\mathrm{GL}_n]$ where $V = \mathrm{SEC}_d \times^{S_d} \mathrm{GL}_n$ is a separated algebraic space. Then $v\colon \operatorname{Spec} \Bbbk \to [V/\mathrm{GL}_n]$ corresponds to a principal GL_n-bundle $P \to \operatorname{Spec} \Bbbk$ and a GL_n-equivariant map $P \to V$. Since principal GL_n-bundles are in bijection to vector bundles (Exercise B.1.56), P is the trivial principal GL_n-bundle and there is a section $\operatorname{Spec} \Bbbk \to P$. The composition $V \to [V/\mathrm{GL}_n] \to \mathcal{X}$ is smooth and the composition $\operatorname{Spec} \Bbbk \to P \to V$ is a lift of x.

As V is a separated algebraic space, we are reduced to showing the second statement, i.e., a $\Bbbk$-point of a Deligne–Mumford stack $\mathcal{X}$ with separated and quasi-compact diagonal lifts to an étale neighborhood by a scheme. We will use the fact that $\mathcal{X}$ has quasi-affine diagonal, which is proven independently later in Corollary 4.5.8 (the proof only uses Proposition 4.3.3 above and the theory of quasi-coherent sheaves in §4.1). We repeat the above argument by choosing an étale map $U \to \mathcal{X}$ from an affine scheme such that the image contains x. Then the space SEC_d of d disjoint sections with respect to $U \to \mathcal{X}$ is a quasi-affine scheme with a free action of S_d. The quotient $\mathrm{ET}_d = \mathrm{SEC}_d / S_d$ is also quasi-affine (Exercise 4.2.14). The induced map $\mathrm{ET}_d \to \mathcal{X}$ is étale and by construction the $\Bbbk$-point x lifts to a $\Bbbk$-point of ET_d. See also [LMB00, Thm. 6.3]. □

There is also a version of the Local Structure Theorem formulated étale locally on the coarse moduli space (see Corollary 4.4.13).

4.4 Coarse moduli spaces and the Keel–Mori Theorem

As for M_g there is virtually no doubt that it can be provided with the structure of an algebraic variety.

André Weil [Wei58, p. 383]

We prove that every separated Deligne–Mumford stack $\mathcal{X}$ of finite type over a noetherian scheme admits a separated coarse moduli space $\pi\colon \mathcal{X} \to X$, which we refer to as the Keel–Mori Theorem (4.4.6). One can view this theorem as a way to remove the stackiness of a Deligne–Mumford stack: at the expense of sacrificing universal properties of $\mathcal{X}$ (e.g., existence of a universal family), one can replace $\mathcal{X}$ with an algebraic space without changing the underlying topological space. Once we introduce the Deligne–Mumford stack $\overline{\mathcal{M}}_g$ of stable curves (Theorem 5.4.14) and show that it is separated (Proposition 5.5.20), the Keel–Mori Theorem implies that there is a coarse moduli space $\overline{\mathcal{M}}_g \to \overline{M}_g$, where $\overline{M}_g$ is a separated algebraic space. Moreover, Semistable Reduction (5.5.9), i.e., the properness of $\overline{\mathcal{M}}_g$, implies further that $\overline{M}_g$ is proper. In §5.8 and §5.9, we give two proofs that $\overline{M}_g$ is projective.

Coarse moduli spaces were introduced in Definition 4.2.1. If G is a finite abstract group acting on an affine scheme $\operatorname{Spec} A$, then $[\operatorname{Spec} A/G] \to \operatorname{Spec} A^G$ is a coarse moduli space (Theorem 4.2.3), or in other words $\operatorname{Spec} A \to \operatorname{Spec} A^G$ is a geometric quotient. To prove the Keel–Mori Theorem (4.4.6), we will apply the Local Structure Theorem (4.3.1) to construct étale neighborhoods $[\operatorname{Spec} A_i/G] \to \mathcal{X}$ and then glue the geometric quotients $\operatorname{Spec} A_i^G$ in the étale topology to construct a coarse moduli space of $\mathcal{X}$.

4.4.1 Descending étale morphisms to geometric quotients

In order to glue geometric quotients in the étale topology, we will need a condition ensuring that an étale G-equivariant morphism $\operatorname{Spec} A \to \operatorname{Spec} B$ induces an étale morphism $\operatorname{Spec} A^G \to \operatorname{Spec} B^G$ on geometric quotients. The following proposition can be viewed as a baby version of Luna's Fundamental Lemma (6.5.28).

Proposition 4.4.1. *Let G be a finite abstract group and $f\colon \operatorname{Spec} A \to \operatorname{Spec} B$ be a G-equivariant morphism of affine schemes of finite type over a noetherian ring R. Let $x \in \operatorname{Spec} A$ be a closed point. Assume that*

(a) f is étale at x and

(b) the induced map $G_x \to G_{f(x)}$ of stabilizer group schemes is bijective.

Then there is an open affine neighborhood $W \subseteq \operatorname{Spec} A^G$ of the image of x such that $W \to \operatorname{Spec} A^G \to \operatorname{Spec} B^G$ is étale and $\pi_A^{-1}(W) \cong W \times_{\operatorname{Spec} B^G} [\operatorname{Spec} B/G]$, where $\pi_A\colon [\operatorname{Spec} A/G] \to \operatorname{Spec} A^G$.

Remark 4.4.2. In other words, after replacing $\operatorname{Spec} A^G$ with an affine neighborhood W of $\pi_A(x)$ and $\operatorname{Spec} A$ with $\pi_A^{-1}(W)$, it can be arranged that the diagram

$$\begin{tikzcd}
{[\operatorname{Spec} A/G]} \arrow[r, "f"] \arrow[d, "\pi_A"] \arrow[dr, phantom, "\square"] & {[\operatorname{Spec} B/G]} \arrow[d, "\pi_B"] \\
\operatorname{Spec} A^G \arrow[r] & \operatorname{Spec} B^G
\end{tikzcd} \tag{4.4.3}$$

is cartesian where both horizontal maps are étale.

Proof of Proposition 4.4.1. Set $y = f(x)$. We first claim that the question is étale local around $\pi_B(y) \in \operatorname{Spec} B^G$. Indeed, if $Y' \to Y := \operatorname{Spec} B^G$ is an affine étale neighborhood of $\pi_B(y)$, we let $X', \mathcal{X}'$ and $\mathcal{Y}'$ denote the base changes of $X := \operatorname{Spec} A^G$, $\mathcal{X} := [\operatorname{Spec} A/G]$, and $\mathcal{Y} := [\operatorname{Spec} B/G]$. By Lemma 4.2.7, we know that $\mathcal{Y}' \cong [\operatorname{Spec} B'/G]$ with $Y' \cong \operatorname{Spec} B'^G$ and similarly for $\mathcal{X}'$ and X'. If the result holds after this base change, there is an open neighborhood $W' \subseteq X'$ containing a preimage of $\pi_A(x)$ such that $W' \hookrightarrow X' \to Y'$ is étale and such that the preimage of W' in $\mathcal{X}'$ is isomorphic to $W' \times_{Y'} \mathcal{Y}'$. Taking W as the image of W' under $X' \to \operatorname{Spec} A^G$ and applying étale descent yields the desired claim.

We now claim that this allows us to assume that B^G is strictly henselian. To see this, define $Y^{\mathrm{sh}} = \operatorname{Spec} \mathcal{O}^{\mathrm{sh}}_{Y,\pi_B(y)}$, and let X^{sh}, $\mathcal{X}^{\mathrm{sh}}$, and $\mathcal{Y}^{\mathrm{sh}}$ be the base changes of X, $\mathcal{X}$, and $\mathcal{Y}$ along $Y^{\mathrm{sh}} \to Y$. We can write $Y^{\mathrm{sh}} = \lim_\lambda Y_\lambda$ as the limit of affine étale neighborhood $Y_\lambda \to Y$ of y, and we set X_λ, $\mathcal{X}_\lambda$, and $\mathcal{Y}_\lambda$ to be the base changes of X, $\mathcal{X}$, and $\mathcal{Y}$ along $Y_\lambda \to Y$. Suppose $U^{\mathrm{sh}} \subseteq X^{\mathrm{sh}}$ is an open affine subscheme of the unique point in X^{sh} over x and the closed point of Y^{sh} such that $U^{\mathrm{sh}} \to Y^{\mathrm{sh}}$ is étale with $\pi^{-1}_{\mathcal{X}^{\mathrm{sh}}}(U^{\mathrm{sh}}) \cong U^{\mathrm{sh}} \times_{Y^{\mathrm{sh}}} \mathcal{Y}^{\mathrm{sh}}$. By Proposition B.3.3, the morphism $U^{\mathrm{sh}} \to X^{\mathrm{sh}}$ descends to $U_\eta \to X_\eta$ for some η. Setting $U_\lambda = U_\eta \times_{X_\eta} X_\lambda$ for $\lambda > \eta$, it follows from Proposition B.3.7 that for $\lambda \gg 0$ (a) $U_\lambda \to X_\lambda$ is an open immersion, (b) the composition $U_\lambda \to X_\lambda \to Y_\lambda$ is étale, and (c) $\pi^{-1}_{\mathcal{X}_\lambda}(U_\lambda) \cong U_\lambda \times_{Y_\lambda} \mathcal{Y}_\lambda$ (by arguing on the étale presentations of $\mathcal{X}$ and $\mathcal{Y}$).

Finally, As $B^G \to B$ is finite (Lemma 4.2.6), $B = B_1 \times \cdots \times B_r$ is a product of strictly henselian local rings (Proposition B.5.10). As in the proof of Theorem 4.2.3, we may replace $[\operatorname{Spec} B/G]$ with $[\operatorname{Spec} B_1/G_y]$ and $[\operatorname{Spec} A/G]$ with $[f^{-1}(\operatorname{Spec} B_1)/G]$ to assume that G fixes x and y while acting trivially on the residue fields $\kappa(x) = \kappa(y)$. Thus $\operatorname{Spec} A \to \operatorname{Spec} B$ has a unique section $s\colon \operatorname{Spec} B \to \operatorname{Spec} A$

taking y to x. The section s is necessarily G-invariant (as in the proof Theorem 4.2.3). Thus s descends to a section of $\operatorname{Spec} A^G \to \operatorname{Spec} B^G$ which gives our desired open and closed subscheme $W \subseteq \operatorname{Spec} A^G$. $\square$

Remark 4.4.4. Here is a conceptual reason why we should expect the induced map of quotients to be étale. For simplicity, assume that $R = \Bbbk$ is an algebraically closed field. Let $\widehat{A}$ and $\widehat{B}$ be the completions of the local rings at x and $f(x)$. The stabilizers G_x and $G_{f(x)}$ act on $\operatorname{Spec} \widehat{A}$ and $\operatorname{Spec} \widehat{B}$, respectively, and the map $\operatorname{Spec} \widehat{A} \to \operatorname{Spec} \widehat{B}$ is equivariant with respect to the map $G_x \to G_{f(x)}$. The completion $\widehat{A^G}$ of A^G at the image of x is isomorphic to $\widehat{A}^{G_x}$ (Exercise 4.2.15) and similarly $\widehat{B^G} = \widehat{B}^{G_{f(x)}}$. Since f is étale at x, $\widehat{B} \to \widehat{A}$ is an isomorphism and since $G_x \to G_{f(x)}$ is bijective, the induced map $\widehat{B^G} \to \widehat{A^G}$ is an isomorphism which shows that $\operatorname{Spec} A^G \to \operatorname{Spec} B^G$ is étale at the image of x.

The above proposition will be applied in the following form in the proof of the Keel–Mori Theorem (4.4.6).

Corollary 4.4.5. *Let G be a finite abstract group and $f\colon \operatorname{Spec} A \to \operatorname{Spec} B$ be a G-equivariant morphism of affine schemes of finite type over a noetherian ring R. Assume that for every closed point $x \in \operatorname{Spec} A$,*

(a) f is étale at x and

(b) the induced map $G_x \to G_{f(x)}$ of stabilizer group schemes is bijective.

Then $\operatorname{Spec} A^G \to \operatorname{Spec} B^G$ is étale and (4.4.3) is cartesian. $\square$

4.4.2 The Keel–Mori Theorem

We now state and prove the Keel–Mori Theorem.

Theorem 4.4.6 (Keel–Mori Theorem)**.** *Let $\mathcal{X}$ be a Deligne-Mumford stack separated and of finite type over a noetherian algebraic space S. Then there exists a coarse moduli space $\pi\colon \mathcal{X} \to X$ with $\mathcal{O}_X = \pi_*\mathcal{O}_{\mathcal{X}}$ such that*

(1) X is separated and of finite type over S,

(2) π is a proper universal homeomorphism, and

(3) for every flat morphism $X' \to X$ of algebraic spaces, the base change $\mathcal{X} \times_X X' \to X'$ is a coarse moduli space.

Moreover, X is proper if and only if $\mathcal{X}$ is.

This theorem extends to (possibly non-separated) Deligne–Mumford stacks $\mathcal{X}$ such that $I_{\mathcal{X}} \to \mathcal{X}$ is finite (Exercise 4.4.10), except that the coarse moduli space may not be separated. More impressively, the Keel–Mori Theorem [KM97] also holds if we replace the hypothesis that $\mathcal{X}$ is Deligne–Mumford with the hypothesis that $\mathcal{X}$ is an algebraic stack with finite inertia $I_{\mathcal{X}} \to \mathcal{X}$; see Remark 4.4.11. This requires all stabilizers to be finite but not necessarily reduced. The theorem also holds without any noetherian or finiteness conditions; see [Con05b], [Ryd13], and [SP, Tag 0DUK].

Remark 4.4.7 (Historical remarks). Around roughly the same time as [KM97] was published, Kollár proved an important special case, which is sufficient for many applications: if G is an affine group scheme over an excellent schemes S acting acting properly on a separated algebraic space U over S such that either $G \to S$ is a reductive group scheme or $S = \operatorname{Spec} \Bbbk$ with $\operatorname{char}(\Bbbk) > 0$, then there exists a separated geometric quotient U/G [Kol97, Thm. 1.5]. Before these articles, the existence of a coarse moduli spaces was viewed as a folklore theorem, but its statement and proof for separated Deligne–Mumford stacks was surely known to experts. For instance, the theorem was was explicitly formulated (but not proven) in [FC90, Thm. I.4.10] and [Kol90, Thm. 2.2]. An arguments using analytic techniques was sketched by Popp in the setting of a group action over $\mathbb{C}$ in [Pop74, Thm. p. 55], and both analytic and algebraic arguments were sketched in [MF82, §5.A].

Remark 4.4.8. We emphasize that this theorem not only produces a coarse moduli space, but that it satisfies properties (1)-(3). It can be difficult to work with the notion of a coarse moduli space that only satisfies the defining properties (Definition 4.2.1). For instance, it is not clear in general that coarse moduli spaces are stable under étale base change (or even open immersions) or that $\pi_* \mathcal{O}_{\mathcal{X}} = \mathcal{O}_X$.

Proof. We first handle the case when $S = \operatorname{Spec} R$ is affine. The question is Zariski-local on $\mathcal{X}$: if $\{\mathcal{X}_i\}$ is a Zariski open covering of $\mathcal{X}$ with coarse moduli spaces $\mathcal{X}_i \to X_i$, then since coarse moduli spaces are unique (Definition 4.2.1(2)), each intersection $\mathcal{X}_i \cap \mathcal{X}_j$ admits a coarse moduli space U_{ij} identified with open subschemes of both X_i and X_j, we may glue to form an algebraic space X and a map $\mathcal{X} \to X$, which is a coarse moduli space by Lemma 4.2.10. It thus suffices to show that every closed point $x \in |\mathcal{X}|$ has an open neighborhood that admits a coarse moduli space.

By the Local Structure Theorem of Deligne–Mumford Stacks (4.3.1), there exists an affine étale morphism

$$f \colon \big(\mathcal{W} = [\operatorname{Spec} A/G_x], w\big) \to (\mathcal{X}, x)$$

such that f induces an isomorphism of geometric stabilizer groups at w. We claim that the locus $\mathcal{U}$ consisting of points $z \in |\mathcal{W}|$, such that f induces an isomorphism of geometric stabilizer groups at z, is open. To establish this, we will exploit that the relative inertia $I_{\mathcal{X}} \to \mathcal{X}$ is finite, which follows from the separatedness of $\mathcal{X}$ (as the diagonal is both proper and quasi-finite). The fiber of the natural morphism $I_{\mathcal{W}} \to I_{\mathcal{X}} \times_{\mathcal{X}} \mathcal{W}$ of relative group schemes over $\mathcal{W}$ over $z \in \mathcal{W}(\Bbbk)$ is precisely the morphism $G_z \to G_{f(z)}$ of stabilizers. We will use the cartesian diagram

$$\begin{tikzcd} I_{\mathcal{W}} \arrow[r, "\Psi"] \arrow[d] \arrow[dr, phantom, "\square"] & I_{\mathcal{X}} \times_{\mathcal{X}} \mathcal{W} \arrow[d] \\ \mathcal{W} \arrow[r] & \mathcal{W} \times_{\mathcal{X}} \mathcal{W} \end{tikzcd}$$

from Exercise 3.2.15. Since $\mathcal{W} \to \mathcal{X}$ is representable, étale, and separated, the diagonal $\mathcal{W} \to \mathcal{W} \times_{\mathcal{X}} \mathcal{W}$ is an open and closed immersion and thus so is Ψ. Since $I_{\mathcal{X}} \to \mathcal{X}$ is finite, so is $p_2 \colon I_{\mathcal{X}} \times_{\mathcal{X}} \mathcal{W} \to \mathcal{W}$. Thus $p_2(|I_{\mathcal{X}} \times_{\mathcal{X}} \mathcal{W}| \smallsetminus |I_{\mathcal{W}}|) \subseteq |\mathcal{W}|$ is closed and its complement, which is identified with the locus $\mathcal{U}$, is open. Let $\pi_{\mathcal{W}} \colon \mathcal{W} \to W = \operatorname{Spec} A^{G_x}$ be the coarse moduli space (Theorem 4.2.3). Choose an affine open subscheme $X_1 \subseteq \pi(\mathcal{U})$ containing $\pi_{\mathcal{W}}(w)$. Then $\mathcal{X}_1 = \pi_{\mathcal{W}}^{-1}(X_1)$ is isomorphic to a quotient stack $[\operatorname{Spec} A_1/G_x]$ such that $X_1 = \operatorname{Spec} A_1^{G_x}$. This provides an affine étale morphism

$$g \colon (\mathcal{X}_1 = [\operatorname{Spec} A_1/G_x], w) \to (\mathcal{X}, x)$$

which induces a bijection on all geometric stabilizer groups.

We now show that the open substack $\mathcal{X}_0 := \operatorname{im}(f)$ admits a coarse moduli space. Since g is affine and $\mathcal{X}_1 = [\operatorname{Spec} A_1/G_x]$, we can express $\mathcal{X}_2 := \mathcal{X}_1 \times_{\mathcal{X}} \mathcal{X}_1$ as $[\operatorname{Spec} A_2/G_x]$, and there is a coarse moduli space $\pi_2 \colon \mathcal{X}_2 \to X_2 = \operatorname{Spec} A_2^{G_x}$. By universality of coarse moduli spaces, there is a diagram

$$
\begin{tikzcd}
\mathcal{X}_2 \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "\pi_2"] \arrow[dr, phantom, "\square"] & \mathcal{X}_1 \arrow[r, "g"] \arrow[d, "\pi_1"] & \mathcal{X}_0 = \operatorname{im}(g) \arrow[d, dashed, "\pi_0"] \\
X_2 \arrow[r, shift left] \arrow[r, shift right] & X_1 \arrow[r, dashed] & X_0
\end{tikzcd}
\tag{4.4.9}
$$

where the natural squares commute. Since g induces bijections of geometric stabilizer groups at all points, the same is true for each projection $\mathcal{X}_2 \to \mathcal{X}_1$. Corollary 4.4.5 implies that each map $X_2 \to X_1$ is étale, and the natural squares of solid arrows in (4.4.9) are cartesian.

The universality of coarse moduli spaces induces an étale groupoid structure $X_2 \rightrightarrows X_1$. To check that this is an étale equivalence relation, it suffices to check that $X_2 \to X_1 \times X_1$ is injective on geometric points, but this follows from the observation the $|\mathcal{X}_2| \to |\mathcal{X}_1| \times |\mathcal{X}_1|$ is injective on closed points. Therefore there is an algebraic space quotient $X_0 := X_1/X_2$ and a map $X_1 \to X_0$. By étale descent along $\mathcal{X}_1 \to \mathcal{X}_0$, there is a map $\pi_0 \colon \mathcal{X}_0 \to X_0$ making the right square in (4.4.9) commute.

To argue that $\pi \colon \mathcal{X}_0 \to X_0$ is a coarse moduli space, we will use the commutative cube

$$
\begin{tikzcd}[row sep=small, column sep=small]
& \mathcal{X}_2 \arrow[rr] \arrow[dl] \arrow[dd] & & \mathcal{X}_1 \arrow[dl] \arrow[dd] \\
\mathcal{X}_1 \arrow[rr, crossing over] \arrow[dd] & & \mathcal{X}_0 & \\
& X_2 \arrow[rr] \arrow[dl] & & X_1 \arrow[dl] \\
X_1 \arrow[rr] & & X_0, \arrow[from=uu, crossing over] &
\end{tikzcd}
$$

where the top, left, and bottom faces are cartesian. Since $\mathcal{X}_1 \to \mathcal{X}_0$ is étale and surjective, and the natural map $\mathcal{X}_1 \to \mathcal{X}_0 \times_{X_0} X_1$ pulls back to an isomorphism, $\mathcal{X}_1 \cong \mathcal{X}_0 \times_{X_0} X_1$ and the right square is also cartesian. Since being a coarse moduli space is étale local on X_0 (Lemma 4.2.10), we conclude that $\pi_0 \colon \mathcal{X}_0 \to X_0$ is a coarse moduli space. The condition that $\mathcal{O}_{X_0} \to \pi_{0,*}\mathcal{O}_{\mathcal{X}_0}$ and each of the properties (1)–(3) are also étale local and since they hold for $[\operatorname{Spec} A_1/G_x] \to \operatorname{Spec} A_1^{G_x}$ by Theorem 4.2.3, they also hold for $\mathcal{X}_0 \to X_0$.

The case when S is a noetherian algebraic space can be reduced to the affine case by imitating the above argument to étale locally construct the coarse moduli space of $\mathcal{X}$. Finally, since $\mathcal{X}_0 \to X_0$ is proper, the separatedness of $\mathcal{X}_0$ is equivalent to the separatedness of X_0. $\square$

Exercise 4.4.10. If $\mathcal{X}$ is a Deligne–Mumford stack of finite type over a noetherian algebraic space S such that $I_{\mathcal{X}} \to \mathcal{X}$ is finite. Then there exists a coarse moduli space $\pi \colon \mathcal{X} \to X$ with $\mathcal{O}_X = \pi_*\mathcal{O}_{\mathcal{X}}$ satisfying properties (1)–(3) of Theorem 4.4.6.

Remark 4.4.11. The more general case when $\mathcal{X}$ is an algebraic stack with finite inertia $I_{\mathcal{X}} \to \mathcal{X}$ (but not necessarily Deligne–Mumford) is proven in an analogous but more technical manner:

Step 1: Construct a quasi-finite and flat surjection $U \to \mathcal{X}$ from a scheme. This is proven as in the Existence of Minimal Presentations (3.6.1); see Remark 3.6.2, [$\mathrm{SGA3_I}$, Lem. V.7.2], [KM97, Lem. 3.3], [Con05b, Lem. 2.1], and [SP, Tag 06MC].

Step 2: Show that every point has an étale neighborhood $\mathcal{W} \to \mathcal{X}$ where $\mathcal{W}$ admits a finite flat presentation $V \to \mathcal{W}$ from an affine scheme. This should be viewed as a replacement to the Local Structure Theorem for Deligne–Mumford Stacks (4.3.1), where instead of $\mathcal{W}$ being the quotient $[\operatorname{Spec} A/G]$ by a finite group, $\mathcal{W}$ is the quotient $[V/R]$ of a finite flat groupoid $R := V \times_{\mathcal{W}} V \rightrightarrows V$ of affine schemes. The proof is also similar: instead of using $\mathrm{ET}_d := [(U \times_{\mathcal{X}} \cdots \times_{\mathcal{X}} U \setminus \Delta)/S_d]$ to parameterize d unordered points in fibres of $U \to \mathcal{X}$, one uses a relative Hilbert stack $\mathcal{H}\mathrm{ilb}_d$ over $\mathcal{X}$, whose objects over $S \to \mathcal{X}$ consist of a closed subscheme $Z \hookrightarrow U \times_{\mathcal{X}} S$ finite and flat (rather than finite and étale) of degree d over S. See [MF82, §5.A], [KM97, Prop. 4.2], [Con05b, Lem. 2.2], and [SP, Tag 0DU4].

Step 3: Prove the existence of groupoids for the quotient $[R/V]$ of a finite flat groupoid $R \rightrightarrows V$ of affine schemes. This is proven analogously to Theorem 4.2.3; see Exercise 4.2.16, [SGA3$_{\mathrm{I}}$, Thm. V.4.1], [KM97, Prop. 5.1], [Con05b, Thm. 3.1], [Ols16, Thm. 6.2.2], and [SP, Tag 0DUR]

Corollary 4.4.12. *If G is an affine fppf group schemes over a noetherian scheme S acting properly on a separated algebraic space U, i.e., the action map $G \times U \to U \times U$ is proper, such that all stabilizer groups are reduced, then there exists a geometric quotient $\pi \colon U \to X$ to an algebraic space X separated and finite type over S such that $\mathcal{O}_X = (\pi_* \mathcal{O}_U)^G$.*

Proof. Because G is affine over S, $\sigma \colon G \times U \to U \times U$ is affine. Hence, the properness of σ is equivalent to the finiteness of U, and it implies that $[U/G]$ has finite diagonal. Since the stabilizer groups are reduced, we see that $[U/G]$ is a separated Deligne–Mumford stack so we may apply the Keel–Mori Theorem (4.4.6). $\square$

The construction of the coarse moduli space in the proof provides a local structure theorem of coarse moduli spaces. (Alteratively, the following theorem also follows directly from the Local Structure Theorem (4.3.1) and Exercise 4.4.14.)

Corollary 4.4.13 (Local Structure of Coarse Moduli Spaces)**.** *Let $\mathcal{X}$ be a Deligne–Mumford stack of finite type and separated over a noetherian algebraic space S, and let $\pi \colon \mathcal{X} \to X$ be its coarse moduli space. For every closed point $x \in |\mathcal{X}|$ with geometric stabilizer group G_x, there exists a cartesian diagram*

$$\begin{tikzcd}
{[\operatorname{Spec} A/G_x]} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{X} \arrow[d, "\pi"] \\
\operatorname{Spec} A^{G_x} \arrow[r] & X
\end{tikzcd}$$

such that $\operatorname{Spec} A^{G_x} \to X$ *is an étale neighborhood of* $\pi(x) \in |X|$. $\square$

Exercise 4.4.14. Establish the following generalization of Proposition 4.4.1: if $f \colon \mathcal{X} \to \mathcal{Y}$ is a morphism of Deligne–Mumford stacks separated and of finite type over a noetherian algebraic space S fitting in a commutative diagram

$$\begin{tikzcd}
\mathcal{X} \arrow[r, "f"] \arrow[d, "\pi_{\mathcal{X}}"] & \mathcal{Y} \arrow[d, "\pi_{\mathcal{Y}}"] \\
X \arrow[r] & Y
\end{tikzcd}$$

where $\pi_{\mathcal{X}} \colon \mathcal{X} \to X$ and $\pi_{\mathcal{Y}} \colon \mathcal{Y} \to Y$ are coarse moduli spaces and $x \in |\mathcal{X}|$ is a closed point such that

(1) f is étale at x and

(2) the induced map $G_x \to G_{f(x)}$ of geometric stabilizer groups is bijective,

then there exists an open neighborhood $U \subseteq X$ of $\pi_{\mathcal{X}}(x)$ such that $U \to X \to Y$ is étale and $\pi_{\mathcal{X}}(U) \cong U \times_Y \mathcal{Y}$.

4.4.3 Examples

The basic example of a coarse moduli space is the map $[\operatorname{Spec} A/G] \to \operatorname{Spec} A^G$ induced by an action of a finite group; see Example 4.2.4 for some elementary examples.

Example 4.4.15 (Elliptic curves)**.** Consider the moduli stack $\mathcal{M}_{1,1} \cong [(\mathbb{A}^2 \smallsetminus V(\Delta))/\mathbb{G}_m]$ of elliptic curves from Exercise 3.1.19, where $\Bbbk$ is a field with $\operatorname{char}(\Bbbk) \neq 2, 3$, $\mathbb{A}^2$ has coordinates a and b with the action given by $t \cdot (a,b) = (t^4a, t^6b)$, and $\Delta = 4a^3 + 27b^2$ is the discriminant. The rings of invariants $\Bbbk[a,b]_\Delta^{\mathbb{G}_m}$ is $\Bbbk[a^3/\Delta]$, where $\beta := b^2/\Delta$ is generated by $\alpha := a^3/\Delta$ under the relation $4\alpha + 27\beta = 1$. Therefore, the coarse moduli space $\mathcal{M}_{1,1} \to \mathbb{A}^1$ is given by $(a,b) \mapsto a^3/\Delta$. An elliptic curve E_λ expressed by its Legendre equation $y^2z = x(x-z)(x-\lambda z)$ in $\mathbb{P}^2$ for $\lambda \in \Bbbk$ is projectively equivalent to $y^2z = x^3 + a_\lambda xz^2 + b_\lambda z^3$ for some $a_\lambda, b_\lambda \in \Bbbk$, and one can check that $a_\lambda^3/\Delta_\lambda$ is a scalar multiple of the j-invariant $2^8\frac{(\lambda^2-\lambda+1)^3}{\lambda^2(\lambda-1)^3}$. Alternatively, using the quotient presentation $\mathcal{M}_{1,1} \cong [V(\Delta - 1)/\mu_{12}]$ from Exercise 3.4.21, the coarse moduli space is again given by $(a,b) \mapsto a^3/\Delta$.

Exercise 4.4.16. (hard) Let $\operatorname{char}(\Bbbk) \neq 2$ and $G = \mathbb{Z}/2$.

(a) Let G act on the non-separated union $X = \mathbb{A}^1 \bigcup_{x\neq 0} \mathbb{A}^1$ by exchanging the copies of $\mathbb{A}^1$. The quotient $[X/G]$ is a Deligne–Mumford stack with quasi-finite but not finite inertia, and in particular non-separated. Show nevertheless that there is a coarse moduli space $[X/G] \to \mathbb{A}^1$.

(b) Let X be the non-separated union $\mathbb{A}^2 \bigcup_{x\neq 0} \mathbb{A}^2$. Let $G = \mathbb{Z}/2$ act on X by simultaneously exchanging the copies of $\mathbb{A}^2$ and by acting via the involution $y \mapsto -y$ on each copy. Show that $[X/G]$ does not admit a coarse moduli space.

Example 4.4.17. Consider the action of PGL_2 on the scheme $\operatorname{Sym}^4 \mathbb{P}^1 \cong (\mathbb{P}^1)^4/S_4$ (which is the coarse moduli space of $[(\mathbb{P}^1)^4/S_4]$) parameterizing four unordered points in $\mathbb{P}^1$. Let $\mathcal{X} \subseteq [\operatorname{Sym}^4 \mathbb{P}^1/\mathrm{PGL}_2]$ be the open substack parameterizing tuples (p_1,p_2,p_3,p_4) where at least three points are distinct. Consider the family $(0,1,\lambda,\infty)$ with $\lambda \in \mathbb{P}^1$. If $\lambda \notin \{0, 1\infty\}$, then we claim that $\operatorname{Aut}(0,1,\lambda,\infty) = \mathbb{Z}/2 \times \mathbb{Z}/2$. To see this, there is a unique element $\sigma \in \mathrm{PGL}_2$ such that $\sigma(0) = \infty$, $\sigma(\infty) = 0$ and $\sigma(1) = \lambda$ which acts on $\mathbb{P}^1$ via $\sigma([x,y]) = [y, \lambda, x]$ and thus $\sigma(\lambda) = 1$. Similarly, there is an element interchanging 0 with 1 and λ with ∞ and an element interchanging 0 with λ and 1 with ∞. However, if $\lambda \in \{0, 1\infty\}$, then $\operatorname{Aut}(0,1,\lambda,\infty) = \mathbb{Z}/2$. We therefore see that the inertia $I_{\mathcal{X}} \to \mathcal{X}$ while quasi-finite is not finite and that $\mathcal{X}$ is not separated. Nevertheless, the map $\mathcal{X} \to \mathbb{P}^1$ taking (p_1,p_2,p_3,p_4) to its cross-ratio is a coarse moduli space.

Exercise 4.4.18. Let $\pi\colon \mathcal{X} \to X$ be the coarse moduli space of a separated and finite type Deligne–Mumford stack $\mathcal{X}$ over a noetherian affine scheme.

Exercise 4.4.19. Let $\mathcal{X}$ be a Deligne–Mumford stack of finite type and separated over a noetherian algebraic space S, and let $\pi\colon \mathcal{X} \to X$ be its coarse moduli space.

(a) If $x \in |\mathcal{X}|$ is a finite type point, show that the residual gerbe $\mathcal{G}_x$ of x is identified with the reduction of the fiber $\mathcal{X}_{\pi(x)}$, and provide an example where the fiber $\mathcal{X}_{\pi(x)}$ is non-reduced.

(b) Show that if $\mathcal{X}$ is normal, then so is X.

(c) If $\mathcal{X}$ is regular, show that $\mathcal{X} \to X$ is flat if and only if X is regular.

(d) Provide an example of a coarse moduli space $\mathcal{X} \to X$ that is *not* flat.

4.4.4 Tame coarse moduli spaces

Definition 4.4.20. A Deligne–Mumford stack $\mathcal{X}$ is *tame* if for every geometric point $x \in \mathcal{X}(\Bbbk)$, the order of $\mathrm{Aut}_{\mathcal{X}(\Bbbk)}(x)$ is invertible in $\Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$. If in addition $\mathcal{X}$ admits a coarse moduli space $\mathcal{X} \to X$, then we say that $\mathcal{X} \to X$ is a *tame coarse moduli space.*

Remark 4.4.21. If $\mathcal{X}$ is defined over a field $\Bbbk$, then this means that the order of every geometric stabilizer group is prime to the characteristic of $\Bbbk$.

Lemma 4.4.22. *Let $\mathcal{X}$ be a Deligne-Mumford stack separated and of finite type over a noetherian algebraic space S. If $\pi\colon \mathcal{X} \to X$ is a tame coarse moduli space, then π_* is exact.*

Proof. The question is étale-local on X: if $g\colon X' \to X$ is an étale cover inducing a cartesian diagram

$$\begin{tikzcd} \mathcal{X}' \arrow[r, "g'"] \arrow[d, "\pi'"'] \arrow[dr, phantom, "\square"] & \mathcal{X} \arrow[d, "\pi"] \\ X' \arrow[r, "g"] & X, \end{tikzcd}$$

then by Flat Base Change (4.1.45), there is an identification $g^*\pi_* = \pi'_* g'^*$ of functors on quasi-coherent sheaves. Since g^* is faithfully exact, we see that π_* is exact if and only π'_* is. We can therefore use the Local Structure Theorem (4.4.13) to reduce to the case that $\mathcal{X} = [\operatorname{Spec} A/G]$ and $X = \operatorname{Spec} A^G$, where the exactness of π_* follows from its interpretation in terms of taking invariants (Exercise 4.1.27) and the linear reductivity of G, i.e., Maschke's Theorem (B.1.38). $\square$

Corollary 4.4.23. *Let $\mathcal{X}$ be a Deligne-Mumford stack separated and of finite type over a noetherian algebraic space S, and $\pi\colon \mathcal{X} \to X$ its coarse moduli space. If F is a quasi-coherent sheaf on $\mathcal{X}$, then $\mathrm{H}^i(\mathcal{X}, F) = \mathrm{H}^i(X, F)$ and is 0 if $i > \dim \mathcal{X}$.*

Proof. The first statement follows from the exactness of π_* and the second from Exercise 4.1.50. $\square$

Exercise 4.4.24 (Base change). Let $\mathcal{X}$ be a Deligne–Mumford stack of finite type and separated over a noetherian algebraic space S. If $\pi\colon \mathcal{X} \to X$ is a *tame* coarse moduli space and $X' \to X$ is a morphism of algebraic spaces, show that $\mathcal{X} \times_X X' \to X'$ is also a coarse moduli space.

On the other hand, Exercise 4.2.8(c) provides an example of a coarse moduli space $\mathcal{X} \to X$ and a map $X' \to X$ such that $\mathcal{X} \times_X X' \to X'$ is *not* a coarse moduli space. In particular, it is does not automatically follow that $\overline{M}_g \times_{\mathbb{Z}} \mathbb{F}_p$ is the coarse moduli space of $\overline{\mathcal{M}}_g \times_{\mathbb{Z}} \mathbb{F}_p$; see Question 5.5.25.

Remark 4.4.25 (Canonical stacks and a bottom-up classification of coarse moduli spaces). The coarse moduli space X of a smooth separated tame Deligne–Mumford stack over a field $\Bbbk$. $\mathcal{X}$ has *tame quotient singularities*, i.e., every point of X has an étale neighborhood which is a geometric quotient V/G of a finite group of order relatively prime to $\mathrm{char}(\Bbbk)$ acting on a smooth scheme V. This follows from the Local Structure of Coarse Moduli Spaces (4.4.13). Conversely, Vistoli showed for every algebraic space X with tame quotient stacks, there is a canonical smooth separated tame Deligne–Mumford stack X^{can} with coarse moduli space X [Vis89, Prop. 2.8]. There is also a relative canonical stack $\mathcal{X}^{\mathrm{can}} \to \mathcal{X}$ for any separated Deligne–Mumford with tame quotient singularities, and it is a theorem of Geraschenko and Satriano that every smooth separated tame Deligne–Mumford stack $\mathcal{X}$ with generically trivial stabilizer can be recovered from its coarse moduli space X as the composition

$$\mathcal{X} \cong (X^{\mathrm{can}}(\sqrt{D}))^{\mathrm{can}} \to X^{\mathrm{can}}(\sqrt{D}) \to X^{\mathrm{can}} \to X,$$

where $X^{\mathrm{can}}(\sqrt{D}) \to X^{\mathrm{can}}$ is a composition of root stacks (Example 3.9.22) along the irreducible components of the ramification divisor of $\mathcal{X} \to X$ where each order is the ramification degree [GS15].

4.4.5 Descending vector bundles to the coarse moduli space

We begin with a Nakayama lemma for coherent sheaves.

Lemma 4.4.26. *Let $\mathcal{X}$ be a Deligne-Mumford stack separated and of finite type over a noetherian algebraic space S, and let $\pi\colon \mathcal{X} \to X$ be its coarse moduli space. Let $x \in |\mathcal{X}|$ be a closed point and $\mathcal{G}_x$ be its residual gerbe.*

(1) If F is a coherent sheaf on $\mathcal{X}$ such that $F|_{\mathcal{G}_x} = 0$, then there exists an open neighborhood $U \subseteq X$ of $\pi(x)$ such that $F|_{\pi^{-1}(U)} = 0$.

(2) If $\phi\colon F \to G$ is a morphism of coherent sheaves (resp., vector bundles of the same rank) on $\mathcal{X}$ such that $\phi|_{\mathcal{G}_x}$ is surjective, then there exists an open neighborhood $U \subseteq X$ of $\pi(x)$ such that $\phi|_{\pi^{-1}(U)}$ is surjective (resp., an isomorphism).

Proof. For (1), the support $\mathrm{Supp}(F) \subseteq |\mathcal{X}|$ of F is a closed subset (which follows from using descent along a presentation) and the open set $U = X \smallsetminus \pi(\mathrm{Supp}(F))$ satisfies the conclusion. For (2), apply (1) to the coherent sheaf $\mathrm{coker}(\phi)$ noting that a surjection of vector bundles of the same rank is an isomorphism. □

We say that a vector bundle F on $\mathcal{X}$ *descends to its coarse moduli space* $\pi\colon \mathcal{X} \to X$ if there exists a vector bundle G on X and an isomorphism $F \cong \pi^*G$. Observe that one necessary condition is that for every geometric point $x\colon \operatorname{Spec} \Bbbk \to \mathcal{X}$ and commutative diagram

$$\begin{tikzcd} BG_x \arrow[r, hook, "i_x"] \arrow[d, "p"'] & \mathcal{X} \arrow[d, "\pi"] \\ \operatorname{Spec} \Bbbk \arrow[r, hook, "j"] & X, \end{tikzcd}$$

the pullback $i_x^*F = i_x^*\pi^*G = p^*j^*G$ is trivial or, in other words ,that G_x acts trivially on the fiber $F \otimes \Bbbk := i_x^*F$. This is equivalent to requiring that the restriciton $F|_{\mathcal{G}_x}$ to the residual gerbe of x is trivial.

Proposition 4.4.27. *Let $\mathcal{X}$ be a tame Deligne-Mumford stack separated and of finite type over a noetherian algebraic space S, and let $\pi\colon \mathcal{X} \to X$ be its coarse moduli space. A vector bundle F on $\mathcal{X}$ descends to a vector bundle on X if and only if for every geometric point $x\colon \operatorname{Spec} \Bbbk \to \mathcal{X}$ with closed image, the action of G_x on the fiber $F \otimes \Bbbk$ is trivial. In this case, $\pi_* F$ is a vector bundle and the adjunction map $\pi^*\pi_* F \to F$ is an isomorphism.*

Proof. To see that the condition is sufficient, consider the commutative diagram

$$\begin{tikzcd} \mathcal{G}_x \arrow[r, hook] \arrow[d, "p"'] & \mathcal{X} \arrow[d, "\pi"] \\ \operatorname{Spec} \kappa(x) \arrow[r, hook] & X \end{tikzcd}$$

where $\mathcal{G}_x$ is the residual gerbe. We break down the proof into three steps.

Step 1: $\pi^*\pi_*F \to F$ is surjective. It suffices by Lemma 4.4.26 to show that $(\pi^*\pi_*F)|_{\mathcal{G}_x} \to F|_{\mathcal{G}_x}$ is surjective for every closed point $x \in |\mathcal{X}|$. Since $F \to F|_{\mathcal{G}_x}$ is surjective and π_* is exact (Lemma 4.4.22), $\pi_*F \to \pi_*(F|_{\mathcal{G}_x})$ is surjective and thus $(\pi^*\pi_*F)|_{\mathcal{G}_x} \to \pi^*(\pi_*(F|_{\mathcal{G}_x}))|_{\mathcal{G}_x} \cong p^*p_*(F|_{\mathcal{G}_x})$ is also surjective. The hypotheses imply that the adjunction $p^*p_*(F|_{\mathcal{G}_x}) \to F|_{\mathcal{G}_x}$ is an isomorphism, and it follows that $(\pi^*\pi_*F)|_{\mathcal{G}_x} \to p^*p_*(F|_{\mathcal{G}_x}) \cong F|_{\mathcal{G}_x}$ is surjective.

Step 2: π_*F is a vector bundle. We can assume that the rank r of F is constant. Since being a vector bundle is an étale-local property, we can assume that $X = \operatorname{Spec} A$. The surjection $\bigoplus_{s \in \Gamma(X, \pi_*F)} A \to \pi_*F$ pulls back to a surjection $\bigoplus_{s \in \Gamma(\mathcal{X}, F)} \mathcal{O}_{\mathcal{X}} \to \pi^*\pi_*F$ and by Step 1, the composition $\bigoplus_{s \in \Gamma(\mathcal{X}, F)} \mathcal{O}_{\mathcal{X}} \to \pi^*\pi_*F \to F$ is surjective. As $F|_{\mathcal{G}_x} \cong \mathcal{O}_{\mathcal{G}_x}^r$ is trivial, for each closed point $x \in |\mathcal{X}|$, we can find r sections $\phi\colon \mathcal{O}_{\mathcal{X}}^{\oplus r} \to F$ such that $\phi|_{\mathcal{G}_x}$ is an isomorphism. By Lemma 4.4.26, there exists an open neighborhood $U \subseteq X$ of $\pi(x)$ such that $\phi|_{\pi^{-1}(U)}$ is an isomorphism. Thus $\pi_*\phi\colon \mathcal{O}_X^{\oplus r} \to \pi_*F$ is an isomorphism over U and we conclude that π_*F is a vector bundle of the same rank as F.

Step 3: $\pi^*\pi_*F \to F$ is an isomorphism. Since $\pi^*\pi_*F \to F$ is a surjection of vector bundles of the same rank, it is an isomorphism. $\square$

Remark 4.4.28. The analogous statement for coherent sheaves is not true. For example, if the characteristic is not 2, then letting $\mathbb{Z}/2$ on $\mathbb{A}^1$ via $x \mapsto -x$, we have a tame coarse moduli space $[\mathbb{A}^1/(\mathbb{Z}/2)] \to \mathbb{A}^1 = \operatorname{Spec} \Bbbk[x^2]$. The inclusion $B(\mathbb{Z}/2) \hookrightarrow [\mathbb{A}^1/(\mathbb{Z}/2)]$ of 0 is a closed substack and $\mathcal{O}_{B(\mathbb{Z}/2)}$ is a coherent sheaf which does not descend.

When $\mathcal{X}$ is not tame, we have the following variant for descending line bundles.

Exercise 4.4.29. Let $\mathcal{X}$ be a Deligne-Mumford stack separated and of finite type over a noetherian algebraic space S, and let $\pi\colon \mathcal{X} \to X$ be its coarse moduli space. Let L be a line bundle on $\mathcal{X}$.

(1) Show that $L^{\otimes N}$ descends to X for N sufficiently divisible.

Hint: Prove that for N sufficiently divisible $\pi_*(L^{\otimes N})$ is a line bundle and the adjunction map $\pi^*\pi_*(L^{\otimes N}) \to L^{\otimes N}$ is an isomorphism. Reduce to the case that $\mathcal{X} = [U/G]$ such that L pulls back to the trivial line bundle on U, and use the spectral sequence $E_{pq}^2 = \mathrm{H}^p(BG, \mathrm{H}^q(U, \mathbb{G}_m)) \Rightarrow \mathrm{H}^{p+q}([U/G], \mathbb{G}_m)$ induced by the composition $[U/G] \to BG \to \operatorname{Spec} \mathbb{Z}$.

(2) Suppose that $\mathcal{X}$ is proper and representable over a Deligne–Mumford stack $\mathcal{Y}$ of finite type over S. If L is relatively ample over $\mathcal{Y}$, show that for N sufficiently divisible $L^{\otimes N}$ descends to a line bundle relatively ample over the coarse moduli space Y of $\mathcal{Y}$.

(Ampleness is an étale local property on the base, and we say that a line bundle L on X is relatively ample over Y if $X \to Y$ is representable by schemes and there exists an étale presentation $V \to Y$ such that the restriction L_V to X_V is relatively ample.)

4.5 Algebraic spaces versus schemes

But who can quantify
the algebra of space,
or weigh those worlds that swim
each in its place?
Who can outdo the dark?
And what computer knows
how beauty comes to birth -
shell star and rose?

Jean Kenward, *Technicians*

We prove various results providing conditions for an algebraic space to be a scheme:

- a quasi-separated algebraic space is a scheme on a dense open subspace (Theorem 4.5.1);
- Zariski's Main Theorem for algebraic spaces: a representable, quasi-finite, and separated morphism factors as a composition of an open immersion and a finite morphism (Theorem 4.5.9). This has some nice consequences:
  - an algebraic space locally quasi-finite and separated over a scheme is a scheme (Corollary 4.5.7),
  - if the diagonal of a Deligne–Mumford stack is separated and quasi-compact, then the diagonal is quasi-affine (Corollary 4.5.8) and in particular representable by schemes, and
  - an algebraic stack with trivial stabilizers is an algebraic space (Theorem 4.5.10) generalizing Theorem 3.6.6;
- Serre's and Chevalley's Criteria for Affineness (Theorems 4.5.16 and 4.5.20) for algebraic spaces;
- if X is a quasi-separated algebraic space locally of finite type over a field $\Bbbk$ such that every finite set of points in $X_{\overline{\Bbbk}}$ is contained in an affine (e.g., $X_{\overline{\Bbbk}}$ is quasi-projective), then X is a scheme (Proposition 4.5.27);
- a quasi-separated group algebraic space locally of finite type over a field is a scheme (Theorem 4.5.28); and
- separated one dimensional algebraic spaces are schemes (Theorem 4.5.32)

4.5.1 Algebraic spaces are schemes over a dense open

Theorem 4.5.1. *Every quasi-separated algebraic space has a dense open subspace which is a scheme.*

Proof. We may assume that X is a quasi-compact and quasi-separated algebraic space. Let $f: V \to X$ be an étale presentation with V an affine scheme. Since X is quasi-separated, $f: V \to X$ is quasi-compact and there exists an open algebraic subspace $U \subseteq X$ such that $f^{-1}(U) \to U$ is finite. By Exercise 4.3.2, U is isomorphic to a quotient stack $[V/G]$ for the free action of a finite abstract group G on a scheme V. If $V_1 \subseteq V$ is a dense affine open subscheme, then $V_2 = \bigcap_{g \in G} gV_1$ is a G-invariant quasi-affine open subscheme of V and in particular separated. Repeating this argument, we can choose a dense affine open subscheme $V_3 \subseteq V_2$ and now $V_4 = \bigcap_{g \in G} gV_3$ is a G-invariant affine open subscheme. Proposition 4.3.3 implies that $V_4/G \cong \operatorname{Spec} A^G$ is a dense affine open algebraic subspace of U. See also [Knu71, II.6.7] and [SP, Tag 06NN]. □

Remark 4.5.2. The above result is not necessarily true if X is not quasi-separated, e.g., $\mathbb{A}^1/\underline{\mathbb{Z}}$ (Example 3.9.35).

Corollary 4.5.3. *An integral quasi-separated algebraic space has a well-defined fraction field.* □

Exercise 4.5.4. Let G be a finite abstract group acting on a quasi-compact and quasi-separated algebraic space U. Show that there is a dense G-invariant affine open subscheme of U.

4.5.2 Zariski's Main Theorem

We prove Zariski's Main Theorem (4.5.9) for Deligne–Mumford stacks, which has the important application that locally quasi-finite and separated morphisms of algebraic spaces are representable by schemes (Corollary 4.5.7). Its proof relies on the theory of quasi-coherent sheaves, and, specifically, on the factorization

$$f: \mathcal{X} \to \mathcal{S}\mathrm{pec}_{\mathcal{Y}}\, f_*\mathcal{O}_{\mathcal{X}} \to \mathcal{Y}$$

of a quasi-compact and quasi-separated morphism See Definition 3.3.34 for the definition of a quasi-finite morphism of algebraic spaces and §A.7 for a discussion of Zariski's Main Theorem for schemes.

Proposition 4.5.5. *A representable, quasi-finite, and separated morphism $f: \mathcal{X} \to \mathcal{Y}$ of Deligne–Mumford stacks factors as the composition of an open immersion $\mathcal{X} \hookrightarrow \mathcal{S}\mathrm{pec}_{\mathcal{Y}}\, f_*\mathcal{O}_{\mathcal{X}}$ and an affine morphism $\mathcal{S}\mathrm{pec}_{\mathcal{Y}}\, f_*\mathcal{O}_{\mathcal{X}} \to \mathcal{Y}$. In particular, f is quasi-affine.*

Proof. Since the construction of $f_*\mathcal{O}_{\mathcal{X}}$ commutes with flat base change on $\mathcal{Y}$ (Exercise 4.1.45), so does the formation of the factorization $f: \mathcal{X} \to \mathcal{S}\mathrm{pec}_{\mathcal{Y}}\, f_*\mathcal{O}_{\mathcal{X}} \to \mathcal{Y}$. The statement is thus étale-local on $\mathcal{Y}$. In particular, we can assume that $\mathcal{Y} = Y$ is an affine scheme and that $\mathcal{X} = X$ is an algebraic space. After replacing Y with $\mathcal{S}\mathrm{pec}_Y\, f_*\mathcal{O}_X$, we can assume that $f_*\mathcal{O}_X = \mathcal{O}_Y$ and we must show that $f: X \to Y$ is an open immersion.

Since X is quasi-compact, there is an étale presentation $\pi: U \to X$ from an affine scheme. Since X is separated, $U \to X$ is also separated. As the composition

$$U \xrightarrow{\pi} X \xrightarrow{f} Y$$

is a quasi-finite morphism of schemes, we can apply Étale Localization of Quasi-Finite Morphisms (A.7.1) around every point $y \in Y$: after replacing Y with an étale neighborhood, we can assume that $U = U_1 \amalg U_2$ with $U_1 \to Y$ finite and $(U_2)_y = \emptyset$. Then $\pi(U_1)$ is open (as π is étale) and closed (as $U_1 \to Y$ is finite and $X \to Y$ is separated). Thus $X = X_1 \amalg X_2$ with $X_1 = \pi(U_1)$ and $(X_2)_y = \emptyset$. This shows that $\mathcal{O}_Y = f_*\mathcal{O}_X$ is the product $\mathcal{A}_1 \times \mathcal{A}_2$ of quasi-coherent $\mathcal{O}_X$-algebras, and thus we can also decompose Y as $Y_1 \amalg Y_2$ such that $y \in Y_1$ and $f(Y_i) \subseteq X_i$ for $i = 1, 2$. After replacing Y with Y_1, the composition $U \to X \to Y$ is finite and Lemma 4.5.6 implies that X is affine. Thus $X = Y = \mathcal{S}\mathrm{pec}_Y\, f_*\mathcal{O}_X$. □

Lemma 4.5.6. *Suppose that $U \to X$ is a surjective étale morphism of algebraic spaces and that $X \to Y$ is a separated morphism of algebraic spaces. If the composition $U \to X \to Y$ is finite, so is $X \to Y$.*

Proof. The statement is étale-local on Y so we can assume that Y and U are affine. As $X \to Y$ is separated, $U \to X$ is also finite. Since X is identified with the quotient U/R of the finite étale equivalence relation $R := U \times_X U \rightrightarrows U$ of affine schemes, Proposition 4.3.3 implies that X is affine. As $U \to Y$ is proper, so is $X \to Y$. As $X \to Y$ is a proper and quasi-finite morphism of schemes, it is finite (Corollary A.7.5). □

Corollary 4.5.7. *A morphism of algebraic spaces which is locally quasi-finite and separated is representable by schemes. In particular, an algebraic space locally quasi-finite and separated over a scheme is a scheme.*

Proof. It suffices to show that if $X \to Y = \operatorname{Spec} A$ is a locally quasi-finite and separated, then X is a scheme. Since being a scheme is a Zariski-local property, we can assume that X is quasi-compact. In this case, Proposition 4.5.5 applies. □

Corollary 4.5.8. *The diagonal of a Deligne–Mumford stack with separated and quasi-compact diagonal is quasi-affine. In particular, a quasi-separated algebraic space has quasi-affine diagonal.*

Proof. Since the diagonal is representable, quasi-finite, and separated, Proposition 4.5.5 applies. Note that the diagonal of an algebraic space is a monomorphism, hence separated. □

As in the case for schemes, we can refine Proposition 4.5.5 to obtain a version of Zariski's Main Theorem.

Theorem 4.5.9 (Zariski's Main Theorem). *A representable, quasi-finite, and separated morphism $f\colon \mathcal{X} \to \mathcal{Y}$ of noetherian Deligne–Mumford stacks factors as the composition of a dense open immersion $\mathcal{X} \hookrightarrow \widetilde{\mathcal{Y}}$ and a finite morphism $\widetilde{\mathcal{Y}} \to \mathcal{X}$.*

Proof. Let $\mathcal{A} \subseteq f_*\mathcal{O}_{\mathcal{X}}$ be the integral closure of $\mathcal{O}_{\mathcal{Y}} \to f_*\mathcal{O}_{\mathcal{X}}$: for an étale map $T \to \mathcal{Y}$ from a scheme, $\mathcal{A}(T \to \mathcal{Y})$ is the integral closure of $\Gamma(T, \mathcal{O}_T) \to \Gamma(\mathcal{X}_T, \mathcal{O}_{\mathcal{X}_T})$. Since the integral closure is compatible with étale extensions (Proposition A.7.4), $\mathcal{A}$ is a quasi-coherent sheaf of $\mathcal{O}_{\mathcal{Y}}$-algebras. Using Exercise 4.1.33, write $\mathcal{A} = \operatorname{colim} \mathcal{A}_\lambda$ as the colimit of finite type $\mathcal{O}_{\mathcal{Y}}$-algebras. As $\mathcal{Y}$ is quasi-compact, there exists an étale presentation $p\colon U \to \mathcal{Y}$ from an affine scheme. Then the base change $\mathcal{X}_U \to U$ is a quasi-finite and separated morphism of algebraic spaces, thus a morphism of schemes by Corollary 4.5.7. Since $p^*\mathcal{A} = \operatorname{colim} p^*\mathcal{A}_\lambda$ is identified with the integral closure of $\mathcal{O}_U \to f_{U,*}\mathcal{O}_{\mathcal{X}_U}$, it follows from Zariski's Main Theorem (A.7) for schemes that for $\lambda \gg 0$ $\mathcal{X}_U \to \mathcal{S}\mathrm{pec}_U\, p^*\mathcal{A}_\lambda$ is an open immersion and $\mathcal{S}\mathrm{pec}_U\, p^*\mathcal{A}_\lambda \to U$ is finite. By étale descent, for $\lambda \gg 0$, $\mathcal{X} \to \mathcal{S}\mathrm{pec}_{\mathcal{Y}}\, \mathcal{A}_\lambda$ is also an open immersion and $\mathcal{S}\mathrm{pec}_{\mathcal{Y}}\, \mathcal{A}_\lambda \to \mathcal{Y}$ is finite. See also [Knu71, II.6.15], [LMB00, Thm. A.2], [Ols16, Thm. 7.2.10], and [SP, Tag 05W7]. □

4.5.3 Characterization of algebraic spaces

We now can remove the hypotheses in Theorem 3.6.6 and Corollary 3.6.9 that the diagonal be representable by schemes.

Theorem 4.5.10 (Characterization of Algebraic Spaces II)**.** *For an algebraic stack $\mathcal{X}$, the following are equivalent:*

(1) the stack $\mathcal{X}$ is an algebraic space,

(2) the diagonal $\mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is a monomorphism, and

(3) every point of $\mathcal{X}$ has a trivial stabilizer.

Proof. We only need to show (2) $\Rightarrow$ (1). As the diagonal of $\mathcal{X}$ is a monomorphism, it is locally quasi-finite and separated. Corollary 4.5.7 implies that the diagonal $\mathcal{X}$ is representable by schemes and thus Theorem 3.6.6 applies. □

Corollary 4.5.11 (Characterization of Representable Morphisms II)**.** *A morphism $\mathcal{X} \to \mathcal{Y}$ of algebraic stacks is representable if and only if for every geometric point $x \in \mathcal{X}(\Bbbk)$, the map $G_x \to G_{f(x)}$ on automorphism groups is injective.* □

If X is a sheaf admitting a surjective étale morphism representable by schemes $U \to X$ from a scheme, then X is an algebraic space (by definition). It turns out that X is also an algebraic sapce if we only require that U is an algebraic space and that $U \to X$ be smooth and *representable (by algebraic spaces)*. Moreover, we can extend the existence of quotients of étale equivalence relations of schemes (Theorem 3.4.13(2)) to étale equivalence relations of *algebraic spaces*.

Corollary 4.5.12.

(1) If X is a sheaf on $\mathrm{Sch}_{\text{ét}}$ such that there exists a surjective, étale (resp., smooth), and representable morphism $U \to X$ from an algebraic space, then X is an algebraic space.

(2) If $R \rightrightarrows U$ is an étale (resp., smooth) equivalence relation of algebraic spaces, then the quotient U/R is an algebraic space.

Remark 4.5.13. The above statement holds with with 'étale' replaced with 'fppf'; see Theorem 6.3.1 and Corollary 6.3.6.

Proof. By Theorem 3.4.13(1), the sheaf X in (1) and the quotient sheaf U/R in (2) are algebraic stacks, and the statement follows from Theorem 4.5.10. □

Corollary 4.5.14. *A proper and quasi-finite morphism (resp., proper monomorphism) of algebraic spaces is finite (resp., a closed immersion).*

Proof. Proper and quasi-finite morphisms are representable by schemes. Thus the statement follows from the corresponding result for schemes (Corollary A.7.5) and étale descent. □

Exercise 4.5.15. Show that the prestack $\underline{\mathrm{AlgSp}}$ over $\mathrm{Sch}_{\text{ét}}$, whose objects over a scheme T are algebraic spaces over T and whose morphisms correspond to cartesian diagrams of algebraic spaces, is a stack.

4.5.4 Affineness criteria

Theorem 4.5.16 (Serre's Criterion for Affineness)**.** *Let X be a quasi-compact and quasi-separated (resp., noetherian) algebraic space. If the functor $\Gamma(X,-)$ is exact on the category of quasi-coherent (resp., coherent) sheaves, then X is an affine scheme.*

Proof. We will show that the canonical morphism $\pi\colon X \to Y := \operatorname{Spec}\Gamma(X, \mathcal{O}_X)$ is a proper monomorphism. This will imply that X is affine as a proper monomorphism is a closed immersion (Corollary 4.5.14). The proof is inspired by properties of good moduli spaces (see §6.5).

Claim: If $g\colon Y' \to Y$ is a morphism of algebraic spaces, then the base change $\pi'\colon X' := X \times_Y Y' \to Y'$ has the following properties:

(a) π'_* induces an equivalence of the categories of quasi-coherent sheaves on X' and Y'.

(b) $\mathcal{O}_{Y'} \to \pi'_*\mathcal{O}_{X'}$ is an isomorphism.

(c) $X' \to Y'$ is a homeomorphism.

By Flat Base Change (Exercise 4.1.45), properties (a) and (b) are étale local on Y' so we may assume $Y' = \operatorname{Spec} B$. We will show that the adjunction morphisms $G \to \pi'_*\pi'^*G$ and $\pi'^*\pi'_*F \to F$ are isomorphisms for $G \in \mathrm{QCoh}(Y')$ and $F \in \mathrm{QCoh}(X')$. This immediately gives (a) and also (b) (by taking $G = \mathcal{O}_{Y'}$). For the first adjunction, choose a free presentation $\mathcal{O}_Y^{\oplus J} \to \mathcal{O}_Y^{\oplus I} \to g_*G \to 0$ of G as an $\mathcal{O}_Y$-module. As π_* is exact, we have a morphism of right exact sequences

$$\begin{tikzcd}
\mathcal{O}_Y^{\oplus J} \arrow[r] \arrow[d] & \mathcal{O}_Y^{\oplus I} \arrow[r] \arrow[d] & g_*G \arrow[r] \arrow[d] & 0 \\
\pi_*\pi^*(\mathcal{O}_Y^{\oplus J}) \arrow[r] & \pi_*\pi^*(\mathcal{O}_Y^{\oplus I}) \arrow[r] & \pi_*\pi^*g_*G \arrow[r] & 0
\end{tikzcd}$$

The two left vertical arrows are isomorphisms since $\pi_*\mathcal{O}_X = \mathcal{O}_Y$. Therefore $g_*G \to \pi_*\pi^*g_*G \cong g_*\pi'_*\pi'^*G$ is an isomorphism. Since g_* is faithfully exact, $G \to \pi'_*\pi'^*G$ is also an isomorphism. To see that $\pi'^*\pi'_*F \to F$ is an isomorphism, let K and Q be the kernel and cokernel of $\pi'^*\pi'_*F \to F$. As π_* is exact and g_* is faithfully exact, it follows that π'_* is exact. Since $\pi'_*\pi'^*\pi'_*F \to \pi'_*F$ is an isomorphism (by applying the first adjunction to π'_*F), we see that $\pi'_*K = \pi'_*Q = 0$. It thus suffices to show that for a quasi-coherent sheaf F' on X', then $F' \neq 0$ implies $\pi'_*F' \neq 0$. If $x\colon \operatorname{Spec}\Bbbk \to X'$ is a geometric point such that $F \otimes \Bbbk := x^*F \neq 0$, then by base changing by the composition $\pi' \circ x\colon \operatorname{Spec}\Bbbk \to Y'$, we may assume that $Y' = \operatorname{Spec}\Bbbk$ and that $x\colon \operatorname{Spec}\Bbbk \to X'$ is a section of π'. Since every $\Bbbk$-point of an algebraic space defined over $\Bbbk$ is a closed point, $x\colon \operatorname{Spec}\Bbbk \to X'$ is a closed immersion, and hence $F \to x_*x^*F = F \otimes \Bbbk$ is surjective. It follows from the exactness of π'_* that $\pi'_*F \to \pi'_*x_*x^*F = F \otimes \Bbbk$ is surjective and hence $\pi'_*F \neq 0$.

To see (c), if $y\colon \operatorname{Spec}\Bbbk \to Y$ is a geometric point, then by (b) $\Gamma(X_y, \mathcal{O}_{X_y}) = \Bbbk$ as thus the fiber X_y is non-empty. On the other hand, if $x, x' \in X_y(\Bbbk)$ were distinct points each necessarily closed, then $\mathcal{O}_{X_y} \to \mathcal{O}_{\{x,x'\}}$ is surjective. Since π_* is exact, we also get a surjection $\Bbbk = \Gamma(X_y, \mathcal{O}_{X_y}) \to \Bbbk \oplus \Bbbk$, a contradiction. To see that π' is closed, let $Z \subseteq X'$ be a closed subspace and $q\colon Z \to \operatorname{im}(Z)$ denote the morphism to its scheme-theoretic image. Then $\mathcal{O}_Z \to q_*\mathcal{O}_{\operatorname{im}(Z)}$ is an isomorphism and q_* is exact. Applying the surjectivity result above to q, we see that q is surjective, and hence $\pi'(Z)$ is closed.

The claim implies that $X \to Y$ is universally closed. To see that $X \to Y$ is a monomorphism, i.e., the diagonal $\Delta\colon X \to X \times_Y X$ is an isomorphism, observe that (b) implies that the pushforward of $\mathcal{O}_{X\times_Y X} \to \Delta_* X$ along the first projection $p_1\colon X\times_Y X \to X$ is an isomorphism. Thus, (a) applied to p_1 shows that $\mathcal{O}_{X\times_Y X} \to \Delta_* X$ is an isomorphism. Zariski's Main Theorem (A.7.3) implies that Δ is an open immersion. Applying (c) to p_1 shows that $p_1\colon |X\times_Y X| \to |X|$ is bijective. Hence, Δ is an isomorphism. It remains to show that $X \to Y$ is of finite type. Let $p\colon U = \operatorname{Spec} A \to X$ be an étale presentation. Since X is separated, $R := U\times_X U$ is a closed subscheme of $U\times_Y U = \operatorname{Spec}(A\otimes_{\Gamma(X,\mathcal{O}_X)} A)$. Hence $R = \operatorname{Spec} B$ is affine. Letting s and t denote the two maps $A \rightrightarrows B$, we have a commutative diagram

$$\begin{tikzcd}
 & A \arrow[dr] \arrow[drr, "s"] & & \\
\Gamma(X,\mathcal{O}_X) \arrow[ur] \arrow[dr] & & A\otimes_{\Gamma(X,\mathcal{O}_X)} A \arrow[r, two heads] & B. \\
 & A \arrow[ur] \arrow[urr, "t"'] & &
\end{tikzcd}$$

Since $p\colon U \to X$ is étale, $t\colon A \to B$ is of finite type and we can choose A-algebra generators $b_1,\dots,b_n \in B$. For each i, choose a preimage $\sum_j a_{ij}\otimes a'_{ij} \in A\otimes_{\Gamma(X,\mathcal{O}_X)} A$ of b_i. Note that $s(a_{ij})$ also generate B as an A-algebra via t. The elements $a_{ij} \in \Gamma(X, p_*\mathcal{O}_U) = A$ define a homomorphism $\mathcal{O}_X[z_{ij}] \to p_*\mathcal{O}_U$ of $\mathcal{O}_X$-algebras taking z_{ij} to a_{ij}. Its pullback via p is identified with $\mathcal{O}_U[z_{ij}] \to p^*p_*\mathcal{O}_U \cong t_*\mathcal{O}_R$, where the last equivalence comes from Flat Base Change (Exercise 4.1.45), and this map is surjective precisely because $s(a_{ij})$ generate B over t. By étale descent, $\mathcal{O}_X[z_{ij}] \to p_*\mathcal{O}_U$ is surjective and therefore so is $\Gamma(X,\mathcal{O}_X)[z_{ij}] \to A$. Thus, $\Gamma(X,\mathcal{O}_X) \to A$ is of finite type and, by étale descent, so is $X \to Y$.

Finally, in the noetherian case, every quasi-coherent sheaf is a colimit of coherent sheaves (Exercise 4.1.28) and $\Gamma(X,-)$ commutes with colimits. Assume that $\Gamma(X,-)$ is exact on coherent sheaves. Given a surjection $p\colon F \twoheadrightarrow G$ of quasi-coherent sheaves on X, write $G = \operatorname{colim}_i G_i$ as a colimit of coherent sheaves and choose coherent subsheaves $F_i \subseteq p^{-1}(G_i)$ surjecting onto G_i. Then $\Gamma(X,F_i) \twoheadrightarrow \Gamma(X,G_i)$ and the composition $\operatorname{colim}_i \Gamma(X,F_i) \to \Gamma(X,F) \to \Gamma(X,G) = \operatorname{colim}_i \Gamma(X,G_i)$ is surjective. Thus $\Gamma(X,F) \to \Gamma(X,G)$ is surjective and we conclude that $\Gamma(X,-)$ is exact on quasi-coherent sheaves. See also [Knu71, Thm. III.2.5], [Ryd15, Thm. 8.7], and [SP, Tag 07V6]. ∎

Corollary 4.5.17. *Let X be a quasi-compact and quasi-separated (resp., noetherian) algebraic space. Then X is an affine scheme if and only if $\mathrm{H}^i(X,F) = 0$ for every quasi-coherent (resp., coherent) sheaf F and $i > 0$.*

Proof. If X is affine, then Theorem 4.1.42 establishes the vanishing of quasi-coherent cohomology. Conversely, the vanishing of quasi-coherent (resp., coherently) cohomology implies that $\Gamma(X,-)$ is exact on the category of quasi-coherent (resp., coherent) sheaves: if $0 \to F_1 \to F_2 \to F_3 \to 0$ is exact, then $\Gamma(X,F_2) \to \Gamma(X,F_3)$ is surjective as $\mathrm{H}^1(X,F_1) = 0$. ∎

Remark 4.5.18. Given a quasi-compact and quasi-separated morphism $f\colon \mathcal{X} \to \mathcal{Y}$ of Deligne–Mumford stacks, the relative versions of Theorem 4.5.16 and Corollary 4.5.17 also hold: f is affine if and only if $f_*\colon \mathrm{QCoh}(\mathcal{X}) \to \mathrm{QCoh}(\mathcal{Y})$ is exact if and only if $R^i f_* F = 0$ for all $i > 0$ and $F \in \mathrm{QCoh}(\mathcal{X})$.

Proposition 4.5.19. *Let X be a noetherian algebraic space. If X_{red} is a scheme (resp., quasi-affine, affine), then so is X.*

Proof. If X_{red} is affine, then one uses Corollary 4.5.17 to show that X is affine exactly as in [Har77, Exc. III.3.1]: if F is a coherent sheaf on X and $I \subseteq \mathcal{O}_X$ denotes the nilpotent ideal defining X_{red}, then one shows the vanishing of $\mathrm{H}^i(X, F)$ using the filtration $0 = I^N F \subseteq I^{N-1}F \subseteq \cdots \subset IF \subseteq F$, whose factors $I^k F/I^{k+1}F$ are supported on X_{red}. If X_{red} is a scheme, then every point $x \in |X|$ has an open neighborhood U such that U_{red} is affine. Thus U is affine and X is a scheme. If X_{red} is quasi-affine, then $X_{\mathrm{red}} \to \operatorname{Spec}\Gamma(X, \mathcal{O}_X)_{\mathrm{red}}$ is an open immersion. Thus $X \to \operatorname{Spec}\Gamma(X, \mathcal{O}_X)$ is an open immersion and X is quasi-affine. □

Theorem 4.5.20 (Chevalley's Criterion for Affineness)**.** *Let $X \to Y$ be a finite surjective morphism of noetherian algebraic spaces. If X is affine, then so is Y.*

Proof. One can argue as in [Har77, Exc. 4.1] using Corollary 4.5.17. □

Exercise 4.5.21. Prove the following generalization: if $X \to Y$ is an *integral* surjective morphism of algebraic spaces with Y noetherian and X is affine, then Y is affine.

Hint: Write $X = \lim_\lambda X_\lambda$ as a limit of finite Y-scheme and use Limit Methods (B.3).

The cohomological criterion for ampleness [Har77, Prop. 5.3] also extends to algebraic spaces. A line bundle L on an algebraic space X is defined to be *ample* if X is a scheme and L is ample in the ordinary sense.

Exercise 4.5.22. Let X be a proper algebraic space over a noetherian ring. For a line bundle L on X, show that L is ample if and only if for every coherent sheaf F on X, there is an integer n_0 such that $\mathrm{H}^i(X, F \otimes L^n) = 0$ for $i > 0$ and $n \geq n_0$. See also [SP, Tag 0D2W].

As in the case of schemes (Proposition B.2.9), ampleness can be checked on a finite cover.

Exercise 4.5.23. Let X be a proper algebraic space over a noetherian ring and L be a line bundle on X. If $f \colon X' \to X$ is a finite surjective morphism, L is ample if and only if f^*L is. See also [SP, Tag 0GFB].

Exercise 4.5.24. A noetherian scheme has the *Chevalley–Kleiman property* if every finite set of points is contained in an affine. Show that if $X \to Y$ is a finite surjective morphism of noetherian algebraic spaces such that X has the Chevalley-Kleiman property, then Y also has the Chevalley-Kleiman property. See [Kol12, Cor. 48].

4.5.5 Effective descent along field extensions

Lemma 4.5.25. *Let X be a quasi-separated algebraic space locally of finite type over a field $\Bbbk$. If $X_{\overline{\Bbbk}}$ is an affine scheme, then so is X.*

Proof. As $X_{\overline{\Bbbk}} \to X$ is an integral surjective morphism, the statement follows Chevalley's Criterion for Affineness (4.5.21). □

Proposition 4.5.26. *Let X be a quasi-separated algebraic space of finite type over a field $\Bbbk$. If $X_{\overline{\Bbbk}}$ is a scheme, then there exists a finite separable field extension $\Bbbk \to K$ such that X_K is a scheme.*

Proof. Choose an étale presentation $U \to X$ from an affine scheme and set $R = U \times_X U$. Write $\overline{\Bbbk} = \operatorname{colim} \Bbbk_\lambda$ as the colimit of finite field extensions, and set $X_\lambda := X_{\Bbbk_\lambda}$, $U_\lambda = U_{\Bbbk_\lambda}$ and $R_\lambda = R_{\Bbbk_\lambda}$. Let $\overline{V} \subseteq X_{\overline{\Bbbk}}$ be an open affine subscheme. We claim that for $\lambda \gg 0$, there exists an open subscheme $V_\lambda \subseteq X_\lambda$ such that $\overline{V} = U_\lambda \times_{\Bbbk_\lambda} \overline{\Bbbk}$. Indeed, the preimage $\overline{Z} \subseteq U_{\overline{\Bbbk}}$ of $\overline{V}$ has the property that its two preimages in $R_{\overline{\Bbbk}}$ are equal. Using Proposition B.3.3 and Proposition B.3.7, for $\lambda \gg 0$ there is an open subscheme $Z_\lambda \subseteq U_\lambda$ with $\overline{Z} = Z_\lambda \times_{\Bbbk_\lambda} \overline{\Bbbk}$ such that the two preimages of Z_λ in R_λ are equal. By étale descent, Z_λ descends to the desired closed subscheme $V_\lambda \subseteq X_\lambda$. Lemma 4.5.25 implies that V_λ is a scheme. By covering $X_{\overline{\Bbbk}}$ with finitely many affines and choosing λ sufficiently large, we obtain a finite field extension $K = \Bbbk_\lambda$ of $\Bbbk$ such that X_λ is a scheme. Now factor $\Bbbk \to \Bbbk_\lambda$ as a separable field extension $\Bbbk \to K$ and a purely inseparable field extension $K \to \Bbbk_\lambda$. Since $X_\lambda \to X_K$ is a finite univeral homemorphism, Chevalley's Theorem for Affineness (4.5.20) implies that the image of each V_λ in X_K is affine, and hence X_K is a scheme. ☐

With an additional condition on $X_{\overline{\Bbbk}}$, we can conclude that X is a scheme.

Proposition 4.5.27. *Let X be a quasi-separated algebraic space locally of finite type over a field $\Bbbk$. If $X_{\overline{\Bbbk}}$ is a scheme such that every finite set of $\overline{\Bbbk}$-points is contained in an affine (e.g., $X_{\overline{\Bbbk}}$ is quasi-projective), then X is a scheme.*

Proof. We may assume that X is quasi-compact. We will show that every closed point $x \in X$ has an affine open neighborhood. Let $\operatorname{Spec} \kappa(x) \hookrightarrow X$ be the inclusion of the residue field of x (Proposition 3.5.16(3)) and consider the cartesian diagram

$$\begin{tikzcd}
\coprod_{i=1}^n \operatorname{Spec} A_i \arrow[r, hook] \arrow[d] & X_{\overline{\Bbbk}} \arrow[r] \arrow[d] & \operatorname{Spec} \overline{\Bbbk} \arrow[d] \\
\operatorname{Spec} \kappa(x) \arrow[r, hook, "x"] & X \arrow[r] & \operatorname{Spec} \Bbbk
\end{tikzcd}$$

where n is the degree of the separable closure of $\Bbbk$ in K, and each A_i is a artinian local $\Bbbk$-algebra. The hypotheses on $X_{\overline{\Bbbk}}$ ensure that there is an affine open subscheme $\overline{U} \subseteq X_{\overline{\Bbbk}}$ containing the images of each $\operatorname{Spec} A_i$. By Proposition 4.5.26, there is a finite field extension $\Bbbk \to K$ such that X_K is a scheme. After enlarging K, we can arrange that $\overline{U}$ descends to an affine open subscheme $U' \subseteq U_K$ by using Proposition B.3.3 to descend the morphism $\overline{U} \to X$, Proposition B.3.7 to arrange that it is an open immersion, and Proposition B.3.5 to arrange affineness. Observe that U' contains all preimages of x under $X_K \to X$. By taking the normal closure of K, we can assume K is normal over $\Bbbk$. Let $G = \operatorname{Aut}(K/\Bbbk)$ so that K^G is a purely inseparable field extension of $\Bbbk$. Then G acts on X_K freely such that $X_K/G = X_{K^G}$. The intersection of the translates of U' by elements of G is a G-invariant quasi-affine variety U''. Choosing an affine open subscheme in U'' containing all of the preimages of x and intersecting again the translates of G, we obtain a G-invariant affine $V \subseteq X_K$ containing the preimages of x. Then the quotient V/G is an affine subscheme of X_{K^G} containing the unique preimage of x (Theorem 4.2.3). Letting W be the image of V/G under the finite universal homeomorphism $X_{K^G} \to X$, Chevalley's Criterion for Affineness (Theorem 4.5.20) implies that W is an affine neighborhood of x. ☐

4.5.6 Group algebraic spaces are schemes

Every quasi-separated group algebraic space over a field $\Bbbk$ is a scheme. When $\Bbbk$ is algebraically closed, this follows easily from Theorem 4.5.1 as we know there is a dense open that is a scheme, and we can translate this around by rational points. The general case relies on Proposition 4.5.27.

Theorem 4.5.28. *A quasi-separated group algebraic space G locally of finite type over a field $\Bbbk$ is a separated scheme. The connected component of the identity G^0 is quasi-projective.*

Remark 4.5.29. If G is not quasi-separated, then the above corollary does not hold, e.g., $G = \mathbb{G}_a/\mathbb{Z}$ over $\Bbbk$ (Example 3.9.35). Note that Proposition 4.5.19 implies that the result also holds over an artinian base. Over a general base scheme, even separated group algebraic spaces need not be schemes; see [Ray70, Lem. X.14].

Proof. It suffices to show that G is a scheme. Indeed, a group scheme locally of finite type over a field is necessarily separated (B.1.16(1)) and the identity component G^0 is quasi-compact (B.1.16(4)), thus quasi-projective (B.1.16(7)). Assume first that $\Bbbk$ is algebraically closed. There is a non-empty open subscheme U of G (Theorem 4.5.1) with a point $h \in U(\Bbbk)$. For every $g \in G(\Bbbk)$, left multiplication by gh^{-1} defines an isomorphism $G \xrightarrow{\sim} G$ and the image $gh^{-1}U$ of U is a scheme containing g. The general case follows from Proposition 4.5.27 using that $G_{\overline{\Bbbk}}$ is a scheme with the property that every finite set of points is contained in an affine (Lemma 4.5.30). See also [Art69b, Lem. 4.2] and [SP, Tag 0B8D]. □

Lemma 4.5.30. *Every group scheme G locally of finite type over an algebraically closed field $\Bbbk$ has the property that every finite set of $\Bbbk$-points is contained in an affine open subscheme.*

Proof. Let $g_1, \ldots, g_n \in G(\Bbbk)$. We first use induction on n to reduce to the case that the elements g_i are in the same connected component. If not, we can write $G = W_1 \amalg W_2$ with r points in W_1 and $n-r$ points in W_2 for $0 < r < n$. By induction, there are affine opens $U_1 \subseteq W_1$ and $U_2 \subseteq W_2$ containing the r and $n-r$ points, respectively. Then $U_1 \amalg U_2$ is an affine containing each g_i. By translating by g_1^{-1}, we may further assume that $g_1, \ldots, g_n \in G^0(\Bbbk)$. Let $U \subseteq G^0$ be an affine open neighborhood of the identity. Since G^0 is irreducible (B.1.16(4)), $Ug_1^{-1} \cap \cdots \cap Ug_n^{-1}$ is non-empty and contains a closed point h. Since $h \in Ug_i^{-1}$, each g_i is contained in the affine open $h^{-1}U$. □

Corollary 4.5.31. *Let $\mathcal{X}$ be an algebraic stack with quasi-separated diagonal. Then the stabilizer of every field-valued point is a group scheme locally of finite type.*

Proof. By Exercise 3.3.5, the diagonal of $\mathcal{X}$ is locally of finite type. As the stabilizer is the base change of the diagonal, the statement follows from Theorem 4.5.28. □

4.5.7 Separated one dimensional algebraic spaces are schemes

Theorem 4.5.32. *A separated noetherian algebraic space X with* $\dim X \le 1$ *is a scheme.*

Proof. We may assume that X has pure dimension 1. By Proposition 4.5.19, we may assume that X is reduced. By Theorem 4.5.1, there exists an open dense subscheme $U \subseteq X$ which is regular. Let $D \subseteq U$ be a union of reduced points such that every irreducible component of X contains at least four points. Then $\mathcal{O}_U(D)$ defines a line bundle on U which extends to a line bundle $\mathcal{O}_X(D)$. Using Le Lemme de Gabber (4.6.1) proved in the next section, there exists a finite surjection $f\colon Z \to X$ from a scheme Z. After replacing Z with its normalization, we may assume that Z is regular.

As Z is a separated scheme of dimension 1, Z is quasi-projective and $f^*\mathcal{O}_X(D)$ is ample. By Exercise 4.5.23, $\mathcal{O}_X(D)$ is also ample. See also [SP, Tags 0ADD and 09NZ]. $\square$

Exercise 4.5.33.

(a) (hard) Show that every smooth separated algebraic space of finite type over a field of dimension two is a scheme.

(b) (moderate) Construct a non-normal separated algebraic space of finite type over a field of dimension two that is *not* a scheme.

(c) (open) Is every separated regular algebraic space of dimension two a scheme?

4.6 Finite covers of Deligne–Mumford stacks

It is obvious if you think about it.

ANGELO VISTOLI

We prove Le Lemme de Gabber (4.6.1): a separated Deligne–Mumford stack has a finite cover by a scheme. As a consequence, we obtain a Chow's Lemma for Algebraic Spaces (4.6.6) and Deligne–Mumford Stacks (4.6.5). We deduce cohomological applications in §4.6.3.

4.6.1 Le Lemme de Gabber

A celebrated result attributed to Gabber asserts that a Deligne–Mumford stack has a finite cover by scheme. Arguments were first published in [Del85] and [Vis89].

Theorem 4.6.1 (Le Lemme de Gabber)**.** *Let $\mathcal{X}$ be a Deligne-Mumford stack separated and of finite type over a noetherian scheme S. Then there exists a finite, generically étale, and surjective morphism $Z \to \mathcal{X}$ from a scheme Z.*

Proof. By replacing $\mathcal{X}$ with the disjoint union of the irreducible components with their reduced stack structure, we may assume that $\mathcal{X}$ is irreducible and reduced. Every étale presentation $U \to \mathcal{X}$ is representable, quasi-finite, and separated, and thus it factors as the composition of an open immersion $U \hookrightarrow \widetilde{\mathcal{X}}$ and a finite morphism $\widetilde{\mathcal{X}} \to \mathcal{X}$ by Zariski's Main Theorem (4.5.9). After replacing $\mathcal{X}$ with $\widetilde{\mathcal{X}}$, we may assume that $\mathcal{X}$ has a dense open subscheme. If $p\colon U \to \mathcal{X}$ is an étale presentation, there is therefore a dense open subscheme $V \subseteq \mathcal{X}$ such that $p^{-1}(V) \to V$ is finite étale of degree d. We may choose a finite étale covering $V' \to V$ such that $p^{-1}(V) \times_V V' \to V'$ is a trivial étale covering; indeed as in Proposition A.3.11, we may take V' to be the complement of the (open and closed) pairwise diagonals in

$$(V'/V)^d = \underbrace{V' \times_V \cdots \times_V V'}_{d}.$$

Applying Zariski's Main Theorem (4.5.9) again, this time to the composition $V' \twoheadrightarrow V \hookrightarrow \mathcal{X}$, gives a finite surjective morphism $\widetilde{\mathcal{X}} \to \mathcal{X}$ restricting to $V' \to V$. Thus after replacing $\mathcal{X}$ with $\widetilde{\mathcal{X}}$, we may further assume that there is an étale presentation $U \to \mathcal{X}$ which over a dense open subscheme $j\colon V \hookrightarrow \mathcal{X}$ is a trivial étale covering, i.e., there is a cartesian diagram

$$\begin{tikzcd} \coprod_{i=1}^d V \arrow[r, hook, "j'"] \arrow[d] & U \arrow[d, "p"] \\ V \arrow[r, hook, "j"] & \mathcal{X}. \end{tikzcd}$$

We will construct a finite surjective morphism $Z \to \mathcal{X}$ from a scheme that is an isomorphism over V. Let $\mathcal{A} \subseteq j_*\mathcal{O}_Z$ be integral closure of $\mathcal{O}_{\mathcal{X}} \to j_*\mathcal{O}_Z$. Then $p^*\mathcal{A}$ is the integral closure of $\mathcal{O}_U$ in $j'_*\mathcal{O}_{\amalg V} = p^*j_*\mathcal{O}_V$ (Proposition A.7.4). The idempotent $e_i \in \Gamma(U, j'_*\mathcal{O}_{\amalg V}) = \Gamma(\amalg V, \mathcal{O}_{\amalg V})$, defining the ith copy of V, is integral over $\mathcal{O}_U$ and thus defines a global section $e_i \in \Gamma(U, p^*\mathcal{A})$. Now write $\mathcal{A} = \operatorname{colim}_\lambda \mathcal{A}_\lambda$ as a filtered colimit of finite type $\mathcal{O}_{\mathcal{X}}$-subalgebras (Exercise 4.1.33). Since $\mathcal{A}$ is integral over $\mathcal{O}_{\mathcal{X}}$, each $\mathcal{A}_\lambda$ is a finite $\mathcal{O}_{\mathcal{X}}$-algebra. For $\lambda \gg 0$, we have $e_i \in \Gamma(U, p^*\mathcal{A}_\lambda)$. The Deligne–Mumford stack $Z := \mathcal{S}\mathrm{pec}_{\mathcal{X}}\, \mathcal{A}_\lambda$ is finite over $\mathcal{X}$ and we claim that Z is a scheme. To see this, consider the cartesian diagram

$$\begin{tikzcd} Z' \arrow[r] \arrow[d] & U \arrow[d, "p"] \\ Z \arrow[r] & \mathcal{X} cartesian, \end{tikzcd}$$

where Z' is a scheme (as it is finite over U). Each idempotent e_i defines a global section of $\mathcal{O}_{Z'}$ and thus yields a decomposition $Z' = \coprod_{i=1}^d Z'_i$. Each morphism $Z'_i \to Z$ is étale, separated and birational, thus an open immersion. Since $Z' \to Z$ is surjective, $\{Z'_i\}$ defines an open covering of Z and Z is a scheme. See also [LMB00, Thm. 16.6]. □

Exercise 4.6.2. Provide an alternative proof following [Vis89, Prop. 2.6]:

(i) Use limit methods to reduce to the case that S is of finite type over $\mathbb{Z}$ (which ensures that normalizations are finite).

(ii) Reduce to the case that $\mathcal{X}$ is irreducible, reduced, and normal.

(iii) Let $\mathcal{X} \to X$ be the coarse moduli space and let $U \to \mathcal{X}$ be an étale presentation by a scheme $U = \coprod_i U_i$, where each U_i is integral and affine. Let L be a normal finite field extension of $K(X)$ containing each function field $K(U_i)$, and let $Y \to X$ and $Y_i \to U_i$ be the normalizations of X and U_i in L. Prove that Y is a scheme and that $Y \to X$ Zariski locally factors through $\mathcal{X}$ by showing that each $Y_i \to Y$ is an open immersion and that Y is the union of the translates $\sigma \cdot Y_i$ over all i and $\sigma \in \operatorname{Aut}(L/K(X))$.

(iv) By replacing X with Y and $\mathcal{X}$ with $\mathcal{X} \times_X Y$, this reduces to the case that the coarse moduli space X is a scheme with an open covering $X = \bigcup X_\alpha$ such that each $X_\alpha \hookrightarrow X$ factors through $\mathcal{X}$. Show that after replacing X with a finite cover, the sections $X_\alpha \to \mathcal{X}$ glue to a global section $X \to \mathcal{X}$.

Remark 4.6.3. More generally, every separated algebraic stack of finite type over S has a proper cover by a quasi-projective scheme [Ols05].

Exercise 4.6.4 (good practice)**.** Let X be a normal algebraic space of finite type over a noetherian scheme S. Show that there is a normal scheme U with an action of a finite abstract group G such that X is the quotient of U by G, i.e., $[U/G] \to X$ is a coarse moduli space.

Hint: After reducing to the case that X is integral, choose a finite, generically étale and surjective morphism $U \to X$ from a scheme. Let L be the Galois closure of $K(U)/K(X)$, and take U to be the integral closure of X in L and take $G = \mathrm{Gal}(L/K(X))$. See also [LMB00, Cor. 16.6.2].

4.6.2 Chow's Lemma

Corollary 4.6.5 (Chow's Lemma for Deligne–Mumford Stacks)**.** *Let $\mathcal{X}$ be a Deligne-Mumford stack separated and of finite type over a noetherian scheme S. Then there exists a projective, generically étale, and surjective morphism $Z \to \mathcal{X}$ from a scheme Z quasi-projective over S.*

Proof. Le Lemme de Gabber (4.6.1) reduces the statement to the case of schemes, c.f., [Har77, Exc. II.4.10]. □

There is also a birational version for algebraic spaces.

Exercise 4.6.6 (Chow's Lemma for Algebraic Spaces)**.** Let $\mathcal{X}$ be an algebraic space separated and of finite type over a noetherian scheme S. Then there exists a projective, birational, and surjective morphism $Z \to \mathcal{X}$ from a scheme Z quasi-projective over S. See also [Knu71, IV.3.1] and [SP, Tag 088U].

Exercise 4.6.7 (Valuative criteria can be checked on dense opens)**.**

(a) Let $\mathcal{X}$ be a Deligne–Mumford stack separated and of finite type over a noetherian scheme S, and let $\mathcal{U} \subseteq \mathcal{X}$ be a dense open substack. Show that $\mathcal{X} \to S$ is proper if and only if for every DVR R defined over S with fraction field K, every map $\operatorname{Spec} K \to \mathcal{U}$ extends to a map $\operatorname{Spec} R \to \mathcal{X}$ over S, after replacing R with an extension of DVRs.

Hint: Use Corollary 4.6.5 to reduce to the case that $\mathcal{X} \to S$ is quasi-projective.

(b) Let $\mathcal{X}$ be a Deligne–Mumford stack with separated and quasi-compact diagonal. Show that a finite type morphism $\mathcal{X} \to S$ to a noetherian scheme is separated if and only if for every DVR R over S with fraction field K and every pair $h, g \colon \operatorname{Spec} R \to \mathcal{X}$ of morphisms over S, any isomorphism $h_K \xrightarrow{\sim} g_K$ of their generic fibers extends to a unique isomorphism $h \xrightarrow{\sim} g$.

Hint: Reduce to Part (a).

4.6.3 Applications to cohomology

Most of the foundational theorems for coherent sheaf cohomology on proper schemes—Finiteness of Cohomology (A.5.3), Cohomology and Base Change (A.6.2), Semicontinuity (A.6.4), Formal Functions (A.5.4), Grothendieck's Existence Theorem (C.5.3), and Stein Factorization [Har77, Cor. III.11.5]— also hold for proper Deligne–Mumford stacks. They imply Zariski's Connectedness Theorem (4.6.15), which is an essential ingredient in the proof of the irreducibility of $\mathcal{M}_g$ in positive characteristic (Theorem 5.7.33).

Theorem 4.6.8 (Finiteness of Cohomology)**.** *Let $f \colon \mathcal{X} \to \mathcal{Y}$ be a proper morphism of noetherian Deligne–Mumford stacks. For any coherent sheaf F on $\mathcal{X}$ and any $i \geq 0$, $\mathrm{R}^i f_* F$ is coherent.*

Proof. By Flat Base Change (4.1.45), we can assume that $\mathcal{Y} = \operatorname{Spec} A$ is an affine scheme, and we need to show that $\mathrm{H}^i(\mathcal{X}, F)$ is a finite A-module for $i \geq 0$. We will use the following generalization of Dévissage (A.5.1), which is proved in the same way: if $\mathcal{P}$ is a property of coherent sheaves on $\mathcal{X}$ such that

(a) in a short exact sequence in $\mathrm{Coh}(\mathcal{X})$, if two out of the three satisfies $\mathcal{P}$, then the third does too, and

(b) for all integral closed substacks $\mathcal{Z} \subseteq \mathcal{X}$ with ideal sheaf $I_{\mathcal{Z}} \subseteq \mathcal{O}_{\mathcal{X}}$ and a coherent sheaf F on $\mathcal{X}$ with $I_{\mathcal{Z}}F = 0$, there exists a coherent sheaf G on $\mathcal{X}$ satisfying $\mathcal{P}$ with $I_{\mathcal{Z}}G = 0$ and a morphism $F \to G$ which is an isomorphism over an open substack of $\mathcal{Z}$,

then every coherent sheaf on $\mathcal{X}$ satisfies $\mathcal{P}$. Taking $\mathcal{P}$ to be the property that $\mathrm{H}^i(\mathcal{X}, F)$ is a finite A-module for $i \geq 0$, we see that (a) holds. For (b), using Le Lemme de Gabber (4.6.1), let $g_0 \colon V_0 \to \mathcal{Z}$ be a finite surjective morphism from a scheme. Letting $V_1 = V_0 \times_{\mathcal{Z}} V_0$ with projection $g_1 \colon V_1 \to \mathcal{Z}$, there is a complex $0 \to F \to g_{0,*}g_0^*F \to g_{1,*}g_1^*F$, which is exact if $V_0 \to \mathcal{Z}$ is flat (Proposition 2.1.1). Defining $G := \ker(g_{0,*}g_0^*F \to g_{1,*}g_1^*F)$, there is a map $F \to G$. By Generic Flatness (3.3.32), $V_0 \to \mathcal{Z}$ is flat over a nonempty open substack $\mathcal{U} \subseteq \mathcal{Z}$, and it follows that $F \to G$ is an isomorphism over $\mathcal{U}$. Since V_0 and V_1 are proper over $\operatorname{Spec} A$, $g_{0,*}g_0^*F$ and $g_{1,*}g_1^*F$ have finite cohomology, and thus so does G. See also [Fal03, Thm. 1], [Ols05, Thm. 1.2], [LMB00, Thm. 15.6], and [Ols16, Thm. 11.6.1]. □

Remark 4.6.9. Unlike schemes, $\mathrm{H}^i(\mathcal{X}, F)$ can be nonzero for arbitrary large i; see Exercise 4.1.49.

Exercise 4.6.10. If X is a noetherian algebraic space, show that there exists an integer N such that for every coherent sheaf F on X, $\mathrm{H}^i(X, F) = 0$ for $i > N$.

Proposition 4.6.11 (Semicontinuity)**.** *Let $\mathcal{X} \to \mathcal{Y}$ be a proper morphism of noetherian Deligne–Mumford stacks, and let F be a coherent sheaf on $\mathcal{X}$ which is flat over $\mathcal{Y}$. For each $i \geq 0$, the function*

$$|\mathcal{Y}| \to \mathbb{Z}, \qquad y \mapsto \dim_K \mathrm{H}^i(\mathcal{X}_K, F_K) \tag{4.6.12}$$

where $\operatorname{Spec} K \to \mathcal{Y}$ *is any field-valued point representing y, is upper semicontinuous.*

Proof. We leave the reader to check that the function (A.6.5) is constructible. It therefore suffices to check that it is upper semicontinuous with respect to a specialization $y \rightsquigarrow y_0$ in $|\mathcal{Y}|$. By Proposition 3.8.20, this specialization is realized by a morphism $\operatorname{Spec} R \to \mathcal{Y}$ from a DVR, and we may therefore assume that $\mathcal{Y} = \operatorname{Spec} R$. Letting K and κ be the fraction and residual field of R, we need to show that $\dim_K \mathrm{H}^i(\mathcal{X}_K, F_K) \leq \dim_\kappa \mathrm{H}^i(\mathcal{X}_\kappa, F_\kappa)$. If $\pi \in R$ denotes a uniformizer, then $\pi \colon F \to F$ is injective since F is flat over R. Apply $\mathrm{H}^i(\mathcal{X}, -)$ to the short exact sequence $0 \to F \to F \to F_\kappa \to 0$, we obtain an exact sequence

$$\cdots \to \mathrm{H}^i(\mathcal{X}, F) \xrightarrow{\pi} \mathrm{H}^i(\mathcal{X}, F) \to \mathrm{H}^i(\mathcal{X}_\kappa, F_\kappa) \to \cdots .$$

This gives an injection $\mathrm{H}^i(\mathcal{X}, F)/\pi\mathrm{H}^i(\mathcal{X}, F) \hookrightarrow \mathrm{H}^i(\mathcal{X}_\kappa, F_\kappa)$. On the other hand, $\mathrm{H}^i(\mathcal{X}, F)$ is a finite R-module by Finiteness of Cohomology (4.6.8) with $\mathrm{H}^i(\mathcal{X}, F) \otimes_R K = \mathrm{H}^i(\mathcal{X}, F_K)$ by Flat Base Change (4.1.45). This gives

$$\dim_K \mathrm{H}^i(\mathcal{X}_K, F_K) \leq \dim_\kappa \mathrm{H}^i(\mathcal{X}, F)/\pi\mathrm{H}^i(\mathcal{X}, F) \leq \dim_\kappa \mathrm{H}^i(\mathcal{X}_\kappa, F_\kappa).$$

□

Exercise 4.6.13 (hard)**.** Let $\mathcal{X}$ be a noetherian Deligne–Mumford stack proper over a complete noetherian local ring $(A, \mathfrak{m})$. Let $\mathcal{X}_n = \mathcal{X} \times_A A/\mathfrak{m}^{n+1}$.

(a) (Formal Functions) If F is a coherent sheaf on $\mathcal{X}$, there is a natural isomorphism

$$\mathrm{H}^i(\mathcal{X}, \mathcal{F}) \xrightarrow{\sim} \varprojlim_n \mathrm{H}^i(\mathcal{X}_n, F|_{\mathcal{X}_n})$$

for every $i \geq 0$.

(b) (Grothendieck's Existence Theorem) There is an equivalence of categories

$$\mathrm{Coh}(\mathcal{X}) \to \varprojlim \mathrm{Coh}(\mathcal{X}_n), \qquad F \mapsto \{F|_{\mathcal{X}_n}\}.$$

See also [Ols05, Thm. 1.4] and [Con05a, Cor. 3.2, Thm. 4.1].

Theorem 4.6.14 (Stein Factorization/Zariski's Connectedness Theorem)**.** *A proper morphism $f\colon \mathcal{X} \to \mathcal{Y}$ of noetherian Deligne–Mumford stacks factors as*

$$f\colon \mathcal{X} \xrightarrow{f'} \mathcal{Y}' = \mathcal{S}\mathrm{pec}_{\mathcal{Y}}\, f_*\mathcal{O}_{\mathcal{X}} \xrightarrow{g} \mathcal{Y},$$

where f' has geometrically connected fibers and g is finite. In particular, if $f_*\mathcal{O}_{\mathcal{X}} = \mathcal{O}_{\mathcal{Y}}$, then f has geometrically connected fibers.

Proof. This is proved just as in the case of schemes—[Har77, Cor. III.11.5], [EGA, III_1.4.3.1], and [SP, Tag 03H2]—using Formal Functions. By Finiteness of Cohomology (4.6.8), $f_*\mathcal{O}_{\mathcal{X}}$ is coherent and g is finite. We are thus reduced to showing that f has geometrically connected fibers if $f_*\mathcal{O}_{\mathcal{X}} = \mathcal{O}_{\mathcal{Y}}$. Since the question is étale local, we can further assume that $\mathcal{Y} = S$ is affine. Moreover, since geometric connectedness of a fiber $\mathcal{X}_s$ can be checked on separable finite field extensions of $\kappa(s)$, it suffices to verify that for every étale morphism $S' \to S$, the fibers of $f'\colon \mathcal{X}_{S'} \to S'$ are connected. By Flat Base Change (4.1.45), $f'_*\mathcal{O}_{\mathcal{X}_{S'}} = \mathcal{O}_{S'}$, so it is enough to verify that f has connected fibers. If a fiber $\mathcal{X}_s$ is disconnected, then setting $S_n = \mathrm{Spec}\, \mathcal{O}_{S,s}/\mathfrak{m}_s^{n+1}$, there are compatible isomorphisms $\mathrm{H}^0(\mathcal{X} \times_S S_n, \mathcal{O}_{\mathcal{X} \times_S S_n}) \cong A_n \times B_n$ for nonzero coherent $\mathcal{O}_{S_n}$-modules A_n and B_n. By Formal Functions (4.6.13(b)), $\widehat{\mathcal{O}}_{S,s} \cong \varprojlim A_n \times \varprojlim B_n$, contradicting that $\widehat{\mathcal{O}}_{S,s}$ is a local ring. □

The following result will be applied to the morphism $\overline{\mathcal{M}}_g \to \mathrm{Spec}\, \mathbb{Z}$ to show that $\mathcal{M}_g \times_{\mathbb{Z}} \overline{\mathbb{F}}_p$ is connected (Theorem 5.7.33). This application requires only the lower semicontinuity below, which relies on Stein Factorization (4.6.14) but not on Semicontinuity (4.6.11).

Corollary 4.6.15 (Zariski's Connectedness Theorem II)**.** *Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a proper flat morphism of noetherian Deligne–Mumford stacks. The function*

$$|\mathcal{Y}| \to \mathbb{Z}, \qquad y \mapsto \#\textit{connected components of } \mathcal{X} \times_{\mathcal{Y}} \mathrm{Spec}\, K, \tag{4.6.16}$$

where $\mathrm{Spec}\, K \to \mathcal{Y}$ is any geometric point representing y, is lower semicontinuous. If in addition f has geometrically reduced fibers, then (4.6.16) is locally constant.

Proof. By étale descent, we can assume that $\mathcal{Y} = S$ is a scheme. Let $\mathcal{X} \xrightarrow{f'} S' \to S$ be the Stein Factorization (4.6.14). As $S' \to S$ is finite, there are only finitely many points $s'_1, \ldots, s'_m$ over a point $s \in S$. By Étale Localization of Quasi-Finite Morphisms (A.7.1), after replacing S with an étale neighborhood of s, we can arrange that $S' = S'_1 \amalg \cdots \amalg S'_m$ with $s'_i \in S'_i$ and $\kappa(s'_i)/\kappa(s)$ purely inseparable. Since $\mathcal{X} \to S'$ has geometrically connected fibers, each fiber $\mathcal{X}_{s'_i}$ is geometrically connected, and thus the geometric fiber of $\mathcal{X} \to S$ over s has precisely m connected components. As $\mathcal{X} \to S$ is flat, the image $U_i \subseteq S$ of $f'^{-1}(S'_i)$ is open. In the open neighborhood $\bigcap_i U_i$ of s, every geometric fiber has at least m connected components.

Assuming now that every fiber is geometrically reduced, the number of geometric components over $s \in S$ is precisely $\dim_{\kappa(s)} \mathrm{H}^0(\mathcal{X}_s, \mathcal{O}_{\mathcal{X}_s})$. By Semicontinuity (4.6.11), (4.6.16) is also upper semicontinuous. See also [EGA, IV$_3$.15.5.3-7] and [SP, Tags 0BUI and 0E0N] for the case of schemes, [SP, Tags 0E1D and 0E1E] for the case of algebraic spaces, and [DM69, Thm. 4.17] (for a statement in the case of Deligne–Mumford stacks). □

We also record the following application to the openness of ampleness, which extends Proposition B.2.10 to algebraic spaces.

Proposition 4.6.17 (Openness of Ampleness)**.** *Let X be an algebraic space proper and finitely presented over a scheme S, and L be a line bundle on X. If for some geometric point $s\colon \operatorname{Spec} \Bbbk \to S$, the restriction L_s of L to the fiber X_s is ample, then there exists an open neighborhood $U \subseteq S$ of s such that X_U is a scheme and the restriction L_U to X_U is relatively ample over U. In particular, for all $u \in U$, L_u is ample on X_u.*

Proof. Apply Le Lemme de Gabber (4.6.1) and the invariance of ampleness under finite covers (Exercise 4.5.23), the statement reduces to the case of schemes (Proposition B.2.10). □

Chapter 5

Moduli of stable curves

The point is, I love maps, that is "maps" in the sense of "maps of the world," "charts of the ocean," "atlases of the sky"! I think one of the key things that attracted me to the group of problems was the hope of making a map of some parts of the world of algebraic varieties. An algebraic variety felt like a tangible thing in the lectures of Oscar Zariski, so why shouldn't you venture out, like Magellan, and uncover their geography?

Mumford [Mum04, p. vii]

This chapter proves Theorem A: for $g \geq 2$, $\overline{\mathcal{M}}_g$ is a smooth, proper, and irreducible Deligne–Mumford stack of dimension $3g-3$ which admits a projective coarse moduli space $\overline{\mathcal{M}}_g \to \overline{M}_g$. In terms of the six-step strategy to construct projective moduli spaces outlined in §0.7.2, we proceed as follows.

① (Algebraicity) We express $\overline{\mathcal{M}}_g$ as a substack of the stack of all curves $\mathcal{M}_g^{\text{all}}$, and we prove that $\mathcal{M}_g^{\text{all}}$ is an algebraic stack locally of finite type over $\mathbb{Z}$ (Theorem 5.4.6).
② (Openness of Stability) $\overline{\mathcal{M}}_g \subseteq \mathcal{M}_g^{\text{all}}$ is an open substack (Proposition 5.3.24).
③ (Boundedness) $\overline{\mathcal{M}}_g$ is finite type (Theorem 5.4.14).
④ (Stable reduction) $\overline{\mathcal{M}}_g$ is proper (Theorem 5.5.23).
⑤ (Existence of a moduli space) Applying the Keel–Mori Theorem (4.4.6) gives a coarse moduli space $\overline{\mathcal{M}}_g \to \overline{M}_g$ with $\overline{M}_g$ a proper algebraic space.
⑥ (Projectivity) Using Kollár's Criterion for Ampleness (5.9.2), we show that $\det \pi_*(\omega_{\mathcal{U}_g/\overline{\mathcal{M}}_g}^{\otimes k})$ is ample for $k \gg 0$, where $\pi\colon \mathcal{U}_g \to \overline{\mathcal{M}}_g$ is the universal family (Corollary 5.9.5).

Along the way, we develop the foundational properties of nodal curves (§5.2) and stable curves (§5.3 and §5.6). The application to moduli theory necessitates that we prove most properties for *families of curves*, which is one of the most technically challenging facets of the exposition. The irreducibility of $\overline{\mathcal{M}}_g$ is established in §5.7, and a GIT construction of $\overline{M}_g$ is sketched in §5.8.

5.1 Review of smooth curves

Everyone knows what a curve is, until he has studied enough mathematics to become confused through the countless number of possible exceptions.

Felix Klein

We review basic properties of smooth curves, which we will generalize to nodal curves in the next section.

5.1.1 Curves

Definition 5.1.1. A *curve over a field* $\Bbbk$ is a one dimensional scheme C of finite type over $\Bbbk$.

Proper curves are projective [Har77, Prop. I.6.7] and moreover every one dimensional proper algebraic space is, in fact, a projective scheme (Theorem 4.5.32). If C is a proper curve over a field $\Bbbk$, we define the *arithmetic genus of* C or simply the *genus of* C as

$$g(C) = 1 - \chi(C, \mathcal{O}_C),$$

which is equal to $\mathrm{h}^1(C, \mathcal{O}_C)$ if C is geometrically connected and reduced. The *geometric genus* of a reduced proper curve C is the (arithmetic) genus of the normalization $\widetilde{C}$.

For a reduced, connected, and projective curve C over an algebraically closed field $\Bbbk$, the degree of a *very ample* line bundle L on C is defined as the number of zeros (counted with multiplicity) of any section of L. In other words, if $C \hookrightarrow \mathbb{P}^n$ is the projective embedding defined by L, then $\deg L = \dim_{\Bbbk} \Gamma(C \cap H, \mathcal{O}_{C \cap H})$, where H is any hyperplane and $C \cap H$ is the scheme-theoretic intersection. Any line bundle on C can be written as the difference of two very ample line bundles: if M is very ample on C, then $M' := L \otimes M^n$ is very ample for $n \gg 0$, and $L \cong M' \otimes (M^{\otimes n})^\vee$. In this way, we can define $\deg L = \deg M' - n \deg M$. Note that the degree is additive under tensor product and that if $C = \bigcup_i C_i$ denotes the irreducible decomposition, then $\deg L = \sum_i \deg L|_{C_i}$.

Theorem 5.1.2 (Riemann–Roch)**.** *Let C be a reduced, connected, and projective curve of genus g over an algebraically closed field $\Bbbk$. If L is a line bundle on C, then*

$$\chi(C, L) = \deg L + 1 - g.$$

Proof. We can write $L = \mathcal{O}_C(D)$ for a divisor D supported on the smooth locus. Since Riemann–Roch holds for $\mathcal{O}_C$ (by the definition of genus), it suffices by adding and subtracting points to show that Riemann–Roch holds for $\mathcal{O}_C(D)$ if and only if it holds for $\mathcal{O}_C(D+p)$ for a smooth point $p \in C(\Bbbk)$. This follows by considering the short exact sequence

$$0 \to \mathcal{O}_C(D) \to \mathcal{O}_C(D+p) \to \kappa(p) \to 0$$

and the identity $\chi(C, \mathcal{O}_C(D+p)) = \chi(C, \mathcal{O}_C(D)) + 1$. See also [Har77, Thm IV.1.3, Exc. IV.1.9] and [Vak17, Exers. 18.4.B and S]. □

5.1.2 Smooth curves

If C is a smooth curve, then the sheaf of differentials Ω_C is a line bundle. Serre Duality is the statement that Ω_C is a dualizing sheaf on C, and is a deep result indispensable in the study of curves.

Theorem 5.1.3 (Serre Duality for Smooth Curves)**.** *If C is a smooth projective curve over a field $\Bbbk$, then Ω_C is a dualizing sheaf, i.e., there is a linear map $\mathrm{tr}\colon \mathrm{H}^1(C, \Omega_C) \to \Bbbk$ such that for every coherent sheaf $\mathcal{F}$, the natural pairing*

$$\mathrm{Hom}_{\mathcal{O}_C}(\mathcal{F}, \Omega_C) \times \mathrm{H}^1(C, \mathcal{F}) \to \mathrm{H}^1(C, \Omega_C) \xrightarrow{\mathrm{tr}} \Bbbk$$

is perfect.

Proof. See [Ser88, Thm. II.8.2] and [Har77, Cor. III.7.12]. □

Remark 5.1.4. The pairing being perfect means that the $\mathrm{Hom}_{\mathcal{O}_C}(\mathcal{F}, \Omega_C)$ is identified with the dual $\mathrm{H}^1(C, \mathcal{F})^\vee$. If $\mathcal{F}$ is a vector bundle, $\mathrm{Hom}_{\mathcal{O}_C}(\mathcal{F}, \Omega_C) \cong \mathrm{H}^0(C, \mathcal{F}^\vee \otimes \Omega_C)$ and Serre Duality gives an isomorphism

$$\mathrm{H}^0(C, \mathcal{F}^\vee \otimes \Omega_C) \cong \mathrm{H}^1(C, \mathcal{F})^\vee.$$

Taking $\mathcal{F} = \Omega_C$, we see that $\mathrm{H}^1(C, \Omega_C) \cong \mathrm{H}^0(C, \mathcal{O}_C)^\vee$ and that $\mathrm{tr}\colon \mathrm{H}^1(C, \Omega_C) \to \Bbbk$ is an isomorphism if C is geometrically connected and reduced.

Serre Duality yields a more powerful version of Riemann–Roch.

Theorem 5.1.5 (Riemann–Roch II)**.** *Let C be a smooth, connected, and projective curve of genus g over an algebraically closed field. If L is a line bundle on C, then*

$$\mathrm{h}^0(C, L) - \mathrm{h}^0(C, \Omega_C \otimes L^\vee) = \deg L + 1 - g. \quad \square$$

Remark 5.1.6. This is often written in divisor form as $\mathrm{h}^0(C, L) - \mathrm{h}^0(C, K - L) = \deg L + 1 - g$, where K denotes a canonical divisor, i.e., $\Omega_C = \mathcal{O}_C(K)$.

The theorem of Riemann–Hurwitz (5.7.4) also plays an essential role in the study of smooth curves, describing how the sheaf of differentials behaves under finite covers. Our treatment is deferred until we discuss branched covers in §5.7.1.

Exercise 5.1.7 (easy)**.** For any $g \geq 0$, show that there exists a smooth, connected, and projective curve of genus g.

Automorphisms. If C is a smooth curve over a field $\Bbbk$ and $p_1, \ldots, p_n \in C(\Bbbk)$, we let $\mathrm{Aut}(C, p_i)$ denote the abstract group of automorphisms of (C, p_i) over $\Bbbk$ consisting of automorphisms $\alpha\colon C \xrightarrow{\sim} C$ such that $\alpha(p_i) = p_i$ for all i.

Proposition 5.1.8. *Let C be a smooth, connected, and projective curve of genus g over an algebraically closed field $\Bbbk$, and let $p_1, \ldots, p_n \in C(\Bbbk)$ be distinct points. The automorphism group $\mathrm{Aut}(C, p_i)$ is finite if $2g - 2 + n > 0$, i.e., either $g \geq 2$, or $g = 1$ and $n \geq 1$, or $g = 0$ and $n \geq 3$. Moreover, if $g \geq 2$ and $\mathrm{char}(\Bbbk) = 0$, then $|\mathrm{Aut}(C)| \leq 84(g - 1)$.*

Proof. This is a classical result of Hurwitz [Hur92]. See also [Har77, Exc. IV.5.2], [ACGH85, Exc. I.F.4], and [Mir95, Thm. 4.18] for the finiteness and [Har77, Exc. IV.2.5], [ACGH85, Exc. I.F.10], and [Mir95, Thm.3.9] for the explicit bound. □

Exercise 5.1.9. Let C be a smooth, connected, and projective curve over an algebraically closed field $\Bbbk$ of genus $g \geq 2$. Show that there are no non-trivial automorphisms of C fixing more than $2g + 2$ points.

While every hyperelliptic curve (and thus every curve of genus 2) has a non-trivial hyperelliptic involution, the following exercise will be applied later to show that general curves of higher genus are automorphism free.

Exercise 5.1.10. Show that if $g \geq 3$ and $\Bbbk$ is an algebraically closed field, there exists a smooth, connected, and projective curve C over $\Bbbk$ with trivial automorphism group.

For further background on smooth curves, we recommend [Har77, §IV], [Vak17, §20], [Liu02, §7], [ACGH85], and [Mir95].

5.1.3 Positivity of divisors on smooth curves

The following consequence of Riemann–Roch provides useful criteria to determine whether a given line bundle is base point free (equivalently globally generated), ample, or very ample.

Corollary 5.1.11. *Let C be a smooth, connected, and projective curve over an algebraically closed field $\Bbbk$, and let L be a line bundle on C.*

(1) if $\deg L < 0$, then $\mathrm{h}^0(C, L) = 0$;

(2) if $\deg L > 0$, then L is ample;

(3) if $\deg L \geq 2g$, then L is base point free; and

(4) if $\deg L \geq 2g+1$, then L is very ample.

Proof. See [Har77, Cor. IV.3.2]. □

Remark 5.1.12. If $g > 1$, we can use Riemann–Roch and Serre Duality to compute that: (a) $\mathrm{h}^0(C, \Omega_C) = \mathrm{h}^1(C, \mathcal{O}_C) = g$, (b) $\mathrm{h}^1(C, \Omega_C) = \mathrm{h}^0(C, \mathcal{O}_C) = 1$, and (c) Ω_C has degree $2g-2$ and is thus ample on C. Similarly, if $k > 1$, we have: (a) $\mathrm{h}^0(C, \Omega_C^{\otimes k}) = (2k-1)(g-1)$, (b) $\mathrm{h}^1(C, \Omega_C^{\otimes k}) = 0$, and (c) $\Omega_C^{\otimes k}$ has degree $2k(g-1)$ and is very ample if $k \geq 3$. Note that Ω_C is not very ample precisely when C is *hyperelliptic*. On the other hand, if $g = 1$ then $\Omega_C \cong \mathcal{O}_C$, and if $g = 0$ then $C = \mathbb{P}^1$ and $\Omega_C = \mathcal{O}(-2)$.

5.1.4 Classification of rational curves

Over an algebraically closed field $\Bbbk$, every connected, smooth, and proper curve of genus 0 is isomorphic to $\mathbb{P}^1_{\Bbbk}$. Over an arbitrary field, the classification is slightly more involved.

Exercise 5.1.13 (Classificiation of the projective line)**.** Show that the following are equivalent for a proper curve C over a field $\Bbbk$:

(a) $C \cong \mathbb{P}^1_{\Bbbk}$;

(b) C is smooth and geometrically irreducible over $\Bbbk$, C has genus 0, and $C(\Bbbk) \neq \emptyset$.

(c) C is Gorenstein and geometrically integral over $\Bbbk$, C has genus 0, and C has a line bundle of odd degree;

(d) $\mathrm{H}^0(C, \mathcal{O}_C) = \Bbbk$, C has genus 0, and C has a line bundle of degree 1;

(e) $\mathrm{H}^1(C, \mathcal{O}_C) = 0$ and C has a line bundle of degree 1.

See also [SP, Tag 0C6U].

There is also a classification of singular rational curves that will be convenient to understand nodal rational curves as well as the classification of rational tails and bridges in prestable curves.

Exercise 5.1.14 (Classification of singular rational curves)**.** Let C be a singular proper curve of genus 0 over a field $\Bbbk$ with $\mathrm{H}^0(C, \mathcal{O}_C) = \Bbbk$, and let $\pi \colon C' \to C$ be the normalization. Show the following:

(a) C has a unique singular point p, which is a $\Bbbk$-rational point;

(b) $\pi \colon C' \to C$ is an isomorphism over $C \smallsetminus p$ and $\pi^{-1}(p) = \{p'\}$ for a $\Bbbk'$-rational point $p' \in C'$;

(c) $C' \cong \mathbb{P}^1_{\Bbbk'}$ for a finite field extension $\Bbbk'/\Bbbk$; and

(d) C is identified with the Ferrand Pushout (B.4.1)

$$\begin{tikzcd} \operatorname{Spec} \Bbbk' \arrow[r, "p'"] \arrow[d] & C' \arrow[d, "\pi"] \\ \operatorname{Spec} \Bbbk \arrow[r, "p"] & C. \end{tikzcd}$$

See also [SP, Tag 0DJB].

5.1.5 Families of smooth curves

Definition 5.1.15. A *family of smooth curves (of genus g)* over a scheme S is a smooth and proper morphism $\mathcal{C} \to S$ of schemes such that every geometric fiber is a connected curve (of genus g).

Recall that the relative sheaf of differentials $\Omega_{\mathcal{C}/S}$ is a line bundle on $\mathcal{C}$ such that for $s \in S$, the restriction $\Omega_{\mathcal{C}/S}|_{\mathcal{C}_s}$ is identified with $\Omega_{\mathcal{C}_s}$. More generally, for every morphism $T \to S$ of schemes, the pullback of $\Omega_{\mathcal{C}/S}$ to $\mathcal{C}_T := \mathcal{C} \times_S T$ is canonically isomorphic to $\Omega_{\mathcal{C}_T/T}$. Generalizing Corollary 5.1.11, $\Omega_{\mathcal{C}/S}^{\otimes k}$ is relatively very ample for $k \geq 3$ and its pushforward is a vector bundle on S.

Proposition 5.1.16 (Properties of Families of Smooth Curves)**.** *Let $\pi\colon \mathcal{C} \to S$ be a family of smooth curves of genus $g \geq 2$.*

(1) $\pi_* \mathcal{O}_{\mathcal{C}} = \mathcal{O}_S$;

(2) The pushforward $\pi_*(\Omega_{\mathcal{C}/S}^{\otimes k})$ is a vector bundle of rank

$$r(k) := \begin{cases} g & \text{if } k = 1 \\ (2k-1)(g-1) & \text{if } k > 1. \end{cases}$$

whose construction commutes with base change (i.e., for a morphism $f\colon T \to S$ of schemes, $f^* \pi_*(\Omega_{\mathcal{C}/S}^{\otimes k}) \cong \pi_{T,*}(\Omega_{\mathcal{C}_T/T}^{\otimes k})$);

(3) $R^1 \pi_* \Omega_{\mathcal{C}/S}^{\otimes k}$ is isomorphic to $\mathcal{O}_S$ if $k = 1$ and zero otherwise; and

(4) For $k \geq 3$, $\Omega_{\mathcal{C}/S}^{\otimes k}$ is relatively very ample. In particular, $\mathcal{C} \to S$ is projective.

Proof. Items (1)–(3) follows from Cohomology and Base Change (A.6.8) as detailed in Proposition A.6.10. For (4), observe that for every point $s \in S$, the fiber $\Omega_{\mathcal{C}/S}^{\otimes k} \otimes \kappa(s) = \Omega_{\mathcal{C}_s}^{\otimes k}$ is very ample by Corollary 5.1.11 as $\deg \Omega_{\mathcal{C}_s}^{\otimes k} = k(2g-2) > 0$. Since $\mathrm{H}^1(\mathcal{C}_s, \Omega_{\mathcal{C}_s}^{\otimes k}) = 0$, we may apply Proposition B.2.10 to conclude that $\Omega_{\mathcal{C}/S}^{\otimes k}$ is relatively very ample. $\square$

It is also true that the relative sheaf of differentials $\Omega_{\mathcal{C}/S}$ is a relative dualizing sheaf, i.e., satisfies a relative version of Serre Duality; see [Har66c] or [Liu02, §6.4].

5.2 Nodal curves

[Doing] mathematics for me is like being on a long hike with no trail and no end in sight.

Maryam Mirzakhami

After providing a Characterization of Nodes (5.2.4), we discuss the Genus Formula (5.2.12), the dualizing sheaf (Definition 5.2.18), and the Local Structure of Nodal Families (5.2.25).

5.2.1 Nodes

Definition 5.2.1 (Nodes)**.** Let C be a curve over a field $\Bbbk$.

- We say that $p \in C(\Bbbk)$ is a *split node* if there is an isomorphism $\widehat{\mathcal{O}}_{C,p} \cong \Bbbk[\![x,y]\!]/(xy)$.
- We say that a closed point $p \in C$ is a *node* if there exists a split node $\overline{p} \in C_{\overline{\Bbbk}}$ over p.

We say that a curve C is a *nodal* (or has *at-worst nodal singularities*) if C is pure one dimensional and every closed point is either smooth or nodal.

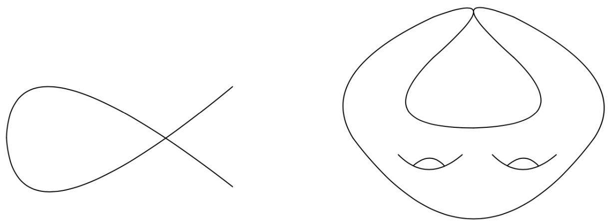


Figure 5.2.2: A node of a curve over $\mathbb{C}$ viewed algebraically (left-hand side) or analytically (right-hand side).

Example 5.2.3.

(1) The curve $C = \operatorname{Spec} \Bbbk[x,y]/(xy)$ has a split node at 0. The normalization $\widetilde{C} \cong \mathbb{A}^1 \amalg \mathbb{A}^1$ has coordinate ring $\Bbbk[x] \times \Bbbk[y]$ and $\Gamma(C, \mathcal{O}_C) = \{(f,g) \in \Gamma(\widetilde{C}, \mathcal{O}_{\widetilde{C}}) \mid f(0) = g(0)\}$, or in other words C is the Ferrand Pushout (B.4.1)

$$\begin{tikzcd} \operatorname{Spec} \Bbbk \amalg \operatorname{Spec} \Bbbk \arrow[r, "0 \amalg 0"] \arrow[d] & \widetilde{C} \arrow[d] \\ \operatorname{Spec} \Bbbk \arrow[r, "0"] & C. \end{tikzcd}$$

(2) The nodal cubic $C = \operatorname{Spec} \Bbbk[x,y]/(y^2 - x^2(x+1))$ has a split node at 0. The normalization $\widetilde{C} \cong \mathbb{A}^1$ has a coordinate $t = y/x$ with $x = t^2 - 1$ and $y = t^3 - t$, and C is again the Ferrand Pushout obtained from $\widetilde{C}$ by gluing the two preimages of 0.

(3) The curve $C = \operatorname{Spec} \mathbb{R}[x,y]/(x^2 + y^2)$ has a node at 0, but it is not a split node as the quadratic form $x^2 + y^2$ does not split into linear factors.

(4) The curve $\operatorname{Spec} \mathbb{Q}[x,y]/(x^2 - 2)(y^2 - 3)$ has a non-split node at the point p defined by the maximal ideal $(x^2 - 2, y^2 - 3)$. Note that unlike the previous examples where the nodes are rational points, the node p is not a rational point and the field extension $\mathbb{Q} \to \kappa(p)$ has degree 4.

(5) Let $\Bbbk \to \Bbbk'$ be a separable field extension of degree 2, and let C be the affine curve over $\Bbbk$ defined by the $\Bbbk$-algebra $\{f \in \Bbbk'[x] \mid f(0) \in \Bbbk\}$. In other words, C is the Ferrand Pushout of the inclusion of the origin $0\colon \operatorname{Spec} \Bbbk' \hookrightarrow \mathbb{A}^1_{\Bbbk'}$ along $\operatorname{Spec} \Bbbk' \to \operatorname{Spec} \Bbbk$. The curve C has a non-split node at a $\Bbbk$-rational point.

5.2.2 Equivalent characterizations of nodes

Recall from §A.3.4 that the singular locus $\mathrm{Sing}(C)$ of a curve C is defined scheme-theoretically as the vanishing of the first fitting ideal sheaf of Ω_C: locally if $C = V(f_1, \ldots, f_m) \subseteq \mathbb{A}^n$, then $\mathrm{Sing}(C)$ is defined by the vanishing of all $(n-1) \times (n-1)$ minors of the Jacobian matrix $J = (\frac{\partial f_j}{\partial x_i})$; note that if $C = V(f) \subseteq \mathbb{A}^2$ is a plane affine curve, then $\mathrm{Sing}(C) = V(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})$. We will also use properties of local complete intersections as discussed in §A.3.5.

Proposition 5.2.4 (Characterization of Nodes)**.** *Let C be a pure one dimensional curve over a field $\Bbbk$, and let $p \in C$ be a closed point with maximal ideal $\mathfrak{m} \subseteq \mathcal{O}_{C,p}$. The following are equivalent:*

(1) $p \in C$ is a node;

(2) C is a local complete intersection at p, and $\mathrm{Sing}(C)$ is unramified over $\Bbbk$ at $p \in \mathrm{Sing}(C)$;

(3) $\Bbbk \to \kappa(p)$ is separable, $\mathcal{O}_{C,p}$ is reduced, $\dim \mathfrak{m}/\mathfrak{m}^2 = 2$, and there is a nondegenerate quadratic form $q \in \mathrm{Sym}^2 \mathfrak{m}/\mathfrak{m}^2$ mapping to 0 in $\mathfrak{m}^2/\mathfrak{m}^3$;

(4) $\Bbbk \to \kappa(p)$ is separable and $\widehat{\mathcal{O}}_{C,p} \cong \kappa(p)[\![x, y]\!]/(q)$ where q is a nondegenerate quadratic form; and

(5) there exists a finite separable field extension $\Bbbk \to \Bbbk'$ and a point $p' \in C_{\Bbbk'}$ such that $\widehat{\mathcal{O}}_{C_{\Bbbk'},p'} \cong \Bbbk'[\![x, y]\!]/(xy)$.

Proof. Assuming (1), let $\overline{p} \in C_{\overline{\Bbbk}}$ be a node over p and let $\mathrm{Sing}(C) \subseteq C$ be the scheme-theoretic singular locus. By properties of fitting ideals (see §A.3.4), $\mathrm{Sing}(C) \times_{\Bbbk} \overline{\Bbbk} = \mathrm{Sing}(C_{\overline{\Bbbk}})$ and the preimage of $\mathrm{Sing}(C_{\overline{\Bbbk}})$ under $\mathrm{Spec}\, \widehat{\mathcal{O}}_{C_{\overline{\Bbbk}},\overline{p}} \to C_{\overline{\Bbbk}}$ is $\mathrm{Sing}(\mathrm{Spec}\, \widehat{\mathcal{O}}_{C_{\overline{\Bbbk}},\overline{p}})$. Since $\widehat{\mathcal{O}}_{C_{\overline{\Bbbk}},\overline{p}} \cong \overline{\Bbbk}[\![x, y]\!]/(xy)$, $\mathrm{Sing}(\mathrm{Spec}\, \widehat{\mathcal{O}}_{C_{\overline{\Bbbk}},\overline{p}}) = V(x, y) = \mathrm{Spec}\, \overline{\Bbbk}$. Therefore $\mathrm{Sing}(C) \to \mathrm{Spec}\, \Bbbk$ is unramified at p. Since $\widehat{\mathcal{O}}_{C_{\overline{\Bbbk}},\overline{p}}$ is a complete intersection, C is a local complete intersection at p (Proposition A.3.16). This gives (2).

Assuming (2), since $\mathrm{Sing}(C)$ is unramified at p, the field extension $\Bbbk \to \kappa(p)$ is separable and there is an open neighborhood $U \subseteq C$ of p such that $\mathrm{Sing}(U) = \mathrm{Sing}(C) \cap U = \{p\}$. In particular, $\mathcal{O}_{C,p}$ is generically reduced. On the other hand, since C is a local complete intersection, $\mathcal{O}_{C,p}$ is a one dimensional Cohen–Macaulay local ring and thus has no embedded primes. It follows that $\mathcal{O}_{C,p}$ is reduced. Using that C is a local complete intersection, we can write $\widehat{\mathcal{O}}_{C,p} = R/(f_1, \ldots, f_{n-1})$ where R is a regular complete local ring. As R contains the base field $\Bbbk$, the Cohen Structure Theorem (B.5.7) implies that $R \cong \kappa(p)[\![x_1, \ldots, x_n]\!]$. Since $\mathrm{Sing}(C)$ is unramified at p, the $(n-1) \times (n-1)$ minors of the Jacobian matrix $\left(\frac{\partial f_j}{\partial x_i}\right)_{i,j}$ generate the maximal ideal $\mathfrak{m} = (x_1, \ldots, x_n) \subseteq \widehat{\mathcal{O}}_{C,p}$. If $\frac{\partial f_j}{\partial x_i} \in R$ is a unit for some i and j, then the sequence $x_1, \ldots, \widehat{x_i}, x_n, f_j$ also generates $\mathfrak{m}/\mathfrak{m}^2$. We may use Complete Nakayama's Lemma (B.5.6(3)) to change coordinates by replacing the generators $x_1, \ldots, x_n$ with $x_1, \ldots, \widehat{x_i}, x_n, f_j$. Eliminating f_j allows us to write $R = \kappa(p)[\![x_1, \ldots, \widehat{x_i}, x_n]\!]/(f_1, \ldots, \widehat{f_j}, \ldots, f_{n-1})$. After finitely many such replacements, we can assume that $\frac{\partial f_j}{\partial x_i} \in \mathfrak{m}$ for every i, j. This implies that every $(n-1) \times (n-1)$ minor is in $\mathfrak{m}^{n-1}$, but since these minors generate $\mathfrak{m}$, we must have that $n = 2$. Therefore, $\widehat{\mathcal{O}}_{C,p} = \kappa(p)[\![x, y]\!]/(f)$ with $f = f_2 + f_3 + \cdots$ and each f_i homogeneous of degree i. Since the partials f_x and f_y generate (x, y), the quadratic form $q := f_2 \in \mathrm{Sym}^2 \mathfrak{m}/\mathfrak{m}^2$ must be nondegenerate. This gives (3).

Assuming (3), we have that $\dim_{\kappa(p)} \mathfrak{m}^d/\mathfrak{m}^{d+1} = 2$ for every $d \geq 1$ since q maps to 0 in $\mathfrak{m}^2/\mathfrak{m}^3$. A choice of elements $x_0, y_0 \in \mathfrak{m}$ mapping to a basis in $\mathfrak{m}/\mathfrak{m}^2$ induces a surjection $\kappa(p)[\![x,y]\!] \to \widehat{\mathcal{O}}_{C,p}$ by Complete Nakayama's Lemma (B.5.6(3)). Since $\mathcal{O}_{C,p}$ is reduced, $\widehat{\mathcal{O}}_{C,p}$ is also reduced (Proposition 2.1.21). Since $\kappa(p)[\![x,y]\!]$ is a UFD, we can write $\kappa(p)[\![x,y]\!]/(f)$ for an element f expressed as a product of distinct irreducible elements such that the quadratic component $q = ax^2 + bxy + cy^2$ of f is a nondegenerate quadratic form. We claim that we can modify our choice of coordinates $x_0, y_0 \in \mathfrak{m}$ so that $f = q$, i.e., $q(x_0, y_0) = 0 \in \widehat{\mathcal{O}}_{C,p}$. We will show inductively that for each N, there exists elements $x_i, y_i \in \mathfrak{m}^{i+1}$ for $i = 0, \ldots, N$ such that $q(x_0 + \cdots + x_N, y_0 + \cdots + y_N) \in \mathfrak{m}^{N+3}$. Since $\widehat{\mathcal{O}}_{C,p}$ is complete, this would enable us to replace x_0 and y_0 with $\sum_i x_i$ and $\sum_i y_i$ and conclude that $\widehat{\mathcal{O}}_{C,p} \cong \kappa(p)[\![x,y]\!]/(q)$. Supposing that we have already chosen $x' = x_0 + \cdots + x_{N-1}$ and $y' = y_0 + \cdots + y_{N-1}$, then for every x_N and $y_N \in \mathfrak{m}^{N+1}$, we have that

$$q(x' + x_N, y' + y_N) = q(x', y') + (2ax_0 + by_0)x_N + (bx_0 + 2cy_0)y_N \mod \mathfrak{m}^{N+3}$$

The nondegeneracy of $q = ax^2 + bxy + y^2$ implies that $2ax_0 + by_0$ and $bx_0 + 2cy_0$ are linearly independent. Since $\dim_{\kappa(p)} \mathfrak{m}^{N+2}/\mathfrak{m}^{N+3} = 2$, we may choose x_N and y_N such that $Q(x' + x_N, y' + y_N) \in \mathfrak{m}^{N+3}$. This completes (4).

Assuming (4) and using that q is nondegenerate, we may choose a degree 2 separable field extension $\kappa(p) \to \Bbbk'$ such that q splits as a product of linear forms. Thus $\widehat{\mathcal{O}}_{C,p} \otimes_{\Bbbk} \Bbbk' \cong \Bbbk'[\![x,y]\!]/(xy)$, yielding (5). Finally, (5) implies that p is a node. See also [SP, Tags 0C49, 0C4D, and 0C4E]. □

Exercise 5.2.5. (easy) Show that $\operatorname{Spec} \Bbbk[x,y]/(f)$ has a node at 0 if and only if $f(0) = f_x(0) = f_y(0) = 0$ and $\det \begin{pmatrix} f_{xx}(0) & f_{xy}(0) \\ f_{yx}(0) & f_{yy}(0) \end{pmatrix}$ is nonzero.

Exercise 5.2.6. (good practice) Let C be a pure one dimensional reduced curve over a field $\Bbbk$ with normalization $\pi \colon \widetilde{C} \to C$. Show that $p \in C$ is a node if and only if $\Bbbk \to \kappa(p)$ is separable, $\dim_{\kappa(p)}(\pi_* \mathcal{O}_{\widetilde{C}}/\mathcal{O}_C) \otimes \kappa(p) = 1$, and $\sum_{\pi(q)=p} [\kappa(q) \colon \kappa(p)] = 2$.

Hint: Identify $(\pi_* \mathcal{O}_{\widetilde{C}}/\mathcal{O}_C) \otimes \kappa(p)$ with the quotient $\widetilde{A}/A$ where $A = \widehat{\mathcal{O}}_{C,p}$ and $\widetilde{A}$ is its normalization, using that normalization commutes with completion (see Remark 2.1.25). To show ($\Leftarrow$), use that $\widetilde{A}$ is a product of complete DVRs to derive the structure of A. See also [SP, Tag 0C4A].

Remark 5.2.7. The quantity $\dim_{\kappa(p)}(\pi_* \mathcal{O}_{\widetilde{C}}/\mathcal{O}_C) \otimes \kappa(p)$ is referred to as the *δ-invariant of p*, and the sum $\sum_{\pi(q)=p} [\kappa(q) \colon \kappa(p)]_{\mathrm{sep}}$ is referred to as the *number of geometric branches over p*. A cusp $\Bbbk[\![x,y]\!]/(y^2 - x^3)$ has δ-invariant one but has only one geometric branch.

Proposition 5.2.8 (Local Structure of Nodes)**.** *Let C be a curve over a field $\Bbbk$. If $p \in C$ is a node, then there exists a finite separable field extension $\Bbbk \to \Bbbk'$ and étale neighborhoods*

$$\begin{tikzcd} & (U,u) \arrow[dl] \arrow[dr] & \\ (C,p) & & (\operatorname{Spec} \Bbbk'[x,y]/(xy), 0). \end{tikzcd} \tag{5.2.9}$$

Proof. Using the Characterization of Nodes (5.2.4(5)), there is a finite separable field extension $\Bbbk \to \Bbbk'$ such that $\widehat{\mathcal{O}}_{C,p} \otimes_{\Bbbk} \Bbbk' \cong \Bbbk'[\![x,y]\!]/(xy)$. The result is now a consequence of Artin Approximation (B.5.21). □

We will shortly generalize this to families of nodal curves (Theorem 5.2.25).

Example 5.2.10. Let us construct an explicit étale neighborhood for the nodal cubic $C = \operatorname{Spec} \mathbb{C}[x,y]/(y^2 - x^2(x-1))$. In fact, in Example 0.5.3 we essentially already showed how do this: if we add a square root $t = \sqrt{x-1}$, then $y^2 - x^3 + x^2 = (y - xt)(y + xt)$. Therefore, we can take the elementary étale neighborhood $U = \operatorname{Spec} \mathbb{C}[x,y,t]_t/(y^2 - x^3 + x^2, t^2 - x + 1) \to C$ given by $(x,y,t) \mapsto (x,y)$ and the étale neighborhood $U \to \operatorname{Spec} \mathbb{C}[x,y]/(xy)$ defined by $(x,y,t) \mapsto (y - xt, y + xt)$.

Exercise 5.2.11. Provide a proof of the Local Structure of Nodes (5.2.8) without appealing to Artin Approximation.

Hint: Use that the normalization of a strict henselization $\mathcal{O}^{\mathrm{sh}}_{C,p}$ has two components to find an affine étale neighborhood $(\operatorname{Spec} R, u) \to (C,p)$ with $\widetilde{R} = R_1 \times R_2$. Use the exact sequence $0 \to R \to R_1 \times R_2 \to \kappa(u) \to 0$ to construct elements $x, y \in R$ mapping to $(1,0)$, $(0,1) \in R_1 \times R_2$, and argue that $\kappa(u)[x,y]/(xy) \to R$ is étale.

5.2.3 Genus formula

Proposition 5.2.12 (Genus Formula)**.** *Let C be a connected, nodal, and projective curve over an algebraically closed field $\Bbbk$ with δ nodes and ν irreducible components. Let $C = \bigcup_i C_i$ be the irreducible decomposition and let $\widetilde{C}_i$ be the normalization of C_i with genus $g(\widetilde{C}_i)$. The genus g of C satisfies*

$$g = \sum_{i=1}^{\nu} g(\widetilde{C}_i) + \delta - \nu + 1.$$

Proof. Let $p_1, \ldots, p_\delta \in C$ denote the nodes of C. We claim that the normalization $\pi \colon \widetilde{C} \to C$ induces a short exact sequence

$$0 \to \mathcal{O}_C \to \pi_* \mathcal{O}_{\widetilde{C}} \to \bigoplus_{j=1}^{\delta} \kappa(p_j) \to 0.$$

It suffices to verify this étale-locally around a node $p_i \in C$, and so by the Local Structure of Nodes (5.2.8), we can assume that $C = \operatorname{Spec} \Bbbk[x,y]/(xy)$. In this case, $\widetilde{C} = \operatorname{Spec}(\Bbbk[x] \times \Bbbk[y])$ and the sequence above corresponds to $0 \to \Bbbk[x,y]/(xy) \to \Bbbk[x] \times \Bbbk[y] \to \Bbbk \to 0$. Alternatively, normalization commutes with completion and a direct calculation as above shows that if $A := \widehat{\mathcal{O}}_{C,p} \cong \Bbbk[\![x,y]\!]/(xy)$, then $\widetilde{A}/A \cong \Bbbk$.

The short exact sequence induces a long exact sequence on cohomology

$$0 \to \underbrace{\mathrm{H}^0(C, \mathcal{O}_C)}_{1} \to \underbrace{\mathrm{H}^0(\widetilde{C}, \mathcal{O}_{\widetilde{C}})}_{\nu} \to \underbrace{\bigoplus_j \kappa(p_j)}_{\delta} \to \underbrace{\mathrm{H}^1(C, \mathcal{O}_C)}_{g} \to \underbrace{\mathrm{H}^1(\widetilde{C}, \mathcal{O}_{\widetilde{C}})}_{\sum_i g(\widetilde{C}_i)} \to 0$$

where the labels indicate the dimensions. The statement follows. $\square$

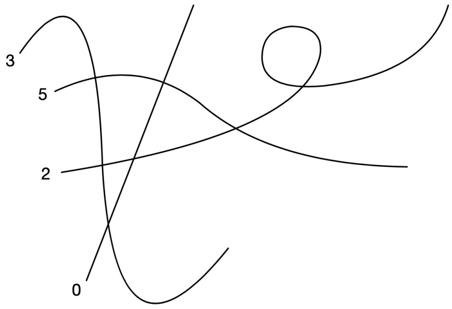


Figure 5.2.13: An example of a nodal curve of genus 14.

Remark 5.2.14. Notice that $\delta - \nu + 1$ is precisely the number of connected regions bounded by the curve C as in Figure 5.2.13. Thus, the genus of a nodal curve can be easily computed from the picture by summing the geometric genera of the irreducible components and adding the number of bounded regions. See Definition 5.3.8 for another interpretation using dual graphs. For non-nodal curves, there is an analogous formula for the genus involving the delta invariants (see Remark 5.2.7) of the singularities.

5.2.4 The dualizing sheaf

Since a nodal curve C over a field $\Bbbk$ is a locally a complete intersection, C is Gorenstein and there is a *dualizing line bundle* ω_C with a trace map $\mathrm{tr}_C \colon \mathrm{H}^1(C, \omega_C) \xrightarrow{\sim} \Bbbk$; see [Har77, III.7.11] or [Ser88, §IV]. In other words, for every coherent sheaf F, the natural pairing

$$\mathrm{Hom}_{\mathcal{O}_C}(F, \omega_C) \times \mathrm{H}^1(C, F) \to \mathrm{H}^1(C, \omega_C) \xrightarrow{\mathrm{tr}} \Bbbk$$

is perfect. Due to its importance in the study of stable curves, we provide an explicit description of ω_C below over an algebraically closed field $\Bbbk$. Let $\Sigma := C^{\mathrm{sing}}$ be the singular locus and $U = C \smallsetminus \Sigma$. Let $\pi \colon \widetilde{C} \to C$ be the normalization of C, and let $\widetilde{\Sigma}$ and $\widetilde{U}$ be the preimages of Σ and U as in the diagram

$$\begin{tikzcd} \widetilde{U} \arrow[r, hook] \arrow[d, "\wr"] & \widetilde{C} \arrow[d, "\pi"] & \widetilde{\Sigma} \arrow[l, hook'] \arrow[d] \\ U \arrow[r, hook] & C & \Sigma. \arrow[l, hook'] \end{tikzcd} \tag{5.2.15}$$

Let $\Sigma = \{z_1, \ldots, z_n\}$ be an ordering of the points and $\pi^{-1}(z_i) = \{p_i, q_i\}$. Since $\widetilde{C}$ is smooth, the sheaf of differentials $\Omega_{\widetilde{C}}$ is a dualizing line bundle. There is a short exact sequence

$$0 \to \Omega_{\widetilde{C}} \to \Omega_{\widetilde{C}}(\widetilde{\Sigma}) \to \mathcal{O}_{\widetilde{\Sigma}} \to 0 \tag{5.2.16}$$

obtained by tensoring the sequence $0 \to \mathcal{O}_{\widetilde{C}}(-\widetilde{\Sigma}) \to \mathcal{O}_{\widetilde{C}} \to \mathcal{O}_{\widetilde{\Sigma}} \to 0$ with $\Omega_{\widetilde{C}}(\widetilde{\Sigma})$. As $\Omega_{\widetilde{C}}(\widetilde{\Sigma})|_{\widetilde{U}} = \Omega_{\widetilde{U}}$, we can interpret sections of $\Omega_{\widetilde{C}}(\widetilde{\Sigma})$ as rational sections of $\Omega_{\widetilde{C}}$ with at worst simple poles along $\widetilde{\Sigma}$. Evaluating (5.2.16) on an open $\widetilde{V} \subseteq \widetilde{C}$ yields

$$\begin{array}{c} 0 \longrightarrow \Gamma(\widetilde{V}, \Omega_{\widetilde{C}}) \longrightarrow \Gamma(\widetilde{V}, \Omega_{\widetilde{C}}(\widetilde{\Sigma})) \longrightarrow \bigoplus_{y \in \widetilde{V} \cap \widetilde{\Sigma}} \kappa(y), \\ \qquad\qquad\qquad\qquad\qquad\qquad s \mapsto (\mathrm{res}_y(s)) \end{array} \tag{5.2.17}$$

where the last map can be interpreted as taking a rational section $s \in \Gamma(\widetilde{V} \cap \widetilde{U}, \Omega_{\widetilde{C}})$ to the tuple whose coordinate at $y \in \widetilde{V} \cap \widetilde{\Sigma}$ is the *residue* $\operatorname{res}_y(s)$ *of* s *at* y.

Definition 5.2.18. Let C be a nodal curve over an algebraically closed field $\Bbbk$. Using the notation of (5.2.15) and (5.2.17), we define the subsheaf $\omega_C \subseteq \pi_*\Omega_{\widetilde{C}}(\widetilde{\Sigma})$ by declaring that sections along $V \subseteq C$ consist of rational sections s of $\Omega_{\widetilde{C}}$ along $\pi^{-1}(V)$ with at worst simple poles along $\widetilde{\Sigma}$ such that for every node $z_i \in V \cap \Sigma$ with preimages $p_i, q_i \in \pi^{-1}(V)$, $\operatorname{res}_{p_i}(s) + \operatorname{res}_{q_i}(s) = 0$.

The definition implies that ω_C sits in the following two exact sequences:

$$0 \longrightarrow \omega_C \longrightarrow \pi_*\Omega_{\widetilde{C}}(\widetilde{\Sigma}) \xrightarrow[s \mapsto (\operatorname{res}_{p_i}(s) + \operatorname{res}_{q_i}(s))]{} \bigoplus_{z_i \in \Sigma} \Bbbk \longrightarrow 0$$

$$0 \longrightarrow \pi_*\Omega_{\widetilde{C}} \longrightarrow \omega_C \xrightarrow[s \mapsto (\operatorname{res}_{p_i}(s))]{} \bigoplus_{z_i \in \Sigma} \Bbbk \longrightarrow 0.$$

Example 5.2.19 (Local calculation)**.** Let $C = \operatorname{Spec} \Bbbk[x,y]/(xy)$. Then $\widetilde{C} = \mathbb{A}^1 \amalg \mathbb{A}^1$ with coordinates x and y respectively. The singular locus of C is $\Sigma = \{0\}$ with preimage $\widetilde{\Sigma}$ consisting of the two origins. Then $\Gamma(\widetilde{C}, \Omega_{\widetilde{C}}) = \Gamma(\mathbb{A}^1, \Omega_{\mathbb{A}^1}) \times \Gamma(\mathbb{A}^1, \Omega_{\mathbb{A}^1})$ and $(\frac{dx}{x}, -\frac{dy}{y})$ is a rational section of $\Omega_{\widetilde{C}}$ with opposite residues at p and q. In fact, every section of $\Gamma(C, \omega_C)$ is of the form

$$\left(f(x)\frac{dx}{x}, g(y)\frac{-dy}{y}\right) = (f(x) + g(y) - f(0)) \cdot (\frac{dx}{x}, \frac{-dy}{y})$$

for polynomials $f(x)$ and $g(y)$ such that $f(0) = g(0)$, which is precisely the condition for $(f,g) \in \Gamma(\widetilde{C}, \mathcal{O}_{\widetilde{C}})$ to descend to a global function $f(x) + g(y) - f(0) \in \Gamma(C, \mathcal{O}_C)$. In other words, $\omega_C \cong \mathcal{O}_C$ with generator $(\frac{dx}{x}, -\frac{dy}{y})$.

Example 5.2.20. Let C be the nodal projective plane cubic and $\mathbb{P}^1 \to C$ be the normalization with coordinates $[x:y]$ such that 0 and ∞ are the fibers of the node. The rational differential $\eta := \frac{dx}{x} = -\frac{dy}{y}$ on $\mathbb{P}^1$ satisfies $\operatorname{res}_0 \eta + \operatorname{res}_\infty \eta = 0$, and it is not hard to see that every local section of ω_C is a multiple of η; in other words, $\omega_C \cong \mathcal{O}_C$ with generator η.

Exercise 5.2.21 (details)**.** Let C be a connected, nodal, and projective curve over an algebraically closed field $\Bbbk$. Define temporarily the subsheaf

$$K_C = \ker\left(\pi_*\Omega_{\widetilde{C}}(\widetilde{\Sigma}) \to \bigoplus_{z_i \in \Sigma} \Bbbk\right)$$

as in Definition 5.2.18.

(a) Show that if $\pi \colon C' \to C$ is an étale morphism, then $\pi^* K_C \cong K_{C'}$.

Hint: Use the fact that normalization commutes with étale base change.

(b) Show that K_C is a line bundle.

Hint: Use the Local Structure of Nodes (5.2.8) and the local calculation in Example 5.2.19.

(c) (hard) Show that K_C is a dualizing sheaf.

Hint: Reduce to the case of a smooth curve by using the normalization:

(i) Use Yoneda's Lemma to show that it suffices to exhibit an isomorphism $\mathrm{Hom}_{\mathcal{O}_C}(-, K_C) \cong \mathrm{H}^1(C, -)^\vee$ *of functors* $\mathrm{Pic}(C) \to \mathrm{Vect}_{\Bbbk}$.

(ii) For any $L \in \mathrm{Pic}(C)$*, show that tensoring the exact sequence defining* K_C *by* $L^\vee$ *and taking cohomology gives identifications*

$$\mathrm{Hom}_{\mathcal{O}_C}(L, K_C) \cong \mathrm{H}^0(C, L^\vee \otimes K_C) \cong \ker\left(\mathrm{H}^0(\widetilde{C}, \pi^* L^\vee \otimes \Omega_{\widetilde{C}}(\widetilde{\Sigma})) \to \bigoplus_{z_i \in \Sigma} \Bbbk\right).$$

(iii) Use the short exact sequence $0 \to \pi_*\pi^*(L(\widetilde{\Sigma})) \to L \to \bigoplus_{z_i \in \Sigma} \Bbbk \to 0$ *to deduce that*

$$\mathrm{H}^1(C, L)^\vee = \ker\left(\mathrm{H}^1(\widetilde{C}, \pi^* L(\widetilde{\Sigma}))^\vee \to \bigoplus_{z_i \in \Sigma} \Bbbk\right).$$

(iv) Show that the isomorphism $\mathrm{H}^0(\widetilde{C}, \pi^* L^\vee \otimes \Omega_{\widetilde{C}}(\widetilde{\Sigma})) \cong \mathrm{H}^1(\widetilde{C}, \pi^* L(\widetilde{\Sigma}))^\vee$ *given by Serre Duality on* $\widetilde{C}$ *commutes with the two maps to* $\bigoplus_{z_i \in \Sigma} \Bbbk$.

See also [Ols16, Prop 13.2.9]. For other approaches, see [Ser88, Ch. IV §3], [Liu02, Lem. 10.3.12], and [ACG11, Ch. X §2].

(d) If $T \subseteq C$ is a subcurve with complement $T^c := \overline{C \smallsetminus T}$, show that

$$\omega_C|_T = \omega_T(T \cap T^c).$$

Exercise 5.2.22 (good practice)**.** Let C be a connected, nodal, and projective curve over an algebraically closed field $\Bbbk$. Let $\pi \colon \widetilde{C} \to C$ be the normalization and $\widetilde{\Sigma} \subseteq \widetilde{C}$ the set of preimages of nodes. Show that there is an identification

$$\pi_* \mathscr{H}om_{\mathcal{O}_{\widetilde{C}}}(\Omega_{\widetilde{C}}(\widetilde{\Sigma}), \mathcal{O}_{\widetilde{C}}) \cong \mathscr{H}om_{\mathcal{O}_C}(\Omega_C, \mathcal{O}_C),$$

or equivalently that $\pi_*(T_{\widetilde{C}}(-\widetilde{\Sigma})) = T_C$. In other words, regular vector fields on C correspond to regular vector fields on $\widetilde{C}$ vanishing at the preimages of nodes.

5.2.5 Nodal families

Recall that the relative singular locus $\mathrm{Sing}(\mathcal{C}/S)$ of a morphism $\mathcal{C} \to S$ with one dimensional fibers is defined by the first fitting ideal sheaf of $\Omega_{\mathcal{C}/S}$; see Definition A.3.14. *Syntomic morphisms* are fppf morphisms (i.e., flat and locally of finite presentation) whose fibers are local complete intersections; see §A.3.5.

Proposition 5.2.23. *Let* $\mathcal{C} \to S$ *be an fppf morphism of schemes and* $s \in S$ *a point such that the fiber* $\mathcal{C}_s$ *is pure one dimensional. A point* $p \in \mathcal{C}_s$ *is a node if and only if* $\mathcal{C} \to S$ *is syntomic at* p *and the relative singular locus* $\mathrm{Sing}(\mathcal{C}/S) \to S$ *is unramified at* p.

Proof. The conditions that $\mathcal{C} \to S$ is syntomic at p and $\mathrm{Sing}(\mathcal{C}) \to S$ is unramified at p are both conditions on the fibers over s. Since $\mathrm{Sing}(\mathcal{C}/S)_s = \mathrm{Sing}(\mathcal{C}_s)$, the result follows from the equivalence of (1) $\Leftrightarrow$ (4) of the Characterization of Nodes (5.2.4). □

The above characterization implies that the property of being a nodal family descends under limits (Definition B.3.6).

Lemma 5.2.24. *The following property of morphisms of schemes descends under limits: an fppf morphism such that every fiber is a pure one dimensional nodal curve.*

Proof. From Descent of Properties of Morphisms under Limits (B.3.7), we know that the properties of being fppf, syntomic, unramified, and having connected pure one dimensional fibers descend under limits. Since the relative singular locus commutes with base change, the result follows from Proposition 5.2.23. □

5.2.6 Local structure of nodal families

By the Local Structure of Smooth Morphisms (A.3.4), if $\mathcal{C} \to S$ is a family of smooth curves, then every point $p \in \mathcal{C}$ over $s \in S$ is étale locally isomorphic to relative one dimensional affine space. More precisely, there is a commutative diagram

$$\begin{tikzcd}
(\mathcal{C}, p) \arrow[d] & (\mathcal{C}', p') \arrow[l, hook', "\text{op}"'] \arrow[r, "\text{ét}"] \arrow[d] & (S' \times_{\mathbb{Z}} \mathbb{A}^1_{\mathbb{Z}}, (s', 0)) \arrow[dl] \\
(S, s) & (S', s'), \arrow[l, hook', "\text{op}"'] &
\end{tikzcd}$$

where the left horizontal maps are open immersions, the right-hand map is étale map, and $S' \times_{\mathbb{Z}} \mathbb{A}^1_{\mathbb{Z}} = \mathbb{A}^1_{S'} \to S'$ is the base change of $\mathbb{A}^1_{\mathbb{Z}} \to \operatorname{Spec} \mathbb{Z}$. We establish an analogous local structure for a family of nodal curves providing a relative version of the Local Structure of Nodes (5.2.8).

Theorem 5.2.25 (Local Structure of Nodal Families)**.** *Let $\pi \colon \mathcal{C} \to S$ be an fppf morphism such that every geometric fiber is a curve. Let $p \in \mathcal{C}$ be a node in the fiber $\mathcal{C}_s$ over a point $s \in S$. There is a commutative diagram*

$$\begin{tikzcd}
(\mathcal{C}, p) \arrow[d] & (\mathcal{C}', p') \arrow[l, "\text{\'et}"'] \arrow[r, "\text{\'et}"] \arrow[d] & (\operatorname{Spec} A[x,y]/(xy - f), (s', 0)) \arrow[dl] \\
(S, s) & (\operatorname{Spec} A, s'), \arrow[l, "\text{\'et}"'] &
\end{tikzcd} \tag{5.2.26}$$

where each horizontal map is étale and $f \in A$ is a function vanishing at s'.

Remark 5.2.27. In other words, every family of nodal curves étale locally on the source and target has the structure of the top horizontal arrow in the fiber product

$$\begin{tikzcd}
\operatorname{Spec} A[x,y]/(xy - f) \arrow[r] \arrow[d] & \operatorname{Spec} A \arrow[d, "f"] \\
\operatorname{Spec} \mathbb{Z}[x,y,t]/(xy - t) \arrow[r] & \operatorname{Spec} \mathbb{Z}[t]
\end{tikzcd}$$

induced by a function $f \in A$.

Proof. We offer a local-to-global proof.

Step 1: Reduce to the case where S is of finite type over $\mathbb{Z}$. Use limit methods and Lemma 5.2.24.

Step 2: Reduce to the case where $\widehat{\mathcal{O}}_{\mathcal{C}_s,p} \cong \kappa(s)[\![x,y]\!]/(xy)$. By Proposition 5.2.4, there is a finite separable field extension $\kappa(s) \to \Bbbk'$ and a point $p' \in \mathcal{C}_s \times_{\kappa(s)} \Bbbk'$

whose completion is isomorphic to $\Bbbk'[[x,y]]/(xy)$. Letting $(S',s') \to (S,s)$ be an étale morphism such that $\kappa(s') \cong \Bbbk'$ over $\kappa(s)$, we replace S with S'.

Step 3: Show that $\widehat{\mathcal{O}}_{\mathcal{C},p} \cong \widehat{\mathcal{O}}_{S,s}[[x,y]]/(xy-\widehat{f})$ for a function $\widehat{f} \in \widehat{\mathfrak{m}}_s \subseteq \widehat{\mathcal{O}}_{S,s}$. We claim that there exists elements $\widehat{x}, \widehat{y} \in \widehat{\mathcal{O}}_{\mathcal{C},p}$ and $\widehat{f} \in \widehat{\mathfrak{m}}_s \subseteq \widehat{\mathcal{O}}_{S,s}$ such that $\widehat{x}\widehat{y} - \widehat{f} = 0$. To achieve this, we will inductively construct elements $x_n, y_n \in \widehat{\mathcal{O}}_{\mathcal{C},p}$ and $f_n \in \widehat{\mathfrak{m}}_s$ for $n \geq 0$ which are compatible (i.e., $x_{n+1} \equiv x_n (\mathrm{mod}\ \mathfrak{m}^{n+1})$, etc.) such that

$$x_n y_n - f_n \in \widehat{\mathfrak{m}}_s^{n+1} \widehat{\mathcal{O}}_{\mathcal{C},p}. \tag{5.2.28}$$

The claim follows by defining $\widehat{x} = \lim_n x_n$, $\widehat{y} = \lim_n y_n$, and $\widehat{f} = \lim_n f_n$.

The base case $n=0$ is handled by Step 2: letting $\overline{x}, \overline{y} \in \widehat{\mathcal{O}}_{\mathcal{C}_s,p}$ be the images of x and y under the isomorphism $\widehat{\mathcal{O}}_{\mathcal{C}_s,p} \cong \kappa(s)[[x,y]]/(xy)$, choose $x_0, y_0 \in \widehat{\mathcal{O}}_{\mathcal{C},p}$ to be any lifts of $\overline{x}$ and $\overline{y}$ under the surjection $\widehat{\mathcal{O}}_{\mathcal{C},p} \to \widehat{\mathcal{O}}_{\mathcal{C}_s,p}$ and set $f_0 = 0$. Assuming that we have constructed x_n, y_n, and f_n satisfying (5.2.28), write

$$x_n y_n - f_n = \sum_i a_i b_i, \qquad \text{where } a_i \in \widehat{\mathfrak{m}}_s^{n+1} \text{ and } b_i \in \widehat{\mathcal{O}}_{\mathcal{C},p}.$$

Since x_n and y_n generate the maximal ideal of p in the fiber $\mathcal{C}_s$, we can use the identity $\kappa(s) = \kappa(p)$ on residue fields to find $a'_i \in \widehat{\mathcal{O}}_{S,s}$ and $b'_i, b''_i \in \widehat{\mathcal{O}}_{\mathcal{C},p}$ such that

$$b_i - (x_n b'_i + y_n b''_i + a'_i) \in \widehat{\mathfrak{m}}_s \widehat{\mathcal{O}}_{\mathcal{C},p}.$$

We then define

$$x_{n+1} = x_n - \sum_i a_i b''_i, \quad y_{n+1} = y_n - \sum_i a_i b'_i, \quad \text{and} \quad f_{n+1} = f_n + \sum_i a_i a'_i,$$

and check that

$$\begin{aligned} x_{n+1}y_{n+1} - f_{n+1} &= (x_n - \sum_i a_i b''_i)(y_n - \sum_i a_i b'_i) - (f_n + \sum_i a_i a'_i) \\ &= (x_n y_n - f_n) - x_n \sum_i a_i b'_i - y_n \sum_i a_i b''_i - \sum_i a_i a'_i + \sum_{i,j} a_i a_j b''_i b'_j \\ &= \sum_i a_i b_i - x_n \sum_i a_i b'_i - y_n \sum_i a_i b''_i - \sum_i a_i a'_i + \sum_{i,j} a_i a_j b''_i b'_j \\ &= \sum_i \underbrace{a_i}_{\widehat{\mathfrak{m}}_s^{n+1}} \underbrace{(b_i - x_n b'_i - y_n b''_i - a'_i)}_{\widehat{\mathfrak{m}}_s \widehat{\mathcal{O}}_{C,p}} + \sum_{i,j} \underbrace{a_i a_j}_{\widehat{\mathfrak{m}}_s^{2(n+1)}} b''_i b'_j \end{aligned}$$

is an element of $\widehat{\mathfrak{m}}_s^{n+2} \widehat{\mathcal{O}}_{C,p}$.

With the claim established, we have a well-defined $\widehat{\mathcal{O}}_{S,s}$-algebra homomorphism

$$\widehat{\mathcal{O}}_{S,s}[[x,y]]/(xy - \widehat{f}) \to \widehat{\mathcal{O}}_{C,p},$$

defined by $x \mapsto \widehat{x}$ and $y \mapsto \widehat{y}$. This map is surjective by Complete Nakayama's Lemma (B.5.6(2)) as it is surjective modulo $\widehat{\mathfrak{m}}_s$, and it is injective by a version of the local criterion for flatness (Lemma A.2.8); it is thus an isomorphism.

Step 4 (formal-to-étale): Extend the isomorphism in Step 3 on formal neighorhoods to étale neighborhoods. From Step 3, we have a diagram

$$\begin{tikzcd} \mathcal{C} \times_S \operatorname{Spec} \widehat{\mathcal{O}}_{S,s} \arrow[dr] & & \operatorname{Spec} \widehat{\mathcal{O}}_{S,s}[x,y]/(xy - \widehat{f}) \arrow[dl] \\ & \operatorname{Spec} \widehat{\mathcal{O}}_{S,s} & \end{tikzcd}$$

such that the points $(p,s) \in \mathcal{C} \times_S \operatorname{Spec} \widehat{\mathcal{O}}_{S,s}$ and $(s,0) \in \operatorname{Spec} \widehat{\mathcal{O}}_{S,s}[x,y]/(xy - \widehat{f})$ have isomorphic completions. A consequence of Artin Approximation (B.5.21) implies that there are étale morphisms

$$\begin{tikzcd}[column sep=tiny] & (U,u) \arrow[dl] \arrow[dr] & \\ (\mathcal{C} \times_S \operatorname{Spec} \widehat{\mathcal{O}}_{S,s}, (p,s)) & & (\operatorname{Spec} \widehat{\mathcal{O}}_{S,s}[x,y]/(xy - \widehat{f}), (s,0)) \end{tikzcd} \tag{5.2.29}$$

defined over $\operatorname{Spec} \widehat{\mathcal{O}}_{S,s}$. After replacing S with an open affine neighborhood of s, we can assume that $S = \operatorname{Spec} A$ is affine. By Neron–Popescu (B.5.15), we may write $\widehat{\mathcal{O}}_{S,s} = \operatorname{colim} B_\lambda$ as a directed colimit of smooth A-algebras. Set $S_\lambda = \operatorname{Spec} B_\lambda$, $\mathcal{C}_\lambda = \mathcal{C} \times_S S_\lambda$, and $U_\lambda = U \times_S S_\lambda$. For $\lambda \gg 0$, $\widehat{f} \in \widehat{\mathcal{O}}_{S,s}$ is the image of an element $f_\lambda \in B_\lambda$, and the pullbacks of x and y to $\Gamma(U, \mathcal{O}_U)$ are the pullbacks of elements in $\Gamma(U_\lambda, \mathcal{O}_{U_\lambda})$ under $U \to U_\lambda$. This yields a commutative diagram

$$\begin{tikzcd} & U_\lambda \arrow[dl] \arrow[dr] & \\ \mathcal{C}_\lambda & & \operatorname{Spec} B_\lambda[x,y]/(xy - f_\lambda) \end{tikzcd}$$

over S_λ which base changes to (5.2.29) under $\operatorname{Spec} \widehat{\mathcal{O}}_{S,s} \to S_\lambda$. By Descent of Properties of Morphisms under Limits (B.3.7), étaleness descends under limits and so the maps $U_\lambda \to \mathcal{C}_\lambda$ and $U_\lambda \to \operatorname{Spec} B_\lambda[x,y]/(xy - f_\lambda)$ are étale for $\lambda \gg 0$. Letting $u_\lambda = (u, s_\lambda) \in U_\lambda$, we have a commutative diagram

$$\begin{tikzcd} (\mathcal{C}, p) \arrow[d] & (U_\lambda, u_\lambda) \arrow[l, "\mathrm{sm}"'] \arrow[r, "\text{ét}"] \arrow[d] & (\operatorname{Spec} B_\lambda[x,y]/(xy - f_\lambda), (s_\lambda, 0)) \arrow[dl] \\ (S, s) & (\operatorname{Spec} B_\lambda, s_\lambda) \arrow[l, "\mathrm{sm}"'] & \end{tikzcd}$$

This gives our desired diagram (5.2.26) except that the left horizontal arrows are smooth rather than étale. Since smooth maps étale locally have sections (Corollary A.3.5), there is an étale map $(\operatorname{Spec} A, s') \to (S, s)$ and a map $(\operatorname{Spec} A, s') \to (S_\lambda, s_\lambda)$ over S. The result follows from setting $\mathcal{C}' := U_\lambda \times_{S_\lambda} \operatorname{Spec} A$ and $p' = (u_\lambda, s')$. See also [SP, Tag 0CBY]. $\square$

Exercise 5.2.30. Provide an alternative proof avoiding Artin Approximation/Neron–Popescu as follows:

(i) Reduce to the case where S is of finite type over $\mathbb{Z}$.

(ii) Reduce to the case that $\mathcal{C} \to S$ is a syntomic morphism of affine schemes whose geometric fibers are connected with at most two irreducible components and that $\operatorname{Sing}(\mathcal{C}/S) \to S$ is either an isomorphism or a closed immersion defined by a nonzerodivisor.

(iii) Handle the two cases that $\operatorname{Sing}(\mathcal{C}/S) = S$ and $\operatorname{Sing}(\mathcal{C}/S) = V(f)$ by using properties of normalization and henselization.

5.3 Stable curves

The only problem is to pick the "boundary" components shrewdly, i.e. to decide which non-generic varieties to allow.

ALAN MAYER AND DAVID MUMFORD [MM64A, P. 3]

Stable curves were introduced in unpublished joint work by Mayer and Mumford [MM64a, MM64b].

5.3.1 Definition and equivalences

An *n-pointed curve* is a curve C over an algebraically closed field $\Bbbk$ together with an ordered collection of $\Bbbk$-points $p_1, \ldots, p_n \in C$; we call the points $p_i \in C$ *marked points*. A point $q \in C$ of an n-pointed curve is called *special* if q is a node or a marked point.

Definition 5.3.1 (Stable, semistable, and prestable curves)**.** An n-pointed connected, nodal, and projective curve $(C, p_1, \ldots, p_n)$ of genus g over an algebraically closed field $\Bbbk$ is *stable* if

(1) $p_1, \ldots, p_n \in C(\Bbbk)$ are distinct smooth points,

(2) C is not of genus 1 without marked points, i.e., $(g, n) \neq (1, 0)$, and

(3) every smooth irreducible rational subcurve $\mathbb{P}^1 \subseteq C$ contains at least 3 special points.

If (1)–(2) hold, and (3) is replaced with the condition that every smooth rational subcurve contains at least 2 (rather than 3) special points, we say that (C, p_i) is *semistable*. If only (1)–(2) hold, we say that (C, p_i) is *prestable*.

We have the implications:

$$\boxed{\text{stable} \Rightarrow \text{semistable} \Rightarrow \text{prestable} \Rightarrow n\text{-pointed nodal.}}$$

In the unpointed case, a curve is a prestable curve if it is nodal. When $\Bbbk$ is not algebraically closed, condition (3) in the definition of stability needs to be amended; see Proposition 5.3.14

Remark 5.3.2. Note that there are no n-pointed stable curves of genus g if $(g, n) \in \{(0, 0), (0, 1), (0, 2), (1, 0)\}$, or equivalently $2g - 2 + n \leq 0$. We often impose the condition that $2g - 2 + n > 0$ to exclude these special cases.

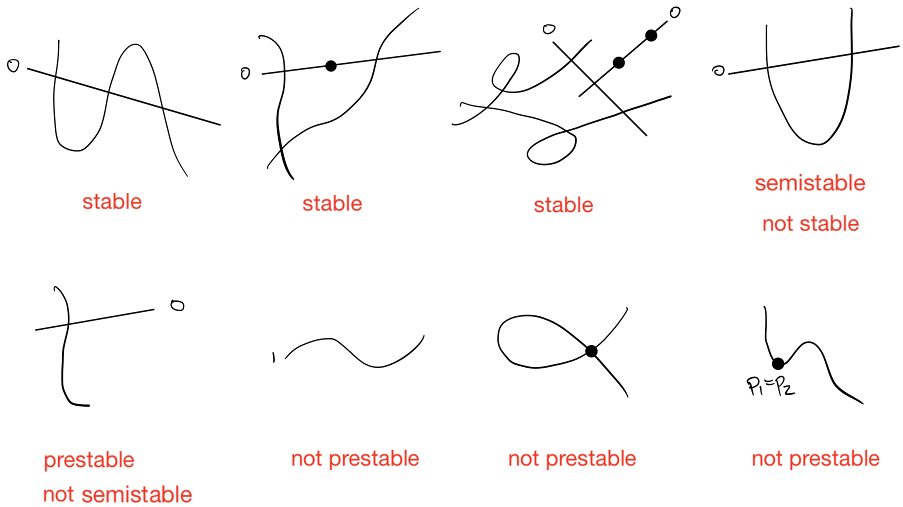


Figure 5.3.3: Examples of stable, semistable, and prestable curves

An automorphism of a stable curve $(C, p_1, \ldots, p_n)$ is an automorphism $\alpha \colon C \xrightarrow{\sim} C$ such that $\alpha(p_i) = p_i$. We denote by $\mathrm{Aut}(C, p_1, \ldots, p_n)$ the abstract group of automorphisms. Recall also that if C is a geometrically smooth, connected, and projective curve of genus $g \geq 2$, then $\mathrm{Aut}(C)$ is finite (Proposition 5.1.8).

Proposition 5.3.4. *Let $(C, p_1, \ldots, p_n)$ be an n-pointed prestable curve over an algebraically closed field $\Bbbk$. The following are equivalent:*

(1) $(C, p_1, \ldots, p_n)$ *is stable,*

(2) $\mathrm{Aut}(C, p_1, \ldots, p_n)$ *is finite, and*

(3) $\omega_C(p_1 + \cdots + p_n)$ *is ample.*

Proof. The equivalence (1) $\iff$ (2) follows from Exercise 5.3.6 and the fact that the only way that a smooth prestable n-pointed curves (C, p_i) can have a positive dimensional automorphism group is if $C = \mathbb{P}^1$ with $n \leq 2$ or if C is a genus 1 curve with $n = 0$ (see Proposition 5.1.8). To see the equivalence with (3), we will use the fact that for a subcurve $T \subseteq C$, we have $\omega_C|_T = \omega_T(T \cap T^c)$ (Exercise 5.2.21). The line bundle $\omega_C(p_1 + \cdots + p_n)$ is ample if and only if its restriction to each irreducible component $T \subseteq C$

$$\omega_C(p_1 + \cdots + p_n)|_T = \omega_T\Big(\sum_{p_i \in T} p_i + (T \cap T^c)\Big) \tag{5.3.5}$$

is ample. If the genus $g(T)$ of T is at least two, then ω_T is ample and thus so is (5.3.5). If $g(T) = 1$, then (5.3.5) is ample if and only if $n \geq 1$ or T must meet the complement T^C. If $g(T) = 0$, then (5.3.5) is ample if and only if T contains at least three special points. $\square$

Exercise 5.3.6 (Pointed normalization)**.** Let $(C, p_1, \ldots, p_n)$ be an n-pointed prestable curve. Let $\pi \colon \widetilde{C} \to C$ be the normalization of C, $\widetilde{p}_i \in \widetilde{C}$ be the unique preimage of p_i, and $\widetilde{q}_1, \ldots, \widetilde{q}_m \in \widetilde{C}$ be an ordering of the preimages of nodes. We call $(\widetilde{C}, \widetilde{p}_i, \widetilde{q}_j)$ the *pointed normalization of* (C, p_i). Show that (C, p_i) is stable if and only if every pointed connected component of $(\widetilde{C}, \widetilde{p}_i, \widetilde{q}_j)$ (i.e., each connected component of $\widetilde{C}$ together with the points $\widetilde{p}_i$ and $\widetilde{q}_j$ lying on it) is stable.

Exercise 5.3.7. Let $(C, p_1, \ldots, p_n)$ be an n-pointed prestable curve.

(a) Show that the automorphism group scheme $\underline{\mathrm{Aut}}(C, p_i)$ is an algebraic group.

(b) Show that $\underline{\mathrm{Aut}}(C, p_i)$ is naturally a closed subgroup of the automorphism group $\underline{\mathrm{Aut}}(\widetilde{C}, \widetilde{p}_i, \widetilde{q}_j)$ of the pointed normalization $(\widetilde{C}, \widetilde{p}_i, \widetilde{q}_j)$.

(c) Show that $\underline{\mathrm{Aut}}(C, p_i)^0 = \underline{\mathrm{Aut}}(\widetilde{C}, \widetilde{p}_i, \widetilde{q}_j)^0$ but that in general $\underline{\mathrm{Aut}}(C, p_i) \neq \underline{\mathrm{Aut}}(\widetilde{C}, \widetilde{p}_i, \widetilde{q}_j)$.

Dual graphs. A *vertex-weighted, n-marked graph* $\Gamma = (G, w, m)$ is the data of a finite, connected, and undirected graph G with vertices $V(G)$ and edges $E(G)$ (where loops and parallel edges are allowed), a weight function $w\colon V(G) \to \mathbb{Z}_{\geq 0}$, and an n-marking $m\colon \{1, \ldots, n\} \to V(G)$. Each vertex $m(i) \in V(G)$ is viewed as a *half-edge*. The *genus of* Γ is

$$g(\Gamma) = g(G) + \sum_{v \in V(G)} w(v),$$

where

$$g(G) = |E(G)| - |V(G)| + 1$$

denotes the genus of G, i.e., the first Betti number of G considered as a 1-dimensional CW complex. We say that a vertex-weighted, n-marked graph Γ is *stable* if for every vertex $v \in V(G)$

$$2w(v) - 2 + \mathrm{val}(v) + |m^{-1}(v)| > 0,$$

where $\mathrm{val}(v)$ is the valence of v, defined as the number of edges containing v with loops counted twice. In other words, stability means that for every vertex v either (a) $w(v) \geq 2$, (b) $w(v) = 1$ and v is contained in an edge or half-edge, or (c) $w(v) = 0$ and $\mathrm{val}(v) + |m^{-1}(v)| \geq 3$.

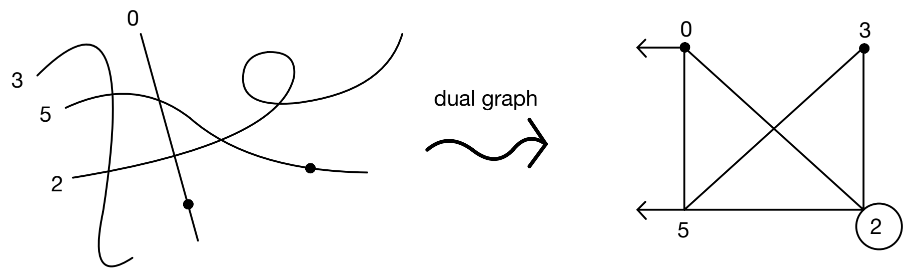


Figure 5.3.7: A genus 13 curve and its dual graph.

Definition 5.3.8 (Dual graph)**.** The *dual graph* of an n-pointed prestable curve $(C, p_1, \ldots, p_n)$ is defined as $\Gamma = (G, w, m)$, where the vertices v_i of G correspond to irreducible components C_i of C and the weight $w(v_i)$ is the geometric genus of C_i. For every node q of C lying on components C_i and C_j there is an edge e_q between v_i and v_j, and the marking $m(i)$ is the vertex v_j with $p_i \in C_j$.

Exercise 5.3.9 (easy)**.** Show that an n-pointed prestable curve (C, p_i) is stable if and only if its dual graph Γ is stable, and that the genus of C is equal to the genus of Γ.

The stack $\overline{\mathcal{M}}_{g,n}$ of stable curves admits a stratification according to the dual graph (Exercise 5.6.23).

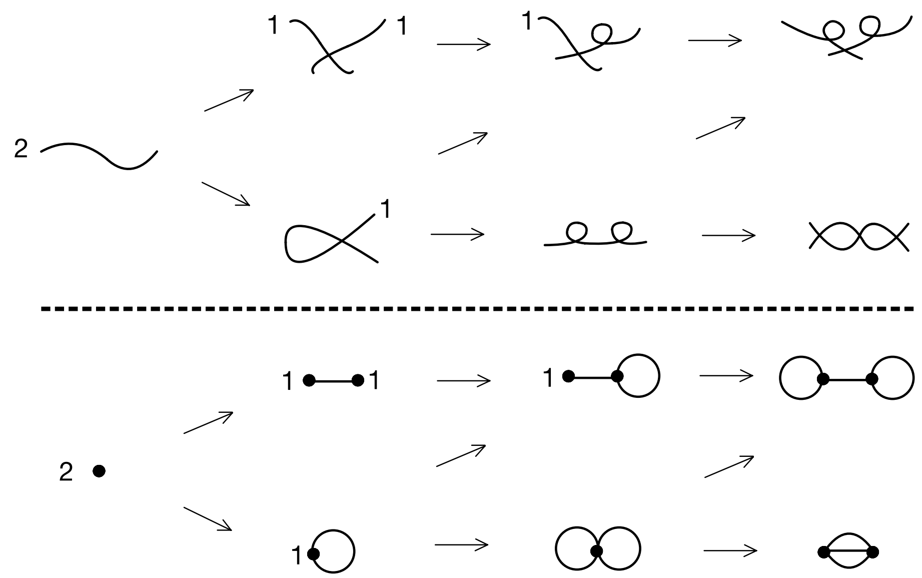


Figure 5.3.9: There are seven genus 2 vertex-weighted graphs with no marked points. Curves are displayed on the top while their dual graphs are displayed below. The arrows indicate specialization. Going from the left column to the right, the dimension of the substacks of $\overline{\mathcal{M}}_2$ parameterizing curves with the given dual graph are 3, 2, 1, and 0.

Exercise 5.3.10.

(a) Classify stable, vertex-weighted 2-marked graphs of genus 1.

(b) Determine the number of isomorphism classes of stable, vertex-weighted 0-marked graphs of genus 3.

5.3.2 Rational tails and bridges

Definition 5.3.11 (Rational tails and bridges)**.** Let $(C, p_1, \ldots, p_n)$ be an n-pointed prestable curve over a field $\Bbbk$. We say that an irreducible smooth subcurve $E \subseteq C$ of genus 0 with nonempty complement $E^c = \overline{C \setminus E}$ is

- a *rational tail* if the scheme-theoretic intersection $E \cap E^c$ is a single reduced point x, $\mathrm{H}^0(E, \mathcal{O}_E) \cong \kappa(x)$, and E contains no marked points, and
- a *rational bridge* if the scheme-theoretic intersection $E \cap E^c$ has degree 2 over $\mathrm{H}^0(E, \mathcal{O}_E)$ and E contains no marked points, or if E contains a single marked point p_j and E is a rational tail of $(C, p_1, \ldots, \widehat{p}_j, \ldots, p_n)$.

Over an algebraically closed field, every irreducible smooth rational subcurve E is isomorphic to $\mathbb{P}^1$. The subcurve E is a rational tail if $E \cdot E^c = 1$ and E has no marked points, and is a rational bridge if either $E \cdot E^c = 2$ and E has no marked points or if $E \cdot E^c = 1$ and E has precisely one marked point.

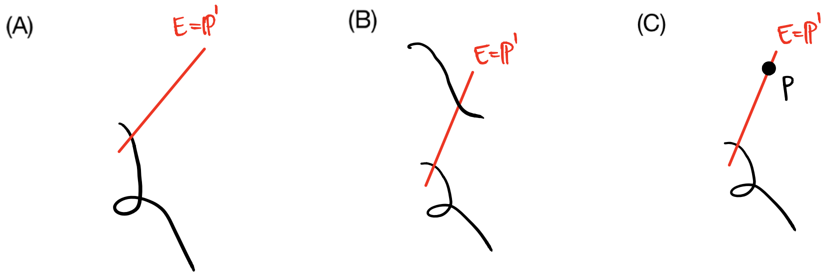


Figure 5.3.11: (A) features a rational tail while (B) and (C) feature rational bridges.

Over an arbitrary field, the classification of rational tails and bridges is more subtle, which is unsurprising given that the classification of rational curves is also more involved (see Exercises 5.1.13 and 5.1.14).

Lemma 5.3.12. *Let (C, p_i) be an n-pointed prestable curve over a field $\Bbbk$.*

(1) If E is a rational tail with $\Bbbk' := \mathrm{H}^0(E, \mathcal{O}_E) = \kappa(x)$, then $\Bbbk'$ is a finite separable field extension of $\Bbbk$ and $E \cong \mathbb{P}^1_{\Bbbk'}$.

(2) If E is a rational bridge with no marked points, then $\Bbbk' := \mathrm{H}^0(E, \mathcal{O}_E)$ is a finite separable field extension of $\Bbbk$ and $\Bbbk'' := \mathrm{H}^0(E \cap E^c, \mathcal{O}_{E\cap E^c})$ is either a degree 2 separable field extension of $\Bbbk'$ or $\Bbbk'' = \Bbbk' \times \Bbbk'$.

Proof. In both cases, since E is reduced and connected, $\Bbbk' = \mathrm{H}^0(E, \mathcal{O}_E)$ is a field. For (1), the intersection $x = E \cap E^c$ is a node and hence $\Bbbk' = \kappa(x)$ is a separable field extension of $\Bbbk$ (Proposition 5.2.4). Viewing E as a scheme over $\Bbbk'$ via structure map $E \to \operatorname{Spec} \Gamma(E, \mathcal{O}_E) = \operatorname{Spec} \Bbbk'$, we see that $E \cong \mathbb{P}^1_{\Bbbk'}$ since it is irreducible, smooth, genus 0, and contains a rational point. For (2), the points of $E \cap E^c$ are nodes in C but smooth points in both E and E^c. As $\dim_{\Bbbk'} \Bbbk'' = 2$, we see that either $\Bbbk''$ is a field, in which case it is separable over $\Bbbk$ by Proposition 5.2.4, or $\Bbbk'' = \Bbbk' \times \Bbbk'$. □

Exercise 5.3.13. If $K/\Bbbk$ is a field extension, (C, p_i) has a rational tail (resp., rational bridge) if and only if $(C \times_{\Bbbk} K, p_i \times_{\Bbbk} K)$ has a rational tail (resp., rational bridge).

Stability is equivalent to not containing a rational tail or bridge.

Proposition 5.3.14. *Let (C, p_i) be an n-pointed prestable curve of genus g over a field $\Bbbk$. If K is an algebraically closed field containing $\Bbbk$, then $(C \times_{\Bbbk} K, p_i \times_{\Bbbk} K)$ is stable (resp., semistable) if and only if (C, p_i) contains no rational tails or rational bridges (resp., no rational tails).*

Proof. Over an algebraically closed field, stability (resp., semistability) is equivalent to not containing a rational tail or bridge (resp., rational tail). Therefore, the statement follows from the fact that having a rational tail or bridge is insensitive to field extensions (Exercise 5.3.13). □

Remark 5.3.15 (Relationship to -1 and -2 curves). Suppose that $\mathcal{C} \to \Delta = \operatorname{Spec} R$ is a family of nodal curves over a DVR R with algebraically closed residue field $\Bbbk$ such that the generic fiber is smooth. If $E \cong \mathbb{P}^1 \subseteq \mathcal{C}_0'$ is a smooth rational subcurve in the central fiber, then $E^2 = -E \cdot E^c$ (which follows from the identity $0 = E \cdot \mathcal{C}_0 = E \cdot E + E \cdot E^c$). Thus if E is a rational tail (resp., rational bridge without a marked point), then $E^2 = -1$ (resp., $E^2 = -2$).

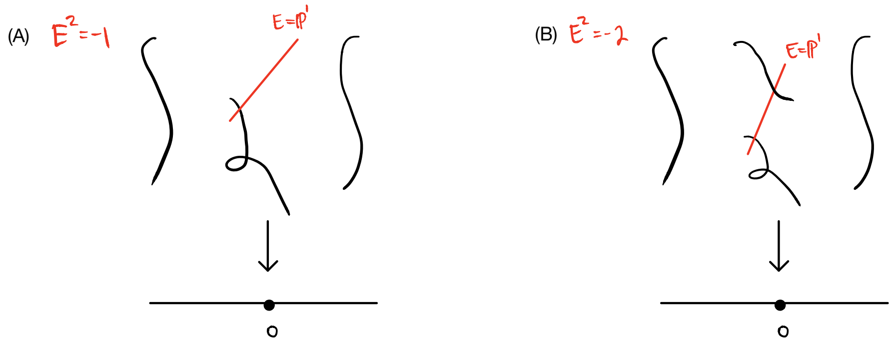


Figure 5.3.15: In (A) (resp., (B)), the exceptional component E meets the rest of the curve at one point (resp., two points) and $E^2 = -1$ (resp., $E^2 = -2$).

The following exercise will be generalized later by the Stable Contraction of a Prestable Family (5.6.6).

Exercise 5.3.16 (hard)**.** Let $\mathcal{C} \to \Delta = \operatorname{Spec} R$ be a family of nodal curves over a DVR R with smooth generic fiber. For any rational tail or bridge $E \subset \mathcal{C}_0$ in the central fiber, show that there is a contraction $\mathcal{C} \to \mathcal{C}'$ of E where $\mathcal{C}' \to \Delta$ is a family of nodal curves.

Hint: Castelnuovo's Contraction Theorem (B.2.6) gives the existence of the contraction $\mathcal{C} \to \mathcal{C}'$. The challenge here is to show that the central fiber $\mathcal{C}'_0$ is nodal. You may want to appeal to a fact in the minimal model program that $\mathcal{C}'_0$ is nodal if and only if the pair $(\mathcal{C}', \mathcal{C}'_0)$ is log canonical [Kol13, Cor. 2.32, Thm. 4.9(2)], and show that the latter property holds because $(\mathcal{C}, \mathcal{C}_0)$ (resp., $(\mathcal{C}, \mathcal{C}_0 - E)$) is log canonical.

5.3.3 Positivity of the dualizing sheaf

Exercise 5.3.17 (moderate)**.** Let $(C, p_1, \dots, p_n)$ be an n-pointed prestable curve over an algebraically closed field $\Bbbk$, and let $L := \omega_C(p_1 + \cdots + p_n)$.

(a) If (C, p_i) is stable, show that $L^{\otimes k}$ is very ample for $k \geq 3$ and that $\mathrm{H}^1(C, (\omega_C(p_1 + \cdots + p_n))^{\otimes k}) = 0$ for $k \geq 2$.

(b) For $k \geq 2$, show that (C, p_i) is semistable if and only if $L^{\otimes k}$ is base point free.

Hint: For (a), show that the global sections of $L^{\otimes k}$ separate points and tangent vectors or, in other words, that the maps

$$\mathrm{H}^0(C, L^{\otimes k}) \to \big(L^{\otimes k} \otimes \kappa(x)\big) \oplus \big(L^{\otimes k} \otimes \kappa(y)\big) \qquad \mathrm{H}^0(C, L^{\otimes k}) \to L^{\otimes k} \otimes \mathcal{O}_{C,x}/\mathfrak{m}_x^2$$

are surjective. Establish this using Serre Duality and a case analysis on whether x, y are smooth or nodal. See also [DM69, Thm. 2], [ACG11, Lem. 10.6.1], [SP, Tag 0E8X], and [Ols16, Prop. 13.2.17].

Exercise 5.3.18 (good practice)**.** If C is the nodal union $C_1 \cup C_2$ of genus i and $g - i$ curves along a single node $p = C_1 \cap C_2$, show that ω_C has a base point at p.

5.3.4 Families of stable curves

Definition 5.3.19. A *family of n-pointed curves* over a scheme S is a proper, flat, and finitely presented morphism $\mathcal{C} \to S$ of *algebraic spaces* together with n sections

$\sigma_1, \dots, \sigma_n \colon S \to \mathcal{C}$ such that every geometric fiber $\mathcal{C}_s$ is a curve over $\operatorname{Spec} \kappa(s)$ (e.g., a finite type $\kappa(s)$-scheme of dimension 1).

A *family of n-pointed stable curves* (resp., *semistable curves*, *prestable curves*, *nodal curves*) is a family of n-pointed curves such that every geometric fiber $(\mathcal{C}_s, \sigma_1(s), \dots, \sigma_n(s))$ is stable (resp., semistable, prestable, nodal).

Caution 5.3.20. If $(\mathcal{C} \to S, \sigma_i)$ is a stable family over a scheme S, $\mathcal{C} \to S$ is necessarily projective (Proposition 5.3.21), hence $\mathcal{C}$ is a scheme. While every one dimensional separated algebraic space of finite type over a field is a scheme (Theorem 4.5.32), in the relative setting, the total family $\mathcal{C}$ may not be a scheme. There exists a families of prestable genus 0 curves [Ful10, Ex. 2.3] and smooth genus 1 curves [Ray70, XIII 3.2] where the total families are not schemes; see also Example 2.1.16. In the next section, we show that the stack of all curves (resp., nodal curves, semistable curves, prestable curves) is algebraic and in particular satisfies the descent condition for families of curves, and this result requires that the total family is an algebraic space.

For a family of n-pointed curves (resp., nodal curves), there is no condition on whether the marked points are distinct or land in the relative smooth locus of $\mathcal{C} \to S$. For stable, semistable, and prestable families, the marked points are distinct and avoid the nodes. If $(\mathcal{C} \to S, \sigma_i)$ is a family of n-pointed prestable curves, then $\mathcal{C} \to S$ is a local complete intersection morphism. Thus there is a relative dualizing line bundle $\omega_{\mathcal{C}/S}$ that is compatible with base change $T \to S$ and in particular restricts to the dualizing line bundle $\omega_{\mathcal{C}_s}$ on every fiber of $\mathcal{C} \to S$; see [Har66c] or [Liu02, §6.4]. The image of each section σ_i is a divisor contained in the smooth locus, and we can form the line bundle $\omega_{\mathcal{C}/S}(\sigma_1 + \cdots + \sigma_n)$.

The following statement extends Proposition 5.3.4 to families. We will denote by $\underline{\mathrm{Aut}}(\mathcal{C}/S, \sigma_1, \dots, \sigma_n)$ the locally of finite type group scheme over S parameterizing automorphisms of a family of n-pointed prestable curves.

Proposition 5.3.21. *Let $(\mathcal{C} \to S, \sigma_i)$ be a family of n-pointed prestable curves of genus g over a scheme S. The following are equivalent:*

(1) $(\mathcal{C} \to S, \sigma_i)$ is a family of stable curves,

(2) $\underline{\mathrm{Aut}}(\mathcal{C}/S, \sigma_1, \dots, \sigma_n) \to S$ is quasi-finite, and

(3) $\omega_{\mathcal{C}/S}(\sigma_1 + \cdots + \sigma_n)$ is relatively ample over S.

In particular, a family of n-pointed stable curves is a projective morphism of schemes.

Proof. Each condition can be checked on geometric fibers so the statement follows from Proposition 5.3.4. ◻

The separatedness of $\overline{\mathcal{M}}_{g,n}$ (Proposition 5.5.20) implies that $\underline{\mathrm{Aut}}(\mathcal{C}/S, \sigma_1, \dots, \sigma_n) \to S$ is, in fact, a finite group scheme.

Exercise 5.3.22. Let $(\pi \colon \mathcal{C} \to S, \sigma_i)$ be a family of n-pointed prestable curves and let $L = \omega_{\mathcal{C}/S}(\sigma_1 + \cdots + \sigma_n)$. For $k \geq 2$, show that $(\mathcal{C} \to S, \sigma_i)$ is semistable if and only if $\pi^*\pi_* L \to L$ is surjective.

The following generalization of Properties of Families of Smooth Curves (5.1.16) is proven in the same way using the very ampleness of $\omega_C(p_1 + \cdots + p_n)^{\otimes 3}$ for a stable n-pointed curve (C, p_i) over an algebraically closed field (Exercise 5.3.17).

Proposition 5.3.23 (Properties of Families of Stable Curves)**.** *Let $(\mathcal{C} \to S, \sigma_i)$ be a family of n-pointed stable curves of genus g, and set $L := \omega_{\mathcal{C}/S}(\sum_i \sigma_i)$. If $k \geq 3$, then $L^{\otimes k}$ is relatively very ample and $\pi_*(L^{\otimes k})$ is a vector bundle of rank $(2k-1)(g-1) + kn$.* ◻

Proposition 5.3.24 (Openness of Stability)**.** *Let $(\pi\colon \mathcal{C} \to S, \sigma_i)$ be a family of n-pointed curves. The locus of points $s \in S$ such that $(\mathcal{C}_s, \sigma_i(s))$ is stable (resp., semistable, prestable, nodal) is open.*

Proof. We first claim that the nodal locus is open. When $\mathcal{C}$ is a scheme, this follows from the Local Structure of Nodes (5.2.25): the locus $\mathcal{C}^{\leq \mathrm{nod}} \subseteq \mathcal{C}$ of points which are smooth or nodal in their fiber is open, and therefore the locus of points $s \in S$ such that $\mathcal{C}_s$ has worse than nodal singularities is identified with the closed locus $\pi(\mathcal{C} \smallsetminus \mathcal{C}^{\leq \mathrm{nod}}) \subseteq S$. In general, we can choose an étale cover $\mathcal{C}' \to \mathcal{C}$ by a scheme and use the fact that a point $p \in \mathcal{C}$ is a node in its fiber $\mathcal{C}_{\pi(p)}$ if and only if a preimage $p' \in \mathcal{C}'$ of p is a node in its fiber $\mathcal{C}'_{\pi(p)}$. Since the locus in S where $\sigma_1(s), \ldots, \sigma_n(s)$ are distinct and smooth points in $\mathcal{C}_s$ is open, the condition that $\mathcal{C}_s$ is prestable is open. For an n-pointed prestable family $(\pi\colon \mathcal{C} \to S, \sigma_i)$, the locus of points $s \in S$ where $\mathcal{C}_s$ is semistable is identified with the open locus over which $\pi^*\pi_*L \to L$ is surjective, where $L := \omega_{\mathcal{C}/S}(\sum_i \sigma_i)$ (see Exercise 5.3.22).

To see that stability is an open condition in an n-pointed prestable family, we provide two arguments. First, observe that the fiber dimension of $\underline{\mathrm{Aut}}(\mathcal{C}/S, \sigma_1, \ldots, \sigma_n) \to S$ is upper semicontinuous (B.1.8), and therefore, the locus of points $s \in S$ such that $\mathrm{Aut}(\mathcal{C}_s, \sigma_1(s), \ldots, \sigma_n(s))$ is finite is open. As this condition categorizes stability (Proposition 5.3.4(2)), this open subset is identified with the stable locus. Second, by Openness of Ampleness (Proposition 4.6.17), the locus of points $s \in S$ such that $\omega_{\mathcal{C}/S}(\sum_i \sigma_i)|_{\mathcal{C}_s} \cong \omega_{\mathcal{C}_s}(\sum_i \sigma_i(s))$ is ample is open, and this condition also categorizes stability (Proposition 5.3.4(3)). $\square$

5.3.5 Deformation theory of stable curves

If C is a smooth curve over a field $\Bbbk$, then every first-order deformation is locally trivial (Proposition C.1.8) and the set $\mathrm{Def}(C)$ of isomorphism classes of first-order deformations is naturally in bijection with $\mathrm{H}^1(C, T_C)$ (Proposition C.1.11). More generally, automorphisms, deformations, and obstructions of higher-order deformations are classified by $\mathrm{H}^i(C, T_C)$ for $i = 0, 1, 2$ (Proposition C.2.4). Nodal singularities, on the other hand, have first-order deformations that are not locally trivial, e.g., $\operatorname{Spec} \Bbbk[x, y, \epsilon]/(xy - \epsilon) \to \operatorname{Spec} \Bbbk[\epsilon]$.

Proposition 5.3.25. *Let $A' \twoheadrightarrow A$ be a surjection of artinian local rings with residue field $\Bbbk$. Suppose that $J = \ker(A' \to A)$ satisfies $\mathfrak{m}_{A'} J = 0$. Let $(\mathcal{C} \to \operatorname{Spec} A, \sigma_i)$ be a family of prestable curves over A, and let (C, p_i) be its base change to the residue field $\Bbbk$.*

(1) The group of automorphisms of a deformation of $(\mathcal{C} \to \operatorname{Spec} A, \sigma_i)$ over A' is $\mathrm{Ext}^0_{\mathcal{O}_C}(\Omega_C(\sum_i p_i), \mathcal{O}_C \otimes_{\Bbbk} J)$.

(2) There is no obstruction to deforming $(\mathcal{C} \to \operatorname{Spec} A, \sigma_i)$ over A'.

(3) The set of isomorphism classes of deformations of $(\mathcal{C} \to \operatorname{Spec} A, \sigma_i)$ over A' is a torsor under $\mathrm{Ext}^1_{\mathcal{O}_C}(\Omega_C(\sum_i p_i), \mathcal{O}_C \otimes_{\Bbbk} J)$ and

$$\dim_{\Bbbk} \mathrm{Ext}^1_{\mathcal{O}_C}(\Omega_C(\textstyle\sum_i p_i), \mathcal{O}_C) - \dim_{\Bbbk} \mathrm{Ext}^0_{\mathcal{O}_C}(\Omega_C(\textstyle\sum_i p_i), \mathcal{O}_C) = 3g - 3 + n.$$

Moreover, if (C, p_i) is stable, then $\mathrm{Ext}^0_{\mathcal{O}_C}(\Omega_C(\sum_i p_i), \mathcal{O}_C) = 0$, i.e., there are no non-trivial automorphisms of a deformation, and $\dim_{\Bbbk} \mathrm{Ext}^1_{\mathcal{O}_C}(\Omega_C(\sum_i p_i), \mathcal{O}_C) = 3g - 3 + n$.

Proof. Since C is generically smooth and a local complete intersection and the marked points $p_i \in C$ are smooth, it is a consequence of Proposition C.2.4 (unpointed case) and Proposition C.2.8 (pointed case) that automorphisms, deformations, and obstructions of a nodal curve C are classified by

$$\mathrm{Ext}^i_{\mathcal{O}_C}(\Omega_C, I)$$

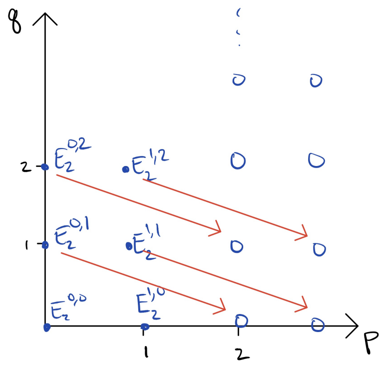
for $i = 0, 1, 2$, where $I \subseteq \mathcal{O}_C$ is the ideal sheaf of $\{p_1, \ldots, p_n\} \subseteq C$. Since I defines a Cartier divisor, $I = \mathcal{O}_C(-\sum_i p_i)$, and thus automorphisms, deformations, and obstructions are classified by

$$\mathrm{Ext}^i_{\mathcal{O}_C}(\Omega_C(\textstyle\sum_i p_i), \mathcal{O}_C).$$

To compute the dimensions of these Ext groups, we first handle the unpointed case. Since $\mathrm{Hom}_{\mathcal{O}_C}(\Omega_C, -)$ is the composition $\Gamma \circ \mathscr{H}om_{\mathcal{O}_C}(\Omega_C, -)$ of left exact functors, there is a Grothendieck spectral sequence

$$E_2^{p,q} = \mathrm{H}^p(C, \mathscr{E}xt^q_{\mathcal{O}_C}(\Omega_C, \mathcal{O}_C)) \Rightarrow \mathrm{Ext}^{p+q}_{\mathcal{O}_C}(\Omega_C, \mathcal{O}_C),$$

c.f., [Wei94, Thm. 5.8.3]. Since $\dim C = 1$, we have that $E_2^{p,q} = 0$ if $p \geq 2$. We can thus draw the E_2-page as:



Setting $T_C = \mathscr{H}om_{\mathcal{O}_C}(\Omega_C, \mathcal{O}_C)$, the associated exact sequence of low-degree terms is

$$0 \to \underbrace{\mathrm{H}^1(C, T_C)}_{E_2^{1,0}} \to \mathrm{Ext}^1_{\mathcal{O}_C}(\Omega_C, \mathcal{O}_C) \to \underbrace{\mathrm{H}^0(C, \mathscr{E}xt^1_{\mathcal{O}_C}(\Omega_C, \mathcal{O}_C))}_{E_2^{0,1}}$$
$$\to \underbrace{\mathrm{H}^2(C, T_C) = 0}_{E_2^{2,0}}. \quad (5.3.26)$$

As Ω_C is locally free away from the nodes, $\mathscr{E}xt^1_{\mathcal{O}_C}(\Omega_C, \mathcal{O}_C)$ and $\mathscr{E}xt^2_{\mathcal{O}_C}(\Omega_C, \mathcal{O}_C)$ are zero-dimensional sheaves supported only at the nodes $\Sigma = \{q_1, \ldots, q_s\}$ of C. It follows that $E_2^{1,1} = \mathrm{H}^1(C, \mathscr{E}xt^1_{\mathcal{O}_C}(\Omega_C, \mathcal{O}_C)) = 0$ and

$$E_2^{0,1} = \mathrm{H}^0(C, \mathscr{E}xt^1_{\mathcal{O}_C}(\Omega_C, \mathcal{O}_C)) = \bigoplus_j \mathrm{Ext}^1_{\widehat{\mathcal{O}}_{C,q_j}}(\Omega_{\widehat{\mathcal{O}}_{C,q_j}}, \widehat{\mathcal{O}}_{C,q_j})$$
$$E_2^{0,2} = \mathrm{H}^0(C, \mathscr{E}xt^2_{\mathcal{O}_C}(\Omega_C, \mathcal{O}_C)) = \bigoplus_j \mathrm{Ext}^2_{\widehat{\mathcal{O}}_{C,q_j}}(\Omega_{\widehat{\mathcal{O}}_{C,q_j}}, \widehat{\mathcal{O}}_{C,q_j}).$$

To compute $\mathrm{Ext}^i_{\widehat{\mathcal{O}}_{C,q_j}}(\Omega^1_{\widehat{\mathcal{O}}_{C,q_j}}, \widehat{\mathcal{O}}_{C,q_j})$, we may assume that $\Bbbk$ is algebraically closed. Consider the locally free resolution of $\widehat{\mathcal{O}}_{C,q_j} = \Bbbk[\![x, y]\!]/(xy)$

$$0 \to \widehat{\mathcal{O}}_{C,q_j} \xrightarrow{\binom{y}{x}} \widehat{\mathcal{O}}_{C,q_j}^{\oplus 2} \xrightarrow{(dx,dy)} \Omega_{\widehat{\mathcal{O}}_{C,q_j}} \to 0.$$

This shows that $\mathrm{Ext}^1_{\widehat{\mathcal{O}}_{C,q_j}}(\Omega_{\widehat{\mathcal{O}}_{C,q_j}}, \widehat{\mathcal{O}}_{C,q_j}) = \mathrm{coker}(\widehat{\mathcal{O}}_{C,q_j}^{\oplus 2} \xrightarrow{(x,y)} \widehat{\mathcal{O}}_{C,q_j}) = \Bbbk$ and $\mathrm{Ext}^2_{\widehat{\mathcal{O}}_{C,q_j}}(\Omega_{\widehat{\mathcal{O}}_{C,q_j}}, \widehat{\mathcal{O}}_{C,q_j}) = 0$. The former implies that $\dim_\Bbbk E_2^{0,1} = |\Sigma|$, the number of nodes, while the latter implies that $\mathrm{Ext}^2_{\mathcal{O}_C}(\Omega_C, \mathcal{O}_C) = 0$ (as $E_2^{0,2} = E_2^{1,1} = E_2^{2,0} = 0$).

By appealing to the short exact sequence (5.3.26) and the identities $H^0(C, T_C) = \mathrm{Ext}^0_{\mathcal{O}_C}(\Omega_C, \mathcal{O}_C)$ and $\mathrm{H}^i(C, T_C) = \mathrm{H}^i(\widetilde{C}, T_{\widetilde{C}}(-\widetilde{\Sigma}))$ (see Exercise 5.2.22), we have that

$$\begin{aligned}\dim_\Bbbk \mathrm{Ext}^1_{\mathcal{O}_C}(\Omega_C, \mathcal{O}_C) - \dim_\Bbbk \mathrm{Ext}^0_{\mathcal{O}_C}(\Omega_C, \mathcal{O}_C) &= |\Sigma| - \chi(C, T_C)\\ &= |\Sigma| - \chi\left(\widetilde{C}, T_{\widetilde{C}}(-\widetilde{\Sigma})\right).\end{aligned}$$

Write the normalization $\widetilde{C} = \coprod_{i=1}^{\nu} \widetilde{C}_i$ as a union of its connected components and set $\widetilde{\Sigma}$ to be the preimage of the nodal locus $\Sigma \subseteq C$ with $\widetilde{\Sigma}_i = \widetilde{C}_i \cap \widetilde{\Sigma}$. We may apply Riemann–Roch to compute that

$$\begin{aligned}|\Sigma| - \chi\left(\widetilde{C}, T_{\widetilde{C}}(-\widetilde{\Sigma})\right) &= |\Sigma| - \sum_{i=1}^{\nu} \chi\left(\widetilde{C}_i, T_{\widetilde{C}_i}(-\widetilde{\Sigma}_i)\right)\\ &= |\Sigma| + \sum_{i=1}^{\nu}\left(3g(\widetilde{C}_i) - 3 + |\widetilde{\Sigma}_i|\right)\\ &= 3\left(\sum_{i=1}^{\nu} g(\widetilde{C}_i) - \nu + |\Sigma|\right)\\ &= 3g - 3,\end{aligned}$$

where we have used the Genus Formula (5.2.12) $g = \sum_{i=1}^{\nu} g(\widetilde{C}_i) - \nu + |\Sigma| + 1$. This completes the proof of (1)–(3) in the case of an unpointed prestable curve.

In the case of marked points, let $D = \{p_1, \ldots, p_n\} \subseteq C$. Applying $\mathrm{Hom}_{\mathcal{O}_C}(\Omega_C, -)$ to the short exact sequence $0 \to \mathcal{O}_C(-\sum_i p_i) \to \mathcal{O}_C \to \mathcal{O}_D \to 0$ gives a long exact sequence

$$\begin{aligned}&0 \to \mathrm{Ext}^0_{\mathcal{O}_C}(\Omega_C(\textstyle\sum_i p_i), \mathcal{O}_C) \to \mathrm{Ext}^0_{\mathcal{O}_C}(\Omega_C, \mathcal{O}_C) \to \mathrm{Ext}^0_{\mathcal{O}_C}(\Omega_C, \mathcal{O}_D) \to\\ &\mathrm{Ext}^1_{\mathcal{O}_C}(\Omega_C(\textstyle\sum_i p_i), \mathcal{O}_C) \to \mathrm{Ext}^1_{\mathcal{O}_C}(\Omega_C, \mathcal{O}_C) \to 0 \to \mathrm{Ext}^2_{\mathcal{O}_C}(\Omega_C(\textstyle\sum_i p_i), \mathcal{O}_C) \to 0,\end{aligned}$$

where we've used that $\mathrm{Ext}^i_{\mathcal{O}_C}(\Omega_C, \mathcal{O}_D) = 0$ for $i = 1, 2$ (since D is zero-dimensional). This yields that $\mathrm{Ext}^2_{\mathcal{O}_C}(\Omega_C(\sum_i p_i), \mathcal{O}_C) = 0$. Moreover, since $\dim_\Bbbk \mathrm{Hom}_{\mathcal{O}_C}(\Omega_C, \mathcal{O}_D) = n$ and since we've already computed the difference of the Ext groups in the unpointed case, the above long exact sequence implies that

$$\begin{aligned}&\dim_\Bbbk \mathrm{Ext}^1_{\mathcal{O}_C}(\Omega_C(\textstyle\sum_i p_i), \mathcal{O}_C) - \dim_\Bbbk \mathrm{Ext}^0_{\mathcal{O}_C}(\Omega_C(\textstyle\sum_i p_i), \mathcal{O}_C)\\ &\qquad = \dim_\Bbbk \mathrm{Ext}^1_{\mathcal{O}_C}(\Omega_C, \mathcal{O}_C) - \dim_\Bbbk \mathrm{Ext}^0_{\mathcal{O}_C}(\Omega_C, \mathcal{O}_C) + n = 3g - 3 + n.\end{aligned}$$

Finally, suppose that (C, p_i) is a *stable* curve with pointed normalization $(\widetilde{C}, \widetilde{p}_i, \widetilde{\Sigma})$. Exercise 5.2.22 implies that

$$\mathrm{Hom}_{\mathcal{O}_C}(\Omega_C(\textstyle\sum_i p_i), \mathcal{O}_C) \cong \mathrm{Hom}_{\mathcal{O}_{\widetilde{C}}}(\Omega_{\widetilde{C}}(\textstyle\sum_i \widetilde{p}_i + \widetilde{\Sigma}), \mathcal{O}_{\widetilde{C}}).$$

Since the pointed normalization is smooth and each pointed connected component is stable (Exercise 5.3.6), the degree of the restriction of $T_{\widetilde{C}}(-\sum_i \widetilde{p}_i - \widetilde{\Sigma})$ to each connected component of $\widetilde{C}$ is strictly negative. Thus, $\mathrm{Hom}_{\mathcal{O}_C}(\Omega_C(\sum_i p_i), \mathcal{O}_C) = 0$. See also [DM69, Prop. 1.5] and [ACG11, §11.3]. □

The proof above shows more, namely it reveals an important relationship between global and local deformations. We denote by $\operatorname{Def}(C, p_i)$ (resp., $\operatorname{Def}^{\mathrm{lt}}(C, p_i)$) the vector space of first-order deformations (resp., locally trivial first-order deformations) of (C, p_i).

Proposition 5.3.27 (Local-to-global Deformation Sequence)**.** *Let (C, p_i) be an n-pointed prestable curve over an algebraically closed field $\Bbbk$, and let $\Sigma \subseteq C$ be the set of nodes. Let $\pi \colon \widetilde{C} \to C$ be the normalization, $\widetilde{p}_i$ the unique preimage of p_i, and $\widetilde{\Sigma} = \pi^{-1}(\Sigma)$. There is an exact sequence*

$$0 \to \operatorname{Def}^{\mathrm{lt}}(C, p_i) \to \operatorname{Def}(C, p_i) \to \bigoplus_{q \in \Sigma} \operatorname{Def}(\widehat{\mathcal{O}}_{C,q}) \to 0 \tag{5.3.28}$$

and identifications

$$\begin{aligned}
\operatorname{Def}^{\mathrm{lt}}(C, p_i) &\cong \mathrm{H}^1(C, T_C) \cong \mathrm{H}^1\big(\widetilde{C}, T_{\widetilde{C}}(-\textstyle\sum_i \widetilde{p}_i - \widetilde{\Sigma}\big) \cong \operatorname{Def}(\widetilde{C}, p_i, \widetilde{\Sigma}) \\
\operatorname{Def}(C, p_i) &\cong \operatorname{Ext}^1_{\mathcal{O}_C}(\Omega_C(\textstyle\sum_i p_i), \mathcal{O}_C) \\
\operatorname{Def}(\widehat{\mathcal{O}}_{C,q}) &\cong \operatorname{Ext}^1_{\widehat{\mathcal{O}}_{C,q}}(\Omega^1_{\widehat{\mathcal{O}}_{C,q}}, \widehat{\mathcal{O}}_{C,q}) \cong \Bbbk, \qquad \textit{for } q \in \Sigma. \quad \square
\end{aligned}$$

Exercise 5.3.29 (Deformation of stable curves revisited)**.**

(a) Provide a direct argument that deformations of an unpointed stable curve are unobstructed.

Hint: Letting $q_i \in C$ be the nodes, choose a cover $C = U_1 \cup U_2$ of affine opens with each $q_i \in U_2 \setminus U_1$. Using that U_1 and $U_{12} = U_1 \cap U_2$ are unobstructed (as they are smooth) and U_2 is also unobstructed (as it is lci and affine, c.f., Proposition A.3.7), show that deformations of U_1, U_{12}, and U_2 can be glued to a deformation of C. See also [SP, Tag 0DZQ].

(b) Extend this argument to the pointed case.

(c) Can you show this instead using properties of the cotangent complex?

Remark 5.3.30 (Consequences of deformation theory). Very shortly we will know that $\overline{\mathcal{M}}_{g,n}$ is an algebraic stack. We will then be able to use deformation theory to argue that $\overline{\mathcal{M}}_{g,n}$ is a smooth Deligne–Mumford stack of dimension $3g - 3 + n$ (see Theorem 5.4.14). Here is the central idea:

- Ext^0: We have already seen that a stable curve (C, p_i) has finitely many automorphism (Proposition 5.3.4). The vanishing of Ext^0 implies that an n-pointed stable curve (C, p_i) has no infinitesimal automorphisms, i.e., the automorphism group scheme $\underline{\operatorname{Aut}}(C, p_i)$ is reduced, finite, and étale. This will allow us to use the Characterization of Deligne–Mumford Stacks (3.6.4) to conclude that $\overline{\mathcal{M}}_{g,n}$ is Deligne–Mumford.
- Ext^1: Since Ext^1 parametrizes isomorphism classes of deformations of a stable curve (C, p_i) over a field $\Bbbk$, it is identified with the Zariski tangent space of $\overline{\mathcal{M}}_{g,n} \times_{\mathbb{Z}} \Bbbk$ at the $\Bbbk$-point corresponding to (C, p_i). The computation of Ext^1 implies that $\overline{\mathcal{M}}_{g,n}$ has relative dimension $3g - 3 + n$ over $\operatorname{Spec} \mathbb{Z}$.
- Ext^2: The vanishing of Ext^2 implies that there are no obstructions to deforming (C, p_i), and thus the Infinitesimal Lifting Criterion (3.7.1) implies that $\overline{\mathcal{M}}_{g,n}$ is smooth over $\operatorname{Spec} \mathbb{Z}$.

5.4 The stack of all curves

It's better to work with a nice category containing some nasty objects, than a nasty category containing only nice objects.

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We show that the stack $\mathcal{M}_{g,n}^{\mathrm{all}}$ of all n-pointed proper curves is algebraic (Theorem 5.4.6) and that the open substack $\overline{\mathcal{M}}_{g,n} \subseteq \mathcal{M}_{g,n}^{\mathrm{all}}$ of stable curves is a quasi-compact Deligne–Mumford stack smooth over $\operatorname{Spec} \mathbb{Z}$ of dimension $3g-3+n$ (Theorem 5.4.14).

5.4.1 Families of arbitrary curves

Recall that a *curve over a field* $\Bbbk$ is a one dimensional scheme of finite type over $\Bbbk$, and that the *genus* of a projective curve C is

$$g(C) = 1 - \chi(C, \mathcal{O}_C).$$

Curves may be non-pure dimensional, non-connected, and arbitrarily singular (e.g., non-reduced). While there is an algebraic stack parameterizing geometrically reduced curves (Remark 5.4.17), once we allow non-reduced curves, we must also allow non-pure dimensional curves, as otherwise the stack would fail to be algebraic. For instance, the plane nodal curve with an embedded point at the node, arising from the flat degeneration of a rational normal curve $\mathbb{P}^1 \hookrightarrow \mathbb{P}^3$ [Har77, Ex. 9.8.4], deforms in a flat family to the disjoint union of a plane nodal cubic and a point in $\mathbb{P}^3$.

A *family of n-pointed curves over a scheme S* is a proper, flat, and finitely presented morphism $\mathcal{C} \to S$ of algebraic spaces together with n sections $\sigma_1, \ldots, \sigma_n \colon S \to \mathcal{C}$ such that every fiber $\mathcal{C}_s$ is a curve. We say that a family $(\mathcal{C} \to S, \sigma_i)$ has *genus g* if every fiber $\mathcal{C}_s$ has genus g. Note that the marked points $\sigma_i(s)$ may be non-distinct and singular in a fiber $\mathcal{C}_s$. For the stack of curves to be algebraic, it is necessary that we allow the total family $\mathcal{C}$ to be an algebraic space (see Caution 5.3.20).

Proposition 5.4.1. *If $\mathcal{C} \to S$ is a family of curves over a scheme S, there exists an étale cover $S' \to S$ such that $\mathcal{C}_{S'} \to S'$ is projective.*

Proof. We offer a local-to-global argument. We first reduce to the case that S is of finite type over $\operatorname{Spec} \mathbb{Z}$. By Noetherian Approximation (B.3.2), we can write $S = \lim_{\lambda \in \Lambda} S_\lambda$ as a limit of finitely presented $\mathbb{Z}$-schemes with affine transition maps. By Descent of Morphisms under Limits (B.3.3), there is an index $0 \in \Lambda$ and a finitely presented morphism $\mathcal{C}_0 \to S_0$ such that $\mathcal{C} \cong \mathcal{C}_0 \times_{S_0} S_\lambda$, and moreover if we set $\mathcal{C}_\lambda = \mathcal{C}_0 \times_{S_0} S$ for $\lambda \geq 0$, then $\mathcal{C} = \lim_{\lambda \geq 0} \mathcal{C}_\lambda$. By Descent of Properties of Morphisms under Limits (B.3.7), $\mathcal{C}_\lambda \to S_\lambda$ is a family of curves for $\lambda \gg 0$. Since projectivity is stable under base change, we may replace S with S_λ.

For a point $s \in S$, define $S_n = \operatorname{Spec} \mathcal{O}_{S,s}/\mathfrak{m}_s^{n+1}$ and $\widehat{S} = \operatorname{Spec} \widehat{\mathcal{O}}_{S,s}$. Consider the cartesian diagram

$$\begin{tikzcd}
\mathcal{C}_s = \mathcal{C}_0 \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{C}_1 \arrow[r, hook] \arrow[d] & \cdots \arrow[r, hook] & \widehat{\mathcal{C}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{C} \arrow[d] \\
\operatorname{Spec} \kappa(s) = S_0 \arrow[r, hook] & S_1 \arrow[r, hook] & \cdots \arrow[r, hook] & \widehat{S} \arrow[r] & S.
\end{tikzcd}$$

Since separated one dimensional algebraic spaces are schemes (Theorem 4.5.32) and proper one dimensional schemes are projective, there exists an ample line bundle L_0 on $\mathcal{C}_0$. By Proposition C.2.11, the obstruction to deforming a line bundle L_n on $\mathcal{C}_n$ to L_{n+1} on $\mathcal{C}_{n+1}$ lives in $H^2(\mathcal{C}_0, \mathcal{O}_{\mathcal{C}_0} \otimes_{\kappa(s)} \mathfrak{m}^n)$. The obstruction vanishes since $\dim \mathcal{C}_0 = 1$, and thus there is a compatible sequence of line bundles L_n on $\mathcal{C}_n$. By Grothendieck's Existence Theorem (C.5.3), there exists a line bundle $\widehat{L}$ on $\widehat{C}$ extending L_n.

Applying Artin Approximation (Theorem B.5.18) to the functor

$$\mathrm{Sch}/S \to \mathrm{Sets}, \qquad (T \to S) \mapsto \mathrm{Pic}(\mathcal{C}_T),$$

we obtain an étale neighborhood $(S', s') \to (S, s)$ of s and a line bundle L' on $\mathcal{C}_{S'}$ extending L_0. Since L_0 is ample and ampleness is an open condition in families (Proposition B.2.10), after replacing S' with an open neighborhood of s', we can arrange that L' is relatively ample over S'. □

Exercise 5.4.2. Provide an alternative argument by explicitly extending an ample line bundle:

(i) Use limit methods to reduce to the case that S is the spectrum of a strictly henselian local ring R with closed point s, and reduce the statement to the surjectivity of $\mathrm{Pic}(\mathcal{C}) \to \mathrm{Pic}(\mathcal{C}_s)$.

(ii) Apply Le Lemma de Gabber (4.6.1) to reduce to the case that $\mathcal{C}$ is a scheme.

(iii) Reduce to the case that $\mathcal{C}$ is reduced, and show that $\mathrm{Pic}(\mathcal{C}) \to \mathrm{Pic}(\mathcal{C}_s)$ is surjective in this case by explicitly extending a line bundle $\mathcal{O}_{\mathcal{C}_s}(-x_s)$ for each closed point $x_s \in \mathcal{C}_s$.

See also [SGA4$\frac{1}{2}$, IV.4.1], [Hal13, Lem. 1.2], and [Ols16, Cor. 13.2.5].

Remark 5.4.3. Raynaud gives an example of a family of smooth $g = 1$ curves over an affine curve which is Zariski-locally projective but not projective [Ray70, XIII 3.1]. The examples in Caution 5.3.20 are not even Zariski-locally projective.

5.4.2 Algebraicity of the stack of all curves

Definition 5.4.4. Let $\mathcal{M}_{g,n}^{\mathrm{all}}$ be the prestack over $\mathrm{Sch}_{\text{ét}}$, where an object over a scheme S is a family of curves $\mathcal{C} \to S$ of genus g with n sections $\sigma_1, \ldots, \sigma_n \colon S \to \mathcal{C}$. A morphism $(\mathcal{C}' \to S', \sigma_1', \ldots, \sigma_n') \to (\mathcal{C} \to S, \sigma_1, \ldots, \sigma_n)$ is the data of a cartesian diagram

$$\begin{tikzcd} \mathcal{C}' \arrow[r, "g"] \arrow[d] & \mathcal{C} \arrow[d] \\ S' \arrow[r, "f"] \arrow[u, bend left, "\sigma_i'"] & S \arrow[u, bend left, "\sigma_i"] \end{tikzcd}$$

such that $g \circ \sigma_i' = \sigma_i \circ f$.

As a stepping stone to the algebraicity of $\mathcal{M}_{g,n}^{\mathrm{all}}$, we show that the diagonal is representable.

Lemma 5.4.5. *The diagonal $\mathcal{M}_{g,n}^{\mathrm{all}} \to \mathcal{M}_{g,n}^{\mathrm{all}} \times \mathcal{M}_{g,n}^{\mathrm{all}}$ is representable.*

Proof. For simplicity, we handle the case when $n = 0$. Let S be a scheme and $S \to \mathcal{M}_g^{\mathrm{all}} \times \mathcal{M}_g^{\mathrm{all}}$ be a morphism corresponding to families of curves $\mathcal{C}_1 \to S$ and

$\mathcal{C}_2 \to S$. Considering the cartesian diagram

$$\begin{tikzcd} \underline{\mathrm{Isom}}_S(\mathcal{C}_1, \mathcal{C}_2) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & S \arrow[d] \\ \mathcal{M}_g^{\mathrm{all}} \arrow[r] & \mathcal{M}_g^{\mathrm{all}} \times \mathcal{M}_g^{\mathrm{all}}, \end{tikzcd}$$

we need to show that $\underline{\mathrm{Isom}}_S(\mathcal{C}_1, \mathcal{C}_2)$ is an algebraic space. By Proposition 5.4.1, there exists an étale cover $S' \to S$ such that $\mathcal{C}_{S'} \to S'$ is projective. Since $\underline{\mathrm{Isom}}_S(\mathcal{C}_1, \mathcal{C}_2) \times_S S' = \underline{\mathrm{Isom}}_{S'}(\mathcal{C}_{1,S'}, \mathcal{C}_{2,S'})$, the morphism $\underline{\mathrm{Isom}}_{S'}(\mathcal{C}_{1,S'}, \mathcal{C}_{2,S'}) \to \underline{\mathrm{Isom}}_S(\mathcal{C}_1, \mathcal{C}_2)$ is surjective, étale, and representable. We may thus assume that $\mathcal{C}_1$ and $\mathcal{C}_2$ are projective over S.

Consider the inclusion of functors:

$$\underline{\mathrm{Isom}}_S(\mathcal{C}_1, \mathcal{C}_2) \subseteq \underline{\mathrm{Mor}}_S(\mathcal{C}_1, \mathcal{C}_2) \subseteq \mathrm{Hilb}(\mathcal{C}_1 \times_S \mathcal{C}_2/S)$$

where the second inclusion assigns a morphism $\alpha\colon \mathcal{C}_1 \to \mathcal{C}_2$ to the graph $\Gamma_\alpha\colon \mathcal{C}_1 \hookrightarrow \mathcal{C}_1 \times_S \mathcal{C}_2$ (and is similarly defined on T-valued points). Since the subfunctor of $\mathrm{Mor}(-, S)$, parameterizing maps $T \to S$ where $(\mathcal{C}_1)_S \to (\mathcal{C}_2)_S$ is an isomorphism, is open (Exercise 0.3.44), the first inclusion is a representable open immersion. Analyzing the second inclusion, we see that a subscheme $[\mathcal{Z} \subseteq \mathcal{C}_1 \times_S \mathcal{C}_2] \in \mathrm{Hilb}(\mathcal{C}_1 \times_S \mathcal{C}_2/S)(S)$ is contained in the image of $\underline{\mathrm{Mor}}(\mathcal{C}_1, \mathcal{C}_2)(S)$ if and only if the composition $Z \hookrightarrow \mathcal{C}_1 \times_S \mathcal{C}_2 \xrightarrow{p_1} \mathcal{C}_1$ is an isomorphism (and similarly for T-valued points). Therefore, Exercise 0.3.44 also establishes that the second inclusion is a representable open immersion. $\square$

Theorem 5.4.6. *$\mathcal{M}_{g,n}^{\mathrm{all}}$ is an algebraic stack locally of finite type over* $\operatorname{Spec} \mathbb{Z}$.

Proof. To see that $\mathcal{M}_{g,n}^{\mathrm{all}}$ is a stack over $\mathrm{Sch}_{\text{ét}}$, suppose that $\{S_i \to S\}$ is an étale cover of schemes, $(\mathcal{C}_i \to S_i, \sigma_{i,1}, \ldots, \sigma_{i,n})$ is a family of n-pointed curves for each i, and $\alpha_{ij}\colon \mathcal{C}_i|_{S_{ij}} \to \mathcal{C}_j|_{S_{ij}}$ is an isomorphism over $S_{ij} = S_i \times_S S_j$ compatible with the sections and satisfying the cocycle condition. The quotient of the étale equivalence relation

$$\coprod_{i,j} \mathcal{C}_{ij} \rightrightarrows \coprod_i \mathcal{C}_i,$$

with maps defined by p_1 and $p_2 \circ \alpha_{ij}$, is an algebraic space $\mathcal{C}$. Moreover, by étale descent of morphisms, there are sections $\sigma_1, \ldots, \sigma_n\colon S \to \mathcal{C}$ such that $\sigma_{k,i} = \sigma_k|_{S_i}$. Thus $(\mathcal{C} \to S, \sigma_k)$ is a n-pointed family of curves that restricts to $(\mathcal{C}_i \to S_i, \sigma_{i,k})$ for each i.

To see algebraicity, we claim that it suffices to handle the $n = 0$ case. Observe that the map $\mathcal{M}_{g,n+1}^{\mathrm{all}} \to \mathcal{M}_{g,n}^{\mathrm{all}}$, defined by forgetting the last marked point, is well-defined and representable: if $S \to \mathcal{M}_{g,n}^{\mathrm{all}}$ is a map corresponding to a n-pointed family $(\mathcal{C} \to S, \sigma_k)$, then $S \times_{\mathcal{M}_{g,n}^{\mathrm{all}}} \mathcal{M}_{g,n+1}^{\mathrm{all}} \cong \mathcal{C}$. (In fact, $\mathcal{M}_{g,n+1}^{\mathrm{all}} \to \mathcal{M}_{g,n}^{\mathrm{all}}$ is identified with the universal family; see Exercise 5.4.8.) Therefore, if $U \to \mathcal{M}_{g,n}^{\mathrm{all}}$ is a smooth presentation by a scheme, then $U' := U \times_{\mathcal{M}_{g,n}^{\mathrm{all}}} \mathcal{M}_{g,n+1}^{\mathrm{all}} \to \mathcal{M}_{g,n+1}^{\mathrm{all}}$ is a surjective, étale, and representable map from an algebraic space U'. Choosing an étale presentation $V \to U'$ by a scheme, the composition $V \to U' \to \mathcal{M}_{g,n+1}^{\mathrm{all}}$ is a smooth presentation.

To show algebraicity of $\mathcal{M}_g^{\mathrm{all}}$, it suffices show that for every projective curve X over a field $\Bbbk$, there exists a smooth representable morphism $U \to \mathcal{M}_g^{\mathrm{all}}$ from a scheme U of finite type over $\mathbb{Z}$ with $[X] \in |\mathcal{M}_g^{\mathrm{all}}|$ in the image. Choose an embedding $X \hookrightarrow \mathbb{P}^N$ such that $\mathrm{H}^1(X, \mathcal{O}_X(1)) = 0$, and let $P(t)$ be its Hilbert polynomial. Let $H := \mathrm{Hilb}^P(\mathbb{P}^N_{\mathbb{Z}}/\mathbb{Z})$ be the Hilbert scheme, which is projective over $\mathbb{Z}$ by Theorem 1.1.2. Considering the universal family

$$\begin{tikzcd} \mathcal{C} \arrow[r, hook] \arrow[d] & \mathbb{P}^N_H \arrow[dl] \\ H, & \end{tikzcd}$$

there is a point $h \in H(\Bbbk)$ such that $\mathcal{C}_h = X$ as closed subschemes of $\mathbb{P}^N_{\Bbbk}$. Cohomology and Base Change (A.6.8) implies that there exists an open neighborhood $H' \subseteq H$ of h such that for all $s \in H'$, $\mathrm{H}^1(\mathcal{C}_s, \mathcal{O}_{\mathcal{C}_s}(1)) = 0$. Consider the morphism

$$H' \to \mathcal{M}_g^{\mathrm{all}}, \qquad [C \hookrightarrow \mathbb{P}^n] \mapsto [C],$$

defined by forgetting the embedding. The representability of the diagonal of $\mathcal{M}_g^{\mathrm{all}}$ (Lemma 5.4.5) implies that $H' \to \mathcal{M}_g^{\mathrm{all}}$ is representable (Corollary 3.2.3).

We claim that $H' \to \mathcal{M}_g^{\mathrm{all}}$ is smooth. Even though we haven't yet established the algebraicity of $\mathcal{M}_g^{\mathrm{all}}$, we do know that $H' \to \mathcal{M}_g^{\mathrm{all}}$ is representable and this is suffices to apply the Infinitesimal Lifting Criterion (3.7.1) to verify smoothness. To this end, let $A' \to A$ be a surjection of artinian local rings with residue field $\Bbbk$ such that $\Bbbk = \ker(A' \to A)$. For every embedded curve $[C \subseteq \mathbb{P}^N] \in H'$ and diagram

$$\begin{tikzcd} \operatorname{Spec} \Bbbk \arrow[r, hook] \arrow[rr, bend left, "{[C \subseteq \mathbb{P}^N_{\Bbbk}]}"] & \operatorname{Spec} A \arrow[r, "{[\mathcal{C} \subseteq \mathbb{P}^N_A]}"] \arrow[d, hook] & H' \arrow[d] \\ & \operatorname{Spec} A' \arrow[r, "{[\mathcal{C}']}"'] \arrow[ur, dashed, "{[\mathcal{C}' \subseteq \mathbb{P}^N_{A'}]}"] & \mathcal{M}_g^{\mathrm{all}}, \end{tikzcd} \tag{5.4.7}$$

we need to show that there is a dotted arrow extending the diagram. This translates to the existence of a dotted arrow in the diagram

$$\begin{tikzcd} & \mathbb{P}^N_{\Bbbk} & \mathbb{P}^N_A & \mathbb{P}^N_{A'} \arrow[ddl] \\ C \arrow[ur, hook] \arrow[r] \arrow[d] & \mathcal{C} \arrow[ur, hook] \arrow[r] \arrow[d] & \mathcal{C}' \arrow[ur, hook, dashed] \arrow[d] & \\ \operatorname{Spec} \Bbbk \arrow[r, hook] & \operatorname{Spec} A \arrow[r, hook] & \operatorname{Spec} A'. & \end{tikzcd}$$

The closed immersion $\mathcal{C} \hookrightarrow \mathbb{P}^N_A$ is defined by a very ample line bundle L on $\mathcal{C}$ and sections $s_0, \ldots, s_N \in \Gamma(\mathcal{C}, L)$. As the obstruction to deforming L to $\mathcal{C}'$ lives in $\mathrm{H}^2(C, \mathcal{O}_C) = 0$ (Proposition C.2.11), we may extend L to a line bundle L' on $\mathcal{C}'$. We now argue that the sections s_i deform to sections $s_i' \in \Gamma(\mathcal{C}', L')$. As $\ker(A' \to A) = \Bbbk$, the ideal sheaf defining $\mathcal{C} \hookrightarrow \mathcal{C}'$ is isomorphic to $\mathcal{O}_C$, and we have a short exact sequence

$$0 \to \mathcal{O}_C \to \mathcal{O}_{\mathcal{C}'} \to \mathcal{O}_{\mathcal{C}} \to 0.$$

Tensoring with L' yields a short exact sequence

$$0 \to L|_C \to L' \to L \to 0.$$

Since $[C \subseteq \mathbb{P}^N] \in H'$, we have that $\mathrm{H}^1(C, L|_C) = 0$. Thus, $\mathrm{H}^0(\mathcal{C}', L') \to \mathrm{H}^0(\mathcal{C}, L)$ is surjective and we may lift the sections s_i to sections s_i'. The sections s_i are base point free, and Nakayama's Lemma implies that so are the sections s_i'. This gives a morphism $j' \colon \mathcal{C}' \to \mathbb{P}^N_{A'}$ restricting to $\mathcal{C} \hookrightarrow \mathbb{P}^N_A$. The map $j' \colon \mathcal{C}' \to \mathbb{P}^N_{A'}$ is proper and quasi-finite, thus finite. To see that it is a closed immersion, it suffices to show that $\mathcal{O}_{\mathbb{P}^N_{A'}} \to j'_* \mathcal{O}_{\mathcal{C}'}$ is surjective. The cokernel is a coherent sheaf whose restriction to A vanishes, and thus Nakayama's lemma implies that the cokernel vanishes. See also [Hal13, Thm. 1.1] and [JHS11, Prop. 3.3]. This can also be established using Artin's Axioms; see Theorem C.7.7 and [SP, Tag 0D5A]. $\square$

Exercise 5.4.8 (easy)**.** Show that map $\mathcal{M}^{\mathrm{all}}_{g,n+1} \to \mathcal{M}^{\mathrm{all}}_{g,n}$ forgetting the last marked point is the universal family as defined in Definition 3.1.27. (We will see in Proposition 5.6.12 that $\overline{\mathcal{M}}_{g,n+1}$ is also the universal family of $\overline{\mathcal{M}}_{g,n}$, but this is a more remarkable fact since an n-pointed stable curve can become unstable if a marked point is forgotten.)

Exercise 5.4.9 (technical)**.** Show that $\mathcal{M}^{\mathrm{all}}_{g,n}$ has quasi-compact and separated diagonal, and in particular $\mathcal{M}^{\mathrm{all}}_{g,n}$ is quasi-separated.

Exercise 5.4.10 (good practice)**.** Show that the stack $\mathcal{P}$, parameterizing a family of curves $\mathcal{C} \to S$ and a line bundle L on $\mathcal{C}$ relatively ample over S, is algebraic, and that the natural map $\mathcal{P} \to \mathcal{M}^{\mathrm{all}}_g$ is smooth and representable.

Remark 5.4.11. Moduli stacks of varieties of higher dimension are often not algebraic. For instance, the stack parameterizing abstract K3 surfaces is not algebraic (see Example C.7.15); on the other hand, there is an analytic stack parameterizing K3 surfaces (see [BHPV04, §VII.12]) and an algebraic stack parameterizing polarized K3 surfaces (i.e., K3 surfaces with a primitive ample line bundle).

5.4.3 Algebraicity and boundedness of $\overline{\mathcal{M}}_{g,n}$

Consider the inclusions of prestacks

$$\mathcal{M}_{g,n} \subseteq \overline{\mathcal{M}}_{g,n} \subseteq \mathcal{M}^{\mathrm{ss}}_{g,n} \subseteq \mathcal{M}^{\mathrm{pre}}_{g,n} \subseteq \mathcal{M}^{\leq \mathrm{nodal}}_{g,n} \subseteq \mathcal{M}^{\mathrm{all}}_{g,n}, \tag{5.4.12}$$

where $\overline{\mathcal{M}}_{g,n}$ (resp., $\mathcal{M}^{\mathrm{ss}}_{g,n}$, $\mathcal{M}^{\mathrm{pre}}_{g,n}$, and $\mathcal{M}^{\leq \mathrm{nodal}}_{g,n}$) denotes the full subcategory of $\mathcal{M}^{\mathrm{all}}_{g,n}$ consisting of n-pointed families $(\mathcal{C} \to S, \sigma_1, \ldots, \sigma_n)$ of stable curves (resp., semistable, prestable, and nodal curves).

Corollary 5.4.13. *The sequence of inclusions in* (5.4.12) *are open immersions, and each prestack is an algebraic stack locally of finite type over* $\operatorname{Spec} \mathbb{Z}$.

Proof. Theorem 5.4.6 establishes that $\mathcal{M}^{\mathrm{all}}_{g,n}$ is algebraic and locally of finite type. Openness of the stable (resp., semistable, prestable, nodal) locus was proved in Proposition 5.3.24, and this implies that each inclusion is an open immersion. $\square$

At this point, we know the following properties of $\overline{\mathcal{M}}_{g,n}$.

Theorem 5.4.14. *Assuming that $2g - 2 + n > 0$, the stack $\overline{\mathcal{M}}_{g,n}$ is a non-empty, quasi-compact, and Deligne–Mumford stack smooth over $\operatorname{Spec} \mathbb{Z}$ such that $\overline{\mathcal{M}}_{g,n} \times_{\mathbb{Z}} \Bbbk$ has pure dimension $3g - 3 + n$ for every field $\Bbbk$.*

Proof. By Corollary 5.4.13, $\overline{\mathcal{M}}_{g,n}$ is an algebraic stack of finite type over $\operatorname{Spec}\mathbb{Z}$. As there are smooth curves of any genus (Exercise 5.1.7), $\overline{\mathcal{M}}_{g,n}$ is non-empty. For the boundedness of $\overline{\mathcal{M}}_{g,n}$ (i.e., finite typeness or equivalently quasi-compactness), we will appeal to the fact that if $(C, p_1, \ldots, p_n)$ is an n-pointed stable curve over a field $\Bbbk$, then $L := (\omega_{C/\Bbbk}(p_1 + \cdots p_n))^{\otimes 3}$ is very ample (Exercise 5.3.17). Let $P(t)$ be the Hilbert polynomial of $C \hookrightarrow \mathbb{P}^N_{\Bbbk}$ embedded via L; this is independent of $[C, p_i] \in \overline{\mathcal{M}}_{g,n}$. Consider the closed subscheme

$$H \subseteq \operatorname{Hilb}^P(\mathbb{P}^N_{\mathbb{Z}}/\mathbb{Z}) \times (\mathbb{P}^N)^n$$

parameterizing pairs $(C \hookrightarrow \mathbb{P}^N, p_1, \ldots, p_n)$ such that $p_i \in C$. Since $\operatorname{Hilb}^P(\mathbb{P}^N_{\mathbb{Z}}/\mathbb{Z})$ is a projective scheme (Theorem 1.1.2) and in particular quasi-compact, so is H. The image of the forgetful morphism

$$H \to \mathcal{M}^{\mathrm{all}}_{g,n} \qquad [C \hookrightarrow \mathbb{P}^N, p_1, \ldots, p_n] \mapsto [C, p_1, \ldots, p_n]$$

contains $\overline{\mathcal{M}}_{g,n}$, and we conclude that $\overline{\mathcal{M}}_{g,n}$ is quasi-compact.

To see the final assertions, we invoke each part of Proposition 5.3.25 characterizing automorphisms, deformations, and obstructions of stable curve. This is analogous to our proof of Proposition 3.7.7 asserting the same properties for the stack $\mathcal{M}_g$ of smooth curves. Let (C, p_i) be an n-pointed stable curve of genus g. Since $\operatorname{Ext}^0_{\mathcal{O}_C}(\Omega_C(\sum_i p_i), \mathcal{O}_C) = 0$, the Lie algebra of $\underline{\operatorname{Aut}}(C, p_i)$ is trivial and $\underline{\operatorname{Aut}}(C, p_i)$ is a finite and reduced group scheme. By the Characterization of Deligne–Mumford stacks (3.6.4), $\overline{\mathcal{M}}_{g,n}$ is Deligne–Mumford. Since $\operatorname{Ext}^2_{\mathcal{O}_C}(\Omega_C(\sum_i p_i), \mathcal{O}_C) = 0$, there are no obstructions to deforming stable curves, and the Infinitesimal Lifting Criterion (3.7.1) implies that $\overline{\mathcal{M}}_{g,n}$ is smooth over $\operatorname{Spec}\mathbb{Z}$. Finally, since the Zariski tangent space of $[C, p_i] \in \overline{\mathcal{M}}_{g,n} \times_{\mathbb{Z}} \Bbbk$ is bijective to $\operatorname{Ext}^1_{\mathcal{O}_C}(\Omega_C(\sum_i p_i), \mathcal{O}_C)$ and this vector space has dimension $3g - 3 + n$, we conclude that $\overline{\mathcal{M}}_{g,n} \to \operatorname{Spec}\mathbb{Z}$ has relative dimension $3g - 3 + n$. □

Exercise 5.4.15. Show that $\overline{\mathcal{M}}_g$ is algebraic by explicitly presenting it as a quotient stack of a locally closed subscheme of the Hilbert scheme.

Hint: Follow the proof of Theorem 3.1.17.

Exercise 5.4.16. Show that each of the stacks $\mathcal{M}^{\mathrm{ss}}_{g,n}$, $\mathcal{M}^{\mathrm{pre}}_{g,n}$, $\mathcal{M}^{\leq\mathrm{nodal}}_{g,n}$, and $\mathcal{M}^{\mathrm{all}}_{g,n}$ are not quasi-compact.

Hint: Use the presence of rational bridges of arbitrary length.

Remark 5.4.17. There are various other open substacks of $\mathcal{M}^{\mathrm{all}}_g$ parameterizing particular classes of curves such as

$$\begin{aligned}
\mathcal{M}^{\mathrm{DM}}_g &= \{\text{curves with finite, reduced automorphism group}\}\\
\mathcal{M}^{\mathrm{CM}}_g &= \{\text{Cohen–Macaulay curves}\}\\
\mathcal{M}^{\text{geom-red}}_g &= \{\text{geometrically reduced curves}\}\\
\mathcal{M}^{\mathrm{Gor}}_g &= \{\text{Gorenstein curves}\}\\
\mathcal{M}^{\mathrm{lci}}_g &= \{\text{local complete intersection curves}\}\\
\mathcal{M}^{\mathrm{iso}}_g &= \{\text{curves with isolated singularities}\};
\end{aligned}$$

see [SP, Tags 0DST–0E0K]. The open substack $\mathcal{M}^{\mathrm{DM}}_g$ is identified with the maximal open Deligne–Mumford substack of $\mathcal{M}^{\mathrm{all}}_g$. For any g, there are curves in $\mathcal{M}^{\mathrm{all}}_g \setminus \mathcal{M}^{\mathrm{DM}}_g$ with positive dimensional and non-reduced automorphism groups. While $\mathcal{M}^{\mathrm{all}}_{g,n}$ is not smooth over $\operatorname{Spec}\mathbb{Z}$, the open substack $\mathcal{M}^{\mathrm{lci}}_g \cap \mathcal{M}^{\mathrm{iso}}_g$ is smooth [SP, Tag 0DZX].

The open substack $\mathcal{M}_g \subseteq \mathcal{M}_g^{\mathrm{all}}$ is not dense: Mumford provided examples of reduced, irreducible proper curves that do not deform to a smooth curve [Mum75b]. However, we show in Proposition 5.7.26 that $\mathcal{M}_g$ is dense in $\overline{\mathcal{M}}_g$, and it is also true that $\mathcal{M}_g$ is dense in $\mathcal{M}_g^{\mathrm{lci}} \cap \mathcal{M}_g^{\mathrm{iso}}$ [SP, Tag 0E86].

Exercise 5.4.18 (not important). Show that $\mathcal{M}_g^{\mathrm{all}} \to \operatorname{Spec} \mathbb{Z}$ satisfies the existence part of the valuative criterion for properness but that it is not universally closed.

Hint: Consider a closed substack of $\mathcal{M}_g^{\mathrm{all}} \times \mathbb{A}^1_{\mathbb{Z}}$ consisting of the disjoint union of $([C_n], n)$ over positive integers n, where C_n is the nodal union of a genus g curve and a tree of $\mathbb{P}^1$'s with n nodes. The argument should also show that $\mathcal{M}_g^{\mathrm{pre}}$ is not universally closed over any field.

5.5 Stable reduction and the properness of $\overline{\mathcal{M}}_{g,n}$

Algebraic curves were created by God and algebraic surfaces by the Devil.

Max Noether

In this section, we discuss Deligne and Mumford's celebrated theorem that $\overline{\mathcal{M}}_{g,n}$ is proper [DM69]. The key ingredient is the stable reduction theorem.

Theorem 5.5.1 (Stable Reduction). *Let R be a DVR with $K = \operatorname{Frac}(R)$, and set $\Delta = \operatorname{Spec} R$ and $\Delta^* = \operatorname{Spec} K$. If $(\mathcal{C}^* \to \Delta^*, \sigma_1^*, \ldots, \sigma_n^*)$ is a family of n-pointed stable curves, then there exists an extension of DVRs $R \to R'$ and an n-pointed family $(\mathcal{C} \to \Delta' = \operatorname{Spec} R', \sigma_1, \ldots, \sigma_n)$ of stable curves extending the base change of $(\mathcal{C}^* \to \Delta^*, \sigma_1^*, \ldots, \sigma_n^*)$ to $K' = \operatorname{Frac} R'$.*

While the theorem holds for any DVR, we give a complete proof in this section only in *characteristic 0*, following [KKMS73, Ch. II], [HM98, §3.C], and [ACG11, §X.4].

Remark 5.5.2. In characteristic 0, the stable reduction theorem was first stated in [MM64b, Lem. A] with a proof given in [May69]. The general version first appeared in Deligne and Mumford's seminar paper [DM69]. Their proof relied on semistable reduction for abelian varieties, which had been established in [SGA7-I, SGA7-II], by embedding the generic fiber $\mathcal{C}^*$ into its Jacobian. Gieseker offerered a different proof using GIT [Gie82], which as expressed in [SP, Tag 0C2Q] "is quite an amazing feat: it seems somewhat counterintuitive that one can prove such a result without ever truly studying families of curves over a positive dimensional base." Later arguments due to Artin–Winters [AW71] and Saito [Sai87] follow roughly the same strategy as we provide here but require additional techniques in positive characteristic; see Remark 5.5.10.

5.5.1 Proof of Stable Reduction

Throughout, we use the notation that $\Delta = \operatorname{Spec} R$ for a DVR R defined over $\mathbb{Q}$, $\Delta^* = \operatorname{Spec} K$ where $K = \operatorname{Frac}(R)$, $t \in R$ is a uniformizer, and $0 = (t) \in \operatorname{Spec} R$ is the unique closed point. We are given an n-pointed stable family $(\mathcal{C}^* \to \Delta^*, \sigma_1^*, \ldots, \sigma_n^*)$ of curves of genus g.

Proof strategy

① Reduce to the case where $\mathcal{C}^* \to \Delta^*$ is smooth.
② Construct a flat extension $\mathcal{C} \to \Delta$.
③ Replace $\mathcal{C}$ with a resolution of singularities such that the reduced central fiber $(\mathcal{C}_0)_{\mathrm{red}}$ is nodal.
④ Take a ramified base extension $\Delta' \to \Delta$ such that the normalization of $\mathcal{C} \times_\Delta \Delta$ has reduced central fiber.
⑤ Arrange that the marked points are smooth and distinct.
⑥ Contract rational tails and bridges in the central fiber.

It is useful to keep examples in mind while reading the proof. For a very simple example, consider the family of elliptic curves $(\mathcal{C}^* \to \Delta^*, \sigma^*)$ defined by the equation

$$y^2 z = x(x-z)(x-tz) \tag{5.5.3}$$

in $\mathbb{P}^2 \times \Delta^*$ and the section $\sigma^*(t) = [0,1,0]$. In this case, the stable limit is transparent from our description of $\mathcal{C}^*$: the family $(\mathcal{C} \to \Delta, \sigma)$, where $\mathcal{C} \subseteq \mathbb{P}^2 \times \Delta$ is defined by (5.5.3) and $\sigma(t) = [0,1,0]$, is a stable family extending $(\mathcal{C}^* \to \Delta^*, \sigma^*)$; see Figure 0.7.3. The stable limit $\mathcal{C}_0$ is the nodal cubic $y^2 z = x^2(x-z)$. Additional examples are given in the proof, and more involved examples are described in §5.5.2.

Proof of Stable Reduction (5.5.1) in characteristic 0.

Step 1: Reduce to the case where $\mathcal{C}^* \to \Delta^*$ is smooth. If $\mathcal{C}^*$ has δ nodes, then after replacing K and R with extensions, we can arrange that the jth node is given by a K-point $n_j^* \in \mathcal{C}^*$ whose preimage under the normalization $\widetilde{\mathcal{C}}^* \to \mathcal{C}^*$ consists of two K-points q_j^* and r_j^*. We call $(\widetilde{\mathcal{C}}^* \to \Delta^*, \sigma_i^*, q_j^*, r_j^*)$ the *pointed normalization*, and we let $(\widetilde{\mathcal{C}}_k^* \to \Delta^*, q_{kl}^*)$ be the pointed connected components, where $\{q_{kl}^*\} = \{\sigma_i^*, q_j^*, r_j^*\} \cap \widetilde{\mathcal{C}}_k^*$. Since $\mathcal{C}^*$ is stable, each $(\widetilde{\mathcal{C}}_k^* \to \Delta^*, q_{kl}^*)$ is also stable (Exercise 5.3.6).

Assuming that stable reduction holds when the generic fiber $\mathcal{C}^*$ is smooth, we can construct stable families $(\widetilde{\mathcal{C}}_k \to \Delta, q_{kl})$ extending $(\widetilde{\mathcal{C}}_k^* \to \Delta^*, q_{kl}^*)$ after replacing K and R with extensions. For each $j = 1, \ldots, \delta$, we use a Ferrand Pushout (B.4.1) to glue the sections q_{il} and $q_{i'l'}$ corresponding to q_j^* and r_j^*. By the étale local structure of smooth morphisms (Proposition A.3.4) and the étale local nature of pushouts (Proposition B.4.8(3)), there is an étale neighborhood of the pushout of the form

$$\begin{tikzcd} \operatorname{Spec} R \amalg \operatorname{Spec} R \arrow[r, hook, "0 \amalg 0"] \arrow[d] & \mathbb{A}^1_R \amalg \mathbb{A}^1_R \arrow[d, dashed] \\ \operatorname{Spec} R \arrow[r, dashed] & \operatorname{Spec} A, \end{tikzcd} \quad \text{where} \quad \begin{aligned} A &= R[x] \textstyle\coprod_{R \times R} R[y] \\ &= \{(f,g) \in R[x] \times R[y] \mid f(0) = g(0)\} \\ &= R[x,y]/(xy). \end{aligned}$$

The sections q_{il} and $q_{i'l'}$ are glued to a node. This produces an n-pointed family $(\mathcal{C} \to \Delta, \sigma_i)$ of nodal curves with δ additional sections picking out the nodes. The pointed normalization of the central fiber $\mathcal{C}_0$ is the disjoint union of the central fibers $(\widetilde{\mathcal{C}}_k)_0$, and it follows from Exercise 5.3.6 that $(\mathcal{C}_0, \sigma_i(0))$ is stable. We conclude that $(\mathcal{C} \to \Delta, \sigma_i)$ is a stable family.

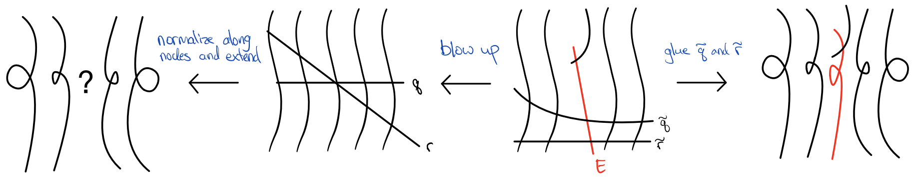

Figure 5.5.3: (Step 1) Consider a family $\mathcal{C}^* \to \Delta^*$ of stable curves with a single node, e.g., a node degenerating to a cusp locally defined by $y^2 = x^3 + tx^2$. The stable limit is obtained by normalizing along the nodes, extending the 2-pointed family, blowing up where the two sections q and r intersect, and then gluing the proper transforms $\widetilde{q}$ and $\widetilde{r}$. If the normalization $\widetilde{\mathcal{C}^*} = C \times \Delta^*$ is a constant family, then the stable limit is the nodal union of C and a rational nodal curve.

Alternatively, using that $\mathcal{M}_{g,n} \subseteq \overline{\mathcal{M}}_{g,n}$ is dense (Theorem 5.7.30) Exercise 4.6.7 asserts that it suffices to check the valuative criterion for properness under the condition that $\Delta^* \to \overline{\mathcal{M}}_{g,n}$ factors through $\mathcal{M}_{g,n}$.

Step 2: Find a flat extension $(\mathcal{C} \to \Delta, \sigma_i)$. Using that $\big(\omega_{\mathcal{C}^*/\Delta^*}(\sum_i \sigma_i^*)\big)^{\otimes 3}$ is very ample (Proposition 5.3.23), we may embed $\mathcal{C}^*$ as a closed subscheme of $\mathbb{P}^N \times \Delta^*$. By the Flatness Criterion over Smooth Curves (A.2.2), the scheme-theoretic image $\mathcal{C}$ of $\mathcal{C}^* \hookrightarrow \mathbb{P}^N \times \Delta$ is flat over Δ (as the closure doesn't introduce any embedded points in the central fiber). This gives a family of curves $\mathcal{C} \to \Delta$ extending $\mathcal{C}^* \to \Delta^*$. (This is the same argument we used in Proposition 1.4.2 to show the properness of the Hilbert scheme using the valuative criterion.) The sections $\sigma_i^* \colon \Delta^* \to \mathcal{C}^*$ extend to sections $\sigma_i \colon \Delta \to \mathcal{C}$ by the Valuative Criterion of Properness (A.4.5).

At this stage, the central fiber $\mathcal{C}_0$ may be very singular and the marked points $\sigma_i(0) \in \mathcal{C}_0$ may be singular and non-distinct.

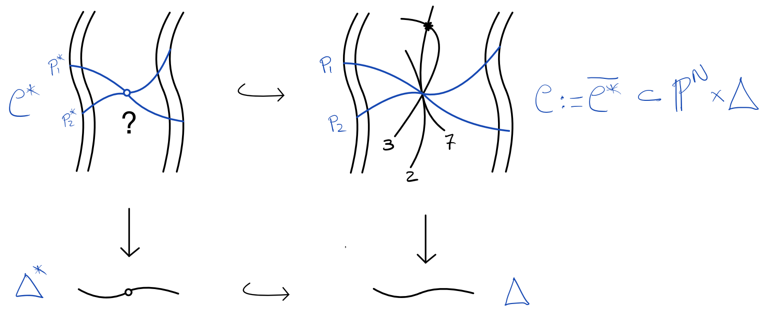

Figure 5.5.4: (Step 2) The closure $\mathcal{C} := \overline{\mathcal{C}^*}$ is a flat family over Δ. The central fiber $\mathcal{C}_0$ may be generically non-reduced with embedded points; the numbers 3, 2, and 7 indicate the multiplicity of the irreducible components. The marked points $\sigma_i(0) \in \mathcal{C}_0$ may be singular and non-distinct.

Step 3: Replace $\mathcal{C}$ *with a resolution of singularities to arrange that the reduced central fiber* $(\mathcal{C}_0)_{\mathrm{red}}$ *is nodal.* By combining Existence of Resolutions (B.2.1) and Existence of Embedded Resolutions (B.2.3), there is a projective birational morphism $\widetilde{\mathcal{C}} \to \mathcal{C}$ from a regular scheme which is an isomorphism over Δ^* and such that the central fiber $\widetilde{\mathcal{C}}_0$ has set-theoretic normal crossings, i.e., $(\widetilde{\mathcal{C}}_0)_{\mathrm{red}}$ is nodal. By the Flatness Criterion over Smooth Curves (A.2.2), $\widetilde{\mathcal{C}} \to \Delta$ is flat. We then replace $\mathcal{C}$ with $\widetilde{\mathcal{C}}$ and the sections σ_i with their strict transform.

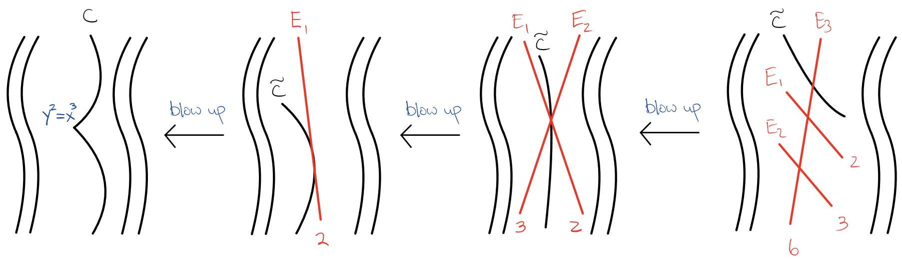

Figure 5.5.5: (Step 3) Suppose $\mathcal{C} \to \Delta = \Bbbk[t]_{(t)}$ is a generically smooth family degenerating to a cusp $y^2 = x^3$ in the central fiber such that the local equation in $\mathcal{C}$ around the singular point is $y^2 = x^3 + t$. We repeatedly blow up the singular point in the central fiber using local coordinates x, y on the original surface and $\widetilde{x}, \widetilde{y}$ on the new surface, where in the first chart of the blowup $\widetilde{x} = x, \widetilde{y} = y/x$ with exceptional divisor $\widetilde{x} = 0$, while in the second chart, $\widetilde{x} = x/y, \widetilde{y} = y$ with exceptional divisor $\widetilde{y} = 0$.

- → For the first blowup, the preimage of the singularity in the first chart is given by $\widetilde{x}^2(\widetilde{y}^2 - \widetilde{x})$ and in the second chart by $\widetilde{y}^2(1 - \widetilde{x}^3\widetilde{y})$. The exceptional divisor E_1 has multiplicity 2. The normalization $\widetilde{C}$ of C has genus $g - 1$.
- → The second blowup has charts defined by $\widetilde{x}^3(\widetilde{x}\widetilde{y}^3 - 1)$ and $\widetilde{x}^2\widetilde{y}^3(\widetilde{y} - \widetilde{x})$, and the new exceptional divisor E_2 has multiplicity 3.
- → The final blowup has charts $\widetilde{x}^6\widetilde{y}^3(\widetilde{y} - 1)$ and $\widetilde{x}^2\widetilde{y}^6(1 - \widetilde{x})$, and the new exceptional divisor E_3 has multiplicity 6. The central fiber is non-reduced and its reduction is nodal.

Step 4: Take a ramified base extension $\Delta' = \operatorname{Spec} R' \to \operatorname{Spec} R = \Delta$ such that the central fiber of the normalization of $\mathcal{C} \times_\Delta \Delta'$ becomes reduced and nodal. This is the most difficult step and is where the characteristic 0 assumption is used. The argument can be viewed as a version of Abhyankar's lemma on tame ramification (see [SGA1, §XIII.5] and [SP, Tag 0EXT]). Since $\mathcal{C}$ is regular and $t \in R$ is a nonzerodivisor, the central fiber $\mathcal{C}_0$ is Cohen–Macaulay, and thus has no embedded points. On the other hand, $(\mathcal{C}_0)_{\mathrm{red}}$ is an effective Cartier divisor and thus is locally cut out by a single equation. For every $p \in \mathcal{C}_0$, we can find an étale neighborhood and local coordinates x, y such that the map $\mathcal{C} \to \Delta$ is given explicitly by:

- if $p \in (\mathcal{C}_0)_{\mathrm{red}}$ is a smooth point, then $(x, y) \mapsto x^a$ and the multiplicity of the irreducible component of $\mathcal{C}_0$ containing p is a, and
- if $p \in (\mathcal{C}_0)_{\mathrm{red}}$ is a node , then $(x, y) \mapsto x^a y^b$ and the two components of $\mathcal{C}_0$ containing p have multiplicities a and b. If $p \in (\mathcal{C}_0)_{\mathrm{red}}$ is a non-separating node (i.e., $\mathcal{C}_0 \smallsetminus p$ is connected), then $a = b$.

Let N be the least common multiple of the multiplicities of the irreducible components of $\mathcal{C}_0$. After replacing R with an extension, we can assume that R contains a primitive Nth root of unity ρ. Let $\Delta' = \operatorname{Spec} R' \to \operatorname{Spec} R = \Delta$ be a totally ramified extension of DVRs of degree N (i.e., the image of the uniformizer $t \in R$ is t'^N for a uniformizer $t' \in R'$). Let $\mathcal{C}' := \mathcal{C} \times_\Delta \Delta'$ and $p' \in \mathcal{C}'$ be the unique preimage of p, and let $\widetilde{\mathcal{C}}'$ be the normalization of $\mathcal{C}'$.

If $p \in (\mathcal{C}_0)_{\mathrm{red}}$ is smooth, then $\mathcal{C}'$ is defined étale locally by $x^a = t^N$ near p'. Since

R is characteristic 0, there is a factorization

$$x^a - t^N = \prod_{i=0}^{a-1} (x - (\rho^i t)^{N/a})$$

into distinct factors. (If the characteristic were positive and divided N, then $\mathcal{C}'$ would be non-reduced.) In the normalization $\widetilde{\mathcal{C}}'$, the point p has a preimages, each etale locally defined by $x = \rho^i t^{N/a}$. Each preimage is a smooth point in both the total family $\widetilde{\mathcal{C}}'$ and the central fiber $\widetilde{\mathcal{C}}'_0$. On the other hand, if $p \in (\mathcal{C}_0)_{\mathrm{red}}$ is a node, then $\mathcal{C}'$ is defined étale locally by $x^a y^b = t^N$ near p'. Let $d = \gcd(a, b)$. If $d > 1$, there is a factorization (again using the characteristic 0 hypothesis)

$$x^a y^b - t^N = \prod_{i=0}^{d-1} (x^{a/d} y^{b/d} - (\rho^i t)^{N/d}).$$

Étale locally near p, the normalization factors as $\widetilde{\mathcal{C}}' \to \mathcal{C}'' \to \mathcal{C}'$ such that p has d preimages in $\mathcal{C}''$, each described by $x^{a/d} y^{b/d} - (\rho^i t)^{N/d}$. Unless $d = a = b$, the central fiber $\mathcal{C}''_0$ is still non-reduced. The reduction $(\mathcal{C}''_0)_{\mathrm{red}}$ is nodal at each preimage of p but now the multiplicities of the two branches are relatively prime. Therefore, we can assume that $\gcd(a, b) = 1$. After possibly exchanging x and y, we can write $1 = \alpha a - \beta b$ for positive integers α and β. The normalization of the integral domain $R[x, y]/(x^a y^b - t^N)$ is given by

$$R[x, y]/(x^a y^b - t^N) \hookrightarrow R[u, v]/(uv - t^{N/(ab)}), \qquad \begin{array}{l} x \mapsto u^b \\ y \mapsto v^a, \end{array}$$

where $u = t^{\alpha N/b}/(x^\beta y^\alpha)$ and $v = x^\beta y^\alpha / t^{\beta N/a}$. The central fiber $\widetilde{\mathcal{C}}'_0$ is reduced with a node at a unique preimage of p. If $N = ab$, $\widetilde{\mathcal{C}}'$ is smooth at this preimage; otherwise $\widetilde{\mathcal{C}}'$ has an A_{n-1}-singularity where $n = N/(ab)$.

We have now arranged the central fiber to be reduced and nodal, but we have also introduced A_{n-1}-singularities (i.e., $xy - t^n$) into the total family. Repeatedly blowing up each A_{n-1}-singularity replaces the singularity with the nodal union of $\lfloor \frac{n}{2} \rfloor$ smooth rational curves; this explicitly describes the minimal resolution of $\widetilde{\mathcal{C}}'$ as in Theorem B.2.2. We now replace $\mathcal{C}$ with the minimal resolution so that the total family $\mathcal{C}$ is regular and the central fiber $\mathcal{C}_0$ is reduced and nodal.

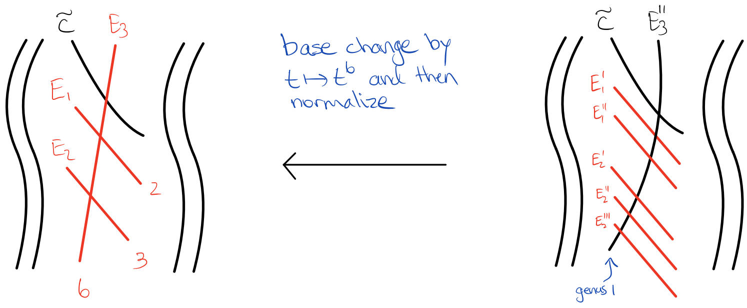

Figure 5.5.6: (Step 4) Continuing from Figure 5.5.5, we base change $\mathcal{C} \to \Delta = \operatorname{Spec} \Bbbk[t]_{(t)}$ by $\Delta' = \operatorname{Spec} \Bbbk[t]_{(t)} \to \operatorname{Spec} \Bbbk[t]_{(t)} = \Delta$, given by $t \mapsto t^6$, and then normalize. The central fiber is now reduced and nodal. Each preimage of E_1 and E_2 are smooth rational curves, while the preimage E_3'' of E_3 is a genus 1 curve. To understand this description, it is convenient to break the base change into the composition of the normalized base change by $t \mapsto t^2$ and the normalized base change by $t \mapsto t^3$, which we describe in Example 5.5.13.

Step 5: Arrange that the marked points $\sigma_i(0) \in \mathcal{C}_0$ are smooth and distinct. After repeatedly blowing up closed points in the central fiber $\mathcal{C}_0$ where the marked points $\sigma_i(0)$ are singular or collide, the strict transform of the sections become distinct and smooth points of the central fiber. After replacing $\mathcal{C}$ with the blowup and the sections σ_i with their strict transform, we have a prestable family $(\mathcal{C} \to \Delta, \sigma_i)$ with regular total family.

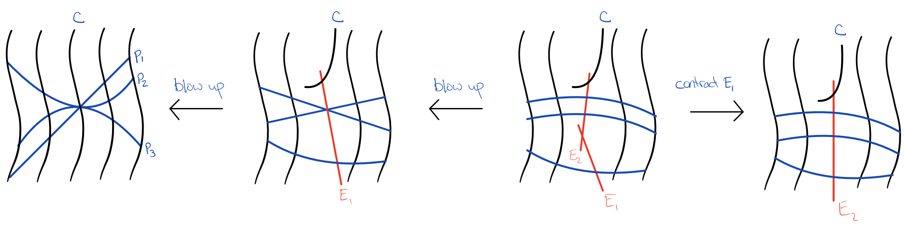

Figure 5.5.7: (Step 5) Consider a constant family $C \times \Delta \to \Delta$ with sections locally defined by $(\sigma_1, \sigma_2, \sigma_3) = (t^2, -t^2, 4t)$. After blowing up twice, the sections become disjoint but the central fiber is not stable as the exceptional component $E_1 \cong \mathbb{P}^1$ only has one node and one marked point. As explained in the next step, the stable limit is obtained by contracting the rational bridge E_1.

Step 6: Contract rational tails and bridges in the central fiber. The central fiber $\mathcal{C}_0$ of the prestable family $(\mathcal{C} \to \Delta, \sigma_i)$ will be stable, unless there are rational tails and bridges (see Definition 5.3.11). If $E \subseteq \mathcal{C}_0$ is a rational tail or rational bridge with a marked point, then $E^2 = -1$, while if E is a rational bridge without a marked point, then $E^2 = -2$ (see Remark 5.3.15). By Exercise 5.3.16, contracting each rational tail and bridge yields a morphism $\mathcal{C} \to \mathcal{C}'$ of families of nodal curves over Δ . Letting $\sigma_i' \colon \Delta \xrightarrow{\sigma_i} \mathcal{C} \to \mathcal{C}'$, the n-pointed family $(\mathcal{C}' \to \Delta, \sigma_i')$ is now stable! Alternatively, we can construct the stable family $(\mathcal{C}' \to \Delta, \sigma_i')$ using the Stable Contraction of a Prestable Family (5.6.6). $\square$

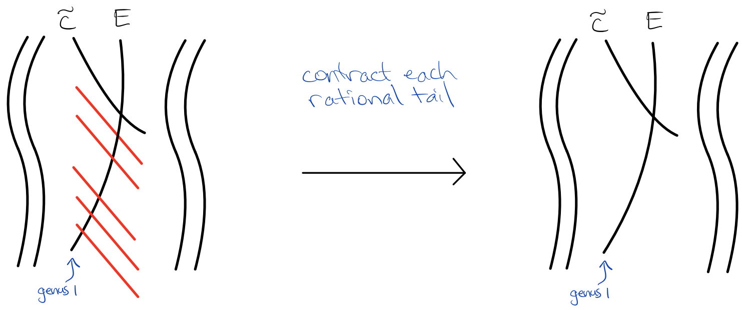

Figure 5.5.8: (Step 6) Continuing from Figure 5.5.6, the contraction of each rational tail in red produces a stable family. The central fiber is the nodal union of the normalization $\widetilde{C}$ of the original central fiber (with a cusp) and an elliptic curve E.

If we proceed with the six-step procedure above, but stop in Step 6 after contracting only rational tails (but not rational bridges), then Castelnuovo's Contraction Theorem (B.2.6) implies that the total family is still regular. This important variant is called *semistable reduction.*

Theorem 5.5.9 (Semistable Reduction)**.** *Let R be a DVR with $K = \operatorname{Frac}(R)$, and set $\Delta = \operatorname{Spec} R$ and $\Delta^* = \operatorname{Spec} K$. If $(\mathcal{C}^* \to \Delta^*, s_1^*, \ldots, s_n^*)$ is a family of n-pointed smooth curves, then there exists an extension of DVRs $R \to R'$ and an n-pointed family $(\mathcal{C} \to \Delta' = \operatorname{Spec} R', s_1, \ldots, s_n)$ of semistable curves with regular total family $\mathcal{C}$ extending the base change of $(\mathcal{C}^* \to \Delta^*, s_1^*, \ldots, s_n^*)$ to $K' = \operatorname{Frac} R'$.* $\square$

Remark 5.5.10 (Proof in characteristic p)*.* Our proof of stable reduction fails in Step 4 if the residue field of R has characteristic $p > 0$ and any of the multiplicities of the components of the central fiber are divisible by p. A different approach is needed in positive characteristic. After resolving the singularities of $\mathcal{C}$ and arranging that $(\mathcal{C}_0)_{\text{red}}$ is nodal (as we do in Step 3 above), Artin and Winters arrange that the l-torsion of $\operatorname{Pic}(\mathcal{C}^*_{K'})$ is isomorphic to $(\mathbb{Z}/l\mathbb{Z})^{2g}$ for a sufficiently large prime $l \neq p$, and they show that this magically forces the central fiber to be reduced and nodal! See [AW71], [Liu02, §10.4], and [SP, Tag 0C2P].

Example 5.5.11 (Base changes are necessary)**.** The stable reduction of the cusp worked out in Figures 5.5.5, 5.5.6 and 5.5.8 demonstrates the necessity of allowing for extensions of the DVR. Indeed, after Step 3, we have a generically smooth family $\mathcal{C} \to \Delta$ with a regular total family $\mathcal{C}$ and with a non-reduced central fiber $\mathcal{C}_0$. Suppose that there is a stable family $\mathcal{C}' \to \Delta$ such that $\mathcal{C}|_{\Delta^*} \cong \mathcal{C}'|_{\Delta^*}$. After resolving the A_n-singularities in $\mathcal{C}'$, there is semistable family $\mathcal{C}'' \to \Delta$ with $\mathcal{C}''$ regular and $\mathcal{C}''_0$ semistable (and in particular reduced). Since $\mathcal{C}$ and $\mathcal{C}''$ are birational, by taking a resolution of singularities of the closure of the graph of a rational map $\mathcal{C} \dashrightarrow \mathcal{C}'$, there is a regular two-dimensional scheme $\mathcal{D}$ and birational morphisms $\mathcal{D} \to \mathcal{C}$ and $\mathcal{D} \to \mathcal{C}''$ over Δ, each which factors as a composition of blowups by Factorization of Birational Maps (B.2.4). However, any blowup of $\mathcal{C}$ will have a non-reduced central fiber, while any blowup of $\mathcal{C}''$ will have a reduced central fiber, and this yields a contradiction.

5.5.2 Explicit stable reduction

In applications to the geometry of curves, it is often essential to explicitly describe the stable limit. While the proof of Stable Reduction offers a strategy, additional care is needed to get an explicit handle. The main challenge is determining the normalization $\widetilde{\mathcal{C}'}$ of the base change $\mathcal{C}' = \mathcal{C} \times_\Delta \Delta'$ by a totally ramified extension $\Delta' \to \Delta$ in Step 4. It is often simpler to factor $\Delta' \to \Delta$ as a composition of prime order base changes, because it is straightforward to determine the ramification locus of prime order normalized base changes $\widetilde{\mathcal{C}'} \to \mathcal{C}$.

Proposition 5.5.12. *Let p be a prime integer. Let $\mathcal{C} \to \Delta = \operatorname{Spec} R$ be a generically smooth family of curves, where R is a DVR of characteristic 0 containing a pth root of unity. Assume that the reduced central fiber $(\mathcal{C}_0)_{\mathrm{red}}$ is nodal. As a divisor on $\mathcal{C}$, we may write $\mathcal{C}_0 = \sum m_i D_i$ where m_i is the multiplicity of the irreducible component D_i. Let $\Delta' = \operatorname{Spec} R' \to \operatorname{Spec} R = \Delta$ be a totally unramified extension of degree p, and set $\mathcal{C}' := \mathcal{C} \times_\Delta \Delta'$ with normalization $\widetilde{\mathcal{C}'}$. Then $\widetilde{\mathcal{C}'} \to \mathcal{C}$ is ramified over a divisor D_i if and only if m_i is relatively prime to p. If $q' \in \widetilde{\mathcal{C}'_0}$ is a preimage of a point $q \in D_i$ which is a smooth point of $(\mathcal{C}_0)_{\mathrm{red}}$, then the multiplicity of $\widetilde{\mathcal{C}'_0}$ around q' is m_i/p if p divides m_i and m_i otherwise.*

Proof. The point $q \in \mathcal{C}$ has an étale neighborhood with local coordinates x and y such that $\mathcal{C} \to \Delta$ is described as $(x, y) \mapsto x^{m_i}$. The base change $\mathcal{C}'$ is described étale locally by $\operatorname{Spec} \Bbbk[x, y, t]/(x^{m_i} - t^p)$ near q'. If p divides m_i,

$$x^{m_i} - t^p = \prod_{i=0}^{p-1} (x^{m_i/p} - \rho^i t), \qquad \text{where } \rho \text{ is a } p\text{th root of unity},$$

and its normalization has p smooth components. The central fiber $\widetilde{\mathcal{C}'_0}$ has multiplicity m_i/p near each of the p preimages of q. If $m_i = pm_i' + r$ with $0 < r < p$, then $\Bbbk[x, y, t]/(x^{m_i} - t^p)$ is an integral domain with normalization

$$\Bbbk[x, y, t']/(x^r - t'^p), \qquad \text{where } t' = t/x^{m_i'}.$$

Thus, $\widetilde{\mathcal{C}'} \to \mathcal{C}$ is ramified at q. The central fiber $\widetilde{\mathcal{C}'} \to \Delta$ is defined by $t = t'x^{m_i'}$ and the multiplicity at the unique preimage of q can be computed as $\dim_\Bbbk \Bbbk[x, t']/(x^r - t'^p, t'x^{m_i'})$. Since the union of $t'^i x^j$ for $i = 0, \ldots, p-1$ and $j = 0, \ldots, m_i' - 1$ and $x^{m_i'}, \ldots, x^{m_i'+r-1}$ forms a basis, the multiplicity is m_i. $\square$

Example 5.5.13 (Stable reduction of a cusp)**.** Let $\mathcal{C} \to \Delta = \operatorname{Spec} \Bbbk[t]_{(t)}$ be a generically smooth family degenerating to a cusp $y^2 = x^3$ in the central fiber $C := \mathcal{C}_0$ such that the local equation in $\mathcal{C}$ around the singular point is $y^2 = x^3 + t$. As described in Figure 5.5.5, after three blowups of $\mathcal{C}$, we obtain a family $\mathcal{C}' \to \Delta$ such that $(\mathcal{C}'_0)_{\mathrm{red}}$ is the nodal union of the normalization $\widetilde{C}$, which has multiplicity 1, and three exceptional components E_1, E_2, and E_3, which have multiplicities 2, 3, and 6.

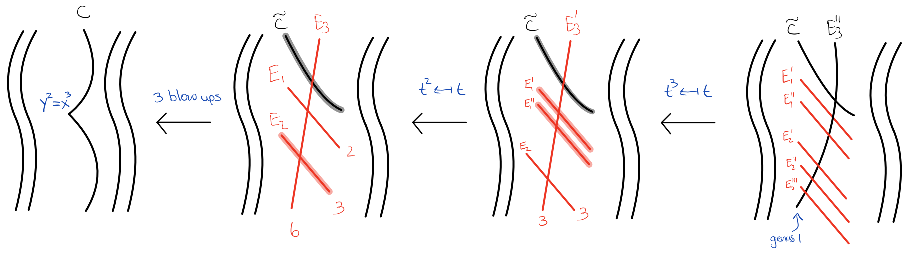

Figure 5.5.14: The numbers indicate the multiplicity of a component; if no number is given, the multiplicity is 1. The thickened components indicate the ramification locus under the normalized base change.

Using Proposition 5.5.12, the ramification locus of the first normalized base change $t \mapsto t^2$ is the union of $\widetilde{C}$ and E_2. The preimage of E_1 is the disjoint union $E_1' \amalg E_1''$ of two smooth rational curves. The preimage E_3' of E_3 is a curve which is a two-to-one cover of $E_3 = \mathbb{P}^1$ ramified at two points, and thus $E_3 = \mathbb{P}^1$ but now with multiplicity 2. For the second normalized base change, the ramification locus is $\widetilde{C} \amalg E_1' \amalg E_1''$ (again by Proposition 5.5.12). The preimage of E_2 is the union of three smooth rational curves, while the preimage E_3'' of E_3' is a smooth curve which is a three-to-one cover of $E_3' = \mathbb{P}^1$ ramified at three points, each with ramification index two. The genus $g_{E_3''}$ of E_3'' can be computed to be 1 using Riemann–Hurwitz (5.7.4): $2g_{E_3''} - 2 = 3(2(0) - 2) + 3(2) = 0$. The final step contracts the five rational tails as pictured in Figure 5.5.8, and the stable limit is the nodal union $\widetilde{C} \cup E_3''$ of a smooth genus $g-1$ curve $\widetilde{C}$ (the normalization of the original central fiber) and a genus 1 curve.

Remark 5.5.15. Precisely which elliptic curve E_3'' appears in the stable limit and how does the stable limit depend on the choice of degeneration? The deformation space of a cusp is $y^2 = x^3 + a_1(t)x + a_0(t)$ and, in other words, we are asking how the stable limit depends on $a_i(t)$. For instance, what happens when the total family of the surface is singular (e.g., $y^2 = x^{2k+1} + t^2$)? These questions are addressed in detail in [HM98, §3.C].

Exercise 5.5.16 (good practice)**.**

(a) Find the stable limit of a generically smooth family degenerating to a tacnode $y^2 = x^4$.

(b) More generally, show that the stable limit of a generically smooth family degenerating to a A_{2k} singularity $y^2 = x^{2k+1}$ (resp., $y^2 = x^{2k+2}$) is the nodal union of a genus $g-k$ curve and a genus k hyperelliptic curve attached at a Weierstrass point (resp., at two Weierstrass conjugate points). Here a *Weierstrass point* of a hyperelliptic curve H is a ramification point under the double cover $H \to \mathbb{P}^1$, while two points are *Weierstrass conjugate points* if their union is a fiber of $H \to \mathbb{P}^1$.

Example 5.5.17 (Stable reduction of a double conic)**.** Consider a generically smooth family $\mathcal{C} \to \Delta = \operatorname{Spec} \Bbbk[t]_{(t)}$, where $\mathcal{C} \subseteq \mathbb{P}^2 \times \Delta$ is defined by $F^2 + tG$ where F is a smooth conic and G is a smooth quartic. The central fiber is the double conic defined by F^2. The total space has an A_1-singularity with local equation $x^2 + yt$ at each of the 8 intersection points $p_1, \ldots, p_8$ of $F \cap G$. Each σ_i is resolved with a single blowup with an exceptional divisor $E_i = \mathbb{P}^1$ of multiplicity 1. This gives a family $\mathcal{C}_2 \to \Delta$ where the central fiber is $2C + \sum_i E_i$ as a divisor. We then take the normalization $\mathcal{C}_3$ of the base change $\mathcal{C}_2 \times_\Delta \Delta'$ by the ramified cover $\Delta' \to \Delta$, $t \mapsto t^2$. By Proposition 5.5.12, $\mathcal{C}_3 \to \mathcal{C}_2$ is ramified over the disjoint union of the E_i's. The preimage of C is a two-to-one cover of $\mathbb{P}^1$ branched over the 8 points σ_i, and hence is *smooth* hyperelliptic genus 3 curve.

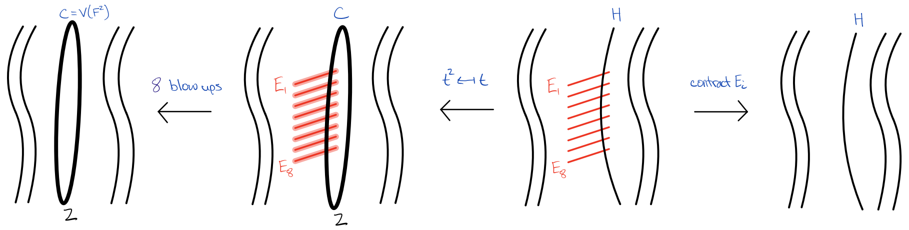

Figure 5.5.18: Stable reduction of a generically smooth family $F^2 + tG$ degenerating to a double conic $C = V(F^2)$.

Meta-exercise 5.5.19. Read the exposition of [HM98, §3.C] and do the exercises.

5.5.3 Properness of $\overline{\mathcal{M}}_{g,n}$

Stable Reduction (5.5.1) implies the existence part of the valuative criterion for properness for $\overline{\mathcal{M}}_{g,n}$. We must also show that the stable limit is unique, i.e., $\overline{\mathcal{M}}_{g,n}$ is separated.

Proposition 5.5.20. *Let R be a DVR with $K = \operatorname{Frac}(R)$, and set $\Delta = \operatorname{Spec} R$ and $\Delta^* = \operatorname{Spec} K$. If $(\mathcal{C} \to \Delta, \sigma_1, \ldots, \sigma_n)$ and $(\mathcal{D} \to \Delta, \tau_1, \ldots, \tau_n)$ are families of n-pointed stable curves, then every isomorphism $\alpha^* \colon \mathcal{C} \times_\Delta \Delta^* \to \mathcal{D} \times_\Delta \Delta^*$ over Δ^* compatible with the restriction of the sections (i.e., $\tau_i^* = \alpha^* \circ \sigma_i^*$) extends to a unique isomorphism $\alpha \colon \mathcal{C} \to \mathcal{D}$ over Δ with $\tau_i = \alpha \circ \sigma_i$.*

Proof. For simplicity, we handle the case without marked points ($n = 0$). We claim that we can reduce to the case where the generic fiber $\mathcal{C}^* \cong \mathcal{D}^*$ is smooth over Δ^*, where $\mathcal{C}^* := \mathcal{C} \times_\Delta \Delta^*$. We will employ the same strategy as in the reduction to the generically smooth case in Step 1 of the proof of Stable Reduction (5.5.1). By flat descent, it suffices to construct $\alpha \colon \mathcal{C} \to \mathcal{D}$ after an extension of DVRs $\Delta' \to \Delta$; indeed, an isomorphism $\alpha' \colon \mathcal{C} \times_\Delta \Delta' \to \mathcal{D} \times_\Delta \Delta'$ will satisfy the cocycle condition (by the separatedness of $\mathcal{D} \to \Delta$) and thus descend to an isomorphism α. Therefore, we may assume that each node of $\mathcal{C}^* \cong \mathcal{D}^*$ is given by a K-point whose preimage under the normalization consists of two K-points. Each node extends to sections $\Delta \to \mathcal{C}$ and $\Delta \to \mathcal{D}$. The pointed normalizations $\widetilde{\mathcal{C}}$ and $\widetilde{\mathcal{D}}$ are each disjoint unions of pointed stable families. If $\widetilde{\mathcal{C}}^* \to \widetilde{\mathcal{D}}^*$ extends to an isomorphism $\widetilde{\alpha} \colon \widetilde{\mathcal{C}} \to \widetilde{\mathcal{D}}$, then α descends to an isomorphism $\mathcal{C} \to \mathcal{D}$. Alternatively, using that $\mathcal{M}_{g,n} \subseteq \overline{\mathcal{M}}_{g,n}$ is dense (proven later in Theorem 5.7.30), it suffices to check the valuative criterion for separatedness under the condition that $\Delta^* \to \overline{\mathcal{M}}_{g,n}$ factors through $\mathcal{M}_{g,n}$ (Exercise 4.6.7).

Let $\widetilde{\mathcal{C}} \to \mathcal{C}$ and $\widetilde{\mathcal{D}} \to \mathcal{D}$ be the minimal resolutions (Theorem B.2.2). Let Γ be the closure of the image of $(\mathrm{id}, \alpha^*) \colon \mathcal{C}^* \to \widetilde{\mathcal{C}} \times_\Delta \widetilde{\mathcal{D}}$, and let $\widetilde{\Gamma} \to \Gamma$ be its minimal resolution. This gives a commutative diagram

$$\begin{tikzcd} & \widetilde{\mathcal{C}} \arrow[r] & \mathcal{C} \arrow[dr] & \\ \widetilde{\Gamma} \arrow[ur] \arrow[dr] & & & \Delta \\ & \widetilde{\mathcal{D}} \arrow[r] & \mathcal{D} \arrow[ur] & \end{tikzcd} \tag{5.5.21}$$

We will give two arguments that there is an isomorphism $\alpha\colon \mathcal{C} \to \mathcal{D}$ extending α^*. First, the local structure of $\mathcal{C}$ (resp., $\mathcal{D}$) around a node in the central fiber is an A_n singularity of the form $xy = t^{n+1}$, where $t \in R$ is a uniformizer. The preimage of each node under $\widetilde{\mathcal{C}} \to \mathcal{C}$ (resp., $\widetilde{\mathcal{D}} \to \mathcal{D}$) is a chain $E_1 \cup \cdots \cup E_n$ of rational bridges with $E_i^2 = -2$. Since $\mathcal{C}$ and $\mathcal{D}$ are families of stable curves, they each have no smooth rational -1 curves and thus neither do $\widetilde{\mathcal{C}}$ and $\widetilde{\mathcal{D}}$. By the Factorization of Birational Maps (Theorem B.2.4), both $\widetilde{\Gamma} \to \widetilde{\mathcal{C}}$ and $\widetilde{\Gamma} \to \widetilde{\mathcal{D}}$ are the compositions of finite sequences of blowups at closed points. Since $\widetilde{\Gamma} \to \Gamma$ is a minimal resolution, $\widetilde{\Gamma}$ has no smooth rational -1 curves that get contracted under both $\widetilde{\Gamma} \to \widetilde{\mathcal{C}}$ and $\widetilde{\Gamma} \to \widetilde{\mathcal{D}}$. We claim that both $\widetilde{\Gamma} \to \widetilde{\mathcal{C}}$ and $\widetilde{\Gamma} \to \widetilde{\mathcal{D}}$ are isomorphisms. This would finish the proof as both $\mathcal{C}$ and $\mathcal{D}$ are obtained by contracting the smooth rational -2 curves of $\widetilde{\mathcal{C}} \cong \widetilde{\mathcal{D}}$.

To see the claim, suppose for instance that $\widetilde{\Gamma} \to \widetilde{\mathcal{C}}$ is not an isomorphism. Then there is a smooth rational -1 curve $E \subseteq \widetilde{\Gamma}$ not contracted under $\widetilde{\Gamma} \to \widetilde{\mathcal{D}}$. Let $E_{\widetilde{\mathcal{D}}} \subseteq \widetilde{\mathcal{D}}$ be its image. Since blowing up only decreases the self-intersection number (indeed, if we write the pre-image of $E_{\widetilde{\mathcal{D}}}$ in $\widetilde{\Gamma}$ as $E + F$, then the projection formula implies that $E^2_{\widetilde{\mathcal{D}}} = E \cdot (E + F) = E^2 + E \cdot F$), we have that $E^2_{\widetilde{\mathcal{D}}} \geq E^2 = -1$. On the other hand, the Hodge Index Theorem for Exceptional Curves (B.2.5) implies that $E^2_{\widetilde{\mathcal{D}}} \leq -1$. Hence $E^2_{\widetilde{\mathcal{D}}} = -1$. But $E_{\widetilde{\mathcal{D}}}$ is not a smooth rational -1 curve so it must be singular, and one of the blowups in the composition $\widetilde{\Gamma} \to \widetilde{\mathcal{D}}$ must be at a singular point of $E_{\widetilde{\mathcal{D}}}$. But this implies that exceptional locus F of $\widetilde{\Gamma} \to \widetilde{\mathcal{D}}$ intersects E non-trivially, and thus $E^2_{\widetilde{\mathcal{D}}} \geq E^2 + 1$, a contradiction.

Alternatively, we could argue as follows. The birational maps $\widetilde{\Gamma} \to \widetilde{\mathcal{C}}$ and $\widetilde{\Gamma} \to \widetilde{\mathcal{D}}$ of smooth projective surfaces are isomorphisms in codimension 2. As the relative dualizing sheaves are line bundles, there are identifications of the pluricanonical sections

$$\Gamma(\widetilde{\mathcal{C}}, \omega^{\otimes k}_{\widetilde{\mathcal{C}}/\Delta}) \cong \Gamma(\widetilde{\Gamma}, \omega^{\otimes k}_{\widetilde{\Gamma}/\Delta}) \cong \Gamma(\widetilde{\mathcal{D}}, \omega^{\otimes k}_{\widetilde{\mathcal{D}}/\Delta})$$

for each nonnegative integer k (c.f., [Har77, Thm. II.8.19]). Using that $\mathcal{C}$ and $\mathcal{D}$ are the stable contraction of the families $\widetilde{\mathcal{C}}$ and $\widetilde{\mathcal{D}}$ over Δ (Theorem 5.6.6) arising as the Proj of the graded ring of pluricanonical sections, we obtain an isomorphism

$$\mathcal{C} \xrightarrow{\sim} \operatorname{Proj} \bigoplus_k \Gamma(\widetilde{\mathcal{C}}, \omega^{\otimes k}_{\widetilde{\mathcal{C}}/\Delta}) \xrightarrow{\sim} \operatorname{Proj} \bigoplus_k \Gamma(\widetilde{\mathcal{D}}, \omega^{\otimes k}_{\widetilde{\mathcal{D}}/\Delta}) \xrightarrow{\sim} \mathcal{D}$$

extending $\alpha^*\colon \mathcal{C}^* \to \mathcal{D}^*$. See also [DM69, Lem. 1.12], [ACG11, Lem. 10.5.1], and [SP, Tag 0E97]. □

Exercise 5.5.22 (details)**.** Extend the proof above to the case of marked points.

Even though we have only proved Stable Reduction (5.5.1) in characteristic 0, we state the next result over $\mathbb{Z}$.

Theorem 5.5.23 (Properness of $\overline{\mathcal{M}}_{g,n}$ and $\overline{M}_{g,n}$)**.** *If $2g - 2 + n > 0$, the Deligne–Mumford stack $\overline{\mathcal{M}}_{g,n}$ is proper over $\operatorname{Spec} \mathbb{Z}$. Moreover, there is a coarse moduli space $\overline{\mathcal{M}}_{g,n} \to \overline{M}_{g,n}$, where $\overline{M}_{g,n}$ is a proper algebraic space over $\operatorname{Spec} \mathbb{Z}$.*

Proof. Using the Valuative Criterion (3.8.7), Stable Reduction (5.5.1) gives the existence of limits, while Proposition 5.5.20 gives the uniqueness of a limit. As $\overline{\mathcal{M}}_{g,n}$ is separated, the Keel–Mori Theorem (4.4.6) gives a coarse moduli space $\overline{\mathcal{M}}_{g,n} \to \overline{M}_{g,n}$, where $\overline{M}_{g,n}$ is an algebraic space separated and of finite type over $\operatorname{Spec} \mathbb{Z}$. As $\overline{\mathcal{M}}_{g,n}$ is proper over $\operatorname{Spec} \mathbb{Z}$, so is $\overline{M}_{g,n}$. □

Exercise 5.5.24. Show that the coarse moduli space of $\overline{\mathcal{M}}_{g,n} \times_{\mathbb{Z}} \mathbb{F}_p$ is the normalization of $\overline{M}_{g,n} \times_{\mathbb{Z}} \mathbb{F}_p$.

Question 5.5.25 (Open)**.** Is $\overline{M}_{g,n} \times_{\mathbb{Z}} \mathbb{F}_p$ the coarse moduli space of $\overline{\mathcal{M}}_{g,n} \times_{\mathbb{Z}} \mathbb{F}_p$?[1]

5.6 Contraction, forgetful, and gluing morphisms

Old theorems never die; they turn into definitions.

Edwin Hewitt

In this section, we construct several important morphisms between moduli spaces of curves.

- (Stable Contraction) There is a morphism $\mathcal{M}_{g,n}^{\mathrm{pre}} \to \overline{\mathcal{M}}_{g,n}$, which maps a prestable family $(\mathcal{C} \to S, \sigma_i)$ to a stable family $(\mathcal{C}^{\mathrm{st}}, \sigma_i^{\mathrm{st}})$ by contracting all rational tails and bridges (Theorem 5.6.6).
- (Forgetful) There is a morphism $\overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ assign an $n+1$-pointed stable family $(\mathcal{C} \to S, \sigma_1, \ldots, \sigma_{n+1})$ to the stable contraction of the n-pointed prestable family $(\mathcal{C} \to S, \sigma_1, \ldots, \sigma_n)$ (Proposition 5.6.10). Moreover, we identify $\overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ with the universal family (Proposition 5.6.12).
- (Gluing) There is a morphism $\overline{\mathcal{M}}_{i,k} \times \overline{\mathcal{M}}_{g-i,n-k+2} \to \overline{\mathcal{M}}_{g,n}$, gluing a k-pointed stable family of genus i curves to a $n-k+2$-pointed stable family of genus $g-i$ curves along the final sections, and a morphism $\overline{\mathcal{M}}_{g-1,n+2} \to \overline{\mathcal{M}}_{g,n}$, gluing the final two sections of an $n+2$-pointed stable family of genus $g-1$ (Corollary 5.6.16).

As with the Local Structure of Nodal Families (5.2.25), the biggest challenge is ensuring that the constructions hold for families over an arbitrary base. We conclude this section in §5.6.4 with a discussion of boundary divisors and line bundles on $\overline{\mathcal{M}}_{g,n}$.

5.6.1 Contracting rational tails and bridges

Rational tails and bridges of a prestable curve over a field were defined in Definition 5.3.11 and characterized in Lemma 5.3.12. We first show that rational tails and bridges can be contracted to a stable curve over a field (Corollary 5.6.3), and then we extend the construction to families (Theorem 5.6.6).

Proposition 5.6.1 (Contracting a rational tail or bridge)**.** *Let (C, p_i) be an n-pointed prestable curve over a field $\Bbbk$, and E be a rational tail or rational bridge. Then there is a canonical morphism*

$$c\colon C \to C'$$

contacting E to a point such that $c_*\mathcal{O}_C = \mathcal{O}_{C'}$ and $\mathrm{R}^1 c_*\mathcal{O}_C = 0$. Moreover, C' is identified with the pushout $\operatorname{Spec}\Gamma(E, \mathcal{O}_E) \amalg_E C$, and the formation of $c\colon C \to C'$ commutes with field extensions of $\Bbbk$.

[1]See https://mathoverflow.net/questions/9981/coarse-moduli-spaces-over-z-and-f-p and https://mathoverflow.net/questions/72903/what-is-m-g-over-a-finite-field-really.

Proof. In both cases, we construct C' as the Ferrand Pushout (B.4.1)

$$\begin{tikzcd} E \cap E^c \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & E^c \arrow[d] \\ \operatorname{Spec} \Bbbk' \arrow[r] & C', \end{tikzcd}$$

where $\Bbbk' = \Gamma^0(E, \mathcal{O}_E)$. Since scheme-theoretic unions are pushouts (Exercise B.4.6), we have that $C = E \amalg_{E \cap E^c} E^c$. This induces a larger commutative diagram

$$\begin{tikzcd} E \cap E^c \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & E^c \arrow[d] \\ E \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & C \arrow[d, "c"] \\ \operatorname{Spec} \Bbbk' \arrow[r] & C', \end{tikzcd}$$

where $c\colon C \to C'$ is the unique map making the diagram commute. Since the top and outer squares are pushouts, so is the bottom square, i.e., $C' = E \amalg_{\operatorname{Spec} \Bbbk'} C$. Since Ferrand pushouts commute with field extension (Proposition B.4.8(4)), so does the construction of $c\colon C \to C'$.

If E is a rational tail or a rational bridge with a marked point, then $E \cap E^c = \operatorname{Spec} \Bbbk' = \operatorname{Spec} \kappa(x)$ for a point $x \in E \cap E^c$ (see Lemma 5.3.12), and thus $C' = E^c$. The map $c\colon C \to C' = E^c$ is the identity on E^c and contracts E to the point $x \in E^c$ via the structure morphism $E \to \operatorname{Spec} \Gamma(E, \mathcal{O}_E) = \operatorname{Spec} \kappa(x)$. Consider the short exact sequence

$$0 \to \mathcal{O}_C \to \mathcal{O}_E \oplus \mathcal{O}_{E^c} \to \kappa(x) \to 0.$$

Applying c_* yields a long exact sequence

$$0 \to c_*\mathcal{O}_C \to i_{x,*}\widetilde{\mathrm{H}^0(E, \mathcal{O}_E)} \oplus \mathcal{O}_{C'} \to \kappa(x) \to \mathrm{R}^1 c_* \mathcal{O}_C \to i_{x,*}\widetilde{\mathrm{H}^1(E, \mathcal{O}_E)} \to 0,$$

where $i_x\colon \operatorname{Spec} \kappa(x) \hookrightarrow C'$ is the inclusion of x. Since $\mathrm{H}^0(E, \mathcal{O}_E) = \kappa(x)$ and $\mathrm{H}^1(E, \mathcal{O}_E) = 0$, we see that $c_*\mathcal{O}_C = \mathcal{O}_{C'}$ and $\mathrm{R}^1 c_* \mathcal{O}_C = 0$.

If E is a rational bridge without marked points, set $\Bbbk'' = \mathrm{H}^0(E \cap E^c, \mathcal{O}_{E \cap E^c})$. By Lemma 5.3.12, $E \cap E^c$ is either a single point with $\Bbbk''$ a degree 2 separable extension of $\Bbbk'$ or two points with $\Bbbk'' = \Bbbk' \times \Bbbk'$. Since the construction of the pushout is étale local (Proposition B.4.8(3)), the affine pushouts

$$\begin{tikzcd} \operatorname{Spec} \Bbbk'' \arrow[r, "0"] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \operatorname{Spec} \Bbbk''[y] \arrow[d] \\ \operatorname{Spec} \Bbbk' \arrow[r] & \operatorname{Spec}\{f \in \Bbbk''[y] \mid f(0) \in \Bbbk'\}, \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \operatorname{Spec} \Bbbk' \times \Bbbk' \arrow[r, "0 \amalg 0"] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \operatorname{Spec} \Bbbk'[y] \times \Bbbk'[z] \arrow[d] \\ \operatorname{Spec} \Bbbk' \arrow[r] & \operatorname{Spec} \Bbbk'[y,z]/(yz) \end{tikzcd}$$

are étale neighborhoods of C at the image of E, depending on whether $E \cap E^c$ is one or two points. In both cases, the image of E is a node. One shows that $c_*\mathcal{O}_C = \mathcal{O}_{C'}$ and $\mathrm{R}^1 c_* \mathcal{O}_C = 0$ as above using the short exact sequence $0 \to \mathcal{O}_C \to \mathcal{O}_E \oplus \mathcal{O}_{E^c} \to \mathcal{O}_{E \cap E^c} \to 0$. See also [SP, Tags 0E3H and 0E3M]. $\square$

A curve can also contain a chain of rational tails and bridges of arbitrary length.

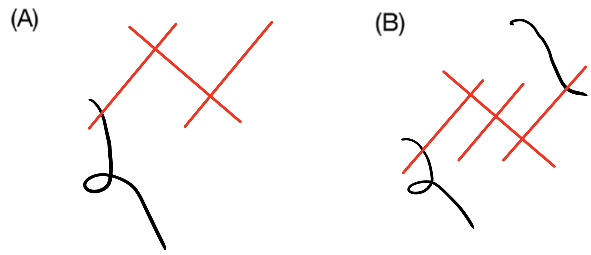

Figure 5.6.2: Chains of rational tails and bridges

Corollary 5.6.3 (Stable Contraction)**.** *Let (C, p_i) be an n-pointed prestable curve of genus g over a field $\Bbbk$ such that $2g-2+n>0$. Then there is a canonical morphism*

$$c\colon C \to C^{\mathrm{st}},$$

called the <u>stable contraction</u>, contacting all rational tails and rational bridges to points, such that $(C^{\mathrm{st}}, c(p_i))$ is an n-pointed stable curve of genus g, $c_*\mathcal{O}_C = \mathcal{O}_{C^{\mathrm{st}}}$, and $\mathrm{R}^1c_*\mathcal{O}_C = 0$. Moreover, the formation of c commutes with field extensions of $\Bbbk$.

Proof. If E denotes the scheme-theoretic union of all rational tails and bridges, then $\mathrm{H}^0(E, \mathcal{O}_E)$ is a finite product of fields. Iteratively applying Proposition 5.6.1 yields a pushout diagram

$$\begin{tikzcd} E \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & C \arrow[d, "c"] \\ \operatorname{Spec} \mathrm{H}^0(E, \mathcal{O}_E) \arrow[r] & C^{\mathrm{st}}, \end{tikzcd}$$

such that $c_*\mathcal{O}_C = \mathcal{O}_{C^{\mathrm{st}}}$ and $\mathrm{R}^1c_*\mathcal{O}_C = 0$, with the construction of c commuting with field extensions. Since $(C^{\mathrm{st}}, c(p_i))$ has no rational tails and bridges, it is stable (Proposition 5.3.14). See also [SP, Tag 0E7Q]. □

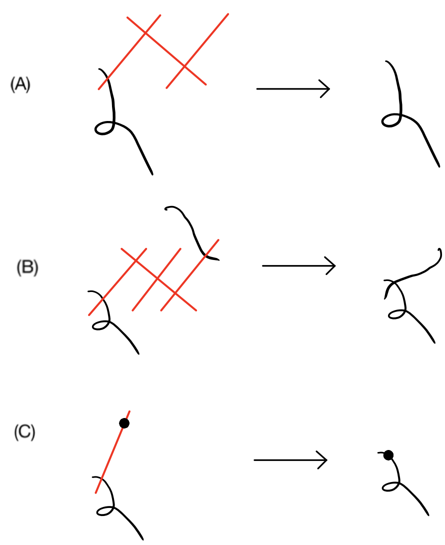

Figure 5.6.4: Stable contractions

Exercise 5.6.5. Let (C, p_i) be an n-pointed prestable curve over a field $\Bbbk$.

(a) Show that the stabilization morphism $c\colon C \to C^{\mathrm{st}}$ is the unique morphism such that $(C^{\mathrm{st}}, c(p_i))$ is stable, $c_*\mathcal{O}_C = \mathcal{O}_{C^{\mathrm{st}}}$, and $\mathrm{R}^1 c_*\mathcal{O}_C = 0$.

(b) If (C, p_i) is semistable and $L := \omega_C(\sum_i p_i)$, show that $C^{\mathrm{st}} \cong \operatorname{Proj} \bigoplus_{d \geq 0} \mathrm{H}^0(C, L^{\otimes d})$ and that $L \cong c^*\omega_{C^{\mathrm{st}}}(\sum_i c(p_i))$.

The construction of the stable contraction extends to families of prestable curves.

Theorem 5.6.6 (Stable Contraction of a Prestable Family)**.** *If $(\mathcal{C} \to S, \sigma_i)$ is a family of n-pointed prestable curves of genus g such that $2g - 2 + n > 0$, then there exists a unique morphism $c\colon \mathcal{C} \to \mathcal{C}^{\mathrm{st}}$ over S such that*

(1) $(\mathcal{C}^{\mathrm{st}} \to S, \sigma_i^{\mathrm{st}})$ is an n-pointed family of stable curves of genus g where $\sigma_i^{\mathrm{st}} = c \circ \sigma_i$;

(2) $\mathcal{O}_{\mathcal{C}^{\mathrm{st}}} = c_*\mathcal{O}_{\mathcal{C}}$ and $R^1 c_*\mathcal{O}_{\mathcal{C}} = 0$;

(3) the construction of $c\colon \mathcal{C} \to \mathcal{C}^{\mathrm{st}}$ is compatible with base change $S' \to S$; and

(4) for each $s \in S$, $(\mathcal{C}_s, \sigma_i(s)) \to (\mathcal{C}_s^{\mathrm{st}}, \sigma_i^{\mathrm{st}}(s))$ is the stable contraction of rational bridges and tails as in Corollary 5.6.3.

Moreover, if $(\pi\colon \mathcal{C} \to S, \sigma_i)$ is a semistable family, then $\omega_{\mathcal{C}/S}(\sum_i \sigma_i)$ is the pullback of the relatively ample line bundle $L := \omega_{\mathcal{C}^{\mathrm{st}}/S}(\sum_i \sigma_i^{\mathrm{st}})$; in particular, $\mathcal{C}^{\mathrm{st}} \cong \mathcal{P}\mathrm{roj}_S \bigoplus_{d \geq 0} \pi_*(L^{\otimes d})$.

Proof. This will be a local-to-global argument. For any $s \in S$, Corollary 5.6.3 yields the stable contraction $c_s\colon \mathcal{C}_s \to Y_0$ over $\kappa(s)$, and satisfies the uniqueness property by Exercise 5.6.5(a). The key idea of the proof is the following: since $c_{s,*}\mathcal{O}_{\mathcal{C}_s} = \mathcal{O}_{Y_0}$ and $\mathrm{R}^1 c_{s,*}\mathcal{O}_{\mathcal{C}_s} = 0$, any infinitesimal deformation of $\mathcal{C}_s$ extends uniquely to a deformation of $\mathcal{C}_s \to Y_0$ (Exercise C.2.9). Setting $S_n = \operatorname{Spec} \mathcal{O}_{S,s}/\mathfrak{m}^{n+1}$, this yields compatible morphisms

$$\mathcal{C} \times_S S_n \to Y_n$$

over S_n. Artin Approximation (B.5.18) then yields a morphism $\mathcal{C} \times_S S' \to Y'$ after an étale cover $S' \to S$, which, by the uniqueness properties, descends to a morphism $\mathcal{C} \to \mathcal{C}^{\mathrm{st}}$.

To make this argument work, we first reduce to the case that S is finite type over $\mathbb{Z}$ using Limit Methods (§B.3). As in the first paragraph, we find compatible morphisms $\mathcal{C} \times_S S_n \to Y_n$ over S_n. The central fibers $\mathcal{C}_s$ and Y_0 are schemes by Proposition 4.5.19, thus so are $\mathcal{C} \times_S S_n$ and Y_n. Since $\widehat{S} := \operatorname{Spec} \widehat{\mathcal{O}}_{S,s}$ is noetherian, Grothendieck's Existence Theorem (Corollary C.5.8) yields a projective morphism $\widehat{Y} \to \widehat{S} := \operatorname{Spec} \widehat{\mathcal{O}}_{S,s}$ extending $Y_n \to S_n$. By the Infinitesimal Criterion for Flatness (A.2.5), $\widehat{Y} \to \widehat{S}$ is flat. A consequence of Grothendieck's Existence Theorem (C.5.11) asserts that there is a morphism $\mathcal{C} \times_S \widehat{S} \to \widehat{Y}$ over $\widehat{S}$ extending $\mathcal{C} \times_S S_n \to Y_n$. Letting $\widehat{\tau}_i$ be the composition $\widehat{S} \xrightarrow{\sigma_i \times_S \widehat{S}} \mathcal{C} \times_S \widehat{S} \to \widehat{Y}$, then $(\widehat{Y} \to \widehat{S}, \widehat{\tau}_i)$ is an n-pointed family of curves. Since the central fiber Y_0 is stable, Openness of Stability (5.3.24) implies that $(\widehat{Y} \to \widehat{S}, \widehat{\tau}_i)$ is a family of stable curves.

Since S is finite type over $\mathbb{Z}$, Artin Approximation (B.5.18) gives an étale neighborhood $(S', s') \to (S, s)$, a family of stable curves $(Y' \to S', \tau_i)$, and a morphism $c'\colon \mathcal{C} \times_S S' \to Y'$ of algebraic spaces over S' whose fiber at s' corresponds to $\mathcal{C}_s \to Y_0$. By Exercise 5.6.8, after replacing S' with an open neighborhood of s', we can assume that $c_*\mathcal{O}_{\mathcal{C} \times_S S'} = \mathcal{O}_{Y'}$ and $\mathrm{R}^1 c_*\mathcal{O}_{\mathcal{C} \times_S S'} = 0$, and that this holds after base change. Therefore, the existence of the desired family of stable curves will follow from étale descent once we establish uniqueness.

To show uniqueness, let $(Y \to S, \tau_i)$ and $(Y' \to S, \tau_i')$ be families of stable curves, let $c\colon \mathcal{C} \to Y$ and $c'\colon \mathcal{C} \to Y'$ be morphisms over S, and suppose that there is an isomorphism $\alpha_s\colon Y_s \xrightarrow{\sim} Y_s'$ compatible with c_s and c_s' such that $c_{s,*}\mathcal{O}_{\mathcal{C}_s} = \mathcal{O}_{Y_s}$ and $\mathrm{R}^1 c_{s,*}\mathcal{O}_{\mathcal{C}_s} = 0$. We need to show that there exists an open neighborhood $U \subseteq S$ of s and an isomorphism $\alpha\colon Y_U \to Y_U'$ extending α_s, which is compatible with c_U and c_U'. By Limit Methods (§B.3), we may assume that $S = \operatorname{Spec} A$ is the spectrum of a noetherian local ring. Letting Y'' be the scheme-theoretic image of $\mathcal{C} \to Y \times_S Y'$, it suffices to show that the projections $Y'' \to Y$ and $Y'' \to Y'$ are isomorphisms. Since the scheme-theoretic image commutes with flat base change, we may further assume that A is complete. By Exercise C.2.9, the restrictions $Y_n'' \to Y_n$ and $Y_n'' \to Y_n'$ to the base change to $\operatorname{Spec} A/\mathfrak{m}^{n+1}$ are isomorphisms. A consequence of Grothendieck's Existence Theorem (C.5.11) yields the desired isomorphism $Y \to Y'$. See also [SP, Tag 0E8A].

If $(\mathcal{C} \to S, \sigma_i)$ is a family of semistable curves, then we claim that the natural map $c^*\omega_{\mathcal{C}^{\mathrm{st}}/S} \to \omega_{\mathcal{C}/S}$ is an isomorphism. Indeed, since the relative dualizing sheaves are line bundles, it suffices to show that this map is surjective. Since the relative dualizing sheaves and the cokernel of the map are compatible with base change, by Nakayama's lemma, it suffices to show that the claim holds when S is the spectrum of a field, which is the assertion in Exercise 5.6.5(b). By the projection formula, $\omega_{\mathcal{C}^{\mathrm{st}}/S} \cong c_* c^* \omega_{\mathcal{C}^{\mathrm{st}}/S}$, and thus the composition $\omega_{\mathcal{C}^{\mathrm{st}}/S} \cong c_* c^* \omega_{\mathcal{C}^{\mathrm{st}}/S} \to c_*\omega_{\mathcal{C}/S}$ is an isomorphism. It follows that $\omega_{\mathcal{C}^{\mathrm{st}}/S}(\sum_i \sigma_i^{\mathrm{st}}) \cong c_*(\omega_{\mathcal{C}/S}(\sum_i \sigma_i))$, which implies the statement as $\omega_{\mathcal{C}^{\mathrm{st}}/S}$ is relatively very ample (Proposition 5.3.21). Alternatively, one can explicitly construct the stable contraction globally by showing that the natural maps $\pi_*(L^{\otimes 2}) \otimes \pi_*(L^{\otimes d}) \to \pi_*(L^{\otimes d+2})$ are surjective for $d \geq 4$. Thus, $\bigoplus_{d \geq 0} \pi_*(L^{\otimes 4d})$ is a finite type $\mathcal{O}_S$-algebra, and we can define $\mathcal{C}^{\mathrm{st}} := \mathcal{P}\mathrm{roj}_S \bigoplus_{d\geq 0} \pi_*(L^{\otimes 4d})$. Since for all $d \geq 0$ the pushforward $\pi_*(L^{\otimes 4d})$ is a vector bundle and its construction commutes with base change (Proposition 5.3.23), it follows that $\mathcal{C} \to \mathcal{C}^{\mathrm{st}}$ is well-defined and $\mathcal{C}^{\mathrm{st}} \to S$ is a family of stable curves. See [Knu83a, Prop. 2.1] and [ACG11, Prop. 10.6.7]. $\square$

Corollary 5.6.7 (Stable Contraction Morphism)**.** *There is a morphism of algebraic stacks*

$$\mathcal{M}_{g,n}^{\mathrm{pre}} \to \overline{\mathcal{M}}_{g,n}, \qquad (C, p_i) \mapsto (C^{\mathrm{st}}, c(p_i)),$$

which is the identity on the open substack $\overline{\mathcal{M}}_{g,n} \subseteq \mathcal{M}_{g,n}^{\mathrm{pre}}$. $\square$

Exercise 5.6.8 (details)**.** Let $f\colon X \to Y$ be a morphism of families of curves over a noetherian scheme S. Suppose that for a point $s \in S$, the morphism $f_s\colon X_s \to Y_s$ satisfies $f_{s,*}\mathcal{O}_{X_s} = \mathcal{O}_{Y_s}$ and $\mathrm{R}^1 f_{s,*}\mathcal{O}_{X_s} = 0$. Show that after replacing S with an open neighborhood of s, $f_*\mathcal{O}_X = \mathcal{O}_Y$ and $\mathrm{R}^1 f_*\mathcal{O}_X = 0$, and that this remains true after base change by a morphism $S' \to S$. See also [SP, Tag 0E88].

5.6.2 The forgetful morphism and the universal family of $\overline{\mathcal{M}}_{g,n}$

We give two consequences of the Stable Contraction Morphism (5.6.7): we show that there is a map $\overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ which is the stable contraction of forgetting the last marked point, and that this map is identified with the universal family.

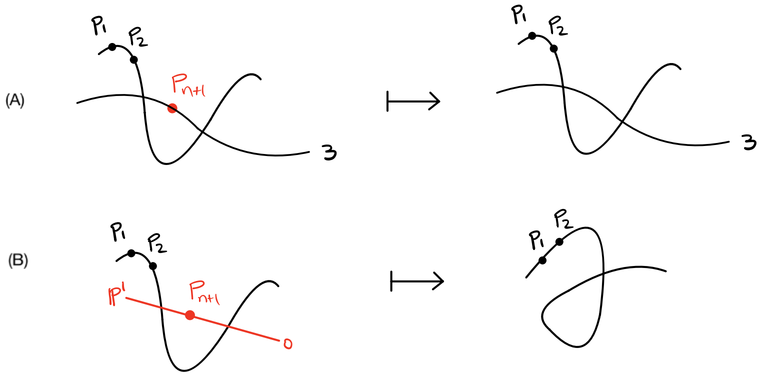

Figure 5.6.9: In (A), the $n+1$th point is simply forgotten. In (B), if p_{n+1} is forgotten, the curve is no longer stable, and we must contract the rational bridge.

Proposition 5.6.10 (Forgetful Morphism)**.** *There is a morphism of algebraic stacks*

$$\overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}, \qquad (C, p_1, \ldots, p_{n+1}) \mapsto (C^{\mathrm{st}}, c(p_1), \ldots, c(p_n)),$$

where $c\colon C \to C^{\mathrm{st}}$ *is the stable contraction of* $(C, p_1, \ldots, p_n)$*.*

Proof. The desired morphism is constructed as the composition

$$\overline{\mathcal{M}}_{g,n+1} \to \mathcal{M}^{\mathrm{pre}}_{g,n} \to \overline{\mathcal{M}}_{g,n},$$

where $\overline{\mathcal{M}}_{g,n+1} \to \mathcal{M}^{\mathrm{pre}}_{g,n}$ is the morphism taking an $n+1$-pointed stable family $(\mathcal{C} \to S, \sigma_1, \ldots, \sigma_{n+1})$ to the n-pointed *prestable* family $(\mathcal{C} \to S, \sigma_1, \ldots, \sigma_n)$ and $\mathcal{M}^{\mathrm{pre}}_{g,n} \to \overline{\mathcal{M}}_{g,n}$ is the Stable Contraction Morphism (5.6.7). $\square$

By the Generalized 2-Yoneda Lemma (3.1.25), the identity morphism $\overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_{g,n}$ corresponds to an object of $\overline{\mathcal{M}}_{g,n}$ over the algebraic stack $\overline{\mathcal{M}}_{g,n}$, which in turn corresponds via descent to an n-pointed family

$$(\mathcal{U}_{g,n} \to \overline{\mathcal{M}}_{g,n}, \sigma_1^{\mathrm{univ}}, \ldots, \sigma_n^{\mathrm{univ}})$$

of stable curves, called the *universal family.* An object of $\mathcal{U}_{g,n}$ over a scheme S is an n-pointed family of stable curves $(\mathcal{C} \to S, \sigma_i)$ with an additional section $\tau\colon S \to \mathcal{C}$ (that may land in the nodal locus $\mathcal{C} \to S$ and may intersection non-trivially with the sections σ_i). Given an n-pointed family of stable curves $(\mathcal{C} \to S, \sigma_i)$, there is a morphism $S \to \overline{\mathcal{M}}_{g,n}$, unique up to unique isomorphism, and a cartesian diagram

$$\begin{tikzcd}
\mathcal{C} \arrow[r] \arrow[d] & \mathcal{U}_{g,n} \arrow[d] \\
S \arrow[r] \arrow[u, bend right, "\sigma_i"'] & \overline{\mathcal{M}}_{g,n}. \arrow[u, bend right, "\sigma_i^{\mathrm{univ}}"']
\end{tikzcd}$$

On the other hand, there is the Forgetful Morphism (5.6.10) $\overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ along with a morphism $\overline{\mathcal{M}}_{g,n+1} \to \mathcal{U}_{g,n}$ of algebraic stacks taking an $n+1$-pointed stable family $(\mathcal{C} \to S, \sigma_i)$ to the $n+1$-pointed family $(\mathcal{C}^{\mathrm{st}}, \sigma_1^{\mathrm{st}}, \ldots, \sigma_n^{\mathrm{st}}, c \circ \sigma_{n+1})$, arising as the Stable Contraction (5.6.6) $c\colon \mathcal{C} \to \mathcal{C}^{\mathrm{st}}$ of the *prestable* family $(\mathcal{C}, \sigma_1, \ldots, \sigma_n)$.

This yields a commutative diagram

$$\begin{tikzcd}
\overline{\mathcal{M}}_{g,n+1} \arrow[rr] \arrow[dr] & & \mathcal{U}_{g,n} \arrow[dl] \\
& \overline{\mathcal{M}}_{g,n}. &
\end{tikzcd}$$

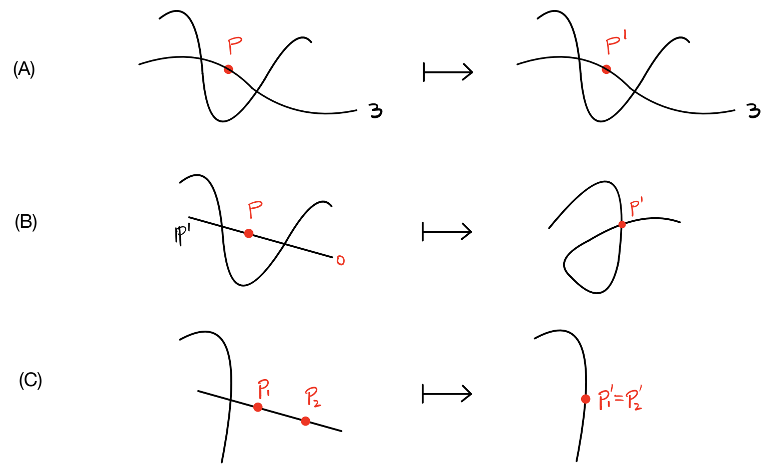

Figure 5.6.11: Examples of the map $\overline{\mathcal{M}}_{g,n} \to \mathcal{U}_{g,n}$.

Proposition 5.6.12. *The morphism $\overline{\mathcal{M}}_{g,n+1} \to \mathcal{U}_{g,n}$ is an isomorphism over $\overline{\mathcal{M}}_{g,n}$. In other words, the forgetful morphism $\overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ is the universal family.*

Proof. The stacks $\overline{\mathcal{M}}_{g,n+1}$ and $\mathcal{U}_{g,n}$ are both proper and representable over $\overline{\mathcal{M}}_{g,n}$ by Stable Reduction (5.5.1). Hence, the morphism $\overline{\mathcal{M}}_{g,n+1} \to \mathcal{U}_{g,n}$ is proper and representable. For every algebraically closed field $\Bbbk$, the induced map $\overline{\mathcal{M}}_{g,n+1}(\Bbbk)/\sim \to \overline{\mathcal{U}}_{g,n}(\Bbbk)/\sim$ on isomorphism classes is bijective. Hence, $\overline{\mathcal{M}}_{g,n+1} \to \mathcal{U}_{g,n}$ is proper and quasi-finite, hence finite. On the other hand, $\overline{\mathcal{M}}_{g,n+1} \to \mathcal{U}_{g,n}$ is birational as it induces an isomorphism between the open substack of $\overline{\mathcal{M}}_{g,n+1}$ parameterizing pointed curves $(C, p_1, \ldots, p_{n+1})$ such that $(C, p_1, \ldots, p_n)$ does *not* contain a rational bridge and the open substack of $\mathcal{U}_{g,n}$ parameterizing pointed curves $(C, p_1, \ldots, p_{n+1})$ such that $p_{n+1} \in C$ is a smooth point that doesn't coincide with p_i for $i = 1, \ldots, n$. By Theorem 5.4.14, $\overline{\mathcal{M}}_{g,n+1}$ is smooth. Since $\mathcal{U}_{g,n} \to \overline{\mathcal{M}}_{g,n}$ has Cohen–Macaulay fibers and $\overline{\mathcal{M}}_{g,n}$ is smooth, $\mathcal{U}_{g,n}$ is Cohen–Macaulay. The stack $\mathcal{U}_{g,n}$ is regular in the open locus where $p_{n+1} \in C$ is a smooth point, and since the complement of this locus is codimension 2, Serre's criterion for normality implies that $\mathcal{U}_{g,n}$ is normal. We thus have a finite birational morphism $\overline{\mathcal{M}}_{g,n+1} \to \mathcal{U}_{g,n}$ between normal Deligne–Mumford stacks, which is necessarily an isomorphism. □

Remark 5.6.13. One can also explicitly construct the inverse morphism $\mathcal{U}_{g,n} \to \overline{\mathcal{M}}_{g,n+1}$. Let $(\mathcal{C} \to S, \sigma_i)$ be an n-pointed family of stable curves and $\tau \colon S \to \mathcal{C}$ be an additional section defined by an ideal sheaf $\mathcal{I}_\tau$. Define the coherent $\mathcal{O}_{\mathcal{C}}$-module $\mathcal{K}$ by

$$0 \to \mathcal{O}_{\mathcal{C}} \xrightarrow{(\alpha,\beta)} \mathcal{I}_\tau^\vee \oplus \mathcal{O}_{\mathcal{C}}(\sigma_1 + \cdots + \sigma_n) \to \mathcal{K} \to 0,$$

where α is the dual of the inclusion $\mathcal{I}_\tau \hookrightarrow \mathcal{O}_\mathcal{C}$ and β is the natural inclusion. Define

$$\widetilde{\mathcal{C}} = \mathcal{P}\mathrm{roj}_S \operatorname{Sym} \mathcal{K} \to \mathcal{C}.$$

With some work, one can prove that $\tau^*(\mathcal{I}_\tau^\vee/\mathcal{O}_\mathcal{C})$ is a line bundle [Knu83a, Lem. 2.2]. The surjection $\tau^*\mathcal{K} \to \tau^*(\mathcal{K}/\mathcal{O}_\mathcal{C}) \cong \tau^*(\mathcal{I}_\tau^\vee/\mathcal{O}_\mathcal{C})$ defines a section $\widetilde{\tau}\colon S \to \widetilde{\mathcal{C}}$. The cokernel of the injection $\mathcal{I}_\tau^\vee \hookrightarrow \mathcal{K}$ is identified with $\mathcal{O}_\mathcal{C}(\sigma_1 + \cdots + \sigma_n)|_{\bigcup_i \sigma_i}$, and the surjections $\sigma_i^*\mathcal{K} \to \sigma_i^*\mathcal{O}_\mathcal{C}(\sigma_1 + \cdots + \sigma_n)$ defines sections $\widetilde{\sigma}_i\colon S \to \widetilde{\mathcal{C}}$. One checks that $(\widetilde{\mathcal{C}} \to S, \widetilde{\sigma}_1, \ldots, \widetilde{\sigma}_n, \widetilde{\tau})$ is an $n+1$-pointed family of stable curves such that $c\colon \widetilde{\mathcal{C}} \to \mathcal{C}$ is the stable contraction of the n-pointed prestable family $(\widetilde{\mathcal{C}}, \widetilde{\sigma}_1, \ldots, \widetilde{\sigma}_n)$ with $\sigma_i = c \circ \widetilde{\sigma}_i$ and $\tau = c \circ \widetilde{\tau}$. In many cases, $\widetilde{\mathcal{C}} \to \mathcal{C}$ can be constructed more directly as a blow up: for instance, if $\mathcal{C} \to S$ is a generically smooth family of stable curves over the spectrum of a DVR such that $\mathcal{C}$ is regular and $\tau\colon S \to \mathcal{C}$ is a section such that $\tau(0) \in \mathcal{C}_0$ is a node, then $\widetilde{\mathcal{C}} \to \mathcal{C}$ is simply the blow up at $\tau(0)$ and $\widetilde{\tau}$ is the strict transform of τ. See [Knu83a, Thm. 2.4] and [ACG11, §X.8].

5.6.3 Gluing morphisms

After showing how sections of families of curves can be glued to nodal families (Proposition 5.6.15), we show that there are well-defined finite morphisms $\overline{\mathcal{M}}_{i,k} \times \overline{\mathcal{M}}_{g-i,n-k+2} \to \overline{\mathcal{M}}_{g,n}$ and $\overline{\mathcal{M}}_{g-1,n+2} \to \overline{\mathcal{M}}_{g,n}$ (Corollary 5.6.16), called *gluing morphisms* (or sometimes *clutching morphisms*).

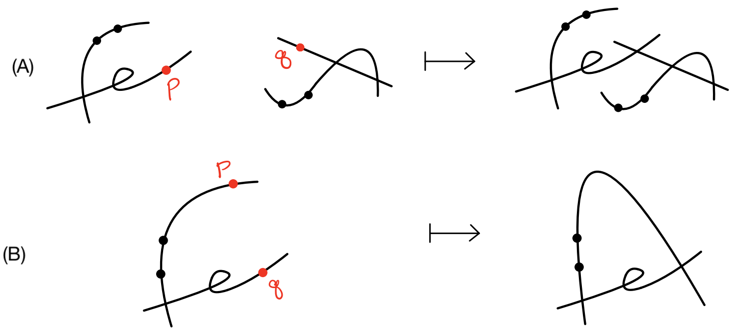

Figure 5.6.14: The nodal gluing of marked points p and q.

Proposition 5.6.15 (Gluing Families Along Sections)**.** *Let $(\mathcal{C} \to S, \sigma, \tau)$ be a 2-pointed projective family of (possibly disconnected) curves over a scheme S such that for every point $s \in S$, $\sigma(s)$ and $\tau(s)$ are distinct smooth points of $\mathcal{C}_s$. Then there is a canonical finite morphism*

$$g\colon \mathcal{C} \to \mathcal{C}'$$

of schemes over S such that

(1) $\mathcal{C}' \to S$ is a family of curves with a section $\nu\colon S \to \mathcal{C}'$ such that $\nu = g \circ \sigma = g \circ \tau$ and $\nu(s) \in \mathcal{C}_s$ is a node for all $s \in S$;

(2) the morphism g restricts to an isomorphism $\mathcal{C} \smallsetminus (\sigma(S) \cup \tau(S)) \xrightarrow{\sim} \mathcal{C}' \smallsetminus \nu(S)$, and there is an identification of the topological space $|\mathcal{C}'|$ with the quotient of $|\mathcal{C}|$ under the equivalence relation $\sigma(s) \sim \tau(s)$ for $s \in S$;

(3) $\mathcal{C}'$ is identified with the Ferrand Pushout (B.4.1) of $\sigma \amalg \tau \colon S \amalg S \to \mathcal{C}$ and the projection $S \amalg S \to S$; in particular, for every open subset $U \subseteq \mathcal{C}'$,

$$\Gamma(U, \mathcal{C}') = \{f \in \Gamma(g^{-1}(U), \mathcal{C}) \mid \sigma^* f = \tau^* f\};$$

and

(4) the construction is compatible with base change $T \to S$.

Proof. Since $\mathcal{C} \to S$ is projective, every two points of a fiber $\mathcal{C}_s$ are contained in an affine open subscheme of $\mathcal{C}$. Therefore, the Ferrand Pushout (B.4.1)

$$\begin{tikzcd} S \coprod S \arrow[r, "\sigma\amalg\tau"] \arrow[d] & \mathcal{C} \arrow[d] \\ S \arrow[r, "\nu"] & \mathcal{C}' \end{tikzcd}$$

exists as a scheme $\mathcal{C}'$. By the universal property of pushouts, there is a natural map $\mathcal{C}' \to S$. Since $S \coprod S \to S$ is finite, so is $\mathcal{C} \to \mathcal{C}'$. Since $S \coprod S \to S$ is flat, the construction is compatible with arbitrary base change by Properties of Pushouts (B.4.8(4)). This gives (2)–(4).

To see (1), first observe that $\mathcal{C}' \to S$ is proper since $\mathcal{C} \to \mathcal{C}'$ is finite. By the Local Structure of Smooth Morphisms (A.3.4), for every $s \in S$, there is an affine open neighborhood $\operatorname{Spec} A \subseteq S$ and étale neighborhoods $\mathcal{C} \to \operatorname{Spec} A[x]$ and $\mathcal{C} \to \operatorname{Spec} A[y]$ of $\sigma(s)$ and $\tau(s)$. By Properties of Pushouts (B.4.8(3)), the pushout $\mathcal{C}'$ has an étale neighborhood of $\nu(s)$ isomorphic to the pushout

$$\begin{tikzcd} \operatorname{Spec} A \times A \arrow[r, "0\amalg 0"] \arrow[d] & \operatorname{Spec} A[x] \times A[y] \arrow[d] \\ \operatorname{Spec} A \arrow[r] & \operatorname{Spec} A[x,y]/(xy). \end{tikzcd}$$

It follows that $\mathcal{C}' \to S$ is flat and $\nu(s) \in \mathcal{C}_s$ is a node.

Alternatively, one can construct $\mathcal{C}'$ using a global Proj construction in the case that $\omega_{\mathcal{C}/S}(\sigma+\tau)$ is a relatively very ample line bundle (as it will be in our application to stable curves). Namely, there are well-defined morphisms $q_\sigma \colon \omega_{\mathcal{C}/S}(\sigma+\tau) \to \mathcal{O}_{\sigma(S)}$ and $q_\tau \colon \omega_{\mathcal{C}/S}(\sigma+\tau) \to \mathcal{O}_{\tau(S)}$. We then define $\mathcal{C}' := \mathcal{P}\mathrm{roj}_S \bigoplus_{d \geq 0} \mathcal{A}_d$, where $\mathcal{A}_d$ is the fiber product

$$\begin{tikzcd} \mathcal{A}_d \arrow[r] \arrow[d] & \mathcal{O}_S \arrow[d, "\Delta"] \\ \pi_*(\omega_{\mathcal{C}/S}(\sigma+\tau)^{\otimes d}) \times \pi_*(\omega_{\mathcal{C}/S}(\sigma+\tau)^{\otimes d}) \arrow[r, "\pi_*(q_\sigma^{\otimes d}) \times \pi_*(q_\tau^{\otimes d})"] & \mathcal{O}_S \times \mathcal{O}_S \end{tikzcd}$$

and $\pi \colon \mathcal{C} \to S$ denotes the structure morphism. One can check $\mathcal{C}' \to S$ satisfies the desired properties by essentially the same argument as above. See [Knu83a, Thm. 3.4] and [ACG11, §X.7]. $\square$

Corollary 5.6.16 (Gluing Morphisms)**.** *Assume that $2g - 2 + n > 0$.*

(1) If $2g_1 - 2 + n_1 > 0$ and we set $g_2 = g - g_1$ and $n_2 = n - n_1 + 2$, there is a finite morphism of algebraic stacks

$$\overline{\mathcal{M}}_{g_1,n_1} \times \overline{\mathcal{M}}_{g_2,n_2} \to \overline{\mathcal{M}}_{g,n}$$
$$\big((C, p_i), (D, q_i)\big) \mapsto (C \amalg_{p_{n_1} \sim q_{n_2}} D, p_1, \dots, p_{n_1 - 1}, q_1, \dots, q_{n_2 - 1}).$$

(2) If $g \geq 1$, there is a finite morphism of algebraic stacks

$$\begin{aligned} \overline{\mathcal{M}}_{g-1,n+2} &\to \overline{\mathcal{M}}_{g,n} \\ (C, p_i) &\mapsto (C/_{p_{n+1} \sim p_{n+2}}, p_1, \ldots, p_{n-2}). \end{aligned}$$

Proof. In both cases, Gluing Families Along Sections (5.6.15) directly yields morphisms of algebraic stacks to $\mathcal{M}^{\mathrm{pre}}_{g,n}$. Since stability can be checked on geometric fibers, it suffices to show that the nodal gluing $C \amalg_{p_{n_1} \sim q_{n_2}} D$ and $C/_{p_{n+1} \sim p_{n+2}}$ are stable, but this is clear as stability can be determined by the pointed normalization (Exercise 5.3.6). Since both maps are quasi-finite and representable morphisms of proper Deligne–Mumford stacks, they are finite. □

More generally, for a finite index set I, we can define the notion of an I-pointed stable curve of genus g, and we denote $\overline{\mathcal{M}}_{g,I}$ as the stack of such curves. A bijection $I \cong \{1, \ldots, n\}$ induces an isomorphism $\overline{\mathcal{M}}_{g,I} \cong \overline{\mathcal{M}}_{g,n}$. With this notation, we have the natural generalization of Corollary 5.6.16: there are finite morphisms

$$\overline{\mathcal{M}}_{g_1,I_1} \times \overline{\mathcal{M}}_{g_2,I_2} \to \overline{\mathcal{M}}_{g_1+g_2,(I_1 \cup I_2) \smallsetminus \{i_1,i_2\}} \quad \text{and} \quad \overline{\mathcal{M}}_{g-1,J} \to \overline{\mathcal{M}}_{g,J \smallsetminus \{j_1,j_2\}},$$

gluing the marked points $i_1 \in I_1$ to $i_2 \in I_2$ (resp., $j_1, j_2 \in J$), subject to the numerical conditions ensuring that each stack is non-empty.

5.6.4 Boundary divisors and line bundles on $\overline{\mathcal{M}}_{g,n}$

We define the boundary divisors $\delta_{i,I}$ of $\overline{\mathcal{M}}_{g,n}$ and show that the total boundary divisor δ is a normal crossings divisor (Proposition 5.6.19).

Definition 5.6.17 (Boundary Divisors)**.** Suppose $2g - 2 + n > 0$. Let $0 \leq i \leq g$ be an integer and $I \subseteq \{1, \ldots, n\}$ be a subset such that $1 - 2i < |I| < 2(g-i) + n - 1 > 0$ (which ensures that both $\overline{\mathcal{M}}_{i,I \cup \{p\}}$ and $\overline{\mathcal{M}}_{g-i,I^c \cup \{q\}}$ are nonempty). The *boundary divisors* are defined as the closed substacks of $\overline{\mathcal{M}}_{g,n}$

$$\begin{aligned} \delta_{i,I} &= \operatorname{im}\left(\overline{\mathcal{M}}_{i,I \cup \{p\}} \times \overline{\mathcal{M}}_{g-i,I^c \cup \{q\}} \to \overline{\mathcal{M}}_{g,I \cup I^c} \cong \overline{\mathcal{M}}_{g,n}\right) \\ \delta_0 &= \operatorname{im}\left(\overline{\mathcal{M}}_{g-1,n+2} \to \overline{\mathcal{M}}_{g,n}\right) \\ \delta &= \delta_0 \cup \bigcup_{i,I} \delta_{i,I} \end{aligned}$$

with the convention that δ_0 is empty if $g = 0$.

Note that $\delta_{i,I} = \delta_{g-i,I^c}$. If $n = 0$, then $\delta = \delta_0 \cup \cdots \cup \delta_{\lfloor \frac{g}{2} \rfloor}$. If $g = 0$, then δ is the union of δ_I over subsets I of size $2 \leq |I| \leq n - 2$.

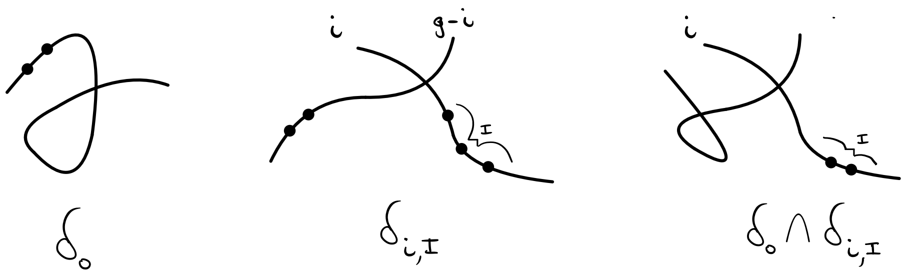

Figure 5.6.18: Examples of stable curves in the boundary.

A closed substack $\mathcal{Z} \subseteq \mathcal{X}$ of a Deligne–Mumford stack is called a *divisor* (resp., *normal crossings divisor*) if there is an étale presentation $U \to \mathcal{X}$ such that $\mathcal{Z} \times_{\mathcal{X}} U \subseteq U$ is a divisor (resp., normal crossings divisor). Recall that a divisor $U \subseteq X$ of a regular scheme has normal crossings if for every $u \in U$, $\widehat{\mathcal{O}}_{U,u} \cong \widehat{\mathcal{O}}_{X,x}/(f_1 \cdots f_k)$ where the sequence $f_1, \ldots, f_k \in \widehat{\mathcal{O}}_{X,x}$ extends to a regular system of parameters $f_1, \ldots, f_n$.

Proposition 5.6.19. *Over a field* $\Bbbk$, $\delta \subseteq \overline{\mathcal{M}}_{g,n}$ *is a normal crossings divisor.*

Proof. An easy dimension count shows that δ has pure dimension $3g - 2 + n$. To see that δ has normal crossings, we may assume that $\Bbbk$ is algebraically closed. Let $(C, p_i) \in \overline{\mathcal{M}}_{g,n}$ be a stable curve with nodes $q_1, \ldots, q_s \in C$, and let $(U, u) \to (\overline{\mathcal{M}}_{g,n}, [C, p_i])$ be an étale neighborhood where U is a scheme. By the Local-to-global Deformation Sequence (5.3.27), there is a surjection

$$\mathrm{Def}(C, p_i) \to \bigoplus_j \mathrm{Def}(\widehat{\mathcal{O}}_{C,q_j}) \tag{5.6.20}$$

of first order deformations spaces with $\mathrm{Def}(\widehat{\mathcal{O}}_{C,q_j}) \cong \Bbbk$.

Letting $\mathrm{Art}_{\Bbbk}$ be the category of local artinian $\Bbbk$-algebras with residue field $\Bbbk$, define the functors

$$F, G_j \colon \mathrm{Art}_{\Bbbk} \to \mathrm{Sets},$$

by setting $F(A)$ to be the set of isomorphism classes of pairs $(\mathcal{C} \to \operatorname{Spec} A, \sigma_i, \alpha)$ where $(\mathcal{C} \to \operatorname{Spec} A, \sigma_i)$ is a family of n-pointed stable curves and $\alpha \colon (C, p_i) \xrightarrow{\sim} (\mathcal{C} \times_A \Bbbk, \sigma_i \times_A \Bbbk)$ is an isomorphism, and by setting $G_j(A)$ to be the set of isomorphism classes of pairs (B, β) where B is a flat A-algebra and $\beta \colon \widehat{\mathcal{O}}_{C,q_j} \to B \otimes_A A/\mathfrak{m}_A$ is an isomorphism. The map $\operatorname{Spec} \widehat{\mathcal{O}}_{U,u} \to \overline{\mathcal{M}}_{g,n}$ induces a miniversal formal deformation of F over $\Bbbk[\![x_1, \ldots, x_{3g-3+n}]\!]$. By Rim–Schlessinger's Criterion (C.4.6), G_j admits a miniversal formal deformation over $\Bbbk[\![z_j]\!]$ (see Exercise C.4.16). Since (5.6.20) is surjective, the natural morphism of functors $F \to \bigoplus_j G_j$ is versal (or formally smooth), or in other words the natural map

$$\widehat{\mathcal{O}}_{U,u} \cong \Bbbk[\![x_1, \ldots, x_{3g-3+n}]\!] \to \Bbbk[\![z_1, \ldots, z_s]\!] \tag{5.6.21}$$

is surjective. We may therefore write $\widehat{\mathcal{O}}_{U,u} \cong \Bbbk[\![z_1, \ldots, z_s, x'_{s+1}, \ldots, x'_{3g-3+n}]\!]$ so that (5.6.21) maps each z_i to itself. Letting $\delta_U = \delta \times_{\overline{\mathcal{M}}_{g,n}} U$, we conclude that $\widehat{\mathcal{O}}_{\delta_U,u} \cong \widehat{\mathcal{O}}_{U,u}/(z_1 \cdots z_s)$. See also [DM69, Thm. 5.2], where it is shown more generally that δ is a relative normal crossings divisor over $\mathbb{Z}$. $\square$

Exercise 5.6.22. Show that $\overline{\mathcal{M}}_{0,n}$ has $2^{n-1} - n - 1$ irreducible boundary divisors.

Exercise 5.6.23. The dual graph $\Gamma = (G, w, m)$ of a stable curve C was defined in Definition 5.3.8.

(a) Show that for every dual graph Γ, there is a locally closed substack $\mathcal{M}_\Gamma \subseteq \overline{\mathcal{M}}_{g,n}$ parameterizing stable curves with dual graph Γ, and conclude that there is a stratification

$$\overline{\mathcal{M}}_{g,n} = \coprod_\Gamma \overline{\mathcal{M}}_\Gamma,$$

called the *dual graph stratification*.

(b) Given two dual graphs Γ and Γ', provide a combinatorial condition for when $\mathcal{M}_{\Gamma'} \subseteq \overline{\mathcal{M}}_\Gamma$.

(c) The dual graph stratification for $\overline{\mathcal{M}}_2$ was described in Figure 5.3.9. Describe the stratifications for $\overline{\mathcal{M}}_{2,1}$ and $\overline{\mathcal{M}}_3$.

Definition 5.6.24 (Hodge line bundle)**.** Letting $\pi\colon \mathcal{U}_{g,n} \to \overline{\mathcal{M}}_{g,n}$ denote the universal family, the *Hodge vector bundle* is defined as the pushforward $E := \pi_*(\omega_{\mathcal{U}_{g,n}/\overline{\mathcal{M}}_{g,n}})$. It is a vector bundle by Properties of Families of Stable Curves (5.3.23). The *Hodge line bundle* is defined as the determinant of the Hodge vector bundle $\lambda := \det E$.

Exercise 5.6.25. Show that under every Forgetful Morphism (5.6.10) and Gluing Morphism (5.6.16), λ pulls back to λ.

Definition 5.6.26 (The Psi line bundles)**.** Let $\sigma_i\colon \overline{\mathcal{M}}_{g,n} \to \mathcal{U}_{g,n}$ be the ith section of the universal family. Define the *line bundle* ψ_i on $\overline{\mathcal{M}}_{g,n}$ as the conormal bundle of σ_i, i.e., $\psi_i = \mathcal{I}_{\sigma_i}/\mathcal{I}_{\sigma_i}^2$ where $\mathcal{I}_{\sigma_i} \subseteq \mathcal{O}_{\mathcal{U}_{g,n}}$ is the ideal sheaf defining σ_i. The *line bundle* ψ is defined (using additive notation) as

$$\psi = \psi_1 + \cdots + \psi_n.$$

There are identifications

$$\psi_i \cong \sigma_i^* \Omega_{\mathcal{U}_{g,n}/\overline{\mathcal{M}}_{g,n}} \cong \sigma_i^* \omega_{\mathcal{U}_{g,n}/\overline{\mathcal{M}}_{g,n}} \cong \sigma_i^* \mathcal{O}_{\mathcal{U}_{g,n}}(-\sigma_i(\overline{\mathcal{M}}_{g,n})).$$

The *Psi classes* are defined as $c_1(\psi_i) \in \mathrm{CH}^1(\overline{\mathcal{M}}_{g,n})$; see §6.1.6 for the definition of the Chow group.

Exercise 5.6.27.

(a) If C is the nodal union $C_1 \cup C_2$ of two smooth curves along $p_1 \in C_1$ and $p_2 \in C_2$, show that there is a natural identification of vector spaces

$$\delta \otimes \kappa([C]) \cong T_{p_1} C_1 \otimes T_{p_2} C_2.$$

(b) Show that δ pulls back under the gluing morphism $\overline{\mathcal{M}}_{g,n+2} \to \overline{\mathcal{M}}_{g,n}$ to $\delta - \psi_{n+1} - \psi_{n+2}$ and pulls back under $\overline{\mathcal{M}}_{g_1,n_1} \times \overline{\mathcal{M}}_{g_2,n_2} \to \overline{\mathcal{M}}_{g,n}$ to $(\delta - \psi_{n_1}) \boxtimes (\delta - \psi_{n_2})$.

(c) Conclude that $\delta - \psi$ pulls back to itself under every gluing morphism.

See also [HM98, Prop. 3.32].

Definition 5.6.28 (Kappa classes)**.** The *ith kappa class* is defined as

$$\kappa_i := \pi_*(\psi_{n+1}^{i+1}) \in \mathrm{CH}^i(\overline{\mathcal{M}}_{g,n}).$$

The first kappa class $\kappa := \kappa_1$ is the class of a line bundle.

Definition 5.6.29 (Canonical line bundle)**.** Over a field $\Bbbk$, let $\Omega_{\overline{\mathcal{M}}_{g,n}/\Bbbk}$ be the sheaf of differentials (Example 4.1.3), which is a vector bundle since $\overline{\mathcal{M}}_{g,n}$ is smooth. Define the *canonical line bundle* as

$$K = K_{\overline{\mathcal{M}}_{g,n}} := \det \Omega_{\overline{\mathcal{M}}_{g,n}/\Bbbk}.$$

Remark 5.6.30 (Mumford's Formula). An application of Grothendeick–Riemann–Roch gives the relations

$$\kappa = 12\lambda - \delta \quad \text{and} \quad K = 13\lambda - 2\delta;$$

see [Mum83, §5] and [HM98, §3.E].

Exercise 5.6.31 (Comparison with coarse moduli space). Let $\Delta_i, \Delta \subseteq \overline{M}_{g,n}$ be the image of $\delta_i, \delta \subseteq \overline{\mathcal{M}}_{g,n}$ under the coarse moduli space map $\pi\colon \overline{\mathcal{M}}_{g,n} \to \overline{M}_{g,n}$. Show that

$$\pi^*\Delta_i = \begin{cases} \delta_i & \text{if } i \neq 1 \\ 2\delta_1 & \text{if} i = 1 \end{cases}$$
$$\pi^*\Delta = \delta + \delta_1$$
$$\pi^* K_{\overline{M}_{g,n}} = K_{\overline{\mathcal{M}}_{g,n}} - \delta_1.$$

5.7 Irreducibility

In your "appendix", you refer to a result of Matsusaka I did not hear of before, namely the connectedness or irreducibility of the variety of moduli for curves of genus g, in any characteristic. I did not know there was any algebraic proof for this (whatever way you state it). Yet I have some hope to prove the connectedness of the $M_{g,n}$ (arbitrary levels) using the transcendental result in char. 0 and the connectedness theorem; but first one should get a natural "compactification" of $M_{g,n}$ which should be simple[2] *over $\mathbb{Z}$.*

Grothendieck, letter to Mumford, 1961 [Mum10, p. 638]

We provide several arguments that $\overline{\mathcal{M}}_{g,n}$ is irreducible:

- the classical topological argument due to Clebsch, Lüroth, and Hurwitz (Theorem 5.7.19),
- a purely algebraic argument in characteristic 0 using degenerations of smooth curves and the inductive nature of the boundary $\delta = \overline{\mathcal{M}}_g \smallsetminus \mathcal{M}_g$ (Theorem 5.7.30), and
- a proof in characteristic $p > 0$ by reduction to characteristic 0 (Theorem 5.7.33)

These proofs rely fundamentally on the theory of branched covering as discussed in §5.7.1.

Equivalences. We begin with a few remarks regarding the equivalences between the connectedness/irreducibility of $\overline{\mathcal{M}}_{g,n}$, $\mathcal{M}_g$, and their coarse moduli spaces. Since $\overline{\mathcal{M}}_{g,n}$ is a smooth algebraic stack over a field $\Bbbk$, its irreducibility is equivalent to its connectedness. Moreover, since $\overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ is the universal family (Proposition 5.6.12) and has connected fibers, it suffices to verify the connectedness of $\overline{\mathcal{M}}_g$. We thus have equivalences

$$\begin{aligned} \overline{\mathcal{M}}_{g,n} \text{ irreducible over } \Bbbk &\iff \overline{\mathcal{M}}_{g,n} \text{ connected over } \Bbbk \\ &\iff \overline{\mathcal{M}}_g \text{ connected over } \Bbbk \\ &\iff \mathcal{M}_g \text{ connected over } \Bbbk \text{ and dense in } \overline{\mathcal{M}}_g. \end{aligned}$$

Finally, we note that since the coarse moduli space $\overline{\mathcal{M}}_{g,n} \to \overline{M}_{g,n}$ induces a homeomorphism $|\overline{\mathcal{M}}_{g,n}| \xrightarrow{\sim} |\overline{M}_{g,n}|$ on topological spaces, each statement can be formulated using the coarse moduli space.

[2] Grothendieck originally called smooth morphisms simple.

5.7.1 Branched coverings

Let $f: C \to D$ be a finite morphism of smooth, connected, and projective curves over an algebraically closed field $\Bbbk$. Assume that f is separable, i.e., the induced map $K(D) \to K(C)$ of functions fields is a separable extension. For a point $p \in C(\Bbbk)$ with image $q \in D(\Bbbk)$, the *ramification index at* p is the integer e_p such that $s \mapsto ut^{e_p}$ under the map $\mathcal{O}_{D,Q} \to \mathcal{O}_{C,p}$, where s and t are uniformizers and u is a unit. We say that f is

$$\begin{cases} \textit{ramified at } p & \text{if } e_p > 1 \\ \textit{tamely ramified at } p & \text{if } e_p > 1 \text{ and either } \operatorname{char}(\Bbbk) = 0 \text{ or } \operatorname{char}(\Bbbk) \nmid e_p \\ \textit{unramified at } p & \text{if } e_p = 1 \end{cases}$$

If f is unramified at p, then the scheme-theoretic fiber over $f(p)$ at p is isomorphic to $\operatorname{Spec} \kappa(p)$, and thus this agrees with the usual definition of unramified (Theorem A.3.3). Moreover, since f is flat, f is unramified at p if and only if f is étale at p.

There is a short exact sequence of differentials

$$0 \to f^*\Omega_D \to \Omega_C \to \Omega_{C/D} \to 0. \tag{5.7.1}$$

Indeed, the sequence above is always right exact. Since $f^*\Omega_D$ and Ω_C are line bundles, the left map is injective if and only if it is nonzero. However, $K(D) \to K(C)$ is separable so $\Omega_{C/D} \otimes K(C) = \Omega_{K(C)/K(D)} = 0$, and thus $f^*\Omega_D \to \Omega_C$ is nonzero at the generic point. Examining the sequence above at the stalks over a point $p \in C(\Bbbk)$, the differential dt maps to $d(us^{e_p}) = eus^{e_p-1}ds + s^{e_p}du$. If f is tamely ramified at p, then $(\Omega_{C/D})_p \cong \mathcal{O}_{C,p}\langle ds\rangle/(s^{e_p-1}ds)$ and $\operatorname{length}(\Omega_{C/D})_p = \dim \Omega_{C/D} \otimes \kappa(p) = e_p - 1$.

Definition 5.7.2. Let $\Bbbk$ be an algebraically closed field.

(1) A *branched covering* is a finite separable morphism $f: C \to D$ of smooth, connected, and projective curves over $\Bbbk$.

(2) A *simply branched covering* is a branched covering such that there is at most one ramification point in every fiber and every ramification point $p \in C(\Bbbk)$ is tamely ramified with ramification index $e_p = 2$.

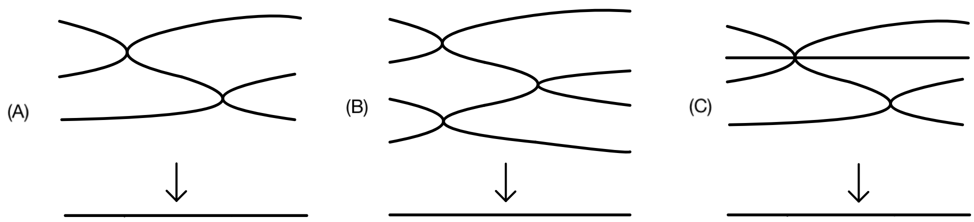

Figure 5.7.3: Examples of branched coverings over $\mathbb{P}^1$: (A) is simply branched while (B) and (C) are not. While the picture may suggest that the source curve C is not smooth, C is, in fact, smooth over the base field $\Bbbk$. However, the map $C \to \mathbb{P}^1$ is not smooth, and the pictures above are designed to reflect the singularities of C over $\mathbb{P}^1$.

If $f: C \to D$ is a branched covering, the *ramification divisor* is defined as

$$R = \sum_{p \in C(\Bbbk)} \operatorname{length}(\Omega_{C/D})_p \cdot p \overbrace{=}^{\substack{\text{if } f \text{ is tamely} \\ \text{ramified}}} \sum_{p \in C(\Bbbk)} (e_p - 1).$$

Theorem 5.7.4 (Riemann–Hurwitz)**.** *If $f\colon C \to D$ is a branched covering with ramification divisor R, then $\Omega_C \cong f^*\Omega_D \otimes \mathcal{O}_C(R)$ and*

$$2g(C) - 2 = \deg(f)(2g(D) - 2) + \deg R.$$

In particular, $f\colon C \to \mathbb{P}^1$ is simply branched, then it is ramified over $2g + 2d - 2$ distinct points.

Proof. This follows directly from the exact sequence (5.7.1). See also [Har77, Prop. IV.2.3] □

Example 5.7.5. For a local model of a branched cover, consider the map $f\colon \mathbb{A}^1 \to \mathbb{A}^1$ defined by $x \mapsto x^n$. The relative sheaf of differentials is $\Omega_{\mathbb{A}^1/\mathbb{A}^1} = \Bbbk[x]\langle dx\rangle/(nx^{n-1}dx)$. Thus, if char($\Bbbk$) does not divide n, f is a branched cover étale over $\mathbb{A}^1 \setminus 0$ and tamely ramified at 0 with index $n - 1$.

Exercise 5.7.6. Show that every branched covering is étale locally isomorphic to $\mathbb{A}^1 \to \mathbb{A}^1, x \mapsto x^n$ around a branch point of index $n - 1$.

Algebraic vs. holomomorphic vs. topological branched covers. *A holomorphic (resp., topological) branched covering of* $\mathbb{P}^1$ is a non-constant and holomorphic (resp., continuous) morphism $f\colon C \to \mathbb{P}^1$ of connected and compact Riemann surfaces (resp., connected, Hausdorff, and compact topological spaces such that f is a covering space over the complement of finitely many points of $\mathbb{P}^1$).

Proposition 5.7.7. *Over $\mathbb{C}$, there are natural bijections*

$$\begin{aligned}\{C \to \mathbb{P}^1 \textit{ algebraic branched coverings}\} &\longleftrightarrow \{C \to \mathbb{P}^1 \textit{ holomorphic branched coverings}\}\\ &\longleftrightarrow \{C \to \mathbb{P}^1 \textit{ topological branched coverings}\}\end{aligned}$$

Proof. An algebraic branched covering is holomorphic and a holomorphic branched covering is topological. Conversely, if $f\colon C \to \mathbb{P}^1$ is a topological covering, then the holomorphic structure on $\mathbb{P}^1$ induces naturally a holomorphic structure on C such that $f\colon C \to \mathbb{P}^1$ is analytic. It is a classical fact relying on the Implicit Function Theorem that there are local charts of C and $\mathbb{P}^1$ such that f is described by $z \mapsto z^k$ (c.f., [Mir95, Prop. II.4.1]), which implies that C is algebraic. □

Monodromy actions. Let $f\colon C \to \mathbb{P}^1$ be a (topological) branched covering of degree d over $\mathbb{C}$ and $B \subsetneq \mathbb{P}^1$ its ramification locus, i.e., the smallest set of points such that $f^{-1}(\mathbb{P}^1 \setminus B) \to \mathbb{P}^1 \setminus B$ is a covering space. Choose a base point $q \in \mathbb{P}^1 \setminus B$. The *monodromy action of* $\pi_1(\mathbb{P}^1 \setminus B, q)$ *on the fiber* $f^{-1}(q)$ is defined as follows: for $\gamma \in \pi_1(\mathbb{P}^1 \setminus B, q)$ and $p \in f^{-1}(q)$, then the path $\gamma\colon [0, 1] \to \mathbb{P}^1$ lifts uniquely to a path $\widetilde{\gamma}\colon [0, 1] \to C$ such that $\widetilde{\gamma}(0) = p$, and the action is defined by $\gamma \cdot p = \widetilde{\gamma}(1)$.

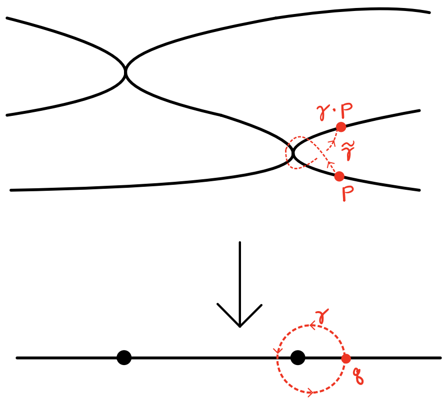

Figure 5.7.8: Monodromy action

A choice of a bijection $f^{-1}(p) \cong \{1, \dots, d\}$ defines a group homomorphism

$$\rho \colon \pi_1(\mathbb{P}^1 \smallsetminus B, q) \to S_d,$$

which we call the *monodromy representation*. The converse is also true: every such group homomorphism is induced by a branched covering.

Proposition 5.7.9. *Let $B \subseteq \mathbb{P}^1$ be a finite subset of size b, $q \in \mathbb{P}^1 \smallsetminus B$ be a point, and $d > 0$ a positive integer. The fundamental group $\pi_1(\mathbb{P}^1 \smallsetminus B, q)$ is identified with the free group generated by the simple loops σ_i for $i = 1, \dots, b$ around the points of B. There is a natural bijection of isomorphism classes*

$$\left\{ \begin{array}{l} \textit{simply branched} \\ \textit{coverings } C \to \mathbb{P}^1 \textit{ of} \\ \textit{of degree } d \\ \textit{branched over } B \end{array} \right\} \Big/ \sim \;\longleftrightarrow\; \left\{ \begin{array}{l} \textit{group homomorphisms} \\ \rho \colon \pi_1(\mathbb{P}^1 \smallsetminus B, q) \to S_d \textit{ such that} \\ \mathrm{im}(\rho) \subseteq S_d \textit{ is a transitive subgroup} \\ \textit{and each } \rho(\sigma_i) \textit{ is a transposition} \end{array} \right\} \Big/ \sim,$$

where two branched covers are equivalent if they are isomorphic over $\mathbb{P}^1$, and two homomorphisms ρ and ρ' are equivalent if they differ by an inner automorphism of S_d, i.e., $\exists\, h \in S_d$ such that $\rho' = \mathrm{h}^{-1}\rho h$.

Proof. We have already explained the natural map from left to right. Conversely, given a group homomorphism $\rho \colon \pi_1(\mathbb{P}^1 \smallsetminus B, q) \to S_d$, we let $H \subseteq \pi_1(\mathbb{P}^1 \smallsetminus B, q)$ be the subgroup containing elements γ fixing 1. Since this subgroup has index d, it corresponds to a covering space $C^* \to \mathbb{P}^1 \smallsetminus B$, which one can show extends to a finite morphism $C \to \mathbb{P}^1$ of degree d. The connectedness of C translates into the condition that $\mathrm{im}(\rho) \subseteq S_d$ is transitive, and the cover $C \to \mathbb{P}^1$ being simply branched translates into each $\rho(\sigma_i)$ being a transposition. See [Mir95, Prop. III.4.9]. $\square$

5.7.2 Hurwitz moduli spaces

For positive integers g and b, we define and study the *Hurwitz moduli space*

$$\mathrm{Hur}_{g,b} := \{\text{simply branched coverings } C \to \mathbb{P}^1 \text{ of genus } g \text{ over } b \text{ ordered points}\}$$

of simply branched covers and its relation to $\mathcal{M}_g$. By Riemann–Hurwitz (5.7.4), if $C \to \mathbb{P}^1$ has degree d, then $b = 2g + 2d - 2$. In the literature, Hurwitz spaces are sometimes indexed as $\mathrm{Hur}_{d,b}$ or $\mathrm{Hur}_{d,g}$. The key diagram is

$$\begin{tikzcd} & \mathrm{Hur}_{g,b} \arrow[dl] \arrow[dr] & \\ \mathcal{M}_g & & M_{0,b} \end{tikzcd} \tag{5.7.10}$$

where a simply branched covering $(C \to \mathbb{P}^1) \in \mathrm{Hur}_{g,b}$ gets mapped to $C \in \mathcal{M}_g$ and the b ordered branch points in $M_{0,b}$.

The moduli space $\mathrm{Hur}_{g,b}$ over $\mathbb{C}$ can be viewed analytically as the topological space with the coarsest topology such that $\mathrm{Hur}_{g,b} \to M_{0,b}$ is continuous. Algebraically, we define it using, as usual, our functorial approach. To this end, we define a *family $(\mathcal{C} \to \mathcal{D} \to S, \sigma_i)$ of coverings of $\mathbb{P}^1$ of genus g simply branched over b ordered points over a scheme S* as a family $\mathcal{C} \to S$ of smooth genus g curves and a b-pointed family $(\mathcal{D} \to S, \sigma_i)$ of smooth genus 0 curves together with a finite, flat morphism $\mathcal{C} \to \mathcal{D}$ of schemes over S of degree d such that for every geometric point $s \in S(\Bbbk)$, $\mathcal{C}_s \to \mathcal{D}_s \cong \mathbb{P}^1_{\Bbbk}$ is simply branched over $\sigma_1(s), \dots, \sigma_b(s)$.

Definition 5.7.11 (Hurwitz functor)**.** For positive integers g and d, set $b = 2g+2d-2$. The *Hurwitz functor* is defined as the functor

$$\begin{aligned} \mathrm{Hur}_{g,b} \colon \mathrm{Sch} &\to \mathrm{Sets} \\ S &\mapsto \left\{ \begin{array}{l} \text{families } (\mathcal{C} \to \mathcal{D} \to S, \sigma_i) \text{ of simply branched coverings} \\ \text{of } \mathbb{P}^1 \text{ of genus } g \text{ and degree } d \end{array} \right\} /\!\sim, \end{aligned}$$

where two families $(\mathcal{C} \to \mathcal{D} \to S, \sigma_i)$ and $(\mathcal{C}' \to \mathcal{D}' \to S, \sigma_i')$ are equivalent if there are isomorphisms $\mathcal{C} \xrightarrow{\sim} \mathcal{C}'$ and $\mathcal{D}' \xrightarrow{\sim} \mathcal{D}'$ compatible with the sections and the structure morphisms $\mathcal{C} \to \mathcal{D}$ and $\mathcal{C}' \to \mathcal{D}'$.

There are morphisms $\mathrm{Hur}_{g,b} \to \mathcal{M}_g$ and $\mathrm{Hur}_{g,b} \to M_{0,b}$ (as in Diagram (5.7.10)) of stacks taking a family of coverings $(\mathcal{C} \to \mathcal{D} \to S, \sigma_i)$ to $\mathcal{C} \to S$ and $(\mathcal{D} \to S, \sigma_i)$ giving Diagram (5.7.10).

Remark 5.7.12 (Variants). There are variants of the above definition that are also referred to as Hurwitz moduli spaces in the literature. First, one can rigidify the moduli problem by considering the functor $\mathrm{Hur}^{\mathrm{rig}}_{g,b} \colon \mathrm{Sch} \to S$, where an object over S is a finite, flat morphism $\mathcal{C} \to \mathbb{P}^1_S$ over S from a family of smooth curves of genus g together with disjoint ordered sections $\sigma_1, \dots, \sigma_b \colon S \to \mathbb{P}^1_S$ such that for every geometric point $s \in S(\Bbbk)$, $\mathcal{C}_s \to \mathbb{P}^1_{\Bbbk}$ is a covering simply branched over $\sigma_1(s), \dots, \sigma_b(s)$. Here two families are equivalent if they are isomorphic over $\mathbb{P}^1_S$ and the sections are equal. There is a morphism $\mathrm{Hur}^{\mathrm{rig}}_{g,b} \to (\mathbb{P}^1)^b \smallsetminus \Delta$, where Δ is the union of all pairwise diagonals. The algebraic group PGL_2 acts freely on $\mathrm{Hur}^{\mathrm{rig}}_{g,b}$ and the quotient is identified with $\mathrm{Hur}_{g,b}$, while the quotient of the PGL_2-equivariant morphism $\mathrm{Hur}^{\mathrm{rig}}_{g,b} \to (\mathbb{P}^1)^b \smallsetminus \Delta$ is identified with $\mathrm{Hur}_{g,b} \to M_{0,b}$

Another common variant, which was the version originally considered by Hurtwitz in [Hur91], is to consider the branch locus as a set of *unordered* points. Let $\mathrm{Hur}^{\mathrm{rig,unord}}_{g,b} \colon \mathrm{Sch} \to \mathrm{Sets}$ be the functor, where an object over S is a finite, flat morphism $f \colon \mathcal{C} \to \mathbb{P}^1_S$ over S from a family of smooth curves of genus g such that for every geometric point $s \in S(\Bbbk)$, $\mathcal{C}_s \to \mathbb{P}^1_{\Bbbk}$ is a covering simply branched over b points. We claim that there is a morphism $\mathrm{Hur}^{\mathrm{rig,unord}}_{g,b} \to \mathrm{Sym}^b \mathbb{P}^1 \smallsetminus \Delta$ to the space parameterizing b unordered distinct points. To see this, observe that the *ramification divisor* $R_f \subseteq \mathcal{C}$, defined as the relative singular locus of $f \colon \mathcal{C} \to \mathbb{P}^1_S$, is finite and étale over S of degree $b = 2g + 2d - 2$. The complement $R_f \times_S \cdots \times_S R_f \smallsetminus \Delta$ of the pairwise diagonals is a principal S_b-torsor over S (see Exercise B.1.52) and the map $R_f \times_S \cdots \times_S R_f \smallsetminus \Delta \to \mathbb{P}^1_S \times_S \cdots \times \mathbb{P}^1_S \smallsetminus \Delta$ is an S_b-equivariant morphism. This defines an S-valued point of the quotient stack $[\big((\mathbb{P}^1)^b \smallsetminus \Delta\big)/S_b]$ over S, which in turn is identified with the scheme $\operatorname{Sym}^b \mathbb{P}^1 \smallsetminus \Delta$. Note also that $\operatorname{Sym}^b \mathbb{P}^1 \smallsetminus \Delta \cong \mathbb{P}^b \smallsetminus \Delta$, where the latter Δ is the discriminant hypersurface. The symmetric group S_b acts freely on $\mathrm{Hur}^{\mathrm{rig}}_{g,b}$ with quotient $\mathrm{Hur}^{\mathrm{rig,unord}}_{g,b}$, and the S_b-quotient of $\mathrm{Hur}^{\mathrm{rig}}_{g,b} \to (\mathbb{P}^1)^b \smallsetminus \Delta$ is identified with $\mathrm{Hur}^{\mathrm{rig,unord}}_{g,b} \to \operatorname{Sym}^b \mathbb{P}^1 \smallsetminus \Delta$.

The following exercise allows us to conclude that the map $\mathrm{Hur}_{g,b} \to M_{0,b}$ is a finite, étale, and surjective morphism, i.e., an algebraic covering space. When working over a field $\Bbbk$, we will abuse notation by also referring to $\mathrm{Hur}_{g,b}$ as the base change $\mathrm{Hur}_{g,b} \times_{\mathbb{Z}} \Bbbk$.

Exercise 5.7.13. Let $\Bbbk$ be an algebraically closed field of characteristic 0. Let g, b, and d be positive integers with $b = 2g + 2d - 2$. Assume that $d > 2$.

(a) Show that the diagonal of $\mathrm{Hur}_{g,b}$ over $\Bbbk$ is representable by schemes.

(b) If $\mathcal{C} \to \mathbb{P}^1_{\Bbbk}$ is a simply branched covering of degree d, show that every automorphism of $\mathcal{C}$ over $\mathbb{P}^1$ is trivial. Conclude that the automorphism group scheme of a family of simply branched coverings of $\mathbb{P}^1$ of degree d is also trivial.

Hint: Use the fact that there are no non-trivial automorphisms of a smooth curve fixing more than $2g + 2$ points (Exercise 5.1.9).

(c) (hard) Show that $\mathrm{Hur}_{g,b} \to M_{0,b}$ is a finite, étale, and representable morphism. In particular, the functor $\mathrm{Hur}_{g,b}$ is representable by a scheme. See also [Ful69, Thm. 7.2], where it is shown more generally that $\mathrm{Hur}_{g,b}$ is representable over $\operatorname{Spec} \mathbb{Z}$, and the map $\mathrm{Hur}_{g,b} \to M_{0,b}$ is étale (resp., finite étale over bases of characteristic greater than d).

(d) Verify that the functors $\mathrm{Hur}^{\mathrm{rig}}_{g,b}$ and $\mathrm{Hur}^{\mathrm{rig,unord}}_{g,b}$ defined in Remark 5.7.12 are also representable by schemes.

Remark 5.7.14. For the proof of the Clebsch–Hurtwitz–Lüroth Theorem (5.7.19) in the next section, we only need to know that $\mathrm{Hur}_{g,b} \to M_{0,b}$ is a topological covering space, and this fact has an elementary argument. Consulting Figure 5.7.15, given a simply branched covering $f\colon C \to \mathbb{P}^1$ and a branched point $p \in C$, we can choose an open neighborhood $U \subseteq \mathbb{P}^1$ around $f(p)$ such that $f^{-1}(U) \to U$ is isomorphic to an open neighborhood of the map $\mathbb{C} \to \mathbb{C}$ given by $x \mapsto x^n$. For every other point $q' \in U$, we can construct a branched cover $C' \to \mathbb{P}^1$ which outside U is the same as $C \to \mathbb{P}^1$ and over U is locally isomorphic to $x \mapsto x^n$ but centered over q' (rather than $f(p)$).

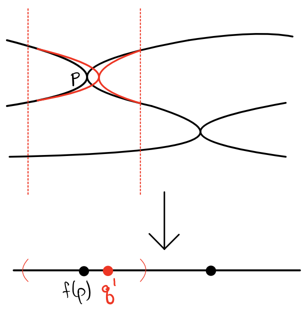

Figure 5.7.15

We turn now to studying the other map: $\mathrm{Hur}_{g,b} \to \mathcal{M}_g$.

Lemma 5.7.16. *Let C be a smooth, connected, and projective curve of genus g over an algebraically closed field $\Bbbk$ of characteristic 0. If L is a line bundle of degree $d \geq 2g+3$ and $V \subseteq \mathrm{H}^0(C,L)$ is a linear system of dimension 2, then a choice of basis of V induces a simply branched covering $C \to \mathbb{P}^1$.*

Proof. We proceed with a dimension count. Since $\mathrm{h}^1(C,L) = \mathrm{h}^0(C, \omega_C \otimes L^\vee) = 0$ as $\deg(\omega_C \otimes L^\vee) < 0$, Riemann–Roch implies that $\mathrm{h}^0(C,L) = d+1-g$, and it follows that the dimension of the Grassmannian $\mathrm{Gr}(2, \mathrm{H}^0(L))$ of 2-dimensional subspaces is $2(d-g-1)$. Since $\mathrm{char}(\Bbbk) = 0$, every finite morphism $C \to \mathbb{P}^1$ is automatically separable. Thus, if V does *not* induce a simply branched covering $C \to \mathbb{P}^1$, then one of the following three conditions must hold:

(a) V has a base point,

(b) there exists a ramification point with index greater than 2, or

(c) there exists two ramification points in the same fiber.

We claim that each condition is closed of codimension at least one. For condition (a), if $p \in C(\Bbbk)$ is a base point, then $V \subseteq \mathrm{H}^0(C, L(-p))$. Since $d \geq 2g$, we have that $\deg(\omega_C \otimes (L(-p))^\vee) < 0$, and thus $\mathrm{h}^0(C, L(-p)) = d-g$ and $\dim \mathrm{Gr}(2, \mathrm{H}^0(C, L(-p))) = 2(d-g-2)$. Varying $p \in C$, we see that the locus defining (a) has dimension $2(d-g-2)+1 = \dim \mathrm{Gr}(2, \mathrm{H}^0(L)) - 1$. We leave cases (b) and (c) for the reader. See also [Sev21] and [Ful69, Prop. 8.1]. $\square$

Exercise 5.7.17 (details)**.**

(a) Verify that conditions (b) and (c) above are closed of codimension at least one.

(b) Show that the argument holds as long as $\mathrm{char}(\Bbbk) \neq 2$ and explain why it fails if $\mathrm{char}(\Bbbk) = 2$.

Corollary 5.7.18. *Let $\Bbbk$ be an algebraically closed field of characteristic 0. If $g \geq 2$ and $d \geq 2g+3$ with $b = 2g+2d-2$, the morphism*

$$\mathrm{Hur}_{g,b} \to \mathcal{M}_g, \qquad (C \to \mathbb{P}^1, \sigma_i) \mapsto C$$

is surjective. $\square$

5.7.3 The Clebsch–Hurwitz–Lüroth Theorem

We provide the classical argument due to Clebsch [Cle73], Hurwitz [Hur91], and Lüroth [Lür71] that the Hurwitz moduli space $\mathrm{Hur}_{g,b}$ is connected over $\mathbb{C}$. From the surjectivity of the map $\mathrm{Hur}_{g,b} \to \mathcal{M}_g$, this allows us to conclude that $\mathcal{M}_g$ is irreducible, a classical result of Klein [Kle82, §19] and Severi [Sev21, §F]. For a modern treatment, see [Ful69, Prop. 1.5].

Theorem 5.7.19 (The Clebsch–Hurwitz–Lüroth Theorem)**.** *For $g \geq 2$ and $d \geq 2$ with $b = 2g+2d-2$, the Hurwitz moduli space $\mathrm{Hur}_{g,b}$ over $\mathbb{C}$ is connected.*

Proof. We utilize the morphism $\beta \colon \mathrm{Hur}_{g,b} \to M_{0,b}$, sending a simply branched cover to the ordered branch locus. By Exercise 5.7.13(c), this map is a covering space. For every finite ordered set $B = \{q_1, \ldots, q_b\} \subseteq \mathbb{P}^1$ of b distinct points and $q \in \mathbb{P}^1 \smallsetminus B$, the fundamental group $\pi_1(\mathbb{P}^1 \smallsetminus B, q) = \langle \sigma_i \,|\, \sigma_1 \cdots \sigma_b = 1 \rangle$, where σ_i is a simple loop around q_i, acts on the fiber $f^{-1}(p)$ of a simply branched covering $f\colon C \to \mathbb{P}^1$. Similarly, since β is a covering space, $\pi_1(M_{0,b}, B)$ acts on the fiber

$$\mathrm{Hur}_{d,B} := \beta^{-1}(B) \subseteq \mathrm{Hur}_{g,b}$$

over $B \in M_{0,b}$. (Note the distinction between the use of the uppercase subscript 'B' and the lowercase 'b'.) Using Proposition 5.7.9, we have bijections

$$\begin{aligned}
\mathrm{Hur}_{g,B} &= \{\text{genus } g \text{ coverings } C \to \mathbb{P}^1 \text{ simply branched over } B\}/\!\sim \\
&\cong \left\{ \begin{array}{l} \text{group homomorphisms } \rho\colon \pi_1(\mathbb{P}^1 \smallsetminus B, q) \to S_d \\ \text{such that } \operatorname{im}(\rho) \subseteq S_d \text{ is a transitive subgroup} \\ \text{and each } \rho(\sigma_i) \text{ is a transposition} \end{array} \right\} \Big/ \text{conjugation by } S_d \\
&\cong \left\{ \begin{array}{l} \text{sequences } (\tau_1, \dots, \tau_b) \in (S_d)^b \text{ of transpositions} \\ \text{with product 1 generating a transitive subgroup} \end{array} \right\} \Big/ \text{conjugation by } S_d.
\end{aligned}$$

The connectedness of $\mathrm{Hur}_{g,b}$ is equivalent to the transitivity of the action of $\pi_1(M_{0,b}, B)$ on the fiber $\mathrm{Hur}_{d,B} = \beta^{-1}(B)$. Consider the sequence

$$\tau := \Big(\underbrace{(12), (12), (13), (13), \dots, (1\,d-1), (1\,d-1)}_{2(d-2)}, \underbrace{(1d), (1d), \dots, (1d)}_{2g+2} \Big) \in \mathrm{Hur}_{d,B}\,.$$

It suffices to show that every orbit of the action of $\pi_1(M_{0,b}, B)$ on the fiber $\mathrm{Hur}_{d,B} = \beta^{-1}(B)$ contains τ, and, to this end, we define the loop

$$\begin{aligned}
\Gamma_i\colon [0,1] &\to M_{0,b} \\
t &\mapsto (q_1, \dots, q_{i-1}, \gamma_i(t), \gamma_i'(t), q_{i+2}, \dots, q_b),
\end{aligned}$$

where γ_i and γ_i' are paths as in Figure 5.7.20.

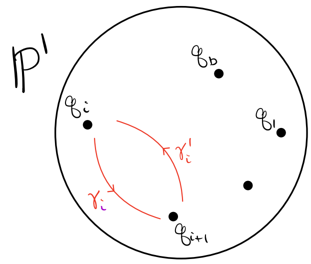

Figure 5.7.20: Paths in $\mathbb{P}^1$

For an element $(\lambda_1, \dots, \lambda_b) \in \mathrm{Hur}_{g,b}$, the action of Γ_i is given by

$$\Gamma_i \cdot (\lambda_1, \dots, \lambda_b) = (\lambda_1, \dots, \lambda_{i-1}, \lambda_i^{-1}\lambda_{i+1}\lambda_i, \lambda_i, \lambda_{i+2}, \dots, \lambda_b).$$

It is now a combinatorial problem that we leave to the reader to show that there exists a sequence $\Gamma_{i_1}, \dots, \Gamma_{i_k}$ of loops such that $\tau = \Gamma_{i_1}\Gamma_{i_2}\cdots\Gamma_{i_k} \cdot (\lambda_1, \dots, \lambda_b)$. $\square$

Exercise 5.7.21. Solve the combinatorial problem at the end of the proof.

Corollary 5.7.22 (Irreducibility of $\mathcal{M}_g$)**.** *For any field $\Bbbk$ of characteristic 0, $\mathcal{M}_g$ is irreducible.*

Proof. Since the morphsim $\mathrm{Hur}_{g,b} \to \mathcal{M}_g$ is surjective (Corollary 5.7.18), the connectedness of $\mathrm{Hur}_{g,b}$ over $\mathbb{C}$ implies the connectedness of $\mathcal{M}_g$ over $\mathbb{C}$. This, in turn, implies that $\mathcal{M}_g$ is geometrically connected over $\mathbb{Q}$, and thus connected over any field $\Bbbk$ of characteristic 0. Since $\mathcal{M}_g$ is smooth over $\Bbbk$ (Theorem 5.4.14), its connectedness is equivalent to its irreducibility. □

The irreducibility of $\mathcal{M}_g$ can also be established using the Teichmüller space $\mathcal{T}_g$ parameterizing complex structures on a topological surface Σ_g of genus g. It was first proved in [Ber60] that $\mathcal{T}_g$ is homeomorphic to a ball in $\mathbb{C}^{3g-3}$, and that $\mathcal{M}_g$ is identified with the quotient $\mathcal{T}_g/\Gamma_g$ by the mapping class group Γ_g.

5.7.4 Irreducibility via degeneration

We now give a completely algebraic argument for the irreducibility of $\overline{\mathcal{M}}_g$ in characteristic 0. The key idea is to show every smooth curve degenerates to a singular stable curve (Proposition 5.7.26)—this is the most challenging part of the argument and is achieved with a technique inspired by the theory of admissible covers. This reduces the connectedness of $\overline{\mathcal{M}}_g$ to the connectedness of the boundary $\delta = \overline{\mathcal{M}}_g \smallsetminus \mathcal{M}_g$, which we establish using the inductive structure and the Gluing Morphisms (5.6.16). This argument is similar in spirit to Deligne and Mumford's first proof of the irreducibility of $\overline{M}_g$ in positive characteristic (see Remark 5.7.34). We follow the treatment in Fulton's appendix of Harris and Mumford's paper [HM82].

We begin with a warmup—the genus 0 case.

Proposition 5.7.23. *For every algebraically field $\Bbbk$ and every integer $n \geq 3$, $\overline{M}_{0,n}$ is irreducible.*

Proof. As $\overline{M}_{0,n}$ is smooth, it suffices to show it is connected. Observe that $M_{0,n}$ is connected as it is identified with $(\mathbb{P}^1 \smallsetminus \{0,1,\infty\})^{n-3} \smallsetminus \Delta$, where Δ is the union of the pairwise diagonals (see Exercise 3.1.20). Therefore, it suffices to show that the boundary $\delta = \overline{M}_{0,n} \smallsetminus M_{0,n}$ does not have a connected component consisting entirely of singular curves. Given a singular curve $(C, p_i) \in \overline{M}_{0,n}(\Bbbk)$, it is easy to directly construct a stable family of curves $(\mathcal{C} \to \operatorname{Spec} R, \sigma_i)$ over a DVR R whose generic fiber is smooth and whose special fiber is (C, p_i): indeed, every node of C has a Zariski-open neighborhood of the form $\operatorname{Spec} \Bbbk[x,y]/(xy)$ and we can glue the deformations $\operatorname{Spec} \Bbbk[x,y]/(xy - \pi)$, where $\pi \in R$ is a uniformizer, to a family $\mathcal{C} \to R$. Alternatively, by deformation theory Proposition 5.3.25, there are compatible stable families $(\mathcal{C}_n \to \operatorname{Spec} \Bbbk[t]/(t^{n+1}), \sigma_{n,i})$ with $(\mathcal{C}_0, \sigma_{0,i}) = (C, p_i)$, which by Grothendieck's Existence Theorem (C.5.8) effectives to a stable family $(\mathcal{C} \to \operatorname{Spec} \Bbbk[\![t]\!]), \sigma_i)$ with smooth generic fiber.

We now give an alternative proof, which uses the identical strategy that we will use shortly to show the connectedness of $\overline{\mathcal{M}}_{g,n}$. First, we claim that an n-pointed curve $(\mathbb{P}^1, p_i)$ degenerates in a one-parameter family to a singular n-pointed stable curve. Indeed, if we let the points p_1 and p_2 approach each other at different rates and blow up the limit in the central fiber where they intersect, then the total family together with the strict transform of the sections defines a family of stable n-pointed genus 0 curves with singular central fiber. It therefore suffices to show that the boundary $\delta = \overline{M}_{0,n} \smallsetminus M_{0,n}$ is connected. The boundary divisor δ decomposes as a union

$$\delta = \bigcup_I \delta_{0,I}, \qquad \text{where } \delta_{0,I} = \operatorname{im}\left(\overline{\mathcal{M}}_{0,I\cup\{p\}} \times \overline{\mathcal{M}}_{0,I^c\cup\{q\}} \to \overline{\mathcal{M}}_{0,n}\right)$$

over subsets $I \subseteq \{1, \ldots, n\}$ of size $2 \le |I| \le n-2$, subject to the relation $\delta_{0,I} = \delta_{0,I^c}$. By induction, we can assume that $\overline{M}_{0,k}$ is irreducible for $k < n$, where the base case holds as $\overline{M}_{0,3} = \operatorname{Spec} \Bbbk$. This implies that each $\delta_{0,I}$ is connected. Let $I, J \subseteq \{1, \ldots, n\}$ be two distinct subsets. After possibly replacing I with I^c, we can assume that $K := I \cap J$ has size at least 2. If $K \subsetneq I$ and $K \subsetneq J$, then by consulting Figure 5.7.24, we see that $\delta_{0,I} \cap \delta_{0,K} \neq \emptyset$ and $\delta_{0,J} \cap \delta_{0,K} \neq \emptyset$. If $I \subseteq J$ or $J \subseteq I$, then $\delta_{0,I} \cap \delta_{0,J} \neq \emptyset$ for the same reason. The connectedness of δ follows. $\square$

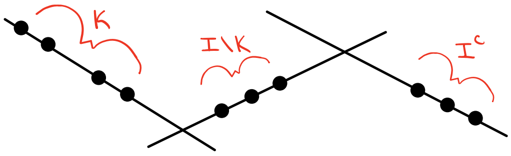

Figure 5.7.24: A curve in $\delta_{0,I} \cap \delta_{0,K}$ as long as $K \subsetneq I$.

Exercise 5.7.25. Show that every stable curve C over an algebraically closed field $\Bbbk$ deforms a smooth curve. More precisely, show that there is a family of stable curves $\pi \colon \mathcal{C} \to T$ over a connected curve[3] T over $\Bbbk$ and a point $t \in T(\Bbbk)$ such that $\mathcal{C}_t \cong C$ and $\pi^{-1}(T \smallsetminus \{t\}) \to T \smallsetminus \{t\}$ is smooth.

The converse to the above exercise, i.e., that every smooth curve *degenerates* to a stable curve, is more difficult.

Proposition 5.7.26. *Let C be a smooth, connected, and projective curve of genus $g \ge 2$ over an algebraically closed field $\Bbbk$ of characteristic 0. There exists a family $\mathcal{C} \to T$ of stable curves over a smooth connected curve T over $\Bbbk$ with points $s, t \in T(\Bbbk)$ such that $\mathcal{C}_s \cong C$ and $\mathcal{C}_t$ is a singular stable curve.*

Proof. The essential strategy is as follows:

① For $d \gg 0$, choose a simply branched covering $C \to \mathbb{P}^1$ of degree d branched over distinct points $q_1, \ldots, q_b \in \mathbb{P}^1$ where $b = 2g + 2d - 2$.

② Deform the covering $C \to \mathbb{P}^1$ branched over q_i to a covering $C' \to \mathbb{P}^1$ branched over *general* points q'_i.

③ Degenerate $(\mathbb{P}^1, q'_i) \in M_{0,b}$ to the stable curve $(D, d_i) \in \overline{M}_{0,b}$ featured in Figure 5.7.27.

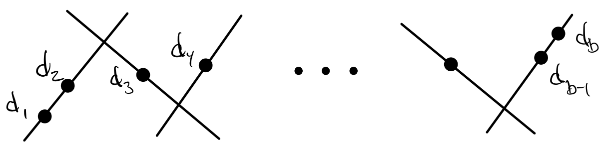

Figure 5.7.27: The degenerate curve $(D, d_i) \in \overline{M}_{0,b}$

④ Degenerate the cover $C' \to \mathbb{P}^1$ to a cover $C'' \to D$ branched over d_i such that C'' is singular and stable.

[3]By our conventions, a curve (Definition 5.1.1) is necessarily of finite type over $\Bbbk$.

For Step 1, Lemma 5.7.16 guarantees the existence of a simply branched covering $C \to \mathbb{P}^1$ of degree $d \gg 0$ simply branched over $b = 2g + 2d - 2$ distinct points q_i. This defines a b-pointed stable curve $(\mathbb{P}^1, q_i) \in M_{0,b}$. For the remaining steps, we first claim that it suffices to find a family $\mathcal{C} \to T$ over a *connected* scheme T of finite type over $\Bbbk$. Indeed, this would show that C is in the same connected (hence irreducible) component $\mathcal{M}' \subseteq \overline{\mathcal{M}}_g$ as a singular stable curve C''. By Le Lemme de Gabber (4.6.1), there is a finite cover $Z \to \overline{\mathcal{M}}_g$ by a scheme, and Z must have an irreducible component Z' surjecting onto $\mathcal{M}'$. As any two $\Bbbk$-points of an irreducible scheme Z' of finite type over $\Bbbk$ are contained in an integral curve $Q \subseteq Z'$, the claim follows by considering the map $\widetilde{Q} \to Z' \to \overline{\mathcal{M}}_g$ from the normalization of Q.

For Step 2, let $H \subseteq \mathrm{Hur}_{g,b}$ be the connected component containing $C \to \mathbb{P}^1$. Since $\mathrm{Hur}_{g,b} \to M_{0,b}$ is finite and étale (Exercise 5.7.13(c)) and thus both an open and closed morphism, the composition $H \hookrightarrow \mathrm{Hur}_{d,b} \to M_{0,b}$ is surjective. As H is connected, it suffices to find a *single* simply branched cover $(C' \to \mathbb{P}^1) \in H$ and a family of stable curves $\mathcal{C} \to T$ over a connected base T connecting C' to a singular stable curve. This brings us to Step 3.

We claim that there is a commutative diagram

$$\begin{tikzcd}
T \smallsetminus \{t\} \arrow[r] \arrow[d, hook] & H \arrow[r, hook, "\text{op \& cl}"] \arrow[drr] & \mathrm{Hur}_{g,b} \arrow[dr, "\text{fin ét}"] & \\
T \arrow[r, "t \mapsto (D, d_i)"'] & \overline{M}_{0,b} & & M_{0,b} \arrow[ll, hook']
\end{tikzcd} \tag{5.7.28}$$

where T is a smooth connected curve and $t \in T(\Bbbk)$ maps to the stable curve (D, d_i) of Figure 5.7.27. To see this, observe that since $\overline{M}_{0,b}$ is irreducible (Proposition 5.7.23), $M_{0,b} \subseteq \overline{M}_{0,b}$ is dense. We can thus choose a map $Y \to \overline{M}_{0,b}$ from a connected reduced curve and a point $y \in Y(\Bbbk)$ such that $y \mapsto (D, d_i)$ and such that the image of every $y' \neq y$ is smooth. Let Y' be a connected component of the base change $(Y \smallsetminus \{y\}) \times_{M_{0,b}} H$, and let T be the normalization of the closure of the image $Y' \to \overline{M}_{0,b}$. There is a point $t \in T$ mapping to $(D, d_i) \in \overline{M}_{0,b}$, and after replacing T with an open neighborhood of t, we can arrange that $T \smallsetminus \{t\} \to \overline{M}_{0,b}$ factors through H.

Diagram 5.7.28 gives a family $(\mathcal{D} \to T, \tau_i)$ of b-pointed stable curves of genus 0 together with a simply branched covering $\mathcal{C}^* \to \mathcal{D}^*$ over $T^* := T \smallsetminus \{t\}$ (where $\mathcal{D}^* \subseteq \mathcal{D}$ is the preimage of T^*) such that the sections $\tau_i^* : T^* \to \mathcal{D}^*$ pick out the branch locus. We now claim that, after replacing (T, t) by a ramified cover, there are dotted arrows completing the cartesian diagram

$$\begin{tikzcd}
\mathcal{C}^* \arrow[r] \arrow[d, dashed] \arrow[dr, phantom, "\square"] & \mathcal{D}^* \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & T^* \arrow[d] \\
\mathcal{C} \arrow[r, dashed, "f"] & \mathcal{D} \arrow[r] & T,
\end{tikzcd}$$

where $\mathcal{C} \to T$ is a family of nodal curves. If we define $\mathcal{C}$ as the integral closure of $\mathcal{O}_{\mathcal{D}}$ in $K(\mathcal{C}^*)$, then $f : \mathcal{C} \to \mathcal{D}$ is a finite morphism. Letting $\mathcal{D}^{\mathrm{sm}} = \mathcal{D} \smallsetminus \{d_i\}$ be the relative smooth locus of $\mathcal{D} \to T$, then Purity of the Branch Locus (A.3.13) implies that the ramification locus of $f^{-1}(\mathcal{D}^{\mathrm{sm}}) \to \mathcal{D}^{\mathrm{sm}}$ is a divisor. Therefore, $\mathcal{C} \to \mathcal{D}$ is ramified only over the sections $\tau_1, \ldots, \tau_b$ and possibly over the nodes d_i of $D = \mathcal{D}_t$. To see that $\mathcal{C} \to S$ is nodal, observe that étale locally around each $d_i \in \mathcal{D}$, $\mathcal{C} \to \mathcal{D}$ has the form of a finite morphism $U \to V := \operatorname{Spec} \Bbbk[x, y]/(xy - \pi^k)$ where $k \leq d$

and π is a local coordinate on an étale neighborhood of $t \in T$. The map $U \to V$ is finite étale over $V \smallsetminus 0$. Since $\operatorname{Spec} \Bbbk[x', y']/(x'^k y'^k - \pi^k) \smallsetminus 0 \to V \smallsetminus 0$, defined by $(x', y') \mapsto (x^k, y^k)$, is the universal cover, we see that the preimage of $V \smallsetminus 0$ in U must be isomorphic to $\operatorname{Spec} \Bbbk[x'', y'']/(x''^l y''^l - \pi^k) \smallsetminus 0$ for some integer l dividing k. Thus, $U = \operatorname{Spec} \Bbbk[x'', y'']/(x''^l y''^l - \pi^k)$, and we see that the reduced scheme structure of the fiber U_t, defined by $\pi = 0$, is nodal. It follows that $(\mathcal{C}_t)_{\mathrm{red}}$ is nodal. By the same argument as in Step 4 of the proof of Stable Reduction (5.5.1), there is ramified cover $T' \to T$ such that the central fiber of the normalization $\widetilde{\mathcal{C}}'$ of $\mathcal{C}' = \mathcal{C} \times_T T'$ is *reduced* and nodal. After replacing $\mathcal{C}$ with $\widetilde{\mathcal{C}}'$ and T with T', we have arranged that $\mathcal{C} \to T$ is a family of nodal curves.

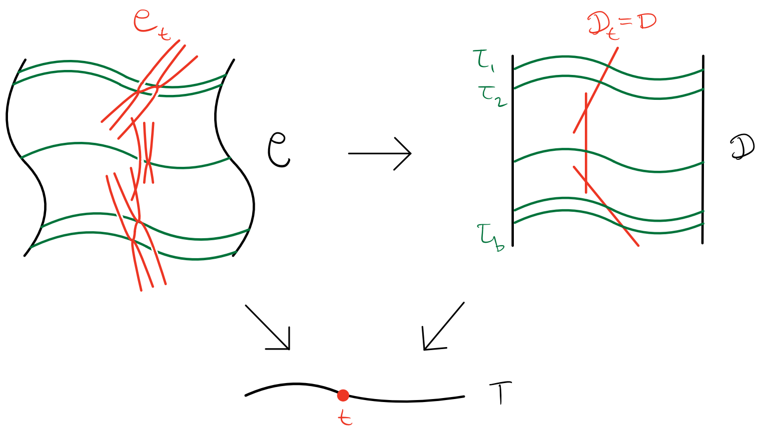

Figure 5.7.29: Picture of $\mathcal{C} \to \mathcal{D} \to T$.

The family $\mathcal{C} \to \mathcal{D}$ may not be stable, but using the Stable Contraction of a Prestable Family (Theorem 5.6.6), we can contract rational tails and bridges to obtain a stable family $\mathcal{C}^{\mathrm{st}} \to T$. To show that $\mathcal{C}_0^{\mathrm{st}}$ is singular, it suffices to show that every smooth irreducible component of $\mathcal{C}_0$ has genus 0. Letting $X \subseteq \mathcal{C}_0$ be such a component, the image of X under $\mathcal{C}_0 \to \mathcal{D}_0 = D$ is one of the $\mathbb{P}^1$'s. Let d' be the degree of the induced map $X \to \mathbb{P}^1$. We know that $X \to \mathbb{P}^1$ is ramified over the marked points of the $\mathbb{P}^1$ and possibly ramified over the nodes of D contained in the $\mathbb{P}^1$ with ramification index at most $d' - 1$. If the $\mathbb{P}^1$ is at the end of the chain in D, it contains two marked points and one node, and Riemann–Hurwitz (5.7.4) implies that $2g_X - 2 \leq -2d' + 2 + (d' - 1)$, which implies that $g_X = 0$. If the $\mathbb{P}^1$ is in the middle, it contains one marked point and two nodes, and Riemann–Hurwitz implies that $2g_X - 2 \leq -2d' + 1 + 2(d' - 1)$, which also implies that $g_X = 0$. $\square$

Theorem 5.7.30 (Irreducibility of $\overline{\mathcal{M}}_{g,n}$)**.** *Let g and n be nonnegative integers satisfying $2g - 2 + n > 0$. For any field $\Bbbk$ of characteristic 0, $\overline{\mathcal{M}}_{g,n}$ is irreducible and contains $\mathcal{M}_{g,n}$ as a nonempty dense open substack.*

Proof. We may assume that $\Bbbk$ is algebraically closed. There exists smooth curves of every genus (Exercise 5.1.7). As $\mathbb{P}^1$ with $n \geq 3$ distinct points is stable and a smooth, connected, and projective genus 1 curve with $n \geq 1$ distinct points is stable, $\mathcal{M}_{g,n}$ is nonempty as long as $2g-2+n > 0$. As $\overline{\mathcal{M}}_{g,n}$ is smooth (Theorem 5.4.14), it suffices to show that $\overline{\mathcal{M}}_{g,n}$ is connected. As $\overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ is the universal family (Proposition 5.6.12), the connectedness of $\overline{\mathcal{M}}_{g,n}$ implies the connectedness of $\overline{\mathcal{M}}_{g,n'}$ for $n' > n$. We already know that $\overline{\mathcal{M}}_{0,n}$ is connected for $n \geq 3$ (Proposition 5.7.23).

We also know that $\overline{\mathcal{M}}_{1,1}$ is isomorphic to $[(\mathbb{A}^2 \smallsetminus 0)/\mathbb{G}_m]$ and thus is connected Exercise 3.1.19(c), and it follows that $\overline{\mathcal{M}}_{1,n}$ is connected for $n \geq 1$.

We are thus reduced to show that $\overline{\mathcal{M}}_g$ is connected for $g \geq 2$, and by induction we can assume that $\overline{\mathcal{M}}_{g'}$ is connected for $g' < g$. Since every smooth curve degenerates to a singular stable curve in the boundary $\delta = \overline{\mathcal{M}}_g \smallsetminus \mathcal{M}_g$ (Proposition 5.7.26), we are further reduced to showing that the boundary δ is connected. We write $\delta = \delta_0 \cup \cdots \cup \delta_{\lfloor g/2 \rfloor}$ where $\delta_0 = \operatorname{im}(\overline{\mathcal{M}}_{g-1,2} \to \overline{\mathcal{M}}_g)$ and $\delta_i = \operatorname{im}(\overline{\mathcal{M}}_{i,1} \times \overline{\mathcal{M}}_{g-i,1} \to \overline{\mathcal{M}}_g)$, using the Gluing Morphisms (5.6.16). By the inductive hypotheses, $\overline{\mathcal{M}}_{g-1,2}$ and $\overline{\mathcal{M}}_{i,1} \times \overline{\mathcal{M}}_{g-i,1}$ are connected, and thus so is each δ_i. But on the other hand, the boundary divisors δ_i intersect! Namely, for every $i, j = 0, \ldots, \lfloor g/2 \rfloor$, the intersection $\delta_i \cap \delta_j$ contains curves as in Figure 5.7.31. See also Fulton's appendix [HM82]. $\square$

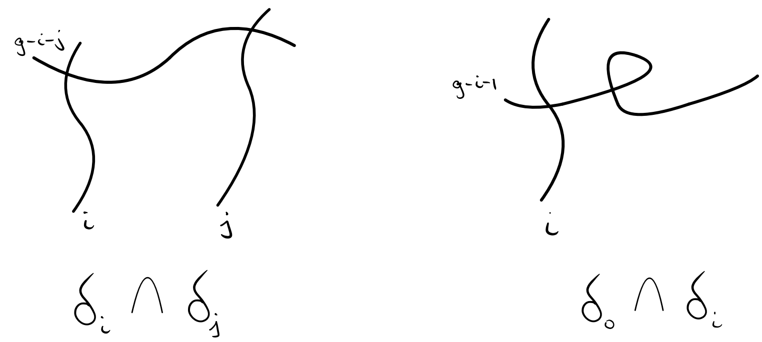

Figure 5.7.31: The boundary divisors δ_i and δ_j have a nonempty intersection.

Remark 5.7.32 (Admissible Covers). The above argument was motivated by the theory of admissible covers introduced by Harris and Mumford [HM82] to compute the Kodaira dimension of $\overline{M}_g$. See also the exposition in [HM98, §3.G]. Admissible covers are a generalization of simply branched covers $C \to \mathbb{P}^1$ where the source *and* target curves may have nodal singularities. The inspiration behind admissible covers is to define a moduli stack $\overline{\mathcal{H}\mathrm{ur}}_{g,b}$ fitting into the diagram

$$\begin{tikzcd} & \overline{\mathcal{H}\mathrm{ur}}_{g,b} \arrow[dl] \arrow[dr] & & \text{compactifying} & & \mathrm{Hur}_{g,b} \arrow[dl] \arrow[dr] & \\ \overline{\mathcal{M}}_g & & \overline{M}_{0,b} & & \mathcal{M}_g & & M_{0,b}. \end{tikzcd}$$

An *admissible cover of genus* g is a surjective finite morphism $f \colon C \to B$ from a prestable genus g curve to a stable b-pointed genus 0 curve $(B, p_1, \ldots, p_b)$ such that

(a) the preimage of the smooth locus B^{sm} under f is the smooth locus C^{sm} and the induced morphism $C^{\mathrm{sm}} \to B^{\mathrm{sm}}$ is simply branched of degree d over the points p_i, i.e., each ramification index is 2 and there is at most one ramification point in every fiber; and

(b) for every node $q \in B$ and every node $r \in C$ over q, the étale local structure of $C \to B$ at r is of the form $\Bbbk[x,y]/(xy) \to \Bbbk[x,y]/(xy)$ defined by $(x, y) \mapsto (x^m, y^m)$ for some m, i.e., the two branches have equal ramification indices at the node.

This definition extends to families and defines a proper Deligne–Mumford stack $\overline{\mathcal{H}\mathrm{ur}}_{g,b}$. (Note that admissible covers may have non-trivial automorphism groups unlike simply branched coverings, and consequently $\overline{\mathcal{H}\mathrm{ur}}_{g,b}$ is not a scheme.) The assignment $(C \to B) \mapsto (B, p_i)$ defines a morphism $\overline{\mathcal{H}\mathrm{ur}}_{g,b} \to \overline{M}_{0,b}$. On the other hand, the assignment $(C \to B) \mapsto C^{\mathrm{st}}$ defines $\overline{\mathcal{H}\mathrm{ur}}_{g,b} \to \mathcal{M}_g^{\mathrm{pre}}$, noting that the source curve C of an admissible cover $C \to B$ need not be stable. Composing with the Stable Contraction Morphism (5.6.7) $\mathcal{M}^{\mathrm{pre}} \to \overline{\mathcal{M}}_g$ gives a morphism $\overline{\mathcal{H}\mathrm{ur}}_{g,b} \to \overline{\mathcal{M}}_g$.

The argument degenerating a smooth curve to a singular stable curve (Proposition 5.7.26) can be rewritten in this language. For $d \gg 0$, given a smooth curve $C \in \mathcal{M}_g$, we choose a preimage $(C \to \mathbb{P}^1) \in \mathrm{Hur}_{g,b}$ (Lemma 5.7.16). Let $H \subseteq \mathrm{Hur}_{g,b}$ be the connected component containing $(C \to \mathbb{P}^1)$, and let $\overline{\mathcal{H}} \subseteq \overline{\mathcal{H}\mathrm{ur}}_{g,b}$ be its closure. Since $\mathrm{Hur}_{g,b} \to M_{0,b}$ is finite and étale (Exercise 5.7.13(c)), $\overline{\mathcal{H}}$ surjects onto $\overline{M}_{0,b}$. Choose a commutative diagram

$$\begin{tikzcd} T \setminus \{t\} \arrow[r] \arrow[d, hook] & \overline{\mathcal{H}} \arrow[d, two heads, "\text{proper}"] \\ T \arrow[r, "t \mapsto (D, d_i)"'] \arrow[ur, dashed] & \overline{M}_{0,b}, \end{tikzcd}$$

as in the proof of Proposition 5.7.26, where T is a smooth connected curve, $t \in T(\Bbbk)$ maps to the stable curve $(D, d_i) \in \overline{M}_{0,b}$ of Figure 5.7.27, and there is some $t' \neq t \in T(\Bbbk)$ mapping to a simply branched covering $(C' \to \mathbb{P}^1) \in H$. By the valuative criterion of properness, after replacing T with a ramified cover, there exists a lift $T \to \overline{\mathcal{H}}$ where t maps to an admissible cover $(C'' \to D)$. One shows that every smooth irreducible component of C'' has genus 0 as in the proof of Proposition 5.7.26. Thus C' degenerates to the singular stable curve $(C'')^{\mathrm{st}}$, and since $(C \to \mathbb{P}^1)$ and $(C' \to \mathbb{P}^1)$ are in the same connected component in $\mathrm{Hur}_{g,b}$, the original smooth curve C also degenerates to $(C'')^{\mathrm{st}}$. (Note that the argument constructing the family $\mathcal{C} \to \mathcal{D} \to T$ of covers in the proof of Proposition 5.7.26 is one of the essential ingredients in the proof of the properness of $\overline{\mathcal{H}\mathrm{ur}}_{g,b}$.)

5.7.5 Irreducibility in positive characteristic

We prove that $\overline{\mathcal{M}}_g$ is irreducible in positive characteristic following the historical arguments of Deligne–Mumford [DM69] and Fulton [Ful69]. A central ingredient in each argument is Zariski's Connectedness Theorem (4.6.15): for a flat, proper morphism of noetherian Deligne–Mumford stacks, the number of geometrically connected components of a fiber is lower semicontinuous. This connectedness theorem, stated but unproven in [DM69, Thm. 4.17], requires a surprisingly large amount of the theory of Deligne–Mumford stacks.

Theorem 5.7.33 (Irreducibility of $\overline{\mathcal{M}}_{g,n}$)**.** *For g and n satisfying $2g - 2 + n > 0$, $\overline{\mathcal{M}}_{g,n}$ is irreducible over any field.*

Proof. As we already know that $\overline{\mathcal{M}}_{g,n}$ is irreducible in characteristic 0 by Theorem 5.7.19 (transcendental proof) or Theorem 5.7.30 (algebraic proof), it suffices to show irreducibility over $\overline{\mathbb{F}}_p$. The morphism $\overline{\mathcal{M}}_{g,n} \to \operatorname{Spec} \mathbb{Z}$ is smooth (Theorem 5.4.14) and proper (Theorem 5.5.23), and thus Zariski's Connectedness Theorem (4.6.15) implies that all geometric fibers have the same number of connected components. As $\overline{\mathcal{M}}_{g,n}$ is connected over $\overline{\mathbb{Q}}$, it is also connected, hence irreducible, over $\overline{\mathbb{F}}_p$. $\square$

Remark 5.7.34 (Deligne and Mumford's proofs). The above proof was the *second* argument given by Deligne and Mumford [DM69, §5]. Keep in mind that this is the same remarkable paper that introduced stacks (now called Deligne–Mumford stacks), introduced stable curves, and proved Stable Reduction (5.5.1). Their first irreducibility argument [DM69, §3], which they called an "elementary derivation of the theorem", was very similar in spirit to the proof of Theorem 5.7.30 using the degeneration of a smooth curve to a singular stable curve and the inductive nature of the boundary. The proof relied on Stable Reduction and the topological irreducibility of $\mathcal{M}_g$. We now sketch their argument that $\mathcal{M}_g \times_{\mathbb{Z}} \Bbbk$ is irreducible for any field $\Bbbk$.

Let H_g (resp., $\overline{H}_g$) denote the locally closed subscheme of $\mathrm{Hilb}^P(\mathbb{P}^{5g-6}_{\mathbb{Z}}/\mathbb{Z})$ parameterizing smooth (resp., stable) curves. In [GIT, Thms. 5.11, 7.13], Mumford had constructed the coarse moduli scheme M_g over $\operatorname{Spec}\mathbb{Z}$ as the geometric quotient $H_g/\operatorname{PGL}_{5g-6}$, and had shown that it is quasi-projective over $\operatorname{Spec}\mathbb{Z}[1/p]$ for every prime p. (It is also true that $\overline{\mathcal{M}}_g$ admits a projective coarse moduli scheme $\overline{M}_g$ over $\operatorname{Spec}\mathbb{Z}$ which is identified with the geometric quotient $\overline{H}_g/\operatorname{PGL}_{5g-6}$, but this was only shown later.)

Step 1: No connected component of $M_g \times_{\mathbb{Z}} \Bbbk$ is proper over $\Bbbk$. Let $W(\Bbbk)$ be the Witt vectors for $\Bbbk$; this is a complete noetherian local ring whose generic point η has characteristic 0 and whose closed point 0 has residue field $\Bbbk$. (For example, $W(\mathbb{F}_p) = \mathbb{Z}_p$ is the ring of p-adics.) As $M_g \times_{\mathbb{Z}} W(\Bbbk)$ is quasi-projective, we can choose a projective compactification

$$\begin{tikzcd} M_g \times_{\mathbb{Z}} W(\Bbbk) \arrow[r, hook] \arrow[dr] & X \arrow[d] \\ & \operatorname{Spec} W(\Bbbk) \end{tikzcd}$$

containing $M_g \times_{\mathbb{Z}} W(\Bbbk)$ as a dense open subscheme. By the characteristic 0 result, we know that the generic fiber X_η is connected. Since X is flat and proper over $W(\Bbbk)$, Zariski's Connectedness Theorem (4.6.15) for schemes implies that the special fiber X_0 is also connected.

Suppose $Y \subseteq M_g \times_{\mathbb{Z}} \Bbbk$ is a connected component proper over $\Bbbk$. Then Y is an open subscheme of X_0, but it's also a closed subscheme as Y is proper. Since X_0 is connected, we conclude that Y must be all of $M_g \times_{\mathbb{Z}} \Bbbk$, hence $M_g \times_{\mathbb{Z}} \Bbbk$ is proper and irreducible. To obtain a contradiction, denote by $A_{g,\Bbbk}$ the moduli space of principally polarized g-dimensional abelian varieties over $\Bbbk$ and consider the morphism

$$\Theta\colon M_g \times_{\mathbb{Z}} \Bbbk \to A_{g,\Bbbk}, \qquad C \mapsto \operatorname{Jac}(C)$$

assigning to a smooth curve C its Jacobian $\operatorname{Jac}(C)$. The properness of $M_g \times_{\mathbb{Z}} \Bbbk$ implies that the image is closed, but it was well known at the time that the closure of the image of Θ contains products of lower dimensional Jacobians.

Step 2: There is no connected component of $\overline{H}_g \times_{\mathbb{Z}} \Bbbk$ consisting entirely of smooth curves. Let $\overline{H}_g^{(1)}, \dots, \overline{H}_g^{(r)}$ be the connected components of $\overline{H}_g \times_{\mathbb{Z}} \Bbbk$. Step 1 implies that $H_g^{(i)} := \overline{H}_g^{(i)} \cap (H_g \times_{\mathbb{Z}} \Bbbk)$ is not proper for each i. Therefore, there is a morphism $\Delta^* = \operatorname{Spec}\Bbbk(\!(t)\!) \to H_g^{(i)}$ that does not extend to $\Delta = \operatorname{Spec}\Bbbk[\![t]\!]$. By Stable Reduction (5.5.1), after possibly replacing Δ with a ramified extension, $\Delta^* \to H_g^{(i)}$ extends to a morphism $\Delta \to \overline{H}_g^{(i)}$. This shows that $\overline{H}_g^{(i)} \smallsetminus H_g^{(i)}$ is non-empty. In other words, we have shown that every smooth curve degenerates to a singular stable curves, giving a proof of Proposition 5.7.26 in positive characteristic.

Step 3: The boundary $\delta \times_{\mathbb{Z}} \Bbbk$ is connected, where $\delta = \overline{M}_g \smallsetminus M_g$. By Step 2, the connectedness of $\overline{M}_g \times_{\mathbb{Z}} \Bbbk$ follows from the connectedness of $\delta \times_{\mathbb{Z}} \Bbbk$. We will show that $\delta_i \times_{\mathbb{Z}} \Bbbk$ is connected for each i and that each pairwise intersection $(\delta_i \cap \delta_j) \times_{\mathbb{Z}} \Bbbk$ is nonempty. This is precisely what we showed in the proof of Theorem 5.7.30 using induction on the genera, the Gluing Morphisms (5.6.16), and the curves in Figure 5.7.31 in $(\delta_i \cap \delta_j) \times_{\mathbb{Z}} \Bbbk$. Deligne and Mumford gave essentially the same argument using ad hoc techniques rather than the formalism of the moduli space $\overline{M}_{g,n}$ of n-pointed stable curves and the Gluing Morphisms (5.6.16), which were only introduced later by Knudsen.

Interestingly, neither of Deligne and Mumford's proofs relies essentially on the theory of the stacks. Their second proof—see Theorem 5.7.33—is valid as long as one knows the existence of a proper and flat coarse moduli space $\overline{M}_{g,n} \to \operatorname{Spec} \mathbb{Z}$. However, both proofs rely fundamentally on the compactification $\overline{\mathcal{M}}_g$ and Stable Reduction (5.5.1).

Remark 5.7.35 (Fulton's proof). In [Ful69], Fulton studied the variant of the Hurwitz moduli space $\mathrm{Hur}_{g,b}^{\mathrm{rig,unord}}$ parameterizing families $\mathcal{C} \to \mathbb{P}^1_S$ of simply branched coverings of degree d over b unordered points (see Remark 5.7.12), and showed that the diagram

$$\begin{tikzcd}
& \mathrm{Hur}_{g,b}^{\mathrm{rig,unord}} \arrow[dl, twoheadrightarrow] \arrow[dr, "\text{ét}"] & \\
\mathcal{M}_g & & \operatorname{Sym}^b \mathbb{P}^1 \smallsetminus \Delta = \mathbb{P}^b \smallsetminus \Delta
\end{tikzcd}$$

is defined over $\mathbb{Z}$ such that $\mathrm{Hur}_{g,b}^{\mathrm{rig,unord}} \to \mathcal{M}_g$ surjective if $d \geq g+1$ (which is a better bound than we proved in Corollary 5.7.18) and that $\mathrm{Hur}_{g,b}^{\mathrm{rig,unord}} \to M_{0,n}$ is finite étale over $\mathbb{Z}_{d!}$, i.e., after inverting all primes $p \leq d$. Fulton established a "reduction theorem": if X is a smooth and projective scheme over a complete DVR R with algebraically closed residue field $\Bbbk$ and characteristic 0 fraction field K and $\Delta \subseteq X$ is relative divisor over R which is *simple* (i.e., each fiber of Δ has no multiple components), and if $Y \to X \smallsetminus \Delta$ is a finite étale covering such that $Y_{\overline{K}}$ is irreducible, then $Y_{\Bbbk}$ is also irreducible. Using the irreducibility of the Hurwitz moduli space in characteristic 0, Fulton's reduction theorem applied to $\mathrm{Hur}_{g,b}^{\mathrm{rig,unord}} \to \operatorname{Sym}^b \mathbb{P}^1 \smallsetminus \Delta$ gives the irreducibility of $\mathrm{Hur}_{g,b}^{\mathrm{rig,unord}}$ over fields of characteristic $p > d$. Taking $d = g+1$ gives the irreducibility of $\mathcal{M}_g$ over fields of characteristic $p > g+1$.

5.8 Projectivity following Mumford

> *It would not be an exaggeration to say that [Theorem A] has played as fundamental a role in the theory of algebraic curves in the last thirty years as the theory of abstract curve did in the preceding sixty.*
>
> Joe Harris and Ian Morrison [HM98, p. 48]

Geometric Invariant Theory (GIT) was developed by Mumford in [GIT] as a method to construct quotients and moduli spaces in algebraic geometry. The central idea is to represent a moduli stack $\mathcal{M}$ (such as $\overline{\mathcal{M}}_g$) into a quotient stack $[U/G]$, where G is a reductive group and $U \subseteq \mathbb{P}(V)$ is a locally closed subscheme of the projectivization of a G-representation V, and to use the Hilbert–Mumford Criterion

(7.4.4) to show that a point $u \in \overline{U}$ is in U if and only if u is GIT stable. Ironically, Mumford's first construction of M_g as a scheme over $\mathbb{Z}$ [GIT, Thms. 5.11 and 7.13] used ad hoc techniques rather than the Hilbert–Mumford Criterion. Shortly after, Mumford [Mum77] and Gieseker [Gie82] constructed $\overline{M}_g$ as a projective variety using GIT. The first proof of projectivity of $\overline{M}_{g,n}$, however, was due to Knudsen and Mumford in [KM76, Knu83a, Knu83b] by relying on Torelli map $\overline{M}_g \to \overline{A}_g$ to the Satake compactification of the moduli space of principally polarized abelian varieties.

In this section, we cover Mumford's construction of $\overline{M}_g$ over an algebraically closed field $\Bbbk$ of characteristic 0[4] by applying GIT, as developed in Chapter 7, to Chow and Hilbert schemes.

5.8.1 GIT outline using Chow and Hilbert schemes

The GIT construction of the Hilbert scheme depends on two integers, while the construction using the Chow scheme depends only on the integer k:

- $k \geq 5$, the multiple of the dualizing sheaf defining the pluricanonical embedding

  $$|\omega_C^{\otimes k}| \colon C \hookrightarrow \mathbb{P}^{N_k}, \qquad \text{where } N_k = (2k-1)(g-1)-1.$$

  We need $k \geq 3$ so that the kth multiple $\omega_C^{\otimes k}$ of the dualizing sheaf of a stable curve C is ample, but we need $k \geq 5$ for the GIT construction using the Chow or Hilbert scheme to yield $\overline{M}_g$.

- $m \gg 0$, the degree of the equations that we use to embed the Hilbert scheme of k-canonically embedded curves into a Grassmannian. We need $m \gg 0$ to obtain an embedding of the Hilbert scheme.

The Hilbert scheme. For $k > 1$, the Hilbert polynomial of a k-pluricanonically embedded stable curve $C \subseteq \mathbb{P}^{N_k}$ of genus g is

$$P(t) = \chi(C, \omega_C^{\otimes kt}) = (2kt-1)(g-1).$$

The Hilbert scheme $\mathrm{Hilb}^P(\mathbb{P}^{N_k})$ is projective over $\Bbbk$ (Theorem 1.1.2) and inherits an action of PGL_{N_k+1}. For $k \geq 3$, there is a locally closed PGL_{N_k+1}-invariant subscheme $U \subseteq \mathrm{Hilb}^P(\mathbb{P}^{N_k})$ such that $\overline{\mathcal{M}}_g \cong [U/\mathrm{PGL}_{N_k+1}]$ (Exercise 5.4.15). Let

$$H := \overline{U} \subseteq \mathrm{Hilb}^P(\mathbb{P}^{N_k}) \tag{5.8.1}$$

be the closure of U. By Theorem 1.4.5, for $m \gg 0$, there is a PGL_{N_k+1}-equivariant closed immersion

$$\begin{aligned} H &\hookrightarrow \mathrm{Gr}\big(P(m), \Gamma(\mathbb{P}^{N_k}, \mathcal{O}(m))\big) \\ [C \subseteq \mathbb{P}^{N_k}] &\mapsto \Big[\underbrace{\Gamma(\mathbb{P}^{N_k}, \mathcal{O}(m))}_{\mathrm{Sym}^m \Gamma(C, \omega_C^{\otimes k})} \twoheadrightarrow \underbrace{\Gamma(C, \mathcal{O}(m))}_{\Gamma(C, \omega_C^{\otimes mk}) \text{ if } C \in U}\Big]. \end{aligned}$$

By Proposition 1.2.8, the Grassmannian is embedded into projective space via the

[4] See Remark 5.8.7 for how the construction extends to positive and mixed characteristics.

Plücker embedding

$$\begin{aligned} \mathrm{Gr}\left(P(m), \Gamma(\mathbb{P}^{N_k}, \mathcal{O}(m))\right) &\hookrightarrow \mathbb{P}\left(\bigwedge^{P(m)} \Gamma(\mathbb{P}^{N_k}, \mathcal{O}(m))\right) \\ \left[\Gamma(\mathbb{P}^{N_k}, \mathcal{O}(m)) \twoheadrightarrow \Gamma(C, \mathcal{O}(m))\right] &\mapsto \left[\bigwedge^{P(m)} \Gamma(\mathbb{P}^{N_k}, \mathcal{O}(m)) \twoheadrightarrow \bigwedge^{P(m)} \Gamma(C, \mathcal{O}(m))\right]. \end{aligned}$$

In fact, for any Hilbert scheme $\mathrm{Hilb}^P(\mathbb{P}^n)$, there are PGL_{n+1}-equivariant closed immersions

$$\mathrm{Hilb}^P(\mathbb{P}^n) \hookrightarrow \mathrm{Gr}(P(m), \Gamma(\mathbb{P}^n, \mathcal{O}(m))) \hookrightarrow \mathbb{P}\left(\bigwedge^{P(m)} \Gamma(\mathbb{P}^n, \mathcal{O}(m))\right)$$

for $m \gg 0$, and it will be just as easy to investigate the stability of curves $C \subseteq \mathbb{P}^n$ that are not necessarily pluricanonically embedded. Let $\mathcal{O}_{\mathrm{Gr}}(1)$ be the very ample line bundle on the Grassmannian, and let

$$L_m = \mathcal{O}_{\mathrm{Gr}}(1)|_{\mathrm{Hilb}^P(\mathbb{P}^n)}$$

be its restriction, noting its dependence on m. For $m \gg 0$, L_m is a very ample line bundle on $\mathrm{Hilb}^P(\mathbb{P}^n)$ which inherits an action by PGL_{n+1}-action, and thus also GL_{n+1} and SL_{n+1}. We will later restrict to one-parameter subgroups of SL_{n+1}, but it will be convenient to also consider one-parameter subgroups $\lambda = \mathrm{diag}(\lambda_i)$ with $\sum \lambda_i \neq 0$.

Definition 5.8.2. We denote the Hilbert–Mumford index (Definition 7.4.1) of $[C \subseteq \mathbb{P}^n] \in \mathrm{Hilb}^P(\mathbb{P}^n)$ as

$$\mu_{L_m}^{\mathrm{Hilb}}(C, \lambda)$$

with respect to a one-parameter subgroup $\lambda \colon \mathbb{G}_m \to \mathrm{GL}_{n+1}$.

To offer an explicit description of $\mu_{L_m}^{\mathrm{Hilb}}(C, \lambda)$, we define the λ-weight of a monomial $x_0^{m_0} \cdots x_n^{m_n}$ as $\sum_i m_i \lambda_i$ and the λ-weight of a polynomial $f(x_0, \ldots, x_m)$ as the largest weight of monomial in f. If $\beta = \{f_1, \ldots, f_p\}$ is a set of polynomials, we write $\lambda\text{-weight}(\beta) = \sum_i \lambda\text{-weight}(f_i)$. With these definitions in place,

$$\mu_{L_m}^{\mathrm{Hilb}}(C, \lambda) = \min_{\beta} \lambda\text{-weight}(\beta), \tag{5.8.3}$$

where the minimum is over subsets $\beta \subseteq \Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(m))$ mapping to a basis of $\Gamma(C, \mathcal{O}_C(m))$.

The Chow scheme. The *Chow form*

$$f_C \in \mathrm{Sym}^d(\Bbbk^{n+1})^{\otimes 2}$$

of an integral curve $C \subseteq \mathbb{P}^n$ of degree d was defined in §1.4.5: it is a multihomogeneous polynomial of degree d in the variables u_{0i} and u_{1i} for $i = 0, \ldots, n$ with the property that $f_C(a_0, b_0, \ldots, a_n, b_n) = 0$ if and only if

$$C \cap \{a_0 x_0 + \cdots + a_n x_n = 0\} \cap \{b_0 x_0 + \cdots + b_n x_n = 0\} \neq \emptyset.$$

The definition of a Chow form extends to effective 1-cycles of degree d, and the Chow scheme

$$\mathrm{Chow}_{1,d}(\mathbb{P}^n) \subseteq \mathbb{P}\left(\mathrm{Sym}^d(\mathbb{k}^{n+1})^{\otimes 2}\right)$$

is defined as the closure of the locus of Chow forms of all effective 1-cycles of degree d.

Definition 5.8.4. We denote the Hilbert–Mumford index of the Chow form of a reduced curve $C \subseteq \mathbb{P}^n$ of degree d as

$$\mu^{\mathrm{Chow}}(C, \lambda)$$

with respect to a one-parameter subgroup $\lambda\colon \mathbb{G}_m \to \mathrm{GL}_{n+1}$.

Given an integral curve $C \subseteq \mathbb{P}^n$ of degree d and its Chow form f_C, for each $t \in \mathbb{G}_m(\mathbb{k})$, we can write

$$f_{\lambda(t)\cdot C} = \lambda(t) \cdot f_C = \sum_{i=\mu}^{\nu} t^i f_{C,i} \in \mathbb{k}[t, u_{i,j}],$$

where $f_{C,i}$ has λ-weight i and where we assume that $f_{C,\mu} \neq 0$. Then $\lim_{t\to 0} \lambda(t)\cdot f_C = f_{C,\mu} \in \mathrm{Chow}_{1,d}(\mathbb{P}^n)$ and $\mu^{\mathrm{Chow}}(C, \lambda) = \mu$.

Chow, Hilbert, and asymptotic Hilbert stability. We say that curve a $C \subseteq \mathbb{P}^n$ of degree d is

- *Chow stable/semistable* if its Chow form in $\mathrm{Chow}_{1,d}(\mathbb{P}^n)$ is stable/semistable with respect to the action by SL_{n+1}.
- *m-Hilbert stable/semistable* if $[C \subseteq \mathbb{P}^n] \in \mathrm{Hilb}^P(\mathbb{P}^n)$ is stable/semistable with respect to the action by SL_{n+1} and the line bundle L_m, and
- *asymptotically Hilbert stable/semistable* if it is m-Hilbert stable/semistable for $m \gg 0$.

The GIT construction $\overline{M}_g$ using the Hilbert and Chow scheme depends on the following stability analysis.

Theorem 5.8.5. *Let $k \geq 5$.*

(1) A smooth curve $C \subseteq \mathbb{P}^{N_k}$ embedded by the complete linear series $|\omega_C^{\otimes k}|$ is Chow stable.

(2) If $[C \subseteq \mathbb{P}^{N_k}] \in \mathrm{Hilb}^P(\mathbb{P}^{N_k})$ is a Chow semistable curve contained in the closure of the locus of smooth k-pluricanonically embedded curves, then C is a k-pluricanonically embedded (Deligne–Mumford) stable curve.

Moreover, there exists a $m \gg 0$ such that every k-pluricanonically embedded smooth curve is m-Hilbert stable and such that if $[C \subseteq \mathbb{P}^{N_k}] \in H$ is m-Hilbert semistable, then C is a k-pluricanonically embedded stable curve.

Some comments are in order:

- Part (1) is conceptually the most challenging ingredient as it requires verifying the negativity of the Hilbert–Mumford index with respect to *every* one-parameter group. Surprising however, this part has a very clean proof due to Mumford's ingenuity in reinterpreting the Hilbert–Mumford index in terms of the multiplicity $e(C, \lambda)$ of an ideal on $\mathbb{A}^1 \times C$ (see §5.8.2). This multiplicity can be bounded in terms of multiplicities on C (see Proposition 5.8.19), which is sufficient to verify the stability of smooth curves (see Theorem 5.8.22).

– Unfortunately, the bounds on the multiplicity are not sufficient to prove the Chow stability of a k-pluricanonically embedded stable curve. Nevertheless, as we show shortly, parts (1) and (2) are sufficient to indirectly conclude that stable curves are Chow stable.

– While (2) is the conceptually easier part (as we just need to exhibit destabilizing one-parameter subgroups), it is nevertheless the most technically demanding. In §5.8.5, we verify only some of the details, while referring the reader to more complete accounts.

Theorem 5.8.5 allows us to wrap up the proof of Theorem A.

Theorem 5.8.6. *Over a field* $\Bbbk$ *of characteristic* 0*, the coarse moduli space* $\overline{M}_{g,n}$ *is projective if* $2g-2+n>0$*.*

Proof. By Theorem 5.5.23, $\overline{\mathcal{M}}_{g,n}$ is a proper Deligne–Mumford stack admitting a proper coarse moduli space $\overline{M}_{g,n}$. Since $\overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ is the universal family (Proposition 5.6.12), it is a family of n-pointed stable curves, and hence there is a line bundle L on $\overline{\mathcal{M}}_{g,n+1}$ which is relatively ample over $\overline{\mathcal{M}}_{g,n}$. (One can take $L = \omega_{\overline{\mathcal{M}}_{g,n+1}/\overline{\mathcal{M}}_{g,n}}(\sum \sigma_i)$ where σ_i are the universal sections.) By Exercise 4.4.29(2), L descends to a line bundle on $\overline{M}_{g,n+1}$ which is relatively ample over $\overline{M}_{g,n}$. It thus suffices to show that $\overline{M}_g$ is projective for $g \geq 2$. Note that we already know that $\overline{M}_{0,3} = \operatorname{Spec} \Bbbk$ and $\overline{M}_{1,1} = \mathbb{P}^1$ are projective.

For $k \geq 5$, let $P(t) = (2kt-1)(g-1)$ be the Hilbert polynomial of a k-pluricanonically embedded stable curve C of genus g and let $N_k + 1 = \mathrm{h}^0(C, \omega_C^{\otimes k}) = (2k-1)(g-1)$. Using Exercise 5.4.15, we can write $\overline{\mathcal{M}}_g \cong [U/\operatorname{PGL}_{N_k+1}]$, where $U \subseteq \operatorname{Hilb}^P(\mathbb{P}^{N_k})$ is a locally closed $\operatorname{PGL}_{N_k+1}$-invariant subscheme. Letting $H = \overline{U} \subseteq \operatorname{Hilb}^P(\mathbb{P}^{N_k})$, we need to show that there is an integer $m \gg 0$ such that $U = H^{\mathrm{s}}_{L_m}$, where $H^{\mathrm{s}}_{L_m}$ is stable locus with respect to the action of $\operatorname{PGL}_{N_k+1}$ and the line bundle L_m. We first note that GIT stability with respect to $\operatorname{PGL}_{N_k+1}$ is the same as stability with respect to $\operatorname{SL}_{N_k+1}$ as the map $\operatorname{SL}_{N_k+1} \to \operatorname{PGL}_{N_k+1}$ has finite cokernel: they have the same one-parameter subgroups up to scaling and therefore the Hilbert–Mumford Criterion (7.4.4) holds for $\operatorname{PGL}_{N_k+1}$ if and only if it holds for $\operatorname{SL}_{N_k+1}$.

After applying Theorem 5.8.5, it remains to show that every k-pluricanonically embedded stable curve C is m-Hilbert stable. Let $\mathcal{C} \to \operatorname{Spec} R$ be a generically smooth family of stable curves over a DVR R such that the central fiber $\mathcal{C}_0$ is isomorphic to C. Since the generic fiber is smooth, it is also m-Hilbert stable. By semistable reduction in GIT (Remark 7.1.4), after possibly replacing R with an extension, there exists a family $\mathcal{C}' \to \operatorname{Spec} R$ of m-Hilbert semistable curves such that the generic fibers $\mathcal{C}_K$ and $\mathcal{C}'_K$ are isomorphic. Since $\mathcal{C}'_0$ is m-Hilbert semistable, it is stable. As $\mathcal{C}$ and $\mathcal{C}'$ are two families of stable curves that are generically isomorphic, the separatedness of $\overline{\mathcal{M}}_g$ (Proposition 5.5.20) implies that $\mathcal{C} \cong \mathcal{C}'$. Thus $C = \mathcal{C}_0 = \mathcal{C}'_0$ is m-Hilbert stable. This establishes the identity $U = H^s_{L_m}$. We now apply the first fundamental theorem of GIT (Theorem 7.2.6) to conclude that there is a geometric quotient $U \to U/\operatorname{PGL}_{N_k+1}$ with $U/\operatorname{PGL}_{N_k+1}$ projective. In other words, $\mathcal{M}_g = [U/\operatorname{PGL}_{N_k+1}] \to U/\operatorname{PGL}_{N_k+1}$ is a coarse moduli space. By uniqueness of coarse moduli spaces, $\overline{M}_g = U/\operatorname{PGL}_{N_k+1}$ is projective.

This argument can also be made using the Chow scheme $\operatorname{Chow}_{1,d}(\mathbb{P}^{N_k})$ with $d = 2k(g-1)$ and the closure $Z \subseteq \operatorname{Chow}_{1,d}(\mathbb{P}^{N_k})$ of the locus of k-pluricanonically embedded smooth curves. One can write $\overline{\mathcal{M}}_g = [U/\operatorname{PGL}_{n+1}]$ where $U \subseteq Z$ is the locus of k-pluricanonically embedded Deligne–Mumford stable curves; this follows for instance from the fact that the Hilbert-to-Chow morphism $Z \to H$ is an isomorphism on the locus of nodal curves,. The same argument shows that $U = Z^{\mathrm{s}}$ is the stable locus and that $\overline{\mathcal{M}}_g \to Z^{\mathrm{s}}/\operatorname{PGL}_{n+1}$ has a projective coarse moduli space. See also [Mum77, Cor. 5.2], [Gie82, Thm. 2.0.2], and [HM98, Cor. 4.42]. $\square$

Remark 5.8.7. The restriction to characteristic 0 in Theorem 5.8.6 is because we have only developed GIT in Chapter 7 for linearly reductive groups. GIT applies equally to reductive groups in positive characteristic (see Remark 6.5.12) and in fact to reductive group schemes over $\mathbb{Z}$ [Ses77]. As Theorem 5.8.5 holds in arbitrary characteristic, this allows one to use GIT to construct $\overline{M}_g$ as a projective scheme over $\mathbb{Z}$.

GIT constructions of $\overline{M}_{g,n}$ directly verifying the Chow stability of a Deligne–Mumford stable curve have been completed more recently in [BS08] and [LW15], which also handle the case of marked points. The stability of arbitrary polarized curves $C \subseteq \mathbb{P}^n$, i.e., curves that are not necessarily k-pluricanonically embedded, has been analyzed in [Cap94] and [BFMV14].

5.8.2 Interpretation of the Hilbert–Mumford indices

We identify the Hilbert–Mumford indices $\mu^{\mathrm{Chow}}(C,\lambda)$ and $\mu^{\mathrm{Hilb}}_{L_m}(C,\lambda)$ with other algebro-geometric invariants. This leads us to a useful criterion for Chow or Hilbert stability (Proposition 5.8.17), which we employ in the next section to prove the stability of smooth curves. For simplicity, we keep the exposition below focused on dimension one, even though much of it extends to arbitrary dimension.

The ideal sheaf $J_\lambda \subseteq \mathcal{O}_{C\times\mathbb{A}^1}$. Let $C \subseteq \mathbb{P}^n$ be a curve; in applications later, we will take $n = N_k$ to be the dimension of the projective space of the kth pluricanonically embedding. Let $\lambda\colon \mathbb{G}_m \to \mathrm{GL}_{n+1}$ be a one-parameter subgroup defined by $\lambda(t) = \operatorname{diag}(t^{\lambda_0}, \ldots, t^{\lambda_n})$ for integers $\lambda_0 \geq \cdots \geq \lambda_n = 0$. Given coordinates x_i on $\mathbb{P}^n$ and t on $\mathbb{A}^1$, we define the ideal sheaf

$$J_\lambda = (t^{\lambda_i} x_i) \subseteq \mathcal{O}_{C\times\mathbb{A}^1}.$$

The rational map $\mathbb{A}^1 \times C \dashrightarrow \mathbb{A}^1 \times \mathbb{P}^n$, where $(t,x) \mapsto \lambda(t)\cdot x$ is resolved by blowing up J_λ, leading to a commutative diagram

$$\begin{tikzcd}[column sep=small]
& \operatorname{Bl}_{J_\lambda}(C\times\mathbb{A}^1) \arrow[dl] \arrow[dr] & \\
\mathbb{A}^1 \times C \arrow[dr] & & \mathcal{X} \arrow[r, hook] \arrow[dl] & \mathbb{A}^1\times\mathbb{P}^n \\
& \mathbb{A}^1, &
\end{tikzcd}$$

where $\mathcal{X}$ is defined as the scheme-theoretic image of the map $\operatorname{Bl}_{J_\lambda}(C\times\mathbb{A}^1) \to \mathbb{A}^1\times\mathbb{P}^n$. The projection $\mathcal{X} \to \mathbb{A}^1$ is flat by construction, and the central fiber $[\mathcal{X}_0 \subseteq \mathbb{P}^n]$ is identified with $\lim_{t\to 0} \lambda(t)\cdot[C\subseteq\mathbb{P}^n] \in \operatorname{Hilb}^P(\mathbb{P}^n)$, which follows from the proof of the valuative criterion for properness of $\operatorname{Hilb}^P(\mathbb{P}^n)$ as in Remark 1.4.4. Observe that $\mathcal{X}_0$ is fixed by $\mathbb{G}_m$, and that $\Gamma(\mathcal{X}_0, \mathcal{O}_{\mathcal{X}_0}(m))$ inherits a $\mathbb{G}_m$-action for each m.

Asymptotic weights of the Hilbert function. The λ*-weight* of a GL_{n+1}-representation V is $\lambda\text{-weight}(V) = \sum_i i\cdot\dim_{\Bbbk} V_i$, where V_i is the ith eigenspace of the induced $\mathbb{G}_m$-action on V.

Definition 5.8.8 ($r(C,\lambda)$)**.** Let $C \subseteq \mathbb{P}^n$ be a curve and $\lambda\colon \mathbb{G}_m \to \mathrm{GL}_{n+1}$ be a one-parameter subgroup defined by $\lambda(t) = \operatorname{diag}(t^{\lambda_0}, \ldots, t^{\lambda_n})$ for integers $\lambda_0, \ldots, \lambda_n$.

We define the *invariant* $r(C,\lambda)$ as the normalized leading coefficient of the quadratic polynomial

$$\lambda\text{-weight}(\Gamma(\mathcal{X}_0, \mathcal{O}_{\mathcal{X}_0}(m))) = r(C,\lambda)\frac{m^2}{2} + c_1 m + c_0$$

for $m \gg 0$.

The fact that $r(C,\lambda)$ is a quadratic polynomial for $m \gg 0$, and therefore that $r(C,\lambda)$ is well-defined, is a consequence of Theorem 5.8.11. The definition of $r(C,\lambda)$ is analogous to the Futaki invariant in K-stability, and it turns out that K-stability corresponds to asymptotic Chow stability, i.e., the Chow stability of $|\omega_C^{\otimes k}| \colon C \subseteq \mathbb{P}^{N_k}$ for $k \gg 0$.

Multiplicity of contact with a weighted flag. If $\mathcal{X}$ is a separated scheme of finite type over $\Bbbk$ of dimension 2, $J \subseteq \mathcal{O}_{\mathcal{X}}$ is a sheaf of ideals defining a subscheme $C \subseteq \mathcal{X}$ proper over $\Bbbk$, and L is a line bundle on $\mathcal{X}$, then the Euler characteristic

$$\chi(\mathcal{X}, L^{\otimes m}/J^m L^{\otimes m}) = e_L(J)\frac{m^2}{2} + (\text{lower terms}) \tag{5.8.9}$$

is a polynomial of degree 2 with rational coefficients. We define the *multiplicity of J measured by L* as the normalized leading coefficient $e_L(J)$. This is a special case of the *Hilbert–Samuel polynomial*; see [Ram74], [Mum77, Prop. 2.1], and [Eis95, §12.1].

If $J = 0$ and L is very ample on $\mathcal{X}$, then $\chi(\mathcal{X}, L^{\otimes m}/J^m L^{\otimes m}) = \chi(\mathcal{X}, L^{\otimes m})$ is the Hilbert polynomial and $e_L(0)$ is the degree of $\mathcal{X}$. Similarly, if $J = 0$ and $L = p^*\mathcal{O}_{\mathbb{P}^m}(1)$ is the pullback under a map $p \colon \mathcal{X} \to \mathbb{P}^n$, then $e_L(0)$ is the degree of the Chow cycle defined by the image $p(\mathcal{X})$ (with the coefficients of the Chow cycle being defined as usual using the degrees of the induced function field extensions). On the other hand, if J is the ideal sheaf of a subscheme supported at a point, then $e_L(J)$ is the multiplicity of the singularity. See also Proposition 5.8.21 for how the multiplicity measures the difference of degrees under a linear projection.

Since the ideal sheaf $J_\lambda = (t^{\lambda_i} x_i)$ is only defined if each $\lambda_i \geq 0$, the definition of $e(C,\lambda)$ below also requires that each $\lambda_i \geq 0$.

Definition 5.8.10 ($e(C,\lambda)$)**.** Let $C \subseteq \mathbb{P}^n$ be a curve and $\lambda \colon \mathbb{G}_m \to \mathrm{GL}_{n+1}$ be a one-parameter subgroup defined by $\lambda(t) = \mathrm{diag}(t^{\lambda_0}, \ldots, t^{\lambda_n})$ for integers $\lambda_0 \geq \cdots \geq \lambda_n \geq 0$. We define

$$e(C,\lambda) = e_{\mathcal{O}_{\mathbb{A}^1} \boxtimes \mathcal{O}_C(1)}(J_\lambda)$$

as the multiplicity of the ideal sheaf $J_\lambda = (t^{\lambda_i} x_i)$ measured by $\mathcal{O}_{\mathbb{A}^1} \boxtimes \mathcal{O}_C(1)$.

The one-parameter subgroup λ defines a weighted flag

$$\underbrace{\{x_1 = \cdots = x_n = 0\}}_{L_0,\ \text{weight } \lambda_0} \subseteq \underbrace{\{x_2 = \cdots = x_n = 0\}}_{L_1,\ \text{weight } \lambda_1} \subseteq \cdots \subseteq \underbrace{\{x_n = 0\}}_{L_{n-1},\ \text{weight } \lambda_{n-1}} \subseteq \mathbb{P}^n.$$

Roughly speaking, the closed subscheme $\mathbb{A}^1 \times C$ defined by the ideal $J_\lambda = (t^{\lambda_i} x_i)$ is a union of the λ_ith-order thickenings of $L_i \cap C$, and $e(C,\lambda)$ measures the degree of contact between this weighted flag and C.

Equivalent interpretations of the Hilbert–Mumford indices.

Theorem 5.8.11. *Let $C \subseteq \mathbb{P}^n$ be a curve and $\lambda \colon \mathbb{G}_m \to \mathrm{GL}_{n+1}$ be a one-parameter subgroup defined by $\lambda(t) = \mathrm{diag}(t^{\lambda_0}, \ldots, t^{\lambda_n})$ for integers $\lambda_0 \geq \cdots \geq \lambda_n$. Then $\mu^{\mathrm{Chow}}(C,\lambda) = r(C,\lambda)$. Moreover, if each $\lambda_i \geq 0$, then*

$$\mu^{\mathrm{Chow}}(C,\lambda) = r(C,\lambda) = e(C,\lambda).$$

Proof. To show that $\mu^{\mathrm{Chow}}(C, \lambda) = r(C, \lambda)$, we will reduce to case of a union of lines and then the simpler case of a line. First, if C is a line, we can change coordinates to assume that $C = V(x_2, \ldots, x_n)$, in which case we can directly compute that its Chow form f_C is $u_{00}u_{11} - u_{01}u_{10}$ and $\mu^{\mathrm{Chow}}(C, \lambda) = \lambda\text{-weight}(f_C) = \lambda_0 + \lambda_1$. Since $\Gamma(C, \mathcal{O}(m)) = \langle x_0^m, x_0^{m-1}x_1, \ldots, x_1^m \rangle$, its λ-weight is $(\lambda_0+\lambda_1)\binom{m+1}{2} = (\lambda_0+\lambda_1)\frac{m^2}{2} +$ (lower terms). Thus $r(C, \lambda) = \lambda_0 + \lambda_1 = \mu^{\mathrm{Chow}}(C, \lambda)$. If C is set-theoretically contained in the union $\bigcup_{ij} L_{ij}$ of the lines $L_{ij} = V(x_0, \ldots, \widehat{x}_i, \ldots, \widehat{x}_j, \ldots, x_n)$. Let a_{ij} be the multiplicity of C at the generic point of L_{ij}. Note that $\mu^{\mathrm{Chow}}(C, \lambda)$ and $r(C, \lambda)$ only depend on the scheme structures at the generic points of L_{ij} and not on the embedded points of C. Since $f_C = \prod_{ij} f_{L_{ij}}^{a_{ij}}$, we see that $\mu^{\mathrm{Chow}}(C, \lambda) = \sum_{ij} a_{ij}\mu^{\mathrm{Chow}}(L_{ij}, \lambda)$. We leave the equality $r(X, \lambda) = \sum_{ij} a_{ij} r(L_{ij}, \lambda)$ to the reader. The case of a line then implies that $\mu^{\mathrm{Chow}}(C, \lambda) = r(C, \lambda)$.

To reduce to the case of a union of lines, observe that since both $\mu^{\mathrm{Chow}}(C, \lambda)$ and $r(C, \lambda)$ are determined by the limit curve $\lim_{t \to 0} \lambda(t) \cdot [C \subseteq \mathbb{P}^n] \in \mathrm{Hilb}^P(\mathbb{P}^n)$, we may assume that C is fixed by λ. The maximal torus $\mathbb{G}_m^{n+1} \subseteq \mathrm{GL}_{n+1}$ acts on the proper scheme $\mathrm{Hilb}^P(\mathbb{P}^n)$, where P is the Hilbert polynomial of C. Since $\mathbb{G}_m^{n+1}$ is solvable, the Borel Fixed Point Theorem (c.f., [Mil17, Cor. 17.3]) implies that the closure of the orbit $\mathbb{G}_m^{n+1} \cdot [C \subseteq \mathbb{P}^n]$ contains a $\mathbb{G}_m^{n+1}$-fixed point $[\mathcal{X}_0 \subseteq \mathbb{P}^{n+1}]$, which is necessarily contained set-theoretically in the union of coordinate lines. This yields a diagram

$$\begin{tikzcd} \mathcal{X} \arrow[r, hook] \arrow[d, "\pi"'] & \mathbb{P}^n \times \mathbb{A}^1 \arrow[dl] \\ \mathbb{A}^1 & \end{tikzcd}$$

where $\mathcal{X} \subseteq \mathbb{P}^n \times \mathbb{A}^1$ is a $\mathbb{G}_m$-invariant closed subscheme (under the λ action on $\mathbb{P}^n$ and the trivial action on $\mathbb{A}^1$) and $\mathcal{X} \to \mathbb{A}^1$ is a $\mathbb{G}_m$-equivariant flat family such that the fiber over 1 is C and over 0 is $\mathcal{X}_0$. We leave the reader to verify that $\mu(C, \lambda) = \mu(\mathcal{X}_0, \lambda)$ and $r(C, \lambda) = r(\mathcal{X}_0, \lambda)$.

We now show that $e(C, \lambda) = r(C, \lambda)$ assuming that each $\lambda_i \geq 0$. We can write

$$\mathcal{X} = \mathrm{Proj} \bigoplus_{m \geq 0} R_m,$$

where R_m is the $\Bbbk[t]$-submodule of $\Bbbk[t] \otimes_{\Bbbk} \mathrm{H}^0(C, \mathcal{O}_C(m))$ generated by degree m polynomials in $t^{\lambda_i}x_i$. By definition, $r(C, \lambda)$ is computed as the normalized leading coefficient of a quadratic polynomial

$$\begin{aligned} r(C, \lambda) &= \text{norm. lead. coef. of } \lambda\text{-weight}\big(\mathrm{H}^0(\mathcal{X}_0, \mathcal{O}_{\mathcal{X}_0}(m))\big) \\ &= \text{norm. lead. coef. of } \lambda\text{-weight}(R_m/tR_m) \\ &= \text{norm. lead. coef. of } \dim_{\Bbbk}(\Bbbk[t] \otimes_{\Bbbk} \mathrm{H}^0(C, \mathcal{O}_C(m))/R_m), \end{aligned} \tag{5.8.12}$$

where the last equality follows the identification from Lemma 5.8.14 that $\lambda\text{-weight}(R_m/tR_m) = \dim_{\Bbbk}(\Bbbk[t] \otimes_{\Bbbk} \mathrm{H}^0(C, \mathcal{O}_C(m))/R_m)$ for all m. On the other hand, letting $X = \mathbb{A}^1 \times C$ and $L = \mathcal{O}_{\mathbb{A}^1} \boxtimes \mathcal{O}_C(1)$, the multiplicity $e(C, \lambda) = e_L(J_\lambda)$ is defined as the normalized leading coefficient of $\chi(X, L^{\otimes m}/J_\lambda^m L^{\otimes m})$. We leave the reader to verify that the quantity can be computed as

$$\begin{aligned} e(C, \lambda) &= \text{norm. lead. coef. of } \mathrm{h}^0\left(X, L^{\otimes m}/J_\lambda^m L^{\otimes m}\right) \\ &= \text{norm. lead. coef. of } \dim_{\Bbbk} \mathrm{H}^0(X, L^{\otimes m})/\mathrm{H}^0(X, J_\lambda^m L^{\otimes m}) \\ &= \text{norm. lead. coef. of } \dim_{\Bbbk}\left(\Bbbk[t] \otimes_{\Bbbk} \mathrm{H}^0(C, \mathcal{O}_C(m))/R_m\right). \end{aligned} \tag{5.8.13}$$

It follows that $r(C,\lambda) = e(C,\lambda)$. See also [Mum77, Prop. 2.6 and Thm. 2.9]. $\square$

A key ingredient of the proof above was the identification of the quantity $\lambda\text{-weight}(R_m/tR_m)$, which depends on the limit *curve* $\mathcal{X}_0$, and the quantity $\dim_{\Bbbk}(\Bbbk[t] \otimes_{\Bbbk} \Gamma(C, \mathcal{O}_C(m))/R_m)$, defined on the *surface* $\mathbb{A}^1 \times C$.

Lemma 5.8.14. *Let V be a $\mathbb{G}_m$-representation of dimension n with weights $\lambda_n \geq \cdots \lambda_0 \geq 0$. Let V_i be the eigenspace of weight λ_i, and define $R \subseteq \Bbbk[t] \otimes V$ as the $\Bbbk[t]$-submodule generated by $t^{\lambda_i} V_i$ for $i = 0, \ldots, n$. Then*

$$\lambda\text{-weight}(R/tR) = \dim_{\Bbbk} \big((\Bbbk[t] \otimes V)/R\big).$$

Proof. Writing $\Bbbk[t] \otimes V = \bigoplus_{i,d} t^d V_i$, R/tR is freely generated by $t^{\lambda_i} V_i$, while $(\Bbbk[t] \otimes V)/R$ is freely generated by $V_i, tV_i, \ldots, t^{\lambda_i - 1} V_i$. Thus

$$\lambda\text{-weight}(R/tR) = \sum_i \lambda_i \dim V_i = \dim_{\Bbbk} \big((\Bbbk[t] \otimes V)/R\big). \quad \square$$

Exercise 5.8.15 (details)**.** Fill in the missing details in the proof of Theorem 5.8.11.

The Hilbert–Mumford index $\mu^{\mathrm{Hilb}}_{L_m}(C,\lambda)$ also has a nice interpretation, allowing us to compare it with $\mu^{\mathrm{Chow}}(C,\lambda)$.

Proposition 5.8.16. *Let $C \subseteq \mathbb{P}^n$ be a curve and $\lambda \colon \mathbb{G}_m \to \mathrm{GL}_{n+1}$ be a one-parameter subgroup defined by $\lambda(t) = \mathrm{diag}(t^{\lambda_0}, \ldots, t^{\lambda_n})$ for integers $\lambda_0 \geq \cdots \geq \lambda_n \geq 0$. Then for $m \gg 0$*

$$\mu^{\mathrm{Hilb}}_{L_m}(C,\lambda) = \dim_{\Bbbk} \mathrm{H}^0\big(\mathbb{A}^1 \times C, \mathcal{O}_{\mathbb{A}^1} \boxtimes \mathcal{O}_C(m)\big)/R_m,$$

where R_m is the subspace generated by monomials of degree m in $t^{\lambda_i} x_i$, while

$$\mu^{\mathrm{Chow}}(C,\lambda) = \textit{norm. lead. coef. of } \dim_{\Bbbk} \mathrm{H}^0\big(\mathbb{A}^1 \times C, \mathcal{O}_{\mathbb{A}^1} \boxtimes \mathcal{O}_C(m)\big)/R_m.$$

In particular, $\mu^{\mathrm{Chow}}(C,\lambda)$ is the normalized leading coefficient of $\mu^{\mathrm{Hilb}}_{L_m}(C,\lambda)$.

Proof. Letting t be the coordinate on $\mathbb{A}^1$, we can write

$$R_m = \bigoplus_{i \geq 0} R_{m,i} \cdot t^i \subseteq \mathrm{H}^0(C, \mathcal{O}_C(m)) \cdot t^i = \mathrm{H}^0(\mathbb{A}^1 \times C, \mathcal{O}_{\mathbb{A}^1} \boxtimes \mathcal{O}_C(m)).$$

By construction, we have inclusions

$$R_{m,0} \subseteq R_{m,1} \subseteq \cdots \subseteq R_{m,I} = \mathrm{H}^0(C, \mathcal{O}_C(m))$$

for $I \gg 0$. We use the description from (5.8.3) of $\mu^{\mathrm{Hilb}}_{L_m}(C,\lambda)$ as the minimum of $\lambda\text{-weight}(\beta)$ over subsets $\beta \subseteq \mathrm{H}^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(m))$ mapping to a basis of $\mathrm{H}^0(C, \mathcal{O}_C(m))$. A basis of minimum weight corresponds to first choosing a basis of $R_{m,0}$ and successively extending it to each $R_{m,i}$. Therefore,

$$\begin{aligned}
\mu^{\mathrm{Hilb}}_{L_m}(C,\lambda) &= 0 \cdot \dim_{\Bbbk} R_{m,0} + 1 \cdot (\dim_{\Bbbk} R_{m,1} - \dim_{\Bbbk} R_{m,0}) + 2 \cdot (\dim_{\Bbbk} R_{m,2} - \dim_{\Bbbk} R_{m,1}) + \cdots \\
&= I \cdot \dim_{\Bbbk} \mathrm{H}^0(C, \mathcal{O}_C(m)) - \dim_{\Bbbk} R_{m,0} - \cdots - \dim_{\Bbbk} R_{m,I-1} \\
&= \sum_{i=0}^{I-1} \dim \mathrm{H}^0(C, \mathcal{O}_C(m))/R_{m,i} \\
&= \dim_{\Bbbk} H^0\big(\mathbb{A}^1 \times C, \mathcal{O}_{\mathbb{A}^1} \boxtimes \mathcal{O}_C(m)\big)/R_m.
\end{aligned}$$

The final statement follows from the equalities of (5.8.12). See also [Mor80, Prop. 3.2]. $\square$

5.8.3 Criteria for Chow and Hilbert stability

Proposition 5.8.17 (Criterion for Chow Stability)**.** *A curve $C \subseteq \mathbb{P}^n$ is Chow stable (resp., semistable) if and only if $r(C,\lambda) < 0$ (resp. $\leq$) for every one-parameter subgroup $\lambda\colon \mathbb{G}_m \to \mathrm{SL}_n$, or equivalently if*

$$e(C,\rho) < \frac{2\deg C}{n+1}\sum_i \rho_i \quad \textit{(resp., } \leq\textit{)}$$

for every one-parameter subgroup $\rho\colon \mathbb{G}_m \to \mathrm{GL}_n$ with weights $\rho_0 \geq \cdots \geq \rho_n = 0$.

Proof. The first equivalence follows directly from the equality $\mu^{\mathrm{Chow}}(C,\lambda) = r(C,\lambda)$ in Theorem 5.8.11 and the Hilbert–Mumford Criterion (7.4.6). Moreover, we know that $r(C,\lambda)$ is the normalized leading coefficient of the λ-weight of $\Gamma(\mathcal{X}_0, \mathcal{O}_{\mathcal{X}_0}(m))$, where $\mathcal{X}_0$ is the limit of C under λ in the Hilbert scheme. After a change of basis, we can write

$$\lambda(t) = t^{\alpha}\underbrace{\begin{pmatrix} t^{\rho_0} & & \\ & \ddots & \\ & & t^{\rho_n}\end{pmatrix}}_{\rho(t)} \qquad \begin{array}{l}\text{with } \alpha = -\frac{\sum_i \rho_i}{n+1} \\ \text{and } \rho_0 \geq \cdots \geq \rho_n = 0\end{array}$$

for a one-parameter subgroup $\rho\colon \mathbb{G}_m \to \mathrm{GL}_n$. Comparing weights under λ and ρ, we have

$$\lambda\text{-weight}\left(\Gamma(\mathcal{X}_0, \mathcal{O}_{\mathcal{X}_0}(m))\right) = \rho\text{-weight}\left(\Gamma(\mathcal{X}_0, \mathcal{O}_{\mathcal{X}_0}(m))\right) + \alpha m \dim_{\Bbbk}\left(\Gamma(\mathcal{X}_0, \mathcal{O}_{\mathcal{X}_0}(m))\right)$$

As the leading coefficient of $\dim_{\Bbbk}\left(\Gamma(\mathcal{X}_0, \mathcal{O}_{\mathcal{X}_0}(m))\right)$ is $\deg(C)$, we obtain that

$$\begin{aligned} r(C,\lambda) &= r(C,\rho) + 2\deg(C)\alpha \\ &= r(C,\rho) - \frac{2\deg(C)}{n+1}\sum_i \rho_i. \quad \square \end{aligned}$$

Proposition 5.8.18. *Let $C \subseteq \mathbb{P}^n$ be a curve. If C is Chow stable (resp., not semistable), then C is asymptotically Hilbert stable (resp., not semistable).*

Proof. If C is Chow stable (resp., not semistable), then $\mu^{\mathrm{Chow}}(C,\lambda) < 0$ (resp., $>$). By Proposition 5.8.16, the normalized leading coefficient of the quadratic polynomial $\mu^{\mathrm{Hilb}}_{L_m}(C,\lambda)$ is strictly negative (resp., strictly positive), and therefore $\mu^{\mathrm{Hilb}}(C,\lambda) < 0$ (resp. $>$) for $d \gg 0$. $\square$

In other words, we have the implications:

Chow stable $\implies$ asymptotically Hilbert stable $\implies$ asymptotically Hilbert semistable $\implies$ Chow semistable.

5.8.4 Chow stability of smooth curves

To verify that the Chow form of a smooth curve is stable, we will exploit the following bound.

Proposition 5.8.19. *Let $C \subseteq \mathbb{P}^n$ be a smooth curve and $\lambda\colon \mathbb{G}_m \to \mathrm{GL}_{n+1}$ be a one-parameter subgroup defined by $\lambda(t) = \mathrm{diag}(t^{\lambda_0}, \ldots, t^{\lambda_n})$ for integers $\lambda_0 \geq \cdots \geq \lambda_n = 0$. For $k = 0, \ldots, n$, let $I_k \subseteq \mathcal{O}_C$ be the ideal generated by $x_k, \ldots, x_n$. Then*

$$e(C,\lambda) \leq \sum_{k=0}^{n-1} (\lambda_k - \lambda_{k+1})\left(e_{\mathcal{O}_C(1)}(I_k) + e_{\mathcal{O}_C(1)}(I_{k+1})\right).$$

Proof. As before, we set $X = \mathbb{A}^1 \times C$, $L = \mathcal{O}_{\mathbb{A}^1} \boxtimes \mathcal{O}_C(1)$, and $J_\lambda \subseteq \mathcal{O}_X$ be the ideal generated by $t^{\lambda_i} x_i$. Recall from the identities (5.8.13) that $e(C, \lambda) = e_L(J_\lambda)$ is the normalized leading coefficient of $\dim_{\Bbbk} \mathrm{H}^0(X, L^{\otimes m})/\mathrm{H}^0(X, J_\lambda^m L^{\otimes m})$. Observe that $J_\lambda = \sum_k t^{\lambda_k} I_k$ and that it contains $t^{\lambda_0} \cdot \mathcal{O}_C$. Setting $J_{m,d} := J_\lambda^m \cap t^d \cdot \mathcal{O}_C \subseteq \mathcal{O}_C$, we have that $J_{m,m\lambda_0} = \mathcal{O}_C$. The expansion $J_\lambda^m = (\sum_k t^{\lambda_k} I_k)^m$ contains the sum

$$\begin{aligned} I_n^m \cdot (t^0 + \cdots + t^{\lambda_{n-1}-1}) + I_n^{m-1} I_{n-1} \cdot (t^{\lambda_{n-1}} + \cdots + t^{2\lambda_{n-1}-1}) + \cdots + \\ I_{n-1}^m \cdot (t^{m\lambda_{n-1}} + \cdots + t^{(m-1)\lambda_{n-1}+\lambda_{n-2}-1}) + \cdots + \\ I_1 I_0^{m-1} \cdot (t^{\lambda_1+(m-1)\lambda_0} + \cdots + t^{m\lambda_0-1}) + \mathcal{O}_C \cdot (t^{m\lambda_0} + \cdots). \end{aligned}$$

In other words, we have

$$J_\lambda^m = \bigoplus_{d\geq 0} t^d \cdot J_{m,d} \supseteq \bigoplus_{k=0}^{n-1} \bigoplus_{j=0}^{m-1} I_{n-k}^{m-j} I_{n-k-1}^j \cdot \left(\bigoplus_{\nu=0}^{\lambda_{n-k-1}-\lambda_{n-k}-1} t^{\lambda_{n-k}(m-j)+\lambda_{n-k-1}j+\nu} \cdot \mathcal{O}_C \right).$$

Each integer $d = 0, \ldots, \lambda_0 m - 1$ can be written uniquely as $\lambda_{n-k}(m-j) + \lambda_{n-k-1} j + \nu$, and $J_{m,d} \supseteq I_{n-k}^{m-j} I_{n-k-1}^j$. Thus,

$$\begin{aligned} \dim_{\Bbbk} \mathrm{H}^0(X, L^{\otimes m})/\mathrm{H}^0(X, J_\lambda^m L^{\otimes m}) &\leq \sum_{k=0}^{n-1} \left[(\lambda_{n-k} - \lambda_{n-k-1}) \sum_{j=0}^{m-1} \mathrm{h}^0\big(C, \mathcal{O}_C(m)/(I_{n-k}^{m-j} I_{n-k-1}^j \mathcal{O}_C(m))\big) \right] \\ &\leq \sum_{k=0}^{n-1} \left[(\lambda_k - \lambda_{k+1}) \sum_{j=0}^{m-1} \mathrm{h}^0\big(C, \mathcal{O}_C(m)/(I_k^{m-j} I_{k+1}^j \mathcal{O}_C(m))\big) \right]. \end{aligned}$$

If $Z = V(I) \subseteq C$ is a finite subscheme defined by an ideal sheaf $I \subseteq \mathcal{O}_C$, then the multiplicity $m_{\mathcal{O}_C(1)}(I)$ is equal to the length $\dim_{\Bbbk} \Gamma(Z, \mathcal{O}_Z)$ of Z, which is also equal to $\Gamma(C, \mathcal{O}_C(m)/I\mathcal{O}_C(m))$ for $m \gg 0$. Therefore, using that C is smooth, we have that

$$h^0\big(C, \mathcal{O}_C(m)/(I_k^{m-j} I_{k+1}^j \mathcal{O}_C(m))\big) = (m-j) e_{\mathcal{O}_C(1)}(I_k) + j e_{\mathcal{O}_C(1)}(I_{k+1})$$

for $m \gg 0$. The sum of these dimensions over $j = 0, \ldots, m-1$ is a quadratic polynomial in m with leading term $(e_{\mathcal{O}_C(1)}(I_k) + e_{\mathcal{O}_C(1)}(I_{k+1}))m^2/2$ and the proposition follows. See [Gie82, Thm. 1.0.0] and [Mum77, Prop. 4.10-11]. □

Remark 5.8.20. A slightly more involved argument establishes the proposition for arbitrary curves (no smoothness required), but the bounds are unfortunately not sufficient to verify the Chow stability of a stable curve. There are also similar bounds for higher dimensional varieties, but they involve the mixed multiplicities $e_{\mathcal{O}_C(1)}(I_{r_k}, I_{r_{k+1}})$.

As a consequence of Proposition 5.8.19, we have an upper bound for $e(C, \lambda)$ depending on multiplicities on the curve C rather than the surface $X = \mathbb{A}^1 \times C$. Multiplicities have the following convenient geometric interpretation, which we state and prove in arbitrary dimension.

Proposition 5.8.21. *Let $Y \subseteq \mathbb{P}^n$ be an integral subscheme. Let $\Lambda \subseteq \Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$ be a linear subspace of codimension r, let $L_\Lambda \subseteq \mathbb{P}^n$ be the r-dimensional hyperplane defined by Λ, and let $I_\Lambda \subseteq \mathcal{O}_Y$ be the ideal sheaf of $Z := Y \cap L_\Lambda$. If*

$$p_\Lambda \colon Y \smallsetminus Z \to \mathbb{P}^r$$

denotes the induced projection and $p_\Lambda(Y)$ denotes the Chow cycle defined by the image of p_Λ, then
$$e_{\mathcal{O}_Y(1)}(I_\Lambda) = \deg Y - \deg p_\Lambda(Y).$$

Proof. Let H be the divisor class of $\mathcal{O}_Y(1)$. There is a commutative diagram

$$\begin{tikzcd} \mathrm{Bl}_{I_\Lambda}(Y) \arrow[d, "\pi"] \arrow[dr, "q"] & \\ Y \arrow[r, dashed, "p_\Lambda"] & \mathbb{P}^r \end{tikzcd}$$

where π is the blowup morphism and q is the morphism extending p_Λ defined by $\pi^{-1}(H) - E$ with E the exceptional divisor. The degree of Y is

$$\deg(Y) = \text{norm. lead. coef. of } \chi(\mathcal{O}_Y(m)),$$

while the degree of $p_\Lambda(Y)$ is

$$\begin{aligned} \deg p_\Lambda(Y) &= \text{norm. lead. coef. of } \chi\left(\mathrm{Bl}_{I_\Lambda}(Y), q^*\mathcal{O}_{\mathbb{P}^r}(m)\right) \\ &= \text{norm. lead. coef. of } \chi\left(\mathrm{Bl}_{I_\Lambda}(Y), \pi^*\mathcal{O}_Y(m)(-mE)\right) \\ &= \text{norm. lead. coef. of } \chi\left(Y, I_\Lambda^m\mathcal{O}_Y(m)\right), \end{aligned}$$

where in the final equality we used the identification $\mathrm{H}^0(\mathrm{Bl}_{I_\Lambda}(Y), \pi^*\mathcal{O}_Y(m)(-mE)) = \mathrm{H}^0(Y, I_\Lambda^m\mathcal{O}_Y(m))$ along with the consequence of Asymptotic Riemann–Roch (B.2.22) that the dimension of the higher cohomology of each sheaf is $O(m^{\dim(Y)-1})$. We conclude that

$$\begin{aligned} e_{\mathcal{O}_Y(1)}(I_\Lambda) &= \text{norm. lead. coef. } \chi(Y, \mathcal{O}_Y(m)/I_\Lambda^m\mathcal{O}_Y(m)) \\ &= \text{norm. lead. coef. } \chi(Y, \mathcal{O}_Y(m)) - \text{norm. lead. coef. } \chi(Y, I_\Lambda^m\mathcal{O}_Y(m)) \\ &= \deg Y - \deg p_\Lambda(Y). \end{aligned}$$

See also [Mum77, Prop. 2.5]. □

Theorem 5.8.22. *Let C be a smooth, connected, and projective curve of genus $g \geq 1$ over an algebraically closed field $\Bbbk$. If $C \subseteq \mathbb{P}^n$ is embedded by a complete linear series of degree $d > 2g$, then $C \subseteq \mathbb{P}^n$ is Chow stable and thus asymptotically Hilbert stable.*

Proof. The proof of Chow stability breaks down into two steps:

(a) A smooth curve $C \subseteq \mathbb{P}^n$ embedded by a complete linear series of degree $d > 2g$ is linearly stable, i.e., for every linear projection $p_\Lambda \colon \mathbb{P}^n \setminus L_\Lambda \dashrightarrow \mathbb{P}^r$,
$$\frac{\deg p_\Lambda(C)}{r} > \frac{\deg C}{n},$$
where $p_\Lambda(C)$ denotes the image Chow cycle. Using the identity $e_{\mathcal{O}_C(1)}(I_\Lambda) = \deg C - \deg p_\Lambda(C)$ of Proposition 5.8.21, where I_Λ is the ideal sheaf of $C \cap L_\Lambda$, this is equivalent to
$$e_{\mathcal{O}_C(1)}(I_\Lambda) < \frac{n-r}{n}\deg C.$$

(b) A linearly stable curve $C \subseteq \mathbb{P}^n$ is Chow stable.

For (a), we consider all morphisms $f\colon C \to \mathbb{P}^N$ whose images are not contained in a hyperplane, and we consider the corresponding pairs (D, N), where D is the degree of the image Chow cycle $f(C)$. If $f^*\mathcal{O}_{\mathbb{P}^N}(1)$ is non-special, i.e., $\mathrm{h}^1(C, f^*\mathcal{O}_{\mathbb{P}^N}(1)) = 0$, then by Riemann-Roch (5.1.2),

$$N = h^0(\mathcal{O}_{\mathbb{P}^N}(1)) - 1 \leq h^0(f^*\mathcal{O}_{\mathbb{P}^N}(1)) - 1 = \deg \varphi^*\mathcal{O}_{\mathbb{P}^N}(1) - g = D - g.$$

If $f^*\mathcal{O}(1)$ is special, Clifford's Theorem (c.f., [Har77, Thm. IV.5.4]) implies that

$$N = h^0(\mathcal{O}_{\mathbb{P}^N}(1)) - 1 \leq h^0(f^*\mathcal{O}_{\mathbb{P}^N}(1)) - 1 \leq (\deg \varphi^*\mathcal{O}(1))/2 = D/2.$$

Since $C \subseteq \mathbb{P}^n$ is embedded by a complete linear series of degree $d > 2g$, $\mathcal{O}_C(1)$ is non-special and Riemann-Roch implies that $n + 1 = \mathrm{h}^0(C, \mathcal{O}_C(1)) = d + 1 - g$, so that $n = d - g > g$. As illustrated by Figure 5.8.23, the point (d, n) lies on the line $N = D - g$ to the right of $(2g, g)$. It follows that the slope of the line to (d, n) is greater than the slope to any point in the shaded area, or, in other words, that $D/N > d/n$ for every map $f\colon C \to \mathbb{P}^N$ of degree D whose image is not contained in a hyperplane. In particular, this implies that C is linearly stable.

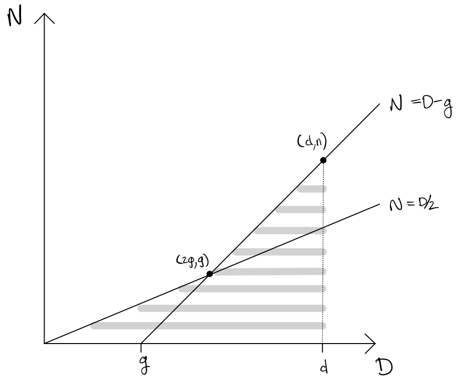

Figure 5.8.23: The slope of the line to (d, n) is greater than the slope to any point in the shaded areas.

For (b), if $\lambda\colon \mathbb{G}_m \to \mathrm{GL}_{n+1}$ be a one-parameter subgroup, we can choose coordinates $x_0, \ldots, x_n$ on $\mathbb{P}^n$ with weights $\lambda_0 \geq \cdots \geq \lambda_n = 0$. For each $k = 0, \ldots, n$, the subspace $\langle x_k, \ldots, x_n \rangle \subseteq \Gamma(C, \mathcal{O}_C(1))$ defines a hyperplane $L_k = V(x_k, \ldots, x_n)$ and induces a linear projection $\mathbb{P}^n \smallsetminus L_k \to \mathbb{P}^{n-k}$. The ideal sheaf $I_k \subseteq \mathcal{O}_C$ defining $C \cap L_k$ is generated by $x_k, \ldots, x_n$. The linear stability of $C \subseteq \mathbb{P}^n$ implies that

$e_{\mathcal{O}_C(1)}(I_k) < \frac{k}{n} \deg C$. The bound of Proposition 5.8.19 yields

$$\begin{aligned}
e(C,\lambda) &\leq \sum_{k=0}^{n-1} (\lambda_k - \lambda_{k+1}) \left(e_{\mathcal{O}_C(1)}(I_k) + e_{\mathcal{O}_C(1)}(I_{k+1})\right) \\
&< \frac{2\deg C}{n} \sum_{k=0}^{n-1} (\lambda_k - \lambda_{k+1}) \left(k + \frac{1}{2}\right) \\
&= \frac{2\deg C}{n} \left(\frac{1}{2}\lambda_0 + \lambda_1 + \cdots + \lambda_{n-1} + \frac{1}{2}\lambda_n\right) \\
&= \frac{2\deg C}{n} \left(\left(\sum_i \lambda_i\right) - \frac{1}{2}(\lambda_0 + \lambda_n)\right) \\
&\leq \frac{2\deg C}{n} \left(\left(\sum_i \lambda_i\right) - \frac{1}{n+1}\left(\sum_i \lambda_i\right)\right) \\
&= \frac{2\deg C}{n+1} \sum_i \lambda_i.
\end{aligned}$$

We may now apply the Criterion for Chow Stability (5.8.17) to conclude that $C \subseteq \mathbb{P}^n$ is Chow stable, and Proposition 5.8.18 to conclude that it is asymptotically Hilbert stable. □

As a corollary, we can deduce that a k-pluricanonical embedded smooth curve is Chow stable.

Proof of Theorem 5.8.5(1). The k-pluricanonical embedded of a smooth curve $|\omega_C^{\otimes k}| \colon C \subseteq \mathbb{P}^{N_k}$ has degree $d = 2k(g-1)$ with $N_k + 1 = (2k+1)(g-1)$. If $k \geq 5$ (in fact $k \geq 3$ suffices), then $d > 2g$ and Theorem 5.8.22 implies that $C \subseteq \mathbb{P}^{N_k}$ is Chow stable and asymptotically Hilbert stable. With more work, it is possible to exhibit a sufficiently large m such that every k-pluricanonical embedded smooth curve is m-Hilbert stable; see [Gie82, Thm. 1.0.0]. □

Exercise 5.8.24. Show that a curve $C \subseteq \mathbb{P}^n$ with a node is not linearly stable.

Remark 5.8.25. There are smooth surfaces that are not Chow stable, e.g., the Steiner surface defined by the image of $\mathbb{P}^2 \to \mathbb{P}^4$, $[x:y:z] \mapsto [xz : yz : x^2 : xy : y^2]$ [Mor80, Ex. 3.6].

5.8.5 Destabilizing non-nodal curves

Proposition 5.8.26. *Let $C \subseteq \mathbb{P}^n$ be a generically reduced curve satisfying $\frac{\deg C}{n+1} < \frac{8}{7}$. If $C \subseteq \mathbb{P}^n$ is Chow semistable, then C is not contained in a hyperplane and C has at worst nodal singularities.*

Proof. If C is contained in a hyperplane H, then we can choose coordinates $x_0, \ldots, x_n$ such that $H = V(x_0)$. Letting $\lambda(t) = \mathrm{diag}(t^{-n}, t, \ldots, t)$, the Hilbert–Mumford index $\mu^{\mathrm{Chow}}(C, \lambda)$ is strictly positive and hence C is not Chow semistable.

Suppose that $p \in C$ is a singularity with $\mathrm{mult}_p(C) \geq 3$. Choose coordinates $x_0, \ldots, x_n$ with $p = (1, 0, \ldots, 0)$, and define $\lambda(t) = (t, 1, \ldots, 1)$. Then $J_\lambda = (tx_0, x_1, \ldots, x_n)$ is the maximal ideal of $(0, p) \in \mathbb{A}^1 \times C$ and

$$e(C,\lambda) = \mathrm{mult}_{(0,p)} \mathbb{A}^1 \times C = \mathrm{mult}_p C \geq 3 > \frac{16}{7} \sum_i \lambda_i > 2\frac{\deg C}{n+1} \sum_i \lambda_i.$$

The Criterion for Chow Stability (5.8.17) shows that C is Chow non-semistable.

Suppose that $p \in C$ is a double point that is *not* a node. Letting $L \subseteq \mathcal{O}_C$ be the ideal defining the reduced tangent line at p, then $\mathfrak{m}_p^2 \subsetneq \mathfrak{m}_p^2 + L \subsetneq \mathfrak{m}_p$, and $\dim_{\Bbbk} \mathfrak{m}_p/\mathfrak{m}_p^2 = 2$. We can choose coordinates $x_0, \ldots, x_n$ such that (i) $x_0(p) \neq 0$, (ii) $v = x_1/x_0$ and $u = x_2/x_0$ span $\mathfrak{m}_p/\mathfrak{m}_p^2$, (iii) $u \in L$ with $u^2 \in \mathfrak{m}_p^3$, and (iv) $x_3/x_0, \ldots, x_n/x_0 \in \mathfrak{m}_p^2$. Setting $\lambda(t) = \operatorname{diag}(t^4, t^2, t, 1, \ldots, 1)$, then $J_\lambda = (t^4 x_0, t^2 x_1, t x_2, x_3, \ldots, x_n)$ defines a closed subscheme of $\mathbb{A}^1 \times C$ supported at $(0, p)$. Thus $e(C, \lambda) = e(J_\lambda)$ is the Hilbert-Samuel multiplicity of $(0, p)$ and it doesn't depend on whether we compute it using the line bundle $\mathcal{O}_{\mathbb{A}^1} \boxtimes \mathcal{O}_C(1)$ or $\mathcal{O}_{\mathbb{A}^1 \times C}$. Defining $J_0 = (t^4, t^2 v, tu, \mathfrak{m}_p^2)$, then $J_\lambda \subseteq J_0$ and thus $e(J_\lambda) \geq e(J_0)$. Consider $J_1 := (t^4, \mathfrak{m}_p^2) \subseteq J_0$. Since

$$(t^2 v)^2 = t^4 v^2 \in J_1^2 \quad \text{and} \quad (tu)^4 = t^4 (u^2)^2 \in t^4 (\mathfrak{m}_p^3)^2 \subseteq J_1^4,$$

we have that $J_0^m = J_1^m$ for $m \gg 0$ and thus $e(J_0) = e(J_1)$. We conclude that

$$e(C, \lambda) = e(J_\lambda) \geq e(J_0) = e(J_1) = 4 \cdot 2 \cdot \operatorname{mult}_p C = 16 = \frac{16}{7} \sum_i \lambda_i > 2 \frac{\deg C}{n+1} \sum_i \lambda_i,$$

and applying the Criterion for Chow Stability (5.8.17) again implies that C is Chow non-semistable. See also [Mum77, Prop. 3.1] and [Gie82, Prop. 1.0.4]. □

Remark 5.8.27. Interestingly, there are low degree Chow stable curves with singularities that are worst than nodes, but these curves become Chow non-semistable when re-embedded under a higher degree embedding. For example, there are Chow stable cuspidal curves $C \subseteq \mathbb{P}^2$ of degree 4 (see Exercise 7.5.4) which are Chow non-semistable when when re-embedded by the degree 8 line bundle $\mathcal{O}_C(2)$.

Proof of Theorem 5.8.5(2). Observe that if $C \subseteq \mathbb{P}^n$ is a stable curve embedded by the complete linear series $|\omega_C^{\otimes k}|$ for $k \geq 5$, then

$$\frac{\deg C}{n+1} = \frac{2k(g-1)}{(2k-1)(g-1)} = \frac{2k}{2k-1} < \frac{8}{7}.$$

Thus Proposition 5.8.26 applies to show that if C is generically reduced and Chow semistable, then it is not contained in a hyperplane and has at worst nodal singularities. While this goes part of the way to establishing that C is a k-pluricanonically embedded stable curve, a rather lengthy and intricate analysis is needed to destabilize curves that are either not generically reduced or not k-pluricanonically embedded. We refer the reader to [Mum77, §5], [Gie82, §1], and [HM98, Thm. 4.45] for details. □

5.9 Projectivity following Kollár

Still you may have a sentimental attachment to familiar old varieties. It would appear especially that projective varieties play such a central technical role in algebraic geometry that it may be virtually impossible to eliminate their use even if you wanted to. In any case, it is very interesting to prove, when possible, that [the space] is a projective variety.

Mumford [Mum76, p. 441]

We offer a second proof that $\overline{M}_{g,n}$ is projective following the approach introduced by Kollár in [Kol90], which builds on earlier ideas of Viehweg [Vie95]. To introduce the general strategy for projectivity, we need some terminology. Let $\pi\colon \mathcal{U}_g \to \overline{\mathcal{M}}_g$ be the universal family over $\operatorname{Spec}\mathbb{Z}$. For each integer $k \geq 1$, define the *kth pluricanonical bundle* as

$$\pi_*(\omega^{\otimes k}_{\mathcal{U}_g/\overline{\mathcal{M}}_g})$$

on $\overline{\mathcal{M}}_g$, which by Proposition 5.3.23 is a vector bundle of rank g if $k=1$ or rank $(2k-1)(g-1)$ if $k>1$. The determinants

$$\lambda_k := \det \pi_*(\omega^{\otimes k}_{\mathcal{U}_g/\overline{\mathcal{M}}_g})$$

are line bundles on $\overline{\mathcal{M}}_g$. For $k=1$, this is the *Hodge line bundle* from Definition 5.6.24.

Projectivity strategy: Show that for $k \gg 0$, a positive power of λ_k descends to an *ample* line bundle on the coarse moduli space $\overline{M}_g$.

5.9.1 Kollár's Criterion and Projectivity of $\overline{M}_{g,n}$

Kollár's strategy is to employ the multiplication map on pluricanonical bundles

$$W := \operatorname{Sym}^m \underbrace{\pi_*(\omega^{\otimes k}_{\mathcal{U}_g/\overline{\mathcal{M}}_g})}_{V} \to \pi_*(\omega^{\otimes mk}_{\mathcal{U}_g/\overline{\mathcal{M}}_g}) =: Q, \qquad (5.9.1)$$

depending on the very same two integers k and m as in the GIT construction using the Hilbert scheme. The fiber of the multiplication map over a stable curve $C \in \overline{\mathcal{M}}_g(\Bbbk)$ is the map $\operatorname{Sym}^m \mathrm{H}^0(C, \omega_C^{\otimes k}) \to \mathrm{H}^0(C, \omega_C^{\otimes mk})$, whose kernel consists of degree m equations cutting out the image of $|\omega_C^{\otimes k}|\colon C \to \mathbb{P}(\mathrm{H}^0(C, \omega_C^{\otimes k}))$. If $k \geq 3$, $\omega_C^{\otimes k}$ is very ample and C can be recovered from the kernel of the multiplication map (5.9.1) for $m \gg 0$, and thus the multiplication map (5.9.1) is surjective. Let $w = \operatorname{rk} W$, $q = \operatorname{rk} Q$, and $v = \operatorname{rk} V$. Since W is expressed as the symmetric product $\operatorname{Sym}^m V$, W has a reduction of the structure group by $\mathrm{GL}_v \to \mathrm{GL}_w$ (see Definition B.1.62).

For a stable curve C, fixing a basis $\mathrm{H}^0(C, \omega_C^{\otimes k}) \cong \Bbbk^v$ defines a quotient $[\operatorname{Sym}^m \Bbbk^v \twoheadrightarrow \mathrm{H}^0(C, \omega_C^{\otimes mk})] \in \operatorname{Gr}(q, \operatorname{Sym}^m \Bbbk^v)$, and modifying the choice of basis changes the quotient up to the action by PGL_v. This constructions extends to families of stable curves and defines a morphism of algebraic stacks, which we call the *classifying map*:

$$\begin{aligned} \overline{\mathcal{M}}_g &\to [\operatorname{Gr}(q, \operatorname{Sym}^m \Bbbk^v)/\,\mathrm{PGL}_v] \\ [C] &\mapsto \left[\operatorname{Sym}^m \Bbbk^v \cong \operatorname{Sym}^m \mathrm{H}^0(C, \omega_C^{\otimes k}) \twoheadrightarrow \mathrm{H}^0(C, \omega_C^{\otimes mk})\right]. \end{aligned}$$

The main idea is to leverage the projectivity of $\operatorname{Gr}(q, \operatorname{Sym}^m \Bbbk^v)$ to show that $\overline{M}_g$ is projective. This will require two properties: the quasi-finiteness of the classifying map and the nefness of the pluricanonical bundle $V = \pi_*(\omega^{\otimes k}_{\mathcal{U}_g/\overline{\mathcal{M}}_g})$.

Theorem 5.9.2 (Kollár's Criterion for Ampleness)**.** *Let X be a proper algebraic space over a field $\Bbbk$. Let $W = \operatorname{Sym}^m V \twoheadrightarrow Q$ be a surjection of vector bundles of rank $w = \operatorname{rk} W$ and $q = \operatorname{rk} Q$ with $v = \operatorname{rk} V$. Suppose that*

(a) the classifying map

$$\begin{aligned} X &\to [\operatorname{Gr}(q, \operatorname{Sym}^m \Bbbk^v)/\,\mathrm{PGL}_v] \\ x &\mapsto [W \otimes \kappa(x) \twoheadrightarrow Q \otimes \kappa(x)] \end{aligned}$$

is quasi-finite, and

(b) *V is nef.*

Then $\det Q$ *is ample.*

We will elaborate on these hypotheses in §5.9.3, where we also prove the criterion, but let's first explain how it implies the projectivity of $\overline{M}_{g,n}$.

Theorem 5.9.3 (Projectivity of $\overline{M}_g$)**.** *For $g \geq 2$ and N sufficiently divisible, the line bundle $\lambda_{18g-18}^{\otimes N}$ on $\overline{\mathcal{M}}_g$ descends to a line bundle on $\overline{M}_g$ which is relatively ample over* $\operatorname{Spec} \mathbb{Z}$*. In particular, $\overline{M}_g$ is projective over* $\operatorname{Spec} \mathbb{Z}$*.*

Proof. By Properness of $\overline{\mathcal{M}}_g$ (5.5.23), $\overline{\mathcal{M}}_g$ is a Deligne–Mumford stack proper over $\operatorname{Spec} \mathbb{Z}$ and the coarse moduli space $\overline{M}_g$ is an algebraic space proper over $\operatorname{Spec} \mathbb{Z}$. By Exercise 4.4.29, $\lambda_{18g-18}^{\otimes N}$ descends to a line bundle L on $\overline{M}_g$ for N sufficiently divisible. Since Ampleness is Open (4.6.17), it suffices show the result in the case that $\overline{\mathcal{M}}_g$ is defined over a field $\Bbbk$. By Proposition 5.3.23, $\omega^{\otimes 3}_{\mathcal{U}_g/\overline{\mathcal{M}}_g}$ is relatively very ample and $\pi_*(\omega_{\mathcal{U}_g/\overline{\mathcal{M}}_g})^{\otimes 3}$ is a vector bundle of rank $5g-5$. For a stable curve C of genus g, the complete linear series $|\omega_C^{\otimes 3}|$ embeds C as a degree $m := 6g-6$ curve in $\mathbb{P}^{5g-6}$. As C is cut out by degree m equations, the multiplication map

$$W := \operatorname{Sym}^m \pi_*(\omega^{\otimes 3}_{\mathcal{U}_g/\overline{\mathcal{M}}_g}) \twoheadrightarrow \pi_*(\omega^{\otimes 3m}_{\mathcal{U}_g/\overline{\mathcal{M}}_g}) =: Q$$

is a surjection of vector bundles on $\overline{\mathcal{M}}_g$ where W has rank $w := \binom{5g-6+m}{m}$ and Q has rank $q := (6m-1)(g-1)$.

The classifying map $\overline{\mathcal{M}}_g \to [\operatorname{Gr}(q, \Bbbk^w)/G]$ is injective on $\Bbbk$-points since the kernel of the multiplication map uniquely determines a stable curve. Since the automorphism group $\operatorname{Aut}(C)$ of a stable curve is identified with the stabilizer of $[C \subseteq \mathbb{P}^{5g-6}] \in \operatorname{Hilb}^P(\mathbb{P}^{5g-6})$ under the action of PGL_{5g-5}, the multiplication map induces isomorphisms of automorphism groups. Thus, the classifying map is quasi-finite. The nefness of V is established in Theorem 5.9.21. By Le Lemme de Gabber (4.6.1), we may choose a finite cover $Z \to \overline{\mathcal{M}}_g$ by a scheme. Applying Kollár's Criterion (5.9.2) on Z shows that the pullback of $\lambda_{18g-18} = \det Q$ to Z is ample. This shows that the pullback of L under the finite morphism $Z \to \overline{\mathcal{M}}_g \to \overline{M}_g$ is ample, and therefore L is ample on $\overline{M}_g$ (see Exercise 4.5.23). □

Remark 5.9.4. In fact, the tricanonical embedding $|\omega_C^{\otimes 3}| \colon C \hookrightarrow \mathbb{P}^{5g-6}$ of a stable curve C is projectively normal and defined by *quadratic equations*; see [Mum70b, p. 58]. Therefore, the multiplication map $\operatorname{Sym}^2 \pi_*(\omega^{\otimes 3}_{\mathcal{U}_g/\overline{\mathcal{M}}_g}) \twoheadrightarrow \pi_*(\omega^{\otimes 6}_{\mathcal{U}_g/\overline{\mathcal{M}}_g})$ is surjective, and the same argument shows that a sufficiently divisible tensor power of λ_6 descend to an ample line bundle on $\overline{M}_g$.

Corollary 5.9.5 (Projectivity of $\overline{M}_{g,n}$)**.** *For integers g and n with $2g-2+n > 0$, $\overline{M}_{g,n}$ is projective over* $\operatorname{Spec} \mathbb{Z}$*.*

Proof. We know that $\overline{M}_{0,3} = \operatorname{Spec} \mathbb{Z}$ and $\overline{M}_{1,1} = \mathbb{P}^1$. By the above theorem, $\overline{M}_g$ is also projective for $g \geq 2$. Since $\overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ is the universal family (Proposition 5.6.12), it is a family of stable curves and $L := \omega_{\overline{\mathcal{M}}_{g,n+1}/\overline{\mathcal{M}}_{g,n}}$ is relatively ample over $\overline{\mathcal{M}}_{g,n}$ (Proposition 5.3.23). Therefore, some power of L descends to a line bundle on $\overline{M}_{g,n+1}$ relatively ample over $\overline{M}_{g,n}$ (Exercise 4.4.29(2)). Thus, $\overline{M}_{g,n+1} \to \overline{M}_{g,n}$ is a projective morphism, and we obtain by induction that $\overline{M}_{g,n}$ is projective if $2g-2+n > 0$. □

Remark 5.9.6. Alternatively, the projectivity of $\overline{M}_{g,n}$ can be shown using Kollár's Criterion (5.9.2) after establishing that $\pi_*(L^{\otimes 3})$ is nef, where $L := \omega_{\mathcal{U}_{g,n}/\overline{\mathcal{M}}_{g,n}}(\sigma_1 + \cdots + \sigma_n)$ and $\sigma_1, \ldots, \sigma_n$ are sections of the universal family; see [Kol90, Prop. 4.7].

In recent years, Kollár's Criterion has been applied in more and more general settings, e.g., Hassett's moduli space of weighted pointed curves [Has03], the moduli of stable varieties of any dimension [KP17], and the moduli of K-polystable Fano varieties [CP21, XZ20]

Outline of this section. We discuss ampleness criteria for algebraic spaces in §5.9.2, we prove Kollar's Criterion for Ampleness in §5.9.3, and establish the nefness of pluricanonical bundles in §5.9.4.

5.9.2 Positivity and ampleness criteria

We extend the notions of positivity for line bundles introduced in §B.2.2 from schemes to algebraic spaces, and we also extend the notion of nef vector bundles from §B.2.4 to algebraic spaces.

Definition 5.9.7. A line bundle L on a quasi-compact algebraic stack $\mathcal{X}$ is called:

(1) *base point free* if for every $x \in |\mathcal{X}|$, there exists $s \in \Gamma(X, L)$ with $s(x) \neq 0$, and

(2) *semiample* if $L^{\otimes n}$ is base point free for some $n > 0$.

A line bundle L on a proper algebraic space X over a field $\Bbbk$ is called:

(3) *ample* if X is a scheme and L is ample in the usual sense (see Proposition B.2.8),

(4) *nef* (resp., *strictly nef*) if for every integral closed curve[5] $C \subseteq X$, $L \cdot C = \deg L|_C \geq 0$ (resp., > 0), and

(5) *big* if there is a constant C such that $\mathrm{h}^0(Z, L|_Z^{\otimes m}) \geq Cm^{\dim(Z)}$ for all $m \geq 0$.

A vector bundle E on a proper algebraic space X is *nef* if $\mathcal{O}_{\mathbb{P}(E)}(1)$ is nef on $\mathbb{P}(E)$.

If $f\colon X' \to X$ is a finite surjective map of algebraic spaces, then a line bundle L on X is ample if and only if f^*L is (Exercise 4.5.23).

Lemma 5.9.8. *Let X be a proper algebraic space over a field $\Bbbk$.*

(1) If $f\colon X' \to X$ is a surjective proper morphism of algebraic spaces, then a vector bundle E is nef if and only if f^*E is.

(2) If a vector bundle E is nef, then so is $\det E$.

(3) If $f\colon X' \to X$ is a generically quasi-finite and proper morphism of integral algebraic spaces, then a line bundle L is big if and only if f^*L is.

Proof. It suffices to prove (1) for a line bundle L. This follows from the equality $\deg(f^*L)|_{C'} = \deg(C' \to C)\deg L|_C$ for an integral subcurve $C' \subseteq X'$ with image $C \subseteq X$. For (2) , we apply Le Lemme de Gabber (4.6.1) to obtain a finite cover $f\colon X' \to X$ by a scheme. By part (1), it suffices to show that $f^*\det(E) = \det(f^*E)$ is nef, but this follows from the case of schemes (Proposition B.2.15). Part (3) can be shown using the same argument as in the case of schemes (Proposition B.2.24). □

The following exercise extending Lemma B.2.26 provides clarity into our ampleness strategy even though we do not apply it.

Exercise 5.9.9 (easy)**.** Let $\mathcal{X}$ be a proper Deligne-Mumford stack over a field $\Bbbk$ with coarse moduli space $\mathcal{X} \to X$. If a line bundle L on $\mathcal{X}$ is semiample and strictly nef, then $L^{\otimes N}$ descends to an ample line bundle on X for some $N > 0$, and X is projective.

[5] By Theorem 4.5.32, every such closed integral curve is a projective scheme.

The semiampleness hypothesis in the above exercise can be very challenging to verify in practice. In the GIT approach, semiampleness is hard-coded into the definition of semistability (Definition 7.2.1): given a reductive group G acting linearly on projective space $\mathbb{P}(V)$, then $v \neq 0 \in V$ is *semistable* if and only if there exists $s \in \Gamma([\mathbb{P}(V)/G], \mathcal{O}(d))$ with $s(v) \neq 0$ and $d > 0$. In other words, the semistable locus $\mathbb{P}(V)^{\mathrm{ss}}$ is the largest open subscheme such that $\mathcal{O}(1)$ is semiample on $[(\mathbb{P}(V) \setminus \mathbb{P}(V)^{\mathrm{ss}})/G]$, i.e., $[\mathbb{P}(V)^{\mathrm{ss}}/G]$ is the stable base locus of $\mathcal{O}(1)$ on $[\mathbb{P}(V)/G]$. The Hilbert–Mumford Criterion (7.4.4) provides an effective way to verify semistability.

In the Nakai–Moishezon Criterion for Ampleness (B.2.28) for a scheme X, the semiampleness of a line bundle $L = \mathcal{O}_X(D)$ is obtained from the nefness of L and from the existence of sections of $L|_Z^{\otimes N}$ for every integral closed subscheme Z. It also holds for algebraic spaces.

Theorem 5.9.10 (Nakai–Moishezon Criterion for Ampleness)**.** *Let X be a proper algebraic space over an algebraically closed field $\Bbbk$, and let L be a line bundle on X. The following are equivalent:*

(1) L is ample,

(2) L is nef and for every integral closed subscheme $Z \subseteq X$, $L|_Z$ is big, and

(3) L is strictly nef and for every integral closed subscheme $Z \subseteq X$, $L|_Z^{\otimes m}$ is effective for some $m > 0$.

Proof. The same argument as in the case of schemes (Theorem B.2.28) proves the non-trivial direction (3) $\Rightarrow$ (1). □

Remark 5.9.11 (Classical formulation). It is not hard to extend the definition of intersection numbers to algebraic spaces. Namely, one shows that $\chi(Z, L|_Z^{\otimes m})$ is a rational polynomial, and one defines $c_1(L)^{\dim Z} \cdot Z$ as the normalized coefficient of $m^{\dim Z}$; see [SP, Tag 0DN3]. For a nef line bundle L, $\chi(Z, L|_Z^{\otimes m}) = \mathrm{h}^0(Z, L|_Z^{\otimes m})$ for $m \gg 0$, and so the bigness of L is equivalent to $c_1(L)^{\dim Z} \cdot Z > 0$. This implies the classical formulation of the Nakai–Moishezon Criterion for an algebraic space X:

$$L \text{ is ample} \iff c_1(L)^{\dim Z} \cdot Z > 0 \text{ for every integral closed subspace } Z \subseteq X.$$

Seshadri's Criterion (B.2.30) holds for proper algebraic spaces [Cor93], while Kleiman's Criterion (B.2.29) is unknown in general for proper algebraic spaces (and even proper schemes).

5.9.3 Proof of Kollár's Criterion for Ampleness

We discuss and prove Kollár's Criterion (5.9.2): if X is a proper algebraic space over a field $\Bbbk$ and $W = \operatorname{Sym}^m V \twoheadrightarrow Q$ is a surjection of vector bundles of rank $w = \operatorname{rk} W$ and $q = \operatorname{rk} Q$ with $v = \operatorname{rk} V$ such that the classifying map

$$\begin{aligned} X &\to [\operatorname{Gr}(q, \operatorname{Sym}^m \Bbbk^v)/\operatorname{PGL}_v] \\ x &\mapsto [W \otimes \kappa(x) \twoheadrightarrow Q \otimes \kappa(x)] \end{aligned}$$

is quasi-finite and V is nef, then $\det Q$ is ample.

Remark 5.9.12 (More general formulation). We offer an alternative formulation, which in addition to giving a more general statement offers a clue into the proof. Suppose that $W \twoheadrightarrow Q$ is a surjection of vector bundles on X of rank w and q such that

(a) W has a reduction of structure group by a homomorphism $G \to \mathrm{GL}_w$ from an affine algebraic group G such that the closure of the image of G in the projective space $\mathbb{P}(\mathrm{Mat}_{w,w})$ of $w \times w$ matrices is normal;

(b) W is nef;

(c) the classifying map $X \to [\mathrm{Gr}(q, \Bbbk^w)/G]$ is quasi-finite; and

(d) either G is linearly reductive (e.g., $\mathrm{char}(\Bbbk) = 0$ and G is reductive), or $G = \mathrm{GL}_{v_1} \times \cdots \times \mathrm{GL}_{v_d}$ with $G \to \mathrm{GL}_w$ a semipositive representation (i.e., it extends to a map $\mathrm{Mat}_{v_1,v_1} \times \cdots \times \mathrm{Mat}_{v_d,v_d} \to \mathrm{Mat}_{w,w}$);

then $\det Q$ is ample [Kol90, Prop. 3.6, Lem. 3.9]. Kollár's Criterion deduces the ampleness of $\det Q$ from the ampleness of the Plücker line bundle on $\mathrm{Gr}(q, \Bbbk^w)$, generalizing the fact that a proper scheme quasi-finite over $\mathrm{Gr}(q, \Bbbk^w)$ is projective. Indeed, when W is the trivial vector bundle, there is a reduction of structure group to the trivial group $G = \{1\}$ and the quasi-finiteness of the classifying map $X \to \mathrm{Gr}(q, \Bbbk^w)$ implies that it is finite, and hence the pullback $\det(Q)$ of the Plücker line bundle is ample. The main idea of the proof is to trivialize W, essentially reducing to this case.

Remark 5.9.13 (Unwinding quasi-finiteness). From Definition 3.3.34, the quasi-finiteness of the classifying map $X \to [\mathrm{Gr}(q, \Bbbk^w)/G]$ translates to: (i) $X(\overline{\Bbbk}) \to \mathrm{Gr}(q, \Bbbk^w)(\overline{\Bbbk})$ has finite fibers for every algebraically closed field $\overline{\Bbbk}$, and (ii) for $x \in X(\overline{\Bbbk})$, only finitely many elements of $G(\overline{\Bbbk})$ leave $\ker(\overline{\Bbbk}^w \cong W \otimes \overline{\Bbbk} \to Q \otimes \overline{\Bbbk})$ invariant.

In fact, as observed in [KP17, Thm. 5.1], the quasi-finiteness of the classifying map in (c) can be replaced with the *set-theoretic* quasi-finiteness of $X(\overline{\Bbbk}) \to \mathrm{Gr}(q, \Bbbk^w)(\overline{\Bbbk})$. Letting $\Gamma \colon X \to X \times [\mathrm{Gr}(q, \Bbbk^w)/G]$ be the graph of the classifying map, the set-theoretic quasi-finite is equivalent to the quasi-finiteness of $\mathrm{im}(\Gamma) \to [\mathrm{Gr}(q, \Bbbk^w)/G]$, which is enough to make the proof below work.

Remark 5.9.14 (Stability). The above theorem does not require that the image of X lands in the G-stable locus of $\mathrm{Gr}(q, \Bbbk^w)$. However, if this happens, then the composition $X \to [\mathrm{Gr}(q, \Bbbk^w)^{\mathrm{s}}/G] \to \mathrm{Gr}(q, \Bbbk^w)/\!\!/G$ is a quasi-finite morphism of proper algebraic spaces. It is hence finite and $\det Q$ is ample as it is the pullback of an ample line bundle on $\mathrm{Gr}(q, \Bbbk^w)/\!\!/G$.

We now prove Kollár's Criterion for Ampleness [Kol90, Lem. 3.9]. See also [KP17, Thm. 5.1] and the excellent exposition in [CLM22]. The proof will proceed by reducing the ampleness of $\det(Q)$ to its bigness, which we establish in Proposition 5.9.15.

Proof of Theorem 5.9.2. By the Nakai–Moishezon Criterion for Ampleness (5.9.10), it suffices to show that $\det(Q)$ is nef and that $\det(Q)|_Z$ is big for each integral subspace $Z \subseteq X$. Since W is nef, so is the quotient Q, and Lemma 5.9.8(1) implies that $\det(Q)$ is also nef. If $Z \subseteq X$ is an integral subspace, by Chow's Lemma (4.6.5), there exists a generically quasi-finite and proper morphism $Z' \to Z$ from a normal projective scheme. Considering the induced surjection $W_{Z'} = \mathrm{Sym}^m V_{Z'} \to Q_{Z'}$ of the restrictions to Z', the classifying map $Z' \to X \to [\mathrm{Gr}(q, \mathrm{Sym}^m \Bbbk^v)/\mathrm{PGL}_v]$ is generically quasi-finite and $V_{Z'}$ is nef. Proposition 5.9.15 implies that $\det(Q_{Z'})$ is big, and Lemma 5.9.8(3) further implies that $\det(Q)$ is big. $\square$

Assuming that X is projective, we establish the weaker conclusion that $\det Q$ is big under the weaker hypothesis that the classifying map is *generically* finite. The proof below exploits the birational invariance of bigness by blowing up a compactification of the projectivized frame bundle $\mathbb{P}\mathrm{Fr}_W$ to $\mathrm{Gr}(q, \Bbbk^w)$ to construct sections.

Proposition 5.9.15. *Let X be a normal projective scheme over a field $\Bbbk$. Let $W = \mathrm{Sym}^m V \twoheadrightarrow Q$ be a surjection of vector bundles of rank $w = \mathrm{rk}\, W$ and $q = \mathrm{rk}\, Q$ with $v = \mathrm{rk}\, V$. Suppose that*

(a) the classifying map $X \to [\mathrm{Gr}(q, \mathrm{Sym}^m \Bbbk^v)/\, \mathrm{PGL}_v]$ is generically quasi-finite, and

(b) V is nef.

Then $\det Q$ is big.

Proof. Step 1: Trivialize V. Define the relative projective space

$$\mathbb{P} := \mathbb{P}(\mathscr{H}om_{\mathcal{O}_X}(V, \mathcal{O}_X^{\oplus v}))$$

over X, and let $\mathbb{P}\mathrm{Fr}_V \subseteq \mathbb{P}$ be the open subscheme defined by the non-vanishing of the determinant of the natural map

$$\mathcal{O}_{\mathbb{P}}^{\oplus v} \to V_{\mathbb{P}} \otimes \mathcal{O}_{\mathbb{P}}(1), \tag{5.9.16}$$

induced from the universal quotient $\mathscr{H}om_{\mathcal{O}_{\mathbb{P}}}(V_{\mathbb{P}}, \mathcal{O}_{\mathbb{P}}^{\oplus w}) \to \mathcal{O}_{\mathbb{P}}(1)$, where $V_{\mathbb{P}}$ is the pullback of V via $\mathbb{P} \to X$. The projection $\mathbb{P}\mathrm{Fr}_V \to X$ is a principal PGL_v-bundle, and in fact is precisely the projectivized frame bundle of V (see Exercise B.1.55). The map (5.9.16) is surjective over $\mathbb{P}\mathrm{Fr}_V$ and defines a morphism $\mathbb{P}\mathrm{Fr}_V \to \mathrm{Gr}(q, \mathrm{Sym}^m \Bbbk^v)$ which sits in a cartesian diagram

$$\begin{tikzcd}
\mathbb{P} \arrow[dr, "p"'] & \mathbb{P}\mathrm{Fr}_W \arrow[l, hook'] \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathrm{Gr}(q, \mathrm{Sym}^m \Bbbk^v) \arrow[d] \\
& X \arrow[r] & {[\mathrm{Gr}(q, \mathrm{Sym}^m \Bbbk^v)/\, \mathrm{PGL}_w]}\,.
\end{tikzcd}$$

Observe that since $\mathbb{P} \cong \mathbb{P}((V^\vee)^{\oplus v})$, the pushforward $p_*\mathcal{O}_{\mathbb{P}}(N)$ is identified with $\mathrm{Sym}^N((V^\vee)^{\oplus v})$. Since V is nef, it follows from properties of nefness (Proposition B.2.33(3)) that $(p_*\mathcal{O}_{\mathbb{P}}(N))^\vee$ is nef, which is how we use hypothesis (b) below.

Step 2: Blow up $\mathbb{P}$ to extend $\mathbb{P}\mathrm{Fr}_W \to \mathrm{Gr}(q, \mathrm{Sym}^m \Bbbk^v)$ to a map $\mathbb{P}' \to \mathrm{Gr}(q, \mathrm{Sym}^m \Bbbk^v)$. If the composition

$$\mathrm{Sym}^m \mathcal{O}_{\mathbb{P}}^{\oplus v} \to \mathrm{Sym}^m V_{\mathbb{P}} \otimes \mathcal{O}_{\mathbb{P}}(m) \to Q_{\mathbb{P}} \otimes \mathcal{O}_{\mathbb{P}}(m)$$

induced from (5.9.16) were surjective, there would be no need to blow up. Otherwise, taking the qth wedge power gives a map $\bigwedge^q \mathrm{Sym}^m \mathcal{O}_{\mathbb{P}}^{\oplus v} \to \det(Q_{\mathbb{P}}) \otimes \mathcal{O}_{\mathbb{P}}(mq)$. Letting I be the image subsheaf, define $\pi\colon \mathbb{P}' \to \mathbb{P}$ as the blowup of the ideal sheaf $I \otimes \det(Q_{\mathbb{P}})^\vee \otimes \mathcal{O}_{\mathbb{P}}(-mq) \subseteq \mathcal{O}_{\mathbb{P}}$. Then $\mathbb{P}' \to \mathbb{P}$ is an isomorphism over $\mathbb{P}\mathrm{Fr}_W$ and there is a map $\mathbb{P}' \to \mathrm{Gr}(q, \mathrm{Sym}^m \Bbbk^v)$ extending $\mathbb{P}\mathrm{Fr}_W \to \mathrm{Gr}(q, \mathrm{Sym}^m \Bbbk^v)$ yielding a commutative diagram

$$\begin{tikzcd}
E \arrow[d, dash] & H \arrow[d, dash] & L \arrow[d, dash] \\
\mathbb{P}' \arrow[r, "\pi"'] \arrow[rr, bend left] & \mathbb{P} \arrow[r, dashed] \arrow[d, "p"] & \mathrm{Gr}(q, \mathrm{Sym}^m \Bbbk^v) \arrow[d] \\
& X \arrow[r] & {[\mathrm{Gr}(q, \mathrm{Sym}^m \Bbbk^v)/\, \mathrm{PGL}_w]},
\end{tikzcd}$$

where $E \subseteq \mathbb{P}'$ is the exceptional divisor, $H \subseteq \mathbb{P}$ is the hyperplane class, and L is the ample divisor on $\mathrm{Gr}(q, \mathrm{Sym}^m \Bbbk^v)$ corresponding to the Plücker embedding. Abusing notation, we also view $\det(Q)$ as a divisor on X. By using subscripts to denote the pullback of divisors, we have the formula

$$L_{\mathbb{P}'} = \det(Q)_{\mathbb{P}'} + mqH_{\mathbb{P}'} - E. \tag{5.9.17}$$

Step 3: Use the generic quasi-finiteness to construct sections of $\mathcal{O}_X(m\det(Q) - A) \otimes \mathrm{Sym}^{Nmq}((V^\vee)^{\oplus v})$ for an ample divisor A on X and some $N > 0$. Since $X \to [\mathrm{Gr}(q, \mathrm{Sym}^m \Bbbk^v)/\mathrm{PGL}_w]$ is generically quasi-finite, so is $\mathbb{P}' \to \mathrm{Gr}(q, \mathrm{Sym}^m \Bbbk^v)$. It follows from Proposition B.2.24 that $L_{\mathbb{P}'}$ is big. For every very ample divisor A on X, the pullback $A_{\mathbb{P}'}$ is effective, and Kodaira's Lemma (B.2.20) implies that there is an $N > 0$ such that $NL_{\mathbb{P}'} - A_{\mathbb{P}'}$ is effective. Using the identity (5.9.17), we obtain an inclusion of line bundles

$$\begin{aligned} \mathcal{O}_{\mathbb{P}'}(NL_{\mathbb{P}'} - A_{\mathbb{P}'}) &\cong \mathcal{O}_{\mathbb{P}'}(N\det(Q)_{\mathbb{P}'} + NmqH_{\mathbb{P}'} - NE_{\mathbb{P}'} - A_{\mathbb{P}'}) \\ &\subseteq \mathcal{O}_{\mathbb{P}'}(N\det(Q)_{\mathbb{P}'} + NmqH_{\mathbb{P}'} - A_{\mathbb{P}'}) \\ &\cong \pi^*\big(p^*\mathcal{O}_X(N\det(Q) - A) \otimes \mathcal{O}_{\mathbb{P}}(Nmq)\big) \end{aligned}$$

Choose a nonzero section $\mathcal{O}_{\mathbb{P}'} \to \pi^*\big(p^*\mathcal{O}_X(m\det(Q) - A) \otimes \mathcal{O}_{\mathbb{P}}(mq)\big)$. Since $\mathbb{P}$ is normal and $\pi \colon \mathbb{P}' \to \mathbb{P}$ is birational, $\pi_*\mathcal{O}_{\mathbb{P}'} = \mathcal{O}_{\mathbb{P}}$. Applying π_* to the chosen section and using the projection formula yields a nonzero section

$$\mathcal{O}_{\mathbb{P}} \to p^*\mathcal{O}_X(N\det(Q) - A) \otimes \mathcal{O}_{\mathbb{P}}(Nmq).$$

Applying p_* gives a nonzero section $\mathcal{O}_X \to \mathcal{O}_X(N\det(Q) - A) \otimes p_*\mathcal{O}_{\mathbb{P}}(Nmq)$, which we rearrange as

$$(p_*\mathcal{O}_{\mathbb{P}}(Nmq))^\vee \to \mathcal{O}_X(N\det(Q) - A). \tag{5.9.18}$$

Step 4: Blow up X so that (5.9.18) is surjective, and use the nefness of V to conclude that $\det(Q)$ is big. As pointed out earlier, the nefness of V implies that $(p_*\mathcal{O}_{\mathbb{P}}(Nmq))^\vee \cong \mathrm{Sym}^{Nmq}((V^\vee)^{\oplus v})^\vee$ is nef by Proposition B.2.33(3). Therefore, if (5.9.18) is surjective, then $N\det(Q) - A$ is nef which expresses $N\det(Q)$ as a nef and ample line bundle, which implies that $\det(Q)$ is even ample. Otherwise, if I is the image subsheaf of (5.9.18), we blow up the ideal sheaf $I \otimes \mathcal{O}_X(N\det(Q) - A)^\vee \subseteq \mathcal{O}_X$ to obtain a birational morphism $f \colon X' \to X$ with exceptional divisor E' and a surjection

$$f^*(p_*\mathcal{O}_{\mathbb{P}}(Nmq))^\vee \twoheadrightarrow M := f^*\big(\mathcal{O}_X(N\det(Q) - A)\big) \otimes \mathcal{O}_{X'}(-E').$$

Since $f^*(p_*\mathcal{O}_{\mathbb{P}}(Nmq))^\vee$ is nef, so is M. This expresses $N\det(Q_{X'})$ as the sum of a nef divisor M, a big divisor $A_{X'}$, and an effective divisor E'. By the Characterization of Bigness (B.2.21), a multiple of $A_{X'}$ is the sum of an ample divisor and effective divisor. Therefore, a multiple of $\det(Q_{X'})$ is the sum of a nef, ample, and effective divisors. Since nef plus ample is ample, and ample plus effective is big, we conclude that $\det(Q_{X'})$ is big. As $X' \to X$ is birational, it follows that $\det(Q)$ is big. □

5.9.4 Nefness of pluricanonical bundles

We prove the nefness of the pluricanonical bundles $\pi_*(\omega_{\mathcal{U}_g/\overline{\mathcal{M}}_g}^{\otimes k})$ on $\overline{\mathcal{M}}_g$ by showing that a negative line bundle quotient contradicts Ekedahl–Mumford Vanishing.

Theorem 5.9.19 (Ekedahl–Mumford Vanishing)**.** *Let X be a smooth projective surface over an algebraically closed field $\Bbbk$ which is minimal and of general type. If $n \geq 1$, then $\mathrm{h}^1(X, K_X^{\otimes -n}) \leq 1$. If $n > 1$ or $\mathrm{char}(\Bbbk) \neq 2$, then $\mathrm{h}^1(X, K_X^{\otimes -n}) = 0$.*

Proof. The characteristic zero version is [Mum67, Thm. 2], while the positive characteristic case is the main theorem of [Eke88]. $\square$

Corollary 5.9.20. *Let X be a smooth projective surface over an algebraically closed field $\Bbbk$ which is minimal and of general type. Let D be a reduced effective Cartier divisor such that each connected component of D has genus at least 2. If $n \geq 2$, then $\mathrm{h}^1(X, K_X^{\otimes n}(D)) \leq 1$. If $n > 2$ or $\mathrm{char}(\Bbbk) \neq 2$, then $h^1(X, K_X^{\otimes n}(D)) = 0$.*

Proof. By the short exact sequence $0 \to K_X^{\otimes n} \to K_X^{\otimes n}(D) \to K_X^{\otimes n}(D)|_D \to 0$ and the inequalities of Theorem 5.9.19, it suffices to show that $\mathrm{h}^1(X, K_X^{\otimes n}(D)|_D) = 0$ for $n \geq 2$. By adjunction, $K_X^{\otimes n}(D)|_D \cong \omega_D^{\otimes n}$. Since each connected component of D has genus at least 2, $\mathrm{h}^1(X, K_X^{\otimes n}(D)|_D) = \mathrm{h}^1(D, \omega_D^{\otimes n}) = 0$. $\square$

The nefness of $\pi_*(\omega_{\mathcal{U}_g/\overline{\mathcal{M}}_g}^{\otimes k})$ for $k \geq 2$ reduces to the following properties of families of stable curves over a smooth curve.

Theorem 5.9.21. *If T is a smooth, connected, and projective curve over a field $\Bbbk$ and $\pi\colon \mathcal{C} \to T$ is a family of stable curves, then $\pi_*(\omega_{\mathcal{C}/T}^{\otimes k})$ is nef for $k \geq 2$.*

Proof. Step 1: Reduction to characteristic p. Assume that $\mathrm{char}(\Bbbk) = 0$. Since T and $\mathcal{C}$ are of finite type over $\Bbbk$, their defining equations involve finitely many coefficients of $\Bbbk$. Thus there exists a finitely generated $\mathbb{Z}$-subalgebra $A \subseteq \Bbbk$ and a cartesian diagram

$$\begin{tikzcd}
\mathcal{C} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \widetilde{\mathcal{C}} \arrow[d] \\
T \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \widetilde{T} \arrow[d] \\
\operatorname{Spec} \Bbbk \arrow[r] & \operatorname{Spec} A
\end{tikzcd}$$

where $\widetilde{C}$ and $\widetilde{T}$ are schemes of finite type over A. By Limit Methods (B.3.7 and 5.2.24), we may further arrange that $\widetilde{T} \to \operatorname{Spec} A$ is a smooth family of curves and that $\widetilde{C} \to \widetilde{T}$ is a family of stable curves. Finally, by restricting along a map $\operatorname{Spec} R \to \operatorname{Spec} A$, we may assume that A is a DVR whose closed and generic points have characteristic $p > 0$ and 0, respectively. Since Nefness for Bundles is Stable under Generization (B.2.34), it suffices to prove the theorem when $\mathrm{char}(\Bbbk) = p > 0$.

Step 2: Further reductions. We claim that we may also assume that

(a) $\pi\colon \mathcal{C} \to T$ is generically smooth,

(b) the genus of T is at least 2, and

(c) $\mathcal{C}$ is a smooth minimal surface of general type.

The reduction to (a) is handled in Exercise 5.9.23. For (b), if $g\colon T' \to T$ is any finite cover where T' is a smooth connected curve of genus $g \geq 2$ and $\mathcal{C}' := \mathcal{C} \times_T T'$, then $g^*\pi_*(\omega_{\mathcal{C}/T}^{\otimes k}) \cong \pi'_*(\omega_{\mathcal{C}'/T'}^{\otimes k})$ by properties of the dualizing sheaf. By Lemma 5.9.8(1), the nefness of $\omega_{\mathcal{C}'/T'}^{\otimes k}$ implies the nefness of $\omega_{\mathcal{C}/T}^{\otimes k}$. For (c), if $f\colon \widetilde{\mathcal{C}} \to \mathcal{C}$ is a resolution of singularities, then $f_*(\omega_{\widetilde{\mathcal{C}}/T}^{\otimes k}) \cong \omega_{\mathcal{C}/T}^{\otimes k}$, and thus we can assume that $\mathcal{C}$ is smooth. If $\mathcal{C}$ contains a smooth rational -1 curve E, then E must be contained in a fiber of $\mathcal{C} \to T$ as otherwise there would be a finite cover $E = \mathbb{P}^1 \to T$ of a genus $g \geq 2$ curve.

By Castelnuovo's Contraction Theorem (B.2.6), there is a morphism $f\colon \mathcal{C} \to \mathcal{C}'$ contracting E. Since $f_*\omega_{\mathcal{C}/T}^{\otimes k} = \omega_{\mathcal{C}'/T}^{\otimes k}$ and the process of contracting smooth rational -1 curves terminates in finitely many steps, we can assume that $\mathcal{C}$ is minimal. As both T and the generic fiber of $\mathcal{C} \to T$ are smooth and of general type, $\mathcal{C}$ is also of general type.

Step 3: Positive characteristic case. If $\pi_*(\omega_{\mathcal{C}/T}^{\otimes k})$ is not nef, then there exists a quotient line bundle $\pi_*(\omega_{\mathcal{C}/T}^{\otimes k}) \twoheadrightarrow M$ where $d = \deg M < 0$. Consider the absolute Frobenius morphisms $F\colon \mathcal{C} \to \mathcal{C}$ and $F\colon T \to T$ which fit into a commutative diagram

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, "\pi"] & \mathcal{C} \arrow[d, "\pi"] \\ T \arrow[r, "F"] & T. \end{tikzcd}$$

Note that $F^*\pi_*(\omega_{\mathcal{C}/T}^{\otimes k}) = \pi_*(\omega_{\mathcal{C}/T}^{\otimes k})$. Since $\deg F^*M = pd$, we can apply Frobenius repeatedly to arrange that d is as small as we want. Specifically, we can arrange that $M^\vee \cong \omega_T^{\otimes k} \otimes L$ where L is a very ample line bundle on T. (This was the entire point of reducing to characteristic p: to repeatedly apply Frobenius to arbitrarily decrease the degree of M.)

The surjection $\pi_*(\omega_{\mathcal{C}/T}^{\otimes k}) \twoheadrightarrow M \cong (\omega_T^{\otimes k} \otimes L)^\vee$ yields a surjection

$$\pi_*(\omega_{\mathcal{C}/T}^{\otimes k}) \otimes \omega_T^{\otimes k} \otimes L \twoheadrightarrow \mathcal{O}_T$$

of vector bundles on T. Since $\dim T = 1$, we obtain that

$$\mathrm{h}^1(T, \pi_*(\omega_{\mathcal{C}/T}^{\otimes k}) \otimes \omega_T^{\otimes k} \otimes L) \geq \mathrm{h}^1(T, \mathcal{O}_T) \geq 2. \tag{5.9.22}$$

To obtain a contraction, we examine $\mathrm{h}^1(\mathcal{C}, \omega_{\mathcal{C}}^{\otimes k} \otimes \pi^*L)$, which by the degeneration of the Leray spectral sequence on the E_2-page is bounded below by $\mathrm{h}^1(T, \pi_*(\omega_{\mathcal{C}}^{\otimes k} \otimes \pi^*L))$. We obtain an inequalities

$$\begin{aligned} \mathrm{h}^1(\mathcal{C}, \omega_{\mathcal{C}}^{\otimes k} \otimes \pi^*L) &\geq \mathrm{h}^1(T, \pi_*(\omega_{\mathcal{C}}^{\otimes k} \otimes \pi^*L)) \\ &= \mathrm{h}^1(T, \pi_*(\omega_{\mathcal{C}/T}^{\otimes k} \otimes \pi^*\omega_T^{\otimes k} \otimes \pi^*L)) \quad && (\text{since } \omega_{\mathcal{C}} \cong \omega_{\mathcal{C}/T} \otimes \pi^*\omega_T) \\ &= \mathrm{h}^1(T, \pi_*(\omega_{\mathcal{C}/T}^{\otimes k}) \otimes \omega_T^{\otimes k} \otimes L) && (\text{projection formula}) \\ &\geq \mathrm{h}^1(T, \mathcal{O}_T) \geq 2 && (\text{by (5.9.22)}). \end{aligned}$$

As L is very ample, there is an effective divisor D which is the union of smooth fibers of $\pi\colon \mathcal{C} \to T$ such that $\mathcal{O}_{\mathcal{C}}(D) \cong \pi^*L$. The above inequality contradicts Corollary 5.9.20 applied to D. See also [Kol90, Thm. 4.3] and [CLM22, Thm. 6.10]. □

Exercise 5.9.23 (details)**.** Let T be a smooth, connected, and projective curve over a field $\Bbbk$. Show that if $\pi_*(\omega_{\mathcal{C}/T}^{\otimes k})$ is nef for every *generically smooth* stable family, then it is also nef for every stable family.

5.10 Glimpse of the geometry of $\overline{M}_g$

When [Zariski] spoke the words algebraic variety, there was a certain resonance in his voice that said distinctly that he was looking into a secret garden. I immediately wanted to be able to do this too… Especially, I became obsessed with a kind of passion flower in this garden, the moduli spaces of Riemann.

DAVID MUMFORD [AI97, P. 225]

The existence, irreducibility, and projectivity (Theorem A) is of course just the beginning in the study of $\overline{M}_g$. We can now begin asking all sorts of geometric questions:

- What is its ample or effective cone?
- What is its Kodaira dimension or canonical model?
- What are the singularities of the coarse moduli space?
- What is its singular, Chow, Hodge, or crystalline cohomology?
- For integers r and d, does a general curve of genus g have a g^r_d?

Note that the projectivity of $\overline{M}_g$ is necessary to even discuss ample line bundles and other birational properties. On the other hand, the irreducibility of $\overline{M}_g$ allows us to translate a property about a general curve into the existence of a non-empty open locus of curves in $\overline{M}_g$ with the given property, and when the property is an open condition, it suffices to exhibit a single smoothable curve with the given property.

We can't possibly give a comprehensive summary; after all, $\overline{M}_g$ is perhaps the single most studied variety over the last sixty years. We refer the reader to [Mum83], [ACGH85], [HM98], [ACG11], and [FM13]. We will however make a few cursory comments.

Ample cone. For each $k \geq 5$ and $m \gg 0$, the GIT construction of $\overline{M}_g$ using the Hilbert scheme constructs a line bundle on $\overline{\mathcal{M}}_g$ which descends to an ample line bundle on $\overline{M}_g$. This class on $\overline{\mathcal{M}}_g$ is proportional to

$$r(k)\lambda_{mk} - r(mk)\lambda_k$$

where if we set $E_k = \pi_*(\omega^{\otimes k}_{\mathcal{U}_g/\overline{\mathcal{M}}_g})$, then $r(k) = \operatorname{rk} E_k$ and $\lambda_k = \det E_k$. Grothendieck–Riemann–Roch can be used to express each of the line bundles $\lambda_k = (6k^2 - 6k + 1)\lambda - \binom{k}{2}\delta$ as a linear combination of $\lambda := \lambda_1$ and δ, the boundary divisor. The asymptotic limit of this class as $d \to \infty$ is proportional to

$$(12 - \frac{4}{k})\lambda - \delta,$$

which is also proportional to the ample class induced by the GIT construction using the Chow scheme; see [Mum77, Thm. 5.15] and [HH13, Prop. 5.2]. Taking $k = 5$, shows that $11.2\lambda - \delta$ is ample. On the other hand, Kollár's construction shows that λ_{km} descends to ample line on $\overline{M}_g$ for $k \geq 3$ and $m \geq 2$, and this shows that $(12 + \epsilon)\lambda - \delta$ is ample for any $0 < \epsilon \ll \infty$.

However, even more is true! By bootstrapping the positivity deduced from GIT, Cornalba and Harris showed that $a\lambda - \delta$ is ample if and only if $a > 11$, thus determining the ample cone of $\overline{M}_g$ in the λ-δ-plane of $\operatorname{Pic}(\overline{M}_g)$ [CH88]. The full ampleness cone is not known, but there is an interesting conjecture called the *F-conjecture*, which provides a finite list of test curves and asserts that a line bundle is ample if and only if it intersects each of these test curves positively. We will not enumerate these test curves here, but it includes for instance the rational curves in $\overline{\mathcal{M}}_g$ parameterizing *elliptic tails* obtained by fixing a one-pointed (C, p) curve of genus $g-1$ and varying the j-invariant of a nodally-attached elliptic curve (E, p).

Kodaira dimension. Severi showed that $\overline{M}_g$ is unirational for $g \leq 10$ and conjectured that it unirational for all g; see the discussion §0.1.2. Harris and Mumford showed that this conjecture not only fails, but that it fails spectacularly [HM98]. Quite the opposite is true: $\overline{M}_g$ is general type for odd $g \geq 25$. Their strategy was to find an ample divisor A and effective divisor D such that the canonical bundle $K_{\overline{M}_g}$ is proportional to $A + D$, as this implies that $K_{\overline{M}_g}$ is big. Defining the *slope of* D as the infimum of a/b such that $a\lambda - b\delta - D$ is effective and appealing to the formula $K_{\overline{\mathcal{M}}_g} = 13\lambda - 2\delta$, it suffices to find an effective divisor of slope less than $13/2$. Writing $g+1$ as a product $(r+1)(s+1)$, they define the *Brill–Noether divisor* $D_{r,s}$ as the closure of curves possessing a g^r_d, and show that its slope is less than $13/2$. Eisenbud and Harris extend this to all genera $g \geq 24$ [EH87].

Given that $\overline{M}_g$ is general type, it is natural to ask what is its canonical model $\overline{M}_g^{\mathrm{can}} = \operatorname{Proj} \bigoplus_d \Gamma(\overline{\mathcal{M}}_g, K_{\overline{\mathcal{M}}_g}^{\otimes d})$, which by [BCHM10], is a projective variety. Given that $\overline{M}_g$ is a particular interesting higher dimensional variety and that its modular description provides a handle on its geometry, it is tempting to test drive the machinery of the minimal model program on $\overline{M}_g$. Quite spectacularly, Hassett and Hyeon realized that the first contraction and flip also carry moduli descriptions [HH09], [HH13]. For instance, the line bundle $K_{\overline{\mathcal{M}}_g} + 9/11\delta$, which is proportional to $11\lambda - \delta$, is nef and has zero intersection with precisely the curves of elliptic tails, and these curves are replaced in the first contraction with cuspidal singularities $y^2 = x^3$.

Singularities of $\overline{M}_g$. First, since $\overline{\mathcal{M}}_g$ is smooth, the Local Structure of Coarse Moduli Spaces (4.4.13) implies that $\overline{M}_g$ has finite quotient singularities. By appealing to the Reid-Tai criterion, Harris and Mumford showed that $\overline{M}_g$ has canonical singularities [HM82], or, in other words, that pluricanonical sections extend to a desingularization of $\overline{M}_g$, a property necessary for their proof that $\overline{M}_g$ is general type.

A general smooth curve $C \in \mathcal{M}_g(\Bbbk)$ of genus $g \geq 3$ has a trivial automorphism group. Since $\mathcal{M}_g$ is irreducible, this follows from exhibiting a single smooth curve with a trivial automorphism groups. It implies that there is an open subset $U \subseteq \mathcal{M}_g$ which is a smooth quasi-projective variety. If $g \geq 4$, the locus of curves with a non-trivial automorphism has codimension at least 2 and a curve $[C] \in M_g$ is a singular point if and only if $\operatorname{Aut}(C) \neq 1$. See [Rau62] and [Pop69].

Chapter 6

Geometry of algebraic stacks

The Red Queen shook her head. "You may call it 'nonsense' if you like," she said, "but I've heard nonsense, compared with which that would be as sensible as a dictionary!"

Lewis Carroll, *Through the Looking-Glass*

6.1 Quasi-coherent sheaves and cohomology theories

We define quasi-coherent sheaves on an algebraic stack in essentially the same way as we did for Deligne–Mumford stacks in §4.1, but using the lisse-étale site on $\mathcal{X}$ instead of the small étale site.

6.1.1 Sheaves and $\mathcal{O}_{\mathcal{X}}$-modules

The smallest site suitable to define sheaves on a general algebraic stack is the lisse-étale topology. This site has some technical issues, most notably the lack of functoriality; see Caution 6.1.5.

Definition 6.1.1 (Lisse-étale site)**.** The *lisse-étale site* $\mathcal{X}_{\text{lis-ét}}$ on an algebraic stack $\mathcal{X}$ is the category of schemes smooth over $\mathcal{X}$, where morphisms are arbitrary maps of schemes over $\mathcal{X}$. A covering $\{U_i \to U\}$ is a collection of morphisms such that $\coprod_i U_i \to U$ is surjective and étale.

We denote by $\mathrm{Sh}(\mathcal{X}_{\text{lis-ét}})$ (resp., $\mathrm{Ab}(\mathcal{X}_{\text{lis-ét}})$) the category of sheaves (resp., sheaves of abelian groups) $\mathcal{X}_{\text{lis-ét}}$. The structure sheaf $\mathcal{O}_{\mathcal{X}}$ is defined as $\mathcal{O}_{\mathcal{X}}(U) = \Gamma(U, \mathcal{O}_U)$, and allows us to define the notion of a $\mathcal{O}_{\mathcal{X}}$-module. For $F \in \mathrm{Ab}(\mathcal{X}_{\text{lis-ét}})$, we can define sections over an arbitrary smooth morphism $\mathcal{U} \to \mathcal{X}$ of algebraic stacks by the formula $F(\mathcal{U} \to \mathcal{X}) = \mathrm{Eq}(F(U \to \mathcal{X}) \rightrightarrows F(R \to \mathcal{X}))$ where $U \to \mathcal{X}$ is a smooth presentation and $R \to U \times_{\mathcal{X}} U$ is an étale presentation by schemes. As in the case of Deligne–Mumford stacks in §4.1.1, this is independent of the choice of presentation. In particular, the global sections $\Gamma(\mathcal{X}, F) := F(\mathcal{X} \xrightarrow{\text{id}} \mathcal{X})$ is well-defined.

Given a morphism $f\colon \mathcal{X} \to \mathcal{Y}$ of algebraic stacks, there are adjoint functors

$$\mathrm{Sh}(\mathcal{X}_{\text{lis-ét}}) \underset{f^{-1}}{\overset{f_*}{\rightleftarrows}} \mathrm{Sh}(\mathcal{Y}_{\text{lis-ét}}) \quad \text{and} \quad \mathrm{Ab}(\mathcal{X}_{\text{lis-ét}}) \underset{f^{-1}}{\overset{f_*}{\rightleftarrows}} \mathrm{Ab}(\mathcal{Y}_{\text{lis-ét}}),$$

where the *pushforward* is defined by $f_*F(V \to \mathcal{Y}) := F(\mathcal{X} \times_{\mathcal{Y}} V \to \mathcal{X})$ and the *inverse image* $f^{-1}G$ is the sheafification of the presheaf

$$(f^{-1}G)^{\text{pre}} \colon (U \to \mathcal{X}) \mapsto \operatorname{colim}_{(U \to V)} G(V \to \mathcal{Y}), \tag{6.1.2}$$

where the colimit is over the category of morphisms $U \to V$ over f such that V is a smooth $\mathcal{Y}$-scheme. See [SP, Tag 00WX] where this is established more generally for any continuous functor of sites.

Exercise 6.1.3. If $f \colon \mathcal{X} \to \mathcal{Y}$ is a *smooth* morphism of algebraic stacks and F is a sheaf on $\mathcal{Y}_{\text{lis-ét}}$, then $f^{-1}F = F|_{\mathcal{X}}$ and f^{-1} is exact.

Exercise 6.1.4. Let $f \colon X \to Y$ be a morphism of schemes.

(a) Let Y' be a smooth Y-scheme and $h_{Y'}$ be the sheaf on $Y_{\text{lis-ét}}$ representable by Y', i.e., $h_{Y'}(V \to Y) = \operatorname{Mor}_Y(V, Y')$. Show that $f^{-1}h_{Y'} = h_{X \times_Y Y'}$.

(b) Show that $f^{-1}\mathcal{O}_{Y_{\text{lis-ét}}} = \mathcal{O}_{X_{\text{lis-ét}}}$.

Hint: use part (a) and the fact that $\mathcal{O}_{Y_{\text{lis-ét}}}$ is representable by $\mathbb{A}^1_Y$.

(c) Conclude that if $i_x \colon \operatorname{Spec} \kappa(x) \to X$ is the inclusion of a point $x \in X$, then $i_x^{-1}\mathcal{O}_{X_{\text{lis-ét}}} = \kappa(x)$. (This should be contrasted to the fact that $i_x^{-1}\mathcal{O}_{X_{\text{Zar}}} = \mathcal{O}_{X,x}$ is the local ring and that $i_x^{-1}\mathcal{O}_{X_{\text{ét}}} = \mathcal{O}^{\text{h}}_{X,x}$ is its henselization.)

(d) Show that f^{-1} is not exact in general.

Hint: consider the case when $f = i_x$ is the inclusion of a residue field and use part (c).

Exercise 6.1.4(b) implies that for a morphism $f \colon \mathcal{X} \to \mathcal{Y}$ of algebraic stacks that $f^{-1}\mathcal{O}_{\mathcal{Y}} = \mathcal{O}_{\mathcal{X}}$. Therefore, the pullback of an $\mathcal{O}_{\mathcal{Y}}$-module G is the $\mathcal{O}_{\mathcal{X}}$-module $f^*G = f^{-1}G \otimes_{f^{-1}\mathcal{O}_{\mathcal{Y}}} \mathcal{O}_{\mathcal{X}} = f^{-1}G$.

Caution 6.1.5 (Lack of functoriality). The non-exactness of f^{-1} is related to the lack of fiber products in the lisse-étale site $\mathcal{X}_{\text{lis-ét}}$: if $U \to V$ and $U' \to V$ are arbitrary morphisms of *smooth* $\mathcal{X}$-schemes, $U \times_V U'$ is not necessarily *smooth* over $\mathcal{X}$. Because there are no fiber products, the category indexing the colimit in (6.1.2) is not filtering. An index category I is filtering if and only if colimits over I commute with finite limits [SP, Tag 002W], and therefore f^{-1} doesn't commute with finite limits or equivalently is not left exact. See also [Beh03, Warning 5.3.12], [Ols07, Ex. 3.4], and [SP, Tag 07BF].

Unfortunately, this means that the lisse-étale site is not functorial: a morphism $f \colon \mathcal{X} \to \mathcal{Y}$ does not necessarily induce a morphism of sites $\mathcal{X}_{\text{lis-ét}} \to \mathcal{Y}_{\text{lis-ét}}$, as the definition [SP, Tag 00X0] of a *morphism of sites* is the data of a functor $\mathcal{Y}_{\text{lis-ét}} \to \mathcal{X}_{\text{lis-ét}}$ (in the opposite direction!) which is continuous, i.e., the preimage of a cover in $\mathcal{Y}_{\text{lis-ét}}$ is a cover in $\mathcal{X}_{\text{lis-ét}}$ (which does hold here), and such that the preimage functor $f^{-1} \colon \operatorname{Sh}(\mathcal{Y}_{\text{lis-ét}}) \to \operatorname{Sh}(\mathcal{X}_{\text{lis-ét}})$ on categories of sheaves is *exact* (which fails here). Because of the lack of functoriality, constructions in the lisse-étale site sometimes require extra care, especially the derived category and the cotangent complex; see [Ols07]. Nevertheless, quasi-coherent sheaves in the lisse-étale site (which is all we will need in this text) can be quickly introduced and enjoy the same familiar properties as in the case of schemes.

Remark 6.1.6 (The big fppf topology). One can define sheaves on an algebraic stack in the big fppf topology. This leads to a different category of sheaves but to an equivalent category of quasi-coherent sheaves, with the same reasoning as in the case of a Deligne–Mumford stack (see Exercise 4.1.25).

The big fppf topology is functorial but has other technical issues. As this is the approach in the *Stacks Project*, we quickly summarize its salient features, beginning with the case of schemes. A morphism $f\colon X \to Y$ of schemes induces a continuous functor $(\mathrm{Sch}/Y)_{\mathrm{fppf}} \to (\mathrm{Sch}/X)_{\mathrm{fppf}}$ between big fppf sites defined by $(V \to Y) \mapsto (V \times_Y X \to X)$. The pushforward is defined as before by $f_*F(V \to Y) = F(V \times_Y X \to X)$, and the pullback is given by the formula $f^{-1}G(U \to X) = F(U \to X \to Y)$. The simplicity of the pullback formula is because we are working in a big site: the identity map $U \to V = U$ is an initial object in the index category of (6.1.2) defining the presheaf $(f^{-1}G)^{\mathrm{pre}}$, and moreover $(f^{-1}G)^{\mathrm{pre}} = f^{-1}G$ is already a sheaf. It is a general fact that a continuous functor $\mathcal{C} \to \mathcal{D}$ commuting with both fiber products and the final objects induces an exact preimage functor $\mathrm{Sh}(\mathcal{C}) \to \mathrm{Sh}(\mathcal{D})$, and therefore defines a morphism of sites $\mathcal{D} \to \mathcal{C}$ [SP, Tag 00X6]. This applies in this case to show that f^{-1} is exact so that $(\mathrm{Sch}/Y)_{\mathrm{fppf}} \to (\mathrm{Sch}/X)_{\mathrm{fppf}}$ defines a morphism of sites $(\mathrm{Sch}/X)_{\mathrm{fppf}} \to (\mathrm{Sch}/Y)_{\mathrm{fppf}}$.[1]

A similar story holds for morphisms of algebraic stacks $f\colon \mathcal{X} \to \mathcal{Y}$. The big fppf site $\mathcal{X}_{\mathrm{fppf}}$ is the category $\mathcal{X}$ of *all* schemes over $\mathcal{X}$ where coverings are fppf coverings. In this case, the pullback $f^{-1}G(U \to \mathcal{X}) = G(U \to \mathcal{X} \to \mathcal{Y})$ is defined as above (no sheafification required), and the pushforward f_*F can be defined as $f_*F(V \to \mathcal{Y}) = \lim_{U \to V} F(U \to \mathcal{X})$, where the index category of the limit is the category of morphisms $U \to V$ of schemes over $\mathcal{X} \to \mathcal{Y}$; if f is representable by schemes, then $V \times_{\mathcal{Y}} \mathcal{X} \to V$ is an final object and $f_*F(V \to \mathcal{Y}) = F(V \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{X})$. Alternatively, we can define sections of F over any morphism $\mathcal{U} \to \mathcal{Y}$ of algebraic stacks by smooth descent, and then define the pushforward using the usual formula $f_*F(V \to \mathcal{Y}) = F(V \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{X})$. Interpreting f_* and f^{-1} via the continuous and cocontinuous functor $f\colon \mathcal{X}_{\mathrm{fppf}} \to \mathcal{Y}_{\mathrm{fppf}}$ of sites as in Footnote 1, it follows that f^{-1} is exact. This means that (f_*, f^{-1}) defines a morphism of sites $\mathcal{X}_{\mathrm{fppf}} \to \mathcal{Y}_{\mathrm{fppf}}$, so that the big fppf site of an algebraic stack is functorial (unlike the lisse-étale site). However, we will see shortly that the pushforward functor on $\mathcal{O}$-modules does not preserve quasi-coherence; see Remark 6.1.9.

6.1.2 Quasi-coherent sheaves

Definition 6.1.7. Let $\mathcal{X}$ be an algebraic stack. An $\mathcal{O}_{\mathcal{X}}$-module F is *quasi-coherent* if

(1) for every smooth $\mathcal{X}$-scheme U, the restriction $F|_{U_{\mathrm{Zar}}}$ to the small Zariski site of U is a quasi-coherent $\mathcal{O}_{U_{\mathrm{Zar}}}$-module, and

(2) for every morphism $f\colon U \to V$ of smooth $\mathcal{X}$-schemes, the natural morphism $f^*(F|_{V_{\mathrm{Zar}}}) \to F|_{U_{\mathrm{Zar}}}$ is an isomorphism.

A quasi-coherent sheaf F on $\mathcal{X}$ is a *vector bundle* (resp., *vector bundle of rank* r, *line bundle*) if $F|_{U_{\mathrm{Zar}}}$ is for every smooth $\mathcal{X}$-scheme U. If in addition $\mathcal{X}$ is locally noetherian, we say F is *coherent* if $F|_{U_{\mathrm{Zar}}}$ is coherent for every smooth $\mathcal{X}$-scheme U.

Equivalently, an $\mathcal{O}_{\mathcal{X}}$-module F is quasi-coherent if it is quasi-coherent as a module in the ringed site $(\mathcal{X}_{\text{lis-ét}}, \mathcal{O}_{\mathcal{X}})$ as formulated in Exercise 4.1.22(iii). If $\mathcal{X}$

[1] The functor $v\colon (\mathrm{Sch}/Y)_{\mathrm{fppf}} \to (\mathrm{Sch}/X)_{\mathrm{fppf}}$ has a continuous left adjoint $u\colon (\mathrm{Sch}/X)_{\mathrm{fppf}} \to (\mathrm{Sch}/Y)_{\text{lis-ét}}$ defined by $(U \to X) \mapsto (U \to X \to Y)$, which can also be used to define f_* and f^{-1}: on the level of presheaves, $f_* = v^p = {}_pu$ and $(f^{-1})^{\mathrm{pre}} = v_p = u^p$; see [SP, Tags 00VC and 00XF] for the general definitions of ${}_pu$, u_p, and u^p, and [SP, Tag 09VQ] for their relationships. The functor u is not only continuous but also cocontinuous (a *cocontinuous* functor $u\colon \mathcal{C} \to \mathcal{D}$ of sites requires that for every object $c \in \mathcal{C}$ and covering $\{d_i \to u(c)\}$, there is a covering $\{c_i \to c\}$ such that the covering $\{u(c_i) \to u(c)\}$ refines $\{d_i \to u(c)\}$). This also automatically implies that f^{-1} is exact [SP, Tag 06NW].

is a scheme (resp., Deligne–Mumford stack), then Exercise 4.1.25 implies that the above definition is consistent with Definition 4.1.17. It is also not hard to see that quasi-coherence in the lisse-étale site of an algebraic stack is equivalent to quasi-coherence in the big fppf topology. In any case, we have well-defined categories $\mathrm{QCoh}(\mathcal{X})$ and $\mathrm{Coh}(\mathcal{X})$ (in the noetherian setting) of quasi-coherent and coherent sheaves. The *tensor product* $F \otimes G := F \otimes_{\mathcal{O}_{\mathcal{X}}} G$ and *Hom sheaf* $\mathscr{H}om_{\mathcal{O}_{\mathcal{X}}}(F, G)$ of quasi-coherent sheaves are defined in the usual way, and are quasi-coherent sheaves that enjoy the familiar properties.

Given a morphism $f\colon \mathcal{X} \to \mathcal{Y}$ of algebraic stacks, the pullback $f^* = f^{-1}$ preserves quasi-coherence. If f is quasi-compact and quasi-separated, the pushforward f_* also preserves quasi-coherence and is a right adjoint to f^*

$$\begin{tikzcd} \mathrm{QCoh}(\mathcal{X}) \arrow[r, shift left, "f_*"] & \mathrm{QCoh}(\mathcal{Y}). \arrow[l, shift left, "f^*"] \end{tikzcd}$$

Exercise 6.1.8 (details)**.** Verify that f_* and f^* preserve quasi-coherence and that they are adjoint functors.

Remark 6.1.9 (Pushforward of quasi-coherent sheaves in the big fppf site). Given a morphism $f\colon \mathcal{X} \to \mathcal{Y}$ of algebraic stacks, the pushforward f_* of sheaves in the big fppf sites as defined in Remark 6.1.6 does *not* preserve quasi-coherence. This even fails in the case of a map of schemes $f\colon X \to Y$. If $F \in \mathrm{QCoh}(X)$ and

$$\begin{tikzcd} U \arrow[r, "f'"] \arrow[d, "g'"'] \arrow[dr, phantom, "\square"] & V \arrow[d, "g"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

is a cartesian diagram of schemes, then it follows from the definitions in the big site that $g^* f_* F = g^{-1} f_* F = f'_* g'^{-1} F = f'_* g'^* F$ but this formula does not hold in general (unless for example g is flat). In other words, $f_* F$ has the correct sections of the desired *quasi-coherent pushforward* $f_{\mathrm{QCoh},*} F$ only over flat maps $V \to Y$. With some care (see [SP, Tag 070A]), the quasi-coherent pushforward can be defined as the composition

$$f_{\mathrm{QCoh},*}\colon \mathrm{QCoh}(\mathcal{X}) \to \mathrm{LQCoh}^{\mathrm{fbc}}(\mathcal{O}_{\mathcal{X}}) \xrightarrow{f_*} \mathrm{LQCoh}^{\mathrm{fbc}}(\mathcal{O}_{\mathcal{Y}}) \xrightarrow{Q} \mathrm{QCoh}(\mathcal{Y}),$$

where $\mathrm{LQCoh}^{\mathrm{fbc}}(\mathcal{O}_{\mathcal{X}})$ denotes the full subcategory of *locally quasi-coherent* $\mathcal{O}_{\mathcal{X}}$-modules (i.e., for every *flat* map $U \to \mathcal{X}$ from a scheme, the restriction $F|_{U_{\text{ét}}}$ is quasi-coherent) which have the *flat base change* property (i.e., for every *flat* morphism $f\colon U \to V$ over $\mathcal{X}$, the induced map $f^*(F|_{V_{\text{ét}}}) \to F|_{U_{\text{ét}}}$ is an isomorphism), and Q is a the *quasi-coherator* functor. The functor $f_{\mathrm{QCoh},*}$ is a right adjoint of $f^{-1} = f^*$. See also [SP, Tag 08MW] for the development of the derived category using the fppf topology.

Exercise 6.1.10. Let X be a scheme and $\mathbb{G}_m$ be the multiplicative group over $\operatorname{Spec} \mathbb{Z}$. Show that a quasi-coherent sheaf on $X \times B\mathbb{G}_m$ is the same data as a $\mathbb{Z}$-graded quasi-coherent sheaf on X.

Hint: Extend the argument of Proposition B.1.15.

Exercise 6.1.11. Let G be an affine algebraic group over a field $\Bbbk$. Recall that a G-representation is a $\Bbbk$-vector space with a dual action $\sigma\colon V \to \Gamma(G, \mathcal{O}_G) \otimes_{\Bbbk} V$ satisfying two natural compatibility conditions (see §B.1.12).

(a) Show that $\mathrm{QCoh}(BG)$ is equivalent to the category $\mathrm{Rep}(G)$ of G-representations. More generally, if $\operatorname{Spec} A$ is an affine $\Bbbk$-scheme with a G-action, show that a quasi-coherent sheaf on $[\operatorname{Spec} A/G]$ is the data of an A-module M together with a coaction $\sigma\colon M \to \Gamma(G, \mathcal{O}_G) \otimes_{\Bbbk} M$ over $\Bbbk$ (i.e., a map of $\Bbbk$-vector spaces giving M the structure of a G-representation) such that multiplication $A \otimes_{\Bbbk} M \to M$ is a map of G-representations. This extends Example 4.1.21 which was the case when G is finite.

(b) Considering the diagram

$$\begin{tikzcd} \operatorname{Spec} A \arrow[r, "p"] & {[\operatorname{Spec} A/G]} \arrow[r, "\pi"] \arrow[d, "q"] & \operatorname{Spec} A^G \\ & BG, & \end{tikzcd}$$

extend Exercise 4.1.27 by providing descriptions of the functors $p_*, p^*, \pi_*, \pi^*, q_*$ and q^* on quasi-coherent sheaves.

(c) If U is a $\Bbbk$-scheme with an action of G, then a *line bundle with a G-action* is a line bundle L on U together with an isomorphism $\alpha\colon \sigma^* L \xrightarrow{\sim} p_2^* L$ satisfying a cocycle condition $p_{23}^*\alpha \circ (\mathrm{id}_G \times \sigma)^*\alpha = (\mu \times \mathrm{id}_U)^*\alpha$; see B.1.28. Show that a line bundle with a G-action is the same as a line bundle on the quotient stack $[U/G]$.

Example 6.1.12. Two non-isomorphism affine algebraic groups G and H over a field $\Bbbk$ can have isomorphic $BG \cong BH$ classifying stacks and thus equivalent categories of representations. For example, if $O(q)$ and $O(q')$ are orthogonal groups with respect to non-degenerate quadratic forms q and q' on an n-dimensional $\Bbbk$-vector space V, then $BO(q) \cong BO(q')$ (see Exercise 3.1.13), and thus $O(q)$ and $O(q')$ have equivalent categories of representations.

We've already seen examples of quasi-coherent sheaves on a Deligne–Mumford stack where the pluricanonical line bundles $\lambda = \det \pi_*(\omega_{\mathcal{U}_g/\overline{\mathcal{M}}_g}^{\otimes k})$ on $\overline{\mathcal{M}}_g$ played a prominent role in the projectivity of $\overline{M}_g$ (see §5.9). Determinantal line bundles play a similar role in the study of the moduli stack of vector bundles.

Example 6.1.13 (Determinantal line bundles)**.** Consider the stack $\mathcal{M} := \mathcal{B}un_{C,r,d}$ of vector bundles on a smooth, connected, and projective curve C over $\Bbbk$. Consider the diagram

$$\begin{tikzcd} & C \times \mathcal{M} \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ C & & \mathcal{M}. \end{tikzcd}$$

The projection $p_2\colon C \times \mathcal{M} \to \mathcal{M}$ is representable, projective, and smooth of relative dimension 1. For every vector bundle F on $C \times \mathcal{M}$, the cohomology $\mathrm{R}^i p_{2,*} F$ (as defined below) is computed as a 2-term complex $[K^0 \to K^1]$ of vector bundles and that the line bundle

$$\det \mathrm{R} p_{2,*} F := \det(K_0) \otimes \det(K_1)^{\vee}$$

is well-defined on $\mathcal{M}$. Note that if $\operatorname{rk} K_0 = \operatorname{rk} K_1$, i.e., $\operatorname{rk} \mathrm{R}p_{2,*}F = 0$, then we have a map $\det K^0 \to \det K^1$ of line bundles and the corresponding map $\mathcal{O}_{\mathcal{M}} \to \det(K^0)^\vee \otimes \det(K_1)$ defines a section of the dual $(\det \mathrm{R}p_{2,*}F)^\vee$.

Let $\mathcal{E}_{\mathrm{univ}}$ be the universal vector bundle on $C \times \mathcal{M}$. For every vector bundle V on C, we define the *determinantal line bundle*

$$\mathcal{L}_V := \big(\det \mathrm{R}p_{2,*}(\mathcal{E}_{\mathrm{univ}} \otimes p_1^* V)\big)^\vee.$$

associated to V.

Example 6.1.14. If $\mathcal{X}$ is an algebraic stack of finite presentation over a scheme S, then the relative sheaf of differentials $\Omega_{\mathcal{X}/S}$ on $\mathcal{X}_{\text{lis-ét}}$, defined on a smooth $\mathcal{X}$-scheme U by $\Omega_{\mathcal{X}/S}(U) = \Omega_{U/S}$, is not quasi-coherent. This is because for a non-étale map $f\colon U \to V$ of smooth $\mathcal{X}$-schemes, $f^*\Omega_{V/S} \to \Omega_{U/S}$ is not necessary an isomorphism. This differs from the Deligne–Mumford case where the sheaf $\Omega_{\mathcal{X}/S}$ on $\mathcal{X}_{\text{ét}}$ is quasi-coherent (Example 4.1.19). When $\mathcal{X}$ is Deligne–Mumford, $\Omega_{\mathcal{X}/S}$ extends to a quasi-coherent sheaf on $\mathcal{X}_{\text{lis-ét}}$ by defining $\Omega_{\mathcal{X}/S}(U)$, for a smooth map $f\colon U \to \mathcal{X}$ from a scheme, to be the global sections of the sheaf $f^*\Omega_{\mathcal{X}/S}$ on $U_{\text{lis-ét}}$.

Exercises 4.1.28 and 4.1.45 generalize to algebraic stacks.

Proposition 6.1.15 (Flat Base Change)**.** *Consider a cartesian diagram*

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] \arrow[dr, phantom, "\square"] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

of algebraic stacks, and let F be a quasi-coherent sheaf on X. If $g\colon Y' \to Y$ is flat and $f\colon X \to Y$ is quasi-compact and quasi-separated, the natural adjunction map

$$g^* f_* F \to f'_* g'^* F$$

is an isomorphism. $\square$

Proposition 6.1.16. *Let $\mathcal{X}$ be a noetherian algebraic stack. Every quasi-coherent sheaf on $\mathcal{X}$ is a directed colimit of its coherent subsheaves. If $\mathcal{U} \subseteq \mathcal{X}$ is an open substack, then every coherent sheaf on $\mathcal{U}$ extends to a coherent sheaf on $\mathcal{X}$.* $\square$

Exercise 6.1.17. Let $\mathcal{X} \to \mathcal{Y}$ be a smooth affine morphism of noetherian algebraic stacks with affine diagonal.

(1) Show that there is a vector bundle $\Omega_{\mathcal{X}/\mathcal{Y}}$ on $\mathcal{X}$ with the property that if $V \to \mathcal{Y}$ is a morphism from a scheme, the pullback of $\Omega_{\mathcal{X}/\mathcal{Y}}$ to $X_V := \mathcal{X} \times_{\mathcal{Y}} V$ is $\Omega_{\mathcal{X}_V/V}$.

(2) Given a commutative diagram

$$\begin{tikzcd} \operatorname{Spec} A_0 \arrow[r, "f_0"] \arrow[d, hook] & \mathcal{X} \arrow[d] \\ \operatorname{Spec} A \arrow[r] \arrow[ur, dashed] & \mathcal{Y} \end{tikzcd}$$

where $A \twoheadrightarrow A_0$ is a surjection of noetherian rings with square-zero kernel J, show that the set of liftings is a torsor under $\operatorname{Hom}_{A_0}(f_0^*\Omega_{\mathcal{X}/\mathcal{Y}}, J)$ and in particular is non-empty.

(3) Can you weaken the hypotheses?

6.1.3 Quasi-coherent constructions

Extending the constructions of §4.1.5 on a Deligne–Mumford stack to an algebraic stack $\mathcal{X}$, a *quasi-coherent $\mathcal{O}_{\mathcal{X}}$-algebra* is a quasi-coherent $\mathcal{O}_{\mathcal{X}}$-module $\mathcal{A}$ with a compatible structure as a ring object. The *relative spectrum* $\mathcal{S}\mathrm{pec}_{\mathcal{X}}\,\mathcal{A}$, defined as the stack of pairs (f, α) where $f\colon S \to \mathcal{X}$ is a morphism from a scheme and $\alpha\colon f^*\mathcal{A} \to \mathcal{O}_S$ is a map of $\mathcal{O}_S$-algebras, is an algebraic stack affine over $\mathcal{X}$. On a noetherian algebraic stack, every quasi-coherent $\mathcal{O}_{\mathcal{X}}$-algebra is a directed colimit of finite type subalgebras.

The *reduction of* $\mathcal{X}$ is $\mathcal{X}_{\mathrm{red}} := \mathcal{S}\mathrm{pec}_{\mathcal{X}}\,\mathcal{O}_{\mathcal{X}}^{\mathrm{red}}$ where $\mathcal{O}_{\mathcal{X}}^{\mathrm{red}}$ is the sheaf of $\mathcal{O}_{\mathcal{X}}$-algebras defined by $\mathcal{O}_{\mathcal{X}}^{\mathrm{red}}(U) = \Gamma(U, \mathcal{O}_U)_{\mathrm{red}}$ for a smooth $\mathcal{X}$-scheme U. If $\mathcal{X}$ is integral, the *normalization of* $\mathcal{X}$ is defined as $\widetilde{\mathcal{X}} := \mathcal{S}\mathrm{pec}_{\mathcal{X}}\,\mathcal{A}$, where $\mathcal{A}$ is the $\mathcal{O}_{\mathcal{X}}$-algebra whose ring of sections over a smooth $\mathcal{X}$-scheme U is the normalization of $\Gamma(U, \mathcal{O}_U)$; this is well-defined since normalization commutes with smooth base change (Proposition A.7.4). For a quasi-compact and quasi-separated morphism $f\colon \mathcal{X} \to \mathcal{Y}$ of algebraic stacks, there is a factorization $f\colon \mathcal{X} \to \mathcal{S}\mathrm{pec}\, f_*\mathcal{O}_{\mathcal{X}} \to \mathcal{Y}$. The morphism f is affine if and only if $\mathcal{X} \to \mathcal{S}\mathrm{pec}\, f_*\mathcal{O}_{\mathcal{X}}$ is an isomorphism, and quasi-affine if and only if $\mathcal{X} \to \mathcal{S}\mathrm{pec}\, f_*\mathcal{O}_{\mathcal{X}}$ is an open immersion. The proof of Zariski's Main Theorem (4.5.9) for Deligne–Mumford stacks extends to algebraic stacks.

Theorem 6.1.18 (Zariski's Main Theorem)**.** *A representable, quasi-finite, and separated morphism $f\colon \mathcal{X} \to \mathcal{Y}$ of noetherian algebraic stacks factors as the composition of a dense open immersion $\mathcal{X} \hookrightarrow \widetilde{\mathcal{Y}}$ and a finite morphism $\widetilde{\mathcal{Y}} \to \mathcal{X}$.* □

6.1.4 Picard groups

If $\mathcal{X}$ is an algebraic stack, we let $\mathrm{Pic}(\mathcal{X})$ denote the set of isomorphism classes of line bundles on $\mathcal{X}$. It is an abelian group under tensor product.

Example 6.1.19. If G is an affine algebraic group over a field $\Bbbk$, then $\mathrm{Pic}(BG)$ is identified with the group $\mathbb{X}_*(G)$ of characters of G. For example, $\mathrm{Pic}(B\mathbb{G}_m) = \mathbb{Z}$, $\mathrm{Pic}(B\mathrm{GL}_n) = \mathbb{Z}$, and $\mathrm{Pic}(\mathrm{PGL}_n) = \{0\}$. More generally, if U is a scheme with an action of G, then $\mathrm{Pic}([U/G])$ is identified with the group $\mathrm{Pic}^G(U)$ of line bundles on U with a G-action (Exercise 6.1.11(c)).

Exercise 6.1.20. Let $\mathcal{X}$ be a smooth and irreducible algebraic stack over a field $\Bbbk$. If $\mathcal{D} \subseteq \mathcal{X}$ is a reduced substack of codimension 1 with complement $\mathcal{U}$, show that there is a naturally defined line bundle $\mathcal{O}(\mathcal{D})$ (generalizing the usual construction for schemes) such that $\mathcal{O}(\mathcal{D})|_{\mathcal{U}} \cong \mathcal{O}_{\mathcal{U}}$.

Exercise 6.1.21. Let $\mathbb{G}_m$ acts on $\mathbb{A}^n$ over a field $\Bbbk$ with weights $d_1, \ldots, d_n$. Let $\mathcal{O}(1)$ be the line bundle on $[\mathbb{A}^n/\mathbb{G}_m]$ corresponding to the projection $[\mathbb{A}^n/\mathbb{G}_m] \to B\mathbb{G}_m$.

(a) Show that $\mathrm{Pic}([\mathbb{A}^n/\mathbb{G}_m]) \cong \mathbb{Z}$ generated by $\mathcal{O}(1)$.

(b) Show that the restriction $\mathrm{Pic}([\mathbb{A}^n/\mathbb{G}_m]) \to \mathrm{Pic}(\mathcal{P}(d_1, \ldots, d_n))$ is an isomorphism, where $\mathcal{P}(d_1, \ldots, d_n)$ is the weighted projective stack (see Example 3.9.14).

(c) If $f \in \Gamma(\mathbb{A}^n, \mathcal{O}_{\mathbb{A}^n})$ is a homogenous polynomial of degree d such that $V(f) \subseteq \mathbb{A}^n$ is reduced, show that $\mathcal{O}(V(f)) \cong \mathcal{O}(d)$ on $[\mathbb{A}^n/\mathbb{G}_m]$.

The following offers an elementary approach to Mumford's calculation that $\mathrm{Pic}(\mathcal{M}_{1,1}) = \mathbb{Z}/12$, first offered as a difficult exercise in Exercise 4.1.35.

Exercise 6.1.22. Let $\Bbbk$ be a field with $\mathrm{char}(\Bbbk) \neq 2, 3$.

(a) Show that $\mathrm{Pic}(\overline{\mathcal{M}}_{1,1}) = \mathbb{Z}$.

Hint: Use the description $\overline{\mathcal{M}}_{1,1} = [(\mathbb{A}^2 \smallsetminus 0)/\mathbb{G}_m]$ *of Exercise 3.1.19(c) where* $\mathbb{G}_m$ *acts with weights 4 and 6. Show that the restriction* $\mathrm{Pic}([\mathbb{A}^2/\mathbb{G}_m]) \to \mathrm{Pic}(\overline{\mathcal{M}}_{1,1})$ *is an equivalence.*

(b) Show that $\mathrm{Pic}(\mathcal{M}_{1,1}) = \mathbb{Z}/12$.

Show that the restriction $\mathrm{Pic}(\overline{\mathcal{M}}_{1,1}) \to \mathrm{Pic}(\mathcal{M}_{1,1})$ *is surjective and that the image of* $\mathcal{O}(\Delta) = \mathcal{O}(12)$ *is trivial. Show that the images of* $\mathcal{O}(4)$ *and* $\mathcal{O}(6)$ *are non-trivial by considering their restrictions to the residual gerbes of the unique elliptic curves with* $\mathbb{Z}/4$ *and* $\mathbb{Z}/6$ *automorphism groups. See also [Mum65] (and [FO10] for a generalization over an arbitrary base).*

6.1.5 Sheaf cohomology

Sheaf cohomology for algebraic stacks can be developed using essentially the same approach as we used in §4.1.7 for Deligne–Mumford stacks.

Lemma 6.1.23. *If* $\mathcal{X}$ *is an algebraic stack, the categories* $\mathrm{Ab}(\mathcal{X}_{\text{lis-ét}})$ *and* $\mathrm{QCoh}(\mathcal{X})$ *have enough injectives.*

Proof. The category of abelian sheaves on any site has enough injectives [SP, Tag 01DP] and $\mathrm{QCoh}(\mathcal{X})$ is a Grothendieck abelian category, which implies that it has enough injectives [SP, Tag 079H]. □

Definition 6.1.24 (Cohomology)**.** Let $\mathcal{X}$ be an algebraic stack and F be a sheaf of abelian groups on $\mathcal{X}_{\text{lis-ét}}$. The *cohomology group* $\mathrm{H}^i(\mathcal{X}_{\text{lis-ét}}, F)$ is defined as the ith right derived functor of the global sections functor $\Gamma\colon \mathrm{Ab}(\mathcal{X}_{\text{lis-ét}}) \to \mathrm{Ab}$.

Čech cohomology is defined just as in the Deligne–Mumford case (Definition 4.1.39).

Definition 6.1.25 (Čech cohomology)**.** Given a smooth covering $\mathcal{U} = \{U_i \to \mathcal{X}\}_{i\in I}$ of algebraic stacks and an abelian sheaf F on $\mathcal{X}_{\text{lis-ét}}$, the *Čech complex of F with respect to $\mathcal{U}$* is $\check{C}^\bullet(\mathcal{U}, F)$ where

$$\check{C}^n(\mathcal{U}, F) = \prod_{(i_0,\dots,i_n)\in I^{n+1}} F(U_{i_0} \times_{\mathcal{X}} \cdots \times_{\mathcal{X}} U_{i_n})$$

with differential

$$d^n\colon \check{C}^n(\mathcal{U}, F) \to \check{C}^{n+1}(\mathcal{U}, F), \qquad (s_{i_0,\dots,i_n}) \mapsto \left(\sum_{k=0}^{n+1} (-1)^k p_{\widehat{k}}^* s_{i_0,\dots,\widehat{i_k},\dots,i_n}\right)_{(i_0,\dots,i_{n+1})}$$

where $p_{\widehat{k}}\colon U_{i_0} \times_{\mathcal{X}} \cdots \times_{\mathcal{X}} U_{i_n} \to U_{i_0} \times_{\mathcal{X}} \cdots \times_{\mathcal{X}} \widehat{U_{i_k}} \times_{\mathcal{X}} \cdots \times_{\mathcal{X}} U_{i_n}$ is the map forgetting the kth component (with indexing starting at 0). The *Čech cohomology of F with respect to $\mathcal{U}$* is

$$\check{\mathrm{H}}^i(\mathcal{U}, F) := \mathrm{H}^i(\check{C}^\bullet(\mathcal{U}, F)).$$

The arguments of Lemma 4.1.41, Theorem 4.1.42, Proposition 4.1.43, and Proposition 4.1.44 extend to algebraic stacks.

Theorem 6.1.26. *For a quasi-coherent* $\mathcal{O}_{X_{\text{lis-ét}}}$*-module F on an affine scheme X,* $\mathrm{H}^i(X_{\text{lis-ét}}, F) = 0$ *for all* $i > 0$. □

Lemma 6.1.27. *Let $\mathcal{X}$ be an algebraic stack with affine diagonal and F be a quasi-coherent sheaf. If $\mathcal{U} = \{U_i \to \mathcal{X}\}$ is an étale covering with each U_i affine, then* $\mathrm{H}^i(\mathcal{X}_{\text{lis-ét}}, F) = \check{\mathrm{H}}^i(\mathcal{U}, F)$. □

Proposition 6.1.28. *If $\mathcal{X}$ is a Deligne–Mumford stack (e.g., a scheme) with affine diagonal and F is a quasi-coherent sheaf, then $\mathrm{H}^i(\mathcal{X}_{\text{ét}}, F) = \mathrm{H}^i(\mathcal{X}_{\text{lis-ét}}, F)$ for all i.* □

A similar argument shows that the cohomology of a quasi-coherent sheaf F can also be computed in the big fppf topology. For this reason, it is customary to drop the subscript and simply write $\mathrm{H}^i(\mathcal{X}, F)$.

Proposition 6.1.29. *If $\mathcal{X}$ is an algebraic stack with affine diagonal of $F \in \mathrm{QCoh}(\mathcal{X})$, then the cohomology $\mathrm{H}^i(\mathcal{X}_{\text{lis-ét}}, F)$ of F as an abelian sheaf agrees with the ith right derived functor of $\Gamma\colon \mathrm{QCoh}(\mathcal{X}) \to \mathrm{Ab}$.*

If $\mathcal{X}$ does not have affine diagonal, then the sheaf cohomology $\mathrm{H}^i(\mathcal{X}, F)$ of a quasi-coherent sheaf may differ from the ith right derived functor of $\Gamma(\mathcal{X}, -)\colon \mathrm{QCoh}(\mathcal{X}) \to \mathrm{Ab}$.

Exercise 6.1.30. Let E be an elliptic curve over a field and BE be its classifying stack. Show that $\mathrm{H}^{2i}(BE, \mathcal{O}_{BE}) \neq 0$ for each i, but that the cohomology is zero if it is computed in $\mathrm{QCoh}(BE)$.

Proposition 6.1.31. *If $\mathcal{X}$ is an algebraic stack and F_i is a directed system of abelian sheaves in $\mathcal{X}_{\text{lis-ét}}$, then $\operatorname{colim}_i \mathrm{H}^i(\mathcal{X}_{\text{lis-ét}}, F_i) \to \mathrm{H}^i(\mathcal{X}_{\text{lis-ét}}, \operatorname{colim}_i F_i)$ is an isomorphism.* □

Example 6.1.32 (Group Cohomology)**.** Group cohomology for an affine algebraic group G over a field $\Bbbk$ can be interpreted as sheaf cohomology on BG just as in the case of a finite group (Example 4.1.47). Indeed, a G-representation V can viewed as a quasi-coherent $\mathcal{O}_{BG}$-module, and by Proposition 6.1.29, the sheaf cohomology $\mathrm{H}^i(BG, V)$ agrees with the *group cohomology* $\mathrm{H}^i(G, V)$, defined as the ith derived functor of $\mathrm{Rep}(G) \to \mathrm{Vect}_{\Bbbk}$, given by $V \mapsto V^G$. Moreover, by Lemma 6.1.27, we may compute the cohomology using the Čech complex $\check{C}^{\bullet}(\mathcal{U}, V)$ of V with respect to the smooth cover $\mathcal{U} = \{\operatorname{Spec} \Bbbk \to BG\}$. This complex has terms $\check{C}^n(\mathcal{U}, V) = \Gamma(G, \mathcal{O}_G)^{\otimes n} \otimes V$ with differentials given by (4.1.48), and corresponds to the *standard* or *bar resolution* in group cohomology.

If G is linearly reductive, i.e., $V \to V^G$ is exact, then $\mathrm{H}^i(G, V) = 0$ for every G-representation V and $i > 0$. If G acts on an affine scheme $\operatorname{Spec} A$ over $\Bbbk$, then it follows that $\mathrm{H}^i([\operatorname{Spec} A/G], F) = 0$ for every quasi-coherent sheaf F and $i > 0$.

Exercise 6.1.33.

(a) If $\Bbbk$ is characteristic 0 and $\Bbbk$ is the trivial $\mathbb{G}_a$-representation, show that $\mathrm{H}^i(B\mathbb{G}_a, \Bbbk) = \Bbbk$ for $i = 0, 1$ and zero otherwise.

(b) If G is an affine algebraic group over a field $\Bbbk$, show that $\mathrm{H}^i(BG, V)$ is finite dimensional for all i and zero for $i \gg 0$.

Hint: Choose an embedding $G \hookrightarrow \mathrm{GL}_n$, use that GL_n is linearly reductive in characteristic 0 (Theorem B.1.43), and consider the composition $BG \to B\mathrm{GL}_n \to \operatorname{Spec} \Bbbk$.

(c) Show that $\mathrm{H}^1(B\mathbb{G}_a, \Bbbk)$ is infinite dimensional.

In positive characteristic, $\mathrm{H}^i(G, V)$ may be nonzero for arbitrary large i; this already occurs in the case of finite groups such as $\mathbb{Z}/p\mathbb{Z}$ (Exercise 4.1.49).

6.1.6 Chow groups

Following [Tot99] and [EG98], we introduce the Chow groups of a quotient stack. Let G be a smooth affine algebraic group over an algebraically closed field $\Bbbk$ of dimension g, and let X be an n-dimensional scheme of finite type over $\Bbbk$. For each i, choose an r-dimensional G-representation V such that there is a nonempty open subscheme $U \subseteq \mathbb{A}(V)$ such that (a) G acts freely on U, (b) the quotient U/G is a scheme, and (c) $\operatorname{codim} \mathbb{A}(V) \smallsetminus U > n - i - g$. It is not too hard to prove that such representations exist. We define the *$(i-g)$th equivariant Chow group of X* or equivariantly the *ith Chow group of $[X/G]$* as

$$\mathrm{CH}^G_{i-g}(X) = \mathrm{CH}_i([X/G]) := \mathrm{CH}_{i+r}(X \times^G U).$$

Exercise 6.1.34.

(a) Show that there exists a representation V with properties (a)–(c).

(b) Check that the definition of $\mathrm{CH}^G_{i-g}(X)$ is independent of the choices.

While this definition may at first appear mysterious, it is actually forced upon us if we require the Chow groups to be invariant under and open immersions of high codimension and under vector bundles:

$$\begin{tikzcd} X \times^G U \arrow[r, hook, "\text{open}"] & {[(X \times \mathbb{A}(V))/G]} \arrow[d, "\text{vect bdl}"] \\ & {[X/G]} \end{tikzcd} \qquad \begin{tikzcd} \mathrm{CH}_{i+r}(X \times^G U) & \mathrm{CH}_{i+r}([(X \times \mathbb{A}(V))/G]) \arrow[l, "\sim"'] \\ & \mathrm{CH}_i([X/G]). \arrow[u, "\wr"'] \end{tikzcd}$$

If $[X/G]$ is smooth of pure dimension $d = n - g$, then we define

$$\mathrm{CH}^i_G(X) = \mathrm{CH}^i([X/G]) := \mathrm{CH}_{d-i}([X/G])$$
$$\mathrm{CH}^*_G(X) = \mathrm{CH}^*([X/G]) := \bigoplus_i \mathrm{CH}^i(([X/G]).$$

The intersection product gives a ring structure, and we call $\mathrm{CH}^*_G(X)$ the *equivariant Chow ring of X* and $\mathrm{CH}^*([X/G])$ the *Chow ring of* $[X/G]$.

Example 6.1.35 ($\mathrm{CH}^*(B\mathbb{G}_m)$)**.** Let V be the r-dimensional $\mathbb{G}_m$-representation over $\Bbbk$ with equal weights 1. Then $\mathbb{G}_m$ acts freely on $\mathbb{A}^n \smallsetminus 0$, and for $-1 \geq i > -r-1$, we have that $\mathrm{CH}_i(B\mathbb{G}_m) = \mathrm{CH}_{i+r}(\mathbb{P}^{r-1}) = \mathbb{Z}$. It follows that $\mathrm{CH}_i(B\mathbb{G}_m) = \mathbb{Z}$ for $i \leq -1$ and is 0 otherwise. Therefore $\mathrm{CH}^j(B\mathbb{G}_m) = \mathbb{Z}$ for $j \geq 0$, and $\mathrm{CH}^*(B\mathbb{G}_m) = \mathbb{Z}[x]$. More generally, if $T \cong \mathbb{G}_m^r$ is a rank r torus, then $\mathrm{CH}^*(BT)$ is isomorphic to the character ring $\mathbb{Z}[x_1, \ldots, x_r]$ of T.

We summarize some of the important properties of equivariant Chow groups.

Properties 6.1.36.

(1) (Independence of quotient presentation) If $[X/G] \cong [X'/G']$, then $\mathrm{CH}^G_i(X) \cong \mathrm{CH}^{G'}_i(X')$, and in particular the definition of $\mathrm{CH}_i([X/G])$ is independent of the quotient presentation.

(2) *(Vector bundle invariance) If $Y \to X$ is G-equivariant and a Zariski-local affine fibration of relative dimension r (e.g., the total space of a rank r vector bundle), then $\mathrm{CH}^G_*(X) \cong \mathrm{CH}^G_{*+r}(Y)$.*

(3) *(Excision sequence) If $\mathcal{Z} \subseteq \mathcal{X} = [X/G]$ is a closed substack with complement $\mathcal{U}$, then there is a right exact sequence*

$$\mathrm{CH}_*(\mathcal{Z}) \to \mathrm{CH}_*(\mathcal{X}) \to \mathrm{CH}_*(\mathcal{U}) \to 0.$$

(4) *(Comparison with coarse moduli space) If $\mathcal{X} \cong [X/G]$ is a separated Deligne–Mumford stack with coarse moduli space X, then $\mathrm{CH}_*(\mathcal{X}) \otimes \mathbb{Q} \cong \mathrm{CH}_*(X) \otimes \mathbb{Q}$.*

(5) *(Functoriality) Flat morphisms induce pullback maps on Chow groups while proper morphisms induce pushforward maps.*

(6) *(Self-intersection) If $\mathcal{X} = [X/G]$ is smooth and $i \colon \mathcal{Z} \hookrightarrow \mathcal{X}$ is a smooth substack of pure codimension d, then there is pullback $i^* \colon \mathrm{CH}^*(\mathcal{X}) \to \mathrm{CH}^*(\mathcal{Z})$ given by intersection with $\mathcal{Z}$ such that $i^* i_* \alpha = c_d(\mathcal{N}_{\mathcal{Z}/\mathcal{X}}) \cap \alpha$ for $\alpha \in \mathrm{CH}^*(\mathcal{Z})$, where c_d is the top Chern class of the normal bundle.*

Exercise 6.1.37. Prove Properties 6.1.36.

Hint: Reduce to the usual properties of Chow groups of schemes, c.f., [Ful98].

Kresch has extended the theory of Chow groups to (not necessarily quotient) algebraic stacks; see [Kre99].

Exercise 6.1.38.

(a) If T is a torus acting trivially on a smooth scheme X, show that $CH^*_T(X) \cong \mathrm{CH}^*(X) \otimes \mathrm{CH}^*(BT)$. is a product of two tori with T_2 acting trivially. Then $\mathrm{CH}^*_T(X) \cong \mathrm{CH}^*_{T_1}(X) \otimes \mathrm{CH}^*(BT_2)$.

(b) If G is a connected reductive group with maximal torus T, and X is a smooth scheme with a G-action, then the Weyl group $W = N_G(T)/T$ acts on $\mathrm{CH}^*_T(X)$. Show that $\mathrm{CH}^*_G(X)_{\mathbb{Q}} = \mathrm{CH}^*_T(X)^W_{\mathbb{Q}}$ and that if in addition G is special (Example B.1.59), then $\mathrm{CH}^*_G(X) = \mathrm{CH}^*_T(X)^W$.

Exercise 6.1.39.

(a) Let $\mathcal{P}(d_0, \ldots, d_n)$ be the weighted projective stack of Example 3.9.14. Show that $\mathrm{CH}^*(\mathcal{P}(d_0, \ldots, d_n)) \cong \mathbb{Z}[x]/(d_1 \cdots d_n x^{n+1})$.

(b) If $\mathrm{char}(\Bbbk) \neq 2, 3$, show that $\mathrm{CH}^*(\mathcal{M}_{1,1}) \cong \mathbb{Z}[x]/(12x)$ and $\mathrm{CH}^*(\overline{\mathcal{M}}_{1,1}) \cong \mathbb{Z}[x]/(24x^2)$. (Compare to the Picard group from Exercise 6.1.22).

(c) Let $\mathbb{G}_m$ act on $\mathbb{P}^n$ with weights $d_0, \ldots, d_n$. Show that $A^*([\mathbb{P}^n/\mathbb{G}_m]) = \mathbb{Z}[h,t]/p(h,t)$, where $p(h,t) = \sum_{i=0}^n h^i e_i(a_0 t, \ldots, a_n t)$ and e_i is the ith symmetric polynomial.

6.1.7 de Rham and singular cohomology

We quickly discuss the de Rham and singular cohomology of an algebraic stack following [Beh04].

Analyticification. If $\mathcal{X}$ is a smooth algebraic stack over $\mathbb{C}$ with affine diagonal, there is an analyticification $\mathcal{X}^{\mathrm{an}}$, analogous to the analyticification of a finite type $\mathbb{C}$-scheme, such that $\mathcal{X}^{\mathrm{an}}$ is a *differentiable stack*. If $U_0 \to \mathcal{X}$ is a smooth presentation by a scheme so that $\mathcal{X}$ is the quotient of the smooth groupoid $U_1 := U_0 \times_{\mathcal{X}} U_0 \rightrightarrows U_0$, then $U_0^{\mathrm{an}} \to \mathcal{X}^{\mathrm{an}}$ is a smooth presentation and $\mathcal{X}^{\mathrm{an}}$ is the quotient of the *Lie groupoid* $U_1^{\mathrm{an}} \rightrightarrows U_0^{\mathrm{an}}$.

De Rham cohomology of a differential stack. Given a differentiable stack $\mathcal{X}$ with a smooth presentation $U_0 \to \mathcal{X}$, we can define a *simplicial manifold*

$$U_\bullet\colon \ \cdots U_3 \mathrel{\substack{\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow}} U_2 \mathrel{\substack{\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow}} U_1 \longrightarrow U_0, \quad \text{where } U_p := \underbrace{U_0 \times_{\mathcal{X}} \cdots \times_{\mathcal{X}} U_0}_{p+1 \text{ times}} \tag{6.1.40}$$

with differential maps $\partial_0, \ldots, \partial_p\colon U_p \to U_{p-1}$ forgetting the ith term and degeneracy maps $s_i\colon U_{p-1} \to U_p$ inserting an identity morphism in the ith term. This defines a double complex $\Omega^q(U_p)$ with differentials given by exterior differentiation $d\colon \Omega^{q-1}(U_p) \to \Omega^q(U_p)$ and $\partial := \sum_{i=0}^p (-1)^i \partial_i^*\colon \Omega^q(U_{p-1}) \to \Omega^q(U_p)$. We define the *de Rham complex* $C^\bullet_{\mathrm{dR}}(\mathcal{X})$ as the total complex

$$C^k_{\mathrm{dR}}(\mathcal{X}) := \bigoplus_{p+q=k} \Omega^q(U_p),$$

with differential $\delta\colon C^k_{\mathrm{dR}}(\mathcal{X}) \to C^{k+1}_{\mathrm{dR}}(\mathcal{X})$ defined by $\delta(\omega) = \partial(\omega) + (-1)^p d(\omega)$ for $\omega \in \Omega^p(U_q)$. The *de Rham cohomology* is

$$\mathrm{H}^n_{\mathrm{dR}}(\mathcal{X}) := \mathrm{H}^n(C^\bullet_{\mathrm{dR}}(\mathcal{X})),$$

and is independent of the choice of presentation.

Singular homology/cohomology of a topological stack. For a *topological stack* $\mathcal{X}$, one can replicate the constructions of singular homology and cohomology. Let $U_0 \to \mathcal{X}$ be a presentation and $U_\bullet$ be the simplicial topological space as in (6.1.40). For each p, we have the singular chain complex $C_\bullet(U_p)$ with differentials $d : C_q(U_p) \to C_{q-1}(U_p)$. This defines a double complex $C_q(U_p)$ using the differential $\partial = \sum_{i=0}^p (-1)^i \partial_j\colon C_q(U_p) \to C_q(U_{p-1})$ induced by the maps $\partial_i\colon U_p \to U_{p-1}$. The *singular chain complex $C_\bullet(\mathcal{X})$ of $\mathcal{X}$* is defined as the total complex

$$C_k(\mathcal{X}) := \bigoplus_{p+q=k} C_q(U_p)$$

with the differential $\delta\colon C_k(\mathcal{X}) \to C_{k-1}(\mathcal{X})$ given by $\delta(\gamma) = (-1)^{p+q}\partial(\gamma) + (-1)^q d(\gamma)$ for $\gamma \in C_q(U_p)$. For an abelian group A, we can therefore define the *singular homology groups of $\mathcal{X}$ with coefficients in A* as

$$\mathrm{H}_n(\mathcal{X}, A) := \mathrm{H}_n(C_\bullet(\mathcal{X}) \otimes_{\mathbb{Z}} A).$$

Dualizing, we define the *singular cochain complex $C^\bullet(\mathcal{X})$* by $C^n(\mathcal{X}) := \mathrm{Hom}(C_n(\mathcal{X}), \mathbb{Z})$ and the *singular cohomology groups of $\mathcal{X}$ with coefficients in A* as

$$\mathrm{H}^n(\mathcal{X}, A) := \mathrm{H}^n(C^\bullet(\mathcal{X}) \otimes_{\mathbb{Z}} A).$$

Comparisons.

– There are pairings $\mathrm{H}_k(\mathcal{X}, \mathbb{Z}) \otimes \mathrm{H}^k(\mathcal{X}, \mathbb{Z}) \to \mathbb{Z}$ which after tensoring with $\mathbb{Q}$ gives identifications $\mathrm{H}^k(\mathcal{X}, \mathbb{Q}) \cong \mathrm{H}_k(\mathcal{X}, \mathbb{Q})^\vee$.

– If G is a topological group acting on a space U, then the *equivariant cohomology* is defined as $\mathrm{H}^*_G(U, A) := \mathrm{H}^*(EG \times^G U, A)$, where EG is a contractible space with a free action of G, and there is an identification $\mathrm{H}^*([U/G], A) = \mathrm{H}^*_G(U, A)$.

– For a differential stack $\mathcal{X}$, there is an identification $\mathrm{H}^*_{\mathrm{dR}}(\mathcal{X})$ with the cohomology $\mathrm{H}^*(\mathcal{X}, \mathbb{R})$ of the constant sheaf $\mathbb{R}$ computed in the big site of manifolds over $\mathcal{X}$ where coverings are usual open coverings.

– If $\mathcal{X}$ is a *topological Deligne–Mumford stack* (e.g., the topological stack associated to a separated Deligne–Mumford stack of finite type over $\mathbb{C}$) with coarse moduli space $\mathcal{X} \to X$, then $\mathrm{H}^*(\mathcal{X}, \mathbb{Q}) = \mathrm{H}^*(X, \mathbb{Q})$.

6.2 Global quotient stacks and the resolution property

Did you ever stop to think, and forget to start again?

WINNIE THE POOH

Quotient stacks of the form $[U/G]$ form an important class of algebraic stacks. Their geometry is readily accessible as the G-equivariant geometry of U (see Table 0.6.15). It is an interesting question on when algebraic stacks are quotient stacks, related to both geometric properties such as the resolution property and to arithmetic properties such as the surjectivity of the Brauer map. In the Local Structure of Deligne–Mumford Stacks (4.3.1), we showed that every Deligne–Mumford stack has an étale stabilizer-preserving neighborhood which a quotient stack $[\operatorname{Spec} A/G]$ by a finite group, and we will shortly prove an analogous result for algebraic stacks around points with linearly reductive stabilizer (see Theorem 6.7.1).

6.2.1 Global quotient stacks

Definition 6.2.1. An algebraic stack $\mathcal{X}$ is a *global quotient stack* if there exists an isomorphism $\mathcal{X} \cong [U/\operatorname{GL}_n]$ where U is an algebraic space.

In other words, $\mathcal{X}$ is a global quotient stack if and only if there is a principal GL_n-bundle $U \to \mathcal{X}$ from an algebraic space, or equivalently a representable morphism $\mathcal{X} \to B\operatorname{GL}_n$.

Exercise 6.2.2 (important)**.** Show that a noetherian algebraic stack $\mathcal{X}$ is a global quotient stack if and only if there exists a vector bundle E on $\mathcal{X}$ such that for every geometric point $x \colon \operatorname{Spec} \Bbbk \to \mathcal{X}$ with closed image, the stabilizer G_x acts faithfully on the fiber $E \otimes \Bbbk$.

Hint: Use the correspondence between principal GL_n*-bundles and vector bundles from Exercise B.1.56.*

Exercise 6.2.3. Show that a smooth Deligne–Mumford stack $\mathcal{X}$ over a noetherian scheme with generically trivial stabilizer is a global quotient stack.

Hint: Consider the tangent bundle and the higher jet bundles.

Exercise 6.2.4. If $\mathcal{X}$ is a reduced noetherian algebraic stack with affine stabilizers, show that there is a stratification $\mathcal{X} = \amalg_i \mathcal{X}_i$, where each $\mathcal{X}_i \hookrightarrow \mathcal{X}$ is a locally closed substack and a quotient stack. If $\mathcal{X}$ is normal, show that $\mathcal{X}_i$ be written as a quotient stack $[U_i/\operatorname{GL}_n]$ where U_i is a quasi-projective scheme U_i.

Hint: Construct a coherent sheaf with the property such that every stabilizer acts faithfully on the fiber, and consider the open locus where it is a vector bundle. For the final statement, apply Sumihiro's Theorem on Linearizations (B.1.30).

In Exercise 4.3.2, we've seen that a Deligne–Mumford stack can be written as a quotient stack $[U/G]$ with G a finite group and U an affine scheme (resp., scheme, algebraic space) if and only if it admits a finite étale cover by an affine scheme (resp., scheme, algebraic space). In a similar spirit, we have:

Exercise 6.2.5. Let $\mathcal{X} \to \mathcal{Y}$ be a surjective, flat, and projective morphism of noetherian algebraic stacks. If $\mathcal{X}$ is a quotient stack, show that $\mathcal{Y}$ is a quotient stack.

Remark 6.2.6 (Quasi-projective stacks). Let $\mathcal{X}$ be a separated Deligne–Mumford of finite type over a field $\Bbbk$ with coarse moduli space $\pi\colon \mathcal{X} \to X$. It is a theorem of Kresch [Kre09, Thm. 5.3] that X is quasi-projective and $\mathcal{X}$ is a global quotient stack if and only if $\mathcal{X} \cong [U/\operatorname{GL}_n]$ with U quasi-projective. Moreover, if $\Bbbk$ is characteristic 0, this if further equivalent to (a) X is quasi-projective and $\mathcal{X}$ has a *generating* coherent sheaf $\mathcal{V}$, i.e., for every quasi-coherent sheaf F on X, $\pi_*\pi^*\mathscr{H}om_{\mathcal{O}_\mathcal{X}}(\mathcal{V}, F) \otimes_{\mathcal{O}_\mathcal{X}} \mathcal{V} \to F$ is surjective, or to (b) there exists a locally closed immersion $\mathcal{X} \hookrightarrow \mathcal{W}$, where $\mathcal{W}$ is a smooth proper Deligne–Mumford stack over $\Bbbk$ with projective coarse moduli space. See also the generalization in [BOW24].

Remark 6.2.7 (Non-quotient stacks). It is not easy to give examples of algebraic stacks with affine stabilizers that are not global quotient stacks. For instance, it is not known whether every separated Deligne–Mumford stack is a global quotient stack. A necessary condition for a noetherian algebraic stack to be a global quotient is that it have quasi-affine diagonal since every quasi-separated algebraic space has quasi-affine diagonal (Corollary 4.5.8). In [Kre13, Prop. 5.2], Kresch showed that the moduli stack $\mathcal{M}_0^{\leq m}$ parameterizing prestable curves of genus 0 with at most $m \geq 2$ does not have quasi-affine diagonal. Thus, $\mathcal{M}_0^{\leq m}$ is not a global quotient. Another necessary condition is that $\mathcal{X}$ has sufficiently many vector bundles (Exercise 6.2.2), and this can also be used to construct counterexamples (see Exercise 6.2.8). Non-quotient stacks can also be constructed using the non-surjectivity of the Brauer map (see Exercise 6.4.38). Give two forward references to examples using Brauer map

Exercise 6.2.8. Consider a split torus $T = \mathbb{G}^r_{m,\mathbb{P}^1}$ of rank at least $r \geq 2$ over $\mathbb{P}^1 = \mathbb{P}^1_\Bbbk$ for a field $\Bbbk$. Show that there is a non-trivial involution $\alpha\colon T_0 = \mathbb{G}^r_{m,\Bbbk} \to \mathbb{G}^r_{m,\Bbbk} = T_\infty$ which glues to a non-split torus T' over the projective nodal cubic $C = \mathbb{P}^1/(0 \sim \infty)$ such that BT' has no vector bundles. Conclude that BT' is not a global quotient stack.

6.2.2 The resolution property and Totaro's Theorem

Being a quotient stack is closely related to the resolution property.

Definition 6.2.9. A noetherian algebraic stack has the *resolution property* if every coherent sheaf is the quotient of a vector bundle.

A smooth or quasi-projective scheme over a field has the resolution property. More generally, a scheme admitting an "ample family" of line bundles has the resolution property, and this implies that every noetherian normal $\mathbb{Q}$-factorial scheme with affine diagonal has the resolution property [BS03].

Proposition 6.2.10. *Let G be an affine algebraic group over a field $\Bbbk$ acting on a quasi-projective $\Bbbk$-scheme U. Assume that there is an ample line bundle L with an action of G (e.g., U is quasi-affine and $L = \mathcal{O}_U$). Then $[U/G]$ has the resolution property.*

Proof. The line bundle L corresponds to a line bundle $\mathcal{L}$ on $[U/G]$ which is relatively ample with respect to the morphism $p\colon [U/G] \to BG$. For a coherent sheaf F on $[U/G]$, the natural map

$$\mathcal{L}^{-N} \otimes p^*p_*(\mathcal{L}^N \otimes F) \twoheadrightarrow F$$

is surjective for $N \gg 0$. The pushforward $p_*(\mathcal{L}^N \otimes F)$ is a quasi-coherent sheaf on BG, i.e., a G-representation, which we can write as a union of finite dimensional G-representations V_i (B.1.17(1)). We therefore obtain a surjection $\operatorname{colim}_i(\mathcal{L}^{-N} \otimes p^*V_i) \twoheadrightarrow F$. Since F is coherent, $\mathcal{L}^{-N} \otimes p^*V_i \twoheadrightarrow F$ is surjective for $i \gg 0$. $\square$

Exercise 6.2.11. Show that $[U/\operatorname{GL}_n]$ has the representation property if U is a quasi-projective scheme over $\mathbb{Z}$ with an action of GL_n such that U has an ample line bundle with a GL_n-action. (See [Tho87, §2] for further generalizations.)

Totaro's Theorem gives a converse.

Theorem 6.2.12 (Totaro's Theorem)**.** *Let $\mathcal{X}$ be a normal noetherian algebraic stack whose stabilizer groups at closed points are affine. Then the following are equivalent*

(1) $\mathcal{X}$ has the resolution property,

(2) $\mathcal{X} \cong [U/\operatorname{GL}_n]$ where U is quasi-affine, and

if in addition $\mathcal{X}$ is of finite type over a field $\Bbbk$, then the above are equivalent to

(3) $\mathcal{X} \cong [\operatorname{Spec} A/G]$ where G is a smooth affine algebraic group over $\Bbbk$.

In particular, $\mathcal{X}$ has affine diagonal.

Remark 6.2.13. This has been extended to non-normal stacks by Gross [Gro17].

Proof. Since $\mathcal{X}$ is normal, Sumihiro's Theorem on Linearizations (B.1.30) implies that $\mathcal{O}_U$ admits a GL_n-action (resp., $\mathcal{O}_{\operatorname{Spec} A}$ admits a G-action), and therefore Exercise 6.2.11 (resp., Proposition 6.2.10) yields the implication (2) $\Rightarrow$ (1) (resp., (3) $\Rightarrow$ (1)).

To see (3) $\Rightarrow$ (2), it suffices to find a faithful representation $G \hookrightarrow \operatorname{GL}_N$ over $\Bbbk$ such that GL_N/G is quasi-affine. Indeed, in this case, $[\operatorname{Spec} A/G] \cong [(\operatorname{Spec} A \times^G \operatorname{GL}_N)/\operatorname{GL}_N]$ (Exercise 3.4.19) and $\operatorname{Spec} A \times^G \operatorname{GL}_N$ is affine over GL_N/G. We begin by choosing any faithful representation $G \subseteq \operatorname{GL}_n$. By B.1.16(7), GL_n/G is quasi-projective, and so there is a GL_n-representation V and a $\Bbbk$-point $x \in \mathbb{P}(V)$ with stabilizer G. Under the action of $\operatorname{GL}_n \times \mathbb{G}_m$ on $\mathbb{A}(V)$ (where $\mathbb{G}_m$ acts via scaling), the stabilizer of a lift $\widetilde{x} \in \mathbb{A}(V)$ of x is G. The map $(\operatorname{GL}_n \times \mathbb{G}_m)/G \hookrightarrow \mathbb{A}(V)$, defined by $g \mapsto g\widetilde{x}$, is a locally closed immersion and thus $(\operatorname{GL}_n \times \mathbb{G}_m)/G$ is quasi-affine. Under the natural inclusion $\operatorname{GL}_n \times \mathbb{G}_m \hookrightarrow \operatorname{GL}_{n+1}$, the quotient $\operatorname{GL}_{n+1}/(\operatorname{GL}_n \times \mathbb{G}_m)$ is affine (and is sometimes called the "Steifel manifold"). The composition $BG \to B(\operatorname{GL}_n \times \mathbb{G}_m) \to B\operatorname{GL}_{n+1}$ is quasi-affine and therefore so is $\operatorname{GL}_{n+1}/G$ and $U := \operatorname{Spec} A \times^G \operatorname{GL}_N$.

Conversely for (2) $\Rightarrow$ (3), we first choose a GL_n-equivariant open immersion $U \hookrightarrow \operatorname{Spec} A$ into an affine scheme of finite type over $\Bbbk$: indeed, since the morphism $p\colon [U/\operatorname{GL}_n] \to B\operatorname{GL}_n$ is quasi-affine, $[U/\operatorname{GL}_n] \to \operatorname{Spec}_{B\operatorname{GL}_n} p_*\mathcal{O}_{[U/\operatorname{GL}_n]}$ is an open immersion, and thus by writing $p_*\mathcal{O}_{[U/\operatorname{GL}_n]} = \operatorname{colim}_\lambda \mathcal{A}_\lambda$ as a colimit of finite type $\mathcal{O}_{B\operatorname{GL}_n}$-algebras, Limit Methods (B.3) imply that $[U/\operatorname{GL}_n] \to \operatorname{Spec}_{B\operatorname{GL}_n} \mathcal{A}_\lambda$ is an open immersion for $\lambda \gg 0$. Letting $Z \subseteq \operatorname{Spec} A$ be the reduced complement of U, choose a GL_n-equivariant morphism $f\colon \operatorname{Spec} A \to \mathbb{A}^r$ such that $f^{-1}(0) = Z$ (Proposition B.1.18(2)). This gives an affine morphism $U \to \mathbb{A}^r \setminus 0$. The complement $\mathbb{A}^r \setminus 0$ can be realized as the quotient GL_r/H where $H \cong \mathbb{G}_a^{r-1} \rtimes \operatorname{GL}_{r-1} \subseteq \operatorname{GL}_r$ is the subgroup consisting of matrices whose last row is $(0, \ldots, 0, 1)$. In the GL_n-equivariant cartesian diagram

$$\begin{tikzcd} P \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \operatorname{GL}_r \arrow[d] \\ U \arrow[r] & \mathbb{A}^r \setminus 0, \end{tikzcd}$$

P is affine over GL_r, thus affine. We conclude using the equivalence $[U/\operatorname{GL}_n] \cong [P/(\operatorname{GL}_n \times H)]$.

It remains to show (1) $\Rightarrow$ (2). We first show that there is a vector bundle on E such that every stabilizer group G_x acts faithfully on the fiber $E \otimes \kappa(x)$. By

Exercise 6.2.2, this implies that $\mathcal{X} = [U/\operatorname{GL}_n]$ where $n = \operatorname{rk} E$ and $U = \operatorname{Fr}(E)$ is the frame bundle. For every closed point $x \in \mathcal{X}$, let $i_x : \mathcal{G}_x \hookrightarrow \mathcal{X}$ be the inclusion of the residual gerbe (Proposition 3.5.16). Choosing a finite type point $\overline{x} \colon \operatorname{Spec} \Bbbk \to \mathcal{X}$ representing x, there is an induced finite morphism $BG_{\overline{x}} \to \mathcal{G}_x$. Since $G_{\widetilde{x}}$ is an affine algebraic group, we can choose a faithful representation W (Proposition B.1.18(2)). Using the resolution property on $\mathcal{X}$, there is a vector bundle E and a surjection $E \twoheadrightarrow (i_x \circ p)_* W$. The kernel subgroup $S_E \subseteq I_{\mathcal{X}}$ of E, i.e., the subgroup stack of the inertia stack parameterizing elements acting trivially on E, is trivial over x. If F is another vector bundle, then $S_{E \oplus F} \subseteq S_E$ is a closed subgroup. Since $I_{\mathcal{X}}$ is noetherian, we can use noetherian induction to choose a vector bundle E with trivial subgroup S_E.

Since $\mathcal{X}$ is normal, so is U, and we may apply Exercise 4.6.4 to arrange that U is the coarse moduli space of the action of a finite group H acting on a normal scheme U'. Let $g \colon U' \to U$ be the quotient morphism, and let $\{U'_i\}$ be a finite affine covering with reduced complements $Z'_i = U' \smallsetminus U'_i$. Then $F := g_*(\bigoplus_i I_{Z'_i})$ is a coherent sheaf on U such that $g^* F \twoheadrightarrow \bigoplus_i I_{Z'_i}$ is surjective. Since $q \colon U \to \mathcal{X}$ is quasi-affine, $q^* q_* F \twoheadrightarrow F$ is surjective, and by writing $q_* F$ as a colimit of coherent sheaves, we may find a coherent sheaf G on $\mathcal{X}$ and a surjection $q^* G \to F$. Since $\mathcal{X}$ has the resolution property, we can arrange that G is a vector bundle. This gives a surjection $g^* q^* G \twoheadrightarrow g^* F \twoheadrightarrow \bigoplus_i I_{Z'_i}$. Letting $V = \operatorname{Fr}(G)$ be the frame bundle, consider the cartesian diagram

$$\begin{tikzcd}
U'_V \arrow[r] \arrow[d, "\beta"] \arrow[dr, phantom, "\square"] & U_V \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & V \arrow[d] \\
U' \arrow[r, "g"] & U \arrow[r, "q"] & \mathcal{X},
\end{tikzcd}$$

where g is a principal H-bundle, q is a principal GL_n-bundle, and $V \to \mathcal{X}$ is a principal GL_m-bundle (with $m = \operatorname{rk} G$). Since the pullback of G to V is trivial, the pullback $\beta^*(\bigoplus_i I_{Z'_i})$ is globally generated. This implies that $\beta^{-1}(Z'_i)$ is defined by global functions on U'_V and that the complement $\beta^{-1}(U'_i)$ is covered by affine opens of the form $\{f \neq 0\}$ for $f \in \Gamma(U'_V, \mathcal{O}_{U'_V})$. In other words, $\mathcal{O}_{U'_V}$ is ample and U'_V is a quasi-affine scheme. Since $\beta \colon U'_V \to U_V$ is the quotient by the finite group H, U_V is also quasi-affine (Exercise 4.2.14). We can write $\mathcal{X} \cong [\operatorname{Fr}(E \oplus F)/\operatorname{GL}_{n+m}]$, where the frame bundle $\operatorname{Fr}(E \oplus F)$ is identified with $U_V \times^{(\operatorname{GL}_n \times \operatorname{GL}_m)} \operatorname{GL}_{n+m}$, which is quasi-affine since U_V is quasi-affine and the quotient $\operatorname{GL}_{n+m}/(\operatorname{GL}_n \times \operatorname{GL}_m)$ is affine. $\square$

6.3 The fppf topology

Waste not thy time in windy argument but let the matter drop.

William Shakespeare

While not essential for the two main theorems of this book, we cover the important facts that algebraic spaces/stacks are sheaves/stacks in the fppf topology and that quotients by fppf groupoids/equivalence relations are algebraic. One upshot is that BG is an algebraic stack for any (non-necessarily smooth) algebraic group, e.g., $\boldsymbol{\mu}_p$ in characteristic p. The fppf topology will allow us to define gerbes in a suitable generality in the next section.

6.3.1 Fppf criterion for algebraicity

Theorem 6.3.1 (Fppf Criterion for Algebraicity)**.**

(1) If X is a sheaf on $\mathrm{Sch}_{\mathrm{fppf}}$ such that there exists an fppf representable morphism $U \to X$ from an algebraic space, then X is an algebraic space.

(2) If $\mathcal{X}$ is a stack over $\mathrm{Sch}_{\mathrm{fppf}}$ such that there exists an fppf representable morphism $U \to \mathcal{X}$ from an algebraic space, then $\mathcal{X}$ is an algebraic stack.

Proof. Part (1) follows from (2) since an algebraic stack that is equivalent to a sheaf is an algebraic space (Theorem 4.5.10). The proof uses a technique analogous to the general Keel–Mori Theorem (see Remark 4.4.11). Since $U \to \mathcal{X}$ is representable, the relative Hilbert stack $\mathcal{H}\mathrm{ilb}_d$ over $\mathcal{X}$, whose objects over $S \to \mathcal{X}$ consist of a closed subscheme $Z \hookrightarrow U \times_{\mathcal{X}} S$ finite and flat of degree d over S, is an algebraic stack. Restricting to the open substack $\mathcal{H}\mathrm{ilb}_{d,\mathrm{lci}}$ parameterizing local complete intersections $Z \hookrightarrow U \times_{\mathcal{X}} S$, one shows that $\coprod_d \mathcal{H}\mathrm{ilb}_d \to \mathcal{X}$ is smooth and surjective using the Infinitesimal Lifting Criterion (3.7.1). For details, see [Art74b, Thm. 6.1], [LMB00, Thm. 10.1], and [SP, Tag 06DC]. □

Remark 6.3.2. The result is not true if you replace 'fppf' with 'fpqc'; see [MO10].

Algebraic spaces are by definition sheaves in the big étale topology but also sheaves in the big fppf topology.

Corollary 6.3.3 (Algebraic Stacks are Fppf Stacks)**.**

(1) An algebraic space X over a scheme S is a sheaf on $(\mathrm{Sch}/S)_{\mathrm{fppf}}$.

(2) An algebraic stack $\mathcal{X}$ over a scheme S is a stack over $(\mathrm{Sch}/S)_{\mathrm{fppf}}$.

Proof. It suffices to show (2). Let $U \to \mathcal{X}$ be a smooth presentation and $\mathcal{X}'$ be the stackification of $\mathcal{X}$ in the fppf topology. Since $U \to \mathcal{X}'$ is fppf and representable, $\mathcal{X}'$ is an algebraic stack by Theorem 6.3.1. As $U \times_{\mathcal{X}} U \cong U \times_{\mathcal{X}'} U$, the natural map $\mathcal{X} \to \mathcal{X}'$ is fully faithful. On the other hand, if $T \to \mathcal{X}'$ is any map from a scheme, the fiber product $U \times_{\mathcal{X}'} T$ is a smooth algebraic space over T. Since smooth morphisms have sections étale locally on the target, there exists an étale cover $T' \to T$ together with a map $T' \to U$ lifting $T' \to T \to \mathcal{X}$. The composition $T' \to U \to \mathcal{X}$ shows that objects of $\mathcal{X}'$ étale locally lift to $\mathcal{X}$, which implies that $\mathcal{X} \to \mathcal{X}'$ is essentially surjective. See [LMB00, Cor. 10.7(a)] and [SP, Tag 076V] □

Remark 6.3.4. In the *Stacks Project* [SP], an algebraic stack is by definition a stack in the fppf topology, and the above result ensures that our definition of an algebraic stack is equivalent to theirs. It is also true that algebraic spaces are sheaves in the fpqc topology, a result attributed to Gabber; see [LMB00, Thm. A.4] and [SP, Tag 0APL]. It is also known that algebraic stacks with quasi-affine diagonal are stacks in the fpqc topology [LMB00, Cor. 10.7(b)].

Exercise 6.3.5. Extend Proposition 3.3.6 by showing that the following properties of morphisms of algebraic stacks are fppf local on the target: representable, isomorphism, open immersion, closed immersion, locally closed immersion, affine, or quasi-affine.

6.3.2 Fppf groupoids and quotient stacks

If $R \rightrightarrows U$ is an fppf equivalence relation of algebraic spaces, we define U/R as the sheafification in big fppf topology $\mathrm{Sch}_{\mathrm{fppf}}$ of the presheaf $T \mapsto U(T)/R(T)$. Likewise, if $s, t \colon R \rightrightarrows U$ is an fppf groupoid of algebraic spaces, we define $[U/R]$ as the stackification in $\mathrm{Sch}_{\mathrm{fppf}}$ of the prestack $[U/R]^{\mathrm{pre}}$, whose fiber category $[U(T)/R(T)]$

over a scheme T is the category of T-points of U where a morphism from $a \in U(T)$ to $b \in U(T)$ is an element $r \in R(R)$ such that $s(r) = a$ and $t(r) = b$. The definitions of U/R and $[U/R]$ are consistent with the quotient of a smooth equivalence relation or groupoid (Definition 3.4.10) as a consequence of Corollary 6.3.3.

Corollary 6.3.6.

(1) If $R \rightrightarrows U$ is an fppf equivalence relation of algebraic spaces, then the quotient U/R is an algebraic space.

(2) If $R \rightrightarrows U$ is an fppf groupoid of algebraic spaces, then the quotient $[U/R]$ is an algebraic stack.

Proof. This follows from the Fppf Criterion for Algebraicity (6.3.1) since $U \to [U/R]$ is fppf and representable. $\square$

As a result, we can construct quotient stacks by fppf group algebraic spaces generalizing the quotient stacks by smooth affine group schemes from Theorem 3.1.10. We first need some definitions.

Definition 6.3.7 (Principal G-bundles)**.** If $G \to S$ is an fppf group algebraic space, then a *principal G-bundle over an S-scheme T* is an algebraic space P with an action of G via $\sigma\colon G \times_S P \to P$ such that $P \to X$ is a G-invariant fppf morphism and $(\sigma, p_2)\colon G \times_S P \to P \times_T P$ is an isomorphism. *Morphisms of principal G-bundles* are G-equivariant morphisms of schemes. We say that a principal G-bundle $P \to T$ is *trivial* if there is a G-equivariant isomorphism $P \cong G \times T$.

In the above definition, it is important to require that P is an *algebraic space* as will want principal G-bundles to satisfy descent in étale topology. Raynaud provides an example of an abelian variety G and a principal G-bundle that is not scheme [Ray70, XIII 3.2]. When $G \to S$ is smooth and affine, this definition agrees with Definition B.1.47: indeed, P is a scheme since it is affine over T and since every principal G-bundle $P \to T$ is trivialized by the smooth cover $P \to T$, we may use the fact that smooth morphisms étale locally have sections to construct an étale cover $T' \to T$ such that $P_{T'}$ is trivial.

Definition 6.3.8 (Quotient and classifying stacks)**.** Let $G \to S$ be an fppf group algebraic space acting on an algebraic space U over S. We define the *quotient stack* $[U/G]$ as the category over Sch/S whose objects over an S-scheme T are diagrams

$$T \leftarrow P \to U \tag{6.3.9}$$

where $P \to T$ is a principal G-bundle and $P \to U$ is a G-equivariant morphism of schemes. A morphism $(T' \leftarrow P' \to U) \to (T \leftarrow P \to U)$ consists of a morphism $T' \to T$ and a G-equivariant morphism $P' \to P$ of schemes such that the diagram

$$\begin{tikzcd} P' \arrow[r] \arrow[d] \arrow[rr, bend left] \arrow[dr, phantom, "\square"] & P \arrow[r] \arrow[d] & U \\ T' \arrow[r] & T & \end{tikzcd}$$

is commutative and the left square is cartesian.

The *classifying stack* BG of G, defined as the quotient stack $[S/G]$, classifies principal G-bundles $P \to T$.

Proposition 6.3.10. *If $G \to S$ is an fppf group algebraic space acting on an algebraic space U over S, then the quotient stack $[U/G]$ is an algebraic stack. In particular, the classifying stack BG is algebraic.*

Proof. Given a map $T \to [U/G]$ corresponding to an object (6.3.9), there is a cartesian diagram

$$\begin{tikzcd} P \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & U \arrow[d] \\ T \arrow[r] & {[U/G]} \end{tikzcd}$$

of stacks over $\mathrm{Sch}_{\mathrm{fppf}}$ (see Exercise 2.4.37). As $P \to T$ is an fppf morphism of algebraic spaces, $U \to [U/G]$ is an fppf representable morphism, the Fppf Criterion for Algebraicity (6.3.1) implies that $[U/G]$ is an algebraic stack. $\square$

Exercise 6.3.11. Let $G \to S$ be an fppf group algebraic space acting on an algebraic space U over S.

(a) Show that the stackification of the prestack $[U/G]^{\mathrm{pre}}$ in the fppf topology is $[U/G]$. (This generalizes Exercise 2.5.21.)

(b) (hard) Provide an example where the stackification of $[U/G]^{\mathrm{pre}}$ in the étale topology is not isomorphic to $[U/G]$.

Recalling that $\boldsymbol{\mu}_{n,\mathbb{Z}}$ is the subgroup of $\mathbb{G}_{m,\mathbb{Z}}$ defined by $\operatorname{Spec} \mathbb{Z}[x]/(x^n - 1)$, we can now deduce that $B\boldsymbol{\mu}_{n,\mathbb{Z}}$ is an algebraic stack. If $\Bbbk$ is a field of characteristic p, then $\boldsymbol{\mu}_n := \boldsymbol{\mu}_{n,\Bbbk}$ is smooth and $B\boldsymbol{\mu}_n$ is Deligne–Mumford if and only if p doesn't divide n.

Exercise 6.3.12. Let $\Bbbk$ be a field.

(a) Exhibit an explicit smooth presentation of $B\boldsymbol{\mu}_n$.

(b) Show that $B\boldsymbol{\mu}_n$ is equivalent to the stack over $(\mathrm{Sch}/\Bbbk)_{\text{ét}}$ whose objects over a scheme T are pairs (L, α) consisting of a line bundle L on T and a trivialization $\alpha \colon \mathcal{O}_T \xrightarrow{\sim} L^{\otimes n}$.

(c) Show that $B\boldsymbol{\mu}_n$ is a smooth and proper algebraic stack of dimension 0.

(d) If $x \colon \operatorname{Spec} \Bbbk \to B\boldsymbol{\mu}_n$ denotes the canonical presentation, compute the tangent space $T_{B\boldsymbol{\mu}_n, x}$.

Hint: The answer depends on the characteristic.

(e) (hard) Show that the stack parameterizing $\boldsymbol{\mu}_n$-bundles that are trivializable in the étale topology is not algebraic.

6.4 Gerbes, rigidification, and Picard schemes

Most hyperelliptic curves are pointless.

DON ZAGIER

Gerbes play a central role in the theory of stacks. For the purposes of this book, we want to know that residual gerbes are gerbes (Proposition 6.4.26) thereby justifying the terminology and that the map $\underline{\mathcal{P}ic}_X \to \underline{\mathrm{Pic}}_X$ from the Picard stack to Picard scheme and $\mathcal{B}un^{\mathrm{s}}_{r,d}(C) \to M^{\mathrm{s}}_{r,d}(C)$ from the stack of *stable* vector bundles to its coarse moduli space is a banded $\mathbb{G}_m$-gerbe (Corollary 8.2.23).

6.4.1 Torsors

For a sheaf G of groups, a G-torsor is a sheaf of sets locally isomorphic to G. When G is representable by an fppf group scheme, this notion is equivalent to that of a principal G-bundle.

Definition 6.4.1 (Torsors)**.** Let $\mathcal{S}$ be a site and G a sheaf of (not necessarily abelian) groups on $\mathcal{S}$. A *G-torsor on $\mathcal{S}$* is a sheaf P of sets on $\mathcal{S}$ with a left action $\sigma\colon G \times P \to P$ of G such that

(1) for every object $T \in \mathcal{S}$, there exists a covering $\{T_i \to T\}$ such that $P(T_i) \neq 0$ for each i, and

(2) the action map $(\sigma, p_2)\colon G \times P \to P \times P$ is an isomorphism.

If $T \in \mathcal{S}$ is an object and G is a sheaf of groups on the restricted site $\mathcal{S}/T$ (Example 2.2.12), then a *G-torsor over T* is by definition a G-torsor on the $\mathcal{S}/T$.

Morphisms of G-torsors are G-equivariant morphisms of sheaves. We say that a G-torsor P is *trivial* if P is G-equivalently isomorphic to G.

Exercise 6.4.2. Show that any morphism of G-torsors is an isomorphism.

Example 6.4.3. Let $\mathcal{X}$ be a stack over a site $\mathcal{S}$, and let $a, b \in \mathcal{X}$ be objects over $S \in \mathcal{S}$. The sheaf $\underline{\mathrm{Isom}}_S(a, b)$ of isomorphisms is a torsor for $\underline{\mathrm{Aut}}(a)$ under the action given by precomposition.

Given a morphism $f\colon T' \to T$ and a G-torsor P over T, the restriction $P|_{T'}$, which is the sheaf on $\mathcal{S}/T'$ whose whose sections over a T'-scheme S are $P(S)$, is naturally a G-torsor over T'.

Exercise 6.4.4. Let $\mathcal{S}$ be a site with a final object S and G be a sheaf of groups on $\mathcal{S}$.

(a) Show that Axiom (1) is equivalent to $P \to S$ being an epimorphism of sheaves.

(b) If P is a G-torsor, show that S is isomorphic to the quotient sheaf P/G.

(c) Show that a G-torsor P is trivial if and only if there exists a section $s\colon S \to P$ of the structure morphism $P \to S$.

(d) Show that a sheaf P of sets on $\mathcal{S}$ with a left action by G is a G-torsor if and only if there exists a covering $\{S_i \to S\}$ and isomorphisms $P|_{S_i} \cong G|_{S_i}$ of $G|_{S_i}$-torsors.

Example 6.4.5 (Principal G-bundles)**.** For an fppf group scheme $G \to S$, there is an equivalence of categories between G-torsors in the fppf topology and principal G-bundles (as defined in Definition 6.3.7). To see this, observe that if $P \to T$ is a principal G-bundle over an S-scheme T, then since algebraic spaces are sheaves in the fppf topology (Corollary 6.3.3), both $G \times_S T$ and P are sheaves on $(\mathrm{Sch}/T)_{\mathrm{fppf}}$. Since $P \to T$ is trivialized by the fppf map $P \to T$, Exercise 6.4.4(d) implies that $P \to T$ is a G-torsor. Conversely, given a G-torsor P on $(\mathrm{Sch}/T)_{\mathrm{fppf}}$ and an fppf cover $T' \to T$ trivializing P, i.e., $P \times_T T' \cong G \times_T T'$, it follows that $P \times_T T'$ is an algebraic space and $P \times_T T' \to T'$ is an fppf and representable morphism. The Fppf Criterion for Algebraicity (6.3.1) implies that P is an algebraic space, and thus $P \to T$ is a principal G-bundle. If in addition $G \to S$ is smooth, then there is an equivalence of categories between G-torsors in the étale topology and principal G-bundles. This holds because smooth morphisms have sections étale locally and thus every principal G-bundle $P \to T$ is étale locally trivial.

6.4.2 Gerbes

Gerbes are a 2-categorical generalization of torsors. While torsors are locally isomorphic to a sheaf of groups G, gerbes are locally isomorphic to classifying stacks BG.

Definition 6.4.6 (Gerbes)**.** A stack $\mathcal{X}$ over a site $\mathcal{S}$ is called a *gerbe* if

(1) for every object $T \in \mathcal{S}$, there exists a covering $\{T_i \to T\}$ in $\mathcal{S}$ such that each fiber category $\mathcal{X}(T_i)$ is non-empty; and

(2) for objects $x, y \in \mathcal{X}$ over $T \in \mathcal{S}$, there exists a covering $\{T_i \to T\}$ and isomorphisms $x|_{T_i} \xrightarrow{\sim} y|_{T_i}$ for each i.

We say that a gerbe $\mathcal{X}$ is *trivial* if there is a section $\mathcal{S} \to \mathcal{X}$ of $\mathcal{X} \to \mathcal{S}$. When $\mathcal{S}$ has a final object S, triviality is equivalent to the non-emptiness of $\mathcal{X}(S)$.

Example 6.4.7. For a sheaf of groups G on a site $\mathcal{S}$, we extend Definition 6.3.8 by defining the *classifying stack of* G as the stack BG over $\mathcal{S}$ consisting of pairs (P, T) where $T \in \mathcal{S}$ and P is G-torsor over $\mathcal{S}/T$ (Definition 6.4.1). A morphism $(P', T') \to (P, T)$ is the data of a morphism $T' \to T$ in $\mathcal{S}$ and an isomorphism $P' \to P|_{T'}$ of G-torsors, where $P|_{T'}$ denotes the restriction of P along $T' \to T$. The classifying stack BG is a gerbe over $\mathcal{S}$ because every G-torsor over T is locally isomorphic to the trivial G-torsor $G \times T$.

Exercise 6.4.8 (Gerbes are locally classifying stacks)**.** Let $\mathcal{S}$ be a site with a final object $S \in \mathcal{S}$, and let $\mathcal{X}$ be a stack over $\mathcal{S}$. Show that $\mathcal{X}$ is a gerbe if and only if there exists a covering $\{S_i \to S\}$ and sheaves of groups G_i on the restricted site $\mathcal{S}/S_i$ (Example 2.2.12) such that there is an isomorphism $\mathcal{X} \times_S S_i \cong BG_i$ over $\mathcal{S}/S_i$.

Exercise 6.4.9. Let S be a scheme and let $\mathcal{X}$ be a gerbe over $(\mathrm{Sch}/S)_{\mathrm{fppf}}$. If the diagonal $\mathcal{X} \to \mathcal{X} \times_S \mathcal{X}$ is representable, show that $\mathcal{X}$ is an algebraic stack.

A *banding* of a gerbe $\mathcal{X}$ by a sheaf of groups G is the additional data of a natural isomorphism $G(T) \to \mathrm{Aut}_T(x)$ for every object $x \in \mathcal{X}(t)$.

Definition 6.4.10 (Banded G-gerbes)**.** Let G be an *abelian* sheaf on a site $\mathcal{S}$. A stack $\mathcal{X}$ over $\mathcal{S}$ is a *gerbe banded by* G (or a *banded* G*-gerbe* or simply a G*-gerbe*) is a gerbe together with the data of isomorphisms $\psi_x \colon G|_T \to \underline{\mathrm{Aut}}_T(x)$ of sheaves for each object $x \in \mathcal{X}(T)$. We require that for each isomorphism $\alpha \colon x \xrightarrow{\sim} y$ over T, the diagram

$$\begin{tikzcd} & G|_T \arrow[dl, "\psi_x"'] \arrow[dr, "\psi_y"] & \\ \underline{\mathrm{Aut}}_T(x) \arrow[rr, "\mathrm{Inn}_\alpha"] & & \underline{\mathrm{Aut}}_T(y). \end{tikzcd} \tag{6.4.11}$$

commutes, where $\mathrm{Inn}_\alpha(\tau) = \alpha\tau\alpha^{-1}$. The data of the isomorphisms ψ_x is called the *band* of $\mathcal{X}$. A *morphism of banded* G*-gerbes* is a morphism of stacks compatible with the bands.

Remark 6.4.12. For another way to think about a band of a gerbe, let $\mathcal{S}/\mathcal{X}$ be the restricted site whose underlying category is $\mathcal{X}$ and where a covering of $a \in \mathcal{X}(S)$ is a covering of S. Then the inertia stack $I_{\mathcal{X}} = \mathcal{X} \times_{\mathcal{X} \times \mathcal{X}} \mathcal{X}$ is a sheaf of groups on $\mathcal{S}/\mathcal{X}$: for $a \in \mathcal{X}(S)$, we have $I_{\mathcal{X}}(a) = \mathrm{Isom}_S(a)$. The compatibility condition (6.4.11) ensures that there is an isomorphism $\psi \colon G|_{\mathcal{X}} \to I_{\mathcal{X}}$ of sheaves on $\mathcal{S}/\mathcal{X}$.

Example 6.4.13 (The trivial banded gerbe)**.** If G is an abelian sheaf on a site $\mathcal{S}$ with a final object S, then the classifying stack BG of Example 6.4.7 is a banded G-gerbe and is trivial. A banded G-gerbe $\mathcal{X}$ over $\mathcal{S}$ is trivial if and only if $\mathcal{X} \cong BG$.

Exercise 6.4.14 (Band associated to a gerbe)**.** Let $\mathcal{S}$ be a site with a final object S. Let $\mathcal{X}$ be an *abelian gerbe* over $\mathcal{S}$, i.e., a gerbe $\mathcal{X}$ such that $\mathrm{Aut}_T(a)$ is abelian for every object $a \in \mathcal{X}(T)$. Show that there is a sheaf of groups G on S such that $\mathcal{X}$ is banded by G.

Hint: Use Axiom (1) of a gerbe to find a covering $\{X_i \to X\}$ and elements $a_i \in \mathcal{X}(X_i)$. Use Axiom (2) to glue the sheaves $G_i := \underline{\mathrm{Aut}}_{X_i}(a_i)$ to a sheaf G.

6.4.3 Algebraic gerbes

Attached to any algebraic stack $\mathcal{Y}$ is the big fppf site $(\mathrm{Sch}/\mathcal{Y})_{\mathrm{fppf}}$ of schemes over $\mathcal{Y}$: the underlying category of $(\mathrm{Sch}/\mathcal{Y})_{\mathrm{fppf}}$ is $\mathcal{Y}$ and a covering of an object $y \in \mathcal{Y}(T)$ is a covering of T. If $\mathcal{X} \to \mathcal{Y}$ is a morphism of algebraic stacks, then $\mathcal{X}$ is a stack over $(\mathrm{Sch}/\mathcal{Y})_{\mathrm{fppf}}$ thanks to Corollary 6.3.3.

Definition 6.4.15 (Gerbes and banded G-gerbes)**.** A morphism $\mathcal{X} \to \mathcal{Y}$ of algebraic stacks is a *gerbe* if $\mathcal{X}$ is a gerbe over the big fppf site $(\mathrm{Sch}/\mathcal{Y})_{\mathrm{fppf}}$. If $G \to S$ is a commutative fppf group scheme, a morphism $\mathcal{X} \to \mathcal{Y}$ of algebraic stacks over S is called a *banded G-gerbe* (or simply *G-gerbe*) if $\mathcal{X}$ is a gerbe over $(\mathrm{Sch}/\mathcal{Y})_{\mathrm{fppf}}$ and is banded by the sheaf of groups $G \times_S \mathcal{Y}$. We say that an algebraic stack $\mathcal{X}$ is a *gerbe* (resp., *banded G-gerbe*) if there exists a morphism $\mathcal{X} \to X$ to an algebraic space which is a gerbe (resp., banded G-gerbe).

A banded G-gerbe $\mathcal{X} \to X$ over an algebraic space X is trivial if and only if $\mathcal{X} \cong BG \times_S X$.

Proposition 6.4.16. *Let $\mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks.*

1. *The morphism $\mathcal{X} \to \mathcal{Y}$ is a gerbe if and only if there exists an fppf morphism $S \to \mathcal{Y}$ from a scheme and an fppf group algebraic space $G \to S$ such that $\mathcal{X} \times_{\mathcal{Y}} S \cong BG$.*
2. *If $\mathcal{X} \to \mathcal{Y}$ is a gerbe, then $\mathcal{X} \to \mathcal{Y}$ is a smooth morphism.*
3. *If $G \to S$ is an fppf group scheme and $\mathcal{X} \to X$ is a banded G-gerbe over an algebraic space X, then there exists an étale cover $X' \to X$ such that $\mathcal{X} \times_X X' \cong X' \times_S BG$.*

Proof. For the ($\Rightarrow$) of (1), by Exercise 6.4.8, there is an fppf morphism $S \to \mathcal{Y}$ and a sheaf of groups G on S such that $\mathcal{X} \times_{\mathcal{Y}} S \cong BG$. Since $\mathcal{X} \times_{\mathcal{Y}} S$ is an algebraic stack, its diagonal is representable, and thus G is an algebraic space. Conversely, if $\mathcal{X} \times_{\mathcal{Y}} S \cong BG$, then Axiom (1) of a gerbe holds: if $a \in (\mathrm{Sch}/\mathcal{Y})$ is an object over a scheme T, then since $S_T := S \times_{\mathcal{Y}} T \to T$ is an fppf covering and $\mathcal{X} \times_{\mathcal{Y}} S_T \cong BG_{S_T}$, there is an object of $\mathcal{X}$ over S_T. Similarly for Axiom (2), if $x_1, x_2 \in \mathcal{X}$ are objects over $y \in \mathcal{Y}(T)$, then pullbacks of x_1 and x_2 become isomorphic under the fppf covering $S_T \to T$. Part (2) follows from (1) since $BG \to S$ is a smooth morphism (even if $G \to S$ is not smooth!): this follows from the considering the composition $S \to BG \to S$ and using that smoothness is an fppf local property on the source (Proposition 2.1.27). For part (3), $\mathcal{X} \to X$ has a section after base changing by the *smooth* and surjective morphism $\mathcal{X} \to X$. Choosing a smooth presentation $U \to \mathcal{X}$, then the composition $U \to \mathcal{X} \to X$ is smooth and the base change $\mathcal{X} \times_X U \to U$ has a section. The statement now follows from the fact that smooth morphisms étale locally have sections (Corollary A.3.5). $\square$

Exercise 6.4.17. Show that a morphism $\mathcal{X} \to \mathcal{Y}$ of algebraic stacks is a gerbe if and only if $\mathcal{X} \to \mathcal{Y}$ and $\mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ are fppf.

In fact, any algebraic stack with flat inertia is a gerbe. The following result will be applied to show that the moduli stack of simple bundles is a $\mathbb{G}_m$-gerbe over an algebraic space (Proposition 8.2.20).

Proposition 6.4.18. *Let $\mathcal{X}$ be an algebraic stack such that $I_{\mathcal{X}} \to \mathcal{X}$ is fppf. Let X be the sheaf on $\mathrm{Sch}_{\mathrm{fppf}}$ defined by the sheafification of the functor assigning a scheme S to the set of isomorphism classes $\mathcal{X}(S)/\sim$ of objects. Then X is an algebraic space and $\mathcal{X} \to X$ is a gerbe.*

Proof. To show that X is an algebraic space, it suffices to show that $\mathcal{X} \to X$ is a smooth representable morphism. Indeed, a smooth presentation $U \to \mathcal{X}$ induces a surjective, smooth, and representable morphism $U \to \mathcal{X} \to X$, and it follows from Corollary 4.5.12 (or Theorem 6.3.1) that X is an algebraic space. As gerbes are smooth morphisms, it suffices to show that for every morphism $S \to X$ from a scheme, the fiber product $\mathcal{X} \times_X S \to S$ is a gerbe. By construction, there is an fppf cover $S' \to S$ and a morphism $a' \colon S' \to \mathcal{X}$ lifting the composition $S' \to S \to X$. Since the property of being a gerbe is fppf local on the target, after replacing S with S', we may assume that $S \to X$ lifts to a map $a \colon S \to \mathcal{X}$. In this case, there is an isomorphism

$$\Psi \colon \mathcal{X} \times_X S \to B\underline{\mathrm{Aut}}_S(a).$$

The map Ψ is defined by sending an object $(b, f) \in \mathcal{X} \times_X S(T)$ to the principal $\underline{\mathrm{Aut}}_S(a)$-bundle $\underline{\mathrm{Isom}}_T(f^*a, b)$. The reader is encouraged to check that Ψ is an isomorphism. Since $\underline{\mathrm{Aut}}_S(a) \to S$ is an fppf group scheme, $B\underline{\mathrm{Aut}}_S(a) \to S$ is a gerbe.

Alternatively, the algebraic space X can be constructed directly. Choosing a smooth presentation $U \to \mathcal{X}$ inducing a smooth groupoid $R \rightrightarrows U$, the stabilizer groupoid scheme $S_U = R \times_{U \times U} U = I_{\mathcal{X}} \times_{\mathcal{X}} U$ is fppf over U. There is an fppf equivalence relation $S_U \times_U R \rightrightarrows R$ where one arrow is given by composition and the other is projection. By applying the algebraicity of quotients by fppf equivalence relations (Corollary 6.3.6) twice, the fppf quotient $R' := R/(S_U \times_U R)$ is an algebraic space, which induces an fppf equivalence relation $R' \rightrightarrows U$, and X is isomorphic to the fppf quotient U/R'. See also [LMB00, Cor. 10.8] and [SP, Tags 06QD and 06QJ]. ☐

Exercise 6.4.19. Show that every gerbe $\mathcal{X} \to X$, where $\mathcal{X}$ is an algebraic stack and X is an algebraic space, is a coarse moduli space (Definition 4.2.1).

Exercise 6.4.20. If $\mathcal{X}$ is a reduced noetherian algebraic stack, show that there is a dense open substack $\mathcal{U}$ which is a gerbe.

Hint: Use Generic Flatness (3.3.32).

Exercise 6.4.21. Show that there is a non-trivial isomorphism

$$\alpha \colon B(\mathbb{Z}/2) \times (\mathbb{A}^1 \smallsetminus 0) \to B(\mathbb{Z}/2) \times (\mathbb{A}^1 \smallsetminus 0)$$

of trivial banded $\mathbb{Z}/2$-gerbes over $\mathbb{A}^1$ which glues to a non-trivial banded $\mathbb{Z}/2$-gerbe over $\mathbb{P}^1$.

Exercise 6.4.22. Let $1 \to K \to G \to Q \to 1$ be a short exact sequence of affine algebraic groups over $\Bbbk$ such that K is commutative. Show that $BG \to BQ$ is a banded K-gerbe which is trivial if and only if the sequence splits.

Exercise 6.4.23. Assume that $\mathrm{char}(\Bbbk) \neq 2, 3$. Recall from Exercise 3.1.19(c) that the moduli stack of stable elliptic curves has a quotient description $\overline{\mathcal{M}}_{1,1} = \mathcal{P}(4,6) := [(\mathbb{A}^2 \setminus 0)/\mathbb{G}_m]$ where $\mathbb{G}_m$ acts with weights 4 and 6.

(a) Show that the j-line $\pi \colon \mathcal{M}_{1,1} \to \mathbb{A}^1$ is a trivial banded $\mathbb{Z}/2$-gerbe over $\mathbb{A}^1 \setminus \{0, 1728\}$.

Hint: Construct a family of elliptic curves over $\mathbb{A}^1_{\Bbbk} \setminus \{0, 1728\}$ *via the Weierstrass equation*

$$y^2z + xyz = x^3 - \frac{36}{t - 1728}xz^2 - \frac{1}{t - 1728}z^3,$$

where t is the coordinate on $\mathbb{A}^1$, where the discriminant $\Delta = t^2/(t - 1728)^3$. See [Sil09, Prop. III.1.4(c)].

(b) Consider the map $\overline{\mathcal{M}}_{1,1} = \mathcal{P}(4,6) \to \mathcal{P}(2,3)$ on weighted projective stacks induced the homomorphism $\mathbb{G}_m \to \mathbb{G}_m$ defined by $t \mapsto t^2$ (which we can view shortly as an example of rigidification). Note that the restriction along $\mathcal{P}(2,3) \setminus \{0, 1728, \infty\}$ is the gerbe from (a). Show that $\overline{\mathcal{M}}_{1,1} \to \mathcal{P}(2,3)$ is non-trivial.

Hint: If it is trivial, show that there are torsion line bundles. This contradicts $\mathrm{Pic}(\overline{\mathcal{M}}_{1,1}) = \mathbb{Z}$ *from Exercise 6.1.22(a).*

(c) Show that $\mathcal{M}_{1,1} \to \mathcal{P}(2,3) \setminus \infty$ is also non-trivial.

Hint: If it is trivial, show that $\mathcal{M}_{1,1}$ has three 2-torsion line bundles contradicting that $\mathrm{Pic}(\mathcal{M}_{1,1}) = \mathbb{Z}/12$ *from Exercise 6.1.22(b).*

Exercise 6.4.24.

(a) Show that every gerbe $\mathcal{X}$ over an algebraic space that is étale locally isomorphic to $B\mathbb{Z}/2$ is, in fact, banded by $\mathbb{Z}/2$.

(b) Give an example of a gerbe over an algebraic space that is étale locally isomorphic to $B\mathbb{G}_m$ but not banded by $\mathbb{G}_m$.

Hint: Consider the classifying stack of a form of $\mathbb{G}_m$.

Exercise 6.4.25 (Root gerbes and root stacks revisited)**.** Root gerbes and stacks were introduced in Examples 3.9.21 and 3.9.22. Since we now know how to construct quotient stacks by actions of $\boldsymbol{\mu}_r$, it follows that Exercise 3.9.23 extends to the case where r may be non-invertible in $\Gamma(S, \mathcal{O}_S)$. Given a scheme X and a line bundle L, show that $X(\sqrt[r]{L}) \to X$ is a banded $\boldsymbol{\mu}_r$-gerbe, which is trivial if and only if L has an rth root.

6.4.4 Residual gerbes revisited

Given an algebraic stack $\mathcal{X}$ and $x \in |\mathcal{X}|$, recall from Definition 3.5.12 that *the residual gerbe at x* (if it exists) is a reduced, locally noetherian algebraic stack $\mathcal{G}_x$ with a monomorphism $\mathcal{G}_x \hookrightarrow \mathcal{X}$ such that $|\mathcal{G}_x|$ is a point mapping to x. We have already shown that the residual gerbe at a finite type point exists and is unique (Proposition 3.5.16).

Proposition 6.4.26 (Existence of Residual Gerbes II)**.** *If $\mathcal{X}$ is a noetherian algebraic stack and $x \in |\mathcal{X}|$ is a point, then the residual gerbe $\mathcal{G}_x$ exists and is a gerbe over a field $\kappa(x)$, called the* residue field of x.

Proof. After replacing $\mathcal{X}$ with the closure $\overline{\{x\}}$, we may assume that $\mathcal{X}$ is reduced and that $x \in |\mathcal{X}|$ is dense. By Exercise 6.4.20 and Theorem 4.5.1, after replacing $\mathcal{X}$ with an open substack, we may assume that $\mathcal{X}$ is a gerbe over a scheme X. Letting $y \in X$ be the image of x, then $\mathcal{G}_x := \mathcal{X} \times_X \kappa(y)$ is a gerbe over $\kappa(x) := \kappa(y)$. The inclusion $\mathcal{G}_x \to \mathcal{X}$ is a monomorphism with image x and $\mathcal{G}_x$ is reduced noetherian substack with $|\mathcal{G}_x|$ a singleton. Therefore, $\mathcal{G}_x$ is a residual gerbe. The uniqueness follows from the same argument as in Proposition 3.5.16 See also [LMB00, Thm. 11.3], [Ryd11, Thm. B.2], or [SP, Tag 06UH]. ☐

If $\mathcal{X}$ is a quasi-separated algebraic stack of finite type over a field $\Bbbk$ and $x \in \mathcal{X}(\Bbbk)$, then $\mathcal{G}_x = BG_x$ (Proposition 3.5.16). More generally, we have:

Exercise 6.4.27. Let $\mathcal{X}$ be a noetherian algebraic stack and $x \in |\mathcal{X}|$ be a finite type point.

(1) For any representative $\overline{x} \colon \operatorname{Spec} \Bbbk \to \mathcal{X}$ of x, there is a cartesian diagram

$$\begin{tikzcd} BG_{\overline{x}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{G}_x \arrow[r, hook] \arrow[d] & \mathcal{X} \\ \operatorname{Spec} \Bbbk \arrow[r] & \operatorname{Spec} \kappa(x). \end{tikzcd}$$

(2) If the stabilizer of x is smooth, show that there is a finite separable extension $\kappa(x) \to \Bbbk$ and a representative of x over $\Bbbk$.

Exercise 6.4.28. Let $\mathcal{X}$ be a separated Deligne–Mumford stack with coarse moduli space $\pi \colon \mathcal{X} \to X$. Show that the residual gerbe of $x \in |\mathcal{X}|$ is reduction of the fiber $\mathcal{X}_{\pi(x)}$.

Exercise 6.4.29. Let $C \subseteq \mathbb{P}^2_{\Bbbk}$ be a non-split quadric over a field $\Bbbk$, and let $\Bbbk \to \Bbbk'$ be a quadratic extension such that $C \times_{\Bbbk} \Bbbk' \cong \mathbb{P}^1_{\Bbbk'}$. Let $D \subseteq C$ be a divisor of degree 6 and let $X \to \mathbb{P}^1_{\Bbbk'}$ be the double cover ramified over $D \times_{\Bbbk} \Bbbk'$. Show that the residual gerbe of $[X] \in \mathcal{M}_2$ is non-trivial and has residue field $\Bbbk$. (In fact, every Deligne–Mumford gerbe over a field can be realized as the residual gerbe of some curve $[C] \in \mathcal{M}_g$ for some g; see [BL24].)

6.4.5 Cohomological characterization

The following exercises provide cohomological characterizations of torsors and gerbes for an abelian sheaf G on the small fppf site S_{fppf} of a scheme S. If $G \to S$ is a *smooth, commutative*, and quasi-projective group scheme, then $\mathrm{H}^p(S_{\text{ét}}, G) = \mathrm{H}^p(S_{\mathrm{fppf}}, G)$ [Mil80, Thm. 3.9]. For $\mathbb{G}_m$, there are identifications $\operatorname{Pic}(S) = \mathrm{H}^1(S_{\mathrm{Zar}}, \mathcal{O}_S^*) = \mathrm{H}^1(S_{\text{ét}}, \mathbb{G}_m) = \mathrm{H}^1(S_{\mathrm{fppf}}, \mathbb{G}_m)$ (Hilbert's Theorem 90, [Mil80, Prop. 4.9]). We also note that if S is a smooth scheme over $\mathbb{C}$ and G is a finite abelian group, then the classical complex cohomology $\mathrm{H}^p(S(\mathbb{C}), G)$ agrees with the étale cohomology $\mathrm{H}^p(S_{\text{ét}}, G)$ of the constant sheaf [Mil80, Thm. 3.12].

For an extra challenge, the reader is encouraged to prove the statements below more generally for abelian sheaves over any site. The reader may consult [Gir71] and [Ols16, §12] for detailed proofs.

Exercise 6.4.30 (Torsors)**.** Let S be a scheme.

(a) If G is an abelian sheaf on S_{fppf}, show that $\mathrm{H}^1(S_{\mathrm{fppf}}, G)$ is in bijective correspondence with isomorphism classes of G-torsors.

Hint: Imitate the proof using Čech cohomology that $\mathrm{H}^1(X, \mathcal{O}_X^*) = \mathrm{Pic}(X)$ *for a scheme* X.

(b) Let $0 \to G' \to G \to G'' \to 0$ be an exact sequence of abelian sheaves on S_{fppf}, and let

$$0 \to \mathrm{H}^0(S_{\mathrm{fppf}}, G') \to \mathrm{H}^0(S_{\mathrm{fppf}}, G) \to \mathrm{H}^0(S_{\mathrm{fppf}}, G'') \xrightarrow{\delta}$$
$$\to \mathrm{H}^1(S_{\mathrm{fppf}}, G') \xrightarrow{\alpha} \mathrm{H}^1(S_{\mathrm{fppf}}, G) \xrightarrow{\beta} \mathrm{H}^1(S_{\mathrm{fppf}}, G'') \to \cdots$$

be the corresponding long exact sequence. Show that under the bijection in (a), the boundary map δ assigns a section $S \to G''$ to the G'-torsor defined by the fiber product $G \times_{G''} S$. Show also that α assigns a G'-torsor P' to the quotient $P' \times^{G'} G := (P' \times G)/G'$ while β assigns a G-torsor P to $P \times^G G''$.

Exercise 6.4.31 (Gerbes)**.** Let S be a scheme.

(a) If G is an abelian sheaf on S_{fppf}, show that $\mathrm{H}^2(S_{\mathrm{fppf}}, G)$ is in bijective correspondence with isomorphism classes of G-banded gerbes.

Hint: Let $0 \to G \to I^0 \xrightarrow{d^0} I^1 \xrightarrow{d^1} I^2 \xrightarrow{d^2} \cdots$ *be an injective resolution. For a cohomology class* $\alpha \in \mathrm{H}^2(S_{\mathrm{fppf}}, G)$*, define a stack* $\mathcal{G}_\alpha$ *over* S_{fppf} *as follows. Choose* $\tau \in \Gamma(S_{\mathrm{fppf}}, I^2)$ *with* $d^2(\tau) = 0$ *representing* α*. Define* $\mathcal{G}_\alpha$ *as the category of pairs* (T, σ) *consisting of an object* $T \in S_{\mathrm{fppf}}$ *and a section* $\sigma \in \Gamma(T, I^1)$ *with* $d^1(\sigma) = \tau|_T$*. A morphism* $(T', \sigma') \to (T, \sigma)$ *is the data of a map* $f \colon T' \to T$ *and an element* $\rho \in \Gamma(T', I^0)$ *such that* $d^0(\rho) = \sigma' - f^*(\sigma)$*. Show that* $\mathcal{G}_\alpha$ *is a* G*-banded gerbe and that the assignment* $\alpha \mapsto \mathcal{G}_\alpha$ *gives the stated bijection. See also [dJ03, §2.5].*

Let $0 \to G' \to G \to G'' \to 0$ be an exact sequence of abelian sheaves on S_{fppf}, and let

$$\cdots \to \mathrm{H}^1(S_{\mathrm{fppf}}, G'') \xrightarrow{\delta} \mathrm{H}^2(S_{\mathrm{fppf}}, G') \xrightarrow{\alpha} \mathrm{H}^2(S_{\mathrm{fppf}}, G) \xrightarrow{\beta} \mathrm{H}^2(S_{\mathrm{fppf}}, G'') \to \cdots$$

be the corresponding long exact sequence.

(b) Show that under the bijection in (a), the boundary map δ assigns a G''-torsor $P'' \to S$ to the *gerbe of trivializations* $\mathcal{G}_{P''}$, where an object over an S-scheme T is a pair (P, α) consisting of a G-torsor $P \to T$ and a trivialization $\alpha \colon P \times^G G'' \cong P'' \times_S T$ of G''-torsors.

(c) Suppose that G', G, and G'' are representable by commutative and affine algebraic groups over a field $\Bbbk$. Show that if $P'' \to S$ is a G''-torsor, then $\mathcal{G}_{P''}$ is identified with both the quotient stack $[P''/G]$ and with the fiber product of the map $S \to BG''$ classifying the G''-torsor P'' and $BG \to BG''$.

(d) In the special case that the exact sequence is $1 \to \boldsymbol{\mu}_r \to \mathbb{G}_m \to \mathbb{G}_m \to 1$ and P'' corresponds to a line bundle L'', show that $\mathcal{G}_{P''}$ is isomorphic to the root stack $X(\sqrt[r]{L})$.

Exercise 6.4.32 (Group structure)**.** Show that the group laws of $\mathrm{H}^1(S_{\mathrm{fppf}}, G)$ and $\mathrm{H}^2(S_{\mathrm{fppf}}, G)$ can be described geometrically as follows:

(a) The product of two G-torsors P_1 and P_2 is the *contracted product* $P_1 \wedge^G P_2$ defined as the sheaf quotient $(P_1 \times P_2)/G$ where $h \cdot (p_1, p_2) = (h^{-1}p_1, hp_2)$ with the G-action specified by $g \cdot (p_1, p_2) = (gp_1, p_2) = (p_1, gp_2)$. The inverse of a G-torsor P is the sheaf P with the inverted G-action: $g \cdot p = g^{-1}p$.

(b) The product of two banded G-gerbes $(\mathcal{X}_1, \psi_{1,x})$ and $(\mathcal{X}_2, \psi_{2,x})$ is the *contracted product* $\mathcal{X}_1 \wedge^G P_2$, which is defined as the rigidification $(\mathcal{X}_1 \times \mathcal{X}_2) /\!\!/ G$ (see Proposition 6.4.18) of the product $\mathcal{X}_1 \times \mathcal{X}_2$ along the subgroup $(\psi_1, \psi_2) \colon G|_{\mathcal{X}_1 \times \mathcal{X}_2} \to$

$I_{\mathcal{X}_1 \times \mathcal{X}_2}$ defined by the bands ψ_1 and ψ_2. The inverse $(\mathcal{X}, \psi_x)^{-1} = (\mathcal{X}, \psi_x^{-1})$ inverts the band.

Remark 6.4.33 (Banded μ_n-gerbes over $\mathbb{P}^1$). Over an algebraically closed field $\Bbbk$, isomorphism classes of banded μ_n-gerbes over $\mathbb{P}^1$ are in bijection with $\mathbb{Z}/n\mathbb{Z}$. To see this, observe that the exact sequence $1 \to \mu_n \to \mathbb{G}_m \xrightarrow{n} \mathbb{G}_m \to 1$ induces an exact sequence on cohomology

$$\mathrm{H}^1(\mathbb{P}^1_{\text{ét}}, \mathbb{G}_m) \xrightarrow{n} \mathrm{H}^1(\mathbb{P}^1_{\text{ét}}, \mathbb{G}_m) \xrightarrow{\delta} \mathrm{H}^2(\mathbb{P}^1_{\text{ét}}, \mu_n) \to \mathrm{H}^2(\mathbb{P}^1_{\text{ét}}, \mathbb{G}_m).$$

Since $\mathrm{H}^1(\mathbb{P}^1_{\text{ét}}, \mathbb{G}_m) = \mathrm{Pic}(\mathbb{P}^1_{\text{ét}}) = \mathbb{Z}$, we can use the fact that $\mathrm{H}^2(\mathbb{P}^1_{\text{ét}}, \mathbb{G}_m) = 0$ to conclude that $\mathrm{H}^2(\mathbb{P}^1_{\text{ét}}, \mu_n) = \mathbb{Z}/n\mathbb{Z}$. The image of a line bundle $\mathcal{O}(d)$ under the boundary map δ is equivalent to the root stack $\mathbb{P}^1(\sqrt[n]{\mathcal{O}(d)})$, and this gerbe is trivial if and only if n divides d.

Exercise 6.4.34 (Forms of group schemes)**.** Let G be an algebraic group over a field $\Bbbk$. We say that a group scheme $H \to \operatorname{Spec} \Bbbk$ is a *form of* G if there is an isomorphism $G_{\overline{\Bbbk}} \cong H_{\overline{\Bbbk}}$. We call G the *trivial form of* G.

(a) Show the algebraic group $H = \operatorname{Spec} \mathbb{R}[x, y]/(x^2 + y^2 - 1)$ over $\mathbb{R}$, with the group structure induced from the embedding $H \subseteq \mathrm{SL}_2$ given by

$$(x, y) \mapsto \begin{pmatrix} x & y \\ -y & x \end{pmatrix},$$

is a non-trivial form of $\mathbb{G}_{m,\mathbb{R}}$.

(b) Assume that $\mathrm{char}(\Bbbk) \neq 2$. Recall the orthogonal groups $O(q)$ defined in Exercise B.1.58 for a non-degenerate quadratic form q on an n-dimensional vector space V. Show that every $O(q)$ is a form of the subgroup $O_n \subseteq \mathrm{GL}_n$ of orthogonal matrices.

(c) If G is smooth and commutative, show that forms of G are classified by $\mathrm{H}^1((\mathrm{Sch}/\Bbbk)_{\text{ét}}, \mathrm{Aut}(G))$.

Exercise 6.4.35 ($\mathbb{G}_m$-gerbes and twisted sheaves)**.** Let $\mathcal{X} \to X$ be a $\mathbb{G}_m$-gerbe over an algebraic space X. We say that a coherent sheaf F on $\mathcal{X}$ is *1-twisted* if for every field-valued point $\operatorname{Spec} \Bbbk \to \mathcal{X}$, the $\mathbb{G}_m$-representation corresponding to the pullback of E under $B\mathbb{G}_m = BG_x \to \mathcal{X}$ decomposes as a direct sum of one dimensional representations of *weight one*. Show that the $\mathbb{G}_m$-gerbe $\mathcal{X} \to X$ is trivial if and only if there exists a 1-twisted line bundle on $\mathcal{X}$.

Exercise 6.4.36 (Azumaya algebras)**.** An *Azumaya algebra of rank* r^2 over a noetherian scheme X is a (possibly non-commutative) associative $\mathcal{O}_X$-algebra A which is coherent as an $\mathcal{O}_X$-module and such that there is an étale covering $X' \to X$ where $A \otimes_{\mathcal{O}_X} \mathcal{O}_{X'}$ is isomorphic to the matrix algebra $\mathrm{Mat}_{r \times r}(\mathcal{O}_X)$. We say that A is trivial if it is isomorphic to $\mathrm{Mat}_{r \times r}(\mathcal{O}_X)$. By Exercise B.1.68, Azumaya algebras are in bijection with principal PGL_r-bundles and with Brauer–Severi schemes of relative dimension r. Let A be an Azumaya algebra over a noetherian scheme X of rank r^2.

(a) Define the *gerbe of trivializations of* A as the stack $\mathcal{G}_A$ over $(\mathrm{Sch}/X)_{\text{ét}}$ where an object over a X-scheme T is a pair (E, α) consisting of a vector bundle E on T of rank r and a trivialization $\alpha \colon \mathrm{End}_{\mathcal{O}_X}(E) \xrightarrow{\sim} A \otimes_{\mathcal{O}_X} \mathcal{O}_T$. Morphisms in $\mathcal{G}_A(T)$ are isomorphisms of vector bundles compatible with the trivializations. Show that $\mathcal{G}_A \to X$ is a banded $\mathbb{G}_m$-gerbe.

(b) Let P_A be the principal PGL_r-bundle corresponding to A. Identify $\mathcal{G}_A$ with the gerbe of trivializations $\mathcal{G}_{P_A}$ defined in Exercise 6.4.31(b) with respect to the PGL_r-torsor P_A and the surjection $\mathrm{GL}_r \to \mathrm{PGL}_r$.

(c) Use the quarternions to construct a non-trivial $\mathbb{G}_m$-gerbe over $\operatorname{Spec} \mathbb{R}$.

Remark 6.4.37 (Brauer groups). Two Azumaya algebras A and A' on a noetherian scheme X are *similar* if there exists vector bundles E and E' on X such that $A \otimes_{\mathcal{O}_X} \mathrm{End}_{\mathcal{O}_X}(E) \cong A \otimes_{\mathcal{O}_X} \mathrm{End}_{\mathcal{O}_X}(A')$. This defines an equivalence relation, and the *Brauer group of* X is the set $\mathrm{Br}(X)$ of Azumaya algebras up to similarity. The set $\mathrm{Br}(X)$ becomes a group under the operators $[A] \cdot [A'] = [A \otimes A']$ and $[A]^{-1} = [A^{\mathrm{op}}]$ (where A^{op} is the opposite algebra with same elements and addition as A but with multiplication reversed: $a \cdot^{A^{\mathrm{op}}} b = b \cdot^{A} a$).

The exact sequence $1 \to \mathbb{G}_{m,X} \to \mathrm{GL}_{r,X} \to \mathrm{PGL}_{r,X} \to 1$ of sheaves on $X_{\text{ét}}$ induces a boundary map

$$\delta\colon \mathrm{H}^1(X_{\text{ét}}, \mathrm{PGL}_r) \to \mathrm{H}^2(X_{\text{ét}}, \mathbb{G}_m), \quad P_A \mapsto \mathcal{G}_A,$$

where P_A is the principal PGL_r-bundle corresponding to A and $\mathcal{G}_A$ is the gerbe of trivializations of A. It is not hard to check that $[\mathcal{G}_A]$ is an r-torsion element and that A is trivial if and only if $\mathcal{G}_A$ is trivial. It follows that there is an injective map

$$\mathrm{Br}(X) \hookrightarrow \mathrm{Br}'(X) := \mathrm{H}^2(X_{\text{ét}}, \mathbb{G}_m)_{\text{tors}}, \qquad A \mapsto \mathcal{G}_A,$$

into the *cohomological Brauer group* $\mathrm{Br}'(X)$. See [Gro68] and [Mil80, §IV.2] for additional background. Grothendieck asked whether $\mathrm{Br}(X) \hookrightarrow \mathrm{Br}'(X)$ is surjective. This is known in some cases. The strongest result is due to Gabber: $\mathrm{Br}(X) = \mathrm{Br}'(X)$ if X admits an ample line bundle (see [dJ03]). It is however open in general, even for smooth separated schemes over a field.

Exercise 6.4.38. Let X be a noetherian scheme and $\mathcal{X} \to X$ be a banded $\mathbb{G}_m$-gerbe corresponding to a cohomology class $[\mathcal{X}] \in \mathrm{H}^2(X_{\text{ét}}, \mathbb{G}_m)$.

(a) Show that the following are equivalent:

(i) There exists an Azumaya algebra A on X such that $\mathcal{X} \cong \mathcal{G}_A$, i.e., $[\mathcal{X}]$ is in the image of $\mathrm{Br}'(X) \to \mathrm{Br}(X)$,

(ii) $\mathcal{X}$ is a global quotient stack, and

(iii) there exists a 1-twisted vector bundle E on $\mathcal{X}$ (see Exercise 6.4.35).

(b) Let X be a normal separated surface over $\mathbb{C}$ such that $\mathrm{H}^2(X, \mathbb{G}_m)$ contains a non-torsion element α; for an example, see [Gro68, II.1.11.b]. Conclude that the banded $\mathbb{G}_m$-gerbe corresponding to α is not a global quotient stack.

(c) Let $Y = \operatorname{Spec} \mathbb{C}[x,y,z]/(xy - z^2)$. Show that there is a non-trivial involution α of $(Y \smallsetminus 0) \times B(\mathbb{Z}/2)$ such that the stack $\mathcal{X}$, obtained by gluing the trivial banded $\mathbb{Z}/2$-gerbes over Y along α, is a banded $\mathbb{Z}/2$-gerbe over the non-separated union $Y \bigcup_{Y \smallsetminus 0} Y$ and that $\mathcal{X}$ is not a global quotient stack.

See also [EHKV01].

6.4.6 Rigidification

Given an algebraic stack $\mathcal{X}$, the inertia stack $I_{\mathcal{X}}$ can be viewed as a group algebraic space over the big étale site $(\mathrm{Sch}/\mathcal{X})_{\text{ét}}$ of $\mathcal{X}$. As a group functor, $I_{\mathcal{X}}$ assigns an object $a \in \mathcal{X}(S)$ to the group $\mathrm{Aut}_S(a)$, and a morphism $\alpha\colon a' \to a$ over $S' \to S$ to the natural pullback map $\alpha^*\colon \operatorname{Aut}_S(a) \to \operatorname{Aut}_{S'}(a')$ (see (3.2.11)). Given $a\colon S \to \mathcal{X}$, there is a canonical isomorphism $I_{\mathcal{X}} \times_{\mathcal{X}} S \cong \underline{\operatorname{Aut}}_S(a)$ of group algebraic spaces over S.

Suppose that $\mathcal{H} \subseteq I_{\mathcal{X}}$ is a closed subgroup space over $\mathcal{X}$ such that $\mathcal{H} \to \mathcal{X}$ is fppf. This is equivalent to requiring that for every $a \in \mathcal{X}(S)$, there is a closed fppf subgroup scheme $\mathcal{H}_a \subseteq \underline{\operatorname{Aut}}_S(a)$ over S such that if $a' \to a$ is a morphism over $S' \to S$, the canonical isomorphism $\underline{\operatorname{Aut}}_{S'}(a') \cong \underline{\operatorname{Aut}}_S(a) \times_S S'$ restricts to an isomorphism $\mathcal{H}_{a'} \cong \mathcal{H}_a \times_S S'$. If $\alpha\colon a \xrightarrow{\sim} a$ is an automorphism over the identity, then the canonical isomorphism $\alpha^*\colon \operatorname{Aut}_S(a) \to \operatorname{Aut}_S(a)$ is conjugation by α. In particular, $\mathcal{H}_a \subseteq \underline{\operatorname{Aut}}_S(a)$ is a *normal* group scheme.

Definition 6.4.39 (Rigidification)**.** Let $\mathcal{X}$ be an algebraic stack and $\mathcal{H} \subseteq I_{\mathcal{X}}$ be an fppf closed subgroup scheme over $\mathcal{X}$. The *rigidification* $\mathcal{X} /\!\!/ \mathcal{H}$ is defined as the stackification in $\mathrm{Sch}_{\mathrm{fppf}}$ of the prestack with the same objects as $\mathcal{X}$ and where the set of morphisms between $a \in \mathcal{X}(S)$ and $b \in \mathcal{X}(T)$ over $f\colon S \to T$ is defined as $\operatorname{Mor}(a, b) = \operatorname{Mor}_{\mathcal{X}(S)}(a, f^*b)/\mathcal{H}(T)$.

One can think of the subgroup $\mathcal{H}$ as giving an action of $B\mathcal{H}$ on $\mathcal{X}$ and the rigidification $\mathcal{X} /\!\!/ \mathcal{H}$ as the quotient $\mathcal{X}/B\mathcal{H}$. Frequently in applications, the closed subgroup $\mathcal{H} \subseteq I_{\mathcal{X}}$ is obtained by the pullback of a fppf group scheme $H \to S$, i.e., $\mathcal{H} = H \times_S \mathcal{X}$. In this case, it is customary to write $\mathcal{X} /\!\!/ H := \mathcal{X} /\!\!/ \mathcal{H}$.

Example 6.4.40. If $I_{\mathcal{X}} \to \mathcal{X}$ is fppf, then we can take $\mathcal{H} = I_{\mathcal{X}}$ and the rigidification $\mathcal{X} /\!\!/ I_{\mathcal{X}}$ is the algebraic space X constructed in Proposition 6.4.18.

Proposition 6.4.41. *Let $\mathcal{X}$ be an algebraic stack and $\mathcal{H} \subseteq I_{\mathcal{X}}$ be an fppf closed subgroup scheme over $\mathcal{X}$. The rigidification $\mathcal{X} /\!\!/ \mathcal{H}$ is an algebraic stack such that*

(1) the natural morphism $\pi\colon \mathcal{X} \to \mathcal{X} /\!\!/ \mathcal{H}$ is a gerbe;

(2) for every object $a \in \mathcal{X}(S)$, the natural map $\operatorname{Aut}_S(a) \to \operatorname{Aut}_S(\pi(a))$ is surjective with kernel $\mathcal{H}(S)$;

(3) a morphism $f\colon \mathcal{X} \to \mathcal{Y}$ factors uniquely through $\mathcal{X} /\!\!/ \mathcal{H}$ if and only if for every object $a \in \mathcal{X}(S)$, the composition $\mathcal{H}(S) \subseteq \ker(\operatorname{Aut}_{\mathcal{X}(S)}(a) \to \operatorname{Aut}_{\mathcal{Y}(S)}(f(a)))$; and

(4) if $\mathcal{H}$ is a commutative group scheme, then $\mathcal{H}$ descends to an fppf group scheme $H \to X$ such that $\mathcal{X} \to X$ is banded H-gerbe. If in addition $\mathcal{H} = H \times_S \mathcal{X}$ is the pullback of a commutative fppf group scheme $H \to S$, then $\mathcal{X} \to X$ is a banded H-gerbe.

Proof. To show that $\mathcal{X}$ is algebraic, it suffices to show that $\pi\colon \mathcal{X} \to \mathcal{X} /\!\!/ \mathcal{H}$ is a smooth representable morphism. Given $g\colon S \to \mathcal{X} /\!\!/ \mathcal{H}$, the definition of $\mathcal{X} /\!\!/ \mathcal{H}$ as the stackification implies that there is an fppf cover $S' \to S$ such that $S' \to S \to \mathcal{X} /\!\!/ \mathcal{H}$ lifts to a map $a'\colon S' \to \mathcal{X}$. By replacing S with S', we may assume that $g\colon S \to \mathcal{X} /\!\!/ \mathcal{H}$ lifts to a morphism $a\colon S \to \mathcal{X}$. As in the proof of Proposition 6.4.18, one shows that there is an isomorphism

$$\Psi\colon \mathcal{X} \times_{\mathcal{X} /\!\!/ \mathcal{H}} S \to B\mathcal{H}_a.$$

The details are left to the reader. See also [ACV03, Thm. 5.1.5], [AGV08, §C], [Rom05, §5], and [AOV08, §A]. □

Example 6.4.42. Every family of stable elliptic curve has a canonical hyperelliptic involution. This yields a subgroup $\mu_2 := \mu_{2,\overline{\mathcal{M}}_{1,1}}$ of the inertia stack $I_{\overline{\mathcal{M}}_{1,1}}$ of the stack of stable elliptic curves $\overline{\mathcal{M}}_{1,1} \cong \mathcal{P}(4,6)$ over a field (assumed as usual to have $\operatorname{char}(\Bbbk) \neq 2, 3$). The rigidification $\overline{\mathcal{M}}_{1,1} \to \overline{\mathcal{M}}_{1,1} /\!\!/ \mu_2$ is identified with the non-trivial μ_2-gerbe $\mathcal{P}(4,6) \to \mathcal{P}(2,3)$ discussed in Exercise 6.4.23.

Example 6.4.43 (Ridigification of $\mathcal{B}un(C)$)**.** Moduli stacks of sheaves provide interesting examples of rigidification since there is a canonical scaling $\mathbb{G}_m$-action on sheaves. For instance, consider the moduli stack $\mathcal{B}un_{r,d}(C)$ of vector bundles over a fixed smooth, connected, and projective curve C over an algebraically closed field $\Bbbk$. For any vector bundle $\mathcal{E}$ on $C \times S$ where S is a $\Bbbk$-scheme, there is a canonical closed immersion $i_{\mathcal{E}} \colon \mathbb{G}_{m,S} \to \underline{\mathrm{Aut}}(\mathcal{E})$ of group schemes over S. Thus, $\mathbb{G}_m := \mathbb{G}_{m,\mathcal{B}un_{r,d}} \subseteq I_{\mathcal{B}un_{r,d}}$ is a closed fppf group scheme of the inertia stack, and we can construct the rigidification

$$\mathcal{B}un(C) /\!\!/ \mathbb{G}_m.$$

Over the open substack $\mathcal{B}un^{\mathrm{simple}}(C)$ of simple bundles (i.e., vector bundles E with $\mathrm{Aut}(E) = \Bbbk^*$), the rigidification $\mathcal{B}un^{\mathrm{simple}}(C) /\!\!/ \mathbb{G}_m$ is an algebraic space so that $\mathcal{B}un^{\mathrm{simple}}(C)$ is $\mathbb{G}_m$-gerbe over its coarse moduli space.

Exercise 6.4.44.

(a) If H is a commutative fppf group scheme over S, show that $BH /\!\!/ H \cong S$. More generally, show that if $\mathcal{X} \to X$ is a banded H-gerbe, then $X \cong \mathcal{X} /\!\!/ H$.

(b) Let $G \to S$ be an fppf group scheme acting on a S-scheme U. Suppose that $H \subseteq G$ is a central commutative fppf subgroup scheme acting trivially on U. Show that $[U/G] /\!\!/ H \cong [U/(G/H)]$.

Exercise 6.4.45. Let $\mathcal{X}$ be a smooth, integral, and separated Deligne–Mumford stack over a scheme S. Let $\operatorname{Spec} K \to \mathcal{X}$ be a representative of the generic point. Show that the closure $\mathcal{H} \subseteq I_{\mathcal{X}}$ of generic fiber $I_{\mathcal{X}} \times_{\mathcal{X}} K$ of the inertia is a closed étale subgroup scheme and that the rigidification $\mathcal{X} /\!\!/ \mathcal{H}$ is a smooth, integral, and separated Deligne–Mumford stack over S with generically trivial inertia.

Exercise 6.4.46. Let $\mathcal{X}$ be an algebraic stack over a scheme S, $H \to S$ be an fppf group scheme, and $H \times_S \mathcal{X} \subseteq I_{\mathcal{X}}$ a closed subgroup scheme. Show that the rigidification $\mathcal{X} /\!\!/ H$ can be given the moduli interpretation where an object over a scheme S is a pair $(\mathcal{G}, f)$, where $\mathcal{G} \to S$ is a banded H-gerbe and $f \colon \mathcal{G} \to \mathcal{X}$ is an H-equivariant morphism.

Exercise 6.4.47. For $g \geq 2$, let $\mathcal{H}_g \subseteq \mathcal{M}_g$ be the closed substack classifying hyperelliptic curves. Show that the rigidification $\mathcal{H}_g \to \mathcal{Y}$ along the hyperelliptic involution is a non-trivial banded $\mathbb{Z}/2$-gerbe.

6.4.7 Picard stacks and spaces

If $X \to S$ is a proper flat morphism of noetherian schemes, define the *Picard stack*

$$\underline{\mathcal{P}\mathrm{ic}}_{X/S}$$

as the stack over $(\mathrm{Sch}/S)_{\text{ét}}$ whose objects over an S-scheme T are line bundles on $X_T = X \times_S T$ and whose morphisms are isomorphisms of line bundles. This is an open substack of $\underline{\mathrm{Coh}}(X/S)$, which is an algebraic stack locally of finite type over S; this follows from an explicit presentation using the Quot scheme (Exercise 3.1.23) in the case that $X \to S$ is strongly projective and as a consequence of Artin's Axioms (Theorem C.7.7) in general. Therefore, $\underline{\mathcal{P}\mathrm{ic}}_{X/S}$ is also algebraic and locally of finite type over S.

On the other hand, there are several candidates for Picard functors:

(1) The *naive Picard functor* (or *absolute Picard functor*) is

$$\underline{\mathrm{Pic}}^{\mathrm{naive}}_{X/S}\colon \mathrm{Sch}/S \to \mathrm{Gps}, \qquad (T \to S) \mapsto \mathrm{Pic}(X_T).$$

(2) The *Picard functor*

$$\underline{\mathrm{Pic}}_{X/S}\colon \mathrm{Sch}/S \to \mathrm{Gps},$$

is the fppf sheafification of $\underline{\mathrm{Pic}}^{\mathrm{naive}}_{X/S}$.

(3) The *relative Picard functor* is

$$\underline{\mathrm{Pic}}^{\mathrm{rel}}_{X/S}\colon \mathrm{Sch}/S \to \mathrm{Gps}, \qquad (T \to S) \mapsto \mathrm{Pic}(X_T)/\operatorname{Pic}(T),$$

where an object over an S-scheme T is a line bundle L on X_T, and two line bundles L and L' on X_T are identified if there exists $M \in \mathrm{Pic}(T)$ such that $L \cong L' \otimes f_T^* M$.

(4) We can define the *rigidification of the Picard stack*

$$\underline{\mathcal{P}\mathrm{ic}}_{X/S} /\!\!/ \mathbb{G}_m$$

under the hypothesis that $\mathcal{O}_T \xrightarrow{\sim} f_{T,*}\mathcal{O}_{X_T}$ is an isomorphism for any map $T \to S$. This hypothesis implies that for a line bundle L on X_T, there is a canonical isomorphism

$$\mathbb{G}_m(T) = \Gamma(T, \mathcal{O}_T)^* \xrightarrow{\sim} \Gamma(X_T, \mathcal{O}_{X_T})^* = \mathrm{Aut}(L),$$

which further implies that the inertia stack $I_{\underline{\mathcal{P}\mathrm{ic}}_{X/S}}$ is isomorphic to fppf group scheme $\mathbb{G}_m := \mathbb{G}_{m,\underline{\mathcal{P}\mathrm{ic}}_{X/S}}$ over $\underline{\mathcal{P}\mathrm{ic}}_{X/S}$.

While the Picard functor $\underline{\mathrm{Pic}}_{X/S}$ and the rigidification $\underline{\mathcal{P}\mathrm{ic}}_{X/S} /\!\!/ \mathbb{G}_m$ are sheaves in the big fppf topology by definition, it may seem surprising that $\underline{\mathrm{Pic}}^{\mathrm{rel}}_{X/S}$ is also a sheaf under relatively mild hypotheses.

Proposition 6.4.48. *Let $f\colon X \to S$ be a proper flat morphism of noetherian schemes such that $\mathcal{O}_S \xrightarrow{\sim} f_*\mathcal{O}_X$ is an isomorphism and this holds after base change, i.e., for every map $g\colon T \to S$, the map $\mathcal{O}_T \xrightarrow{\sim} f_{T,*}\mathcal{O}_{X_T}$ is an isomorphism. Then*

(1) $\underline{\mathrm{Pic}}_{X/S}$ is representable by an algebraic space locally of finite type over S,

(2) the map $\underline{\mathcal{P}\mathrm{ic}}_{X/S} \to \underline{\mathrm{Pic}}_{X/S}$ from the Picard stack is a banded $\mathbb{G}_m$-gerbe and $\underline{\mathrm{Pic}}_{X/S} \cong \underline{\mathcal{P}\mathrm{ic}}_{X/S} /\!\!/ \mathbb{G}_m$, and

(3) if in addition there is a section $s\colon S \to X$, then $\underline{\mathrm{Pic}}_{X/S} \cong \underline{\mathrm{Pic}}^{\mathrm{rel}}_{X/S}$.

If in addition the geometric fibers of f are integral, then $\underline{\mathrm{Pic}}_{X/S}$ is separated over S.

Remark 6.4.49. If $f\colon X \to S$ is a proper flat proper morphism with geometrically connected and reduced fibers, then $\mathcal{O}_S \xrightarrow{\sim} f_*\mathcal{O}_X$ is an isomorphism (and remains so after base chagne) by Lemma A.6.12. In [FGAV, Thm. 3.1], Grothendieck proved that $\underline{\mathrm{Pic}}_{X/S}$ is a scheme in the case that $X \to S$ is projective. See [Mum66a, §20-21], [AK80], and [Kle05, §9.4] for alternative expositions and various generalizations. The representability as an algebraic space above was first established by Artin [Art69b, Thm. 7.3], and this holds with the slightly weaker hypothesis that f is *cohomologically flat in dimension* 0, i.e., the formation of $f_*\mathcal{O}_X$ commutes with base change.

Proof. As pointed out above, the condition that $\mathcal{O}_S \xrightarrow{\sim} f_*\mathcal{O}_X$ holds after base change implies that the inertia stack of $\underline{\mathcal{P}\mathrm{ic}}_{X/S}$ is isomorphic to the fppf group scheme $\mathbb{G}_m :=$

$\mathbb{G}_{m,\underline{\mathcal{P}\mathrm{ic}}_{X/S}}$. Therefore Proposition 6.4.18 implies that the rigidification $\underline{\mathcal{P}\mathrm{ic}}_{X/S} /\!\!/ \mathbb{G}_m$ is an algebraic space locally of finite type over S. Moreover, $\underline{\mathcal{P}\mathrm{ic}}_{X/S} /\!\!/ \mathbb{G}_m$ is identified with the Picard functor $\underline{\mathrm{Pic}}_{X/S}$ by the definition of rigidification. This gives both (1) and (2). When the fibers are geometrically integral, the separatedness of $\underline{\mathrm{Pic}}_{X/S}$ over S follows from Proposition A.6.17 and Remark A.6.19.

To identify $\underline{\mathrm{Pic}}^{\mathrm{rel}}_{X/S}$ with $\underline{\mathrm{Pic}}_{X/S}$, it suffices to prove that $\underline{\mathrm{Pic}}^{\mathrm{rel}}_{X/S}$ is a sheaf in the big fppf topology. To this end, it will be convenient to identify $\underline{\mathrm{Pic}}^{\mathrm{rel}}_{X/S}$ with the prestack $\mathcal{P}^{s-\mathrm{rig}}$, called the *rigidification by the section s*, whose fiber category over an S-scheme T is

$$\mathcal{P}^{s-\mathrm{rig}}(T) = \big\{(L, \alpha) \mid L \in \mathrm{Pic}(X_T) \text{ and } \alpha\colon \mathcal{O}_T \xrightarrow{\sim} s_T^* L\big\},$$

where a morphism $(L, \alpha) \sim (L', \alpha')$ is the data of an isomorphism $\beta\colon L \to L'$ of line bundles such that $\alpha' = s_T^*\beta \circ \alpha$. The advantage of considering $\mathcal{P}^{s-\mathrm{rig}}$ is that it is a straightforward application of fppf descent of quasi-coherent sheaves to check that $\mathcal{P}^{s-\mathrm{rig}}$ is a stack over the big fppf topology $(\mathrm{Sch}/S)_{\mathrm{fppf}}$. It therefore suffices to verify that the natural map

$$\mathcal{P}^{s-\mathrm{rig}} \to \underline{\mathrm{Pic}}^{\mathrm{rel}}_{X/S}, \qquad (L, \alpha) \mapsto L \tag{6.4.50}$$

is an equivalence.

We first check that (6.4.50) is faithful and in particular that $\mathcal{P}^{s-\mathrm{rig}}$ is equivalent to a functor. We must show that if $\beta\colon L \xrightarrow{\sim} L$ is an automorphism with $s_T^*\beta = \mathrm{id}_{s_T^* L}$, then $\beta = \mathrm{id}_L$. Since $\mathcal{O}_T \to f_{T,*}\mathcal{O}_{X_T}$ is an isomorphism, the pullback map $f_T^*\colon \mathrm{H}^0(T, \mathcal{O}_T) \to \mathrm{H}^0(X_T, \mathcal{O}_{X_T})$ is an isomorphism; as s_T is a section, s_T^* is the inverse of f_T^*. The composition

$$s_T^*\colon \mathrm{Hom}_{\mathcal{O}_{X_T}}(L, L) \cong \mathrm{H}^0(X_T, \mathcal{O}_{X_T}) \xrightarrow{s_T^*} \mathrm{H}^0(T, \mathcal{O}_T) \cong \mathrm{Hom}_{\mathcal{O}_T}(s_T^* L, s_T^* L)$$

is an isomorphism of groups, and thus $\beta = \mathrm{id}_L$. To see that (6.4.50) is full (i.e., the induced map on the functors of isomorphism classes is injective), let $(L, \alpha) \in \mathcal{P}^{s-\mathrm{rig}}(T)$ be such that there is a line bundle M on T and an isomorphism $\beta\colon L \xrightarrow{\sim} f_T^* M$. The isomorphism

$$\gamma\colon \mathcal{O}_{X_T} = f_T^*\mathcal{O}_T \xrightarrow{f_T^*\alpha} f_T^* s_T^* L \xrightarrow{f_T^* s_T^* \beta} f_T^* s_T^* f_T^* M = f_T^* M$$

satisfies $s_T^*\gamma = s_T^*\beta \circ \alpha$ and shows that (L, α) is isomorphic to $(\mathcal{O}_{X_T}, \mathrm{id}) \in \mathcal{P}^{s-\mathrm{rig}}(T)$. Finally, to see that the functor (6.4.50) is essentially surjective, let $L \in \mathrm{Pic}(X_T)$ and define

$$L' = L \otimes (f_T^* s_T^* L)^\vee.$$

The images of L and L' are equal in $\underline{\mathrm{Pic}}^{\mathrm{rel}}_{X/S}(T)$, and $s_T^* L' \cong s_T^* L \otimes (s_T^* f_T^* s_T^* L)^\vee \cong \mathcal{O}_T$ defines an isomorphism $\alpha'\colon \mathcal{O}_T \xrightarrow{\sim} s_T^* L'$ such that $L \in \underline{\mathrm{Pic}}^{\mathrm{rel}}_{X/S}(T)$ is the image of $(L', \alpha') \in \mathcal{P}^{s-\mathrm{rig}}(T)$. □

Over an algebraically closed field $\Bbbk$, it is remarkably easy to verify that the Picard functor $\underline{\mathrm{Pic}}_X := \underline{\mathrm{Pic}}_{X/\Bbbk}$ is a scheme.

Theorem 6.4.51. *Let X be a proper integral scheme over an algebraically closed field $\Bbbk$.*

(1) $\underline{\mathrm{Pic}}_X$ is a group scheme locally of finite type over $\Bbbk$, and in particular the connected component $\underline{\mathrm{Pic}}^0_X$ of the identity is quasi-projective.

(2) $\underline{\mathrm{Pic}}_X \cong \underline{\mathrm{Pic}}_X^{\mathrm{rel}}$ *and* $\underline{\mathcal{P}\mathrm{ic}}_X \to \underline{\mathrm{Pic}}_X$ *is a banded* $\mathbb{G}_m$*-gerbe,*

(3) *If* X *is smooth,* $\underline{\mathrm{Pic}}_X^0$ *is projective.*

(4) *If* $\mathrm{char}(\Bbbk) = 0$*, then* $\underline{\mathrm{Pic}}_X$ *is smooth of dimension* $\mathrm{h}^0(X, \mathcal{O}_X)$*. In particular,* $\underline{\mathrm{Pic}}_X^0$ *is an abelian variety.*

Proof. As $\Bbbk$ is algebraically closed and X is integral, the structure map $f\colon X \to \operatorname{Spec} \Bbbk$ has a section and $\mathcal{O}_T \xrightarrow{\sim} f_{T,*}\mathcal{O}_{X_T}$ is an isomorphism for any $\Bbbk$-scheme T. Proposition 6.4.48 implies that $\underline{\mathrm{Pic}}_X$ is an algebraic space locally of finite type over $\Bbbk$, and that (2) holds. The Picard stack $\underline{\mathcal{P}\mathrm{ic}}_X$ is quasi-separated (it even has affine diagonal) and it follows that $\underline{\mathrm{Pic}}_X$ is also quasi-separated. This is enough to show that $\underline{\mathrm{Pic}}_X$ is a separated scheme and that $\underline{\mathrm{Pic}}_X^0$ is quasi-projective. This is a direct consequence of Theorem 4.5.28, but it is worth recalling the argument: $\underline{\mathrm{Pic}}_X$ has a dense open subspace which is a scheme (Theorem 4.5.1), the group structure $\underline{\mathrm{Pic}}_X$ allows us to translate this open to show that $\underline{\mathrm{Pic}}_X$ is an algebraic group, which is automatically separated with quasi-projective connected components (Proposition B.1.16). This gives (1).

For (3), it suffices to show that $\underline{\mathrm{Pic}}_X^0$ is proper. As we already know it is separated, we only need to verify the existence part of the valuative criterion for properness: let R be a DVR over $\Bbbk$ with fraction field K and L be a line bundle on X_K. As X_R is regular, the line bundle L extends to a line bundle $\widetilde{L}$ on X_R (for instance, if $L = \mathcal{O}(D)$ for a divisor $D \subseteq X_K$, then take $\widetilde{L} = \mathcal{O}(\overline{D})$). The smoothness in (4) follows from the fact that algebraic groups are smooth in characteristic 0 (Proposition B.1.16). If L is a line bundle on X, then the Zariski tangent spaces of the Picard stack and Picard scheme agree, and deformation theory (Proposition C.1.18) implies that $T_{\underline{\mathrm{Pic}}_X, L} \cong \mathrm{H}^1(X, \mathcal{O}_X)$. □

Remark 6.4.52*.* As a consequence of the representability of $\underline{\mathrm{Pic}}_X \cong \underline{\mathrm{Pic}}_X^{\mathrm{rel}}$, there is a *universal family* (or *Poincaré family*) $\mathcal{P}$ on $X \times \underline{\mathrm{Pic}}_X$ that satisfies the following: for any $\Bbbk$-scheme T and any line bundle L on X_T, there is a unique morphism $T \to \underline{\mathrm{Pic}}_X$ such that

$$L \cong \mathcal{P}|_{X \times T} \otimes p_2^* M$$

for some line bundle M on T. The connected component of the identity $\underline{\mathrm{Pic}}_X^0$ has the functorial description of parameterizing line bundles L *algebraically equivalent* to $\mathcal{O}_X$ (i.e., there is a connected $\Bbbk$-scheme T with points $t_0, t_1 \in T(\Bbbk)$ and a family of line bundles $\mathcal{L}$ on X_T such that $L_{t_0} \cong L$ and $L_{t_1} \cong \mathcal{O}_X$). When X is a smooth curve, $\underline{\mathrm{Pic}}_X^0$ parameterizes degree 0 line bundles.

Remark 6.4.53*.* In characteristic p, Igusa showed that $\mathrm{Pic}(X)$ may fail to be reduced [Igu55]. We also note that when X is not normal (e.g., a nodal or cuspidal curve), then $\underline{\mathrm{Pic}}^0(X)$ is not proper. Altman and Kleiman [AK80] provide a compactification of $\underline{\mathrm{Pic}}^0(X)$ by classifying rank 1 torsion free sheaves. Picard functors and schemes have a fascinating history as they were one of the first examples of moduli spaces constructed in algebraic geometry. See Kleiman's article [Kle05] for a beautiful account of the history and a broader discussion of the properties of Picard schemes.

6.5 Affine GIT and good moduli spaces

Das ist nicht Mathematik, das ist Theologie.

PAUL GORDAN

The main theorem of affine GIT (Corollary 6.5.8) asserts that if G is a linearly reductive group acting on an affine scheme $\operatorname{Spec} A$, then the map $\operatorname{Spec} A \to \operatorname{Spec} A^G$ is a quotient where closed points of $\operatorname{Spec} A^G$ are identified with closed G-orbits in $\operatorname{Spec} A$. Recall that an affine algebraic group G over a field $\Bbbk$ is *linearly reductive* if the functor $\operatorname{Rep}(G) \to \operatorname{Vect}_{\Bbbk}$, taking a G-representation V to its G-invariants V^G, is exact. Examples include:

- finite discrete groups G whose order is not divisible by $\operatorname{char}(\Bbbk)$ (Maschke's Theorem (B.1.38));
- tori $\mathbb{G}_m^n$ and diagonalizable group schemes (Proposition B.1.15); and
- reductive groups (e.g., GL_n, SL_n and PGL_n) in $\operatorname{char}(\Bbbk) = 0$ (Theorem B.1.43).

See §B.1.6 for further equivalences, properties, and a discussion of linearly reductive groups.

Given an action of G on an affine $\Bbbk$-scheme $\operatorname{Spec} A$, the inclusion $A^G \hookrightarrow A$ induces a commutative diagram

$$\begin{tikzcd} \operatorname{Spec} A \arrow[d] \arrow[dr, "\widetilde{\pi}"] & \\ {[\operatorname{Spec} A/G]} \arrow[r, "\pi"] & \operatorname{Spec} A^G. \end{tikzcd}$$

Let us observe that $\pi \colon [\operatorname{Spec} A/G] \to \operatorname{Spec} A^G$ enjoys the following two properties:

(1) $\Gamma([\operatorname{Spec} A/G], \mathcal{O}_{[\operatorname{Spec} A/G]}) = A^G$; this follows from the definition of global sections.

(2) The functor $\pi_* \colon \operatorname{QCoh}([\operatorname{Spec} A/G]) \to \operatorname{QCoh}(\operatorname{Spec} A^G)$ is exact. This holds because functor π_* takes a quasi-coherent $\mathcal{O}_{\mathcal{X}}$-module $\widetilde{M}$, corresponding to an A-module M with a G-action, to $\widetilde{M^G}$ (Exercise 6.1.11) and is therefore exact by the defining property of linear reductivity.

These are the defining properties of a *good moduli space* (Definition 6.5.1), and it turn out most of the arguments showing for affine GIT quotients extend easily to good moduli spaces. We discuss the projective case of GIT in Chapter 7.

6.5.1 Good moduli spaces

The definition of a good moduli space is inspired by the properties of GIT quotients and specifically by the properties of the morphisms $\pi \colon [\operatorname{Spec} A/G] \to \operatorname{Spec} A^G$ and $\pi \colon [X^{\mathrm{ss}}/G] \to X^{\mathrm{ss}}/\!\!/G := \operatorname{Proj} \bigoplus_{d \geq 0} \Gamma(X, \mathcal{O}_X(d))^G$, where G is linearly reductive and $X \subseteq \mathbb{P}(V)$ is a G-invariant closed subscheme of a projectivized G-representation.

Definition 6.5.1 (Good moduli spaces)**.** A quasi-compact and quasi-separated morphism $\pi \colon \mathcal{X} \to X$ from an algebraic stack $\mathcal{X}$ to an algebraic space X is a *good moduli space* if

(1) $\mathcal{O}_X \to \pi_* \mathcal{O}_{\mathcal{X}}$ is an isomorphism, and

(2) $\pi_* \colon \operatorname{QCoh}(\mathcal{X}) \to \operatorname{QCoh}(X)$ is exact.

Example 6.5.2 (Basic example: affine GIT)**.** If G is a linearly reductive group over a field $\Bbbk$ acting on an affine $\Bbbk$-scheme $\operatorname{Spec} A$, then $[\operatorname{Spec} A/G] \to \operatorname{Spec} A^G$ is a good moduli space. We will prove later (Corollary 6.7.4) that every good moduli space is étale locally on X of this form.

Example 6.5.3 (Concrete examples)**.** If $\mathbb{G}_m$ acts on $\mathbb{A}^n$ over a field $\Bbbk$ via $t \cdot (x_1, \ldots, x_n) = (tx_1, \ldots, tx_n)$, then $[\mathbb{A}^n/\mathbb{G}_m] \to \operatorname{Spec} \Bbbk$ is a good moduli space. While $0 \in [\mathbb{A}^n/\mathbb{G}_m](\Bbbk)$ is a closed point, every other $\Bbbk$-point is not closed and contains 0 in its closure (which is simply the stack-theoretic translation of every $\mathbb{G}_m$-orbit containing 0 in its closure. Note also that $[\mathbb{A}^n/\mathbb{G}_m] \smallsetminus 0 = \mathbb{P}^{n-1}$.

If $\mathbb{G}_m$ acts on $\mathbb{A}^2$ via $t \cdot (x, y) = (tx, t^{-1}y)$, then $[\mathbb{A}^2/\mathbb{G}_m] \to \operatorname{Spec} \Bbbk[xy] = \mathbb{A}^1$ is a good moduli space. The fiber over $a \neq 0 \in \mathbb{A}^1$ under the good quotient $\mathbb{A}^2 \to \mathbb{A}^1$ is the hyperbola $xy = a$ in $\mathbb{A}^2$ and the fiber under the good moduli space $[\mathbb{A}^2/\mathbb{G}_m] \to \mathbb{A}^1$ is the point $\operatorname{Spec} \Bbbk \cong [V(xy - a)/\mathbb{G}_m)$. The fiber over the origin is the union of the three orbits $\{(x, 0) | x \neq 0\} \cup \{(0, y) | y \neq 0\} \cup \{0, 0\}$ in $\mathbb{A}^2$. Note that $[\mathbb{A}^2/\mathbb{G}_m] \smallsetminus 0 = \mathbb{A}^1 \bigcup_{\mathbb{A}^1 \smallsetminus 0} \mathbb{A}^1$ is the non-separated affine line. See Example 3.9.28 for an illustration.

Example 6.5.4 (Conjugation action of $n \times n$ matrices)**.** Under the conjugation action of GL_n on $\mathrm{Mat}_{n,n}$, the trace and determinant are obvious invariant functions. More generally, if $X = (X_{ij})$ is a generic $n \times n$ matrix, then the coefficients $a_0, \ldots, a_{n-1}$ of the characteristic polynomial $\det(X - \lambda I)$ are invariant functions, and it is a classical fact that they freely generate the ring of invariants. Therefore, the map

$$[\mathrm{Mat}_{n,n} / \mathrm{GL}_n] \to \mathrm{Mat}_{n,n} /\!\!/ \mathrm{GL}_n = \operatorname{Spec} \Bbbk[a_0, \ldots, a_{n-1}],$$

taking a matrix to its characteristic polynomial, is a good moduli space. Two matrices are identified if they have the same Jordan block decomposition, and the unique closed point in the a fiber corresponds to a diagonal matrix. For instance, the matrix $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$ has $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$ in its GL_2-orbit closure, and these matrices are identified in $\mathrm{Mat}\, 2, 2 /\!\!/ \mathrm{GL}_2$.

Example 6.5.5 (Tame coarse moduli spaces)**.** If $\mathcal{X}$ is a separated Deligne–Mumford stack of finite type over a noetherian scheme S, then the Keel–Mori Theorem (4.4.6) implies that there exists a coarse moduli space $\pi \colon \mathcal{X} \to X$. We say that the coarse moduli space $\mathcal{X} \to X$ is *tame* if every automorphism group has order prime to the characteristic, i.e., invertible in $\Gamma(S, \mathcal{O}_S)$. A tame coarse moduli space is a good moduli space (Lemma 4.4.22). If $\mathcal{X}$ has quasi-finite stabilizers, then in fact every good moduli space $\pi \colon \mathcal{X} \to X$ is a tame coarse moduli space with π separated; see Proposition 6.5.33.

The goal of this section is to establish the following theorem.

Theorem 6.5.6. *If $\pi \colon \mathcal{X} \to X$ is a good moduli space with X quasi-separated, then*

(1) π is surjective and universally closed;

(2) For closed substacks $\mathcal{Z}_1, \mathcal{Z}_2 \subseteq \mathcal{X}$, $\mathrm{im}(\mathcal{Z}_1 \cap \mathcal{Z}_2) = \mathrm{im}(\mathcal{Z}_1) \cap \mathrm{im}(\mathcal{Z}_2)$. For geometric points $x_1, x_2 \in \mathcal{X}(\Bbbk)$, $\pi(x_1) = \pi(x_2) \in X(\Bbbk)$ if and only if $\overline{\{x_1\}} \cap \overline{\{x_2\}} \neq \emptyset$ in $|\mathcal{X} \times_S \Bbbk|$. In particular, π induces a bijection between closed points in $\mathcal{X}$ and closed points in X;

(3) If $\mathcal{X}$ is noetherian, so is X. If $\mathcal{X}$ is of finite type over a noetherian algebraic space S, then X is of finite type over S and π_* preserves coherence, i.e., for $F \in \mathrm{Coh}(\mathcal{X})$, $\pi_* F \in \mathrm{Coh}(X)$; and

(4) If $\mathcal{X}$ is noetherian, then π is universal for maps to algebraic spaces.

Remark 6.5.7. In (2), the images and intersections are taken scheme-theoretically. Note that since π is closed, the set-theoretic image of a closed substack $\mathcal{Z}$ is identified with the topological space of its scheme-theoretic image $\mathrm{im}(\mathcal{Z})$. If $I \subseteq \mathcal{O}_{\mathcal{X}}$ is the sheaf of ideals defining $\mathcal{Z}$, the image $\mathrm{im}(\mathcal{Z})$ is defined by $\pi_* I \subseteq \pi_* \mathcal{O}_{\mathcal{X}} = \mathcal{O}_X$.

In the case of affine GIT where we have a good moduli space $\pi \colon [\operatorname{Spec} A/G] \to \operatorname{Spec} A^G$ and a good quotient $\widetilde{\pi} \colon \operatorname{Spec} A \to \operatorname{Spec} A^G$, this theorem translates to:

Corollary 6.5.8 (Affine GIT). *Let G be a linearly reductive group over an algebraically closed field $\Bbbk$. Then $\widetilde{\pi} \colon U = \operatorname{Spec} A \to U /\!\!/ G := \operatorname{Spec} A^G$ satisfies:*

(1) $\widetilde{\pi}$ is surjective and for every G-invariant closed subscheme $Z \subseteq U$, $\operatorname{im}(Z) \subseteq U/\!\!/G$ is closed. The same holds for the base change $T \to U/\!\!/G$ by a morphism from a scheme;

(2) For closed G-invariant closed subschemes $Z_1, Z_2 \subseteq U$, $\operatorname{im}(Z_1 \cap Z_2) = \operatorname{im}(Z_1) \cap \operatorname{im}(Z_2)$. For $x_1, x_2 \in X(\Bbbk)$, $\widetilde{\pi}(x_1) = \widetilde{\pi}(x_2)$ if and only if $\overline{Gx_1} \cap \overline{Gx_2} \neq \emptyset$. In particular, $\widetilde{\pi}$ induces a bijection between closed G-orbits of $\Bbbk$-points in U and $\Bbbk$-points of $U/\!\!/G$.

(3) If A is noetherian, so is A^G. If A is finitely generated over $\Bbbk$, then A^G is also finitely generated over $\Bbbk$ and for every finitely generated A-module M with a G-action, M^G is a finitely generated A^G-module; and

(4) If A is noetherian, then $\widetilde{\pi}$ is universal for G-invariant maps to algebraic spaces. $\square$

Remark 6.5.9. If $Z \subseteq U = \operatorname{Spec} A$ is defined by a G-invariant ideal I, then (1) implies that $\pi(Z)$ is defined by $I^G \subseteq A^G$. If Z_1, Z_2 are defined by G-invariant ideals $I_1, I_2 \subseteq A$, then (2) implies that $(I_1 + I_2)^G = I_1^G + I_2^G$. In particular, if Z_1 and Z_2 are disjoint, then so are $\operatorname{im}(Z_1)$ and $\operatorname{im}(Z_2)$ and we can write $1 = f_1 + f_2$ with $f_1 \in I_1^G$ and $f_2 \in I_2^G$; the function f_1 restricts to 0 on Z_1 and 1 on Z_2. We see that G-invariant functions separate disjoint G-invariant closed subschemes.

Remark 6.5.10 (Hilbert's 14th problem). Hilbert's 14th problem asks when the invariant ring A^G is finitely generated. While this is not always true, Hilbert showed it is true when G is linearly reductive, which is precisely the assertion of (3) above. Hilbert's original argument in [Hil90] is so elegant and played such an important role in the development of modern algebra that we reproduce it here. Our proof of Theorem 6.5.6(3), while similar in spirit, will not be as explicit.

Let $f_1, \ldots, f_n$ be $\Bbbk$-algebra generators of A and let $V \subseteq A$ be a finite dimensional G-invariant subspace containing each f_i (Proposition B.1.17(1)). Then we have a surjection $\operatorname{Sym}^* V = \Bbbk[x_1, \ldots, x_m] \twoheadrightarrow A$ of $\Bbbk$-algebras with G-actions and we set $I = \ker(\Bbbk[x_1, \ldots, x_m] \to A)$. Since G is linearly reductive, $A^G = (\Bbbk[x_1, \ldots, x_m]/I)^G = \Bbbk[x_1, \ldots, x_m]^G / I^G$ and we can assume that $A = \Bbbk[x_1, \ldots, x_m]$ is the polynomial ring so that A^G is a graded $\Bbbk$-algebra whose degree 0 component is $\Bbbk$. It therefore suffices to show that the ideal $J_+ := \sum_{d>0} A_d^G \subseteq A^G$ is finitely generated, as it is not hard to see that ideal generators of J_+ are also $\Bbbk$-algebra generators of A^G.

Hilbert first showed that every ideal in $A = \Bbbk[x_1, \ldots, x_n]$ is finitely generated—this is what is referred to today as *Hilbert's Basis Theorem*, which was proved by Hilbert precisely to make this argument. It follows that $J_+ A \subseteq A$ is finitely generated by homogenous invariants $f_1, \ldots, f_n \in A^G$. We will show that they also generate J_+ as an ideal in A^G. For $f \in A_d^G$, we can write

$$f = \sum_{i=1}^{n} f_i g_i \tag{6.5.11}$$

with $g_i \in A$ a homogeneous (not necessarily invariant) function of degree $d - \deg f_i$ (with $g_i = 0$ if $\deg f_i > d$). Since G is linearly reductive, there is a $\Bbbk$-linear map $R \colon A \to A^G$ called the *Reynolds operator* (see Remark B.1.42), which is the identity on A^G, respects the grading, and satisfies $R(xy) = xR(y)$ for $x \in A^G$ and $y \in A$. Applying R to (6.5.11) shows that $f = R(f) = \sum_i f_i R(g_i)$ with $R(g_i) \in A^G$ and thus f lies in the ideal in A^G generated by the f_i.[2]

Hilbert gave a constructive proof of this theorem in [Hil93], where in the course of the argument, he established the Syzygy Theorem, the Nullstellensatz, a version of Noether normalization, and a version of the Hilbert–Mumford criterion. These results spurred the development of modern commutative algebra. We strongly encourage you to read [Hil90] and [Hil93] (or Hilbert's translated lecture notes [Hil93]).

Remark 6.5.12 (Reductivity in positive characteristic). In characteristic p, every smooth linearly reductive group is an extension of a torus by a finite étale group scheme prime to the characteristic. In particular, GL_n is not linearly reductive (see Example B.1.44). In characteristic p, there are the following variants for an affine algebraic group G over an algebraically closed field $\Bbbk$:

(1) *G is reductive* if G is smooth and every smooth, connected, unipotent, and normal subgroup of G is trivial, and

(2) *G is geometrically reductive* if for every surjection $V \to W$ of G-representations and $w \in W^G$, there exists $n > 0$ such that w^{p^n} is in the image of $\mathrm{Sym}^{p^n} V \to \mathrm{Sym}^{p^n} W$.

Mumford's conjecture [GIT, Preface]—now Haboush's theorem [Hab75]—that these notions are equivalent when G is smooth (see Theorem B.1.46). See also §B.1.6 for a further discussion.

Affine GIT was originally developed for linearly reductive groups, but the defining property of geometrically reductive groups allows one to prove that $\operatorname{Spec} A \to \operatorname{Spec} A^G$ enjoys similar properties to affine GIT quotients by linearly reductive groups. Namely, Corollary 6.5.8(1)–(4) hold (with the exception that the noetherianness of A does not necessarily imply the noetherianness of A^G). While the arguments are not substantially more complicated than the linearly reductive case, we only prove them for linearly reductive groups for the sake of simplicity. We refer the reader to [Nag64], [MFK94, App. 1.C], [New78, §3], [Dol03, §3.4], [Spr77, §2] and [DC71, §2].

Likewise, the notion of a good moduli space can be extended to characterize quotients by geometrically reductive groups: in [Alp14], a quasi-compact and quasi-separated morphism $\pi\colon \mathcal{X} \to X$, from an algebraic stack to an algebraic space, is called an *adequate moduli space* if (1) $\mathcal{O}_X \to \pi_*\mathcal{O}_{\mathcal{X}}$ is an isomorphism and (2) for every surjection $\mathcal{A} \to \mathcal{B}$ of quasi-coherent $\mathcal{O}_{\mathcal{X}}$-algebras, then every section s of $\pi_*(\mathcal{B})$ over a smooth morphism $\operatorname{Spec} A \to \mathcal{Y}$ has a positive power that lifts to a section of $\pi_*(\mathcal{A})$). An adequate moduli space satisfies Theorem 6.5.6(1)–(4) (except again for the noetherian implication). Moreover, if G is geometrically reductive, then $\pi\colon [\operatorname{Spec} A/G] \to \operatorname{Spec} A^G$ is an adequate moduli space. In characteristic 0, an adequate moduli space is necessarily good. Again, it is not substantially harder to establish the properties of adequate moduli spaces, but we choose to only develop the theory of good moduli spaces as it is more streamlined and probably best seen first by a student.

[2] For an alternative argument that A^G is noetherian, linear reductivity can be used to show that $JA \cap A^G = J$ for every ideal $J \subseteq A^G$ (see Lemma 6.5.24(5)). If $J_1 \subseteq J_2 \subseteq \cdots \subseteq A^G$ is an ascending chain of ideals, then the ascending chain $J_1 A \subseteq J_2 A \subseteq \cdots \subseteq A$ terminates, which implies that the original sequence $J_1 = J_1 A \cap A^G \subseteq J_2 = J_2 A \cap A^G \subseteq \cdots \subseteq A^G$ also terminates.

6.5.2 Cohomologically affine morphisms

The exactness condition on the pushforward π_* in the definition of a good moduli space (Definition 6.5.1(2)) is a non-representable analog of affineness.

Definition 6.5.13 (Cohomologically affine)**.** A quasi-compact and quasi-separated morphism $f\colon \mathcal{X} \to \mathcal{Y}$ of algebraic stacks is *cohomologically affine* if

$$f_*\colon \operatorname{QCoh}(\mathcal{X}) \to \operatorname{QCoh}(\mathcal{Y})$$

is exact. A quasi-compact and quasi-separated algebraic stack $\mathcal{X}$ is *cohomologically affine* if $\mathcal{X} \to \operatorname{Spec} \mathbb{Z}$ is.

Example 6.5.14. An affine algebraic group G over a field $\Bbbk$ is linearly reductive (Definition B.1.34) if and only if BG is cohomologically affine. If G is linearly reductive and acts on an affine $\Bbbk$-scheme $\operatorname{Spec} A$, then $[\operatorname{Spec} A/G]$ is cohomologically affine as the structure map $[\operatorname{Spec} A/G] \to \operatorname{Spec} \Bbbk$ is the composition of the affine morphism $[\operatorname{Spec} A/G] \to BG$ and the cohomologically affine morphism $BG \to \operatorname{Spec} \Bbbk$.

Remark 6.5.15. By Serre's Criterion for Affineness (4.5.16), an algebraic space is cohomologically affine if and only if it is an affine scheme. An algebraic stack $\mathcal{X}$ with affine diagonal is cohomologically affine if and only if $\mathrm{H}^i(\mathcal{X}, F) = 0$ for all $i > 0$ and every quasi-coherent sheaf F; this follows because the cohomology $\mathrm{H}^i(\mathcal{X}, F)$ can be computed in $\operatorname{QCoh}(\mathcal{X})$ for such stacks $\mathcal{X}$ by Proposition 6.1.29. This is not true for algebraic stacks with non-affine diagonal, e.g., BE for an elliptic curve E.

Likewise, a morphism $f\colon \mathcal{X} \to \mathcal{Y}$ of algebraic stacks, each affine diagonal, is cohomologically affine if and only if $\mathrm{R}^i f_*(F) = 0$ for all $i > 0$ and every quasi-coherent sheaf F. If in addition f is representable, then f is cohomologically affine if and only if it is affine (see Corollary 6.5.18 below).

Remark 6.5.16 (Noetherian case). If $\mathcal{X}$ is noetherian, then a quasi-compact, quasi-separated morphism $f\colon \mathcal{X} \to \mathcal{Y}$ is cohomologically affine if and only if $f_*\colon \operatorname{Coh}(\mathcal{X}) \to \operatorname{QCoh}(\mathcal{Y})$ is exact. This holds because every quasi-coherent sheaf is a colimit of coherent sheaves (Proposition 6.1.16) and f_* commutes with colimits. Since cohomology also commutes with colimits (Proposition 6.1.31), a morphism $f\colon \mathcal{X} \to \mathcal{Y}$ of noetherian algebraic stacks, both with affine diagonal, is cohomologically affine if and only if $\mathrm{R}^i f_*(F) = 0$ for all $i > 0$ and every *coherent* sheaf F.

Lemma 6.5.17. *Consider a cartesian diagram*

$$\begin{tikzcd} \mathcal{X}' \arrow[r, "g'"] \arrow[d, "\pi'"'] \arrow[dr, phantom, "\square"] & \mathcal{X} \arrow[d, "\pi"] \\ \mathcal{Y}' \arrow[r, "g"] & \mathcal{Y}. \end{tikzcd}$$

of algebraic stacks.

1. *If g is faithfully flat and π' is cohomologically affine, then π is cohomologically affine.*
2. *If $\mathcal{Y}$ has quasi-affine diagonal (e.g., a quasi-separated algebraic space) and π is cohomologically affine, then π' is cohomologically affine.*

Proof. For (1), by Flat Base Change (6.1.15) there is an equivalence $g^*\pi_* \simeq \pi'_* g'^*$ of functors defined on categories of quasi-coherent sheaves. Since π'_* and g'^* are exact and g^* is faithfully exact, π_* is exact.

For (2), we first show that if g is quasi-affine and π is cohomologically affine, then π' is also cohomologically affine. It suffices to handle the cases that g is an open immersion and g is affine. If g is an open immersion and $F' \twoheadrightarrow G'$ is a surjection in $\mathrm{QCoh}(\mathcal{X}')$, we define $G = \mathrm{im}(g'_*F' \to g'_*G')$. Note that $g'^*G \cong G'$. Since π_* is exact, $\pi_* g'_* F \twoheadrightarrow \pi_* G$ is surjective. If we apply g^* and use the identities $g^*\pi_* \simeq \pi'_* g'^*$ and $g'^* g'_* \simeq \mathrm{id}$, we obtain a surjection $\pi'_* F' \twoheadrightarrow \pi'_* g'^* G \cong \pi'_* G'$. On the other hand, if g is affine then g_* is faithfully exact. Since π_* and g'_* are are exact, the identity $g_*\pi'_* \simeq \pi_* g'_*$ implies that π'_* is also exact. To show (2), we may assume that $\mathcal{Y}$ and $\mathcal{Y}'$ are quasi-compact. Choose smooth presentations $Y = \operatorname{Spec} A \to \mathcal{Y}$ and $Y' = \operatorname{Spec} A' \to \mathcal{Y}' \times_{\mathcal{Y}} Y$. Since $\mathcal{Y}$ has quasi-affine diagonal, $Y \to \mathcal{Y}$ is quasi-affine, and thus $\mathcal{X}_Y \to Y$ is cohomologically affine. As $Y' \to Y$ is affine, $\mathcal{X}_{Y'} \to Y'$ is also cohomologically affine. By (1), we conclude that $\pi' \colon \mathcal{X}' \to \mathcal{Y}'$ is cohomologically affine. □

Corollary 6.5.18. *Let $f \colon \mathcal{X} \to \mathcal{Y}$ be a representable morphism of algebraic stacks where $\mathcal{Y}$ has quasi-affine diagonal. Then f is affine if and only if f is cohomologically affine.*

Proof. Under the hypotheses, both affine and cohomologically affine morphisms are local properties in the fppf topology. The statement therefore reduces to the case where $\mathcal{Y}$ is an affine scheme and $\mathcal{X}$ is an algebraic space, which is precisely the content of Serre's Criterion for Affineness (4.5.16). □

6.5.3 Properties of linearly reductive groups

Recall that an affine algebraic group G over a field $\Bbbk$ is *linearly reductive* if the functor $\mathrm{Rep}(G) \to \mathrm{Vect}_{\Bbbk}$, defined by $V \mapsto V^G$, is exact (Definition B.1.34), or equivalently if $BG \to \operatorname{Spec} \Bbbk$ is cohomologically affine.

Proposition 6.5.19. *Let $1 \to K \to G \to Q \to 1$ be an exact sequence of affine algebraic groups over a field $\Bbbk$. Then G is linearly reductive if and only if both K and Q are.*

Proof. We will use the cartesian diagram

$$\begin{tikzcd} Q \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & BK \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \operatorname{Spec} \Bbbk \arrow[d] \\ \operatorname{Spec} \Bbbk \arrow[r] & BG \arrow[r] & BQ \end{tikzcd}$$

of Exercise 2.4.40(c). For ($\Leftarrow$), fppf descent (Lemma 6.5.17(2)) along the morphism $\operatorname{Spec} \Bbbk \to BQ$ implies that $BG \to BQ$ is cohomologically affine, and thus so is the composition $BG \to BQ \to \operatorname{Spec} \Bbbk$. Conversely, since Q is affine, fppf descent along $\operatorname{Spec} \Bbbk \to BG$ implies that $BK \to BG$ is affine. Therefore, the composition $BK \to BG \to \operatorname{Spec} \Bbbk$ is cohomologically affine, and K is linearly reductive. To see that Q is linearly reductive, we claim that for every Q-representation V, the adjunction $V \to q_* q^* V$ is an isomorphism. Indeed, q^*V is a G-representation in which K acts trivially, and since the pushforward of a G-representation W under $q \colon BG \to BQ$ is the Q-representation W^K, we see that $V = q_* q^* V$. It follows that $\Gamma(BQ, -) = \Gamma(BG, q^* -)$ is exact. □

Proposition 6.5.20. *Let H be a linearly reductive group over an algebraically closed field $\Bbbk$. If H acts freely on an affine scheme U over $\Bbbk$, then the algebraic space quotient U/H is affine.*

Proof. The algebraic space U/H and the good quotient $\operatorname{Spec} A^H$ are both universal for maps to algebraic spaces (Theorem 6.5.6(4)). Alternatively, the composition $U/H \to BH \to \operatorname{Spec} \Bbbk$ is an affine morphism followed by a cohomologically affine morphism. It follows from Serre's Criterion for Affineness (4.5.16) that U/H is affine. $\square$

In particular, if H is a linearly reductive subgroup of an affine algebraic group G, then the quotient G/H is affine. Matsushima's Theorem provides a converse.

Proposition 6.5.21 (Matsushima's Theorem)**.** *Let G be a linearly reductive group over an algebraically closed field $\Bbbk$.*

(1) A subgroup H of G is linearly reductive if and only if G/H is affine.

(2) Given an action of G on an algebraic space U of finite type over $\Bbbk$ and a $\Bbbk$-point $u \in U$ with stabilizer G_u, then G_u is linearly reductive if and only if the orbit Gu is affine. In particular, if U is affine, then the stabilizer of a point with closed orbit is necessarily linearly reductive.

Proof. Part (2) follows from (1) since $Gu = G/G_u$. For (1), the ($\Rightarrow$) implication follows from Proposition 6.5.20. For the converse, consider the cartesian diagram

$$\begin{tikzcd} G/H \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \operatorname{Spec} \Bbbk \arrow[d] \\ BH \arrow[r] & BG. \end{tikzcd}$$

If G/H is affine, then $BH \to BG$ is affine is affine by fppf descent along $\operatorname{Spec} \Bbbk \to BG$. Therefore, $BH \to BG \to \operatorname{Spec} k$ is cohomologically affine, i.e., H is linearly reductive. $\square$

6.5.4 First properties of good moduli spaces

Lemma 6.5.22. *Consider a cartesian diagram*

$$\begin{tikzcd} \mathcal{X}' \arrow[r, "g'"] \arrow[d, "\pi'"] \arrow[dr, phantom, "\square"] & \mathcal{X} \arrow[d, "\pi"] \\ X' \arrow[r, "g"] & X \end{tikzcd}$$

of algebraic stacks where X and X' are quasi-separated algebraic spaces.

(1) If g is faithfully flat and π' is a good moduli space, then π is a good moduli space.

For the remaining statements, assume in addition that π is a good moduli space.

(2) The morphism π' is a good moduli space.

(3) For $F \in \operatorname{QCoh}(\mathcal{X})$ and $G \in \operatorname{QCoh}(X)$, the adjunction map $\pi_* F \otimes G \to \pi_*(F \otimes \pi^* G)$ is an isomorphism. In particular, $G \xrightarrow{\sim} \pi_* \pi^* G$ is an isomorphism.

(4) For $F \in \operatorname{QCoh}(\mathcal{X})$, then the adjunction map $g^* \pi_* F \xrightarrow{\sim} \pi'_* g'^* F$ is an isomorphism.

(5) For a quasi-coherent sheaf of ideals $J \subseteq \mathcal{O}_X$, the natural map $J \to \pi_*(\pi^{-1} J \cdot \mathcal{O}_{\mathcal{X}})$ is an isomorphism.

Proof. If $g\colon X' \to X$ is flat, then the pullback of the natural map $\mathcal{O}_X \to \pi_*\mathcal{O}_{\mathcal{X}}$ under g is the map $\mathcal{O}_{X'} \to \pi'_*\mathcal{O}_{\mathcal{X}'}$. Thus, Lemma 6.5.17 and fppf descent imply both (1) and the case of (2) when g is flat. Also note that since X' is quasi-separated, it has quasi-affine diagonal (Corollary 4.5.8). Before proving the general case of (2), we prove (3). Choose an étale presentation $U \to X$ with U the disjoint union of affine schemes. The base change $\pi_U\colon \mathcal{X}_U \to U$ is also a good moduli space by the flat case of (2). Moreover, by Flat Base Change (6.1.15). the adjunction map $\pi_*F \otimes G \to \pi_*(F \otimes \pi^*G)$ pulls back under $U \to X$ to the adjunction map $\pi_{U,*}F_U \otimes G_U \to \pi_{U,*}(F_U \otimes \pi_U^* G_U)$, where F_U and G_U are the pullbacks to $\mathcal{X}_U$ and X_U respectively. We may therefore assume that $X = \operatorname{Spec} A$ is affine. If $G_2 \to G_1 \to G \to 0$ is a free presentation, then the maps $\pi_*F \otimes G_i \to \pi_*(F \otimes \pi^*G_i)$ are isomorphisms for $i = 1, 1$. Since $\pi_*F \otimes -$ and $\pi_*(F \otimes \pi^*-)$ are right exact, we have a commutative diagram

$$\begin{tikzcd}
\pi_*F \otimes G_2 \arrow[r] \arrow[d] & \pi_*F \otimes G_1 \arrow[r] \arrow[d] & \pi_*F \otimes G \arrow[r] \arrow[d] & 0 \\
\pi_*(F \otimes \pi^*G_2) \arrow[r] & \pi_*(F \otimes \pi^*G_1) \arrow[r] & \pi_*(F \otimes \pi^*G) \arrow[r] & 0
\end{tikzcd}$$

of right exact sequences. Since the left two vertical maps are isomorphisms, so is the right one.

For (2), it remains show that $\mathcal{O}_{X'} \to \pi'_*\mathcal{O}_{\mathcal{X}'}$ is an isomorphism as Lemma 6.5.17(2) already established that π'_* is exact. We can assume that X and X' are affine. In this case, g_* is faithfully exact so it suffices to show that $g_*\mathcal{O}_{X'} \to g_*\pi'_*\mathcal{O}_{\mathcal{X}'}$ is an isomorphism. We may use the identity $g'_*\pi'^*\mathcal{O}_{X'} \cong \pi^*g_*\mathcal{O}_{X'}$ following from the affineness of g to identify this map with

$$g_*\mathcal{O}_{X'} \to g_*\pi'_*\mathcal{O}_{\mathcal{X}'} \cong \pi_*g'_*\mathcal{O}_{\mathcal{X}'} \cong \pi_*\pi^*g_*\mathcal{O}_{X'}.$$

The composition is an isomorphism as it is identified with the adjunction isomorphism of (3) applied to $F = g_*\mathcal{O}_{X'}$.

For (4), we know by Flat Base Change (6.1.15) that (4) is fppf local on X and X' and that it holds when g is flat. We may therefore reduce to when $X' \to X$ is a morphism of affine schemes. By factoring $X' \to X$ as a closed immersion followed by a flat morphism, we can further reduce to the case that $X' \hookrightarrow X$ is a closed immersion defined by a quasi-coherent sheaf of ideals $J \subseteq \mathcal{O}_X$. We aim to show that $\pi_*F/J\pi_*F \cong \pi_*(F/(\pi^{-1}J \cdot \mathcal{O}_{\mathcal{X}})F)$. Using the exactness of π_*, this is equivalent to the inclusion $J\pi_*F \hookrightarrow \pi_*((\pi^{-1}J \cdot \mathcal{O}_{\mathcal{X}})F)$ being surjective. The sheaf $(\pi^{-1}J \cdot \mathcal{O}_{\mathcal{X}})F$ is the image of $\pi^*J \otimes F \to F$. By the exactness of π_*, the pushforward $\pi_*((\pi^{-1}J \cdot \mathcal{O}_{\mathcal{X}})F)$ is the image of $\pi_*(\pi^*J \otimes F) \to \pi_*F$, but by (3) this is identified with the image of $J \otimes \pi_*F \to \pi_*F$.

For (5), if $Z \subseteq X$ is the closed subspace defined by J, then the preimage ideal sheaf $\pi^{-1}J \cdot \mathcal{O}_{\mathcal{X}}$ defines the preimage $\pi^{-1}(Z)$. The exactness of π_* implies that there is a commutative diagram of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & J \arrow[r] \arrow[d] & \mathcal{O}_X \arrow[r] \arrow[d] & \mathcal{O}_Z \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \pi_*(\pi^{-1}J \cdot \mathcal{O}_{\mathcal{X}}) \arrow[r] & \pi_*\mathcal{O}_{\mathcal{X}} \arrow[r] & \pi_*\mathcal{O}_{\pi^{-1}(Z)} \arrow[r] & 0.
\end{tikzcd}$$

As $\mathcal{X} \to X$ and $\pi^{-1}(Z) \to Z$ are good moduli spaces, the right two vertical arrows are isomorphisms, and thus so is the left arrow. $\square$

Remark 6.5.23. The isomorphism $\pi_* F \otimes G \to \pi_*(F \otimes \pi^* G)$ in (3) is similar to the projection formula, except that it holds even if G is not locally free!

Lemma 6.5.24. *Let $\pi\colon \mathcal{X} \to X$ be a good moduli space with X quasi-separated.*

(1) If $\mathcal{A}$ is a quasi-coherent sheaf of $\mathcal{O}_{\mathcal{X}}$-algebras, then $\mathcal{S}\mathrm{pec}_{\mathcal{X}}\, \mathcal{A} \to \mathcal{S}\mathrm{pec}_X\, \pi_* \mathcal{A}$ is a good moduli space.

(2) If $\mathcal{Z} \subseteq \mathcal{X}$ is a closed substack defined by a sheaf of ideals I and $\operatorname{im} \mathcal{Z} \subseteq X$ is the scheme-theoretic image, i.e., the closed subspace defined by $\pi_* I \subseteq \mathcal{O}_X$, then $\mathcal{Z} \to \operatorname{im} \mathcal{Z}$ is a good moduli space.

Proof. For (1), since $\mathcal{X} \times_X \mathcal{S}\mathrm{pec}_X\, \pi_* \mathcal{A} \to \mathcal{S}\mathrm{pec}_X\, \pi_* \mathcal{A}$ is cohomologically affine by Lemma 6.5.17(2) and $\mathrm{Spec}_{\mathcal{X}}\, \mathcal{A} \to \mathcal{X} \times_X \mathcal{S}\mathrm{pec}_X\, \pi_* \mathcal{A}$ is affine, it follows that $\mathcal{S}\mathrm{pec}_{\mathcal{X}}\, \mathcal{A} \to \mathcal{S}\mathrm{pec}_X\, \pi_* \mathcal{A}$ is cohomologically affine. Since the pushforward of $\mathcal{O}_{\mathcal{S}\mathrm{pec}_{\mathcal{X}} \mathcal{A}}$ is $\mathcal{O}_{\mathcal{S}\mathrm{pec}_X \pi_* \mathcal{A}}$ by construction, $\mathcal{S}\mathrm{pec}_{\mathcal{X}}\, \mathcal{A} \to \mathcal{S}\mathrm{pec}_X\, \pi_* \mathcal{A}$ is a good moduli space. Applying (1) to $\mathcal{Z} = \mathcal{S}\mathrm{pec}_{\mathcal{X}}(\mathcal{O}_{\mathcal{X}}/I)$ and using the identity $\pi_*(\mathcal{O}_{\mathcal{X}}/I) = \mathcal{O}_X/\pi_* I$. recovers (2). ☐

The above lemmas allow for quick proofs of the first two parts of Theorem 6.5.6.

Proof of Theorem 6.5.6(1). For every field-valued point $x \in X(\Bbbk)$, the base change $\mathcal{X}_x := \mathcal{X} \times_X \operatorname{Spec} \Bbbk \to \operatorname{Spec} \Bbbk$ is a good moduli space (Lemma 6.5.22(2)). In particular, $\Gamma(\mathcal{X}_x, \mathcal{O}_{\mathcal{X}_x}) = \Bbbk$, and it follows that $\mathcal{X}_x$ is non-empty and that $\pi\colon \mathcal{X} \to X$ is surjective. For a closed substack $\mathcal{Z} \subseteq \mathcal{X}$, Lemma 6.5.24(2) implies that $\mathcal{Z} \to \operatorname{im} \mathcal{Z}$ is a good moduli space and therefore also surjective. Thus, the set-theoretic image $\pi(\mathcal{Z})$ is identified with the scheme-theoretic image $\operatorname{im} \mathcal{Z}$ and is therefore closed. Since good moduli spaces are stable under base change, they are universally closed. ☐

Proof of Theorem 6.5.6(2). For two substacks $\mathcal{Z}_1, \mathcal{Z}_2 \subseteq X$ defined by ideal sheaves $I_1, I_2 \subseteq \mathcal{O}_X$, we apply the exact functor π_* to the short exact sequence $0 \to I_1 \to I_1 + I_2 \to I_2/I_1 \cap I_2 \to 0$ and surjection $I_2 \to I_2/I_1 \cap I_2$ to obtain a commutative diagram

$$\begin{tikzcd} & & \pi_* I_2 \arrow[d] \arrow[dr, two heads] & & \\ 0 \arrow[r] & \pi_* I_1 \arrow[r] & \pi_*(I_1 + I_2) \arrow[r, two heads] & \pi_* I_2/\pi_*(I_1 \cap I_2) \arrow[r] & 0. \end{tikzcd}$$

It follows that the natural inclusion $\pi_* I_1 + \pi_* I_2 \to \pi_*(I_1 + I_2)$ is surjective. ☐

Exercise 6.5.25. Let $\mathcal{X}$ be an algebraic stack with affine diagonal and $\mathcal{X} \to X$ be a good moduli space. Show that for every field valued point $x \in \mathcal{X}(\Bbbk)$ whose image in $|\mathcal{X}|$ is a closed point, the stabilizer G_x is linearly reductive.

6.5.5 Finite typeness of good moduli spaces

To show that a good moduli space $\mathcal{X} \to X$ preserves finite typeness, i.e., we will use that $\mathcal{X} \to X$ is universally submersive, and prove that finite typeness descends under universally submersive maps. Recall from §A.4.2 that a morphism $f\colon X \to Y$ of schemes is *universally submersive* if f is surjective and Y has the quotient topology, and that these properties are stable under base change. Both fppf morphisms and universally closed morphisms of noetherian schemes are universally submersive. This notion extends to morphisms of algebraic stacks and since good moduli spaces are universally closed (Theorem 6.5.6(1)), they are universally submersive.

Proposition 6.5.26 (Universally Submersive Descent for Finite Typeness)**.** *Let $X' \to X$ be a universally submersive morphism of noetherian schemes. If $X \to Y$ is a morphism of noetherian schemes and $X' \to X \to S$ is of finite type, then so is $X \to S$.*

Proof. We can assume that $S = \operatorname{Spec} A$ and $X = \operatorname{Spec} B$ are affine. Since a noetherian ring B is of finite type over A if and only if the reductions of the irreducible components of $\operatorname{Spec} B$ are of finite type over A, we can assume that B is an integral domain. By Generic Flatness (A.2.13) and Raynaud-Gruson Flatification (A.2.18), there is a commutative diagram

$$\begin{tikzcd} Z' \arrow[r] \arrow[d] & X' \arrow[d, "f"] \arrow[dr] & \\ Z \arrow[r, "g"] & X \arrow[r] & S \end{tikzcd}$$

where $g\colon Z = \operatorname{Bl}_I X \to X$ is the blowup along an ideal $I \subseteq B$, Z' is the strict transform of X', i.e., the closure of $(Z \smallsetminus g^{-1}(V(I))) \times_X X'$ in the base change $Z \times_X X'$, and $Z' \to Z$ is flat. We claim the universal submersiveness of $X' \to X$ implies that $Z' \to Z$ is surjective. As $g\colon Z \to X$ is an isomorphism over $U = X \smallsetminus V(I)$ and $f\colon X' \to X$ is surjective, we know that $g^{-1}(U) \subseteq Z$ is contained in the image. If $z \in Z$ is a point, we can choose a map $\operatorname{Spec} R \to Z$ from a DVR whose generic point maps to $g^{-1}(U)$ and whose special point maps to z. Since $X' \to X$ is universally submersive, there exists an extension of DVRs $R \to R'$ and a lift $\operatorname{Spec} R' \to X'$ (see Exercise A.4.9). The induced map $\operatorname{Spec} R' \to Z \times_X X'$ factors through Z', and we see that z is thus in the image of Z'.

Since X' is of finite type over S, so is Z'. By faithfully flat descent (Proposition 2.1.27(1)), $Z \to S$ is also of finite type. To show that $X \to S$ is of finite type, we may choose generators $f_1, \ldots, f_n \in I$ so that $Z = \bigcup_i \operatorname{Spec} B_i$ where $B_i = B\langle f_j/f_i\rangle \subseteq K = \operatorname{Frac}(B)$ is the subalgebra generated by B and the elements f_j/f_i for $j \neq i$. Write $B = \bigcup_\lambda B_\lambda$ as a union of its finitely generated A-subalgebras. For $\lambda \gg 0$, each $f_i \in B_\lambda$ and we set $I_\lambda = (f_1, \ldots, f_n) \subseteq B_\lambda$. Since Z is finite type over S, each B_i is finitely generated over B, and thus for $\lambda \gg 0$, we see that in the diagram

$$\begin{tikzcd} B_{\lambda,i} \arrow[r, equal] \arrow[d, hook] & B_\lambda\langle f_j/f_i\rangle \arrow[r, hook] & \operatorname{Frac}(B_\lambda) \arrow[d, hook] \\ B_i \arrow[r, equal] & B\langle f_j/f_i\rangle \arrow[r, hook] & \operatorname{Frac}(B) \end{tikzcd}$$

the inclusion $B_{\lambda,i} \hookrightarrow B_i$ is surjective. It follows that $Z = \operatorname{Bl}_I \operatorname{Spec} B = \operatorname{Bl}_{I_\lambda} \operatorname{Spec} B_\lambda$ for $\lambda \gg 0$. Considering the composition

$$g_\lambda\colon Z \xrightarrow{g} X = \operatorname{Spec} B \xrightarrow{p_\lambda} \operatorname{Spec} B_\lambda,$$

the pushforward of the injection $\mathcal{O}_X \hookrightarrow g_*\mathcal{O}_Z$ along p_λ yields an inclusion $p_{\lambda,*}\mathcal{O}_X \hookrightarrow g_{\lambda,*}\mathcal{O}_Z$. But $g_{\lambda,*}\mathcal{O}_Z$ is coherent, hence so is $p_{\lambda,*}\mathcal{O}_X$. This shows that B is a finite B_λ-module and thus finitely generated as an A-algebra. $\square$

Proof of Theorem 6.5.6(3). If $\mathcal{X}$ is noetherian and $I_1 \subseteq I_2 \subseteq \cdots$ is an ascending chain of ideal sheaves of $\mathcal{O}_X$, then $\pi^{-1}I_1 \cdot \mathcal{O}_{\mathcal{X}} \subseteq \pi^{-1}I_2 \cdot \mathcal{O}_{\mathcal{X}} \subseteq$ is an ascending chain of ideal sheaves of $\mathcal{O}_{\mathcal{X}}$ which terminates. By Lemma 6.5.24(5), $I_n = \pi_*(\pi^{-1}I_n \cdot \mathcal{O}_{\mathcal{X}})$ and therefore the chain $I_1 \subseteq I_2 \subseteq \cdots$ terminates and X is noetherian.

Assume now that S is noetherian and $\mathcal{X}$ is of finite type over S. As $\mathcal{X} \to X$ is universally closed (Theorem 6.5.6(1)), it is also universally submersive. Choose a smooth presentation $U \to \mathcal{X}$ from a scheme. Since $U \to \mathcal{X}$ is universally submersive, so is the composition $U \to \mathcal{X} \to X$. Since $U \to S$ is of finite type and X is noetherian, Proposition 6.5.26 implies that $X \to S$ is also of finite type.

Given a coherent sheaf F on $\mathcal{X}$, to show that the pushforward $\pi_* F$ is coherent, we may assume that $X = \operatorname{Spec} A$ is affine and that $\mathcal{X}$ is irreducible. We claim we can further reduce to the case that $\mathcal{X}$ is reduced. Let $I \subseteq \mathcal{O}_{\mathcal{X}}$ be the ideal sheaf defining $\mathcal{X}_{\mathrm{red}} \hookrightarrow \mathcal{X}$. Then for some $N > 0$, we have that $I^N = 0$. By examining the exact sequences $0 \to \pi_*(I^{k+1}F) \to \pi_*(I^k F) \to \pi_*(I^k F/I^{k+1}F) \to 0$ and using that $\pi_*(I^k F/I^{k+1}F)$ is coherent (since $I^k F/I^{k+1}F$ is supported on $\mathcal{X}_{\mathrm{red}}$), we conclude by induction that $\pi_* F$ is coherent.

By noetherian induction, we can assume that $\pi_* F$ is coherent if $\operatorname{Supp}(F) \subsetneq \mathcal{X}$. The maximal torsion subsheaf $F_{\mathrm{tors}} \subseteq F$ has support strictly contained in $\mathcal{X}$. Using the exact sequence $0 \to F_{\mathrm{tors}} \to F \to F/F_{\mathrm{tors}} \to 0$ and the exactness of π_*, we see the coherence of $\pi_*(F/F_{\mathrm{tors}})$ implies the coherence of $\pi_* F$. In other words, we can assume that F is torsion free. In this case, every section $s\colon \mathcal{O}_{\mathcal{X}} \to F$ is injective. We now argue by induction on the dimension of the vector space $\xi^* F$, where $\xi\colon \operatorname{Spec} K \to \mathcal{X}$ is a field-valued point whose image is the generic point. If F has no sections, then $\pi_* F = 0$ is coherent. Otherwise, a section induces a short exact sequence $0 \to \mathcal{O}_{\mathcal{X}} \to F \to F/\mathcal{O}_{\mathcal{X}} \to 0$ and $\xi^*(F/\mathcal{O}_{\mathcal{X}})$ has strictly smaller dimension. By again appealing to the exactness of π_*, we see that the coherence of $\pi_*(F/\mathcal{O}_{\mathcal{X}})$ implies the coherence of $\pi_* F$. □

6.5.6 Universality of good moduli spaces

We now complete the proof of Theorem 6.5.6 by showing that $\pi\colon \mathcal{X} \to X$ is universal for maps to algebraic spaces. Our argument follows the same logic as for coarse moduli spaces in Theorem 4.2.3.

Proof of Theorem 6.5.6(4). We need to show that every diagram

$$\begin{tikzcd} \mathcal{X} \arrow[d, "\pi"'] \arrow[dr, "f"] & \\ X \arrow[r, dashed] & Y \end{tikzcd} \tag{6.5.27}$$

has a unique filling, or in other words that the natural map $\operatorname{Mor}(X, Y) \to \operatorname{Mor}(\mathcal{X}, Y)$ is bijective. The uniqueness follows as in the proof of Theorem 4.2.3 and uses only that $\pi\colon \mathcal{X} \to X$ is universally closed, schematically dominant, and surjective: if $h_1, h_2\colon X \to Y$ are two fillings of (6.5.27), then $\pi\colon \mathcal{X} \to X$ factors through the equalizer $E \to X$ of h_1 and h_2. Since $E \to X$ is universally closed, locally of finite type, surjective, and a monomorphism, it is an isomorphism.

For existence, the case when Y is affine is easy:

$$\begin{aligned} \operatorname{Mor}(X, Y) = \operatorname{Hom}(\Gamma(Y, \mathcal{O}_Y), \Gamma(X, \mathcal{O}_X)) = \\ \operatorname{Hom}(\Gamma(Y, \mathcal{O}_Y), \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})) = \operatorname{Mor}(\mathcal{X}, Y). \end{aligned}$$

(Although unnecessary for the argument below, the case when Y is a scheme is also straightforward: if $\{Y_i\}$ is an affine cover of Y and we set $\mathcal{X}_i := f^{-1}(Y_i) \subseteq \mathcal{X}$ with complement $\mathcal{Z}_i$, then $\mathcal{X} \setminus \pi^{-1}(\pi(\mathcal{Z}_i)) \to X \setminus \pi(\mathcal{Z}_i)$ is a good moduli space and $\mathcal{X} \setminus \pi^{-1}(\pi(\mathcal{Z}_i)) \subseteq \mathcal{X}_i$. By the affine case, we have unique factorizations

$X \smallsetminus \pi(\mathcal{Z}_i) \to Y_i$ and since $\bigcap_i \pi(\mathcal{Z}_i) = \emptyset$, these maps glue to the desired map $X \to Y$; see also [GIT, §0.6].)

For the general case, since $\mathcal{X}$ is quasi-compact, the map $\mathcal{X} \to Y$ factors through a quasi-compact subspace, so we can further assume that Y is quasi-compact. We can also use étale descent and limit methods to reduce to the case that $X = \operatorname{Spec} A$ where A is a strictly henselian local ring. This reduction works just as in the case of coarse moduli spaces (Theorem 4.2.3). Since A is local, there is a unique closed point $x \in |\mathcal{X}|$, and we let $\mathcal{G}_x \hookrightarrow \mathcal{X}$ be the closed immersion of the residual gerbe (Proposition 3.5.16). Let $(Y' = \operatorname{Spec} B, y') \to (Y, f(x))$ be an étale presentation. The base change $\mathcal{X}' := \mathcal{X} \times_Y Y' \to \mathcal{X}$ is an étale, separated, surjective, and representable morphism. Let $x' \in |\mathcal{X}'|$ be a preimage of $x \in |\mathcal{X}|$ and $\mathcal{U}' \subseteq \mathcal{X}'$ be a quasi-compact open substack containing x'. In the diagram,

$$\begin{tikzcd}
\mathcal{U}' \arrow[r, hook] & \mathcal{X}' \arrow[r] \arrow[dd] & \mathcal{X} \arrow[d, "\pi"] \arrow[dd, bend left, "f"] \\
 & & X \\
 & Y' \arrow[r] & Y
\end{tikzcd}$$

$\mathcal{U}' \to \mathcal{X}$ is a representable, quasi-finite, and separated morphism, and Zariski's Main Theorem (6.1.18) implies that there is a factorization $\mathcal{U}' \to \widetilde{\mathcal{X}} \to \mathcal{X}$ with $\mathcal{U}' \hookrightarrow \widetilde{\mathcal{X}}$ an open immersion and $\widetilde{\mathcal{X}} \to \mathcal{X}$ a finite morphism. Writing $\widetilde{\mathcal{X}} = \mathcal{S}\mathrm{pec}_{\mathcal{X}}\, \mathcal{A}$ for a coherent sheaf of algebras $\mathcal{A}$, Lemma 6.5.24(1) implies that $\widetilde{\pi}\colon \widetilde{\mathcal{X}} \to \widetilde{X} := \operatorname{Spec}_X \pi_*\mathcal{A}$ is a good moduli space and we know from Theorem 6.5.6(3) that $\pi_*\mathcal{A}$ is coherent. Thus, $\widetilde{X} \to X = \operatorname{Spec} A$ is finite with A henselian, and we can apply Proposition B.5.9 to write $\widetilde{X} = \coprod_i \operatorname{Spec} A_i$ with each A_i a henselian local ring. Replace $\widetilde{\mathcal{X}}$ with the copy of $\widetilde{\mathcal{X}}_i := \widetilde{\pi}^{-1}(\operatorname{Spec} A_i)$ containing x' and replace $\mathcal{U}'$ with $\widetilde{\mathcal{X}}_i \cap \mathcal{U}'$. Then $\widetilde{\mathcal{X}}$ has a unique closed point (i.e., $x' \in |\mathcal{U}'|$) and thus the complement $\widetilde{\mathcal{X}} \smallsetminus \mathcal{U}'$ is empty (i.e., $\mathcal{U}' = \widetilde{\mathcal{X}}$). We conclude that $\mathcal{U}' \to \mathcal{X}$ is a finite étale morphism, and since it induces an isomorphism of residual gerbes at x', the map has degree one, i.e., $\mathcal{U}' \to \mathcal{X}$ is an isomorphism. Finaly, since Y' is affine, the morphism $\mathcal{X} \cong \mathcal{U}' \to Y'$ factors through a map $X \to Y'$, and thus $f\colon \mathcal{X} \to Y$ factors through the composition $X \to Y' \to Y$. □

6.5.7 Luna's Fundamental Lemma

We will apply the following result in our construction of good moduli spaces (Theorem 6.10.1), in the Local Structure Theorem for Good Moduli Spaces (6.7.3), and in the proof of Luna's Étale Slice Theorem (6.7.5), but it also used in many other arguments as well.

Theorem 6.5.28 (Luna's Fundamental Lemma)**.** *Consider a commutative diagram*

$$\begin{tikzcd}
\mathcal{X}' \arrow[r, "f"] \arrow[d, "\pi'"] & \mathcal{X} \arrow[d, "\pi"] \\
X' \arrow[r, "g"] & X,
\end{tikzcd} \tag{6.5.29}$$

where $f\colon \mathcal{X}' \to \mathcal{X}$ is a separated and representable morphism of noetherian algebraic stacks, each with affine diagonal, and where π and π' are good moduli spaces. Let $x' \in |\mathcal{X}'|$ be a point such that

(a) f is étale at x',

(b) f induces an isomorphism of stabilizer groups at x', and

(c) $x' \in |\mathcal{X}'|$ and $x = f(x') \in |\mathcal{X}|$ are closed points.

Then there is an open neighborhood $U' \subseteq X'$ of $\pi'(x')$ such that $U' \to X$ is étale and such that $U' \times_X \mathcal{X} \cong \pi'^{-1}(U')$.

Remark 6.5.30. This result is really saying two things: first, g is étale at $\pi'(x')$ and second, after replacing X' with an open neighborhood of $\pi'(x')$ the diagram (6.5.29) is cartesian. In the case of quotients by finite groups, this was established in Proposition 4.4.1. Luna's original formulation [Lun73, p. 94] was the case when $\mathcal{X}' \cong [\operatorname{Spec} A'/G]$ and $\mathcal{X} \cong [\operatorname{Spec} A/G]$ with G linearly reductive and where $\mathcal{X}' \to \mathcal{X}$ is induced by a G-equivariant map $\operatorname{Spec} A' \to \operatorname{Spec} A$.

Proof. We will adapt the argument of Theorem 6.5.6(4). Since the question is étale local on X, limit methods (see the proof of Proposition 4.4.1) allow us to assume that $X = \operatorname{Spec} A$ with A a strictly henselian local ring. If $\mathcal{U}' \subseteq \mathcal{X}'$ is the étale locus of f, then $\mathcal{X}' \smallsetminus \pi'^{-1}(\pi'(\mathcal{X}' \smallsetminus \mathcal{U}'))$ contains x' as Theorem 6.5.6(2) implies that $\pi'(x')$ and $\pi'(\mathcal{X}' \smallsetminus \mathcal{U}')$ are disjoint. We can therefore replace $\mathcal{X}'$ with $\mathcal{X}' \smallsetminus \pi'^{-1}(\pi'(\mathcal{X}' \smallsetminus \mathcal{U}'))$ and assume that f is étale. By Zariski's Main Theorem (6.1.18), we may choose a factorization $\mathcal{X}' \to \widetilde{\mathcal{X}} = \mathcal{S}\mathrm{pec}_{\mathcal{X}}\, \mathcal{A} \to \mathcal{X}$ with $\mathcal{X}' \hookrightarrow \widetilde{\mathcal{X}}$ an open immersion and $\widetilde{\mathcal{X}} \to \mathcal{X}$ a finite morphism. Then $\widetilde{\mathcal{X}} \to \widetilde{X} := \operatorname{Spec}_X \pi_* \mathcal{A}$ is a good moduli space and $\widetilde{X} \to X$ is finite. As A is henselian, we can write $\widetilde{X} = \coprod_i \operatorname{Spec} A_i$ with each A_i a henselian local ring. If $U' = \operatorname{Spec} A_i$ denotes the connected component containing the image of x', then $\widetilde{\pi}^{-1}(U') \subseteq \widetilde{\mathcal{X}}$ is an open substack containing a unique closed point, which is necessarily x'; it follows that $\mathcal{X}' = \pi^{-1}(U')$. Since $\mathcal{X}' \to \mathcal{X}$ is a finite étale morphism of degree one (as it preserves residual gerbes at x'), we see that $f\colon \mathcal{X}' \to \mathcal{X}$ is an isomorphism and thus so is $g\colon X' \to X$. □

Corollary 6.5.31. *With the same hypotheses as Theorem 6.5.28, suppose that f is étale and that for all closed points $x' \in |\mathcal{X}'|$*

(a) $f(x') \in |\mathcal{X}|$ is closed, and

(b) f induces an isomorphism of stabilizer groups at x'.

Then $g\colon X' \to X$ is étale and (6.5.29) is cartesian. □

6.5.8 Finite covers of good moduli spaces

Proposition 6.5.32. *Consider a commutative diagram*

$$\begin{tikzcd} \mathcal{X}' \arrow[r, "f"] \arrow[d, "\pi'"] & \mathcal{X} \arrow[d, "\pi"] \\ X' \arrow[r, "g"] & X \end{tikzcd}$$

where $\mathcal{X}$ and $\mathcal{X}'$ are noetherian algebraic stacks with affine diagonal, and π and π' are good moduli spaces. Assume that

(a) $f\colon \mathcal{X}' \to \mathcal{X}$ is representable, quasi-finite, and separated

(b) f maps closed points to closed points, and

(c) g is finite.

Then f is finite.

Proof. By Zariski's Main Theorem (6.1.18), there is a factorization $\mathcal{X}' \to \widetilde{\mathcal{X}} = \mathcal{S}\mathrm{pec}_{\mathcal{X}} \mathcal{A} \to \mathcal{X}$ with $\mathcal{X}' \hookrightarrow \widetilde{\mathcal{X}}$ an open immersion and $\widetilde{\mathcal{X}} \to \mathcal{X}$ a finite morphism. Then $\widetilde{X} = \mathrm{Spec}_X \pi_* \mathcal{A}$ is a finite over X and $\widetilde{\mathcal{X}} \to \widetilde{X}$ is a good moduli space. By replacing $\mathcal{X} \to X$ with $\widetilde{\mathcal{X}} \to \widetilde{X}$, we can assume that f is an open immersion. By replacing $\mathcal{X}$ with the fiber product $X' \times_X \mathcal{X}$, we can further reduce to the case that $X' = X$. For every closed point $x \in X$, let $x' \in |\mathcal{X}'|$ be the unique closed point over x. By (b), $f(x') \in |\mathcal{X}|$ is the unique closed point over x. Since $\mathcal{X}'$ contains all the closed points of $\mathcal{X}$, $f\colon \mathcal{X}' \to \mathcal{X}$ is an isomorphism. $\square$

Proposition 6.5.33. *Suppose that $\mathcal{X}$ is a noetherian algebraic stack with affine diagonal admitting a good moduli space $\pi\colon \mathcal{X} \to X$. If the diagonal $\mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is quasi-finite, then it is finite (i.e., $\pi\colon \mathcal{X} \to X$ is separated).*

Proof. We claim that $\mathcal{X} \times_X \mathcal{X} \to X$ is a good moduli space. By Lemma 6.5.17, the projection $p_1\colon \mathcal{X} \times_X \mathcal{X} \to \mathcal{X}$ is cohomologically affine and therefore so is the composition $\mathcal{X} \times_X \mathcal{X} \xrightarrow{p_1} \mathcal{X} \xrightarrow{\pi} X$. On the other hand, if $U \to \mathcal{X}$ is a smooth presentation, then $p_1\colon U \times_X \mathcal{X} \to U$ is a good moduli space (Lemma 6.5.22) and in particular $\mathcal{O}_U \xrightarrow{\sim} p_{1,*}\mathcal{O}_{U \times_X \mathcal{X}}$. It follows from descent that $\mathcal{O}_{\mathcal{X}} \xrightarrow{\sim} p_{1,*}\mathcal{O}_{\mathcal{X} \times_X \mathcal{X}}$ and thus $\mathcal{O}_X \xrightarrow{\sim} (\pi \circ p_1)_* \mathcal{O}_{\mathcal{X} \times_X \mathcal{X}}$. This establishes the claim. The diagonal $\mathcal{X} \to \mathcal{X} \times_X \mathcal{X}$ is a representable, quasi-finite, and separated morphism that sends closed points to closed points and induces an isomorphism on good moduli spaces. Proposition 6.5.32 implies that $\mathcal{X} \to \mathcal{X} \times_X \mathcal{X}$ is finite. Note that since $\mathcal{X}$ has affine diagonal, the finiteness of the diagonal is equivalent to its properness. $\square$

6.5.9 Descending vector bundles

Proposition 6.5.34. *Let $\mathcal{X}$ be a noetherian algebraic stack and $\pi\colon \mathcal{X} \to X$ be a good moduli space. A vector bundle F on $\mathcal{X}$ descends to a vector bundle on X if and only if for every field-valued point $x\colon \mathrm{Spec}\, \Bbbk \to \mathcal{X}$ with closed image, the action of G_x on the fiber $F \otimes \Bbbk$ is trivial. In this case, $\pi_* F$ is a vector bundle and the adjunction map $\pi^*\pi_* F \to F$ is an isomorphism.*

Proof. We follow the argument in the case of a tame coarse moduli space (Proposition 4.4.27). The condition is clearly necessary. To see that the condition is sufficient, consider the commutative diagram

$$\begin{tikzcd} \mathcal{G}_x \arrow[r, hook] \arrow[d, "p"'] & \mathcal{X} \arrow[d, "\pi"] \\ \mathrm{Spec}\, \kappa(x) \arrow[r, hook] & X. \end{tikzcd}$$

We first claim that $\pi^*\pi_* F \to F$ is surjective. For every closed point $x \in |\mathcal{X}|$, the hypotheses imply that $p^* p_* (F|_{\mathcal{G}_x}) \cong F|_{\mathcal{G}_x}$. Applying $\pi^*\pi_*(-)|_{\mathcal{G}_x}$ to the surjection $F \to F|_{\mathcal{G}_x}$ and using the exactness of π_*, we obtain that $(\pi^*\pi_* F)|_{\mathcal{G}_x} \to \pi^*(\pi_*(F|_{\mathcal{G}_x}))|_{\mathcal{G}_x} \cong p^*p_*(F|_{\mathcal{G}_x}) \cong F|_{\mathcal{G}_x}$ is surjective. The claim now follows from Lemma 6.5.35(2).

To show that $\pi_* F$ is a vector bundle, we may assume that $X = \mathrm{Spec}\, A$ is affine and that the rank r of F is constant. The surjection $\bigoplus_{s \in \Gamma(X, \pi_* F)} A \to \pi_* F$ pulls back to a surjection $\bigoplus_{s \in \Gamma(\mathcal{X}, F)} \mathcal{O}_{\mathcal{X}} \to \pi^*\pi_* F$, and by the above claim, the composition $\bigoplus_{s \in \Gamma(\mathcal{X}, F)} \mathcal{O}_{\mathcal{X}} \to \pi^*\pi_* F \to F$ is surjective. As $F|_{\mathcal{G}_x} \cong \mathcal{O}_{\mathcal{G}_x}^r$ is trivial, for each closed point $x \in |\mathcal{X}|$, we can find r sections $\phi\colon \mathcal{O}_{\mathcal{X}}^{\oplus r} \to F$ such that

$\phi|_{\mathcal{G}_x}$ is an isomorphism. By Lemma 6.5.35(2), there exists an open neighborhood $U \subseteq X$ of $\pi(x)$ such that $\phi|_{\pi^{-1}(U)}$ is an isomorphism. Thus $\pi_*\phi\colon \mathcal{O}_X^{\oplus r} \to \pi_* F$ is an isomorphism over U and we conclude that $\pi_* F$ is a vector bundle of the same rank as F. Finally, since $\pi^*\pi_* F \to F$ is a surjection of vector bundles of the same rank, it is an isomorphism. See also [Alp13, Thm. 10.3] and [Ryd20, Thm. B]. The case of a good quotient is due to Kempf; see [KKV89, Prop. 4.2]. □

Lemma 6.5.35. *Let $\mathcal{X}$ be a noetherian algebraic stack and $\pi\colon \mathcal{X} \to X$ be a good moduli space. Let $x \in |\mathcal{X}|$ be a closed point.*

1. *If F is a coherent sheaf on $\mathcal{X}$ such that $F|_{\mathcal{G}_x} = 0$, then there exists an open neighborhood $U \subseteq X$ of $\pi(x)$ such that $F|_{\pi^{-1}(U)} = 0$.*
2. *If $\phi\colon F \to G$ is a morphism of coherent sheaves (resp., vector bundles of the same rank) on $\mathcal{X}$ such that $\phi|_{\mathcal{G}_x}$ is surjective, then there exists an open neighborhood $U \subseteq X$ of $\pi(x)$ such that $\phi|_{\pi^{-1}(U)}$ is surjective (resp., an isomorphism).*

Proof. The same (elementary) argument of Lemma 4.4.26 applies. □

6.5.10 Finiteness of cohomology

Proposition 6.5.36. *Let $\mathcal{X}$ be an algebraic stack proper over a noetherian affine scheme $\operatorname{Spec} A$ and let $\pi\colon \mathcal{X} \to X$ be a good moduli space. There exists an integer N such that for every coherent sheaf F on $\mathcal{X}$, $\mathrm{H}^i(\mathcal{X}, F) = 0$ for $i > N$ and such that $\mathrm{H}^i(\mathcal{X}, F)$ is a finitely generated A-module for every i.*

Proof. Since π_* is exact, the Grothendieck spectral sequence implies that $\mathrm{H}^i(\mathcal{X}, F) = \mathrm{H}^i(X, \pi_* F)$. Since $\pi_* F$ is coherent (Theorem 6.5.6(3)), the statement follows from the finiteness of cohomology on X (Theorem 4.6.8) and Exercise 4.6.10. □

6.5.11 Singularities of good moduli spaces

If $\mathcal{X} \to X$ is a tame coarse moduli space of a separated noetherian algebraic stack and $\mathcal{X}$ is smooth, then X has tame quotient singularities (see Remark 4.4.25). Here we consider analogous properties of good moduli spaces, focusing first on the case of an action of a linearly reductive group G over a field $\Bbbk$ on a noetherian affine $\Bbbk$-scheme $X = \operatorname{Spec} A$. First, it is a classical theorem of Hochster and Roberts that if A is regular, then A^G is Cohen–Macaulay [HR74]. Interestingly, it is not true that if A is Cohen–Macaulay, then so is A^G [HR74, Ex. 2.1]. It is a theorem of Boutot that, assuming X is of finite type over $\Bbbk$ and $\operatorname{char}(\Bbbk) = 0$, then if X has rational singularities, so does $X/\!\!/G = \operatorname{Spec} A^G$ [Bou87]. Since smooth varieties are rational and rational varieties are Cohen–Macaulay, Boutot's theorem implies the Hochster–Robert's theorem in characteristic zero.

These results also hold for a good moduli space $\mathcal{X} \to X$ of an algebraic stack $\mathcal{X}$ of finite type over a field $\Bbbk$ of characteristic zero and with affine diagonal. This is a consequence of the Local Structure for Good Moduli Spaces (6.7.3). We also mention the recent generalization: if $\mathcal{X}$ has klt singularities, then so does X [BGLM24, Thm. 5]. To summarize these implications:

$$\begin{aligned} \mathcal{X} \text{ smooth} &\implies X \text{ Cohen–Macaulay} \\ \mathcal{X} \text{ rational} &\implies X \text{ rational} \\ \mathcal{X} \text{ klt} &\implies X \text{ klt.} \end{aligned}$$

6.6 Coherent Tannaka duality and coherent completeness

Perhaps the purpose of categorical algebra is to show that which is trivial is trivially trivial.

Peter Freyd

We prove a version of Tannaka duality for noetherian algebraic stacks with affine diagonal (Theorem 6.6.1). After introducing the notion of coherent completeness of an algebraic stack $\mathcal{X}$ (Definition 6.6.5), we show that certain quotient stacks with a unique closed point are coherently complete (Theorem 6.6.13). This includes the important examples of Θ_R and ϕ_R defined in §6.9.2 for a complete DVR R.

The combined power of Coherent Tannaka Duality and coherent completeness allows us to extend compatible maps $\mathcal{X}_n \to \mathcal{Y}$ from the nth nilpotent thickenings of $\mathcal{X}_0$ to a morphism $\mathcal{X} \to \mathcal{Y}$ (Corollary 6.6.9). This technique is used in an essential way in the proof of the Local Structure Theorem for Algebraic Stacks (6.7.1), and also appears in many other arguments.

6.6.1 Coherent Tannaka Duality

A classical theorem of Gabriel [Gab62] states that two noetherian schemes X and Y are isomorphic if and only if their abstract categories $\mathrm{Coh}(X)$ and $\mathrm{Coh}(Y)$ of coherent sheaves are equivalent, or in other words that a scheme X can be recovered from the category $\mathrm{Coh}(X)$. In representation theory, classical Tannaka duality by Saavedra Rivano [SR72] (see also Deligne and Milne's article [DMOS82, Ch. II]) asserts that an affine group scheme G over a field $\Bbbk$ can be recovered from the tensor category $\mathrm{Rep}^{\mathrm{fd}}(G)$ of finite dimensional representations and its forgetful functor $\mathrm{Rep}^{\mathrm{fd}}(G) \to \mathrm{Vect}_{\Bbbk}$.

Combining these two facts, one might hope that an algebraic stack $\mathcal{X}$ is recovered by the tensor category $\mathrm{Coh}(\mathcal{X})$.[3] Following a brilliant observation of Lurie [Lur04], we will not only confirm this expectation, but we will show that a tensor functor $\mathrm{Coh}(\mathcal{Y}) \to \mathrm{Coh}(\mathcal{X})$ is enough to recover a morphism $\mathcal{X} \to \mathcal{Y}$ of algebraic stacks.

Theorem 6.6.1 (Coherent Tannaka Duality)**.** *For noetherian algebraic stacks $\mathcal{X}$ and $\mathcal{Y}$ with affine diagonal, the functor*

$$\mathrm{Mor}(\mathcal{X},\mathcal{Y}) \to \mathrm{Mor}^{\otimes}(\mathrm{Coh}(\mathcal{Y}),\mathrm{Coh}(\mathcal{X})), \qquad f \mapsto f^* \tag{6.6.2}$$

is an equivalence of categories, where $\mathrm{Mor}^{\otimes}(\mathrm{Coh}(\mathcal{Y}),\mathrm{Coh}(\mathcal{X}))$ denotes the category of right exact additive tensor functors $\mathrm{Coh}(\mathcal{Y}) \to \mathrm{Coh}(\mathcal{X})$ of symmetric monoidal abelian categories where morphisms are tensor natural transformations.

Remark 6.6.3. A *symmetric monoidal category* is a category $\mathcal{A}$ endowed with a bifunctor $\otimes : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ and a unit $1 \in \mathcal{A}$ together with associativity isomorphisms $\alpha_{A,B,C} : A \otimes (B \otimes C) \xrightarrow{\sim} (A \otimes B) \otimes C$, left and right unit isomorphisms $l_A : 1 \otimes A \xrightarrow{\sim} A \xrightarrow{\sim} A$ and $r_A : A \otimes 1 \xrightarrow{\sim} A$, and commutativity isomorphisms $s_{A,B} : A \otimes B \cong B \otimes A$ (with $s_{A,B} \circ s_{B,A} = \mathrm{id}$) satisfying certain coherence conditions [Mac71, §XI.1]. A *tensor functor* $F : \mathcal{A} \to \mathcal{B}$ between symmetric monoidal abelian categories is a functor equipped with isomorphisms $\Phi_{A,B} : F(A) \otimes F(B) \xrightarrow{\sim} F(A \otimes B)$ and $\varphi : 1_{\mathcal{B}} \xrightarrow{\sim} F(1_{\mathcal{A}})$ compatible with the isomorphisms $\alpha_{A,B,C}$, l_A, r_A and $s_{A,B}$ [Mac71, §XI.2]. A

[3] The structure as an abelian category is not enough, e.g., $\mathrm{Coh}(B(\mathbb{Z}/2)) \cong \mathrm{Coh}(\operatorname{Spec} \Bbbk \coprod \operatorname{Spec} \Bbbk)$.

tensor natural transformation between tensor functors is a natural transformation of functors compatible with the isomorphisms $\Phi_{A,B}$ and φ [Mac71, §XI.2]. A *symmetric monoidal abelian category* (resp., *symmetric monoidal R-linear abelian category* for a ring R) is a symmetric monoidal (resp., R-linear) abelian category $\mathcal{A}$ such that $\otimes : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ is additive (resp., R-linear) in each variable. A tensor functor is *additive* or *R-linear* if the underlying functor is. When $\mathcal{X}$ and $\mathcal{Y}$ are defined over a noetherian ring R, then Theorem 6.6.1 induces an equivalence

$$\mathrm{Mor}_R(\mathcal{X}, \mathcal{Y}) \xrightarrow{\sim} \mathrm{Mor}_R^{\otimes}(\mathrm{Coh}(\mathcal{Y}), \mathrm{Coh}(\mathcal{X}))$$

between morphisms over R and right exact R-linear tensor functors.

Proof. Since every quasi-coherent sheaf on a noetherian algebraic stack is a colimit of its coherent subsheaves (Proposition 6.1.16), every right exact tensor functor $F\colon \mathrm{Coh}(\mathcal{Y}) \to \mathrm{Coh}(\mathcal{X})$ extends to a tensor functor $F\colon \mathrm{QCoh}(\mathcal{Y}) \to \mathrm{QCoh}(\mathcal{X})$ preserving colimits. Likewise every tensor natural transformation between functors of coherent sheaves extends uniquely to one defined on quasi-coherent sheaves.

Fully faithfulness: Let $f, g\colon \mathcal{X} \to \mathcal{Y}$. Choose a smooth presentation $p\colon U \to \mathcal{Y}$ where U is an affine scheme. Since the question is smooth local on $\mathcal{X}$, after replacing $\mathcal{X}$ with $\mathcal{X} \times_{f,\mathcal{Y},p} U$, we may assume there is a factorization $f\colon \mathcal{X} \xrightarrow{\widetilde{f}} U \xrightarrow{p} \mathcal{Y}$. Likewise, we may assume there is a factorization $g\colon \mathcal{X} \xrightarrow{\widetilde{g}} V \xrightarrow{q} \mathcal{Y}$ where V is an affine scheme. Since $\mathcal{Y}$ has affine diagonal, $p\colon U \to \mathcal{Y}$ is affine and we have identifications

$$\mathrm{Mor}_{\mathcal{Y}}(\mathcal{X}, U) \cong \mathrm{Hom}_{\mathcal{O}_{\mathcal{Y}}-\mathrm{alg}}(p_*\mathcal{O}_U, f_*\mathcal{O}_{\mathcal{X}}) \cong \mathrm{Hom}_{\mathcal{O}_{\mathcal{X}}-\mathrm{alg}}(f^*p_*\mathcal{O}_U, \mathcal{O}_{\mathcal{X}})$$

Therefore $\widetilde{f}$ and $\widetilde{g}$ correspond to sections $s_{\widetilde{f}}\colon f^*p_*\mathcal{O}_U \to \mathcal{O}_{\mathcal{X}}$ and $s_{\widetilde{g}}\colon g^*q_*\mathcal{O}_V \to \mathcal{O}_{\mathcal{X}}$. A 2-isomorphism $\alpha\colon f \to g$ is identified with a factorization

$$\begin{tikzcd}
\mathcal{X} \arrow[r, "(\widetilde{f},\widetilde{g},\alpha)"] \arrow[dr, "\mathrm{id}"'] & U \times_{\mathcal{X}} V \arrow[d, "\pi"] \\
& \mathcal{X},
\end{tikzcd}$$

which is the same data as a section s_α of $\mathcal{O}_{\mathcal{X}} \to f^*\pi_*\mathcal{O}_{U\times_{\mathcal{X}}V}$. Letting $\alpha^*\colon f^* \to g^*$ be the image of α under (6.6.2), i.e., the pullback tensor natural transformation, the section s_α can be written as

$$f^*\pi_*\mathcal{O}_{U\times_{\mathcal{X}}V} \cong f^*(p_*\mathcal{O}_U) \otimes f^*(q_*\mathcal{O}_V) \xrightarrow{\mathrm{id}\otimes\alpha^*_{q_*\mathcal{O}_V}} f^*(p_*\mathcal{O}_U) \otimes g^*(q_*\mathcal{O}_V) \xrightarrow{s_{\widetilde{f}}\otimes s_{\widetilde{g}}} \mathcal{O}_S.$$

To see the faithfulness of (6.6.2), if $\alpha, \alpha'\colon f \to g$ are 2-isomorphisms with $\alpha^* = \alpha'^*$, then $\alpha^*_{q_*\mathcal{O}_V} = \alpha'^*_{q_*\mathcal{O}_V}$. Therefore, the two sections s_α and $s_{\alpha'}$ are equal and $\alpha = \alpha'$. For the fullness of (6.6.2), let $\beta\colon f^* \to g^*$ be a tensor natural transformation. Then $\mathrm{id}\otimes\beta_{q_*\mathcal{O}_V}$ defines a section $f^*\pi_*\mathcal{O}_{U\times_{\mathcal{X}}V} \to \mathcal{O}_S$ and thus a 2-isomorphism $\alpha\colon f \to g$ such that $\beta_{q_*\mathcal{O}_V} = \alpha^*_{q_*\mathcal{O}_V}$. To see that $\beta_E = \alpha^*_E$ for every $E \in \mathrm{QCoh}(\mathcal{Y})$, note that the factorization $g = q \circ \widetilde{g}$ yields a splitting of $g^*E \to g^*(q_*q^*E)$. Since f^* and g^* commute with direct sums, it suffices to assume that $E = q_*G$ for $G \in \mathrm{QCoh}(V)$. Write $G = \mathrm{colim}(\mathcal{O}_V^{\oplus I} \to \mathcal{O}_V^{\oplus J})$ as a colimit of free $\mathcal{O}_V$-modules. Since f^* and g^* commute with colimits and q_* is exact, we conclude that $\beta_{q_*G} = \alpha^*_{q_*G}$.

Essential surjectivity (affine case): Let $F\colon \mathrm{QCoh}(\mathcal{Y}) \to \mathrm{QCoh}(\mathcal{X})$ be a tensor functor preserving colimits. Assuming that $\mathcal{X} = \operatorname{Spec} A$ and $\mathcal{Y} = \operatorname{Spec} B$ are noetherian affine schemes, there is a map

$$\phi\colon B \cong \mathrm{End}(\mathcal{O}_{\mathcal{Y}}) \xrightarrow{F} \mathrm{End}(\mathcal{O}_{\mathcal{X}}) \cong A.$$

We claim that ϕ is a ring homomorphism and that there is a functorial isomorphism $F(N) = N \otimes_B A$ for $N \in \mathrm{Mod}_B$. For $b, b' \in B$, consider the commutative diagrams

$$\begin{tikzcd}
\mathcal{O}_{\mathcal{Y}} \otimes \mathcal{O}_{\mathcal{Y}} \arrow[r] \arrow[d, "b\otimes b'"] & \mathcal{O}_{\mathcal{Y}} \arrow[d, "bb'"] & & A \otimes A \arrow[r] \arrow[d, "\phi(b)\otimes\phi(b')"] & A \arrow[d, "\phi(bb')"] \\
\mathcal{O}_{\mathcal{Y}} \otimes \mathcal{O}_{\mathcal{Y}} \arrow[r] & \mathcal{O}_{\mathcal{Y}} \arrow[r, phantom, "\overset{F}{\longmapsto}"] & & A \otimes A \arrow[r] & A,
\end{tikzcd}$$

where the horizontal maps correspond to multiplication. The commutativity of the right square is implied by the fact that F preserves tensor products. This shows that $\phi(b)\phi(b') = \phi(bb')$. For a B-module N, choose a free presentation $B^{\oplus J} \to B^{\oplus I} \to N \to 0$. Since both F and $- \otimes_B A$ are right exact and preserve direct sums, applying them to the free presentation yields an identification $F(N) \cong N \otimes_B A$ as both are cokernels of $A^{\oplus J} \to A^{\oplus I}$. One checks similarly that this identification is functorial.

Reduction to the case that $\mathcal{X}$ is affine: Choose a smooth presentation $g\colon U \to \mathcal{X}$ from an affine scheme and consider the diagram

$$\begin{tikzcd}
U \times_X U \arrow[d, shift right, "p_2"'] \arrow[d, shift left, "p_1"] & \\
U \arrow[d, "g"] \arrow[dr, dotted, "g^* \circ F"] & \\
\mathcal{X} \arrow[r, dotted, "F"] & \mathcal{Y}
\end{tikzcd}$$

where the dashed arrow $\mathcal{X} \dashrightarrow \mathcal{Y}$ is denoting the tensor functor $F\colon \mathrm{QCoh}(\mathcal{Y}) \to \mathrm{QCoh}(\mathcal{X})$ in the other direction. Assuming that the result holds when U is affine, there is a morphism $h\colon U \to \mathcal{Y}$ and an isomorphism $h \xrightarrow{\sim} g^* \circ F$ of functors. By full faithfulness, there is an isomorphism $p_1 \circ h \xrightarrow{\sim} p_2 \circ h$ satisfying the cocycle condition, and thus smooth descent implies that there is a unique morphism $f\colon \mathcal{X} \to \mathcal{Y}$ with $F \simeq f^*$.

Reduction to the case that $\mathcal{Y}$ is affine: Let $\mathcal{X} = \operatorname{Spec} A$ and choose a smooth presentation $q\colon V = \operatorname{Spec} C \to \mathcal{Y}$. Since $\mathcal{Y}$ has affine diagonal, q is an affine morphism. Define $B := F(q_*\mathcal{O}_V)$ which is an A-algebra since $q_*\mathcal{O}_V$ is an $\mathcal{O}_{\mathcal{Y}}$-algebra. Consider the diagram

$$\begin{tikzcd}
\operatorname{Spec} B \arrow[r, dotted, "F'"] \arrow[d] & V = \operatorname{Spec} C \arrow[d, "q"] \\
\operatorname{Spec} A = \mathcal{X} \arrow[r, dotted, "F"] & \mathcal{Y}
\end{tikzcd}$$

where $F'\colon \mathrm{Mod}_C \to \mathrm{Mod}_B$ is the right exact tensor functor sending M to $F(q_*\widetilde{M})$ (which is a module over $B = F(q_*\mathcal{O}_V)$ because $q_*\widetilde{M}$ is a $q_*\mathcal{O}_V$-module). By the affine case, F' is induced by a morphism $f'\colon \operatorname{Spec} B \to \operatorname{Spec} C$. We can extend the above diagram to:

$$\begin{tikzcd}
\operatorname{Spec} B \otimes_A B \arrow[r, "f''"] \arrow[d, shift left] \arrow[d, shift right] & V \times_{\mathcal{Y}} V \arrow[d, shift left] \arrow[d, shift right] \\
\operatorname{Spec} B \arrow[r, "f'"] \arrow[d] & V = \operatorname{Spec} C \arrow[d, "q"] \\
\operatorname{Spec} A = \mathcal{X} \arrow[r, dotted, "F"] & \mathcal{Y}.
\end{tikzcd}$$

Since q is affine, $V \times_{\mathcal{Y}} V$ is affine and the top square (under either set of projections) is cartesian.

If we could show that $A \to B$ is faithfully flat, we would be done as the full faithfulness in the affine case would imply that f' descends to our desired morphism $f\colon \mathcal{X} \to \mathcal{Y}$. This seems hard to directly check, but we do know already that the maps $B \rightrightarrows B \otimes_A B$ are faithfully flat as they correspond to base changes of the smooth maps $V \times_{\mathcal{Y}} V \rightrightarrows V$. We will show instead that $A \to B$ is universally injective. Since faithful flatness descends under universal injectivity maps (Proposition A.2.24(4)), the faithful flatness of $A \to B$ follows from the universal injectivity.

Universal injectivity of $A \to B$: Recall from Definition A.2.22 that an injective map of A-modules is called universally injective if it remains injective after tensoring by every A-module. By Proposition A.2.24(3), this notion is local under faithfully flat morphisms and thus extends to morphisms $F \to G$ of quasi-coherent sheaves on an algebraic stack. Since $q\colon V \to \mathcal{Y}$ is faithfully flat, $\mathcal{O}_{\mathcal{Y}} \to q_*\mathcal{O}_V$ is universally injective (Proposition A.2.24(1)). We write $q_*\mathcal{O}_V = \operatorname{colim} Q_i$ as a colimit of coherent subsheaves (Proposition 6.1.16), and we may assume that each Q_i contains the image of $\mathcal{O}_{\mathcal{Y}} \to q_*\mathcal{O}_V$. Then $\mathcal{O}_{\mathcal{Y}} \to Q_i$ is also universally injective. Since Q_i is coherent, $\mathcal{O}_{\mathcal{Y}} \to Q_i$ is a split injection smooth locally on $\mathcal{Y}$ (Proposition A.2.24(2)). Applying F to $\mathcal{O}_{\mathcal{Y}} \to q_*\mathcal{O}_V = \operatorname{colim} Q_i$ and using that F preserves colimits, we have a factorization

$$\begin{tikzcd}
 & F(Q_i) \arrow[d] \\
A = F(\mathcal{O}_{\mathcal{Y}}) \arrow[ur] \arrow[r] & B = F(q_*\mathcal{O}_V) = \operatorname{colim} F(Q_i).
\end{tikzcd}$$

It suffices to show that $A \to F(Q_i)$ is universally injective. We will show in fact that it is a split injection. As $\mathcal{O}_{\mathcal{Y}} \to Q_i$ is smooth locally split, the map on duals $Q_i^\vee \to \mathcal{O}_{\mathcal{Y}}^\vee = \mathcal{O}_{\mathcal{Y}}$ is surjective. Applying F, we have a surjection $F(Q_i^\vee) \to F(\mathcal{O}_{\mathcal{Y}}) = A$ (using right exactness) and we can choose an element $\lambda \in F(Q_i^\vee)$ mapping to 1. Under the natural map $F(Q_i^\vee) \to F(Q_i)^\vee$, the element λ is sent to a map $F(Q_i) \to A$, which is a section of the given map $A \to F(Q_i)$. See also [Lur04], [HR19b], [BHL17] and [SP, Tag 0GRR]. □

Remark 6.6.4 (Relation to classical Tannaka duality). If G is an affine group scheme over a field $\Bbbk$, then the category $\mathcal{C} = \operatorname{Rep}^{\mathrm{fd}}(G)$ of finite dimensional representations is a symmetric monoidal $\Bbbk$-linear category and there is a tensor functor $\omega\colon \operatorname{Rep}^{\mathrm{fd}}(G) \to \operatorname{Vect}_{\Bbbk}$. For $\Bbbk$-algebra R, define ω_R as the composition

$$\omega_R\colon \operatorname{Rep}^{\mathrm{fd}}(G) \to \operatorname{Vect}_{\Bbbk} \xrightarrow{-\otimes_{\Bbbk} R} \operatorname{Mod}_R,$$

and let $\mathrm{Aut}^{\otimes}(\omega_R)$ denote the group of tensor natural isomorphisms of ω_R. Then G is recovered as the functor $\underline{\mathrm{Aut}}^{\otimes}(\omega)$ on affine $\Bbbk$-schemes assigning R to $\mathrm{Aut}^{\otimes}(\omega_R)$ [DMOS82, II.2.8]. On the other hand, Coherent Tannaka Duality (6.6.1) asserts that for every noetherian $\Bbbk$-algebra R, there is an equivalence of categories

$$\mathrm{Mor}_{\Bbbk}(\operatorname{Spec} R, BG) \xrightarrow{\sim} \mathrm{Mor}^{\otimes}(\mathrm{Rep}(G)^{\mathrm{fd}}, \mathrm{Mod}_R).$$

In this way, we see that $\mathrm{Rep}(G)^{\mathrm{fd}}$ determines BG. To recover G, the fiber functor $\omega\colon \mathrm{Rep}^{\mathrm{fd}}(G) \to \mathrm{Vect}_{\Bbbk}$ corresponds to a morphism $p\colon \operatorname{Spec} \Bbbk \to BG$ and $G = \underline{\mathrm{Aut}}_{\Bbbk}(p)$. For instance, consider orthogonal groups $O(q)$ and $O(q')$ with respect to non-degenerate quadratic forms q and q' of the same dimension. Then $\mathrm{Rep}(O(q)) \cong \mathrm{Rep}(O(q'))$ even though $O(q)$ and $O(q')$ may not be isomorphic; in this case the two maps $\operatorname{Spec} \Bbbk \to BO(q)$ and $\operatorname{Spec} \Bbbk \to BO(q')$ define two different fiber functors on the same category.

The classical version also provides conditions when the data of $(\mathcal{C}, \omega)$ is isomorphic to the category of representations of a group scheme. Namely, $\mathcal{C}$ is called *rigid* if for every object of $X \in \mathcal{C}$, there is a *dual* $X^{\vee} \in \mathcal{C}$, i.e., an object $X^{\vee}$ such that $X^{\vee} \otimes - \colon \mathcal{C} \to \mathcal{C}$ is right adjoint to $X \otimes - \colon \mathcal{C} \to \mathcal{C}$. If $\mathcal{C}$ is a rigid symmetric monoidal $\Bbbk$-linear abelian category with $\mathrm{End}(1) = \Bbbk$ and $\omega\colon \mathcal{C} \to \mathrm{Vect}_{\Bbbk}$ is an exact faithful $\Bbbk$-linear tensor functor, then $\underline{\mathrm{Aut}}^{\otimes}(\omega)$ is representable by an affine group scheme G over $\Bbbk$ and there is a tensor equivalence $\mathcal{C} \cong \mathrm{Rep}^{\mathrm{fd}}(G)$ under which ω corresponds to the forgetful functor [DMOS82, II.2.11]. Moreover, G is of finite type over $\Bbbk$ if and only if $\mathcal{C}$ has a tensor generator.

6.6.2 Coherent completeness

The combination of Coherent Tannaka Duality and coherent completeness (Corollary 6.6.9) will become a valuable new tool at our disposal.

Definition 6.6.5. A noetherian algebraic stack $\mathcal{X}$ is *coherently complete along a closed substack* $\mathcal{X}_0$ if the natural functor

$$\mathrm{Coh}(\mathcal{X}) \to \varprojlim \mathrm{Coh}(\mathcal{X}_n), \quad F \mapsto (F_n)$$

is an equivalence of categories, where $\mathcal{X}_n$ denotes the nth nilpotent thickening of $\mathcal{X}_0$ and F_n is the pullback of F to $\mathcal{X}_n$.

Remark 6.6.6. If $\mathcal{I} \subseteq \mathcal{O}_{\mathcal{X}}$ is the coherent sheaf of ideals defining $\mathcal{X}_0$, then $\mathcal{X}_n$ is defined by $\mathcal{I}^{n+1}$. Letting $i_n\colon \mathcal{X}_n \hookrightarrow \mathcal{X}_{n+1}$ denote the natural inclusion, an object in $\varprojlim \mathrm{Coh}(\mathcal{X}_n)$ corresponds to a sequence $F_n \in \mathrm{Coh}(\mathcal{X}_n)$ of coherent sheaves together with maps $\alpha_n\colon i_{n,*}F_n \to F_{n+1}$ inducing isomorphism $F_n \to i_n^* F_{n+1}$. A morphism $(F_n, \alpha_n) \to (F'_n, \alpha'_n)$ is a sequence of maps $\phi_n\colon F_n \to F'_n$ such that $\phi_{n+1} \circ \alpha_n = \alpha'_{n+1} \circ i_{n,*}\phi_n$.

Example 6.6.7. If $(R, \mathfrak{m})$ is a complete noetherian local ring, then the Artin–Rees Lemma (B.5.4) implies that $\operatorname{Spec} R$ is coherently complete along $\operatorname{Spec} R/\mathfrak{m}$. The same is true if $R = \lim R/I^n$ is a noetherian I-adically complete ring.

Example 6.6.8. Grothendieck's Existence Theorem (C.5.3) asserts that if X is a proper scheme over a complete local ring $(R, \mathfrak{m})$ and $X_0 = X \times_R R/\mathfrak{m}$, then X is coherently complete along X_0. If $\mathcal{X}$ is a proper Deligne–Mumford stack over $\operatorname{Spec} R$, the same is true (Exercise 4.6.13(b)) The result also holds if $\mathcal{X}$ is a proper *algebraic* stack over any I-adically complete noetherian ring. See [Ols05, Thm. 1.4] or [Con05a, Thm. 4.1].

Corollary 6.6.9 (Coherent Tannaka Duality)**.** *Let $\mathcal{X}$ and $\mathcal{Y}$ be noetherian algebraic stacks with affine diagonal. Suppose that $\mathcal{X}$ is coherently complete along $\mathcal{X}_0$. Then there is an equivalence of categories*

$$\mathrm{Mor}(\mathcal{X}, \mathcal{Y}) \to \varprojlim \mathrm{Mor}(\mathcal{X}_n, \mathcal{Y}), \quad f \mapsto (f_n),$$

where $f_n \colon \mathcal{X}_n \to Y$ denotes the restriction of f to the nth nilpotent thickening $\mathcal{X}_n$ of $\mathcal{X}_0$.

Proof. This follows from the equivalences

$$\begin{aligned}
\mathrm{Mor}(\mathcal{X}, \mathcal{Y}) &\simeq \mathrm{Mor}^{\otimes}\big(\mathrm{Coh}(\mathcal{Y}), \mathrm{Coh}(\mathcal{X})\big) && \text{(Coherent Tannaka Duality)} \\
&\simeq \mathrm{Mor}^{\otimes}\big(\mathrm{Coh}(\mathcal{Y}), \varprojlim \mathrm{Coh}(\mathcal{X}_n)\big) && \text{(coherent completeness)} \\
&\simeq \varprojlim \mathrm{Mor}^{\otimes}\big(\mathrm{Coh}(\mathcal{Y}), \mathrm{Coh}(\mathcal{X}_n)\big) && \\
&\simeq \varprojlim \mathrm{Mor}\big(\mathcal{X}_n, \mathcal{Y}\big) && \text{(Coherent Tannaka Duality).}
\end{aligned}$$

□

Remark 6.6.10. If $\mathcal{X}$ and $\mathcal{Y}$ are defined over a noetherian ring R, then there is an equivalence $\mathrm{Mor}_R(\mathcal{X}, \mathcal{Y}) \to \varprojlim \mathrm{Mor}_R(\mathcal{X}_n \mathcal{Y})$. This follows in the same way using the Tannaka duality equivalence between the category of morphisms $\mathcal{X} \to \mathcal{Y}$ over R and the category of right exact R-linear tensor functors (Remark 6.6.3).

Exercise 6.6.11. Give a direct proof $\mathrm{Mor}(\operatorname{Spec} R, \mathcal{Y}) \to \varprojlim \mathrm{Mor}(\operatorname{Spec} R/\mathfrak{m}^n, \mathcal{Y})$ is an equivalence when $(R, \mathfrak{m})$ is a complete noetherian local ring.

Hint: Use Proposition 4.3.4 to lift a map $\operatorname{Spec} R/\mathfrak{m} \to \mathcal{Y}$ *to a smooth presentation and the Infinitesimal Lifting Criterion (3.7.1) to reduce to the case that $\mathcal{Y}$ is a scheme.*

Exercise 6.6.12. Let G be an affine algebraic group acting on a separated noetherian algebraic space W over $\Bbbk$. Let $W_0 \subseteq W$ be a G-invariant closed subspace and let W_n be its nth nilpotent thickenings. Suppose that $[W/G]$ is coherently complete along $[W_0/G]$. For every noetherian algebraic space X over $\Bbbk$ with affine diagonal equipped with an action of G, the natural map on equivariant maps

$$\mathrm{Mor}^G(W, X) \to \varprojlim_n \mathrm{Mor}^G\big(W_n, X\big)$$

is bijective.

Hint: reduce to Corollary 6.6.9 by using that a G-equivariant map $W \to X$ corresponds to a morphism $[W/G] \to [X/G]$ over BG.

6.6.3 Coherent completeness of quotient stacks

The coherent completeness result that we will apply, e..g, in the proof of the Local Structure Theorem for Algebraic Stacks (6.7.1), is the following:

Theorem 6.6.13. *Let $\Bbbk$ be an algebraically closed field and R be a complete noetherian local $\Bbbk$-algebra with residue field $\Bbbk$. Let G be a linearly reductive group over $\Bbbk$ acting on an affine scheme* $\operatorname{Spec} A$ *of finite type over R. Suppose that $A^G = R$ and that there is a G-fixed $\Bbbk$-point $x \in \operatorname{Spec} A$. Then* $[\operatorname{Spec} A/G]$ *is coherently complete along the closed substack BG defined by x.*

Example 6.6.14. If $\mathbb{G}_m$ acts diagonally on $\mathbb{A}^r$, then $[\mathbb{A}^r/\mathbb{G}_m]$ is coherently complete along the origin $B\mathbb{G}_m$. In other words a $\mathbb{G}_m$-equivariant module over $\Bbbk[x_1, \ldots, x_r]$ is equivalent to a compatible family of $\mathbb{G}_m$-equivariant modules over $\Bbbk[x_1, \ldots, x_r]/(x_1, \ldots, x_r)^{n+1}$.

Remark 6.6.15. Consider the diagram

$$\begin{tikzcd} BG \arrow[r, hook] & {[\operatorname{Spec} A/G] \times_{A^G} \Bbbk} \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & {[\operatorname{Spec} A/G]} \arrow[d] \\ & \operatorname{Spec} \Bbbk \arrow[r, hook] & \operatorname{Spec} A^G. \end{tikzcd}$$

A formal consequence of the above theorem is that $[\operatorname{Spec} A/G]$ is also coherently complete with respect to the fiber $[\operatorname{Spec} A/G] \times_{A^G} \Bbbk$. This version is analogous to Grothendieck's Existence Theorem (6.6.8), but the coherent completeness along BG is a substantially stronger statement, e.g., for $[\mathbb{A}^n/\mathbb{G}_m]$ where the fiber of $[\mathbb{A}^n/\mathbb{G}_m] \to \operatorname{Spec} \Bbbk$ is everything.

Proof of Theorem 6.6.13. We need to show that $\operatorname{Coh}(\mathcal{X}) \to \varprojlim \operatorname{Coh}(\mathcal{X}_n)$ is an equivalence of categories, where $\mathcal{X} = [\operatorname{Spec} A/G]$ and $\mathcal{X}_n$ is the nth nilpotent thickening of $BG \hookrightarrow \mathcal{X}$ of the inclusion of the residual gerbe at x.

Full faithfulness: Suppose that F and F' are coherent $\mathcal{O}_{\mathcal{X}}$-modules, and let F_n and F'_n denote the restrictions to $\mathcal{X}_n$, respectively. We need to show that

$$\operatorname{Hom}_{\mathcal{O}_{\mathcal{X}}}(F, F') \to \varprojlim \operatorname{Hom}_{\mathcal{O}_{\mathcal{X}}}(F_n, F'_n)$$

is bijective. Since $\mathcal{X}$ has the resolution property (Proposition 6.2.10), we can find a resolution $F_2 \to F_1 \to F \to 0$ by vector bundles. This induces a diagram

$$\begin{tikzcd} 0 \arrow[r] & \operatorname{Hom}_{\mathcal{O}_{\mathcal{X}}}(F, F') \arrow[r] \arrow[d] & \operatorname{Hom}_{\mathcal{O}_{\mathcal{X}}}(F_1, F') \arrow[r] \arrow[d] & \operatorname{Hom}_{\mathcal{O}_{\mathcal{X}}}(F_2, F') \arrow[d] \\ 0 \arrow[r] & \varprojlim \operatorname{Hom}_{\mathcal{O}_{\mathcal{X}}}(F_n, F'_n) \arrow[r] & \varprojlim \operatorname{Hom}_{\mathcal{O}_{\mathcal{X}}}(F_{1,n}, F'_n) \arrow[r] & \varprojlim \operatorname{Hom}_{\mathcal{O}_{\mathcal{X}}}(F_{2,n}, F'_n) \end{tikzcd}$$

with exact rows, and we see that it suffices to prove full faithfulness in the case that F is a vector bundle. In this case,

$$\operatorname{Hom}_{\mathcal{O}_{\mathcal{X}}}(F, F') = \operatorname{Hom}_{\mathcal{O}_{\mathcal{X}}}(\mathcal{O}_{\mathcal{X}}, F^\vee \otimes F')$$
$$\operatorname{Hom}_{\mathcal{O}_{\mathcal{X}}}(F_n, F'_n) = \operatorname{Hom}_{\mathcal{O}_{\mathcal{X}}}\big(\mathcal{O}_{\mathcal{X}_n}, (F_n^\vee \otimes F'_n)\big).$$

Therefore, we can also assume that $F = \mathcal{O}_{\mathcal{X}}$ and we are reduced to showing that

$$\Gamma(\mathcal{X}, F') \to \varprojlim \Gamma\big(\mathcal{X}_n, F'_n\big) \tag{6.6.16}$$

is an isomorphism. Writing $F' = \widetilde{M}$ where M is a finitely generated A-module with an action of G and letting $\mathfrak{m} \subseteq A$ be the maximal ideal for x, then $\Gamma(\mathcal{X}_n, F'_n) = M^G/(\mathfrak{m}^n M)^G$ since G is linearly reductive. We must therefore verify that

$$M^G \to \varprojlim M^G/(\mathfrak{m}^n M)^G \tag{6.6.17}$$

is an isomorphism. To this end, we first show that

$$\bigcap_{n \geq 0} \big(\mathfrak{m}^n M\big)^G = 0, \tag{6.6.18}$$

or in other words that (6.6.17) is injective. Let $N := \bigcap_{n \geq 0} \mathfrak{m}^n M$. The Artin–Rees Lemma (B.5.4) applied to $N \subseteq M$ implies that there exists an integer c such that $\mathfrak{m}^n M \cap N = \mathfrak{m}^{n-c}(\mathfrak{m}^c M \cap N)$ for all $n \geq c$. Taking $n = c+1$, we see that $N = \mathfrak{m}N$ so $N \otimes_A A/\mathfrak{m} = 0$. Since the support of N is a closed G-invariant subscheme of $\operatorname{Spec} A$ which does not contain x, it follows that $N = 0$.

Note also that since G is linearly reductive, M^G is a finitely generated A^G-module (Corollary 6.5.8(3)). We next establish that (6.6.17) is an isomorphism if A^G is artinian. In this case, $((\mathfrak{m}^n M)^G)$ automatically satisfies the Mittag–Leffler condition (it is a sequence of artinian A^G-modules). Taking the inverse limit of the exact sequences $0 \to (\mathfrak{m}^n M)^G \to M^G \to M^G/(\mathfrak{m}^n M)^G \to 0$ and applying (6.6.18) yields an exact sequence

$$0 \to 0 \to M^G \to \varprojlim M^G/(\mathfrak{m}^n M)^G \to 0,$$

and implies that (6.6.17) is an isomorphism. To establish (6.6.17) in the general case, let $J = \mathfrak{m}^G A \subseteq A$ and observe that

$$M^G \cong \varprojlim M^G/\big(\mathfrak{m}^G\big)^n M^G \cong \varprojlim \big(M/J^n M\big)^G, \tag{6.6.19}$$

since G is linearly reductive. For each n, we know that

$$\big(M/J^n M\big)^G \cong \varprojlim_l M^G/\big((J^n + \mathfrak{m}^l)M\big)^G \tag{6.6.20}$$

by the artinian case. Finally, combining (6.6.19) and (6.6.20) together with the observation that $J^n \subseteq \mathfrak{m}^l$ for $n \geq l$, we conclude that

$$\begin{aligned} M^G &\cong \varprojlim_n \big(M/J^n M\big)^G \\ &\cong \varprojlim_n \varprojlim_l M^G/\big((J^n + \mathfrak{m}^l)M\big)^G \\ &\cong \varprojlim_l M^G/\big(\mathfrak{m}^l M\big)^G. \end{aligned}$$

Essential surjectivity (explicit construction): The linear reductivity of G implies that every coherent sheaf $F = \widetilde{M}$ on $[\operatorname{Spec} A/G]$ decomposes as a direct sum

$$M = \bigoplus_{\rho \in \Gamma} M^{(\rho)}, \tag{6.6.21}$$

where Γ denotes the set of isomorphism classes of irreducible representations of G and $M^{(\rho)}$ is the isotypic component corresponding to ρ; explicitly, if W_ρ denotes the irreducible representation corresponding to ρ, then $M^{(\rho)} = \operatorname{Hom}^G_{\Bbbk}(W_\rho, M) \otimes W_\rho$. Moreover, the decomposition (6.6.21) is compatible with the A-module structure of M and the decomposition $A = \bigoplus_{\rho \in \Gamma} A^{(\rho)}$.

Let us also note that if $F = \widetilde{M} \in \operatorname{Coh}(\mathcal{X})$ with restrictions $M_n = M/\mathfrak{m}^{n+1}M$, then applying (6.6.17) to $M \otimes W_\rho^\vee$ shows that $M^{(\rho)} = \varprojlim M_n^{(\rho)}$. By Theorem 6.5.6(3), we also know that $M^{(\rho)}$ is a finitely generated A^G-module. In particular, $A^{(\rho)} = \varprojlim (A/\mathfrak{m}^{n+1})^{(\rho)}$ is a finitely generated A^G-module.

This suggests that if $F_n = \widetilde{M_n}$ is a compatible system of coherent $\mathcal{O}_{\mathcal{X}_n}$-modules with $M_n = \bigoplus_\rho M_n^{(\rho)}$, we define

$$M^{(\rho)} := \varprojlim M_n^{(\rho)} \qquad \text{and} \qquad M := \bigoplus_{\rho \in \Gamma} M^{(\rho)}. \tag{6.6.22}$$

To see that M is an A-module with a G-action, let $\rho, \gamma \in \Gamma$ be irreducible representations and let $\Lambda \subseteq \Gamma$ denote the finite set of nonzero irreducible representations appearing $W_\rho \otimes W_\gamma$. Taking limits of the maps $A_n^{(\rho)} \otimes_{(A/\mathfrak{m}^{n+1})^G} M_n^{(\gamma)} \to \bigoplus_{\lambda \in \Lambda} M_n^{(\lambda)}$, defines multiplication

$$A^{(\rho)} \otimes_A M^{(\gamma)} \to \varprojlim \left(A_n^{(\rho)} \otimes_{(A/\mathfrak{m}^{n+1})^G} M_n^{(\gamma)}\right) \to \varprojlim \Big(\bigoplus_{\lambda \in \Lambda} M_n^{(\lambda)}\Big) \cong \bigoplus_{\lambda \in \Lambda} M^{(\lambda)}.$$

Note that $M/\mathfrak{m}^{n+1}M \cong M_n$ by construction.

It remains to show that the A-module M of (6.6.22) is finitely generated. The coherent sheaf $F_0 = \widetilde{M_0}$ on $\mathcal{X}_0 = BG$ is a finite dimensional G-representation and we can consider the coherent $\mathcal{O}_\mathcal{X}$-module $F_0 \otimes_{\mathcal{O}_{\mathcal{X}_0}} \mathcal{O}_\mathcal{X}$ or equivalently the A-module $M_0 \otimes_\Bbbk A$ with its natural G-action. Since $\mathcal{X}$ is cohomologically affine, the functor

$$\mathrm{Hom}_{\mathcal{O}_\mathcal{X}}(F_0 \otimes_{\mathcal{O}_{\mathcal{X}_0}} \mathcal{O}_\mathcal{X}, -) = \Gamma(\mathcal{X}, (F_0^\vee \otimes_{\mathcal{O}_{\mathcal{X}_0}} \mathcal{O}_\mathcal{X}) \otimes_{\mathcal{O}_\mathcal{X}} -)$$

is exact. Apply the functor to the surjection $M \twoheadrightarrow M_0$ induces a map

$$M_0 \otimes_\Bbbk A \to M \tag{6.6.23}$$

which we would like to show is surjective. We already know that the restrictions $M_0 \otimes_\Bbbk (A/\mathfrak{m}^{n+1}) \to M_n$ are surjective as its cokernel is a coherent module on $\mathcal{X}_n$ not supported at the unique closed point.

As above, we first handle the case that A^G is artinian. Since $A^{(\rho)} \xrightarrow{\sim} \varprojlim (A/\mathfrak{m}^n)^{(\rho)}$ is a finitely generated A^G-module, it follows that $(A/\mathfrak{m}^n)^{(\rho)}$ stabilizes to $A^{(\rho)}$ for $n \gg 0$. Since (6.6.23) induces surjections $M_0 \otimes_\Bbbk (A/\mathfrak{m}^{n+1}) \to M_n$, the modules $M_n^{(\rho)}$ stabilize to $M_\infty^{(\rho)}$ for $n \gg 0$ and $M = \bigoplus_\rho M_\infty^{(\rho)}$ is finitely generated. In the general case, let $X_m = \operatorname{Spec} A^G/(\mathfrak{m} \cap A^G)^{m+1}$ and consider the cartesian diagram

$$\begin{tikzcd}
\mathcal{X} \times_X X_m \arrow[r, hook, "i_m"] \arrow[d, "\pi_m"] \arrow[dr, phantom, "\square"] & \mathcal{X} \times_X X_{m+1} \arrow[r, hook] \arrow[d, "\pi_{m+1}"] & \cdots \arrow[r, hook] & \mathcal{X} \arrow[d, "\pi"] \\
X_m \arrow[r, hook, "j_m"] & X_{m+1} \arrow[r, hook] & \cdots \arrow[r, hook] & X.
\end{tikzcd}$$

For each m, we may consider the nth nilpotent thickenings $\mathcal{Z}_{m,n}$ of $\mathcal{X}_0 \hookrightarrow \mathcal{X} \times_X X_m$ which is a closed substack of $\mathcal{X}_n$. Since X_m is the spectrum of artinian ring, the restrictions $F_n|_{\mathcal{Z}_{m,n}}$ extend to a coherent sheaf $H_m = \widetilde{N}_m$ on $\mathcal{X} \times_X X_m$. Moreover, there is a canonical isomorphism between H_m and the restriction of H_{m+1} to $\mathcal{X} \times_X X_m$. By Lemma 6.5.22(4), the adjunction morphism $j_m^* \pi_{m+1,*} \xrightarrow{\sim} \pi_{m,*} i_m^*$ is an isomorphism on quasi-coherent sheaves. This implies that $N_{m+1}^{(\rho)} = \Gamma(\mathcal{X} \times_X X_{m+1}, H_{m+1} \otimes W_\rho^\vee)$ restricts to $N_m^{(\rho)}$ and that $M^{(\rho)} = \varprojlim N_m^{(\rho)}$ is a finitely generated A^G-module. The map (6.6.23) is surjective as it surjective on each ρ-isotypical component by Nakayama's lemma.

Essential surjectivity (indirect construction): For an alternative argument, use the resolution property of $\mathcal{X}$ (Proposition 6.2.10) to choose a surjection $E \twoheadrightarrow F_0$ from a vector bundle E on $\mathcal{X}$. Since each $F_{n+1} \to F_n$ is surjective and $\mathrm{Hom}_{\mathcal{O}_\mathcal{X}}(E, -) = \Gamma(\mathcal{X}, E^\vee \otimes_{\mathcal{O}_\mathcal{X}} -)$ is exact, we can lift $E \twoheadrightarrow F_0$ to compatible maps $E \to F_n$, each which is surjective (Nakayama's lemma). The sequence $(\ker(E_n \to F_n))$ is not necessarily an adic sytem of coherent sheaves as the restriction $\ker(E_{n+1} \to F_{n+1})$ to $\mathcal{X}_n$ may not be $\ker(E_n \to F_n)$. But we can modify it as follows: for each $l \geq m \geq n$, the images of $\ker(E_l \to F_l)$ in E_m stabilize to K'_m for $l \gg m$ and $K'_m/\mathfrak{m}^{n+1}K'_m$ stabilize to K_n for $m \gg n$ (see also [SP, Tag 087X]). Then $(K_n) \in \varprojlim \mathrm{Coh}(\mathcal{X}_n)$ is an adic sequence. Repeating the construction, we can find a vector bundle E' on $\mathcal{X}$ and compatible surjections $E' \to K_n$. By full faithfulness, there is a morphism $E' \to E$ extending the maps $E'_n \to E_n$. Then $\mathrm{coker}(E' \to E)$ is a coherent $\mathcal{O}_{\mathcal{X}}$ extending (F_n). See also [AHR20, Thm. 1.3] and [AHR19, Thm. 1.6]. $\square$

Exercise 6.6.24. If S is a noetherian affine scheme, show that $[\mathbb{A}^1/\mathbb{G}_m]_S$ is coherently complete along $BG_{m,S}$.

6.7 Local structure of algebraic stacks

All human knowledge thus begins with intuitions, proceeds thence to concepts, and ends with ideas.

Emmanual Kant

We prove the Local Structure Theorem for Algebraic Stacks (6.7.1) around points with linearly reductive stabilizer. Philosophically, this means that quotient stacks of the form $[\operatorname{Spec} A/G]$, where G is linearly reductive, are the building blocks of algebraic stacks near points with linearly reductive stabilizers in a similar way to how affine schemes are the building blocks of schemes and algebraic spaces. This allows one to reduce many properties of algebraic stacks to the well-understood case of $[\operatorname{Spec} A/G]$. This theorem is a generalization of the Local Structure Theorem for Deligne–Mumford Stacks (4.3.1) and will be applied in a similar way. For example, we will apply it to construct good moduli spaces (Theorem 6.10.1) in an analogous way to how the Deligne–Mumford case was applied to construct coarse moduli spaces in the proof of the Keel–Mori Theorem (4.4.6).

Theorem 6.7.1 (Local Structure Theorem for Algebraic Stacks)**.** *Let $\mathcal{X}$ be an algebraic stack of finite type over an algebraically closed field $\Bbbk$ with affine diagonal. For every point $x \in \mathcal{X}(\Bbbk)$ with linearly reductive stabilizer G_x, there exists an affine étale morphism*

$$f\colon ([\operatorname{Spec} A/G_x], w) \to (\mathcal{X}, x)$$

which induces an isomorphism of stabilizer groups at w.

Remark 6.7.2. In the case that $x \in |\mathcal{X}|$ is a smooth point, then one can say more: there is also an étale morphism

$$([\operatorname{Spec} A/G_x], w) \to ([T_{\mathcal{X},x}/G_x], 0)$$

where $T_{\mathcal{X},x}$ is the Zariski tangent space equipped as a G_x-representation. This addendum follows from the proof but also follows from applying Luna's Étale Slice Theorem (6.7.5) to $[\operatorname{Spec} A/G_x]$. The upshot is that we can reduce étale local properties of $\mathcal{X}$ to G_x-equivariant properties of $T_{\mathcal{X},x}$; for moduli problems, this translates into studying the first-order deformation space as a representation under the automorphism group.

By combining this theorem with Luna's Fundamental Lemma (6.5.28), we obtain the following result.

Corollary 6.7.3 (Local Structure for Good Moduli Spaces)**.** *Let $\mathcal{X}$ be an algebraic stack of finite type over an algebraically closed field $\Bbbk$ with affine diagonal. Suppose that there exists a good moduli space $\pi\colon \mathcal{X} \to X$. Then for every closed point $x \in |\mathcal{X}|$, there exists an étale neighborhood $W \to X$ of $\pi(x)$ and a cartesian diagram*

$$\begin{tikzcd} [\operatorname{Spec} A/G_x] \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{X} \arrow[d, "\pi"] \\ W = \operatorname{Spec} A^{G_x} \arrow[r] & X. \end{tikzcd}$$

$\square$

A nice upshot is that we can provide an equivalent and more intuitive definition of a good moduli space.

Corollary 6.7.4 (Characterization of Good Moduli Spaces)**.** *Let $\mathcal{X}$ be an algebraic stack of finite type over an algebraically closed field $\Bbbk$ with affine diagonal. A morphism $\pi\colon \mathcal{X} \to X$ to an algebraic space is a good moduli space if and only if every point $x \in X$ has an étale neighborhood* $\operatorname{Spec} B \to X$ *such that* $\mathcal{X} \times_X \operatorname{Spec} B \cong [\operatorname{Spec} A/G]$ *with G linearly reductive and $B = A^G$.* $\square$

Section outline: We first discuss Luna's Étale Slice Theorem (6.7.5), a beautiful argument providing an explicit construction of an étale neighborhood in the case that $\mathcal{X}$ is already known to have the form $[\operatorname{Spec} B/G]$ with G linearly reductive. The proof of the Local Structure Theorem (6.7.1) is far less explicit requiring: (1) deformation theory, (2) coherent completeness, (3) Coherent Tannaka Duality and (4) Artin Approximation or Equivariant Artin Algebraization.

Letting $\mathcal{T} = [T_{\mathcal{X},x}/G_x]$, deformation theory produces an embedding $\mathcal{X}_n \hookrightarrow \mathcal{T}_n$ of the nth nilpotent thickenings of x and 0. The key step in the proof is to show that the system of closed morphisms $(\mathcal{X}_n \to \mathcal{T}_n)$ algebraizes. The first step is effectivization: the fiber product $\widehat{\mathcal{T}} := \mathcal{T} \times_T \operatorname{Spec} \widehat{\mathcal{O}}_{T,\pi(0)}$, where $\pi\colon \mathcal{T} \to T := T_{\mathcal{X},x}/\!\!/G_x$, is coherently complete (Theorem 6.6.13). We can thus construct a closed substack $\widehat{\mathcal{X}} \hookrightarrow \widehat{\mathcal{T}}$ extending $\mathcal{X}_n \hookrightarrow \mathcal{T}$ and then apply Coherent Tannaka Duality (6.6.9) to construct a morphism $\widehat{\mathcal{X}} \to \mathcal{X}$ extending $\mathcal{X}_n \to \mathcal{X}$. If $x \in |\mathcal{X}|$ is smooth, Artin Approximation over the GIT quotient $T_{\mathcal{X},x}/\!\!/G_x$ produces an étale neighborhood $U \to T_{\mathcal{X},x}/\!\!/G_x$ such that $\pi^{-1}(U) \to \mathcal{X}$ algebraizes $\widehat{\mathcal{T}} \to \mathcal{X}$. The general case is more involved, requiring an equivariant version of Artin Algebraization (Theorem 6.7.18).

6.7.1 Luna's Étale Slice Theorem

The Local Structure Theorem (6.7.1) is inspired by Luna's Étale Slice Theorem in equivariant geometry.

Theorem 6.7.5 (Luna's Étale Slice Theorem)**.** *Let G be a linearly reductive group over an algebraically closed field $\Bbbk$ and let X be an affine scheme of finite type over $\Bbbk$ with an action of G. If $x \in X(\Bbbk)$ has linearly reductive stabilizer, then there exists a G_x-invariant, locally closed, and affine subscheme $W \subseteq X$ such that the induced map*

$$[W/G_x] \to [X/G] \tag{6.7.6}$$

is affine and étale. If in addition the orbit $Gx \subseteq X$ is closed, then there is a cartesian diagram

$$\begin{tikzcd} {[W/G_x]} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & {[X/G]} \arrow[d] \\ W/\!\!/G_x \arrow[r] & X/\!\!/G \end{tikzcd}$$

where $W/\!\!/G_x \to X/\!\!/G$ is also étale.

Moreover, if $x \in X$ is a smooth point and if we denote by $N_x = T_{X,x}/T_{Gx,x}$ the normal space to the orbit, then we can arrange that there is a cartesian diagram

$$\begin{tikzcd} {[N_x/G_x]} \arrow[d] \arrow[dr, phantom, "\square"] & {[W/G_x]} \arrow[l, "g"'] \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & {[X/G]} \arrow[d] \\ N_x/\!\!/G_x & W/\!\!/G_x \arrow[l] \arrow[r] & X/\!\!/G_x \end{tikzcd}$$

where all horizontal arrows are étale and g is induced from a G_x-invariant étale morphism $W \to N_x$.

Remark 6.7.7. If the orbit Gx is closed, then Matsushima's Theorem (6.5.21) implies that the stabilizer G_x is linearly reductive.

Remark 6.7.8. One can also formulate the statement G-equivariantly: G acts naturally on the quotient $G \times^{G_x} W := (G \times W)/G_x$ and there is an identification $[W/G_x] \cong [(G \times^{G_x} W)/G]$ and likewise $W/\!\!/G_x \cong (G \times^{G_x} W)/\!\!/G$ (see Exercise 3.4.19). The morphism (6.7.6) corresponds to an étale G-equivariant morphism $G \times^{G_x} W \to X$.

The proof will rely on the existence of a G_x-invariant morphism $X \to T_{X,x}$, which we refer to as the *Luna map*. The use of $T_{X,x}$ here is an abuse of notation—$T_{X,x}$ is a vector space over $\Bbbk$ and we view it as a scheme via $\operatorname{Spec}(\operatorname{Sym}^* T_{X,x}^\vee)$.

Lemma 6.7.9 (Luna map)**.** *Let G be a linearly reductive group over an algebraically closed field $\Bbbk$ and let X be an affine scheme of finite type over $\Bbbk$ with an action of G. If $x \in X(\Bbbk)$ has linearly reductive stabilizer, there exists a G_x-equivariant morphism*

$$f\colon X \to T_{X,x} \tag{6.7.10}$$

sending x to the origin. If X is smooth at x, then f is étale at x.

Proof. Letting $X = \operatorname{Spec} A$ and $\mathfrak{m} \subseteq A$ be the maximal ideal of x, then $\mathfrak{m}$ and $\mathfrak{m}/\mathfrak{m}^2$ are G_x-representations, and thus G_x acts naturally on the tangent space $T_{X,x} := \operatorname{Spec}(\operatorname{Sym}^* \mathfrak{m}/\mathfrak{m}^2)$. Since G_x is linearly reductive, the surjection $\mathfrak{m} \to \mathfrak{m}/\mathfrak{m}^2$ of G_x-representations has a section $\mathfrak{m}/\mathfrak{m}^2 \hookrightarrow \mathfrak{m}$. This induces a G_x-equivariant ring map $\operatorname{Sym}^* \mathfrak{m}/\mathfrak{m}^2 \to A$ and thus a G_x-equivariant morphism $f\colon \operatorname{Spec} A \to T_{X,x}$ sending x to the origin. If $x \in X$ is smooth, then since f induces an isomorphism of tangent spaces at x, we conclude that f is étale at x (Theorem A.3.2). $\square$

Proof of Theorem 6.7.5. Since X is affine and of finite type, we can choose a finite dimensional G-representation V and a G-equivariant closed immersion $X \hookrightarrow \mathbb{A}(V)$ (Proposition B.1.18). If $W \subseteq \mathbb{A}(V)$ is an affine G_x-invariant locally closed subscheme such that $[W/G_x] \to [\mathbb{A}(V)/G]$ is étale, then the same is true for $W' := W \cap X \subseteq X$ and $[W'/G_x] \to [X/G]$. We can therefore immediately reduce to the case that $x \in X$ is smooth. In this case, the Luna map $f\colon X \to T_{X,x}$ is G_x-invariant, étale at x, and with $f(x) = 0$ (Lemma 6.7.9). The subspace $T_{Gx,x} \subseteq T_{X,x}$ is G_x-invariant and using again that G_x is linearly reductive, the surjection $T_{X,x} \to N_x = T_{X,x}/T_{Gx,x}$ has a section $N_x \hookrightarrow T_{X,x}$. We define W as the preimage of N_x under f:

$$\begin{tikzcd} W \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\square"] & N_x \arrow[d, hook] \\ X \arrow[r, "f"] & T_{X,x}. \end{tikzcd}$$

Since the maps $f\colon [W/G_x] \to [X/G]$ and $g\colon [W/G_x] \to [N_x/G_x]$ induce an isomorphism of tangent spaces and stabilizer groups at w, they are both étale at $x \in W$ (or equivalently the G-equivariant maps $G \times^{G_x} W \to X$ and $G \times^{G_x} W \to G \times^{G_x} N_x$ are étale at (id, x)). We have a commutative diagram

$$\begin{tikzcd} {[N_x/G_x]} \arrow[d] & {[W/G_x]} \arrow[l, "g"'] \arrow[r, "f"] \arrow[d] & {[X/G]} \arrow[d] \\ N_x/\!\!/G_x & W/\!\!/G_x \arrow[l] \arrow[r] & X/\!\!/G_x \end{tikzcd}$$

where both f and g are étale at x, preserve stabilizer groups at x, and map x to closed points. We can therefore apply Luna's Fundamental Lemma (6.5.28) to replace W with a G_x-equivariant, open, and affine neighborhood of x so that the above squares are cartesian. $\square$

When $\mathcal{X}$ is already known to be a quotient stack of a normal quasi-projective scheme, the Local Structure Theorem follows from a direct argument. This case is sufficient to handle many moduli problems, e.g., $\mathcal{B}un^{\mathrm{ss}}_{r,d}(C)$ in characteristic 0.

Exercise 6.7.11. If G is a connected affine algebraic group over an algebraically closed field $\Bbbk$ acting on a normal finite type $\Bbbk$-scheme X, and $x \in X(\Bbbk)$ has linearly reductive stabilizer, show that there is a G_x-invariant, locally closed, and affine subscheme $W \hookrightarrow X$ such that $[W/G_x] \to [X/G]$ is étale.

Hint: Use Sumihiro's Theorem on Linearizations (B.1.30) to reduce to the case that $X = \mathbb{P}(V)$. Choose a homogenous polynomial f not vanishing at x such that $\mathbb{P}(V)_f$ is G_x-invariant and then argue as in the proof of Luna's Étale Slice Theorem by considering the G_x-equivariant étale map $\mathbb{P}(V)_f \to T_x\mathbb{P}(V)$.

6.7.2 Deformation theory

In our proof of the Local Structure Theorem (6.7.1), we will need some deformation theory of algebraic stacks.

Proposition 6.7.12. *Consider a commutative diagram*

$$\begin{tikzcd} \mathcal{W} \arrow[r, "f"] \arrow[d, hook] & \mathcal{X} \arrow[d] \\ \mathcal{W}' \arrow[r] \arrow[ur, dashed] & \mathcal{Y} \end{tikzcd}$$

of noetherian algebraic stacks with affine diagonal, where $\mathcal{X} \to \mathcal{Y}$ is smooth and affine and $\mathcal{W} \hookrightarrow \mathcal{W}'$ is a closed immersion defined by a square-zero sheaf of ideals. If $\mathcal{W}$ is cohomologically affine, there exists a lift in the above diagram.

Proof. When $\mathcal{W}$ is affine, the statement follows from the Infinitesimal Lifting Criterion (A.3.1). To reduce to this case, let $U' \to \mathcal{W}'$ be a smooth presentation with U' an affine scheme, and define $U := U' \times_{\mathcal{W}'} \mathcal{W}$. Since $\mathcal{W}$ has affine diagonal, each n-fold fiber product $(U/\mathcal{W})^n := U \times_{\mathcal{W}} \cdots \times_{\mathcal{W}} U$ is affine. There is a commutative diagram

$$\begin{tikzcd}
(U/\mathcal{W})^2 \arrow[r, shift left] \arrow[r, shift right] \arrow[d, hook] & U \arrow[r, "q_1"] \arrow[d, hook] & \mathcal{W} \arrow[r, "f"] \arrow[d, hook] & \mathcal{X} \arrow[d] \\
(U'/\mathcal{W}')^2 \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & U' \arrow[r] \arrow[urr, "f'_U"] & \mathcal{W}' \arrow[r] \arrow[ur, dashed, "f'"'] & \mathcal{Y},
\end{tikzcd}$$

where we have chosen a lift $f'_U \colon U' \to \mathcal{X}$. Define the coherent sheaf $F = f^*(\Omega^\vee_{\mathcal{X}/\mathcal{Y}}) \otimes J$ on $\mathcal{W}$, where J is the ideal sheaf defining $\mathcal{W} \hookrightarrow \mathcal{W}'$. We know by Exercise 6.1.17 that the set of lifts $U' \to \mathcal{X}$ is a torsor under $\Gamma(U, q_1^* F)$, i.e., any other lift differs from f'_U by an element of $\Gamma(U, q_1^* F)$. Because $\mathcal{X} \to \mathcal{Y}$ is representable, to check that f'_U descends to a morphism $f' \colon \mathcal{W}' \to \mathcal{X}$, we need to arrange that $f'_U \circ p_1 = f'_U \circ p_2$. Let $q_n \colon (U/\mathcal{W})^n \to \mathcal{W}$. The difference $f'_U \circ p_1 - f'_U \circ p_2$ can be viewed an element of $\Gamma((U/\mathcal{W})^2, q_2^* F)$. Since $q_1 \colon U \to \mathcal{W}$ is a surjective, smooth, and affine morphism, there is an exact sequence of quasi-coherent sheaves

$$0 \to F \to q_{1,*} q_1^* F \to q_{2,*} q_2^* F \to q_{3,*} q_3^* F \to \cdots;$$

see Exercise 2.1.3. Since $\mathcal{W}$ is cohomologically affine, taking global sections yields an exact sequence

$$\begin{array}{ccccc}
\Gamma(U, q_1^* F) & \xrightarrow{d_0} & \Gamma((U/\mathcal{W})^2, q_2^* F) & \xrightarrow{d_1} & \Gamma((U/\mathcal{W})^3, q_3^* F) \\
s \longmapsto p_1^* s - p_2^* s, & & t \longmapsto p_{12}^* t - p_{13}^* t + p_{23}^* t. & &
\end{array}$$

One checks that $d_1(f'_U \circ p_1 - f'_U \circ p_2) = 0$ so there exists an element $s \in \Gamma(U, q_1^* F)$ with $d_0(s) = f'_U \circ p_1 - f'_U \circ p_2$. After modifying the lift f'_U by s, we see that $f'_U \circ p_1 - f'_U \circ p_2 = 0$ so that f'_U descends to $f' \colon \mathcal{W}' \to \mathcal{X}$. $\square$

Remark 6.7.13. Alternatively, one can show that the obstruction to this deformation problem lies in $\mathrm{Ext}^1_{\mathcal{O}_{\mathcal{W}}}(f^* \Omega_{\mathcal{X}/\mathcal{Y}}, J) = \mathrm{H}^1(\mathcal{W}, f^*(\Omega^\vee_{\mathcal{X}/\mathcal{Y}}) \otimes J)$, which vanishes since $\mathcal{W}$ is cohomologically affine. The above result holds more generally [Ols06, Thm. 1.5].

Exercise 6.7.14. Let $\mathcal{W} \hookrightarrow \mathcal{W}'$ be a closed immersion of algebraic stacks of finite type over $\Bbbk$ with affine diagonal defined by a square-zero sheaf of ideals. Let G be an affine algebraic group over $\Bbbk$. If $\mathcal{W}$ is cohomologically affine, show that every principal G-bundle $\mathcal{P} \to \mathcal{W}$ extends to a principal G-bundle $\mathcal{P}' \to \mathcal{W}'$.

Hint: Use smooth descent and the deformation theory of principal G-bundles over schemes (Exercise C.2.6).

Remark 6.7.15. The deformation question in Exercise 6.7.14 is equivalent to deforming the morphism $f \colon \mathcal{W} \to BG$ to a morphism $\mathcal{W}' \to BG$, which is analogous to Proposition 6.7.12 except that $\mathcal{X} = BG \to \mathcal{Y} = \operatorname{Spec} \Bbbk$ is not affine. Since maps to BG correspond to principal G-bundles, this question is equivalent to deformation a principal G-bundle $P \to \mathcal{W}$ to $P' \to \mathcal{W}'$, and just like the case when $\mathcal{W}$ is a scheme (Exercise C.2.6), the obstruction lies in the group $\mathrm{H}^2(\mathcal{W}, \mathfrak{g} \otimes J)$, where J is the ideal sheaf defining $\mathcal{W} \hookrightarrow \mathcal{W}'$. Indeed, the general theory of the cotangent complex (see Remarks C.3.7 and C.7.6) implies that the obstruction lies in $\mathrm{Ext}^1_{\mathcal{O}_{\mathcal{W}}}(Lf^*\mathrm{L}_{BG/\Bbbk}, J)$. Under the composition $\operatorname{Spec}\Bbbk \xrightarrow{p} BG \to \operatorname{Spec}\Bbbk$, we have an exact triangle $p^*\mathrm{L}_{BG/\Bbbk} \to \mathrm{L}_{\Bbbk/\Bbbk} \to \mathrm{L}_{\Bbbk/BG}$. Since $\mathrm{L}_{\Bbbk/\Bbbk} = 0$, we obtain that $p^*\mathrm{L}_{BG/\Bbbk} = \mathrm{L}_{\Bbbk/BG}[-1] \cong \mathfrak{g}^\vee[-1]$ and $\mathrm{L}_{BG/\Bbbk} \cong \mathfrak{g}^\vee[-1]$, where the Lie algebra $\mathfrak{g}$ is equipped with the adjoint representation. Thus $\mathrm{Ext}^1_{\mathcal{O}_{\mathcal{W}}}(Lf^*\mathrm{L}_{BG/\Bbbk}, J) = \mathrm{H}^1(\mathcal{W}, f^*\mathfrak{g}[1] \otimes J) = \mathrm{H}^2(\mathcal{W}, \mathfrak{g} \otimes J)$. Since $\mathcal{W}$ is cohomologically affine with affine diagonal, this cohomology group is 0 and the obstruction vanishes.

We will also need the following criteria for morphisms to be closed immersions or isomorphisms.

Lemma 6.7.16. *Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a representable morphism of algebraic stacks of finite type over an algebraically closed field $\Bbbk$, each with affine diagonal. Assume that $|\mathcal{X}| = \{x\}$ and $|\mathcal{Y}| = \{y\}$ consist of a single point and that f induces an isomorphism between $\mathcal{X}_0 := BG_x$ and $\mathcal{Y}_0 := BG_y$. Let $\mathfrak{m}_x \subseteq \mathcal{O}_{\mathcal{X}}$ and $\mathfrak{m}_y \subseteq \mathcal{O}_{\mathcal{Y}}$ be the ideal sheaves defining $\mathcal{X}_0$ and $\mathcal{Y}_0$, and let $f_1\colon \mathcal{X}_1 \to \mathcal{Y}_1$ be the induced morphism between the first nilpotent thickenings of $\mathcal{X}_0$ and $\mathcal{Y}_0$.*

(1) If f_1 is a closed immersion, then so is f.

(2) If f_1 is a closed immersion and there is an isomorphism $\bigoplus_{n\geq 0} \mathfrak{m}_y^n/\mathfrak{m}_y^{n+1} \cong \bigoplus_{n\geq 0} \mathfrak{m}_x^n/\mathfrak{m}_x^{n+1}$ of graded $\mathcal{O}_{\mathcal{X}_0}$-modules, then f is an isomorphism.

Proof. Choose a smooth presentation $V = \operatorname{Spec} B \to \mathcal{Y}$ from an affine scheme such that $V \times_{\mathcal{Y}} \mathcal{Y}_0 \cong \operatorname{Spec}\Bbbk$ (Theorem 3.6.1). Then B is a local artinian $\Bbbk$-algebra as $\mathcal{Y}$ consists of only one point. The base change $U = V \times_{\mathcal{Y}} \mathcal{X}$ is an algebraic space and since $U_{\mathrm{red}} = V_{\mathrm{red}}$ is a point, it follows from Proposition 4.5.19 that $U = \operatorname{Spec} A$ with A a local artinian $\Bbbk$-algebra. We can therefore assume that $f\colon \operatorname{Spec} A \to \operatorname{Spec} B$ is a morphism of local artinian schemes. For (1), we need to show that if $B/\mathfrak{m}_B^2 \to A/\mathfrak{m}_A^2$ is surjective, so is $B \to A$. We first claim that the inclusion $\mathfrak{m}_B A \hookrightarrow \mathfrak{m}_A$ is surjective. By Nakayama's Lemma, it suffices to show that $\mathfrak{m}_B A/\mathfrak{m}_A\mathfrak{m}_B A \to \mathfrak{m}_A/\mathfrak{m}_A^2$ is surjective, but this follows from the hypothesis that the composition $\mathfrak{m}_A/\mathfrak{m}_A^2 \to \mathfrak{m}_B A/\mathfrak{m}_A\mathfrak{m}_B A \to \mathfrak{m}_A/\mathfrak{m}_A^2$ is surjective. Since $B/\mathfrak{m}_B \to A/\mathfrak{m}_B A = A/\mathfrak{m}_A$ is surjective, another application of Nakayama's Lemma shows that $B \to A$ is surjective. See also [Har77, Lem. II.7.4] for a related criterion. For (2), since $\dim_\Bbbk \mathfrak{m}_B^n/\mathfrak{m}_B^{n+1} = \dim_\Bbbk \mathfrak{m}_A^n/\mathfrak{m}_A^{n+1}$, the surjections $\mathfrak{m}_B^n/\mathfrak{m}_B^{n+1} \to \mathfrak{m}_A^n/\mathfrak{m}_A^{n+1}$ are isomorphisms and it follows that f is an isomorphism. □

6.7.3 Proof of the Local Structure Theorem—smooth case

In the case that $x \in |\mathcal{X}|$ is a smooth point with linearly reductive stabilizer, we will show that there are étale morphisms

$$[T_{\mathcal{X},x}/G_x] \leftarrow [\operatorname{Spec} A/G_x] \to \mathcal{X}.$$

The proof of the general case (which does not rely on the smooth case) is technically much more involved than the smooth case but nevertheless follows the same logic. For this reason, we recommend the reader to first digest the proof in the smooth case.

Proof of Theorem 6.7.1—smooth case. Since the $\Bbbk$-point $x \in |\mathcal{X}|$ is locally closed (Proposition 3.5.16), by replacing $\mathcal{X}$ by an open substack we may assume that $x \in |\mathcal{X}|$ is a closed point. Let $\mathcal{I}$ be the coherent sheaf of ideals defining $\mathcal{X}_0 := BG_x \hookrightarrow \mathcal{X}$ and set $\mathcal{X}_n$ to be the nth nilpotent thickening defined by $\mathcal{I}^{n+1}$. The Zariski tangent space $T_{\mathcal{X},x}$ can be identified with the normal space $(\mathcal{I}/\mathcal{I}^2)^\vee$ to the orbit, viewed as a G_x-representation. (Note that when $\mathcal{X} = [X/G]$ with G a smooth affine algebraic group, then $T_{\mathcal{X},x}$ is identified with the normal space to the orbit $T_{X,\widetilde{x}}/T_{Gx,\widetilde{x}}$ for a point $\widetilde{x} \in X(\Bbbk)$ over x.)

Define the quotient stack $\mathcal{T} := [T_{\mathcal{X},x}/G_x]$. Let $\mathcal{T}_0 = BG_x$ be the closed substack of $\mathcal{T}$ supported at the origin and $\mathcal{T}_n$ its nth nilpotent thickenings. We claim that there are compatible isomorphisms $\mathcal{X}_n \cong \mathcal{T}_n$. Since G_x is linearly reductive, $\mathcal{X}_0 = BG_x$ is cohomologically affine. By the deformation theory of principal G_x-bundles (Exercise 6.7.14), we can inductively extending the principal G_x-bundle $\operatorname{Spec} \Bbbk \to \mathcal{X}_0$ to principal G_x-bundles $\operatorname{Spec} A_n \to \mathcal{X}_n$. This yields isomorphisms $\mathcal{X}_n \cong [\operatorname{Spec} A_n/G_x]$ and affine morphisms $\mathcal{X}_n \to BG_x$. We have a closed immersion $\mathcal{X}_0 \hookrightarrow \mathcal{T}$ and we can inductively find lifts

$$\begin{tikzcd}
\mathcal{X}_n \arrow[r] \arrow[d, hook] & \mathcal{T} \arrow[d] \\
\mathcal{X}_{n+1} \arrow[r] \arrow[ur, dashed] & BG_x
\end{tikzcd}$$

since $\mathcal{T} \to BG_x$ is smooth and affine (Proposition 6.7.12). The induced morphism $\mathcal{X}_1 \to \mathcal{T}_1$ is an isomorphism

Since $\mathcal{X}_1 \to \mathcal{T}_1$ is a morphism between deformations of BG_x by the coherent sheaf $\mathcal{I}/\mathcal{I}^2$, it is necessarily an isomorphism (see Lemma C.1.7). (In fact, both $\mathcal{X}_1$ and $\mathcal{T}_1$ are trivial deformations as they admit retractions to BG_x.) Lemma 6.7.16(2) now implies that the maps $\mathcal{X}_n \to \mathcal{T}_n$ are isomorphisms.

Let $\pi\colon \mathcal{T} \to T = T_{\mathcal{X},x}/\!\!/G_x$ be the morphism to the GIT quotient. The fiber product $\widehat{\mathcal{T}} := \operatorname{Spec} \widehat{\mathcal{O}}_{T,\pi(0)} \times_T \mathcal{T}$ is a quotient stack of the form $[\operatorname{Spec} B/G]$ where B is of finite type over the complete noetherian local $\Bbbk$-algebra $B^G = \widehat{\mathcal{O}}_{T,\pi(0)}$. Therefore $\widehat{\mathcal{T}}$ is coherently complete along $\mathcal{T}_0$ (Theorem 6.6.13) and $\operatorname{Mor}(\mathcal{T}, \mathcal{X}) \xrightarrow{\sim} \varprojlim \operatorname{Mor}(\mathcal{T}_n, \mathcal{X})$ is an equivalence by Coherent Tannaka Duality (6.6.9). It follows that the morphisms $\mathcal{X}_n \cong \mathcal{T}_n \hookrightarrow \mathcal{X}$ extend to a morphism $\widehat{\mathcal{T}} \to \mathcal{X}$ filling in the diagram

$$\begin{tikzcd}
\mathcal{X}_n \cong \mathcal{T}_n \arrow[r] \arrow[rrr, bend left] & \widehat{\mathcal{T}} \arrow[r] \arrow[d] \arrow[rr, dashed, bend left] \arrow[dr, phantom, "\square"] & \mathcal{T} \arrow[d] & \mathcal{X} \\
& \operatorname{Spec} \widehat{\mathcal{O}}_{T,\pi(0)} \arrow[r] & T. &
\end{tikzcd}$$

The functor parameterizing isomorphism classes of morphisms

$$F\colon \operatorname{Sch}/T \to \operatorname{Sets}, \qquad (T' \to T) \mapsto \{T' \times_T \mathcal{T} \to \mathcal{X}\}/\sim$$

is limit preserving as $\mathcal{X}$ is of finite type over $\Bbbk$ (see Exercise 3.3.33). The morphism $\widehat{\mathcal{T}} \to \mathcal{X}$ yields an element of F over $\operatorname{Spec} \widehat{\mathcal{O}}_{T,\pi(0)}$. By Artin Approximation (B.5.18), there exists an étale morphism $(U, u) \to (T, 0)$ where U is an affine scheme with a $\Bbbk$-point $u \in U$ and a morphism $(U \times_T \mathcal{T}, (u, 0)) \to (\mathcal{X}, x)$ agreeing with $(\widehat{\mathcal{T}}, 0) \to (\mathcal{X}, x)$ to first order. Observe that $U \times_T \mathcal{T} \cong [\operatorname{Spec} A/G_x]$ for a finitely generated $\Bbbk$-algebra A such that $U = \operatorname{Spec} A^{G_x}$. Since $U \times_T \mathcal{T}$ is smooth at $(u, 0)$ and $\mathcal{X}$ is smooth at x, and since $U \times_T \mathcal{T} \to \mathcal{X}$ induces an isomorphism of tangent spaces and stabilizer groups at $(u, 0)$, the morphism $U \times_T \mathcal{T} \to \mathcal{X}$ is étale at $(u, 0)$. After replacing U with an affine open subscheme, we can arrange that $U \times_T \mathcal{T} \to \mathcal{X}$ is affine (by Proposition 6.7.17) and étale. $\square$

Proposition 6.7.17. *Let $\mathcal{X}$ be an algebraic stack of finite type over an algebraically closed field with affine diagonal. Let $f\colon [\operatorname{Spec} A/G] \to \mathcal{X}$ be a finite type morphism* *with G linearly reductive. If $w \in \operatorname{Spec} A$ has closed G-orbit and f induces an isomorphism of stabilizer groups at w, then there exists a G-invariant affine open subscheme $U \subseteq \operatorname{Spec} A$ containing w such that $f|_{[U/G]}$ is affine.*

Proof. Set $\mathcal{W} = [\operatorname{Spec} A/G]$ with $\pi\colon \mathcal{W} \to \operatorname{Spec} A^G$. Since $f\colon \mathcal{W} \to \mathcal{X}$ induces an isomorphism of stabilizer groups at w, after replacing $\operatorname{Spec} A^G$ with an affine open containing the image w, we can use upper semicontinuity to arrange that the inertia stack $I_{\mathcal{W}/\mathcal{X}} \to \mathcal{W}$ is quasi-finite. Choose a smooth presentation $V = \operatorname{Spec} B \to \mathcal{X}$ and consider the fiber product

$$\begin{tikzcd} & \mathcal{W}_V \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & V = \operatorname{Spec} B \arrow[d] \\ {[\operatorname{Spec} A/G]} = & \mathcal{W} \arrow[r, "f"] & \mathcal{X}. \end{tikzcd}$$

Since $\mathcal{X}$ has affine diagonal, $\operatorname{Spec} B \to \mathcal{X}$ is affine and therefore $\mathcal{W}_V$ is cohomologically affine. As $\mathcal{W}_V$ has quasi-finite diagonal, Proposition 6.5.33 implies that $\mathcal{W}_V \to V$ is separated, and it follows from descent that $\mathcal{W} \to \mathcal{X}$ is also separated. Therefore, the relative inertia $I_{\mathcal{W}/\mathcal{X}} \to \mathcal{W}$ is both proper and quasi-finite, hence finite. Since the fiber over $w \in |\mathcal{W}|$ is trivial, there is an open neighborhood $\mathcal{U}$ over which the relative inertia is trivial. Therefore, after replacing $\operatorname{Spec} A^G$ with an affine open, we can arrange that f is representable. Since f is also cohomologically affine, Serre's Criterion for Affineness (6.5.18) implies that f is affine. □

6.7.4 Equivariant Artin Algebraization

The smoothness hypothesis of $x \in |\mathcal{X}|$ was used above to establish that $\mathcal{T}_n \cong \mathcal{X}_n$ and that $U \times_T \mathcal{T} \to \mathcal{X}$ is étale. More critically, it implied that $\varprojlim \Gamma(\mathcal{X}_n, \mathcal{O}_{\mathcal{X}_n})$, which is identified with the G_x-invariants of a miniversal deformation space, is the completion of a finitely generated $\Bbbk$-algebra, namely $\widehat{\mathcal{O}}_{T,0}$. If $x \in |\mathcal{X}|$ is not smooth, it seems difficult to directly establish that $\varprojlim \Gamma(\mathcal{X}_n, \mathcal{O}_{\mathcal{X}_n})$ is the completion of a finitely generated $\Bbbk$-algebra. Recall that we encountered a similar issue when discussing Artin Algebraization (C.6.8): when the complete local ring R is known to be the completion of a finitely generated algebra, then Artin Algebraization is an easy consequence of Artin Approximation (see Remark C.6.9). To circumvent this issue in our general proof of Artin Algebraization, we wrote $R = \widehat{\mathcal{O}}_{V,v}/I$ where V is a finite type $\Bbbk$-scheme (e.g., $V = \mathbb{A}^n_{\Bbbk}$ and $v = 0$) and used Artin Approximation to simultaneously approximate both the given object over R and the equations defining I. We follow a similar strategy but proceed G-equivariantly.

We will use the following extension of the notion of formal versality introduced in Definition C.4.2: for an algebraic stack $\widehat{\mathcal{T}}$ with a unique closed point t, a morphism $\widehat{\xi}\colon \widehat{\mathcal{T}} \to \mathcal{X}$ of prestacks over Sch is *formally versal at t* if every commutative diagram

$$\begin{tikzcd} \mathcal{Z} \arrow[r] \arrow[d, hook] & \widehat{\mathcal{T}} \arrow[d, "\widehat{\xi}"] \\ \mathcal{Z}' \arrow[r] \arrow[ur, dashed] & \mathcal{X} \end{tikzcd}$$

has a lift, where $\mathcal{Z} \hookrightarrow \mathcal{Z}'$ is a closed immersion of noetherian algebraic stacks with affine diagonal, $|\mathcal{Z}| = |\mathcal{Z}'|$ consists of a single point, and the image of $\mathcal{Z} \to \widehat{\mathcal{T}}$ is t.

Theorem 6.7.18 (Equivariant Artin Algebraization)**.** *Let $\Bbbk$ be an algebraically closed field and R be a complete noetherian local $\Bbbk$-algebra with residue field $\Bbbk$. Let $\widehat{\mathcal{T}} = [\operatorname{Spec} B/G]$ be an algebraic stack of finite type over $R = B^G$, where G is linearly reductive. Assume that the unique closed point $t \in |\widehat{\mathcal{T}}|$ has stabilizer equal to G. If $\mathcal{X}$ is a limit preserving prestack over $\mathrm{Sch}/\Bbbk$ and $\eta\colon \widehat{\mathcal{T}} \to \mathcal{X}$ is a morphism of prestacks formally versal at t, then there exists*

(1) an algebraic stack $\mathcal{W} = [\operatorname{Spec} A/G]$ of finite type over $\Bbbk$ and a closed point $w \in |\mathcal{W}|$;

(2) morphisms $f\colon \mathcal{W} \to \mathcal{X}$ and $\varphi\colon \widehat{\mathcal{T}} \to \mathcal{W}$ such that in the diagram

$$\begin{tikzcd} \widehat{\mathcal{T}} \arrow[d, "\varphi"'] \arrow[dr, "\eta"] & \\ \mathcal{W} \arrow[r, "f"] & \mathcal{X} \end{tikzcd} \tag{6.7.19}$$

the induced morphisms $\varphi_n\colon \widehat{\mathcal{T}}_n \to \mathcal{W}_n$ between the nth nilpotent thickenings of t and w are isomorphisms, and there exists compatible 2-isomorphisms $\eta_n \xrightarrow{\sim} f_n \circ \varphi_n$.

Moreover, if $\mathcal{X}$ is an algebraic stack of finite type over $\Bbbk$ with affine diagonal, then it can be arranged that (6.7.19) is commutative and that φ induces an isomorphism $\widehat{\mathcal{T}} \to \widehat{\mathcal{W}} := \mathcal{W} \times_W \operatorname{Spec} \widehat{\mathcal{O}}_{W,\pi(w)}$, where $\pi\colon \mathcal{W} \to W = \operatorname{Spec} A^G$.

Remark 6.7.20. If one takes G to be the trivial group, one recovers the classical version of Artin Algebraization (C.6.8).

As in the proof of Artin Algebraization, we will apply Artin Approximation to a well-chosen integer N to construct $\mathcal{W}$ such that there are isomorphisms $\mathcal{W}_n \cong \widehat{\mathcal{T}}_n$ for $n \leq N$ and such that the Artin–Rees Lemma (B.5.4) implies that there are also isomorphisms $\mathcal{W}_n \cong \widehat{\mathcal{T}}_n$ for $n > N$. To get control over the constant in the Artin–Rees Lemma, we need to generalize Definition C.6.3: for a noetherian algebraic stack $\mathcal{X}$ with a closed point x defined by a a sheaf of ideals $\mathfrak{m}_x$ and an integer $c \geq 0$, we say that $(\mathrm{AR})_c$ *holds at* x for a map $\varphi\colon E \to F$ of coherent sheaves on $\mathcal{X}$ if

$$\varphi(E) \cap \mathfrak{m}_x^n F \subseteq \varphi(\mathfrak{m}_x^{n-c} E), \quad \forall n \geq c.$$

When $\mathcal{X}$ is a scheme, $(\mathrm{AR})_c$ holds for all sufficiently large c by the Artin–Rees Lemma, even if $\{x\}$ is replaced with an arbitrary closed subscheme. By smooth descent, $(\mathrm{AR})_c$ also holds for algebraic stacks for $c \gg 0$.

Proof. The morphism $\eta\colon \widehat{\mathcal{T}} \to \mathcal{X}$ and $\widehat{\mathcal{T}} \to BG$ induce a morphism $\widehat{\mathcal{T}} \to \mathcal{X} \times BG$. Let $\widehat{T} = \operatorname{Spec} R$ be the GIT quotient of $\widehat{\mathcal{T}} = [\operatorname{Spec} B/G]$ with $R = B^G$. Since R is the colimit of its finitely generated $\Bbbk$-subalgebras and $\mathcal{X} \times BG$ is limit preserving, limit methods (§B.3) imply that there is a commutative diagram

$$\begin{tikzcd} \widehat{\mathcal{T}} \arrow[r] \arrow[rr, bend left] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{S} \arrow[r] \arrow[d] & \mathcal{X} \times BG \arrow[d] \\ \widehat{T} \arrow[r] & S \arrow[r] & \operatorname{Spec} \Bbbk \end{tikzcd}$$

where $S = \operatorname{Spec} R'$ is an affine scheme of finite type over $\Bbbk$ and $\mathcal{S}$ is an algebraic stack of finite type over S with affine diagonal such that $\widehat{\mathcal{T}} = \widehat{T} \times_S \mathcal{S}$. Moreover, we can arrange that $\mathcal{S} \to BG$ is affine. Let $\widetilde{s} \in |\mathcal{S}|$ and $s \in S$ be the images of t. After possibly adding generators to R', we can arrange that $R' \to R \to R/\mathfrak{m}_R^2$ is surjective. By Complete Nakayama's Lemma (B.5.6(3)), $\widehat{\mathcal{O}}_{S,s} \to R$ is surjective. In particular, both $\widehat{T} \hookrightarrow \widehat{S} := \operatorname{Spec} \widehat{\mathcal{O}}_{S,s}$ and $\widehat{\mathcal{T}} \hookrightarrow \mathcal{S} \times_S \widehat{S}$ are closed immersions.

By choosing a resolution $\mathcal{O}_{\widehat{S}}^{\oplus r} \to \mathcal{O}_{\widehat{S}} \twoheadrightarrow R$ and pulling it back to $\mathcal{S} \times_S \widehat{S}$, we obtain a resolution

$$\ker(\beta) \xhookrightarrow{\alpha} \mathcal{O}_{\mathcal{S}\times_S \widehat{S}}^{\oplus r} \xrightarrow{\beta} \mathcal{O}_{\mathcal{S}\times_S \widehat{S}} \twoheadrightarrow \mathcal{O}_{\widehat{\mathcal{T}}}. \tag{6.7.21}$$

Consider the functor $F\colon \mathrm{Sch}/S \to \mathrm{Sets}$ assigning an S-scheme U to the set of isomorphism classes of complexes

$$L \xrightarrow{\alpha} \mathcal{O}_{\mathcal{S}\times_S U}^{\oplus r} \xrightarrow{\beta} \mathcal{O}_{\mathcal{S}\times_S U}$$

of finitely presented, quasi-coherent $\mathcal{O}_{\mathcal{S}\times_S U}$-modules. By limit arguments, F is limit preserving. The complex (6.7.21) defines an element $(\alpha, \beta) \in F(\widehat{S})$ such that $\operatorname{coker}(\beta) = \mathcal{O}_{\widehat{\mathcal{T}}}$. Let N be an integer such that $(\mathrm{AR})_N$ holds for α and β at $(\widetilde{s}, s)$. Artin Approximation (B.5.18) gives an étale neighborhood $(S', s') \to (S, s)$ and an element $(\alpha', \beta') \in F(S')$ such that $(\alpha, \beta) = (\alpha', \beta')$ in $F(\mathcal{O}_{S,s}/\mathfrak{m}_s^{N+1})$. We let $\mathcal{W} \hookrightarrow \mathcal{S} \times_S S'$ be the closed substack defined by $\operatorname{coker}(\beta')$ and set $w = (\widetilde{s}, s') \in |\mathcal{W}|$. Letting S_n, S'_n and $\widehat{T}_n$ be the nth nilpotent thickenings of S, S' and $\widehat{T}$ at the images of $t \in |\mathcal{T}|$, we have that $\widehat{\mathcal{T}} \times_{\widehat{T}} \widehat{T}_N$ and $\mathcal{W} \times_{S'} S'_N$ are equal as closed substacks of $\mathcal{S} \times_S S_N$. This gives (1)–(2) for $n \leq N$. In particular, we have an isomorphism $\varphi_N\colon \widehat{\mathcal{T}}_N \to \mathcal{W}_N$ and we let $\psi_N\colon \mathcal{W}_N \to \widehat{\mathcal{T}}_N$ be its inverse.

Using that $\eta\colon \widehat{\mathcal{T}} \to \mathcal{X}$ is formally versal, we can inductively find compatible lifts for $n \geq N$

$$\begin{tikzcd}
\mathcal{W}_n \arrow[r, "\psi_n"] \arrow[d, hook] & \widehat{\mathcal{T}} \arrow[d, "\eta"] \\
\mathcal{W}_{n+1} \arrow[r] \arrow[ur, dashed, "\psi_{n+1}"] & \mathcal{X}.
\end{tikzcd}$$

On the other hand, applying Lemma C.6.4 (generalized to stacks by smooth descent) on $\mathcal{S} \times_S \widehat{S}$ with $c = N$ to the complex (6.7.21) and the restriction of the complex defined by (α', β'), we obtain an isomorphism $\operatorname{Gr}_{\mathfrak{m}_t} \mathcal{O}_{\widehat{\mathcal{T}}} \cong \operatorname{Gr}_{\mathfrak{m}_w} \mathcal{O}_{\mathcal{W}}$ of graded $\mathcal{O}_{BG}$-modules. By Lemma 6.7.16, the induced morphisms $\psi_n\colon \mathcal{W}_n \to \widehat{\mathcal{T}}_n$ are isomorphisms for all n. As $\widehat{\mathcal{T}}$ is coherently complete (Theorem 6.6.13), Coherent Tannaka Duality (6.6.9) implies that the inverses $\varphi_n = \psi_n^{-1}\colon \widehat{\mathcal{T}}_n \to \mathcal{W}_n$ effectivize to a morphism $\varphi\colon \widehat{\mathcal{T}} \to \mathcal{W}$. This completes (1)–(2).

For the final statement, when $\mathcal{X}$ is algebraic, we again apply Coherent Tannaka Duality, using the coherent completeness of both $\widehat{\mathcal{T}}$ and $\mathcal{W}$. By applying Corollary 6.6.9 to the inverses $\psi_n = \varphi_n^{-1}$, we can construct an inverse $\psi\colon \widehat{\mathcal{W}} \to \widehat{\mathcal{T}}$ of φ. Thus $\varphi\colon \widehat{\mathcal{T}} \to \widehat{\mathcal{W}}$ is an isomorphism. Using the fully faithfulness of Corollary 6.6.9, there is a 2-isomorphism $\eta \to f \circ \varphi$ extending the given 2-isomorphisms $\eta_n \xrightarrow{\sim} f_n \circ \varphi_n$ and thus $\widehat{\mathcal{T}} \to \widehat{\mathcal{W}}$ is a morphism over $\mathcal{X}$. $\square$

6.7.5 Proof of the Local Structure Theorem—general case

Proof of Theorem 6.7.1. We may assume that $x \in |\mathcal{X}|$ is a closed point. Let $\mathcal{T} := [T_{\mathcal{X},x}/G_x]$, let $\pi\colon \mathcal{T} \to T = T_{\mathcal{X},x}/\!\!/G_x$ be the morphism to the GIT quotient, and let $\widehat{\mathcal{T}} := \operatorname{Spec} \widehat{\mathcal{O}}_{T,\pi(0)} \times_T \mathcal{T}$. Let $\mathcal{T}_0 = BG_x$ be the closed substack supported at the origin and $\mathcal{T}_n$ its nth nilpotent thickenings. We will construct compatible closed immersions $\mathcal{X}_n \hookrightarrow \mathcal{T}_n$. Since G_x is linearly reductive, $\mathcal{X}_0 = BG_x$ is cohomologically affine. By deforming the principal G_x-bundle $\operatorname{Spec} \Bbbk \to \mathcal{X}_0$ using Exercise 6.7.14, we can inductively construct isomorphisms $\mathcal{X}_n \cong [\operatorname{Spec} A_n/G_x]$. By the deformation theory of the smooth and affine morphism $\widehat{\mathcal{T}} \to BG_x$ (Proposition 6.7.12), we can inductively find lifts

$$\begin{tikzcd}
\mathcal{X}_n \arrow[r] \arrow[d, hook] & \mathcal{T} \arrow[d] \\
\mathcal{X}_{n+1} \arrow[r] \arrow[ur, dashed] & BG_x.
\end{tikzcd}$$

As in the smooth case, $\mathcal{X}_1 \to \mathcal{T}_1$ is an isomorphism. By Lemma 6.7.16(1), each morphism $\mathcal{X}_n \to \mathcal{T}_n$ is a closed immersion.

If I_n denotes the ideal sheaf defining $\mathcal{X}_n \hookrightarrow \mathcal{T}_n$, then $\mathcal{O}_{\mathcal{T}_n}/I_n$ is a system of coherent $\mathcal{O}_{\mathcal{T}_n}$-modules. Since $\widehat{\mathcal{T}}$ is coherently complete (Theorem 6.6.13), there exists a coherent sheaf of ideals $I \subseteq \mathcal{O}_{\widehat{\mathcal{T}}}$ such that the surjection $\mathcal{O}_{\widehat{\mathcal{T}}} \to \mathcal{O}_{\widehat{\mathcal{T}}}/I$ extends the surjections $\mathcal{O}_{\mathcal{T}_n} \to \mathcal{O}_{\mathcal{X}_n}$. The closed immersion $\widehat{\mathcal{X}} \hookrightarrow \widehat{\mathcal{T}}$ defined by I extends the given closed immersions $\mathcal{X}_n \hookrightarrow \mathcal{T}_n$ yielding a commutative diagram

$$\begin{tikzcd}
\mathcal{X}_n \arrow[r, hook] \arrow[d, hook] \arrow[rr, bend left] & \widehat{\mathcal{X}} \arrow[r, dashed, "\eta"] \arrow[d, hook] & \mathcal{X} \\
\mathcal{T}_n \arrow[r, hook] & \widehat{\mathcal{T}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{T} \arrow[d] \\
& \operatorname{Spec} \widehat{\mathcal{O}}_{T,0} \arrow[r] & T
\end{tikzcd}$$

of solid arrows. Since $\widehat{\mathcal{X}}$ is also coherently complete, Coherent Tannaka Duality (6.6.9) gives a morphism $\eta\colon \widehat{\mathcal{X}} \to \mathcal{X}$ extending the above diagram. Since $\widehat{\mathcal{X}}$ has the same nilpotent thickenings of $\widehat{\mathcal{X}}$, the morphism $\eta\colon \widehat{\mathcal{X}} \to \mathcal{X}$ is formally versal at 0. By Equivariant Artin Algebraization (6.7.18) with $G = G_x$, we obtain a morphism $f\colon \mathcal{W} = [\operatorname{Spec} B/G_x] \to \mathcal{X}$ from an algebraic stack $\mathcal{W}$ of finite type over $\Bbbk$ with a closed point $w \in |\mathcal{W}|$ and a morphism $\varphi\colon \widehat{\mathcal{X}} \to \mathcal{W}$ over $\mathcal{X}$ inducing an isomorphism $\widehat{\mathcal{X}} \to \mathcal{W} \times_W \operatorname{Spec} \widehat{\mathcal{O}}_{W,\pi(w)}$ where $\pi\colon \mathcal{W} \to \operatorname{Spec} B^{G_x}$. Since $f\colon \mathcal{W} \to \mathcal{X}$ induces isomorphisms $\mathcal{W}_n \to \mathcal{X}_n$, f is étale at w. After replacing $\mathcal{W}$ with an open substack, we can arrange that f is étale everywhere. By Proposition 6.7.17, we can also arrange that f is affine. See also [AHR20, AHR19, AHHLR24]. $\square$

6.7.6 The coherent completion at a point

We say that $(\mathcal{X}, x)$ is a *complete local stack* if $\mathcal{X}$ is a noetherian algebraic stack with affine stabilizers and with a unique closed point x such that $\mathcal{X}$ is coherently complete along the residual gerbe $\mathcal{G}_x$. Important examples are the quotient stack $([\operatorname{Spec} A/G], x)$ of Theorem 6.6.13, e.g., $([\mathbb{A}^n/\mathbb{G}_m], 0)$ is complete local. The *coherent completion* of a noetherian algebraic stack $\mathcal{X}$ at a point x is a complete local stack $(\widehat{\mathcal{X}}_x, \widehat{x})$ together with a morphism $\eta\colon (\widehat{\mathcal{X}}_x, \widehat{x}) \to (\mathcal{X}, x)$ inducing isomorphisms of nth infinitesimal neighborhoods of $\widehat{x}$ and x. If $\mathcal{X}$ has affine diagonal, then the pair $(\widehat{\mathcal{X}}_x, \eta)$ is unique up to unique 2-isomorphism by Coherent Tannaka Duality (6.6.9).

Theorem 6.7.22. *Let $\mathcal{X}$ be an algebraic stack of finite type over an algebraically closed field $\Bbbk$ with affine diagonal. For every point $x \in \mathcal{X}(\Bbbk)$ with linearly reductive stabilizer G_x, the coherent completion $\widehat{\mathcal{X}}_x$ exists. Moreover,*

(1) The coherent completion is a quotient stack $\widehat{\mathcal{X}}_x = [\operatorname{Spec} B/G_x]$ such that the invariant ring B^{G_x} is the completion of a finite type $\Bbbk$-algebra and $B^{G_x} \to B$ is of finite type.

(2) If $f\colon (\mathcal{W}, w) \to (\mathcal{X}, x)$ is an étale morphism where $\mathcal{W} = [\operatorname{Spec} A/G_x]$, the point $w \in |\mathcal{W}|$ is closed, and f induces an isomorphism of stabilizer groups at w, then $\widehat{\mathcal{X}}_x \cong \mathcal{W} \times_W \operatorname{Spec} \widehat{\mathcal{O}}_{W,\pi(w)}$, where $\pi\colon \mathcal{W} \to W = \operatorname{Spec} A^{G_x}$ is the morphism to the GIT quotient.

(3) If $\pi\colon \mathcal{X} \to X$ is a good moduli space, then $\widehat{\mathcal{X}}_x = \mathcal{X} \times_X \operatorname{Spec} \widehat{\mathcal{O}}_{X,\pi(x)}$.

Proof. The Local Structure Theorem (6.7.1) gives an étale morphism $f\colon (\mathcal{W}, w) \to (\mathcal{X}, x)$, where $\mathcal{W} = [\operatorname{Spec} A/G_x]$ and f induces an isomorphism of stabilizer groups at the closed point w. The main statement as well as (1) and (2) follow by taking $\widehat{\mathcal{X}}_x = \mathcal{W} \times_W \operatorname{Spec} \widehat{\mathcal{O}}_{W,\pi(w)}$ and $B = A \otimes_{A^{G_x}} \widehat{A^{G_x}}$ because $\widehat{\mathcal{X}}_x = [\operatorname{Spec} B/G_x]$ is coherently complete (Theorem 6.6.13). Part (3) follows from (2) using Corollary 6.7.3. $\square$

The fact that completions determine the étale local structure of finite type schemes (Corollary B.5.21) generalizes to stacks.

Theorem 6.7.23. *Let $\mathcal{X}$ and $\mathcal{Y}$ be algebraic stacks of finite type over an algebraically closed field $\Bbbk$ with affine diagonal. Suppose $x \in |\mathcal{X}|$ and $y \in |\mathcal{Y}|$ are $\Bbbk$-points with linearly reductive stabilizer group schemes G_x and G_y, respectively. Then the following are equivalent:*

(1) There exist compatible isomorphisms $\mathcal{X}_n \to \mathcal{Y}_n$.

(2) There exists an isomorphism $\widehat{\mathcal{X}}_x \to \widehat{\mathcal{Y}}_y$.

(3) There exist an affine scheme $\operatorname{Spec} A$ with an action of G_x, a point $w \in \operatorname{Spec} A$ fixed by G_x, and a diagram of étale morphisms

$$\begin{tikzcd} & {[\operatorname{Spec} A/G_x]} \arrow[dl, "f"'] \arrow[dr, "g"] & \\ \mathcal{X} & & \mathcal{Y} \end{tikzcd}$$

such that $f(w) = x$ and $g(w) = y$, and both f and g induce isomorphisms of stabilizer groups at w.

If, in addition, the points $x \in |\mathcal{X}|$ and $y \in |\mathcal{Y}|$ are smooth, then the conditions above are equivalent to the existence of an isomorphism $G_x \to G_y$ of group schemes and an isomorphism $T_{\mathcal{X},x} \to T_{\mathcal{Y},y}$ of tangent spaces which is equivariant under $G_x \to G_y$.

Proof. The implications (3)⇒(2)⇒(1) are immediate. We also have (1)⇒(2) by Coherent Tannaka Duality (6.6.9) To show that (2)⇒(3), let $(\mathcal{W} = [\operatorname{Spec} A/G_x], w) \to (\mathcal{X}, x)$ be an étale neighborhood as given by the Local Structure Theorem (6.7.1). Let $\pi\colon \mathcal{W} \to W = \operatorname{Spec} A^{G_x}$ denote the good moduli space. Then $\widehat{\mathcal{X}}_x \cong \mathcal{W} \times_W \operatorname{Spec} \widehat{\mathcal{O}}_{W,\pi(w)}$. The functor

$$F\colon \mathrm{Sch}/W \to \mathrm{Sets}, \quad (T \to W) \mapsto \operatorname{Mor}_{\Bbbk}(\mathcal{W} \times_W T, \mathcal{Y})$$

is locally of finite presentation. Artin Approximation (B.5.18) applied to F and $\alpha \in F(\operatorname{Spec} \widehat{\mathcal{O}}_{W,\pi(w)})$ provides an étale morphism $(W', w') \to (W, w)$ and a morphism

$\varphi\colon \mathcal{W}' := \mathcal{W} \times_W W' \to \mathcal{Y}$ such that $\varphi|_{\mathcal{W}'_1}\colon \mathcal{W}'_1 \to \mathcal{Y}_1$ is an isomorphism. Since $\widehat{\mathcal{W}'}_{w'} \cong \widehat{\mathcal{X}}_x \cong \widehat{\mathcal{Y}}_y$, it follows that φ induces an isomorphism $\widehat{\mathcal{W}'} \to \widehat{\mathcal{Y}}$ by Lemma 6.7.16. After replacing W' with an open neighborhood, we obtain an étale morphism $(\mathcal{W}', w') \to (\mathcal{Y}, y)$. The final statement follows from Luna's Etale Slice Theorem (6.7.5). □

6.7.7 Applications to equivariant geometry

Sumihiro's Theorem on Torus Action (B.1.31) asserts that for a normal scheme of finite type over $\Bbbk$ with the action of a torus T, every $\Bbbk$-point has a T-invariant affine open neighborhood. If X is not normal, there are not necessarily T-invariant affine open neighborhoods, e.g., the node in the projective nodal cubic C with its $\mathbb{G}_m$-action. However, there is always a T-equivariant affine *étale* neighborhood, e.g., there is a $\mathbb{G}_m$-equivariant étale neighborhood $\operatorname{Spec} \Bbbk[x,y]/(xy) \to C$ of the node.

Theorem 6.7.24. *Let X be an algebraic space locally of finite type over an algebraically closed field $\Bbbk$ with affine diagonal. Suppose that X has an action of an affine algebraic group G. If $x \in X(\Bbbk)$ has linearly reductive stabilizer, then there exists a G-equivariant étale neighborhood $(\operatorname{Spec} A, u) \to (X, x)$ inducing an isomorphism of stabilizer groups at u.*

If G is a torus, then every point $x \in X(\Bbbk)$ has a G-invariant étale neighborhood $(\operatorname{Spec} A, u) \to (X, x)$ inducing an isomorphism of stabilizer groups at u.

Proof. By the Local Structure Theorem (6.7.1), there is an étale neighborhood $([\operatorname{Spec} A/G_x], u) \to ([X/G], x)$ such that u is a closed point and f induces an isomorphism of stabilizer groups at u. By Proposition 6.7.17, after replacing $\operatorname{Spec} A$ with a G_x-invariant open affine neighborhood of u, we can arrange that the composition $[\operatorname{Spec} A/G_x] \to [X/G] \to BG$ is affine. Therefore, $W := [\operatorname{Spec} A/G_x] \times_{[X/G]} X$ is an affine scheme and $W \to X$ is a G-equivariant étale neighborhood of x. If G is a torus, then any subgroup of G is linearly reductive. □

6.8 $\mathbb{G}_m$-actions and the Białynicki-Birula Stratification

We show that the fixed locus of a linearly reductive group action on a smooth variety is smooth (Theorem 6.8.2). We also prove the representability and properties of the attractor locus with respect to a $\mathbb{G}_m$-action (Theorem 6.8.9), which will be useful in understanding the valuative criteria of Θ- and S-completeness in the next section. Finally, after establishing a general version of the Białynicki-Birula Stratification (Theorem 6.8.14), we discuss applications to computing cohomology (§6.8.20).

6.8.1 Fixed loci

Definition 6.8.1 (Fixed locus)**.** If X is an algebraic space over a field $\Bbbk$ equipped with an action of an affine algebraic group G, we define the *fixed locus* as the functor

$$X^G := \underline{\operatorname{Mor}}^G_{\Bbbk}(\operatorname{Spec} \Bbbk, X)\colon \mathrm{Sch}/\Bbbk \to \mathrm{Sets}$$

assigning a $\Bbbk$-scheme S to the set $\operatorname{Mor}^G_{\Bbbk}(S, X)$ of G-equivariant maps $S \to X$, where S is endowed with the trivial G action.

Theorem 6.8.2. *Let X be a separated algebraic space of finite type over an algebraically closed field $\Bbbk$ with affine diagonal equipped with an action of a linearly reductive group G. Then the fixed locus X^G is representable by a closed algebraic subspace of X. If X is smooth, so is X^G.*

Proof. If G is connected and $U \to X$ is a G-invariant étale morphism, we claim that

$$\begin{tikzcd} U^G \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & U \arrow[d] \\ X^G \arrow[r, hook] & X \end{tikzcd} \tag{6.8.3}$$

is cartesian. Indeed, suppose $S \to U$ is a map such that $S \to U \to X$ is G-invariant. Let $U_S \to S$ be the base change of $U \to X$ by $S \to X$. Since $U_S \to S$ is G-invariant, it suffices to show that the section $j \colon S \to U_S$ is G-invariant. As $U \to X$ is étale, $j \colon S \to U_S$ is an open immersion. Because G is connected, for each point $s \in S$, the G-orbit $Gj(s) \subseteq U_S$ is connected and thus contained in S.

Given a fixed point $x \in X^G(\Bbbk)$, Theorem 6.7.24 produces a G-invariant étale neighborhood $(U,u) \to (X,x)$ with U affine and $u \in U^G(\Bbbk)$. If G is connected, then $U^G \to X^G$ is étale and representable by (6.8.3), and it suffices to show that U^G is representable by a closed subscheme of U. Since U is affine, we can choose a G-equivariant embedding $U \hookrightarrow \mathbb{A}(V)$ into a finite dimensional G-representation. In this case, $\mathbb{A}(V)^G = \mathbb{A}(V^G)$ and thus $U^G = U \cap \mathbb{A}(V)^G \subseteq U$ is a closed subscheme is affine. In general, let $G^0 \subseteq G$ be the connected component of the identity, and let $g_1, \ldots, g_n \in G(\Bbbk)$ be representatives of the finitely many cosets $G(\Bbbk)/G^0(\Bbbk)$. Then G/G_0 acts on X^{G_0} and then $X^G = \bigcap_i (X^{G_0})^{g_i}$ is a closed subscheme of X.

If in addition x is a smooth point of X, then $u \in U$ is smooth and the Luna map (6.7.9) gives a a G-invariant étale morphism $U \to T_{U,u}$. Since $T_{U,u}^G$ is a linear subspace, it is smooth. Since $U^G \to X^G$ and $U^G \to T_{U,u}^G$ are étale at u, then étale descent implies that $x \in X^G$ is smooth. See also [Ive72, Prop. 1.3] and [Mil17, Thm. 13.1]. $\square$

6.8.2 Limits under $\mathbb{G}_m$-actions and attractor loci

Definition 6.8.4 (Limits)**.** Given a $\mathbb{G}_m$-action on an algebraic space U over a field $\Bbbk$ and a point $u \in U(\Bbbk)$, we say that the *limit* $\lim_{t\to 0} t \cdot u$ *exists* if there exists an extension of the diagram

$$\begin{tikzcd} \mathbb{G}_m \arrow[r, "t \mapsto t \cdot u"] \arrow[d, hook] & U \\ \mathbb{A}^1. \arrow[ur, dashed] & \end{tikzcd}$$

Valuative Criteria (3.8.7 imply that the limit is unique if X is separated and that there exists a unique limit if X is proper.

Example 6.8.5. If $X = \mathbb{P}(V)$ where V is a finite dimensional representation of G and $\lambda \colon \mathbb{G}_m \to G$ is a one-parameter subgroup, then we can choose a basis of V such that $\lambda(t) \cdot (x_1, \ldots, x_n) = (t^{d_1}x_1, \ldots, t^{d_n}x_n)$ with $d_1 \le \cdots \le d_n$. If $d = \min\{d_i \mid x_i \ne 0\}$, then $\lim_{t\to 0} \lambda(t) \cdot [x_0 : \cdots : x_n] = [x'_0 : \cdots : x'_n]$ where $x'_i = x_i$ for all i such that $d_i = d$ and is 0 otherwise.

Definition 6.8.6 (Attractor locus)**.** Let X be a separated algebraic space of finite type over $\Bbbk$ equipped with an action of $\mathbb{G}_m$. Define the *attractor locus* as the functor

$$X^+ := \underline{\mathrm{Mor}}_{\Bbbk}^{\mathbb{G}_m}(\mathbb{A}^1, X) \colon \mathrm{Sch}/\Bbbk \to \mathrm{Sets}$$

assigning a $\Bbbk$-scheme S to the set $\mathrm{Mor}_{\Bbbk}(S \times \mathbb{A}^1, X)$ of $\mathbb{G}_m$-equivariant maps $S \times \mathbb{A}^1 \to X$, where $\mathbb{G}_m$ acts trivially on S and with the usual scaling action on $\mathbb{A}^1$

Evaluation at 0 and 1 define morphism of functors

$$\mathrm{ev}_0 \colon X^+ \to X^{\mathbb{G}_m} \quad \text{and} \quad \mathrm{ev}_1 \colon X \to X.$$

On $\Bbbk$-points, $X^+(\Bbbk)$ is the set of points $x \in X(\Bbbk)$ such that $\lim_{t \to 0} t \cdot x$ exists, and $\mathrm{ev}_0(x)$ is this limit. Since X is separated, the limit is unique if it exists. If X is proper, the limit always exists and $X^+(\Bbbk) = X(\Bbbk)$. The functorial definition of X^+ endows it with an interesting scheme-structure, e.g., when $\mathbb{G}_m$ acts on $X = \mathbb{P}^1$ via $t \cdot [x : y] = [tx : y]$, then $X^+ = \mathbb{A}^1 \coprod \{\infty\}$.

Exercise 6.8.7 (Affine case)**.** If $X = \operatorname{Spec} A$ is affine, then the $\mathbb{G}_m$-action induces a $\mathbb{Z}$-grading $A = \bigoplus_{d \in \mathbb{Z}} A_d$. Show that the functors $X^{\mathbb{G}_m}$ and X^+ are representable by the closed subschemes of X defined by the ideals $\sum_{d \neq 0} A_d$ and $\sum_{d < 0} A_d$.

Example 6.8.8 (Centralizers and parabolics)**.** Let G be an affine algebraic group over an algebraically closed field $\Bbbk$. A one-parameter subgroup $\lambda \colon \mathbb{G}_m \to G$ induces a $\mathbb{G}_m$-action on G via conjugation $t \cdot g := \lambda(t) g \lambda(t)^{-1}$. Under this action, the fixed locus $G^{\mathbb{G}_m} = C_\lambda$ is identified with the centralizer of λ and the attractor locus $G_\lambda^+ = P_\lambda$ is identified with the subgroup consisting of elements $g \in G$ such that $\lim_{t \to 0} \lambda(t) g \lambda(t)^{-1}$ exists. The unipotent subgroup U_λ is identified with kernel of $\mathrm{ev}_0 \colon P_\lambda \to C_\lambda$. When G is reductive, $P_\lambda \subseteq G$ is a parabolic subgroup or in other words G/P_λ is projective. See §B.1.4 for more properties of these subgroups.

We say that a map $X \to Y$ is an *affine fibration* (resp., *Zariski-local affine fibration*) if there exists an étale (resp., Zariski) cover $\{Y_i \to Y\}$ such that $X \times_Y Y_i \cong \mathbb{A}^n_{Y_i}$ over Y_i. Since the transition functions are not required to be linear, this notion is more general than a vector bundle.

Theorem 6.8.9. *Let X be a separated algebraic space of finite type over an algebraically closed field $\Bbbk$ equipped with an action of $\mathbb{G}_m$. The functor X^+ is representable by an algebraic space of finite type over $\Bbbk$, $\mathrm{ev}_1 \colon X^+ \to X$ is a monomorphism, and $\mathrm{ev}_0 \colon X^+ \to X^{\mathbb{G}_m}$ is an affine morphism.*

Assume in addition that X is smooth (resp., smooth scheme). Then $X^{\mathbb{G}_m}$ is also smooth and $\mathrm{ev}_0 \colon X^+ \to X^{\mathbb{G}_m}$ is an affine fibration (resp., Zariski-local affine fibration), and in particular X_λ^+ is smooth. If $x \in X^{\mathbb{G}_m}$ and $T_{X,x} = T_{>0} \oplus T_0 \oplus T_{<0}$ is the $\mathbb{G}_m$-equivariant decomposition into nonnegative, zero, and nonpositive weights, then $T_{X_i,x} = T_0 \oplus T_{>0}$, $T_{F_i,x} = T_0$, and $X_i \to F_i$ has relative dimension $\dim T_{>0}$.

Proof. If $X = \operatorname{Spec} A$ is affine, then $X^{\mathbb{G}_m}$ and X^+ are closed subschemes of X (Exercise 6.8.7). In the special case that $X = \mathbb{A}(V)$ where V is a finite dimensional G-representation, then $X^{\mathbb{G}_m} = \mathbb{A}(V^{\mathbb{G}_m})$ and $X^+ = \mathbb{A}(V_{\geq 0})$ where $V_{\geq 0}$ is the direct sum of the nonnegative isotypic components, and moreover $\mathrm{ev}_0 \colon X^+ \to X^{\mathbb{G}_m}$ is a relative affine space. We claim that if $U \to X$ is a $\mathbb{G}_m$-invariant étale morphism, then the diagram

$$\begin{tikzcd} U^+ \arrow[r, "\mathrm{ev}_0"] \arrow[d] \arrow[dr, phantom, "\square"] & U^{\mathbb{G}_m} \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & U \arrow[d] \\ X^+ \arrow[r, "\mathrm{ev}_0"] & X^{\mathbb{G}_m} \arrow[r, hook] & X \end{tikzcd} \tag{6.8.10}$$

is cartesian. The right square was verified in the proof of Theorem 6.8.2. For the left square, we need to show that there exists a unique $\mathbb{G}_m$-equivariant morphism filling in a $\mathbb{G}_m$-equivariant diagram

$$\begin{tikzcd} \operatorname{Spec} \Bbbk \times S \arrow[r] \arrow[d] & U \arrow[d] \\ \mathbb{A}^1 \times S \arrow[r] \arrow[ur, dashed] & X \end{tikzcd} \tag{6.8.11}$$

for every $\Bbbk$-scheme S, where the vertical left arrow is the inclusion of the origin. By uniqueness, we can assume that S is affine, and by Limit Methods (B.3), we can further assume that S is of finite type over k. For each $n \geq 1$, the Infinitesimal Lifting Criterion for Étaleness (3.7.1) yields a unique $\mathbb{G}_m$-equivariant map $\operatorname{Spec} \Bbbk[x]/x^n \times S \to U$ such that

$$\begin{tikzcd} \operatorname{Spec} \Bbbk \times S \arrow[r] \arrow[d] & U \arrow[d] \\ \operatorname{Spec}(\Bbbk[x]/x^n) \times S \arrow[r] \arrow[ur, dashed] & X \end{tikzcd}$$

commutes. As $[\mathbb{A}^1/\mathbb{G}_m] \times S$ is coherently complete along $B\mathbb{G}_{m,S}$ (Exercise 6.6.24), Coherent Tannaka Duality in the form of Exercise 6.6.12 yields a unique $\mathbb{G}_m$-equivariant morphism $\mathbb{A}^1 \times S \to U$ such that (6.8.11) commutes.

Choose a $\mathbb{G}_m$-invariant étale surjective morphism $U \to X$ from an affine scheme (Theorem 6.7.24). Then (6.8.10) implies that $U^+ \to X^+$ is etale and representable, and since U^+ is an affine scheme of finite type, it follows that X^+ is an algebraic space of finite type. Since X is separated, $\mathrm{ev}_1 \colon X^+ \to X$ is a monomorphism. Since $U^+ \to U^{\mathbb{G}_m}$ is affine, étale descent implies that $\mathrm{ev}_0 \colon X^+ \to X^{\mathbb{G}_m}$ is also affine. If X is smooth, then $X^{\mathbb{G}_m}$ is smooth by Theorem 6.8.2. As U is also smooth, for each $u \in U^{\mathbb{G}_m}(\Bbbk)$, there is a $\mathbb{G}_m$-equivariant morphism $U \to T_{U,u}$ étale at u with $f(u) = 0$ (Lemma 6.7.9). Then $U^+ \to T^+_{U,u}$ is also étale at u. Let $V \subseteq U^+$ be the open locus where $U^+ \to T^+_{U,u}$ is étale. Since V is $\mathbb{G}_m$-equivariant, if $v \in V^{\mathbb{G}_m}$, then $\mathrm{ev}_0^{-1}(v) \subseteq V$. Choosing an affine subscheme $V' \subseteq V^{\mathbb{G}_m}$ containing u and replacing U^+ with $\mathrm{ev}_0^{-1}(V')$, we may assume that $U^+ \to T^+_{U,u}$ is everywhere étale. By (6.8.10), we have a cartesian diagram

$$\begin{tikzcd} T^+_{U,u} \arrow[d] \arrow[dr, phantom, "\square"] & U^+ \arrow[l] \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X^+ \arrow[d] \\ T^{\mathbb{G}_m}_{U,u} & U^{\mathbb{G}_m} \arrow[l] \arrow[r] & X^{\mathbb{G}_m} \end{tikzcd} \tag{6.8.12}$$

where the horizontal arrows are étale. With $T_{X,x} = T_{>0} \oplus T_0 \oplus T_{<0}$, there are identifications $T^{\mathbb{G}_m}_{X,x} = T_0$ and $T^+_{X,x} = T_{>0} \oplus T_0$. Since $T^+_{U,u} \to T^{\mathbb{G}_m}_{U,u}$ a surjection of vector spaces, $U^+ \to U^{\mathbb{G}_m}$ is a Zariski-local affine fibration. By étale descent, $X^+ \to X^{\mathbb{G}_m}$ is an affine fibration of relative dimension $\dim T_{>0}$.

If X is a smooth *scheme*, then by Sumihiro's Theorem on Torus Actions (B.1.31), we may choose $U = \coprod_i U_i \to X$ such that $\{U_i\}$ is a $\mathbb{G}_m$-invariant affine open covering. Then (6.8.12) implies that $X^+ \to X^{\mathbb{G}_m}$ is a Zariski-local affine fibration. See also [Dri13, Prop. 1.2.2, Thm. 1.4.2] and [AHR20, Thm. 5.16]. □

Remark 6.8.13. For an alternative approach to the algebraicity of X^+, consider an algebraic stack $\mathcal{X}$ of finite type over $\Bbbk$ with affine diagonal and the stack

$\underline{\mathrm{Mor}}_{\Bbbk}([\mathbb{A}^1/\mathbb{G}_m], \mathcal{X})$ of morphisms, whose objects over a $\Bbbk$-scheme S are morphisms $[\mathbb{A}^1/\mathbb{G}_m]_S \to \mathcal{X}$. It can be verified using Artin's Axioms (C.7.4) that $\underline{\mathrm{Mor}}_{\Bbbk}([\mathbb{A}^1/\mathbb{G}_m], \mathcal{X})$ is algebraic, where the crucial step is to verify the effectivity condition (AA_5): this follows from the coherent completeness of $[\mathbb{A}^1/\mathbb{G}_m]_R$, where R is a noetherian local $\Bbbk$-algebra, along the unique closed point (Theorem 6.6.13) together with Coherent Tannaka Duality (6.6.9).

Taking $\mathcal{X} = [X/\mathbb{G}_m]$, a $\mathbb{G}_m$-equivariant morphism $\mathbb{A}^1 \to X$ corresponds to a morphism $[\mathbb{A}^1/\mathbb{G}_m] \to [X/\mathbb{G}_m]$ over $B\mathbb{G}_m$ (Exercise 3.1.16), and there is a cartesian diagram

$$\begin{tikzcd}
\underline{\mathrm{Mor}}_{\Bbbk}^{\mathbb{G}_m}(\mathbb{A}^1, X) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \underline{\mathrm{Mor}}_{\Bbbk}([\mathbb{A}^1/\mathbb{G}_m], [X/\mathbb{G}_m]) \arrow[d] \\
\operatorname{Spec} \Bbbk \arrow[r] & \underline{\mathrm{Mor}}_{\Bbbk}([\mathbb{A}^1/\mathbb{G}_m], B\mathbb{G}_m).
\end{tikzcd}$$

Therefore, the algebraicity of the stacks of morphisms on the right imply that $\underline{\mathrm{Mor}}_{\Bbbk}^{\mathbb{G}_m}(\mathbb{A}^1, X)$ is an algebraic space.

6.8.3 The Białynicki-Birula Stratification

Theorem 6.8.14 (Białynicki-Birula Stratification[4]). *Let X be a separated algebraic space of finite type over an algebraically closed field $\Bbbk$ with an action of $\mathbb{G}_m$. Let $X^{\mathbb{G}_m} = \coprod_{i=1}^n F_i$ be the decomposition of the fixed locus into connected components F_i. Then there exists an affine morphism $X_i \to F_i$ for each i and a monomorphism $\coprod_i X_i \to X$. Moreover,*

(1) If X is proper, then $\coprod_i X_i \to X$ is surjective.

(2) If X is smooth (resp., smooth scheme), then F_i is smooth and $X_i \to F_i$ is a (resp., Zariski-local) affine fibration. If $x \in F_i$ and $T_{X,x} = T_{x,>0} \oplus T_{x,0} \oplus T_{x,<0}$ is the $\mathbb{G}_m$-equivariant decomposition into nonnegative, zero, and nonpositive weights, then $T_{X_i,x} = T_{x,>0} \oplus T_{x,0}$, $T_{F_i,x} = T_{x,0}$, and $X_i \to F_i$ has relative dimension $\dim T_{x,>0}$.

(3) The map $X_i \hookrightarrow X$ is a locally closed immersion under any of the following conditions:

(a) X is affine,

(b) X is a smooth scheme, or

(c) there exists a $\mathbb{G}_m$-equivariant locally closed immersion $X \hookrightarrow \mathbb{P}(V)$ where V is a $\mathbb{G}_m$-representation (e.g., X is a normal quasi-projective variety).

(4) If X is smooth, irreducible, and quasi-projective, then the stratification $X^+ = \coprod_i X_i$ is filterable, *i.e., there is an ordering of the indices such that $X_{\geq i} := \bigcup_{j \geq i} X_j$ is closed for each i. If in addition there are finitely many fixed points $\{x_1, \ldots, x_n\}$, then $T_{x_i,0} = 0$ and $X_i = \mathbb{A}(T_{x_i,>0})$ is an affine space; in particular,*

$$X^+ = X_{\geq 1} \supseteq X_{\geq 2} \supseteq \cdots \supseteq X_{\geq n} \supseteq \emptyset$$

is a cell decomposition, *i.e., each $X_i = X_{\geq i} \setminus X_{\geq i-1}$ is an affine space.*

Proof. By Theorem 6.8.9, X^+ is representable by an algebraic space of finite type over $\Bbbk$, $\mathrm{ev}_1 \colon X^+ \to X$ is a monomorphism, and $\mathrm{ev}_0 \colon X^+ \to X^{\mathbb{G}_m}$ is affine. We define

[4]This is frequently referred to as the 'Białynicki-Birula Decomposition' as some authors prefer to reserve the term 'stratification' to a decomposition where each stratum has a neighborhood which is topologically locally trivial.

X_i as the preimage $\mathrm{ev}_0^{-1}(F_i)$. This gives the main statement, and the conclusion of (2) also follows directly from Theorem 6.8.9. If X is proper, then $X^+ \to X$ is surjective (i.e., (1) holds) as $\lim_{t\to 0} t\cdot x$ exists for every $x \in X(\Bbbk)$.

For (3), if $X = \operatorname{Spec} A$ and $A = \bigoplus_d A_d$ is the $\mathbb{Z}$-grading induced by the $\mathbb{G}_m$-action, then X^+ is the closed subscheme defined by the ideal $\sum_{d<0} A_d$ (Exercise 6.8.7) and in particular affine. If X is a smooth scheme, then there exists a $\mathbb{G}_m$-invariant affine open cover (Theorem B.1.30). For any point $x \in X^+$, let x_0 be the image of x under $\mathrm{ev}_0 \colon X^+ \to X^0$, and choose a $\mathbb{G}_m$-invariant affine open neighborhood $U \subseteq X$ of x_0. This induces a diagram

$$\begin{tikzcd} U^+ \arrow[r, hook] \arrow[dr, hook] & \mathrm{ev}_1^{-1}(U) \arrow[r] \arrow[d, hook] & U \arrow[d, hook] \\ & X^+ \arrow[r, "\mathrm{ev}_1"] & X. \end{tikzcd} \tag{6.8.15}$$

Since $U^+ \to U$ is a closed immersion (as U is affine) and $X^+ \to X$ is separated (it is a monomorphism), $U^+ \to \mathrm{ev}_1^{-1}(U)$ is a closed immersion. Since $U^+ = X^+ \times_{X^0} U^0$ (see (6.8.10)), $x \in U^+$ and $U^+ \to X^+$ is an open immersion. In particular, $U^+ \subseteq \mathrm{ev}_1^{-1}(U)$ is an open and closed subscheme containing x. On the other hand, X_i is smooth and connected (as $X_i \to F_i$ is an affine fibration), thus irreducible. It follows that $X_i \cap U^+ = X_i \cap \mathrm{ev}_1^{-1}(U)$ and that $X_i \cap \mathrm{ev}_1^{-1}(U) \to U$ is a closed immersion which in turn implies that $X_i \to X$ is a locally closed immersion. The final case (3)(c) easily reduces to the case of $X = \mathbb{P}(V)$ in which a direct calculation shows that each X_i is of the form $\mathbb{P}(W) \smallsetminus \mathbb{P}(W')$ for linear subspaces $W' \subseteq W \subseteq V$. See also [BB73, Thm. 4.1], [Hes81, Thm. 4.5,p. 69], [Dri13, Thm. B.0.3], [AHR20, Thm. 5.27], and [JS21, Thm. 1.5].

For (4), by Sumihiro's Theorem on Linearizations (B.1.30), we can choose a G-equivariant locally closed immersion $X \hookrightarrow \mathbb{P}^n$, where $\mathbb{G}_m$ acts on $\mathbb{P}^n$ via $t \cdot [x_0 : \cdots : x_n] = [t^{d_0}x_0 : \cdots : t^{d_n}x_n]$ with $d_0 \le \cdots \le d_n$. Let $D_1, \ldots, D_s$ be the distinct weights and set $J_i = \{j \mid d_j = D_i\}$ so that $J_1 \cup \cdots \cup J_s$ is a partition of $\{0, 1, \ldots, n\}$. Then $(\mathbb{P}^n)^{\mathbb{G}_m} = \coprod i = 1^s F_i$ where $F_i = V(x_j \mid j \in J_i)$. The preimage of F_i under the morphism $\mathrm{ev}_0 \colon \mathbb{P}^n \to (\mathbb{P}^n)^{\mathbb{G}_m}$, given by $p \mapsto \lim_{t\to 0} t \cdot p$, is

$$P_i := \mathrm{ev}_0^{-1}(F_i) = \left\{ [x_0 : \cdots : x_n] \;\middle|\; \begin{array}{l} x_j = 0 \text{ for all } j \in J_1 \cup \cdots \cup J_{i-1} \\ x_k \neq 0 \text{ for some } k \in J_i \end{array} \right\},$$

and the union

$$P_{\ge i} := \bigcup_{j \ge i} P_j = V(x_k \mid k \in J_1 \cup \cdots \cup J_{i-1}) \subseteq \mathbb{P}^n$$

is closed. The fixed locus for X is $X^{\mathbb{G}_m} = (\mathbb{P}^n)^{\mathbb{G}_m} \cap X = \coprod_i F_i \cap X$. For each i, we write $F_i \cap X = \coprod_{j=1}^{l_i} F_{ij}$ and $P_i \cap X = \coprod_{j=1}^{l_i} X_{ij}$ as the irreducible decompositions. Then $\mathrm{ev}_0 \colon \mathbb{P}^n \to (\mathbb{P}^n)^{\mathbb{G}_m}$ restricts to morphisms $\mathrm{ev}_0 \colon X_{ij} \to F_{ij}$. For $j \neq k$, the strata X_{ij} and X_{ik} are disjoint, and thus $\overline{X}_{ij} \cap \overline{X}_{ik} \subseteq P_{\ge i+1} \cap X$. It follows that

$$(P_{\ge i+1} \cap X) \cup X_{i1} \cup \cdots \cup X_{ij} \subseteq X$$

is closed for each $j = 1, \ldots, s$. Ordering the strata as $X_{11}, \ldots, X_{1l_1}, \ldots, X_{s1}, \ldots, X_{sl}$ establishes the claim. See also [BB76, Thm. 3]. $\square$

Remark 6.8.16. It is not true in general that $X_i \hookrightarrow X$ is a locally closed immersion. Based on Hironaka's example of a proper, non-projective, smooth 3-fold, Sommese constructed a smooth algebraic space X such that $X_i \hookrightarrow X$ is not a locally closed immersion [Som82]. Konarski provided an example of a normal proper toric variety X such that $X_i \hookrightarrow X$ is not a locally closed immersion [Kon82].

Remark 6.8.17 (Morse stratifications). The Białynicki-Birula stratification of X can be obtained as the Morse stratification corresponding to the non-degenerate Morse function $\mu\colon X \to \mathrm{Lie}(S^1)^\vee = \mathbb{R}$: a point $x \in X$ lies in X_i if only if the limit of its forward trajectory under the gradient flow of μ lies in F_i. See [CS79].

Example 6.8.18. Suppose $\mathbb{G}_m$ acts on $X = \mathbb{P}^2$ via $t \cdot [x : y : z] = [x : ty : t^2]$. Then $X^{\mathbb{G}_m} = F_1 \amalg F_2 \amalg F_3$ where $F_1 = \{[1 : 0 : 0]\}$, $F_2 = \{[0 : 1 : 0]\}$, and $F_3 = \{[0 : 0 : 1]\}$, and $X_1 = \{x \neq 0\} = \mathbb{A}^2$, $X_2 = \{[0 : y : z] \mid y \neq 0\} = \mathbb{A}^1$ and $X_3 = F_3$.

Let $\widetilde{X}$ be the blowup $\mathrm{Bl}_p X$ at the fixed point $p = [0 : 1 : 0]$. Then $\mathbb{G}_m$ acts on the exceptional divisor $E \cong \mathbb{P}^1$ via $t \cdot [u : v] = [u : t^2 v]$ with fixed points $q_1 = [1 : 0]$ and $q_2 = [0 : 1]$. The fixed locus $\widetilde{X}^{\mathbb{G}_m}$ contains four points $\widetilde{F}_1 = \{[1 : 0 : 0]\}$, $\widetilde{F}_2 = \{q_1\}$, $\widetilde{F}_3 = \{q_2\}$, and $\widetilde{F}_4 = \{[0 : 0 : 1]\}$. We have that $\widetilde{X}_1 = X_1 \cong \mathbb{A}^2$, $\widetilde{X}_2 = X_2 \cong \mathbb{A}^1$, $X_3 = E \smallsetminus \{q_2\} \cong \mathbb{A}^1$, and $\widetilde{X}_4 = X_4 = \widetilde{F}_4$ as illustrated in Figure 6.8.19. Observe that $\overline{X}_3 \smallsetminus X_3 = \{q_2\}$ is not the union of other strata.

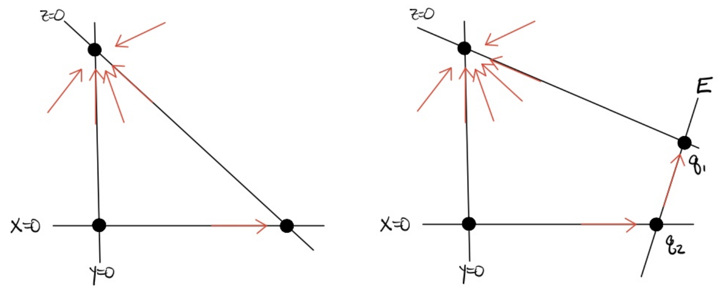

Figure 6.8.19: Białynicki-Birula stratifications for $\mathbb{P}^2$ (left) and $\mathrm{Bl}_p\,\mathbb{P}^2$ (right).

6.8.4 Applications of the Białynicki-Birula Stratification to cohomology

Proposition 6.8.20. *Let X be a smooth, irreducible, and quasi-projective scheme over an algebraically closed field $\Bbbk$ with a $\mathbb{G}_m$-action of with only finitely many fixed points. Then the Chow group $CH_i(X)$ is a free $\mathbb{Z}$-module generated by the closures of the i-dimensional cells. If in addition $\Bbbk = \mathbb{C}$, then the cycle map* $\mathrm{CH}_i(X) \to \mathrm{H}^{\mathrm{BM}}_{2i}(X, \mathbb{Z})$ *to Borel–Moore homology is an isomorphism and* $\mathrm{H}^{\mathrm{BM}}_{2i+1}(X, \mathbb{Z}) = 0$.

Remark 6.8.21. When X is compact (e.g., projective), then $H^{\mathrm{BM}}_{2i}(X, \mathbb{Z})$ is ordinary integral singular homology.

Proof. The Białynicki-Birula Stratification (6.8.14(4)) implies that X has a cell decomposition, and the statement follows from [Ful98, Ex. 19.1.11]. See also [Bri97, §3.2]. ☐

Example 6.8.22 (Chow groups of $\mathrm{Hilb}^n(\mathbb{A}^2)$)**.** Let $X = \mathrm{Hilb}^n(\mathbb{A}^2)$ be the Hilbert scheme of n points; this is a smooth irreducible scheme (see 1.5.3). The natural action of $\mathbb{G}_m^2$ induces a $\mathbb{G}_m^2$-action on X. Under the $\mathbb{G}_m$-action induced by $\mathbb{G}_m \to \mathbb{G}_m^2, t \mapsto (t, t)$, the evaluation map $\mathrm{ev}_0\colon X^+ \to X$ is surjective, and since the $\mathbb{G}_m$-fixed points correspond to subschemes $Z = V(I) \subseteq \mathbb{A}^2$ supported at the origin where I is a monomial ideal, there are only finitely many $\mathbb{G}_m$-fixed points. We may therefore use Proposition 6.8.20 to compute $\mathrm{CH}^*(X)$.

For a monomial ideal $I \subseteq R := \Bbbk[x, y]$ and an integer $i \geq 0$, define

$$a_i := \min\{j \mid x^i y^j \in I\},$$

and let r be the largest integer such that $a_r > 0$. Then $a_0 \geq \cdots \geq a_r$ is a partition of n and $I = (y^{a_0}, xy^{a_1}, \dots, x^{r+1})$. We need to compute the dimension of the positive weight space $T_{I,>0}$ of the $\mathbb{G}_m$-action on the tangent space

$$T_I = \mathrm{Hom}_R(I, R/I)$$

of X at the monomial ideal I, which is identified with $\mathrm{Hom}_R(I, R/I)$ by Exercise 1.5.6. To accomplish this, we first argue that

$$T_I = \sum_{0 \leq i \leq j \leq r} \sum_{s=a_{j+1}}^{a_j - 1} (\chi_1^{i-j-1} \chi_2^{a_i - s - 1} + \chi_1^{j-i} \chi_2^{s - a_i}), \tag{6.8.23}$$

as $\mathbb{G}_m^2$ representations, where $\chi_i \colon \mathbb{G}_m^2 \to \mathbb{G}_m$ denotes the one dimensional representation giving by $(t_1, t_2) \mapsto t_i^{-1}$. There are

$$\sum_{0 \leq i \leq j \leq r} 2(a_j - a_{j+1}) = 2 \sum_{0 \leq i \leq r} a_i = 2n = \dim T_I$$

one dimensional representations appearing on the right-hand side, and they are linearly independent. It thus suffices to show that each of them occurs in T_I. An R-module map $\phi \colon I \to R/I$ is given by the values $\phi(x^i y^{a_i})$ subject to the relations

$$\phi(x^{i+1} y^{a_i}) = x \phi(x^i a_i) \quad \text{and} \quad \phi(x^i y^{a_{i-1}}) = y^{a_{i-1} - a_i} \phi(x^i a_i).$$

Let $0 \leq i \leq j \leq r$ and $a_{j+1} \leq s < a_j$. Defining

$$\begin{aligned} \phi_{i,j,s} \colon I \to R/I, \quad x^l y^{a_l} &\mapsto \begin{cases} x^{l+j-i} y^{a_l + s - a_i} & \text{if } l \leq i \\ 0 & \text{otherwise} \end{cases} \\ \psi_{i,j,s} \colon I \to R/I, \quad x^l y^{a_l} &\mapsto \begin{cases} x^{l+i-j-1} y^{a_l + s - a_i} & \text{if } l \geq j+1 \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

one checks that $\phi_{i,j,s}$ and $\psi_{i,j,s}$ are R-module maps that are eigenvectors for $\chi_1^{j-i} \chi_2^{s-a_i}$ and $\chi_1^{i-j-1} \chi_2^{a_i - s - 1}$. Thus (6.8.23) holds.

Choose $\lambda = (\lambda_1, \lambda_2) \colon \mathbb{G}_m \to \mathbb{G}_m^2$ with $\lambda_1 \gg \lambda_2$. Under our sign conventions, a character $\chi_1^a \chi_2^b$ appearing in (6.8.23) has positive weight with respect to λ if $a < 0$, or if $a = 0$ and $b < 0$. Thus

$$T_{I,>0} = \sum_{0 \leq i \leq j \leq r} \sum_{s=a_{j+1}}^{a_j - 1} \chi_1^{i-j-1} \chi_2^{a_i - s - 1} + \sum_{j=0}^{r} \sum_{s=a_{j+1}}^{a_j - 1} \chi_1^{j-i} \chi_2^{s - a_i}$$

and

$$\begin{aligned} \dim T_{I,>0} &= \left(\sum_{i=0}^{r} \sum_{j=i}^{r} (a_j - a_{j+1}) \right) + \left(\sum_{j=0}^{r} (a_j - a_{j+1}) \right) \\ &= \left(\sum_{i=0}^{r} a_i \right) + a_0 = n + a_0 \end{aligned}$$

Since there is a bijection between monomial ideals $I \subseteq R = \Bbbk[x,y]$ with $\dim_{\Bbbk} R/I = n$ and partitions $a_0 \geq \cdots \geq a_r$ of n, for every $d \geq 0$, the number of monomial ideals I such that $\dim T_{I,>0} = d$ is equal to

$$P(2n-d, d-n) := \#\,\{\text{partitions } a_1 \geq \cdots \geq a_r \text{ of } 2n-d \text{ with each } a_i \leq d-n\}. \tag{6.8.24}$$

It follows from Proposition 6.8.20 that

$$\dim \mathrm{CH}_d(\mathrm{Hilb}^n(\mathbb{A}^2))_{\mathbb{Q}} = P(2n-d, d-n).$$

See also [ES87, Thm. 1.1] and [Göt94, §2.2].

Exercise 6.8.25 (Chow groups of $\mathrm{Hilb}^n(\mathbb{P}^2)$)**.** Follow the above strategy to show that the dth Betti number b_d of $\mathrm{Hilb}^n(\mathbb{P}^2)$ (or equivalently $\dim \mathrm{CH}_d(\mathrm{Hilb}^n(\mathbb{P}^2))$) is equal to

$$b_d = \sum_{n_0+n_1+n_2=n} \sum_{p+r=d-n_1} P(p, n_0-p)P(n_1)P(2n_2-r, r-n_2),$$

where $P(a)$ is the number of partitions of a and $P(a,b)$ is defined by (6.8.24).

Remark 6.8.26. Göttsche used the Weil conjectures in [Göt90, Thm. 0.1] (see also [Göt94, Thm. 2.3.10]) to show that for any smooth projective surface S over $\mathbb{C}$ or $\overline{\mathbb{F}}_q$ that the Poincaré polynomial $p(S^{[n]}, z) = \sum_i b_i(S^{[n]})z^n$ of $S^{[n]} := \mathrm{Hilb}^n(S)$ satisfies

$$\sum_{n=0}^{\infty} p(S^{[n]}, z)t^n = \prod_{m=1}^{\infty} \frac{(1+z^{2m-1}t^m)^{b_1(S)}(1+z^{2m+1}t^m)^{b_3(S)}}{(1-z^{2m-2}t^m)^{b_0(S)}(1-z^{2m}t^m)^{b_2(S)}(1-z^{2m+2}t^m)^{b_4(S)}}$$

In particular, the Betti numbers of $S^{[n]}$ only depend on the Betti numbers of S. While each term $p(S^{[n]}, z)$ does not admit a particularly nice expression, the generating function involving *all* n does. On the other hand, Nakajima constructed an action of the Heisenberg algebra on $\mathrm{H}_*(S^{[n]})$, which can be used to recover the above formula as well as additional properties of the cohomology ring [Nak97] (see also [Nak99b]).

The Białynicki-Birula Stratification can also be applied to compute equivariant Chow rings $\mathrm{CH}^*_G(X)$; see Exercise 7.7.12.

6.9 Valuative criteria for algebraic stacks: Θ- and S-completeness

You might underestimate the importance of definitions in mathematics, but you shouldn't.

Claire Voisin

We discuss the valuative criteria of Θ- and S-completeness (see Definitions 6.9.10 and 6.9.14) which play a central role in the Existence Theorem of Good Moduli Spaces (6.10.1) in the next section.

6.9.1 Maps from $\Theta = [\mathbb{A}^1/\mathbb{G}_m]$

We define the stack 'Theta' as

$$\Theta := [\mathbb{A}^1/\mathbb{G}_m]$$

over $\operatorname{Spec}\mathbb{Z}$.[5] When we are working over a field $\Bbbk$, we will abuse notation by also using Θ to denote $\Theta_{\Bbbk} = [\mathbb{A}^1_{\Bbbk}/\mathbb{G}_{m,\Bbbk}]$. While a map $\mathcal{X} \to \Theta$ from an algebraic stack is classified by a line bundle and a section (Example 3.9.25), maps $\Theta \to \mathcal{X}$ *from* Θ often also have geometric significance, e.g., in the case that $\mathcal{X}$ is a quotient stack, a stack of coherent sheaves, and the stack of all curves. These descriptions will be useful to interpret the valuative criteria of Θ- and S-completeness introduced in §6.9.2.

Quotient stacks. Given a quotient stack $[X/G]$, a one-parameter subgroup $\lambda\colon \mathbb{G}_m \to G$, and a point $x \in X(\Bbbk)$ such that $\lim_{t\to 0}\lambda(t)\cdot x \in X$ exists, the $\mathbb{G}_m$-equivariant extension $\mathbb{A}^1 \to X$ induces a morphism $[\mathbb{A}^1/\mathbb{G}_m] \to [X/G]$ of algebraic stacks. The next proposition asserts that the converse is also true, i.e., any map $[\mathbb{A}^1/\mathbb{G}_m] \to [X/G]$ is induced by a one-parameter subgroup λ and a point $x \in X(\Bbbk)$.

Recall that C_λ denotes the centralizer of a one-parameter subgroup λ and P_λ is the subgroup of elements $g \in G$ such that $\lim_{t\to 0}\lambda(t)g\lambda(t)^{-1}$ exists; see §B.1.4.

Proposition 6.9.1. *If G is a smooth affine algebraic group over an algebraically closed field $\Bbbk$ acting on a separated algebraic space X of finite type over $\Bbbk$, then there is an equivalence of groupoids*

$$\operatorname{Mor}_{\Bbbk}(\Theta,[X/G]) \xrightarrow{\sim} \bigl\{(x \in X(\Bbbk), \lambda\colon \mathbb{G}_m \to G) \;\big|\; \lim_{t\to 0}\lambda(t)\cdot x \in X(\Bbbk) \text{ exists}\bigr\},$$

where a morphism $(x,\lambda) \to (x',\lambda')$ is an isomorphism class of a pair (g,h) with $g \in P_\lambda(\Bbbk)$ and $h \in G(\Bbbk)$ such that $x' = hgx$ and $\lambda' = h\lambda h^{-1}$, with $(g,h) \sim (c^{-1}g, hc)$ for $c \in C_\lambda$. Under this correspondence, the morphism $\Theta \to [X/G]$ sends 1 to x and 0 to $\lim_{t\to 0}\lambda(t)\cdot x$.

Remark 6.9.2. Observe that the automorphism group of (x,λ) is $P_\lambda \cap G_x$.

Proof. Given (x,λ), the $\mathbb{G}_m$-equivariant map $m_{x,\lambda}\colon \mathbb{G}_m \to X$ defined by $t \mapsto \lambda(t)\cdot x$ extends to a commutative diagram

$$\begin{tikzcd}
\mathbb{G}_m \arrow[r, "m_{x,\lambda}"] \arrow[d, hook] & X \\
\mathbb{A}^1 \arrow[ur, dashed, "\widetilde{m}_{x,\lambda}"'] &
\end{tikzcd}$$

The extension is $\mathbb{G}_m$-equivariant and induces a morphism of quotient stacks $f_{x,\lambda}\colon \Theta \to [X/G]$. We first show that this defines a functor

$$\begin{aligned}\bigl\{(x,\lambda) \;\big|\; \lim_{t\to 0}\lambda(t)\cdot x \text{ exists}\bigr\} &\to \operatorname{Mor}_{\Bbbk}(\Theta,[X/G]) \\ (x,\lambda) &\mapsto f_{x,\lambda}.\end{aligned} \tag{6.9.3}$$

Given a morphism $(g,h)\colon (x,\lambda) \to (x',\lambda')$, we need to define a 2-morphism $f_{x,\lambda} \xrightarrow{\sim} f_{x',\lambda'}$. Since h determines a canonical isomorphism $f_{x',\lambda'} \xrightarrow{\sim} f_{h^{-1}x',\lambda}$, it suffices to define a 2-morphism $f_{x,\lambda} \xrightarrow{\sim} f_{h^{-1}x',\lambda}$. Since $g \in P_\lambda(\Bbbk)$, the map $t \mapsto \lambda(t)g\lambda(t)^{-1}$ extends to a map $\widetilde{g}\colon \mathbb{A}^1 \to G$ such that $\widetilde{m}_{h^{-1}x',\lambda} = \widetilde{g}\cdot \widetilde{m}_{x,\lambda}$ (as $h^{-1}x' = gx$). The element $\widetilde{g}$ defines an isomorphism $f_{x,\lambda} \xrightarrow{\sim} f_{h^{-1}x',\lambda}$. For $c \in C_\lambda$, the pairs (g,h) and $(c^{-1}g, hc)$ define the same isomorphism: indeed this follows from the observation that if $c \in G_x$, then (c^{-1},c) defines the identify automorphism of $f_{x,\lambda}$.

[5] The symbol Θ is used as it resembles the picture of the two orbits of $\mathbb{G}_m$ on the complex plane.

Conversely, any isomorphism $f_{x,\lambda} \xrightarrow{\sim} f_{x',\lambda}$ is an induced by element $\widetilde{g} \in G(\mathbb{A}^1)$ satisfying $\widetilde{m}_{x',\lambda} = \widetilde{g} \cdot \widetilde{m}_{x,\lambda}$, and we conclude that (6.9.3) is a fully faithful functor.

For the essential surjectivity of (6.9.3), let $f\colon \Theta \to [X/G]$ be a morphism. In the fiber diagram

$$\begin{tikzcd} \mathcal{P} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X \arrow[d] \\ \Theta \arrow[r, "f"] & {[X/G],} \end{tikzcd}$$

$\mathcal{P} \to \Theta$ is a principal G-bundle. The restriction $\mathcal{P}|_{B\mathbb{G}_m}$ along the unique closed point $0 \in \Theta$ corresponds to a $\mathbb{G}_m$-equivariant principal G-bundle P on $\operatorname{Spec} \Bbbk$. After choosing an isomorphism $P \cong G$, $\mathcal{P}$ corresponds to a one-parameter subgroup $\lambda\colon \mathbb{G}_m \to G$. This induces a $\mathbb{G}_m$-action on the product $\mathbb{A}^1 \times G$ by $t \cdot (x, g) = (tx, g\lambda(t)^{-1})$ and thus a principal G-bundle

$$\mathcal{P}_\lambda := [(\mathbb{A}^1 \times G)/\mathbb{G}_m] \to \Theta.$$

We claim that there is an isomorphism $\alpha\colon \mathcal{P} \to \mathcal{P}_\lambda$ of principal G-bundles. By construction, we have an isomorphism $\alpha_0\colon \mathcal{P}|_{B\mathbb{G}_m} \to \mathcal{P}_\lambda|_{B\mathbb{G}_m}$. Since $\underline{\mathrm{Isom}}_\Theta(\mathcal{P}, \mathcal{P}_\lambda) \to \Theta$ is smooth (as it is a principal G-bundle and G is smooth), we may use deformation theory (Proposition 6.7.12) to construct compatible isomorphisms $\alpha_n\colon \mathcal{P}|_{\mathcal{X}_n} \to \mathcal{P}_\lambda|_{\mathcal{X}_n}$ over the nilpotent thickenings $\mathcal{X}_n$ of $B\mathbb{G}_m \hookrightarrow \Theta$. Coherent Tannaka Duality (6.6.9) coupled with the coherent completeness of Θ along $B\mathbb{G}_m$ (Theorem 6.6.13) implies that the isomorphisms α_n extend to an isomorphism $\alpha\colon \mathcal{P} \to \mathcal{P}_\lambda$. Restricting the composition

$$\mathbb{A}^1 \times G \to \mathcal{P}_\lambda \xrightarrow{\alpha^{-1}} \mathcal{P} \to X$$

to the identity in G yields a $\mathbb{G}_m$-equivariant morphism $\mathbb{A}^1 \to X$, which in turn induces a map $f\colon \Theta \to [X/G]$ Letting $x \in X(\Bbbk)$ be the image of 1, one checks that $f_{x,\lambda}$ is 2-isomorphic to f. $\square$

Exercise 6.9.4 (details)**.** Letting R be a complete noetherian local $\Bbbk$-algebra, upgrade the proposition to an equivalence between $\mathrm{Mor}_\Bbbk(\Theta_R, [X/G])$ and pairs $(x \in X(R), \lambda : \mathbb{G}_m \to G)$.

Remark 6.9.5. Proposition 6.9.1 can be further upgraded to a description of the stack of morphisms from $[\mathbb{A}^1/\mathbb{G}_m]$ to $[X/G]$. Namely, there is a decomposition

$$\underline{\mathrm{Mor}}([\mathbb{A}^1/\mathbb{G}_m], [X/G]) \cong \coprod_\lambda [X_\lambda^+/P_\lambda]$$

where λ varies over conjugacy classes of one-parameter subgroups. The algebraicity of this stack was discussed already in Remark 6.8.13. See also [HL14, Thm. 1.37].

Stacks of coherent sheaves. Given a projective scheme X, recall that $\underline{\mathrm{Coh}}(X)$ denotes the algebraic stack of coherent sheaves on X (see Exercise 3.1.23).

Proposition 6.9.6. *Let X be a projective scheme over an algebraically closed field $\Bbbk$. For a noetherian $\Bbbk$-algebra R, $\mathrm{Mor}_\Bbbk(\Theta_R, \underline{\mathrm{Coh}}(X))$ is equivalent to the groupoid of pairs $(E, E_\bullet)$ where E is a coherent sheaf on X_R flat over R and*

$$E_\bullet : 0 \subseteq \cdots \subseteq E_{i-1} \subseteq E_i \subseteq E_{i+1} \subseteq \cdots \subseteq E$$

is a $\mathbb{Z}$-graded filtration such that $E_i = 0$ for $i \ll 0$, $E_i = E$ for $i \gg 0$, and each factor E_i/E_{i-1} is flat over R. A morphism $(E, E_\bullet) \to (E', E'_\bullet)$ is an isomorphism $E \to E'$

of coherent sheaves compatible with the filtration. Under this correspondence, the morphism $\Theta_R \to \underline{\mathrm{Coh}}(X)$ *sends 1 to* E *and* 0 *to the associated graded* $\operatorname{gr} E_\bullet := \bigoplus_i E_i/E_{i-1}$*, and factors through* $\mathcal{B}un(X) \subseteq \underline{\mathrm{Coh}}(X)$ *if and only if* E *and each factor* E_i/E_{i-1} *is a vector bundle.*

Proof. The reader may prefer to read the proof in the special (but important) case that $R = \Bbbk$. A morphism $\Theta_R \to \underline{\mathrm{Coh}}(X)$ corresponds to a coherent sheaf F on $X \times \Theta_R$ flat over Θ_R, or alternatively using smooth descent to a coherent sheaf on $X \times \mathbb{A}^1_R$ flat over $\mathbb{A}^1_R$. Since $f\colon X \times \Theta_R \to X \times B\mathbb{G}_{m,R}$ is an affine morphism, F can be recovered from the data of the quasi-coherent sheaf f_*F on $X \times B\mathbb{G}_{m,R}$ together with the structure of a $f_*\mathcal{O}_{X\times\Theta_R}$-module. Using that quasi-coherent sheaves on $X \times B\mathbb{G}_{m,R}$ are $\mathbb{Z}$-graded quasi-coherent sheaves on X_R (see Exercise 6.1.10), it follows that F corresponds to a $\mathbb{Z}$-graded $\mathcal{O}_{X_R}[x]$-module flat over $R[x]$. Writing $F = \bigoplus_{i\in\mathbb{Z}} E_i$ with each E_i a coherent sheaf on X_R, then multiplication by x induces maps $x\colon E_i \to E_{i+1}$ which are necessarily injective as F is flat over $R[x]$, hence torsion free. Since F is finitely generated as a $\mathbb{Z}$-graded $R[x]$-module, there exists finitely many homogeneous generators with bounded degree. Thus $E_i = E$ for $i \gg 0$. On the other hand, considering the $\mathcal{O}_{X_R}[x]$-submodule $E_{\geq d} := \bigoplus_{i\geq d} E_i \subseteq F$, the ascending chain $\cdots \subseteq E_{\geq d} \subseteq E_{\geq d-1} \subseteq \cdots \subseteq F$ must terminate as F is noetherian. It follows that $E_i = 0$ for $i \ll 0$. Since F is flat as an $R[x]$-module, the quotient $F/xF = \bigoplus_i E_i/E_{i-1}$ is flat as an R-module and thus each factor E_i/E_{i-1} is flat over R.

Conversely, given E and a filtration $E_\bullet$ satisfying the above conditions, consider the graded $\mathcal{O}_{X_R}[x]$-module $F := \bigoplus_i E_i$, which is often referred to as the *Rees construction*. We will show by induction that $E_{\geq d} := \bigoplus_{i\geq d} E_i$ is flat and finitely generated over $R[x]$; this implies that F is flat and finitely generated over $R[x]$ since $E_i = 0$ for $i \ll 0$. For $d \gg 0$, $E_{\geq d}$ is isomorphic to the graded $R[x]$-module $(E \otimes_R R[x])\langle d\rangle$, where $\langle d\rangle$ denotes the grading shift, and is thus flat and finitely generated. For every d, we have an exact sequence

$$0 \to (E_d \otimes_R R[x])\langle d\rangle \to E_{\geq d} \to ((E_{d+1}/E_d) \otimes_R R[x])\langle d+1\rangle \to 0.$$

The flatness of E and the quotients E_{d+1}/E_d implies the flatness of each E_d. Thus the left and right term above are flat and finitely generated as $R[x]$-modules, and thus so is the middle term. □

Stack of all curves.

Proposition 6.9.7. *Let* $\mathcal{M}_g^{\mathrm{all}}$ *be the algebraic stack of all proper curves (Theorem 5.4.6) over an algebraically closed field* $\Bbbk$*. For every* $\Bbbk$*-algebra* R*,* $\mathrm{Mor}_{\Bbbk}(\Theta_R, \mathcal{M}_g^{\mathrm{all}})$ *is the groupoid whose objects are* $\mathbb{G}_m$*-equivariant families of proper curves* $\mathcal{C} \to \mathbb{A}^1_R$*, where* $\mathbb{G}_m$ *acts on* $\mathbb{A}^1_R$ *with the usual scaling action. Morphisms are* $\mathbb{G}_m$*-equivariant morphisms.*

Proof. The statement follows from smooth descent applied to $\mathbb{A}^1_R \to \Theta_R$. □

Remark 6.9.8. A similar description holds for other moduli stacks of varieties. These $\mathbb{G}_m$-equivariant families are often called *test configurations* in the literature.

6.9.2 The valuative criteria: Θ- and S-completeness

We maintain the notation that $\Theta := [\mathbb{A}^1/\mathbb{G}_m]$ over $\operatorname{Spec}\mathbb{Z}$. Let R be a DVR with fraction field K and residue field κ. Define $\Theta_R := \Theta \times \operatorname{Spec} R$ and set $0 \in \Theta_R$ to be the unique closed point. Observe that Θ_R is a local model of the quotient stack $[\mathbb{A}^2/\mathbb{G}_m]$ with weights $0, 1$ as it is identified with the base change of the good moduli space $[\mathbb{A}^2/\mathbb{G}_m] \to \operatorname{Spec} \Bbbk[x]$ along the map $\operatorname{Spec} R \to \operatorname{Spec} \Bbbk[x]$ where x maps to a uniformizer π in R. The following cartesian diagram gives a schematic picture of Θ_R (where x is the coordinate on $\mathbb{A}^1$ and $\pi \in R$ is the uniformizer).

$$\begin{tikzcd}
& \operatorname{Spec} R \arrow[dr, hook, "x\neq 0"] & & B\mathbb{G}_{m,R} \arrow[dl, hook', "x=0"'] & \\
\operatorname{Spec} K \arrow[ur, hook] \arrow[dr, hook] & & \Theta_R & & B\mathbb{G}_{m,\kappa} \arrow[ul, hook'] \arrow[dl, hook'] \\
& \Theta_K \arrow[ur, hook, "\pi\neq 0"] & & \Theta_\kappa \arrow[ul, hook', "\pi=0"'] &
\end{tikzcd} \tag{6.9.9}$$

where the maps to the left are open immersions and to the right are closed immersions. In particular, a morphism $\Theta_R \smallsetminus 0 = \operatorname{Spec} R \bigcup_{\operatorname{Spec} K} \Theta_K \to \mathcal{X}$ to an algebraic stack is the data of morphisms $\operatorname{Spec} R \to \mathcal{X}$ and $\Theta_K \to \mathcal{X}$ together with an isomorphism of their restrictions to $\operatorname{Spec} K$.

Definition 6.9.10. A noetherian algebraic stack $\mathcal{X}$ is Θ-*complete*[6] DVR R, every commutative diagram

$$\begin{tikzcd}
\Theta_R \smallsetminus 0 \arrow[r] \arrow[d] & \mathcal{X} \\
\Theta_R \arrow[ur, dashed] &
\end{tikzcd} \tag{6.9.11}$$

of solid arrows can be uniquely filled in.

Remark 6.9.12. Equivalently, using the stack $\underline{\operatorname{Mor}}(\Theta, \mathcal{X})$ classifying morphisms $\Theta \to \mathcal{X}$, evaluation at 1 gives a morphism

$$\operatorname{ev}_1 \colon \underline{\operatorname{Mor}}(\Theta, \mathcal{X}) \to \mathcal{X}, \quad f \mapsto f(1),$$

$\mathcal{X}$ is Θ-complete if and only if ev_1 satisfies the valuative criterion for properness. If $\mathcal{X}$ is of finite type over an algebraically closed field $\Bbbk$, then $\underline{\operatorname{Mor}}(\Theta, \mathcal{X})$ is an algebraic stack *locally* of finite type over $\Bbbk$; see Remark 6.9.5 where an explicit description is given when $\mathcal{X}$ is a quotient stack. The stack $\underline{\operatorname{Mor}}(\Theta, \mathcal{X})$ is however rarely quasi-compact, e.g., for $\mathcal{X} = B\mathbb{G}_m$, and ev_1 is thus rarely proper.

For a DVR R with fraction field K, residue field κ, and uniformizer π, we define

$$\phi_R := [\operatorname{Spec}\big(R[s,t]/(st - \pi)\big)/\mathbb{G}_m], \tag{6.9.13}$$

where s and t have $\mathbb{G}_m$-weights 1 and -1 respectively.[7] The quotient ϕ_R is a local model of the quotient stack $[\mathbb{A}^2/\mathbb{G}_m]$ with weights $1, -1$ as it is identified with the base change of the good moduli space $[\mathbb{A}^2/\mathbb{G}_m] \to \operatorname{Spec} \Bbbk[xy]$ along the map $\operatorname{Spec} R \to \operatorname{Spec} \Bbbk[xy]$ given by $xy \mapsto \pi$.

[6]This notion was introduced in [HL14], where it was called 'Θ-reductive'.

[7]The symbol ϕ is used because it looks like the non-separated affine line with an additional origin. This stack was introduced in [Hei17, §2.B] using the notation $\overline{\mathrm{ST}}_R$, as it is a compactification of the 'standard test' scheme $\mathrm{ST}_R = \overline{\mathrm{ST}}_R \smallsetminus 0 = \operatorname{Spec} R \bigcup_{\operatorname{Spec} K} \operatorname{Spec} R$ for for separatedness.

The locus where $s \neq 0$ in ϕ_R is isomorphic to $[\operatorname{Spec}\big(R[s,t]_s/(t-\pi/s)\big)/\mathbb{G}_m] \cong [\operatorname{Spec}(R[s]_s)/\mathbb{G}_m] \cong \operatorname{Spec} R$ and the locus where $t \neq 0$ has a similar description. We thus have cartesian diagrams analogous to (6.9.9)

$$\begin{tikzcd}
 & \operatorname{Spec} R \arrow[dr, hook, "s\neq 0"] & & \Theta_\kappa \arrow[dl, hook', "s=0"'] & \\
\operatorname{Spec} K \arrow[ur, hook] \arrow[dr, hook] & & \phi_R & & B\mathbb{G}_{m,\kappa} \arrow[ul, hook'] \arrow[dl, hook] \\
 & \operatorname{Spec} R \arrow[ur, hook, "t\neq 0"'] & & \Theta_\kappa \arrow[ul, hook', "t=0"] &
\end{tikzcd}$$

where the maps to the left are open immersions and to the right are closed immersions. In particular, a morphism $\phi_R \setminus 0 = \operatorname{Spec} R \bigcup_{\operatorname{Spec} K} \operatorname{Spec} R \to \mathcal{X}$ to an algebraic stack is the data of two morphisms $\xi, \xi' \colon \operatorname{Spec} R \to \mathcal{X}$ together with an isomorphism $\xi_K \simeq \xi'_K$ over $\operatorname{Spec} K$.

Definition 6.9.14. A noetherian algebraic stack $\mathcal{X}$ is S*-complete* if for every DVR R, every commutative diagram

$$\begin{tikzcd}
\phi_R \setminus 0 \arrow[r] \arrow[d] & \mathcal{X} \\
\phi_R \arrow[ur, dashed] &
\end{tikzcd} \tag{6.9.15}$$

of solid arrows can be uniquely filled in.[8]

The definition of Θ-completeness and S-completeness naturally extends to morphisms $f \colon \mathcal{X} \to \mathcal{Y}$, but we will not use this.

Exercise 6.9.16. Let X be a $\Bbbk$-scheme and R be a DVR over $\Bbbk$.

(1) Show that a quasi-coherent sheaf $\mathcal{F}$ on $X \times \phi_R$ corresponds to a $\mathbb{Z}$-graded quasi-coherent sheaf $\bigoplus_{n \in \mathbb{Z}} \mathcal{F}_n$ on X_R together with a diagram of maps

$$\begin{tikzcd}
\cdots \arrow[r, bend left, "s"] & \mathcal{F}_{n-1} \arrow[l, bend left, "t"] \arrow[r, bend left, "s"] & \mathcal{F}_n \arrow[l, bend left, "t"] \arrow[r, bend left, "s"] & \mathcal{F}_{n+1} \arrow[l, bend left, "t"] \arrow[r, bend left, "s"] & \cdots \arrow[l, bend left, "t"]
\end{tikzcd}$$

such that $st = ts = \pi$.

(2) Show that $\mathcal{F}$ is coherent if each $\mathcal{F}_n$ is coherent, $s \colon \mathcal{F}_{n-1} \to \mathcal{F}_n$ is an isomorphism for $n \gg 0$ and $t \colon \mathcal{F}_n \to \mathcal{F}_{n-1}$ is an isomorphism for $n \ll 0$. Show that the converse holds if the maps s and t are injective.

(3) Show that $\mathcal{F}$ is flat over ϕ_R if and only if the maps s and t are injective, and the induced map $t \colon \mathcal{F}_{n+1}/s\mathcal{F}_n \to \mathcal{F}_n/s\mathcal{F}_{n-1}$ is injective.

(4) Assuming that $\mathcal{F}$ is coherent and flat over ϕ_R, characterize the restrictions of $\mathcal{F}$ as follows:

(i) $\mathcal{F}|_{\{s\neq 0\}} \cong \operatorname{colim}(\cdots \xrightarrow{s} \mathcal{F}_n \xrightarrow{s} \mathcal{F}_{n+1} \xrightarrow{s} \cdots)$ on X_R,

(ii) $\mathcal{F}|_{\{t\neq 0\}} \cong \operatorname{colim}(\cdots \xrightarrow{t} \mathcal{F}_{n+1} \xrightarrow{t} \mathcal{F}_n \xrightarrow{t} \cdots)$ on X_R

[8] The 'S' stands for 'Seshadri' as S-completeness is a geometric property reminiscent of how the S-equivalence relation on sheaves implies separatedness of the moduli space.

(iii) $\mathcal{F}|_{\{s=0\}}$ on Θ_κ corresponds under Proposition 6.9.6 to the $\mathbb{Z}$-filtration $\cdots \xrightarrow{t} \mathcal{F}_{n+1}/s\mathcal{F}_{n+2} \xrightarrow{t} \mathcal{F}_n/s\mathcal{F}_{n-1} \xrightarrow{t} \cdots$, and

(iv) $\mathcal{F}|_{\{t=0\}}$ on Θ_κ corresponds to the $\mathbb{Z}$-filtration $\cdots \xrightarrow{s} \mathcal{F}_n/t\mathcal{F}_{n+1} \xrightarrow{s} \mathcal{F}_{n+1}/t\mathcal{F}_{n+2} \xrightarrow{s} \cdots$.

6.9.3 Properties of Θ- and S-completeness

Lemma 6.9.17. *A noetherian algebraic stack with affine diagonal is Θ-complete (resp., S-complete), if and only if every diagram (6.9.11) (resp., (6.9.15)), there exists a lift after an extension of DVRs $R \subseteq R'$. In particular, Θ- and S-completeness can be verified on complete DVRs with algebraically closed residue fields.*

Proof. We begin with the observation that if $\mathcal{X} \to \mathcal{Y}$ has affine diagonal and $j\colon \mathcal{U} \to \mathcal{T}$ is an open immersion of algebraic stacks over $\mathcal{Y}$ with $j_*\mathcal{O}_{\mathcal{U}} = \mathcal{O}_{\mathcal{T}}$, then two extensions $f_1, f_2\colon \mathcal{T} \to \mathcal{X}$ of a $\mathcal{Y}$-morphism $\mathcal{U} \to \mathcal{X}$ are canonically 2-isomorphic. Indeed, since $\underline{\mathrm{Isom}}_{\mathcal{T}}(f_1, f_2) \to \mathcal{T}$ is affine, the section over $\mathcal{U}$ induced by the 2-isomorphism $f_1|_{\mathcal{U}} \xrightarrow{\sim} f_2|_{\mathcal{U}}$ extends uniquely to a section of $\mathcal{T}$.

Consider a diagram (6.9.11), an extension of DVRs $R \subseteq R'$, and a lifting $\Theta_{R'} \to \mathcal{X}$. The open immersion $j\colon \Theta_R \smallsetminus 0 \to \Theta_R$ satisfies $j_*\mathcal{O}_{\Theta_R \smallsetminus 0} = \mathcal{O}_{\Theta_R}$ and by Flat Base Change (6.1.15) the same property holds for the morphisms obtained by base changing j along $\Theta_{R'} \to \Theta_R$, $\Theta_{R'} \times_{\Theta_R} \Theta_{R'} \to \Theta_R$, and $\Theta_{R'} \times_{\Theta_R} \Theta_{R'} \times_{\Theta_R} \Theta_{R'} \to \Theta_R$. By the above observation, there exists a canonical 2-isomorphism between the two extensions $\Theta_{R'} \times_{\Theta_R} \Theta_{R'} \rightrightarrows \Theta_{R'} \to \mathcal{X}$ which necessarily satisfies the cocycle condition. By fpqc descent, the lifting $\Theta_{R'} \to \mathcal{X}$ descends to a lifting $\Theta_R \to \mathcal{X}$. The same argument works for S-completeness. $\square$

Remark 6.9.18. It is also true that when $\mathcal{X}$ is of finite type over $\Bbbk$, these criteria can be verified on DVRs essentially of finite type over $\Bbbk$; see [AHLH23, §4]. We will not use this fact.

Lemma 6.9.19. *Let $f\colon \mathcal{X} \to \mathcal{Y}$ be an affine morphism of noetherian algebraic stacks. If $\mathcal{Y}$ is Θ-complete (resp., S-complete), so is $\mathcal{X}$.*

Proof. Since Θ_R is regular and $0 \in \Theta_R$ is codimension 2, the pushforward of the structure sheaf along $\Theta_R \smallsetminus 0 \to \Theta_R$ is the structure sheaf. We therefore have canonical equivalences

$$\begin{aligned}
\mathrm{Mor}_{\mathcal{Y}}(\Theta_R \smallsetminus 0, \mathcal{X}) &\cong \mathrm{Mor}_{\mathcal{O}_{\mathcal{Y}}-\mathrm{alg}}(f_*\mathcal{O}_{\mathcal{X}}, (\Theta_R \smallsetminus 0 \to \mathcal{Y})_*\mathcal{O}_{\Theta_R \smallsetminus 0}) \\
&\cong \mathrm{Mor}_{\mathcal{O}_{\mathcal{Y}}-\mathrm{alg}}(f_*\mathcal{O}_{\mathcal{X}}, (\Theta_R \to \mathcal{Y})_*\mathcal{O}_{\Theta_R}) \\
&\cong \mathrm{Mor}_{\mathcal{Y}}(\Theta_R, \mathcal{X}).
\end{aligned}$$

The case of S-completeness is identical. $\square$

Proposition 6.9.20. *If G is a reductive group over an algebraically closed field $\Bbbk$, then every quotient stack $[\operatorname{Spec} A/G]$ is Θ-complete and S-complete.*

Proof. We first show that $B\mathrm{GL}_n$ is Θ-complete. A morphism $\Theta_R \smallsetminus 0 \to \mathcal{X}$ corresponds to a vector bundle E on $\Theta_R \smallsetminus 0$. The algebraic stack Θ_R is regular and $0 \in \Theta_R$ is a codimension 2 point. If $\widetilde{E}$ is a coherent sheaf on Θ_R extending E, then the double dual $\widetilde{E}^{\vee\vee}$ is a vector bundle extending E. (In fact, the pushforward of E along $\Theta_R \smallsetminus 0 \hookrightarrow \Theta_R$ is a vector bundle.) This provides the desired extension $\Theta_R \to \mathcal{X}$. As G is affine, we can choose a faithful representation $G \subseteq \mathrm{GL}_n$. As G is reductive, the quotient GL_n/G is affine by Matsushima's Theorem (B.1.45). Using the cartesian diagram

$$\begin{tikzcd} \mathrm{GL}_n/G \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \operatorname{Spec} \Bbbk \arrow[d] \\ BG \arrow[r] & B\mathrm{GL}_n \end{tikzcd}$$

and smooth descent, we see that $BG \to B\mathrm{GL}_n$ is affine. We conclude that BG and $[\operatorname{Spec} A/G]$ are Θ-complete by Lemma 6.9.19. $\square$

For a ring R, the map $\Theta_R \to \operatorname{Spec} R$ is a good moduli space, and thus every map $\Theta_A \to X$ to an algebraic space factors through $\operatorname{Spec} R$ by the universality of good moduli spaces (Theorem 6.5.6(4)). A similar result holds if X is a Deligne–Mumford stack.

Lemma 6.9.21. *Let $\mathcal{X}$ be a noetherian algebraic stack with affine and quasi-finite diagonal. If R is a complete DVR, every map $\Theta_R \to \mathcal{X}$ (resp., $\phi_R \to \mathcal{X}$) factors through $\Theta_R \to \operatorname{Spec} R$ (resp., $\phi_R \to \operatorname{Spec} R$).*

Proof. We will reduce to the case when $\mathcal{X}$ is affine, where the statement follow easily from the fact that $\Gamma(\Theta_R, \mathcal{O}_{\Theta_R}) = \Gamma(\phi_R, \mathcal{O}_{\phi_R}) = R$. Let $\kappa = R/\mathfrak{m}$ and $x \in \mathcal{X}(\kappa)$ be the image of $0 \in \Theta_R$. Since $\mathbb{G}_m$ has no non-trivial finite quotients, the induced map $\mathbb{G}_m \to G_x$ on stabilizers is trivial. By Proposition 4.3.4, we may find a smooth presentation $U \to \mathcal{X}$ from an affine scheme together with a lift $u \in U(\kappa)$ of x. The map $B\mathbb{G}_{m,\kappa} \to \mathcal{X}$ factors through $u\colon \operatorname{Spec} \kappa \to U$ and thus lifts to a map $B\mathbb{G}_{m,\kappa} \to \operatorname{Spec} \kappa \xrightarrow{u} U$. Letting $\mathcal{T}_n$ be the nth nilpotent thickening of $B\mathbb{G}_{m,\kappa} \hookrightarrow \Theta_R$, deformation theory (Proposition 6.7.12) implies that we may find compatible lifts $\mathcal{T}_n \to U$ of $\mathcal{T}_n \hookrightarrow \Theta_R \to \mathcal{X}$. By Coherent Tannaka Duality (6.6.9), there is an extension $\Theta_R \to U$. Since $\Theta_R \to U$ factors through $\operatorname{Spec} R$, so does $\Theta_R \to \mathcal{X}$. $\square$

Proposition 6.9.22. *Every noetherian algebraic stack $\mathcal{X}$ with affine and quasi-finite diagonal (e.g., a Deligne–Mumford stack with affine diagonal) is Θ-complete. Moreover, $\mathcal{X}$ is S-complete if and only if it is separated.*

Proof. By Lemma 6.9.17, Θ-completeness and S-completeness can be tested on a complete DVR R. Lemma 6.9.21 implies that $\mathcal{X}$ is Θ-complete and also implies that $\mathcal{X}$ is S-complete if only if every diagram

$$\begin{tikzcd} \operatorname{Spec} R \bigcup_{\operatorname{Spec} K} \operatorname{Spec} R \arrow[r] \arrow[d] & \mathcal{X} \\ \operatorname{Spec} R \arrow[ur, dashed] & \end{tikzcd}$$

has a lift, which is the usual valuative criterion for separatedness. $\square$

Θ- and S-completeness are necessary for the existence of a separated good moduli space.

Proposition 6.9.23. *Let $\mathcal{X}$ be an algebraic stack of finite type over an algebraically closed field $\Bbbk$ with affine diagonal. If $\pi\colon \mathcal{X} \to X$ be a good moduli space, then $\mathcal{X}$ is Θ-complete. Moreover, $\mathcal{X}$ is S-complete if and only if X is separated.*

Proof. Since X is Θ-complete, to see that $\mathcal{X}$ is Θ-complete, it suffices to find a lift of every commutative diagram

$$\begin{tikzcd} \Theta_R \setminus 0 \arrow[r] \arrow[d] & \mathcal{X} \arrow[d, "\pi"] \\ \operatorname{Spec} R \arrow[r] \arrow[ur, dashed] & X \end{tikzcd}$$

of solid arrows. By the Local Structure for Good Moduli Spaces (6.7.3), there exists an étale morphism $\operatorname{Spec} B \to X$ containing the image of $\operatorname{Spec} R$ such that $\mathcal{X} \times_X \operatorname{Spec} B \cong [\operatorname{Spec} A/G]$ with G linearly reductive and $B = A^G$. Since $\operatorname{Spec} R \to X$ lifts to $\operatorname{Spec} B$ after an extension of DVRs and since Θ-completeness can be checked after an extension (Lemma 6.9.17), we are reduced to the case of $[\operatorname{Spec} A/G]$. This is Proposition 6.9.20.

If X is separated, then X is S-complete as $\phi_R \setminus 0 = \operatorname{Spec} R \cup_{\operatorname{Spec} K} \operatorname{Spec} R \to X$ factors through $\operatorname{Spec} R$ by the valuative criterion for separatedness. The same argument as above shows that $\mathcal{X}$ is S-complete. Conversely, suppose $f, g \colon \operatorname{Spec} R \to X$ are two maps such that $f|_K = g|_K$. After possibly an extension of R, we may choose a lift $\operatorname{Spec} K \to \mathcal{X}$ of $f|_K = g|_K$. Since $\mathcal{X} \to X$ is universally closed (Theorem 6.5.6(1)), after possibly further extensions of R, we may choose lifts $\widetilde{f}, \widetilde{g} \colon \operatorname{Spec} R \to \mathcal{X}$ of f, g such that $\widetilde{f}|_K \cong \widetilde{g}|_K$ by the Valuative Criterion for Universal Closedness (3.8.7). Since $\mathcal{X}$ is S-complete, we can extend $\widetilde{f}$ and $\widetilde{g}$ to a morphism $\phi_R \to \mathcal{X}$. As $\phi_R \to \operatorname{Spec} R$ is a good moduli space and hence universal for maps to algebraic spaces, the morphism $\phi_R \to \mathcal{X}$ descends to a unique morphism $\operatorname{Spec} R \to X$ which necessarily must be equal to both f and g. We conclude that X is separated by the Valuative Criterion for Separatedness. $\square$

6.9.4 Examples of Θ- and S-completeness

We discuss Θ- and S-completeness for quotient stacks, stacks of coherent sheaves, and the stack of all curves.

Quotient stacks. By Proposition 6.9.1, a map $\Theta \to [U/G]$ is classified by a point $u \in U$ and a one-parameter subgroup $\lambda \colon \mathbb{G}_m \to G$ such that $\lim_{t\to 0} \lambda(t) \cdot u$ exists. We apply this to provide a geometric characterization of Θ-completeness for quotient stacks. Under the $\mathbb{G}_m$-action induced by λ, the attractor locus U_λ^+ is defined as the algebraic space representing the functor $\operatorname{Mor}_{\Bbbk}^{\mathbb{G}_m}(\mathbb{A}^1, U)$ (see Theorem 6.8.9). The evaluation map $\mathrm{ev}_1 \colon U_\lambda^+ \to U$ is defined by sending $f \colon \mathbb{A}^1 \to U$ to $f(1)$.

Proposition 6.9.24. *Let G be a smooth linearly reductive group over an algebraically closed field $\Bbbk$, and U be a separated algebraic space of finite type over $\Bbbk$ with an action of G. Then*

$[U/G]$ is Θ-complete	$\iff$	*for every map $u \colon \operatorname{Spec} R \to U$ from a complete DVR over $\Bbbk$ with algebraically closed residue field and one-parameter subgroup $\lambda \colon \mathbb{G}_m \to G$ such that $\lim_{t\to 0} \lambda(t) \cdot u_K \in U(K)$ exists, then $\lim_{t\to 0} \lambda(t) \cdot u \in U(R)$ also exists;*
	$\iff$	*for every one-parameter subgroup $\lambda \colon \mathbb{G}_m \to G$, the morphism $\mathrm{ev}_1 \colon U_\lambda^+ \to U$ is a closed immersion.*

Proof. Since G is linearly reductive, BG is Θ-complete (Proposition 6.9.20). Therefore Θ-completeness of $[U/G]$ is equivalent to the existence of a lift in every diagram

$$\begin{tikzcd} \Theta_R \setminus 0 \arrow[r] \arrow[d, hook] & {[U/G]} \arrow[d] \\ \Theta_R \arrow[r] \arrow[ur, dashed] & BG \end{tikzcd} \tag{6.9.25}$$

where R is a complete DVR with algebraically closed residue field (Lemma 6.9.17). By Proposition 6.9.1 and Exercise 6.9.4, the map $\Theta_R \to BG$ corresponds to a one-parameter subgroup $\lambda\colon \mathbb{G}_m \to G$ while $\Theta_R \setminus 0 \to [U/G]$ corresponds to a map $u\colon \operatorname{Spec} R \to U$ such that $\lim_{t\to 0} \lambda(t)\cdot u_K \in U(K)$ exists. In other words, we have a commutative diagram

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & U_\lambda^+ \arrow[d, "\mathrm{ev}_1"] \\ \operatorname{Spec} R \arrow[r, "u"] \arrow[ur, dashed] & U \end{tikzcd} \tag{6.9.26}$$

of solid arrows. A lift of (6.9.25) corresponds to the existence of $\lim_{t\to 0}\lambda(t)\cdot u \in U(R)$ or equivalently to a lift of (6.9.26). Since $\mathrm{ev}_1\colon U_\lambda^+ \to U$ is a monomorphism of finite type, it is closed immersion if and only if it is proper if and only if satisfies the existence part of the valuative criterion. □

Example 6.9.27. When $U = \operatorname{Spec} A$ is affine, a one-parameter subgroup $\lambda\colon \mathbb{G}_m \to G$ induces a grading $A = \bigoplus_{d\in\mathbb{Z}} A_d$, and U_λ^+ is representable by $V(\sum_{d<0} A_d)$ (Exercise 6.8.7). We see thus that $\mathrm{ev}_1\colon U_\lambda^+ \hookrightarrow U$ is a closed immersion; this recovers the fact that $[U/G]$ is Θ-complete (Proposition 6.9.20).

Example 6.9.28. Consider the $\mathbb{G}_m$-action on $\mathbb{P}^1$ given by $t\cdot[x:y] = [tx:y]$. Taking $\lambda = \mathrm{id}$, then $(\mathbb{P}^1)_\lambda^+ = \mathbb{A}^1 \coprod \{\infty\}$, and we see that $[\mathbb{P}^1/\mathbb{G}_m]$ is not Θ-complete. Similarly, the quotient stack $[C/\mathbb{G}_m]$ of the nodal cubic C is not Θ-complete: letting $\mathbb{P}^1 \to C$ be the normalization where the fiber over the node is $\{0,\infty\}$, then $C_\lambda^+ = \mathbb{P}^1 \setminus \infty$ for $\lambda = \mathrm{id}$. Finally, if $\mathbb{G}_m$ acts on $X = \mathbb{A}^2 \setminus 0$ via $t\cdot(x,y) = (tx,y)$ then $[X/\mathbb{G}_m]$ is not Θ-complete as $X_\lambda^+ = \{y \neq 0\}$.

Remark 6.9.29. Consider the algebraic stack $\underline{\mathrm{Mor}}(\Theta, [X/G])$ of morphisms, which decomposes as a disjoint union $\coprod_\lambda [X_\lambda^+/P_\lambda]$, where λ varies over conjugation classes of one-parameter subgroups $\lambda\colon \mathbb{G}_m \to G$ (Remark 6.9.5). The evaluation morphism $\mathrm{ev}_0\colon [X_\lambda^+/P_\lambda] \to [X/G]$ is induced by the inclusion $X_\lambda^+ \to X$. The Θ-completeness of $[X/G]$ is therefore equivalent to the properness of the maps $[X_\lambda^+/P_\lambda] \to [X/G]$.

It is also possible to give a criterion for when $[U/G]$ is S-complete in terms of one-parameter subgroups $\lambda\colon \mathbb{G}_m \to G$ and properties of the morphism $\mathrm{Mor}^{\mathbb{G}_m}_{\mathbb{A}^1}(\mathbb{A}^2, U\times\mathbb{A}^1) \to U \times U \times \mathbb{A}^1$, where the maps to U are obtained by restricting along the two maps $\mathbb{A}^1 \to \mathbb{A}^2$ given by $x \mapsto (x,1)$ and $x \mapsto (1,x)$. We resist giving this criterion here as it is more involved and not needed for later.

Stacks of coherent sheaves. Recall that maps $\Theta \to \underline{\mathrm{Coh}}(X)$ correspond to filtrations (Proposition 6.9.6).

Proposition 6.9.30. *For every projective scheme X over an algebraically closed field $\Bbbk$, the algebraic stack $\underline{\mathrm{Coh}}(X)$ is Θ- and S-complete.*

Proof. Given a DVR R, Proposition 6.9.6 implies that a map $\Theta_R \setminus 0 \to \underline{\mathrm{Coh}}(X)$ corresponds to a coherent sheaf E on X_R flat over R and a $\mathbb{Z}$-graded filtration $F_\bullet\colon \subseteq \cdots F_{i-1} \subseteq F_i \subseteq \cdots \subseteq E_K$ such that $F_i = E_K$ for $i \gg 0$, $F_i = 0$ for $i \ll 0$, and F_i/F_{i-1} is flat over R. Viewing E as a subsheaf of E_K, we define $E_i := F_i \cap E$ as the intersection in E_K. Since E_i/E_{i-1} is a subsheaf of F_i/F_{i-1}, it is torsion free, hence flat as an R-module. The filtration $E_\bullet$ defines an extension $\Theta_R \to \underline{\mathrm{Coh}}(X)$. (Aside: this is exactly the argument for the valuative criterion of properness of the Quot scheme (Proposition 1.4.2). Note also that if we let $j\colon \Theta_R \setminus 0 \hookrightarrow \Theta_R$, let $j_x\colon \operatorname{Spec} R \hookrightarrow \Theta_R$ and let $j_\pi\colon \Theta_K \to \Theta_R$ denote the open immersions, and let $\mathcal{E}$ be the coherent sheaf on $C \times (\Theta_R \setminus 0)$ denoting the union of E and $F_\bullet$, then the extension is given by $(\mathrm{id} \times j)_* \mathcal{E} = j_{x,*}E \cap j_{\pi,*}F_\bullet = E[x^{\pm 1}] \cap F_\bullet = E_\bullet$, where $E[x^{\pm 1}]$ is the $\mathbb{Z}$-graded filtration given by placing E in every degree.)

For S-completeness, suppose we are given a map $\phi_R \setminus 0 \to \underline{\mathrm{Coh}}(X)$ corresponding to coherent sheaves E and F flat over R and an isomorphism $\alpha\colon E_K \to F_K$. Recalling the quotient presentation $\phi_R = [\operatorname{Spec}(R[s,t]/(st-\pi))/\mathbb{G}_m]$, we have several natural open immersions: $j\colon \phi_R \setminus 0 \hookrightarrow \phi_R$, $j_s, j_t\colon \operatorname{Spec} R \hookrightarrow \phi_R$ (with $s \neq 0$ and $t \neq 0$), and $j_{st}\colon \operatorname{Spec} K \to \phi_R$ (with $st \neq 0$). We compute the pushforward as the equalizer

$$
\begin{array}{ccccccc}
0 \longrightarrow (\mathrm{id} \times j)_* \mathcal{E} \longrightarrow & (\mathrm{id} \times j_s)_* E \oplus (\mathrm{id} \times j_t)_* F & \longrightarrow & (\mathrm{id} \times j_{st})_* F_K \\
& (a,b) & \longmapsto & a - \alpha(b).
\end{array}
$$

The pushforwards can be computed as graded modules over $R[s,t]/(st-\pi)$:

$$
\begin{aligned}
(\mathrm{id} \times j_{st})_* F_K &= F_K \otimes_R R[t^{\pm 1}] = \bigoplus_{n \in \mathbb{Z}} F_K t^n, \\
(\mathrm{id} \times j_s)_* E &= E \otimes_R R[t^{\pm 1}] = \bigoplus_{n \in \mathbb{Z}} E t^n, \\
(\mathrm{id} \times j_t)_* F &= F \otimes_R R[s^{\pm 1}] \cong \bigoplus_{n \in \mathbb{Z}} (\pi^{-n} \cdot F) t^n \subseteq (\mathrm{id} \times j_{st})_* F_K
\end{aligned}
$$

where we have used that $s = t^{-1}\pi$. Thus

$$
j_* \mathcal{E} \cong \bigoplus_{n \in \mathbb{Z}} \big(E \cap (\pi^{-n} \cdot F)\big) t^n \subseteq (\mathrm{id} \times j_{st})_* F_K.
$$

Each R-module $E \cap (\pi^{-n} \cdot F) \subseteq E$ is finitely generated since E is. Since the ascending chain $\cdots \subseteq E \cap (\pi^{-n} \cdot F) \subseteq E \cap (\pi^{-n-1} \cdot F) \subseteq \cdots$ terminates to E, $j_* \mathcal{E}$ is coherent. To show that $j_* \mathcal{E}$ is flat over ϕ_R, we only need to check that it is flat over 0. By the Local Criterion for Flatness (Theorem A.2.5), we need to show that $\mathrm{Tor}_1^A(A/\mathfrak{m}, j_* \mathcal{E}) = 0$ where $A = R[s,t]/(st - \pi)$ and $\mathfrak{m} = (s,t)$. The Koszul complex gives a resolution of the residue field $\kappa = A/\mathfrak{m} = R/\pi$:

$$
0 \to A \xrightarrow{(t,-s)} A \oplus A \xrightarrow{(s,t)} A \to \kappa \to 0.
$$

Tensoring with $j_* \mathcal{E}$ yields a complex

$$
0 \to j_* \mathcal{E} \xrightarrow{(t,-s)} j_* \mathcal{E} \oplus j_* \mathcal{E} \xrightarrow{(s,t)} j_* \mathcal{E}. \tag{6.9.31}
$$

The pushforward of the exact sequence

$$
0 \to \mathcal{O}_{\phi_R \setminus 0} \xrightarrow{(t,-s)} \mathcal{O}_{\phi_R \setminus 0} \oplus \mathcal{O}_{\phi_R \setminus 0} \xrightarrow{(s,t)} \mathcal{O}_{\phi_R \setminus 0} \to 0
$$

along $\mathrm{id} \times j \colon C \times \phi_R \smallsetminus 0 \hookrightarrow C \times \phi_R$ is a left exact sequence of vector bundles, and tensoring with $j_*\mathcal{E}$ yields a left exact sequence, which identified with (6.9.31). Thus $\mathrm{Tor}_1^A(A/\mathfrak{m}, j_*\mathcal{E}) = 0$. □

In Proposition 8.3.6, we provide a criterion for when an open substack of $\underline{\mathrm{Coh}}(X)$ is Θ- and S-complete. For $X = C$ is a smooth curve, we apply this criterion in Proposition 8.3.8 to establish the Θ- and S-completeness of $\mathcal{B}un^{\mathrm{ss}}_{r,d}$, from which we conclude the existence of a good moduli space.

Stack of all curves. The stacks $\mathcal{M}_g$ and $\overline{\mathcal{M}}_g$ of smooth and stable curves are both Θ- and S-complete as they are separated Deligne–Mumford stacks. While maps from Θ to the stack of all curves correspond to test configurations (Proposition 6.9.7), there is unfortunately no known simple criteria—similar to the above criteria for quotient stacks and stacks of coherent sheaves—to verify whether a given substack of the stack $\mathcal{M}_g^{\mathrm{all}}$ of all curves is Θ-complete or S-complete.

6.10 Existence of good moduli spaces

I believe that GIT did not fulfill all the early expectations. The first successes leading to the construction of the moduli spaces of curves, Abelian varieties, vector bundles and sheaves were not carried much further. For instance, GIT never came up with a good approach to compactify the moduli space of surfaces of general type, and it did not tackle the moduli problem for higher dimensional varieties.

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We prove necessary and sufficient conditions for the existence of a separated good moduli space in characteristic 0.

Theorem 6.10.1 (Existence Theorem of Good Moduli Spaces)**.** *Let $\mathcal{X}$ be an algebraic stack, of finite type over an algebraic closed field $\Bbbk$ of characteristic 0, with affine diagonal. There exists a good moduli space $\pi\colon \mathcal{X} \to X$ with X a separated algebraic space if and only if $\mathcal{X}$ is Θ-complete (Definition 6.9.10) and S-complete (Definition 6.9.14). Moreover, X is proper if and only if $\mathcal{X}$ satisfies the existence part of the valuative criterion for properness.*

6.10.1 Strategy for constructing good moduli spaces

The Local Structure Theorem for Algebraic Stacks (6.7.1) gives us a natural strategy to construct the good moduli space X. For each closed point $x \in \mathcal{X}$, we have an étale quotient presentation

$$\begin{array}{ccc} \mathcal{W} = [\mathrm{Spec}\, A/G_x] & \xrightarrow{f} & \mathcal{X} \\ \downarrow & & \\ W = \mathrm{Spec}\, A^{G_x} & & \end{array}$$

where f is affine and étale, and there is a preimage $w \in \mathcal{W}$ of x such that f induces an isomorphism of stabilizer groups at w. We want to show that the GIT quotients $W = \mathrm{Spec}\, A^{G_x}$, as x ranges over closed points, provide étale models that can be glued to a good moduli space of $\mathcal{X}$. To this end, we need to construct an étale equivalence relation on W. Since f is affine, the fiber product $\mathcal{R} := \mathcal{W} \times_{\mathcal{X}} \mathcal{W}$ is isomorphic to a quotient stack $[\operatorname{Spec} B/G_x]$ and we have a diagram

$$\begin{tikzcd} \mathcal{R} \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] \arrow[d] & \mathcal{W} \arrow[r, "f"] \arrow[d] & \mathcal{X} \\ R \arrow[r, shift left, "q_1"] \arrow[r, shift right, "q_2"'] & W & \end{tikzcd}$$

where $R = \operatorname{Spec} B^{G_x}$. If $q_1, q_2 \colon R \rightrightarrows W$ defines an étale equivalence relation, the algebraic space quotient W/R gives a candidate for a good moduli space of $f(\mathcal{W}) \subseteq \mathcal{X}$. Luna's Fundamental Lemma (6.5.31) provides condition on when $q_1, q_2 \colon R \rightrightarrows W$ are étale: we need that for all closed points $r \in \mathcal{R}$ that

(a) $p_1(r), p_2(r) \in \mathcal{W}$ are closed points; and

(b) p_1 and p_2 induce isomorphisms of stabilizer groups at r.

On the other hand, we know that $f(w) \in \mathcal{X}$ is closed and f induces an isomorphism of stabilizer groups at the given preimage w of x. We want to show that there is an open neighborhood $\mathcal{U}$ of w such that the restriction $f|_{\mathcal{U}}$ satisfies: (a) $f|_{\mathcal{U}}$ sends closed points map to closed points and (b) $f|_{\mathcal{U}}$ induces isomorphisms of stabilizer groups at closed points, and moreover that these conditions are stable under base change.[9] While property (b) is stable under base change, property (a) is not, and we will introduce a stronger condition below—called Θ-surjectivity (Definition 6.10.8)—which is stable under base change and implies (a).

In the construction, the role of Θ-completeness and S-completeness is the following: the Θ-completeness of $\mathcal{X}$ implies that Θ-surjectivity holds (and thus condition (a) and its base changes hold) in an open neighborhood of w (Proposition 6.10.13), while S-completeness implies that condition (b) holds in an open neighborhood of w (Proposition 6.10.20).

6.10.2 Counterexamples

The following examples do not admit good moduli spaces. As the proof strategy above fails, these examples are instructive for understanding the role of Θ- and S-completeness. We work over an algebraically closed field $\Bbbk$.

The following three examples—$[\mathbb{P}^1/\mathbb{G}_m]$, $[C/\mathbb{G}_m]$, and $[(\mathbb{A}^2 \setminus 0)/\mathbb{G}_m]$—violate Θ-completeness as demonstrated in Example 6.9.28.

Example 6.10.2. The quotient stack $[\mathbb{P}^1/\mathbb{G}_m]$ does not admit a good moduli space, where $\mathbb{G}_m$ acts on $\mathbb{P}^1$ via $t\cdot[x:y] = [tx:y]$. For a stack admitting a good moduli space, every $\Bbbk$-point has a unique closed point in its closure (Theorem 6.5.6(2)). This fails here as $[1:1]$ specializes to two closed points $[1:0]$ and $[0:1]$. Alternatively, if there were a good moduli space, it would have to be $[\mathbb{P}^1/\mathbb{G}_m] \to \operatorname{Spec} \Bbbk$ (which is universal for maps to algebraic spaces), but then the composition $\mathbb{P}^1 \to [\mathbb{P}^1/\mathbb{G}_m] \to \operatorname{Spec} \Bbbk$ would be affine by Serre's Criterion for Affineness (4.5.16), a contradiction.

There are two open substacks $\mathcal{U}_1, \mathcal{U}_2 \subseteq [\mathbb{P}^1/\mathbb{G}_m]$ isomorphic to $[\mathbb{A}^1/\mathbb{G}_m]$ each which admits a good moduli space $\pi_i \colon \mathcal{U}_i \to \operatorname{Spec} \Bbbk$ but they do not glue to a good moduli space of $[\mathbb{P}^1/\mathbb{G}_m]$: the intersection $\mathcal{U}_1 \cap \mathcal{U}_2$ is the open point in both $\mathcal{U}_1$ and $\mathcal{U}_2$ and not the preimage of an open subscheme under π_i. To see how the approach

[9] By Luna's Fundamental Lemma (6.5.28), we do know that there is an open neighborhood $\mathcal{R}_1$ (resp., $\mathcal{R}_2$) of $\rho = (w, w, \mathrm{id}) \in \mathcal{R}$ where p_1 (resp., p_2) satisfies (a) and (b), but it is not clear that these loci are $\mathcal{R}$-invariant, i.e., the preimage of an open substack of $\mathcal{W}$.

above fails, observe that the étale presentation $f\colon \mathcal{W} := \mathcal{U}_1 \coprod \mathcal{U}_2 \to [\mathbb{P}^1/\mathbb{G}_m]$ satisfies (a) and (b) but the base changes $p_1, p_2\colon \mathcal{W} \times_{[\mathbb{P}^1/\mathbb{G}_m]} \mathcal{W} = \mathcal{U}_1 \coprod \mathcal{U}_2 \coprod \mathcal{U}_1 \cap \mathcal{U}_2 \to \mathcal{W}$ fails (b), i.e., the closed point in $\mathcal{U}_1 \cap \mathcal{U}_2$ is mapped to a non-closed point under either projection.

Example 6.10.3. For a related example, let C be the projective nodal cubic with its $\mathbb{G}_m$-action. The quotient $[C/\mathbb{G}_m]$ has two points—one open and one closed—but while there is no topological obstruction as above, $[C/\mathbb{G}_m]$ again does not admit a good moduli space because C is projective, not affine. Viewing the nodal cubic as the quotient of nodal union X' of two $\mathbb{P}^1$'s along 0 and ∞ modulo the rotation action of $\mathbb{Z}/2$, we have a finite étale cover $[X'/\mathbb{G}_m] \to [X/\mathbb{G}_m]$. Removing one of the origins, we have an affine étale cover $\mathcal{W} = [\operatorname{Spec}(\Bbbk[x,y]/xy)/\mathbb{G}_m] \to [C/\mathbb{G}_m]$ where $\mathbb{G}_m$ acts via $t \cdot (x,y) = (tx, t^{-1}y)$. Again, this map sends closed points to closed points, but the projections $\mathcal{W} \times_{[C/\mathbb{G}_m]} \mathcal{W} \rightrightarrows \mathcal{W}$ do not.

Example 6.10.4. Let $\mathbb{G}_m$ act on $\mathbb{A}^2$ via $t \cdot (x,y) = (tx, y)$ and set $\mathcal{X} = [\mathbb{A}^2/\mathbb{G}_m] \setminus 0$. The point $p = (1,0) \in \mathcal{X}$ is closed with trivial stabilizer, and the open immersion $f\colon \mathbb{A}^1 \hookrightarrow \mathcal{X}$, sending z to $(z,1)$, is an étale quotient presentation. Note that while $f(0)$ is closed, the image $f(z)$ is not closed for $z \neq 0$. The map $\mathcal{X} \to \mathbb{A}^1$ defined by $(x,y) \mapsto y$ is not a good moduli space as $\mathbb{A}^2 \setminus 0$ is not affine.

Examples violating Θ-completeness naturally occur in moduli, e.g., by removing a single polystable but not stable vector bundle from $\mathcal{B}un_{r,d}(C)^{\mathrm{ss}}$. The next three examples violate S-completeness.

Example 6.10.5. Suppose $\operatorname{char}(\Bbbk) \neq 2$ and let $G = \mathbb{Z}/2$ act on the non-separated union $U = \mathbb{A}^1 \bigcup_{x \neq 0} \mathbb{A}^1$ by exchanging the copies of $\mathbb{A}^1$. The quotient stack $[U/G]$ has a $\mathbb{Z}/2$ stabilizer everywhere except at the origin. This is a Deligne–Mumford stack with quasi-finite but not finite inertia, and is not S-complete. In fact, we have seen this before in Exercise 4.4.16, when we illustrated the necessity of finite inertia hypothesis in the Keel–Mori Theorem (4.4.6). By precomposing by the inclusion of one of the $\mathbb{A}^1$'s, we have an affine étale morphism $\mathbb{A}^1 \to [U/G]$ which is stabilizer preserving at 0 but not in any open neighborhood of 0.

Example 6.10.6. The Deligne–Mumford locus $\mathcal{X}^{\mathrm{DM}}$ in the moduli stack $\mathcal{X} = [\operatorname{Sym}^4 \mathbb{P}^1/\operatorname{PGL}_2]$ of four unordered points in $\mathbb{P}^1$ is not separated (see Example 4.4.17) and thus not S-complete. Note however that the stable locus $\mathcal{X}^{\mathrm{s}}$ consisting of four distinct points is separated and the semistable locus $\mathcal{X}^{\mathrm{ss}} = \mathcal{X}^{\mathrm{DM}} \cup \{[0:0:\infty:\infty]\}$ has a projective good moduli space.

Example 6.10.7. Consider the action of $G = \mathbb{G}_m \rtimes \mathbb{Z}/2$ on $X = \mathbb{A}^2 \setminus 0$ via $t \cdot (a,b) = (ta, t^{-1}b)$ and $-1 \cdot (a,b) = (b,a)$. Note that every point $(a,b) \in X$ with $ab \neq 0$ is fixed by the order 2 element $(a/b, -1) \in G$. The quotient stack $[X/G]$ is a non-separated Deligne–Mumford stack that is not S-complete and that does not admit a good moduli space; note however that $[\mathbb{A}^2/G] \to \operatorname{Spec} \Bbbk[xy]$ is a good moduli space.

6.10.3 Θ-completeness and Θ-surjectivity

The property that a morphism $\mathcal{X} \to \mathcal{Y}$ sends closed points to closed points is not stable under base change (see Examples 6.10.2 and 6.10.3). We introduce a stronger and better behaved property called Θ-surjectivity. The main result of this section is that an étale quotient presentation $([\operatorname{Spec} A/G_x], w) \to (\mathcal{X}, x)$ is Θ-surjective in an open neighborhood of w as long as $\mathcal{X}$ is Θ-complete (Proposition 6.10.13). As motivated in §6.10.1, this result will be crucial in proving the main existence theorem (Theorem 6.10.1) of this chapter.

Definition 6.10.8. Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks and $x \in \mathcal{X}(\Bbbk)$ be a geometric point. We say that f is *Θ-surjective at x* if every diagram

$$\begin{tikzcd} \operatorname{Spec} \Bbbk \arrow[r, "x"] \arrow[d, hook, "1"] & \mathcal{X} \arrow[d, "f"] \\ \Theta_{\Bbbk} \arrow[r] \arrow[ur, dashed] & \mathcal{Y} \end{tikzcd} \tag{6.10.9}$$

has a lift. We say that f is *Θ-surjective* if it is Θ-surjective at every geometric point.

This notion is clearly stable under base change. Every morphism $f\colon \mathcal{X} \to \mathcal{Y}$ of noetherian algebraic stacks, where $\mathcal{Y}$ has affine and quasi-finite diagonal, is Θ-surjective since in this case every map $\Theta_{\Bbbk} \to \mathcal{Y}$ factors through $\operatorname{Spec} \Bbbk$ (Lemma 6.9.21). The next lemma gives conditions for when the lift is unique and when the definition is independent of the choice of geometric point.

Lemma 6.10.10. *Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a separated, representable, and finite type morphism of noetherian algebraic stacks.*

(1) Every lift of (6.10.9) is unique.

(2) If f is Θ-surjective at a geometric point $x \in \mathcal{X}(\Bbbk)$, then f is Θ-surjective at every other geometric point $x' \in \mathcal{X}(\Bbbk')$ representing the same point in $|\mathcal{X}|$ as x.

Proof. Part (1) follows from descent and the valuative criterion for separatedness. To show (2), it suffices to show that given an extension $\Bbbk \to \Bbbk'$ of algebraically closed fields, a lift $\Theta_{\Bbbk'} \to \mathcal{X}$ implies the existence of a lift $\Theta_{\Bbbk} \to \mathcal{X}$. We write $\Bbbk' = \bigcup_{\lambda} A_{\lambda}$ as a union of finitely generated $\Bbbk$-subalgebras. By Limit Methods (§B.3), there exists a lift $\Theta_{A_{\lambda}} \to \mathcal{X}$ of $\operatorname{Spec} A_{\lambda} \to \mathcal{X}$. Restricting along a closed point of $\operatorname{Spec} A_{\lambda}$ provides a lift over $\Bbbk$. □

Proposition 6.10.11. *Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks, each of finite type over an algebraically closed field $\Bbbk$ with affine diagonal. Suppose that the closed points of $\mathcal{Y}$ have linearly reductive stabilizer. If f is Θ-surjective, then f sends closed points to closed points.*

Proof. Let $x \in |\mathcal{X}|$ be a closed point. Let $f(x) \rightsquigarrow y_0$ be a specialization to a closed point. By the Destabilization Theorem (7.3.8), this specialization can be realized by a map $\Theta \to \mathcal{Y}$. Since f is Θ-surjective, this can be lifted to a map $g\colon \Theta \to \mathcal{X}$ with $g(1) = x$. But $x \in |\mathcal{X}|$ is a closed point, so this lift must correspond to the trivial specialization $x \rightsquigarrow x$. It follows that $f(x) = y_0$ is a closed point. □

Remark 6.10.12. The converse is not true. In Example 6.10.3, where C is the nodal cubic with $\mathbb{G}_m$-action, the étale morphism $[\operatorname{Spec}(\Bbbk[x,y]/(xy))/\mathbb{G}_m] \to [C/\mathbb{G}_m]$ sends closed points to closed points but is not Θ-surjective.

Proposition 6.10.13. *Let $\mathcal{X}$ be an algebraic stack of finite type over an algebraically closed field $\Bbbk$ with affine diagonal such that the closed points of $\mathcal{X}$ have linearly reductive stabilizers. Let $x \in |\mathcal{X}|$ be a closed point, let $f\colon ([\operatorname{Spec} A/G_x], w) \to (\mathcal{X}, x)$ be an affine étale morphism inducing an isomorphism of stabilizer groups at w, and let $\pi\colon [\operatorname{Spec} A/G_x] \to \operatorname{Spec} A^{G_x}$. If $\mathcal{X}$ is Θ-complete, there exists an open affine neighborhood $U \subseteq \operatorname{Spec} A^{G_x}$ of $\pi(w)$ such that $f|_{\pi^{-1}(U)}\colon \pi^{-1}(U) \to \mathcal{X}$ is Θ-surjective.*

Proof. Let $\mathcal{W} = [\operatorname{Spec} A/G_x]$ and define $\Sigma_f \subseteq |\mathcal{W}|$ as the set of points $y \in |\mathcal{W}|$ such that f is Θ-surjective at y. We first show that $\Sigma_f \subseteq \mathcal{W}$ is open in the case that $\mathcal{X} \cong [\operatorname{Spec} B/G]$ with G linearly reductive. Zariski's Main Theorem (6.1.18) provides a factorization

$$f\colon \mathcal{W} \overset{j}{\hookrightarrow} \widetilde{\mathcal{X}} \xrightarrow{\nu} \mathcal{X}$$

where j is an open immersion and ν is a finite morphism. By Lemma 6.9.19, $[\operatorname{Spec} B/G]$ is Θ-complete. By Proposition 6.9.20, since ν is affine, $\widetilde{\mathcal{X}}$ is also Θ-complete. The finiteness of ν implies that $\Sigma_j = \Sigma_f$, and we may thus assume that f is an open immersion. Let $\mathcal{Z} \subseteq \mathcal{X}$ be the reduced complement of $\mathcal{W}$ and let $\pi\colon \mathcal{X} \to \operatorname{Spec} B^G$ denote the good moduli space. We claim that $|\mathcal{W}| \smallsetminus \Sigma_f = \pi^{-1}(\pi(|\mathcal{Z}|)) \cap |\mathcal{W}|$. The inclusion "$\subseteq$" is clear: the morphism $\mathcal{X} \smallsetminus \pi^{-1}(\pi(|\mathcal{Z}|)) \hookrightarrow \mathcal{X}$ is the base change of the Θ-surjective morphism $X \smallsetminus \pi(|\mathcal{Z}|) \hookrightarrow X$ of algebraic spaces. Conversely, let $y \in \pi^{-1}(\pi(|\mathcal{Z}|)) \cap |\mathcal{W}|$ represented by a geometric point $\operatorname{Spec} \mathbb{F} \to \mathcal{X}$. Let $z \in |\mathcal{Z}_{\mathbb{F}}|$ be the unique closed point in the closure of $y \in |\Theta_{\mathbb{F}}|$ and let $\Theta_{\mathbb{F}} \to \mathcal{X}_{\mathbb{F}}$ be a morphism representing the specialization $y \rightsquigarrow z$ (Corollary 7.3.8). Since $\Theta_{\mathbb{F}} \to \mathcal{X}$ does not lift to $\mathcal{W}$, $y \notin \Sigma_f$.

We now claim that $\Sigma_f \subseteq \mathcal{W}$ is constructible. Use the Local Structure Theorem (6.7.1) to choose an affine, étale, and surjective morphism $g\colon \mathcal{X}' = [\operatorname{Spec} B/G] \to \mathcal{X}$ with G linearly reductive. Let $\mathcal{W}' = \mathcal{W} \times_{\mathcal{X}} \mathcal{X}'$ with projections $g'\colon \mathcal{W}' \to \mathcal{W}$ and $f'\colon \mathcal{W}' \to \mathcal{X}'$. Since we already know that $\Sigma_{f'}$ is open, the claim follows from Chevalley's Theorem (3.3.31) once we show that $\mathcal{W} \smallsetminus \Sigma_f = g'(\mathcal{W}' \smallsetminus \Sigma_{f'})$. To see this, it suffices to show that for an algebraically closed field $\mathbb{F}$, every map $h\colon \Theta_{\mathbb{F}} \to \mathcal{X}$ lifts to a map $h'\colon \Theta_{\mathbb{F}} \to \mathcal{X}'$. Let $x' \in \mathcal{X}'(\mathbb{F})$ be a preimage of $h(0) \in \mathcal{X}(\mathbb{F})$. Since g is representable and étale, the induced map $G_{x'} \to G_{h(0)}$ on stabilizers is injective with finite cokernel. Thus the map $\mathbb{G}_{m,\mathbb{F}} \to G_{h(0)}$ on stabilizers induced by $h\colon \Theta_{\mathbb{F}} \to \mathcal{X}$ factors through $G_{x'}$. We may therefore lift the map $h|_{B\mathbb{G}_{m,\mathbb{F}}}$ to a map $B\mathbb{G}_{m,\mathbb{F}} \to \mathcal{X}'$. Letting $\mathcal{X}_n$ be the nth nilpotent thickening of $B\mathbb{G}_{m,\mathbb{F}} \hookrightarrow \Theta_{\mathbb{F}}$, there are compatible lifts $\mathcal{X}_n \to \mathcal{X}'$ of $\mathcal{X}_n \to \mathcal{X}$ by deformation theory (Proposition 6.7.12) which extends to a lift $\Theta_{\mathbb{F}} \to \mathcal{X}'$ by Coherent Tannaka Duality (6.6.9).

Since $\Sigma_f \subseteq \mathcal{W}$ is constructible and $w \in \Sigma_f$, to show that Σ_f is open, it suffices to show that every generization $\xi \rightsquigarrow w$ of w is contained in Σ_f. Let $h\colon \operatorname{Spec} R \to \mathcal{W}$ be a morphism from a complete DVR representing the specialization $\xi \rightsquigarrow w$. Letting K and κ be the fraction and residue field of R, we claim that there exists a lift (necessarily unique as f is separated)

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r, "h_{\mathbb{F}}"] \arrow[d, hook] & \mathcal{W} \arrow[d, "f"] \\ \Theta_K \arrow[r, "g"] \arrow[ur, dashed, "\widetilde{g}"] & \mathcal{X}. \end{tikzcd} \tag{6.10.14}$$

This claim implies that f is Θ-surjective at ξ, i.e., $\xi \in \Sigma_f$. To show the claim, we first apply the Θ-completeness of $\mathcal{X}$ to construct a lift

$$\begin{tikzcd} \Theta_R \smallsetminus 0 \arrow[r, "(f\circ h)\cup g"] \arrow[d, hook] & \mathcal{X} \\ \Theta_R. \arrow[ur, dashed, "q"'] & \end{tikzcd}$$

Since $\mathcal{W} \to \mathcal{X}$ is stabilizer preserving at w, we have a lift $B\mathbb{G}_{m,\kappa} \to \mathcal{W}$ of $q|_{B\mathbb{G}_{m,\kappa}}$. Since Θ_R is coherently complete along $B\mathbb{G}_{m,\kappa}$ (6.6.13), we may apply deformation theory (Proposition 6.7.12) and Coherent Tannaka Duality (6.6.9) to construct a lift

$$\begin{tikzcd} B\mathbb{G}_{m,\kappa} \arrow[r] \arrow[d, hook] & \mathcal{W} \arrow[d, "f"] \\ \Theta_R \arrow[r, "q"] \arrow[ur, dashed, "\widetilde{q}"] & \mathcal{X} \end{tikzcd}$$

The restriction $\widetilde{q}|_{\operatorname{Spec} R}$ is 2-isomorphic to h since it agrees at the closed point and f is étale. It follows that $\widetilde{g} := \widetilde{q}|_{\Theta_K}$ is a lift of (6.10.14). $\square$

The topology of $\Bbbk$-points of Θ-complete stacks is analogous to the topology of quotient stacks arising from GIT.

Exercise 6.10.15. Let $\mathcal{X}$ be an algebraic stack of finite type over an algebraically closed field $\Bbbk$ with affine diagonal. Assume that $\mathcal{X}$ is Θ-complete and that the closed points of $\mathcal{X}$ have linearly reductive stabilizers. Show that the closure of every $\Bbbk$-point contains a unique closed point.

Exercise 6.10.16. With the hypotheses of Exercise 6.10.15, show that if in addition $\mathcal{X}$ has a unique closed point, then $\mathcal{X} \cong [\operatorname{Spec}(A)/G_x]$ such that A^{G_x} is an artinian local $\Bbbk$-algebra with residue field $\Bbbk$.

6.10.4 Unpunctured inertia

We introduce the concept of *unpunctured inertia* for an algebraic stack $\mathcal{X}$, and show that it implies that every étale quotient presentation $f\colon ([\operatorname{Spec} A/G_x], w) \to (\mathcal{X}, x)$ is stabilizer preserving in an open neighborhood of w (Proposition 6.10.20).

Definition 6.10.17. A noetherian algebraic stack has *unpunctured inertia* if for every closed point $x \in |\mathcal{X}|$ and for every formally versal morphism $p\colon (T,t) \to (\mathcal{X}, x)$, where T is the spectrum of a local ring with closed point t, every connected component of the inertia group scheme $\underline{\operatorname{Aut}}_{\mathcal{X}}(p) \to T$ has non-empty intersection with the fiber over t.

Remark 6.10.18. Here $(T,t) \to (\mathcal{X}, x)$ is formally versal if the map $\widehat{T} \to \mathcal{X}$ from the completion is is formally versal at t as in Definition C.4.2.

Remark 6.10.19. Unpuncturedness is related to the purity of the morphism $\underline{\operatorname{Aut}}_{\mathcal{X}}(p) \to T$ as defined in [GR71, §3.3] (see also [SP, Tag 0CV5]). If T is the spectrum of a strictly henselian local ring, then purity requires that if $s \in T$ is an *arbitrary* point and γ is an associated point in the fiber $\underline{\operatorname{Aut}}_{\mathcal{X}}(p)_s$, then the closure of γ in $\underline{\operatorname{Aut}}_{\mathcal{X}}(p)$ has non-empty intersection with the fiber over the closed point t of T.

Proposition 6.10.20. *Let $\mathcal{X}$ be an algebraic stack of finite type over an algebraically closed field $\Bbbk$ with affine diagonal. Let $x \in |\mathcal{X}|$ be a closed point with linearly reductive stabilizer G_x. Let $f\colon ([\operatorname{Spec} A/G_x], w) \to (\mathcal{X}, x)$ be an affine étale morphism inducing an isomorphism of stabilizer groups at w, and let $\pi\colon [\operatorname{Spec} A/G_x] \to \operatorname{Spec} A^{G_x}$. If $\mathcal{X}$ has unpunctured inertia, there exists an open affine neighborhood $U \subseteq \operatorname{Spec} A^{G_x}$ of $\pi(w)$ such that $f|_{\pi^{-1}(U)}\colon \pi^{-1}(U) \to \mathcal{X}$ induces isomorphisms of stabilizer groups at all points.*

Proof. Set $\mathcal{W} = [\operatorname{Spec} A/G_x]$. It suffices to find an open neighborhood $\mathcal{U} \subseteq \mathcal{W}$ of w such that $f|_{\mathcal{U}}\colon \mathcal{U} \to \mathcal{X}$ induces an isomorphism $I_{\mathcal{U}} \to \mathcal{U} \times_{\mathcal{X}} I_{\mathcal{X}}$. Consider the cartesian diagram

$$\begin{tikzcd} I_{\mathcal{W}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{W} \times_{\mathcal{X}} I_{\mathcal{X}} \arrow[d] \\ \mathcal{W} \arrow[r] & \mathcal{W} \times_{\mathcal{X}} \mathcal{W}; \end{tikzcd}$$

see Exercise 3.2.15. Since f is separated and étale, the morphism $I_{\mathcal{W}} \to \mathcal{W} \times_{\mathcal{X}} I_{\mathcal{X}}$ is finite and étale. We set $\mathcal{Z} \subseteq \mathcal{W} \times_{\mathcal{X}} I_{\mathcal{X}}$ to be the open and closed substack over which $I_{\mathcal{W}} \to \mathcal{W} \times_{\mathcal{X}} I_{\mathcal{X}}$ is *not* an isomorphism. Since f is stabilizer preserving at w, the point w is not contained in the image of $\mathcal{Z}$ under $p_1 \colon \mathcal{W} \times_{\mathcal{X}} I_{\mathcal{X}} \to \mathcal{W}$.

Consider a formally smooth morphism $(T, t) \to (\mathcal{X}, x)$ from the spectrum of a local ring with closed point t. Since $\mathcal{X}$ has unpunctured inertia, the preimage of $\mathcal{Z}$ in $\mathcal{W} \times_{\mathcal{X}} I_{\mathcal{X}} \times_{\mathcal{X}} T$ is empty; indeed, if there were a non-empty connected component of this preimage, it must intersect the fiber over t non-trivially contradicting that $w \notin p_1(\mathcal{Z})$. This in turn implies that $w \notin \overline{p_1(\mathcal{Z})}$. Therefore, if we set $\mathcal{U} = \mathcal{W} \smallsetminus \overline{p_1(\mathcal{Z})}$, the induced morphism $I_{\mathcal{U}} \to \mathcal{U} \times_{\mathcal{X}} I_{\mathcal{X}}$ is an isomorphism. □

Proposition 6.10.21. *Let $\mathcal{X}$ be a noetherian algebraic stack.*

(1) If $\mathcal{X}$ has quasi-finite inertia, then $\mathcal{X}$ has unpunctured inertia if and only if $\mathcal{X}$ has finite inertia.

(2) If $\mathcal{X}$ has connected stabilizer groups, then $\mathcal{X}$ has unpunctured inertia.

Proof. If $\mathcal{X}$ has finite inertia, then $\underline{\mathrm{Aut}}_{\mathcal{X}}(p) \to T$ is finite, so clearly the image of each connected component contains the unique closed point $t \in T$. For the converse, we may assume that T is the spectrum of a Henselian local ring, in which case $\underline{\mathrm{Aut}}_{\mathcal{X}}(p) = G \coprod H$ where $G \to T$ finite and the fiber of $H \to T$ over t is empty (Proposition B.5.9). If T is nonempty (i.e., $\underline{\mathrm{Aut}}_{\mathcal{X}}(p) \to T$ is not finite), then any connected component of T doesn't meet the central fiber and thus $\mathcal{X}$ does not have unpunctured inertia.

For (2), by definition, all fibers of $\underline{\mathrm{Aut}}_{\mathcal{X}}(p) \to U$ are connected, so every connected component of $\underline{\mathrm{Aut}}_{\mathcal{X}}(p)$ intersects the component containing the identity section. □

Remark 6.10.22. For algebraic stacks with connected stabilizer groups (e.g., the moduli stack $\mathcal{B}un^{\mathrm{ss}}_{r,d}(C)$ of semistable vector bundles on a curve), Proposition 6.10.21(2) implies unpunctured inertia. The deeper result below (Theorem 6.10.23) is therefore unneeded in the proof of the existence of a good moduli space of $\mathcal{B}un^{\mathrm{ss}}_{r,d}(C)$.

6.10.5 S-completeness implies unpunctured inertia

Theorem 6.10.23. *Let $\mathcal{X}$ be an algebraic stack of finite type over an algebraically closed field $\Bbbk$ with affine diagonal. Assume that the closed points have linearly reductive stabilizers. If $\mathcal{X}$ is S-complete, then $\mathcal{X}$ has unpunctured inertia.*

Proof. Let $x \in |\mathcal{X}|$ be a closed point, let $p \colon (U, u) \to (\mathcal{X}, x)$ be a formally smooth morphism from the spectrum of a local ring, and let $H \subseteq \underline{\mathrm{Aut}}_{\mathcal{X}}(p)$ be a connected component. The image of the projection $H \to U$ is a constructible set whose closure contains u. Choose a DVR R with residue field $\Bbbk$ and a map $\operatorname{Spec} R \to U$ whose special point maps to u and whose generic point lies in the image of $H \to U$. Let $\xi \colon \operatorname{Spec} R \to U \xrightarrow{p} \mathcal{X}$ denote the composition. After a residually-trivial extension of DVRs, we may assume that the generic point $\operatorname{Spec} K \to U$ lifts to H. This gives a commutative diagram

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & \operatorname{Spec} R \arrow[d] \arrow[dr, "\xi"] & \\ H \arrow[r] & U \arrow[r, "p"] & \mathcal{X}. \end{tikzcd}$$

Let H_K be the base change of $H \to U$ along $\operatorname{Spec} K \to U$. We claim we can choose a finite type point $g \in H_K$ of finite order. If $g \in H_K$ is any finite type point, then after replacing K with a finite field extension, we can decompose $g = g_s g_u$ under the Jordan decomposition, where g_s is semisimple and g_u is unipotent (see §B.1.3). Now consider the reduced closed K-subgroup $H' \subseteq \underline{\operatorname{Aut}}_{\mathcal{X}}(p)_K$ generated by g_s. Because g_s is semisimple, H' is a diagonalizable group scheme over K, and we may replace g_s with a finite order element in H' which still commutes with g_u. If $\operatorname{char}(K) > 0$, then g_u has finite order and we are finished. If $\operatorname{char}(K) = 0$, then g_u lies in the identity component of G, so g lies on the same component as the finite order element g_s. This gives the desired element.

We claim that after replacing R with a residually-trivial extension, there is a map $\xi' \colon \operatorname{Spec} R \to \mathcal{X}$ such that $\xi'_K \simeq \xi_K$ and $g \in H_K$ extends to an automorphism of ξ'. This would finish the proof: since the closure of g meets the fiber of $\underline{\operatorname{Aut}}_{\mathcal{X}}(p) \to U$ over u, the component H must also meet the central fiber. If $\mathcal{X} \cong [\operatorname{Spec} A/\operatorname{GL}_n]$, then this claim is precisely the content of Proposition 6.10.24 below. We use the Local Structure Theorem (6.7.1) to reduce to this case: let $f \colon ([\operatorname{Spec} A/G_x], w) \to (\mathcal{X}, x)$ be an étale quotient presentation. After replacing R with a residually-trivial extension, we may lift ξ to a map $\widetilde{\xi} \colon \operatorname{Spec} R \to [\operatorname{Spec} A/G_x]$ such that $\widetilde{\xi}(0) = w$. To show that g lifts to an element $\widetilde{g} \in \operatorname{Aut}(\widetilde{\xi}_K)$, we use S-completeness. We may glue ξ to itself along g to define a morphism

$$\operatorname{Spec} R \bigcup_{\operatorname{Spec} K} \operatorname{Spec} R = \phi_R \smallsetminus 0 \to \mathcal{X}.$$

Since $\mathcal{X}$ is S-complete, this map extends to a morphism $h \colon \phi_R \to \mathcal{X}$. Since $\xi(0) = x$ and x is a closed point, the image $h(0)$ of $0 \in \phi_R$ is also x. Since f is stabilizer preserving at w, we may lift $h|_{B\mathbb{G}_m}$ to a map $\widetilde{h}_0 \colon B\mathbb{G}_m \to [\operatorname{Spec} A/G_x]$ with image w. By Deformation Theory (6.7.12), we may find compatible lifts to $[\operatorname{Spec} A/G_x]$ of the restrictions of h to the nilpotent thickenings of ϕ_R along 0, and by Coherent Tannaka Duality (6.6.9), we may find construct a lift $\widetilde{h}$ below

$$\begin{tikzcd} B\mathbb{G}_m \arrow[r, "\widetilde{h}_0"] \arrow[d, hook] & {[\operatorname{Spec} A/G_x]} \arrow[d, "f"] \\ \phi_R \arrow[r, "h"] \arrow[ur, dashed, "\widetilde{h}"] & \mathcal{X}. \end{tikzcd}$$

Since f is affine and étale, both restrictions $\widetilde{h}|_{s\neq 0}$ and $\widetilde{h}|_{t\neq 0}$ to $\operatorname{Spec} R$ are isomorphic to $\widetilde{\xi}$ and thus $\widetilde{h}|_{\phi_R \smallsetminus 0}$ gives a lift $\widetilde{g} \in \operatorname{Aut}(\widetilde{\xi}_K)$ of g. Finally, we apply Proposition 6.10.24 to construct a map $\widetilde{\xi}' \colon \operatorname{Spec} R \to [\operatorname{Spec} A/G_x]$ with $\widetilde{\xi}'(0) = w$ such that $\widetilde{\xi}_K \simeq \widetilde{\xi}'_K$ and $\widetilde{g}$ extends to an automorphism of $\widetilde{\xi}'$. The composition $\xi'' := f \circ \widetilde{\xi}'' \colon \operatorname{Spec} R \to \mathcal{X}$ then satisfies the claim. See also [AHLH23, Thm. 5.2]. $\square$

Our proof used the following valuative criterion for a quotient stack.

Proposition 6.10.24. *Let $\mathcal{X} = [\operatorname{Spec} A/G]$ where $\operatorname{Spec} A$ is an affine scheme of finite type over an algebraically closed field $\Bbbk$ equipped with an action by a linearly reductive group G. Let $x \in |\mathcal{X}|$ be a closed point. Then $\mathcal{X}$ satisfies the following property:*

($\star$) *For every DVR R with residue field $\Bbbk$ and fraction field K, for every morphism $\xi\colon \operatorname{Spec} R \to \mathcal{X}$ with $\xi(0) \simeq x$, and for every K-pint $g \in \operatorname{Aut}_{\mathcal{X}}(\xi_K)$ of finite order, there is an extension $R \to R'$ of DVRs (with $K' = \operatorname{Frac}(R')$) and a morphism $\xi'\colon \operatorname{Spec} R' \to \mathcal{X}$ such that $\xi'(0) \simeq x$, $\xi'_{K'} \simeq \xi_{K'}$ and $g|_{K'}$ extends to an automorphism of ξ'.*

Remark 6.10.25. In other words, for every map $\xi\colon \operatorname{Spec} R \to \operatorname{Spec} A$ and element $g \in G_{\xi_K} \subseteq G(K)$ of finite order, there exists after an extension $R \subseteq R'$ of DVRs and an element $h \in G(K')$ such that $h \cdot \xi_{K'}$ extends to a map $\xi'\colon \operatorname{Spec} R' \to \operatorname{Spec} A$ with $\xi'(0) \in Gx$ and such that $h^{-1} g|_{K'} h$ extends to an R'-point of G. To illustrate this criterion, consider the action of $G = \mathbb{G}_m \rtimes \mathbb{Z}/2$ on $\mathbb{A}^2$ via $t \cdot (a,b) = (ta, t^{-1}b)$ and $-1 \cdot (a,b) = (b,a)$. Note that every point $(a,b) \in \mathbb{A}^2$ with $ab \neq 0$ is fixed by the order 2 element $(a/b, -1) \in G$. Consider $\xi\colon \operatorname{Spec} R = \Bbbk[\![z]\!] \to \mathbb{A}^2$ via $z \mapsto (z^2, z)$. The element $g = (z^{-1}, -1) \in G(\Bbbk(\!(z)\!))$ stabilizes ξ_K but does not extend to $G(\Bbbk[\![z]\!])$. However, we may take the degree 2 ramified extension $\Bbbk[\![z]\!] \to \Bbbk[\![\sqrt{z}]\!]$ and define $\xi'\colon \operatorname{Spec} \Bbbk[\![\sqrt{z}]\!] \to \mathbb{A}^2$ by $\sqrt{z} \mapsto ((\sqrt{z})^3, (\sqrt{z})^3)$. Over the generic point, there is an isomorphism $\xi'_{\Bbbk(\!(\sqrt{z})\!)} \simeq \xi_{\Bbbk(\!(\sqrt{z})\!)}$ given by $h = (\sqrt{z}, -1) \in G(\Bbbk(\!(\sqrt{z})\!))$ and the element $g|_{\Bbbk(\!(\sqrt{z})\!)} = (\sqrt{z}, -1)^{-1} \cdot g|_{K'} \cdot (\sqrt{z}, -1) = (1, -1) \in G(\Bbbk(\!(\sqrt{z})\!))$ extends to an element of $G(\Bbbk[\![\sqrt{z}]\!])$-point.

Proof. After choosing an embedding $G \hookrightarrow \mathrm{GL}_n$ and replacing $[\operatorname{Spec} A/G]$ with $[(\operatorname{Spec} A \times^{G_x} \mathrm{GL}_n)/\mathrm{GL}_n]$, we may assume that $G = \mathrm{GL}_n$. We first verify ($\star$) for quotient stacks $[\operatorname{Spec} A/G] = \operatorname{Spec} A \times BG$ with a trivial action. As R is local and $G = \mathrm{GL}_n$, the composition $\operatorname{Spec} R \to [\operatorname{Spec} A/G] \to BG$ corresponds to the trivial G-bundle. We need to prove that every finite order element $g \in G(K)$ is conjugate to an element of $G(R)$ after passing to an extension of the DVR R. We can conjugate g to its Jordan canonical form (after an extension of R). Since g has finite order, the diagonal entries of the resulting matrix are rth roots of unity for some r. Because the group $\boldsymbol{\mu}_r$ of r^{th} roots of unity is a finite group scheme over $\operatorname{Spec} R$, the entries of the Jordan canonical form must lie in R.

If $\mathcal{X} = [X/G]$ with X a proper scheme over $\Bbbk$, we show that ($\star$) holds except that $\xi'(0)$ may not be isomorphic to x. Since $p\colon \mathcal{X} \to BG$ is proper and representable, for every morphism $\xi\colon \operatorname{Spec} R \to \mathcal{X}$ from a DVR, we have a closed immersion $\underline{\mathrm{Aut}}_{\mathcal{X}}(\xi) \hookrightarrow \underline{\mathrm{Aut}}_{BG}(p \circ \xi)$ of group schemes over $\operatorname{Spec} R$. Moreover, any lift of the generic point of a morphism $\operatorname{Spec} R \to BG$ to $[X/G]$ extends to a unique morphism $\operatorname{Spec} R \to BG$. Therefore, given an element $g \in \operatorname{Aut}_{\mathcal{X}}(\xi_K)$, we use that ($\star$) holds for BG to find (after replacing R with an extension) a morphism $\eta\colon \operatorname{Spec} R \to BG$ such that $\eta_K \simeq (p \circ \xi)_K$ and $g|_K$ extends to a R-point of $\operatorname{Aut}_{BG}(\eta)$. If we lift η to a morphism $\xi'\colon \operatorname{Spec} R \to [X/G]$ such that $\xi'_K \simeq \xi_K$, then the element $g|_K$ extends to an automorphism of ξ'.

In verifying ($\star$) for $[\operatorname{Spec} A/G]$, we may assume that A is reduced. Viewing $[\operatorname{Spec} A/G]$ as an algebraic stack which is affine and of finite type over $\operatorname{Spec} A^G \times BG$, we can choose a vector bundle $\mathcal{E}$ on $\operatorname{Spec} A^G \times BG$ and a G-equivariant embedding $\operatorname{Spec} A \hookrightarrow \mathbb{A}(\mathcal{E}|_{\operatorname{Spec} A^G})$ over A^G. Viewing $\mathbb{A}(\mathcal{E}|_{\operatorname{Spec} A^G})$ as an open subscheme of $\mathbb{P}(\mathcal{E}|_{\operatorname{Spec} A^G} \oplus \mathcal{O}_{\operatorname{Spec} A^G})$, we let X be the closure of $\operatorname{Spec} A$ in $\mathbb{P}(\mathcal{E}|_{\operatorname{Spec} A^G} \oplus \mathcal{O}_{\operatorname{Spec} A^G})$.

This gives a G-equivariant diagram

$$\begin{tikzcd} \operatorname{Spec} A \arrow[r, hook] \arrow[dr] & X \arrow[d] \\ & \operatorname{Spec} A^G \end{tikzcd} \tag{6.10.26}$$

where X is a reduced projective scheme and the complement $X \setminus \operatorname{Spec} A$ is the support of an ample G-invariant Cartier divisor E. We also claim that $\operatorname{Spec} A$ is precisely the semistable locus of X with respect to $\mathcal{O}_X(E)$ in the sense of Exercise 7.2.11. Indeed the tautological invariant section $s \colon \mathcal{O}_X \to \mathcal{O}_X(E)$ restricts to an isomorphism over $\operatorname{Spec} A$ and thus $\operatorname{Spec} A \subseteq X^{\mathrm{ss}}$. Conversely, s^n defines an isomorphism

$$A^G \xrightarrow{\sim} \Gamma(X, \mathcal{O}_X(nE))^G$$

for all $n \geq 0$. Under this isomorphism, for every invariant global section $f \in \Gamma(X, \mathcal{O}_X(nE))^G$, the restriction $f|_{\operatorname{Spec} A}$ agrees with a section of the form gs^n, where g is the pullback of a function under the map $X \to \operatorname{Spec} A^G$. It follows that $f = g \cdot s^n$ because X is reduced. This shows that $X^{\mathrm{ss}} \subseteq \operatorname{Spec} A$.

We now verify that $(\star)$ holds for $[\operatorname{Spec} A/G]$. Let $\xi \colon \operatorname{Spec} R \to [\operatorname{Spec} A/G]$ be a map with $\xi(0) \simeq x$, and let $g \in \operatorname{Aut}_{\mathcal{X}}(\xi_K)$ be a finite order K-point. By applying the above result to $[X/G]$, there exists (after an extension of R) a map $\xi' \colon \operatorname{Spec} R \to [X/G]$ such that $\xi'_K \simeq \xi_K$ and g extends to an element of $\operatorname{Aut}_{\mathcal{X}}(\xi')$ but where $\xi'(0)$ may not be isomorphic to x. The stabilizer group scheme $\operatorname{Stab}_G(X) \subseteq X \times G$ is a closed subscheme equivariant with respect to the product action of G on $X \times G$ where G acts on itself via conjugation. The pair (ξ', g) defines a morphism

$$\eta \colon \operatorname{Spec} R \to [\operatorname{Stab}_G(X)/G].$$

We will show that after an extension of R, there is a map $\eta' \colon \operatorname{Spec} R \to [\operatorname{Stab}_G(\operatorname{Spec} A)/G]$ with $\eta'_K \simeq \eta_K$. Similar to (6.10.26), we have a G-equivariant diagram

$$\begin{tikzcd} \operatorname{Stab}_G(\operatorname{Spec} A) \arrow[r, hook] \arrow[dr] & \operatorname{Stab}_G(X) \arrow[d] \\ & \operatorname{Spec} A^G \times G \end{tikzcd}$$

with $\operatorname{Stab}_G(X)$ projective over $\operatorname{Spec} A^G \times G$. We claim that the semistable locus of $\operatorname{Stab}_G(X)$ for the action of G with respect to the pullback of $\mathcal{O}_X(E)$ is precisely $\operatorname{Stab}_G(\operatorname{Spec} A)$. The invariant section $s \in \Gamma(X, \mathcal{O}_X(E))^G$ pulls back to an invariant section on $\operatorname{Stab}_G(X)$ and thus $\operatorname{Stab}_G(\operatorname{Spec} A) \subseteq \operatorname{Stab}_G(X)^{\mathrm{ss}}$. To see the converse, suppose that $(y, h) \in \operatorname{Stab}_G(X)$ with $y \notin X^{\mathrm{ss}} = \operatorname{Spec} A$. Applying Kempf's Optimal Destabilizing Theorem (7.6.6) to a lift $\widehat{y}$ of y to the affine cone $\widehat{X} \to \operatorname{Spec} A^G$ of X yields a one-parameter subgroup $\lambda \colon \mathbb{G}_m \to G$ such that $\lim_{t \to 0} \lambda(t) \cdot \widehat{y} \in \widehat{X}$ exists and is contained in the zero section $\operatorname{Spec} A^G$. Moreover, since $G_y \subseteq P_\lambda$ (Exercise 7.6.13), $\lim_{t \to 0} \lambda(t) \cdot (\widehat{y}, h)$ also exists and is contained in the zero section of the affine cone over of $\operatorname{Stab}_G(X)$; thus (y, h) is not semistable.

The induced morphism $\operatorname{Stab}_G(\operatorname{Spec} A)/\!\!/G \to (\operatorname{Spec} A^G \times G)/\!\!/G$ of GIT quotients is proper, and the good moduli space $[\operatorname{Stab}_G(\operatorname{Spec} A)/G] \to \operatorname{Stab}_G(\operatorname{Spec} A)/\!\!/G$ is universally closed. By the valuative criterion, after an extension of R, there exists a lift

$$\begin{tikzcd}
 & & {[\mathrm{Stab}_G(\operatorname{Spec} A)/G]} \arrow[r] & \mathrm{Stab}_G(\operatorname{Spec} A)/\!\!/G \arrow[d] \\
\operatorname{Spec} R \arrow[r, "\eta"] \arrow[urr, dashed, "\chi"] & \mathrm{Stab}_G(X) \arrow[r] & \operatorname{Spec} A^G \times G \arrow[r] & (\operatorname{Spec} A^G \times G)/\!\!/G
\end{tikzcd}$$

such that $\chi_K \simeq \xi_K$. Under the composition $\xi' \colon \operatorname{Spec} R \xrightarrow{\chi} [\mathrm{Stab}_G(\operatorname{Spec} A)/G] \to [\operatorname{Spec} A/G]$, $\xi'_K \simeq \xi_K$ and g extends to an element of $\mathrm{Aut}_{\mathcal{X}}(\xi')$. To arrange that $\xi'(0) \simeq x$, we apply Lemma 6.10.27 below. $\square$

Lemma 6.10.27. *Let $\mathcal{X} = [\operatorname{Spec} A/G]$ where $\operatorname{Spec} A$ is an affine scheme of finite type over an algebraically closed field $\Bbbk$ equipped with an action of a reductive group G. Let $\xi, \xi' \colon \operatorname{Spec} R \to \mathcal{X}$ be morphisms from a DVR with residue field $\Bbbk$ such that $\xi_K \simeq \xi'_K$ and $\xi(0) \in |\mathcal{X}|$ is a closed point. For every element $g \in \mathrm{Aut}_{\mathcal{X}}(\xi')$, there exists (after replacing R with an extension) a morphism $\xi'' \colon \operatorname{Spec} R \to \mathcal{X}$ such that $\xi''_K \simeq \xi'_K$, $g|_K$ extends to an automorphism of ξ'', and $\xi''(0) \simeq \xi(0)$.*

Proof. Since $\xi(0)$ and $\xi'(0)$ lie in the same fiber of $\mathcal{X} \to \operatorname{Spec} A^G$, the closure of $\xi'(0)$ in $|\mathcal{X}|$ must contain $\xi(0)$. Kempf's Criterion (7.6.5) yields a canonical map $f \colon \Theta \to [\operatorname{Spec} A/G]$ with $f(1) \simeq \xi'(0)$ and $f(0) \simeq \xi(0)$. Since f is canonical, every automorphism of $f(1)$ extends to an automorphism of the map f. In particular the restriction of $g \in \mathrm{Aut}_{\mathcal{X}}(\xi')$ to $f(1) = \xi'(0)$ extends uniquely to an automorphism g_f of f.

We now apply the Strange Gluing Lemma (6.10.28), which after replacing R with $R[\pi^{1/N}]$ and precomposing f with the map $\Theta \to \Theta$ defined by $x \mapsto x^N$ for $N \gg 0$, yields a *unique* map $\gamma \colon \phi_R \to \mathcal{X}$, such that $\gamma|_{s=0} \simeq f$ and $\gamma|_{t \neq 0} \simeq \xi'$. The uniqueness γ guarantees that the automorphism g of ξ' and g_f of f extends uniquely to an automorphism g_γ of γ. Finally, we construct the desired map ξ'' as the composition

$$\xi'' \colon \operatorname{Spec}(R[\sqrt{\pi}]) \xrightarrow{q} \phi_R \xrightarrow{\gamma} \mathcal{X},$$

where in (s, t, π) coordinates the first map q is defined by $(\sqrt{\pi}, \sqrt{\pi}, \pi)$. Under q, the special point of $\operatorname{Spec}(R[\sqrt{\pi}])$ maps to the point $0 \in \phi_R$. By construction, $\xi''(0) \simeq \xi(0)$ and the automorphism g of γ restricts to an automorphism of ξ'' extending $g|_{K(\sqrt{\pi})}$. $\square$

Lemma 6.10.28 (Strange Gluing Lemma)**.** *Let $\mathcal{X}$ be an algebraic stack of finite type over an algebraically closed field $\Bbbk$ with affine diagonal. Let R be a DVR with residue field $\Bbbk$. Let $f \colon \Theta \to \mathcal{X}$ and $\xi \colon \operatorname{Spec} R \to \mathcal{X}$ be morphisms with an isomorphism $f(1) \simeq \xi(0)$. For $n \gg 0$, after replacing R with $R[\pi^{1/n}]$ and f with the composition $\Theta \xrightarrow{n} \Theta \xrightarrow{f} \mathcal{X}$, there is a unique morphism $\gamma \colon \phi_R \to \mathcal{X}$ such that $\gamma|_{s=0} \simeq f$ and $\gamma|_{t \neq 0} \simeq \xi$.*

Proof. For $n > 0$, define

$$\phi_R^{n,1} = [\operatorname{Spec}(R[s,t]/(st^n - \pi))/\mathbb{G}_m]$$

where the $\mathbb{G}_m$-acts with weight n on s and -1 on t. We have a closed immersion $\Theta \hookrightarrow \phi_R^{n,1}$ defined by $s = 0$ and an open immersion $\operatorname{Spec} R \hookrightarrow \phi_R^{n,1}$ defined by $t \neq 0$. Note that any morphism $\phi_R^{n,1} \to \mathcal{X}$ restricts to morphisms $f \colon \Theta \to \mathcal{X}$ and $\xi \colon \operatorname{Spec} R \to \mathcal{X}$ along with an isomorphism $\xi(0) \simeq f(1)$. We will show conversely that for $n \gg 0$, any $f\colon \Theta \to \mathcal{X}$ and $\xi\colon \operatorname{Spec} R \to \mathcal{X}$ with $\xi(0) \simeq f(1)$ extends canonically to a map $\phi_R^{n,1} \to \mathcal{X}$.

Letting $C = R[t, \pi/t, \pi/t^2, \dots] \subseteq R[t]_t$, the diagram

$$\begin{tikzcd} \operatorname{Spec} \Bbbk[t]_t \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \operatorname{Spec} \Bbbk[t] \arrow[d] \\ \operatorname{Spec} R[t]_t \arrow[r] & \operatorname{Spec} C \end{tikzcd}$$

is a pushout in the category of schemes (Theorem B.4.1). This diagram is $\mathbb{G}_m$-equivariant, and the diagram obtained by taking the fiber product with $\mathbb{G}_m$ is also a pushout. By Properties of Pushouts (B.4.8), taking quotients by $\mathbb{G}_m$ yields a pushout square

$$\begin{tikzcd} \operatorname{Spec} \Bbbk \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \Theta \arrow[d] \arrow[ddr, bend left, "f"] & \\ \operatorname{Spec} R \arrow[r] \arrow[drr, bend right, "\xi"'] & {[\operatorname{Spec}(C)/\mathbb{G}_m]} \arrow[dr, dashed, "\Psi"] & \\ & & \mathcal{X} \end{tikzcd} \tag{6.10.29}$$

in the category of algebraic stacks with affine diagonal. This induces the dotted arrow Ψ. We can write C as a union $C = \bigcup C_n$ where $C_n := R[t, \pi/t^n] \subseteq R[t]_t$. Note that $C_n \cong R[s,t]/(st^n - \pi)$ so in particular $[\operatorname{Spec}(C_n)/\mathbb{G}_m] \cong \phi_R^{n,1}$. As $\mathcal{X} \to S$ is locally of finite presentation, for $n \gg 0$ the morphism Ψ factors uniquely as $[\operatorname{Spec}(C)/\mathbb{G}_m] \to \phi_R^{n,1} \to \mathcal{X}$ (see Exercise 3.3.33).

To finish the proof, compose the uniquely defined map $\phi_R^{n,1} \to \mathcal{X}$ with the canonical map $\phi_{R[\pi^{1/n}]} \to \mathcal{X}$ induced by the map of graded algebras $R[s,t]/(st^n - \pi) \to R[\pi^{1/n}][s^{1/n}, t]/(s^{1/n}t - \pi)$, where $s^{1/n}$ has weight 1. $\square$

6.10.6 S-completeness and reductivity

We have already seen that S-completeness characterizes separatedness (Proposition 6.9.22 and Proposition 6.9.23). We have also seen that it implies unpunctured inertia (Theorem 6.10.23) and therefore implies the existence of stabilizer preserving local quotient presentations (Proposition 6.10.20). We now prove a third remarkable property of S-completeness: it characterizes reductivity. More precisely, a smooth affine algebraic group G is reductive if and only if BG is S-complete if and only if G has Cartan Decompositions. The existence of Cartan Decompositions is discussed further in §7.3, where we use it to prove the Hilbert–Mumford Criterion (7.4.4)

Proposition 6.10.30. *Let G be a smooth affine algebraic group over an algebraically closed field $\Bbbk$. The following are equivalent:*

(1) G is reductive,

(2) BG is S-complete, and

(3) G satisfies the Cartan Decomposition: for every complete DVR R over $\Bbbk$ with residue field $\Bbbk$ and fraction field K and for every element $g \in G(K)$, there exists elements $h_1, h_2 \in G(R)$ and a one-parameter subgroup $\lambda\colon \mathbb{G}_m \to G$ such that

$$g = h_1 \lambda|_K h_2.$$

Proof. We have already seen that (1) $\Rightarrow$ (2) in Proposition 6.9.20. For (2) $\Rightarrow$ (3), observe that since $\phi_R \smallsetminus 0 = \operatorname{Spec} R \bigcup_{\operatorname{Spec} K} \operatorname{Spec} R$, an element $g \in G(K)$ determines a morphism

$$\rho_g \colon \phi_R \smallsetminus 0 \to BG$$

by gluing two trivial G-torsors over $\operatorname{Spec} R$ via the isomorphism of their restrictions to $\operatorname{Spec} K$ determined by g. Since BG is S-complete, we have a lift

$$\begin{tikzcd} \phi_R \smallsetminus 0 \arrow[r, "\rho_g"] \arrow[d] & BG \\ \phi_R. \arrow[ur, dashed, "h"] & \end{tikzcd} \tag{6.10.31}$$

The restriction $h_{B\mathbb{G}_m} \colon B\mathbb{G}_m \to BG$ corresponds to a one-parameter subgroup $\lambda \colon \mathbb{G}_m \to G$ (up to conjugation), which is our candidate for the decomposition $g = h_1 \lambda|_K h_2$. We make two observations:

- If $g, g' \in G(K)$ are elements, the morphisms $\rho_g, \rho_{g'} \colon \phi_R \smallsetminus 0 \to BG$ are isomorphic if and only if there are elements $h, h' \in G(R)$ such that $hg = g'h'$.
- If $\lambda \colon \mathbb{G}_m \to G$ is a one-parameter subgroup and $\lambda|_{\phi_R \smallsetminus 0}$ denotes the composition $\phi_R \smallsetminus 0 \hookrightarrow \phi_R \to B\mathbb{G}_m \xrightarrow{\lambda} BG$, then $\lambda|_{\phi_R \smallsetminus 0}$ and $\rho_{g'}$, where $g' = \lambda|_K$, are isomorphic.

It therefore suffices to show that the extension h in (6.10.31) is isomorphic to $\lambda|_{\phi_R} \colon \phi_R \to B\mathbb{G}_m \xrightarrow{\lambda} BG$. To see this, let $\mathcal{P}$ and $\mathcal{P}'$ denote the principal G-bundles over ϕ_R classifying h and $\lambda|_{\phi_R}$. Since G is smooth and affine, $\underline{\operatorname{Isom}}_{\phi_R}(\mathcal{P}, \mathcal{P}') \to \phi_R$ is smooth and affine. We have a section over the inclusion $\mathcal{X}_0 := B\mathbb{G}_m \hookrightarrow \phi_R$ of 0. Letting $\mathcal{X}_n$ denote the nth nilpotent thickening, deformation theory (Proposition 6.7.12) and the cohomological affineness of $\mathcal{X}_n$ imply that there are compatible sections over $\mathcal{X}_n$. Coherent Tannaka Duality (6.6.9) and the coherent completeness of ϕ_R along $B\mathbb{G}_m$ (Theorem 6.6.13) imply that

$$\operatorname{Mor}_{\phi_R}(\phi_R, \underline{\operatorname{Isom}}_{\phi_R}(\mathcal{P}, \mathcal{P}')) \to \varprojlim \operatorname{Mor}_{\phi_R}(\mathcal{X}_n, \underline{\operatorname{Isom}}_{\phi_R}(\mathcal{P}, \mathcal{P}'))$$

is an equivalence. We thus obtain a section of $\underline{\operatorname{Isom}}_{\phi_R}(\mathcal{P}, \mathcal{P}') \to \phi_R$, i.e., an isomorphism between $\mathcal{P}$ and $\mathcal{P}'$. (The logic in this implication is reversible.)

For (3) $\Rightarrow$ (1), if G is not reductive, there is a normal subgroup $\mathbb{G}_a \trianglelefteq R_u(G)$ of the unipotent radical. As $G/R_u(G)$ and $R_u(G)/\mathbb{G}_a$ are both affine, the composition $B\mathbb{G}_a \to BR_uG \to BG$ is affine. By Lemma 6.9.19, this would imply that $B\mathbb{G}_a$ is S-complete but this is a contradiction: taking $R = \Bbbk[\![x]\!]$ and $K = \Bbbk(\!(x)\!)$ the element $x \in \mathbb{G}_a(K)$ cannot be written as $h_1 \lambda|_K h_2$. See also [AHHL21, Thm. A]. $\square$

In particular, if $\mathcal{X}$ is an S-complete algebraic stack and $x \in |\mathcal{X}|$ is a closed point with smooth affine stabilizer G_x, then G_x is reductive. With the stronger hypothesis that $\mathcal{X}$ admits a good moduli space, this was established in Exercise 6.5.25.

6.10.7 Proof of the Existence Theorem of Good Moduli Spaces

The necessity of Θ- and S-completeness for the existence of a good moduli space was established in Proposition 6.9.23. We now prove the sufficiency following the strategy outlined in §6.10.1.

Proof of Theorem 6.10.1. Since $\mathcal{X}$ is S-complete and $\mathrm{char}(\Bbbk) = 0$, the stabilizer G_x of every closed point $x \in |\mathcal{X}|$ is linearly reductive (Proposition 6.10.30). By the Local Structure Theorem (6.7.1), there exists an affine étale morphism $f : ([\operatorname{Spec} A/G_x], w) \to (\mathcal{X}, x)$ inducing an isomorphism of stabilizer groups at x. Since $\mathcal{X}$ is Θ- and S-complete, we may assume that f is Θ-surjective and stabilizer preserving at all points after replacing $[\operatorname{Spec} A/G_x]$ with an open neighborhood of x (Propositions 6.10.13 and 6.10.20). Since $\mathcal{X}$ is quasi-compact, there exists finitely many closed points $x_i \in |\mathcal{X}|$ and morphisms $f_i : [\operatorname{Spec} A_i/G_{x_i}] \to \mathcal{X}$ as above whose images cover $\mathcal{X}$. Choosing embeddings $G_{x_i} \hookrightarrow \mathrm{GL}_n$ for some n, there are equivalence $[\operatorname{Spec} A_i/G_{x_i}] \cong [(\operatorname{Spec} A_i \times^{G_{x_i}} \mathrm{GL}_n)/\operatorname{GL}_n]$. Setting $A = \prod_i (A_i \times^{G_{x_i}} \mathrm{GL}_n)$, there is an surjective, affine, and étale morphism

$$f : \mathcal{X}_1 := [\operatorname{Spec} A/\operatorname{GL}_n] \to \mathcal{X}$$

which is Θ-surjective and stabilizer preserving at all points. Since $\mathrm{char}(\Bbbk) = 0$, there is a good moduli space $\mathcal{X}_1 \to X_1 := \operatorname{Spec} A^{\mathrm{GL}_N}$. Set $\mathcal{X}_2 = \mathcal{X}_1 \times_{\mathcal{X}} \mathcal{X}_1$. The projections $p_1, p_2 : \mathcal{X}_2 \to \mathcal{X}_1$ are also affine, étale, Θ-surjective, and stabilizer preserving. Since f is affine, $\mathcal{X}_2 \cong [\operatorname{Spec} B/\operatorname{GL}_n]$ and there is a good moduli space $\mathcal{X}_2 \to X_2 := \operatorname{Spec} B^{\mathrm{GL}_n}$. This provides a commutative diagram

$$\begin{tikzcd}
\mathcal{X}_2 \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] \arrow[d] & \mathcal{X}_1 \arrow[r, "f"] \arrow[d] & \mathcal{X} \arrow[d, dashed] \\
X_2 \arrow[r, shift left, "q_1"] \arrow[r, shift right, "q_2"'] & X_1 \arrow[r, dashed] & X
\end{tikzcd} \tag{6.10.32}$$

which each square on the left is cartesian by Luna's Fundamental Lemma (6.5.31). Moreover, by the universality of good moduli spaces (Theorem 6.5.6(4)), the étale groupoid structure on $\mathcal{X}_2 \rightrightarrows \mathcal{X}_1$ induces a étale groupoid structure on $X_2 \rightrightarrows X_1$.

We claim that $X_2 \rightrightarrows X_1$ is an étale equivalence relation, i.e., that the quotient stack $[X_1/X_2]$ is an algebraic space. By the Characterization of Algebraic Spaces (3.6.6), it suffices to show that if $x_1 \in X_1$ is a $\Bbbk$-point, then (x_1, x_1) has a unique preimage under $(q_1, q_2) : X_2 \to X_1 \times X_1$. Let $x_2, x_2' \in X_2$ be two points mapping to $(x_1, x_1) \in X_1 \times X_1$, and let $\widetilde{x}_2, \widetilde{x}_2' \in |\mathcal{X}_2|$ be the unique closed points in their preimages. Since f is Θ-surjective, the images $p_1(\widetilde{x}_2)$, $p_2(\widetilde{x}_2)$, $p_1(\widetilde{x}_2')$, and $p_2(\widetilde{x}_2')$ are all closed points of $\mathcal{X}_1$ over x_1, and therefore they are all identified with the unique closed point $\widetilde{x}_1$ over x_1. On the other hand, since f is stabilizer preserving, the stabilizer groups of $\widetilde{x}_2$ and $\widetilde{x}_2'$ are the same as the stabilizer groups of $\widetilde{x}_1$ and of its image in $\mathcal{X}$. Let us denote this stabilizer group by G. It follows that the fiber product of $(p_1, p_2) : \mathcal{X}_2 \to \mathcal{X}_1 \times \mathcal{X}_1$ along the inclusion of the residual gerbe $\mathcal{G}_{(\widetilde{x}_1, \widetilde{x}_1)} = BG \times BG \to \mathcal{X}_1 \times \mathcal{X}_1$ is isomorphic to BG and thus identified with the residual gerbe of a *unique* closed point. Therefore $x_2 = x_2'$. Since $X_2 \rightrightarrows X_1$ is an étale equivalence relation, the quotient $X = X_1/X_2$ is an algebraic space. From étale descent, there is a morphism $\mathcal{X} \to X$ which pulls back under $X_1 \to X$ to the good moduli space $\mathcal{X}_1 \to X_1$. By descent of good moduli spaces (Lemma 6.5.22(2)), $\mathcal{X} \to X$ is a good moduli space. Finally, we use that $\mathcal{X}$ is S-complete to conclude that X is separated (Proposition 6.9.23). $\square$

Chapter 7

Geometric Invariant Theory

It is sometimes hard to appreciate how transformative some definitions and theorems were 60 years ago. Several of Mumford's ideas seem to have been quite alien to his predecessors, but once Mumford introduced them, they quickly became viewed as the "obvious approach."

János Kollár [Kol21, p. 623]

Geometric Invariant Theory (GIT) was developed by Mumford in [GIT] as a means to construct quotients and moduli spaces in algebraic geometry. For other expository accounts, we recommend [New78], [Kra84], [Dol03], [Muk03], and [Stu08].

7.1 Good quotients

The basic problem that I wish to discuss is this: if V is a variety, or scheme, parameterizing the set, or functor, of all structures of some type in projective n-space $\mathbb{P}^n$, then the group $\mathrm{PGL}(n)$ of automorphisms of $\mathbb{P}^n$ acts on V. Then under what conditions does there exist a quotient or orbit space $V/\mathrm{PGL}(n)$, i.e. when can we construct enough "projective invariants" for these structures?

David Mumford [Mum63, p. 526]

Let G be an affine algebraic group over an algebraically closed field $\Bbbk$ acting on an algebraic space U of finite type over $\Bbbk$. For a finite abstract group G viewed as a group scheme over $\Bbbk$, we have already established the existence of a *geometric quotient* U/G (Definition 4.2.1) in the following cases:

- If G acts freely on U (i.e., the action map $G \times U \to U \times U$ is a monomorphism), then $U/G := [U/G]$ exists as an algebraic space of finite type over $\Bbbk$ (Corollary 3.1.14). This also holds in the non-finite case: if G is an algebraic group and the action is free, then $[U/G]$ is an algebraic stack (Proposition 6.3.10) such that $[U/G] \to [U/G] \times [U/G]$ is a monomorphism and therefore $U/G := [U/G]$ is an algebraic space (Theorem 4.5.10).
- If $U = \operatorname{Spec} A$ is affine, then $U/G := \operatorname{Spec} A^G$ is a geometric quotient (Theorem 4.2.3). Moreover, if U is projective (resp., quasi-projective, quasi-affine), then the quotient U/G exists as a projective (resp., quasi-projective, quasi-affine) $\Bbbk$-scheme (Exercise 4.2.14).

- If U is separated, then U/G exists as a separated algebraic space as a consequence of the Keel–Mori Theorem (4.4.6). This also holds in the non-finite case: if G is an affine algebraic group, the stabilizers of the action are finite and reduced, and the action map $G \times U \to U \times U$ is proper, then $[U/G]$ is a separated Deligne–Mumford stack (Theorem 3.6.4) and the existence of a geometric quotient follows from the Keel–Mori Theorem.

GIT studies the case where G is linearly reductive[1] but not necessarily finite. GIT allows for the possibility of points $u \in U$ where the stabilizer G_u may not be finite and the orbit Gu may not be closed, e.g., $\mathbb{G}_m$ acting on $\mathbb{A}^1$. We have already covered Affine GIT (6.5.8): if G on an affine $\Bbbk$-scheme $\operatorname{Spec} A$, then there is a commutative diagram

$$\begin{tikzcd} \operatorname{Spec} A \arrow[d] \arrow[dr, "\widetilde{\pi}"] & \\ {[\operatorname{Spec} A/G]} \arrow[r, "\pi"] & (\operatorname{Spec} A)/\!\!/G := \operatorname{Spec} A^G \end{tikzcd}$$

where $\pi\colon [\operatorname{Spec} A/G] \to \operatorname{Spec} A^G$ is a good moduli space. The map $\widetilde{\pi}\colon \operatorname{Spec} A \to \operatorname{Spec} A^G$ is a categorical quotient and identifies two $\Bbbk$-points if and only if their orbit closures intersect; we will call this map a *good quotient*.

7.1.1 The definition and first properties

Definition 7.1.1 (Good quotients)**.** Given an action of a linearly reductive group G over an algebraically closed field $\Bbbk$ on an algebraic space U over $\Bbbk$, a G-invariant map $\widetilde{\pi}\colon U \to X$ is a *good quotient*[2] if

(1) $\mathcal{O}_X \to (\pi_* \mathcal{O}_U)^G$ is an isomorphism (where $(\pi_*\mathcal{O}_U)^G(V) = \Gamma(U_V, \mathcal{O}_{U_V})^G$ for an étale X-scheme V) and

(2) $\widetilde{\pi}$ is affine.

The good quotient of U by G is often denoted as $U/\!\!/G = X$.

Remark 7.1.2. The map $\widetilde{\pi}\colon U \to X$ is a good quotient if and only if $\pi\colon [U/G] \to X$ is a good moduli space. To see the equivalence, we may assume that $X = \operatorname{Spec} B$ is affine since both properties are étale local (Lemma 6.5.22(1)). For $(\Rightarrow)$, $U = \operatorname{Spec} A$ is also affine and $B = A^G$, and thus $[\operatorname{Spec} A/G] \to \operatorname{Spec} A^G$ is a good moduli space. To see $(\Leftarrow)$, observe that since $U \to [U/G]$ is affine and π_* is exact on quasi-coherent sheaves, the pushforward $\widetilde{\pi}_*$ is exact on quasi-coherent sheaves and thus $\widetilde{\pi}$ is affine by Serre's Criterion for Affineness (4.5.16).

Proposition 7.1.3. *Let G be a linearly reductive group over an algebraically closed field $\Bbbk$ acting on an algebraic space U over $\Bbbk$. If $\widetilde{\pi}\colon U \to X$ is a good quotient, then*

(1) $\widetilde{\pi}$ is surjective and the image of a closed G-invariant subscheme is closed. The same holds for the base change $T \to X$ by a morphism from a scheme;

(2) for closed G-invariant closed subschemes $Z_1, Z_2 \subseteq U$, $\operatorname{im}(Z_1 \cap Z_2) = \operatorname{im}(Z_1) \cap \operatorname{im}(Z_2)$. In particular, for $u_1, u_2 \in U(\Bbbk)$, $\widetilde{\pi}(u_1) = \widetilde{\pi}(u_2)$ if and only if $\overline{Gu_1} \cap$

[1] GIT can be developed in the more general setting of actions by *reductive* algebraic groups; see Remark 6.5.12.

[2] In [Ses72, Def. 1.5], for any affine algebraic group G, a *good quotient* by G is defined as an affine G-invariant morphism $\widetilde{\pi}\colon U \to X$ such that $\mathcal{O}_X \xrightarrow{\sim} (\pi_*\mathcal{O}_U)^G$ and properties Proposition 7.1.3(1)–(2) holds. When G is linearly reductive, these notions coincide.

$\overline{Gu_2} \neq \emptyset$, *and* $\widetilde{\pi}$ *induces a bijection between closed* G*-orbits in* U *and* $\Bbbk$*-points of* X*;*

(3) if U *is noetherian, so is* X*. If* U *is finite type over* $\Bbbk$*, then so is* X*, and for every coherent* $\mathcal{O}_U$*-module* F *with a* G*-action,* $(\pi_* F)^G$ *is coherent; and*

(4) $\widetilde{\pi}$ *is universal for* G*-invariant maps to algebraic spaces.*

Proof. This follows from Theorem 6.5.6 as $[U/G] \to X$ is a good moduli space. □

Remark 7.1.4 (Semistable reduction in GIT). Since $[U/G] \to X$ is universally closed (Theorem 6.5.6(1)), it satisfies the valuative criterion for universal closedness (Theorem 3.8.7). This translates into the following: for every DVR R over $\Bbbk$ with fraction field K and every map $\operatorname{Spec} R \to X$ with a lift $\eta\colon \operatorname{Spec} K \to U$, there exists an extension $R \to R'$ of DVRs, an element $g' \in G(K')$ over the fraction field of R', and a lift in the commutative diagram

$$\begin{tikzcd}
\operatorname{Spec} K' \arrow[d] \arrow[rr, bend left, "g'\cdot\eta|_{K'}"] & \operatorname{Spec} K \arrow[r, "\eta"] \arrow[d] & U \arrow[d, "\widetilde{\pi}"] \\
\operatorname{Spec} R' \arrow[r] \arrow[urr, dashed] & \operatorname{Spec} R \arrow[r] & X.
\end{tikzcd}$$

In fact, if $R = \Bbbk[\![x]\!]$, it can be arranged that $R \to R'$ is finite; see [Mum77, Lem. 5.3] and [AHLH23, Thm. A.8].

7.2 Projective GIT

Ich habe mich überzeugt, dass auch die Theologie ihre Vorzüge hat.

Paul Gordan

We cover the projective case of GIT: if G acts on a G-invariant closed subscheme $U \subseteq \mathbb{P}(V)$, then the semistable locus (Definition 7.2.1) has a projective good quotient $U^{\mathrm{ss}} /\!\!/ G$ (Theorem 7.2.6).

7.2.1 Semistability and stability

Definition 7.2.1. Let G be a linearly reductive group acting on a projective $\Bbbk$-scheme U and let $\mathcal{O}_U(1)$ be an ample line bundle with a G-action. The *semistable* and *stable* loci are defined as

$$U^{\mathrm{ss}} := \{u \in U \mid \text{there exists } f \in \Gamma(U, \mathcal{O}_U(d))^G \text{ with } d > 0 \text{ such that } f(u) \neq 0\},$$

$$U^{\mathrm{s}} := \left\{ u \in U \;\middle|\; \begin{array}{l} \text{there exists } f \in \Gamma(U, \mathcal{O}_U(d))^G \text{ with } d > 0 \text{ such that} \\ \quad -f(u) \neq 0, \\ \quad -\text{the orbit } Gu \subseteq U_f \text{ is closed, and} \\ \quad -\text{the function } U \to \mathbb{Z},\ x \mapsto \dim G_x \text{ is constant in an open} \\ \qquad \text{neighborhood of } u^{3} \end{array} \right\}.$$

[3] Since the function $x \mapsto \dim G_x$ is upper semicontinuous, this condition is automatic if $\dim G_u = 0$.

A point $u \in U$ is called *semistable* (resp., *stable*) if $u \in U^{\mathrm{ss}}$ (resp., $u \in U^{\mathrm{s}}$).[4]

Remark 7.2.2 (G-equivariant projective embeddings). If U is proper over $\Bbbk$, the data an ample line bundle $\mathcal{O}_U(1)$ with a G-action, i.e., a very ample G-linearization (see §B.1.28), is equivalent to the data of a G-equivariant embedding $U \hookrightarrow \mathbb{P}(V)$ where V is a finite dimensional G-representation. We stress that the stable and semistable loci depend on the choice of G-equivariant embedding $U \hookrightarrow \mathbb{P}(V)$: different embeddings give different semistable locus U^{ss} and different quotients $U^{\mathrm{ss}}/\!\!/G$. In fact, the same very ample line bundle on U may have different G-actions and embeddings. When G is semisimple, then G has no non-trivial characters $G \to \mathbb{G}_m$, and a line bundle on U has a unique G-action if it has any G-action. On the other hand, when U is a normal projective variety, then every line bundle L has a positive tensor power $L^{\otimes n}$ that has a G-action by Sumihiro's Theorem on Linearizations (B.1.30). For example, $\mathcal{O}(1)$ on $\mathbb{P}^n$ does not have a PGL_{n+1}-action, but $\mathcal{O}(n+1)$ does.

Remark 7.2.3 (Non-projective case). If U is an arbitrary $\Bbbk$-scheme (or $\Bbbk$-algebraic space) and L is a line bundle with G-action, then the definition of semistability and stability, with the additional requirement that each complement U_f is affine, for each invariant section f in Definition 7.2.1 extends. If U is projective and $\mathcal{O}_U(1)$ is ample, the U_f is automatically affine.

Remark 7.2.4 (Nullcone). The *nullcone* $\widehat{N} \subseteq \mathbb{A}(V)$ is by definition the affine cone over $U \smallsetminus U^{\mathrm{ss}}$: it is the closed G-invariant locus consisting of points $\widehat{u} \in \widehat{U}$ in the affine cone such that $f(\widehat{u}) = 0$ for every non-constant G-invariant polynomial on $\mathbb{A}(V)$.

7.2.2 The first fundamental theorem of GIT

Given a G-equivariant closed subscheme $U \subseteq \mathbb{P}(V)$, we consider the graded ring $R = \bigoplus_{d \geq 0} \Gamma(U, \mathcal{O}_U(d))$ and the morphism

$$\widetilde{\pi} \colon U^{\mathrm{ss}} \to U^{\mathrm{ss}}/\!\!/G := \operatorname{Proj} R^G. \tag{7.2.5}$$

Note that U^{ss} may be empty in which case $\operatorname{Proj} R^G$ is the empty scheme. If U^{ss} is non-empty, it is precisely the locus where the rational map $\operatorname{Proj} R \dashrightarrow \operatorname{Proj} R^G$ is defined.

Theorem 7.2.6. *Let G be a linearly reductive group over an algebraically closed field $\Bbbk$. Let $U \subseteq \mathbb{P}(V)$ be a G-equivariant closed subscheme where V is a finite dimensional G-representation. Then there is a cartesian diagram*

$$\begin{tikzcd} U^{\mathrm{s}} \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & U^{\mathrm{ss}} \arrow[r, hook] \arrow[d, "\widetilde{\pi}"] & U \\ U^{\mathrm{s}}/G \arrow[r, hook] & U^{\mathrm{ss}}/\!\!/G & \end{tikzcd}$$

where $U^{\mathrm{s}}/G \subseteq U^{\mathrm{ss}}/\!\!/G$ is an open subscheme, the map $\widetilde{\pi}$ of (7.2.5) is a good quotient, and the restriction $\widetilde{\pi}|_{U^{\mathrm{s}}} \colon U^{\mathrm{s}} \to U^{\mathrm{s}}/G$ is a geometric quotient. Moreover, $U^{\mathrm{ss}}/\!\!/G$ is projective with an ample line bundle L such that $\widetilde{\pi}^* L \cong \mathcal{O}_U(N)$ for some N.

If in addition the action of G on U has generically finite stabilizers, then the action of G on U^{s} is proper (i.e., the action map $G \times U^{\mathrm{s}} \to U^{\mathrm{s}} \times U^{\mathrm{s}}$ is proper) or in other words $[U^{\mathrm{s}}/G]$ is separated.

[4] In the literature, a non-semistable point $u \in U \smallsetminus U^{\mathrm{ss}}$ is sometimes called 'unstable' if it is not semistable; we avoid this potentially misleading terminology.

Proof. Since U is projective, $R = \bigoplus_{d \geq 0} \Gamma(U, \mathcal{O}_U(d))$ is finitely generated over $\Bbbk$. Thus by Corollary 6.5.8(3), R^G is also finitely generated over $\Bbbk$ and $U^{\mathrm{ss}} /\!\!/ G = \operatorname{Proj} R^G$ is projective. As localization commutes with taking invariants, $(R^G)_{(f)} = (R_{(f)})^G$ for every homogeneous element $f \in R^G$ of positive degree. We thus have a cartesian diagram

$$\begin{tikzcd} U_f = \operatorname{Spec} R_{(f)} \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & U^{\mathrm{ss}} \arrow[r, hook] \arrow[d, "\widetilde{\pi}"] & U \\ U_f /\!\!/ G = (U^{\mathrm{ss}} /\!\!/ G)_f \arrow[r, hook] & U^{\mathrm{ss}} /\!\!/ G. & \end{tikzcd}$$

Since the property of being a good quotient is Zariski local and since the loci $(U^{\mathrm{ss}} /\!\!/ G)_f$ cover $U^{\mathrm{ss}} /\!\!/ G$, we conclude that $\widetilde{\pi} \colon U^{\mathrm{ss}} \to U^{\mathrm{ss}} /\!\!/ G$ is a good quotient. By construction, there is an integer N such that $L := \mathcal{O}_{U^{\mathrm{ss}} /\!\!/ G}(N)$ is an ample line bundle which pulls back to $\mathcal{O}_U(N)|_{U^{\mathrm{ss}}}$.

To show that $U^{\mathrm{s}} \to U^{\mathrm{s}}/G$ is a geometric quotient, it suffices to show that every G-orbit in U^{s} is closed. Since the dimension of the stabilizer increases under orbit degeneration, it in fact suffices to show that the dimension of the stabilizers in U^{s} is locally constant. Every point $u \in U^{\mathrm{s}}$ has by definition an open neighborhood $V \subseteq U$ such that $\dim G_v = \dim G_u$ for all $v \in V$. Since $\dim G = \dim G_v + \dim Gv$, we see that the dimension of the orbit is constant on V. Finally, if there is a dense open subset of U which has dimension 0 stabilizers, then it follows from the definition of stability that every $u \in U^{\mathrm{s}}$ has a finite (possibly non-reduced) stabilizer. Since $[U^{\mathrm{s}}/G] \to U^s/G$ is also a good moduli space and $[U^{\mathrm{s}}/G]$ has quasi-finite diagonal, Proposition 6.5.33 implies that $[U^{\mathrm{s}}/G]$ is separated. $\square$

7.2.3 First examples

Example 7.2.7. Given $\mathbb{G}_m$ acting on $\mathbb{P}^2$ via $t \cdot [x : y : z] = [tx : t^{-1}y : z]$, the semistable locus is the complement of $V(xy, z) = \{[0 : 1 : 0], [1 : 0 : 0]\}$ and the good quotient is $(\mathbb{P}^2)^{\mathrm{ss}} \to \operatorname{Proj} \Bbbk[xy, z] = \mathbb{P}^1$. The fiber over $xy = 0$ is the union of three orbits and its complement in $(\mathbb{P}^2)^{\mathrm{ss}}$ is the stable locus. Observe that the restriction to $z \neq 0$ is the good quotient $\mathbb{A}^2 \to \mathbb{A}^1$, given by $(x, y) \mapsto xy$, while the fiber over $z = 0$ is the line at infinity with $[0 : 1 : 0]$ and $[1 : 0 : 0]$ removed.

Example 7.2.8. Consider the diagonal action of SL_2 on $X = (\mathbb{P}^1)^4$ and the SL_2-equivariant Segre embedding

$$(\mathbb{P}^1)^4 \to \mathbb{P}^{15}, \quad ([x_1 : y_1], \ldots, [x_4 : y_4]) \mapsto [x_1x_2x_3x_4 : \cdots : y_1y_2y_3y_4].$$

This corresponds to the SL_2-linearization of $L := \mathcal{O}(1) \boxtimes \cdots \boxtimes \mathcal{O}(1)$. The invariant ring $\bigoplus_{d \geq 0} \Gamma(X, L^{\otimes d})$ is generated in degree 1 by the *generalized cross ratios*

$$\begin{aligned} I_1 &= (x_1y_2 - x_2y_1)(x_3y_4 - x_4y_3) \\ I_2 &= (x_1y_3 - x_3y_1)(x_2y_4 - x_4y_2) \\ I_3 &= (x_1y_4 - x_4y_1)(x_2y_3 - x_3y_2) \end{aligned}$$

with the linear relation $I_1 - I_2 + I_3 = 0$. The invariant ring is $\Bbbk[I_1, I_2]$ and the quotient $X^{\mathrm{ss}} /\!\!/ \mathrm{SL}_2 = \mathbb{P}^1$. The semistable locus X^{ss} consists of tuples where at most two points are equal, while the stable locus consists of tuples of distinct points.

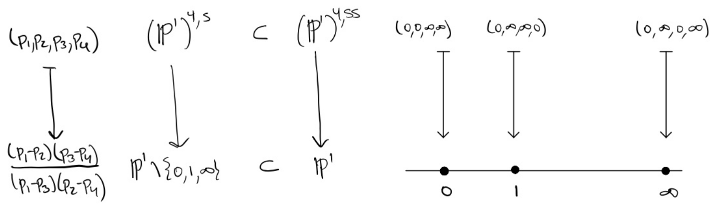


Figure 7.2.9: 4 unordered points up to projective equivalence

An ordered tuple $(p_1, \ldots, p_4)$ of distinct points is mapped to the *cross ratio*

$$\frac{(p_1 - p_2)(p_3 - p_4)}{(p_1 - p_3)(p_2 - p_4)}.$$

In particular, two stable tuples are projectively equivalent (i.e., in the same SL_2 orbit) if and only if they have the same cross ratio. The complement $X^{\mathrm{ss}} \setminus X^{\mathrm{s}}$ contains 3 closed orbits: the SL_2-orbits of $(0, 0, \infty, \infty)$, $(0, \infty, 0, \infty)$, and $(0, \infty, \infty, 0)$. Tuples such as $(0, 0, 1, \infty)$ or $(1, \infty, 0, 0)$ have non-closed SL_2-orbits in X^{ss} with $\mathrm{SL}_2 \cdot (0, 0, \infty, \infty)$ in the orbit closure. See Example 7.5.1 for the more general case of n ordered points in $\mathbb{P}^1$.

7.2.4 Other GIT setups

Exercise 7.2.10 (Affine GIT with respect to a character)**.** Let $U = \operatorname{Spec} A$ be a finite type scheme over an algebraically closed field $\Bbbk$ with an action of a linearly reductive group G specified by a coaction $\sigma \colon A \to \Gamma(G, \mathcal{O}_G) \otimes A$. Let $\chi \colon G \to \mathbb{G}_m = \operatorname{Spec} \Bbbk[t]_t$ be a character and let $\mathcal{O}_U(\chi)$ be the G-action on the trivial line bundle $\mathcal{O}_U$ determined by χ. The complement U_f of any invariant section is affine, and thus the definitions of semistability and stability (Definition 7.2.1 and Remark 7.2.3) with respect to $\mathcal{O}_U(\chi)$ translate to:

$$U^{\mathrm{ss}} := \left\{ u \in U \;\middle|\; \begin{array}{l} \text{there exists } f \in A \text{ such that } f(u) \neq 0 \\ \text{and } \sigma(f) = \chi^*(t)^d \otimes f \text{ for } d > 0 \end{array} \right\}$$

$$U^{\mathrm{s}} := \left\{ u \in U \;\middle|\; \begin{array}{l} \text{there exists } f \in A \text{ such that } f(u) \neq 0,\, \sigma(f) = \chi^*(t)^d \otimes f \\ \text{for } d > 0, \text{ the orbit } Gu \subseteq U_f \text{ is closed, and the function} \\ x \mapsto \dim G_x \text{ is constant in an open neighborhood of } u \end{array} \right\}.$$

Defining $U^{\mathrm{ss}} /\!\!/ G := \operatorname{Proj} \bigoplus_{d \geq 0} A_d$ where $A_d = \{ f \in A \mid \sigma(f) = \chi^*(t)^d \otimes f \}$, show that the conclusion of Theorem 7.2.6 holds except that $U^{\mathrm{ss}} /\!\!/ G$ is projective over $A^G = A_0$ (rather than $\Bbbk$).

For example, under the scaling $\mathbb{G}_m$-action on $U = \mathbb{A}^n$ and with respect to the identity character $\chi = \mathrm{id}$, then $U^{\mathrm{ss}} = U^{\mathrm{s}} = \mathbb{A}^n \setminus 0$ and the quotient is $\mathbb{P}^{n-1}$.

Exercise 7.2.11 (Projective GIT over an affine)**.** Let U be a projective scheme over an affine finite type $\Bbbk$-scheme $\operatorname{Spec} B$, where $\Bbbk$ is an algebraically closed field. Let G be a linearly reductive group acting compatibly on U and $\operatorname{Spec} B$. Suppose that there is a G-equivariant embedding $U \hookrightarrow \mathbb{P}(\mathcal{E})$, where $\mathcal{E}$ is a vector bundle on $\operatorname{Spec} B$ with a G-action. Show that the conclusion of Theorem 7.2.6 holds except that $U^{\mathrm{ss}} /\!\!/ G$ is projective over B^G (rather than $\Bbbk$).

Remark 7.2.12 (Symplectic reduction). There is an interesting connection between GIT and symplectic geometry. Let G be a reductive group over $\mathbb{C}$ acting on a smooth projective variety $U \subseteq \mathbb{P}(V)$, where V is an $n+1$ dimensional G-representation. Let ω be a symplectic form on U, and let $K \subseteq G$ be a maximal compact subgroup K and $\mathfrak{k}$ its Lie algebra. There is a *moment map*

$$\mu\colon U \to \mathfrak{k}^\vee$$

which is K-equivariant with respect to the coadjoint action on $\mathfrak{k}^\vee$ and satisfies $d\mu(x)(\xi) \cdot a = \omega_x(\xi, v_x)$ for $u \in U$, $\xi \in T_x U$, and $a \in \mathfrak{k}$, where v_x is the vector field on U obtained by the infinitesimal action of K on U. Then

$$u \in U \text{ is semistable } \iff \overline{Gu} \cap \mu^{-1}(0) \neq \emptyset$$

and the inclusion $\mu^{-1}(0) \hookrightarrow U$ induces a homeomorphism $\mu^{-1}(0)/K \to U^{\mathrm{ss}}/\!\!/G$. See [MFK94, §8].

7.3 The Cartan Decomposition and the Destabilization Theorem

The Cartan Decomposition (7.3.1) is the key algebraic input to prove the Destabilization Theorem (7.3.6), which in turn will be applied in the next section to prove the Hilbert–Mumford Criterion (7.4.4).

7.3.1 The Cartan Decomposition

The *Cartan Decomposition*–sometimes referred to as the *Iwahori Decomposition* or the *Cartan–Iwahori–Matsumoto Decomposition*—states than an element $g \in G(K)$ over the fraction field K of a DVR R can be multiplied on the left and right by elements of $G(R)$ such that it is induced from a one-parameter subgroup. For a one-parameter subgroup $\lambda\colon \mathbb{G}_m \to G$, we denote by $\lambda|_K \in G(K)$ the image of the composition

$$\operatorname{Spec} K \to \mathbb{G}_m \xrightarrow{\lambda} G$$

where the first map is defined by the $\Bbbk$-algebra map $\Bbbk[t]_t \to K$ taking t to a uniformizer in R.

Theorem 7.3.1 (Cartan Decomposition). *Let G be a reductive group over an algebraically closed field $\Bbbk$. Let R be a complete DVR over $\Bbbk$ with residue field $\Bbbk$ and fraction field K. Then for every element $g \in G(K)$, there exists $h_1, h_2 \in G(R)$ and a one-parameter subgroup $\lambda\colon \mathbb{G}_m \to G$ such that*

$$g = h_1 \lambda|_K h_2.$$

Proof. We have already proved this in Proposition 6.10.30, where we also showed that the existence of Cartan Decompositions characterizes reductivity. For classical proofs, see [IM65, Cor. 2.17], [Ses72, Thm. 2.1], and [BT72, §4]. □

Remark 7.3.2 (Equivalent formulation). Let $T \subseteq G$ be a maximal torus. The above theorem is equivalent to the identity

$$G(K) = G(R)T(K)G(R).$$

To see how the decomposition implies the above identity, choose $h \in G(R)$ such that $h\lambda|_K h^{-1} \in T(K)$. Then

$$g = h_1 \lambda|_K h_2 = \underbrace{(h_1 h^{-1})}_{\in G(R)} \underbrace{(h\lambda|_K h^{-1})}_{\in T(K)} \underbrace{(hh_2)}_{\in G(R)}.$$

Conversely, suppose $g = h_1 t h_2$ for $h_1, h_2 \in G(R)$ and $t \in T(K)$. If we write $T \cong \mathbb{G}_m^r$ and $\pi \in R$ as the uniformizing parameter, then $t = (u_1 \pi^{d_1}, \dots, u_r \pi^{d_r})$ for units $u_i \in R^\times$ and integers $d_i \in \mathbb{Z}$. After replacing h_1 with $h_1 \cdot (u_1, \dots, u_r)$, we can write $g = h_1 \lambda|_K h_2$ where $\lambda\colon \mathbb{G}_m \to T \subseteq G$ is the one-parameter subgroup given by $t \mapsto (t^{d_1}, \dots, t^{d_r})$.

Remark 7.3.3 (Case of GL_n). The Cartan Decomposition for GL_n can be established by an elementary linear algebra argument. Let $g = (g_{ij}) \in \mathrm{GL}_n(K)$. After performing row and column operations, we can assume that $g_{1,1} = \pi^d$ has minimal valuation among the g_{ij}, where $\pi \in R$ is a uniformizer. For each $k \geq 2$, we write $g_{k,1} = u\pi^e$. Now perform the row operations where the nth row r_n is exchanged for $r_n - u\pi^{e-d} r_1$. In this way, we can arrange that $g_{k,1} = 0$ for $k \geq 2$. By performing analogous column operations, we can also arrange that $g_{1,k} = 0$ for $k \geq 2$. The statement is thus established by induction.

Exercise 7.3.4.

(a) Let $X \subseteq \mathbb{P}(V)$ be a $\mathbb{G}_m$-equivariant locally closed subscheme where V is a finite dimensional $\mathbb{G}_m$-representation. Show that $[X/\mathbb{G}_m]$ is separated if and only if X has no $\mathbb{G}_m$-fixed points. In other words, the diagonal $[X/\mathbb{G}_m] \to [X/\mathbb{G}_m] \times [X/\mathbb{G}_m]$ is finite if and only if it is quasi-finite.

(b) Let G be a reductive group acting on an algebraic space X over $\Bbbk$. Show that $[X/G]$ is separated if and only if for every one-parameter subgroup $\lambda\colon \mathbb{G}_m \to G$, the induced quotient stack $[X/\mathbb{G}_m]$ is separated.

Hint: Verify the valuative criterion by applying the Cartan Decomposition.

Remark 7.3.5. Exercise 7.3.47.3.4 does not holf if $\mathbb{G}_m$ is replaced with a general linearly reductive group; a counterexample is given by a free action of SL_2 on a quasi-affine variety (see Exercise 3.9.4(d)).

7.3.2 The Destabilization Theorem

Theorem 7.3.6 (Destabilization Theorem). *Let G be a reductive group over an algebraically closed field $\Bbbk$ acting on an affine scheme X of finite type over $\Bbbk$. Given $x \in X(\Bbbk)$, there exists a one-parameter subgroup $\lambda\colon \mathbb{G}_m \to G$ such that $x_0 := \lim_{t \to 0} \lambda(t) \cdot x$ exists and has closed G-orbit.*

Proof. Let $R = \Bbbk[\![t]\!]$ with fraction field $K = \Bbbk(\!(t)\!)$. We can choose an element $g \in G(K)$ and a commutative diagram

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d, hook] & Gx \arrow[d, hook] \\ \operatorname{Spec} R \arrow[r, "\widetilde{g}"] & X \end{tikzcd}$$

where the top map is given by the composition $\operatorname{Spec} K \xrightarrow{g} G \to Gx$ and such that $y := \widetilde{g}(0)$ has closed G-orbit. By the Cartan Decomposition (7.3.1), there exists

$h_1, h_2 \in G(R)$ and a one-parameter subgroup $\lambda\colon \mathbb{G}_m \to G$ such that $h_1 g = \lambda|_K h_2$. By applying the general fact that for $a \in G(R)$ and $b \in X(R)$, $(a \cdot b)(0) = a(0) \cdot b(0)$ to $h_1 \in G(R)$ and $\widetilde{g} \in X(R)$, we obtain that

$$\lim_{t\to 0} \lambda(t)h_2(t) \cdot x = \lim_{t \to 0} h_1(t)g(t) \cdot x = h_1(0) \cdot \widetilde{g}(0) = h_1(0) \cdot y \in Gy.$$

We claim that the related but *possibly different* limit $\lim_{t\to 0} \lambda(t)h_2(0) \cdot x$ exists and is also contained in the closed orbit Gy. Once this is established, the theorem would be established by using the one-parameter subgroup $h_2(0)^{-1}\lambda h_2(0)$:

$$\lim_{t\to 0}(h_2(0)^{-1}\lambda h_2(0))(t) \cdot x = h_2^{-1}(0) \cdot \lim_{t\to 0} \lambda(t)h_2(0) \cdot x \in Gy.$$

First, to see that $\lim_{t\to 0} \lambda(t)h_2(0) \cdot x$ exists, we may apply Proposition B.1.18(1) below to reduce to the case that $X = \mathbb{A}(V)$ is a G-representation. We may choose a basis of $V \cong \Bbbk^n$ such that the λ-action has weights $\lambda_1, \ldots, \lambda_n$. We may also write $h_2 \cdot x = (a_1, \ldots, a_n) \in X(R)$ with each $a_i \in \Bbbk[\![t]\!]$ and further decompose $a_i = a_i(0) + a_i'$ with $a_i' \in (t)$. Since

$$\lim_{t\to 0} \lambda(t)h_2(t) \cdot x = \lim_{t\to 0} \big(t^{\lambda_1}(a_1(0) + a_1'), \ldots, t^{\lambda_n}(a_n(0) + a_n')\big) \tag{7.3.7}$$

exists, we see that for each i with $\lambda_i < 0$, we must have that $a_i(0) = 0$, which in turn implies that $\lim_{t\to 0} \lambda(t)h_2(0) \cdot x$ exists.

Finally, to see that this limit lies in Gy, we may apply Proposition B.1.18(2) to obtain a G-equivariant map $f\colon X \to \mathbb{A}(W)$ such that $f^{-1}(0) = Gy$. We are thus reduced to showing that $\lim_{t\to 0} \lambda(t)h_2(0) \cdot f(x) = 0$. By computing the limit in (7.3.7), the same argument as above shows that that since $\lim_{t\to 0} \lambda(t)h_2(t) \cdot f(x) = 0$ implies $\lim_{t\to 0} \lambda(t)h_2(0) \cdot f(x) = 0$. See also [GIT, p. 53] and [Kem78, Thm. 1.4]. ☐

Corollary 7.3.8 (Destabilization Theorem II)**.** *Let $\mathcal{X}$ be an algebraic stack of finite type over an algebraically closed field $\Bbbk$ with affine diagonal. Let $x \rightsquigarrow x_0$ be a specialization of $\Bbbk$-points such that the stabilizer G_{x_0} is linearly reductive. Then there exists a morphism $[\mathbb{A}^1/\mathbb{G}_m] \to \mathcal{X}$ representing the specialization $x \rightsquigarrow x_0$.*

Proof. The Local Structure Theorem (6.7.1) yields an étale morphism $[\operatorname{Spec} A/G_{x_0}] \to \mathcal{X}$ and a point w_0 mapping to x_0. After possibly replacing $\operatorname{Spec} A$ with a G_{x_0}-invariant affine subscheme, we can assume that w_0 is a closed point. The specialization $x \rightsquigarrow x_0$ lifts a specialization $w \rightsquigarrow w_0$ in $[\operatorname{Spec} A/G_{x_0}]$, and we can choose a representative $\widetilde{w} \in \operatorname{Spec} A$ of the orbit corresponding to w. The Destabilization Theorem (7.3.6) gives a one-parameter subgroup $\lambda\colon \mathbb{G}_m \to G$ such that $\widetilde{w}_0 = \lim_{t\to 0} \lambda(t) \cdot \widetilde{w}$ exists and has closed orbit. By Affine GIT (6.5.8), there is a unique closed orbit in $\overline{G\widetilde{w}}$ and thus $\widetilde{w}_0 \in \operatorname{Spec} A$ maps to w_0. The $\mathbb{G}_m$-equivariant extension $\mathbb{A}^1 \to X$ of $t \mapsto \lambda(t) \cdot \widetilde{w}$ defines a morphism of algebraic stacks $[\mathbb{A}^1/\mathbb{G}_m] \to [\operatorname{Spec} A/G_{x_0}]$ such that the image of the specialization $1 \rightsquigarrow 0$ is $w \rightsquigarrow w_0$. The composition $[\mathbb{A}^1/\mathbb{G}_m] \to [\operatorname{Spec} A/G_{x_0}] \to \mathcal{X}$ yields the desired map. ☐

7.4 The Hilbert–Mumford Criterion

> *A word of warning - and apology. There are several thousand formulas in this paper which allow one or more 'sign-like ambiguities': i.e., alternate and symmetric but non-equivalent reformulations. These occur in definitions and theorems. I have made a superhuman effort[5] to achieve consistency and even to make correct statements: but I still cannot guarantee the result.*
>
> David Mumford [Mum66b, p. 288]

7.4.1 The Hilbert–Mumford index

The stable and semistable locus can often be effectively computed using the Hilbert–Mumford Criterion (7.4.4). To set up the formulation, let $U \subseteq \mathbb{P}(V)$ be a G-equivariant closed subscheme where V is a finite dimensional G-representation, and let $u \in U$ be a $\Bbbk$-point with a lift $\widetilde{u} \in \mathbb{A}(V)$. Given a one-parameter subgroup $\lambda\colon \mathbb{G}_m \to G$, we can choose a basis $V \cong \Bbbk^n$ such that $\lambda(t) \cdot (v_1, \dots, v_n) = (t^{d_1} v_1, \dots, t^{d_n} v_n)$.

Definition 7.4.1 (Hilbert–Mumford index)**.** The *Hilbert–Mumford index of u with respect to λ* is

$$\mu(u, \lambda) := \max_{i, \widetilde{u}_i \neq 0} -d_i. \tag{7.4.2}$$

This definition depends on the ample G-line bundle L defining the projective embedding $U \subseteq \mathbb{P}(V)$. To emphasize this dependence, we sometimes write the Hilbert–Mumford index as $\mu_L(u, \lambda)$. The Hilbert–Mumford index can be equivalently defined as follows: if $u_0 = \lim_{t \to 0} \lambda(t) \cdot u \in \mathbb{P}(V)$ (which exists since $\mathbb{P}(V)$ is proper), then $\mathbb{G}_m$ fixes u_0 and $\mu(u, \lambda)$ is the opposite of the weight of the induced $\mathbb{G}_m$-action on the line $L_{u_0} \subseteq V$ classified by u_0.

Remark 7.4.3. From the definition of the Hilbert–Mumford index, we see that

(a) $\lim_{t \to 0} \lambda(t) \cdot \widetilde{u}$ exists if and only if $\mu(u, \lambda) \leq 0$,

(b) $\lim_{t \to 0} t \cdot \widetilde{u} = 0$ if and only if $\mu(u, \lambda) < 0$, and

(c) $\mu(gx, g\lambda g^{-1}) = \mu(x, \lambda)$.

7.4.2 The second fundamental theorem in GIT

Theorem 7.4.4 (Hilbert–Mumford Criterion)**.** *Let G be a linearly reductive group over an algebraically closed field $\Bbbk$ acting on a G-equivariant closed subscheme $U \subseteq \mathbb{P}(V)$, where V is a finite dimensional G-representation. Let $u \in \mathbb{P}(V)$ be a $\Bbbk$-point with a lift $\widetilde{u} \in \mathbb{A}(V)$. Then*

$$\begin{aligned} u \in U^{\mathrm{ss}} &\iff 0 \notin \overline{G\widetilde{u}} \\ &\iff \lim_{t \to 0} \lambda(t) \cdot \widetilde{u} \neq 0 \text{ for all } \lambda\colon \mathbb{G}_m \to G \\ &\iff \mu(u, \lambda) \geq 0 \text{ for all } \lambda\colon \mathbb{G}_m \to G. \end{aligned}$$

If in addition the action of G on U has generically finite stabilizers, then

$$\begin{aligned} u \in U^{\mathrm{s}} &\iff G\widetilde{u} \subseteq \mathbb{A}(V) \text{ is closed} \\ &\iff \mu(u, \lambda) > 0 \text{ for all non-trivial } \lambda\colon \mathbb{G}_m \to G. \end{aligned}$$

Remark 7.4.5. The criterion that is now referred to as the *Hilbert–Mumford Criterion* was first developed by Hilbert in [Hil93, § 15-16] and then adapted by Mumford in [GIT, p. 53]. It holds more generally when G is reductive.

Proof. For semistability, the first ($\Rightarrow$) implication is clear: if $0 \in \overline{G\widetilde{u}}$, then for every non-constant invariant function, we have that $f(\widetilde{u}) = f(0) = 0$; hence $u \notin U^{\mathrm{ss}}$. For the converse, if $0 \notin \overline{G\widetilde{u}}$, then 0 and $\overline{G\widetilde{u}}$ are disjoint closed G-invariant subschemes of $\mathbb{A}(V)$. Therefore their images in $\mathbb{A}(V)/\!\!/G = \operatorname{Spec}(\operatorname{Sym}^* V^\vee)^G$ are disjoint (Corollary 6.5.8(2)). We may thus find an invariant function $f \in (\operatorname{Sym}^* V^\vee)^G$ with $f(0) = 0$ and $f(\widetilde{u}) \neq 0$ which we may assume to be homogeneous of positive degree, i.e $f \in \operatorname{Sym}^d V^\vee = \Gamma(\mathbb{P}(V), \mathcal{O}(d))$ for $d > 0$. In the second equivalence, ($\Rightarrow$)

[5] I have not made such a superhuman effort and also don't guarantee the correctness of all signs.

is again clear: if there is a λ such that $\lim_{t\to 0}\lambda(t)\cdot\widetilde{u}=0$, then $0\in\overline{G\widetilde{u}}$. Conversely, if $0\in\overline{G\widetilde{u}}$, the Destabilization Theorem (7.3.6) provides a one-parameter subgroup λ such that the limit of u under λ is 0. The third equivalence follows from the definition of the Hilbert–Mumford index (see Remark 7.4.3).

For stability, we may assume that $u\in U^{\mathrm{ss}}$; otherwise 0 is in the closure of $G\widetilde{u}$ and thus $G\widetilde{u}$ is not closed. By definition, there is an invariant section $f\in\Gamma(U,\mathcal{O}(d))^G$ of positive degree not vanishing at u. After possibly increasing d, we can arrange that f extends to an invariant section $f\in\Gamma(\mathbb{P}(V),\mathcal{O}(d))^G$: this follows from the exact sequence $0\to I_U\to\mathcal{O}_{\mathbb{P}(V)}\to\mathcal{O}_U\to 0$ using the vanishing of $\mathrm{H}^1(\mathbb{P}(V),I_U(N))$ for $N\gg 0$ and the exactness of taking invariants (i.e., the linear reductivity of G). We may thus view f as a homogeneous polynomial of degree d on $\mathbb{A}(V)$. Letting $\alpha=f(\widetilde{u})$, we have a commutative diagram

$$\begin{tikzcd} G \arrow[r, "\Psi_{\widetilde{u}}"] \arrow[dr, "\Psi_u"'] & V(f-\alpha) \arrow[r, hook] \arrow[d] & \mathbb{A}(V) \\ & \mathbb{P}(V)_f & \end{tikzcd}$$

where $\Psi_u(g)=g\cdot u$ and $\Psi_{\widetilde{u}}(g)=g\cdot\widetilde{u}$. By assumption, we have that $\dim G_u=\dim G_{\widetilde{u}}=0$ so both stabilizers are finite, thus proper. By Exercise 3.8.6(b), $Gu\subseteq\mathbb{P}(V)_f$ (resp., $G\widetilde{u}\subseteq\mathbb{A}(V)$) is closed if and only if Ψ_u (resp., $\Psi_{\widetilde{u}}$) is proper. On the other hand, $V(f-\alpha)\to\mathbb{P}(V)_f$ is proper, and thus Ψ_u is proper if and only if $\Psi_{\widetilde{u}}$ is. Thus $Gu\subseteq U_f$ is closed if and only if $G\widetilde{u}\subseteq\mathbb{A}(V)$ is closed giving the first equivalence. For the second equivalence, if $G\widetilde{u}$ is not closed, then there exists a one-parameter subgroup $\lambda\colon\mathbb{G}_m\to G$ such that $\lim_{t\to 0}\lambda(t)\cdot\widetilde{u}$ exists and is not contained in $G\widetilde{u}$. This gives a non-trivial λ with $\mu(u,\lambda)\le 0$. Conversely, if $G\widetilde{u}$ is closed, then $\Psi_{\widetilde{u}}$ is proper and therefore for every non-trivial λ, the map $\mathbb{G}_m\to\mathbb{A}(V)$, defined by $t\mapsto\lambda(t)\cdot\widetilde{u}$, is also proper, which implies that $\lim_{t\to 0}\lambda(t)\widetilde{u}$ does not exist and thus $\mu(u,\lambda)>0$. $\square$

7.4.3 Variants of the Hilbert–Mumford Criterion

The Hilbert–Mumford Criterion (7.4.4) translates to a stack-theoretic criterion for a point $u\in[U/G]$ to be *semistable*, i.e., u is contained in the open substack $[U^{\mathrm{ss}}/G]$. The data of a G-equivariant embedding $U\subseteq\mathbb{P}(V)$ corresponds to a line bundle L with on $[U/G]$ such that the pullback of L under $U\to[U/G]$ is very ample. Since the stable and semistable locus are G-invariant, they define open substacks of $[U/G]$. The data of a point $u\in U(\Bbbk)$ and a one-parameter subgroup $\lambda\colon\mathbb{G}_m\to G$ up to conjugation is classified by a map $f_{u,\lambda}\colon[\mathbb{A}^1/\mathbb{G}_m]\to[U/G]$ such that the induced map

$$B\mathbb{G}_m\xhookrightarrow{0}[\mathbb{A}^1/\mathbb{G}_m]\xrightarrow{f_{u,\lambda}}[U/G]\to BG$$

corresponds λ (see Proposition 6.9.1). The Hilbert–Mumford index is $\mu(u,\lambda)=-\operatorname{wt}(f_{u,\lambda}^*L)|_{B\mathbb{G}_m}$.

Corollary 7.4.6 (Hilbert–Mumford Criterion). *Let G be a linearly reductive group over an algebraically closed field $\Bbbk$ acting on a projective $\Bbbk$-scheme U. Let L be a line bundle on $[U/G]$ whose pullback to U is very ample. Then $u\in[U/G]$ is semistable if and only if $\operatorname{wt}((f^*L)|_{B\mathbb{G}_m})\ge 0$ for all maps*

$$f\colon[\mathbb{A}^1/\mathbb{G}_m]\to[U/G],\quad \textit{with } f(1)\simeq u.$$

If in addition the action of G on U has generically finite stabilizers, then u is stable if and only if $\operatorname{wt}((f^*L)|_{B\mathbb{G}_m}) > 0$ for all maps $f\colon [\mathbb{A}^1/\mathbb{G}_m] \to [U/G]$ such that $f(1) \simeq u$ and the induced map $\mathbb{G}_m \to G_{f(0)}$ on stabilizers is non-trivial. □

Exercise 7.4.7 (Affine Hilbert-Mumford Criterion)**.** Let G be a linearly reductive group over an algebraically closed field $\Bbbk$ acting on an affine scheme $U = \operatorname{Spec} A$ of finite type. Let $\chi\colon G \to \mathbb{G}_m$ be a character, and let U^{ss} and U^{s} be the semistable and stable locus with respect to χ as defined in Exercise 7.2.10. For $u \in U(\Bbbk)$, show that

$$u \in U^{\mathrm{ss}} \iff \text{for all one-parameter subgroups } \lambda\colon \mathbb{G}_m \to G \text{ such that } \lim_{t\to 0} \lambda(t)\cdot u \text{ exists, } \langle \chi, \lambda\rangle \geq 0\ ,$$

where $\langle -, -\rangle$ is the natural pairing of characters and one-parameter subgroups. If in addition the action of G on U has generically finite stabilizers, show that $u \in U^{\mathrm{s}}$ if and only if the same condition holds with strict inequality $\langle \chi, \lambda\rangle > 0$.

Hint: Consider the action of G on $U \times \mathbb{A}^1$ induced by χ defined by $g \cdot (u, z) = (g\cdot u, \chi(g)^{-1}\cdot z)$, and show that $u \notin U^{\mathrm{ss}}$ if and only if $\overline{G\cdot(u,1)} \cap (U \times \{0\}) \neq \emptyset$. Use the Destabilization Theorem (7.3.6) to show that this is equivalent to the existence of a one-parameter subgroup λ such that

$$\lim_{t\to 0} \lambda(t)\cdot(u,1) = \lim_{t\to 0}(\lambda(t)\cdot u, t^{-\langle\chi,\lambda\rangle}) \in U \times \{0\}.$$

7.5 Examples in GIT

> *A good stock of examples, as large as possible, is indispensable for a thorough understanding of any concept, and when I want to learn something new, I make it my first job to build one.*
>
> Paul Halmos

See Example 6.5.3–6.5.4 for examples of Affine GIT, and Example 7.2.8–7.2.7 for elementary examples of Projective GIT.

7.5.1 Ordered points in projective space

Example 7.5.1. Consider the diagonal action of SL_2 on $X = (\mathbb{P}^1)^n$, and consider the SL_2-equivariant Segre embedding $(\mathbb{P}^1)^n \to \mathbb{P}^{2^n-1}$ defined by the line bundle $\mathcal{O}(1) \boxtimes \cdots \boxtimes \mathcal{O}(1)$ with SL_2-action. We claim that

$$X^{\mathrm{s}} = \{(p_1,\ldots,p_n) \mid \text{for all } q \in \mathbb{P}^1,\ \#\{i \mid p_i = q\} < n/2\},$$
$$X^{\mathrm{ss}} = \{(p_1,\ldots,p_n) \mid \text{for all } q \in \mathbb{P}^1,\ \#\{i \mid p_i = q\} \leq n/2\}.$$

To see this, let $(p_1,\ldots,p_n) \in X(\Bbbk)$ and $\lambda\colon \mathbb{G}_m \to \mathrm{SL}_2$ be a one-parameter subgroup. There exists $g \in \mathrm{SL}_2(\Bbbk)$ such that $g\lambda g^{-1} = \lambda_0^d$ for some $d \in \mathbb{Z}$ where $\lambda_0(t) = \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix}$. We can assume $d \geq 0$ as the case $d < 0$ is handled similarly. Since $\mu(x,\lambda) = \mu(gx, \lambda_0^d) = d\mu(gx,\lambda_0)$, it suffices to compute $\mu(gx,\lambda_0)$. Since $\mu(-,\lambda_0)$ is symmetric with respect to the S_n-action, we can assume that $gx = (0,\ldots,0,p_k,\ldots,p_n)$ with $p_k,\ldots,p_n \neq 0$. A coordinate of the Segre embedding is of the form $(\prod_{i\in\Sigma} x_i)(\prod_{i\notin\Sigma} y_i)$ for a subset $\Sigma \subseteq \{1,\ldots,n\}$, and its weight is $n - 2(\#\Sigma)$. The coordinate, where gx is nonzero, that has largest weight is $y_1 \cdots y_k x_{k+1} \cdots x_n$ with weight $2k - n$. Thus $\mu(gx, \lambda_0) = n - 2k$. Therefore, if no more than (resp., less than) $n/2$ of the points p_i are the same, then x is semistable (resp., stable) if and only if $n \geq 2k$ (resp., $n > 2k$). Conversely, if more than (resp., at least) $n/2$ of the same, then after translating by an element of SL_2 and using the symmetry of the S_n-action, we can write $u = (0, \ldots, 0, p_k, \ldots, p_n)$ with $k > n/2$ (resp., $k \geq n/2$) and $\lambda_0 = \mathrm{diag}(t^{-1}, t)$ destabilizes u.

If n is odd, then $X^{\mathrm{ss}} = X^{\mathrm{s}}$ and $X^{\mathrm{ss}} \to X^{\mathrm{ss}} /\!\!/ \mathrm{SL}_2$ is a geometric quotient. If n is even, the map $X^{\mathrm{ss}} \to X^{\mathrm{ss}} /\!\!/ \mathrm{SL}_2$ identifies $(p_1, \ldots, p_n)$ and $(q_1, \ldots, q_n)$ if there is a subset $\Sigma \subseteq \{1, \ldots, n\}$ of size $n/2$ such that $p_i = p_j$ and $q_i = q_j$ for all $i, j \in \Sigma$; in this case, the unique closed orbit in fiber is the orbit of the n-tuple with 0's in positions in Σ and ∞'s elsewhere. The complement $X^{\mathrm{ss}} \smallsetminus X^{\mathrm{s}}$ has precisely $\frac{1}{2}\binom{n}{n/2}$ closed orbits. Since $\mathrm{Aut}(\mathbb{P}^1) = \mathrm{PGL}_2$, the quotient $X^{\mathrm{ss}} /\!\!/ \mathrm{SL}_2 = X^{\mathrm{ss}} /\!\!/ \mathrm{PGL}_2$ can be viewed as a compactification of the moduli of n ordered points in $\mathbb{P}^1$ up to projective equivalence.

Exercise 7.5.2.

(a) Under the action of SL_2 on the projectivization $\mathbb{P}(\Gamma(\mathbb{P}^1, \mathcal{O}(n))) \cong \mathbb{P}^n$ of binary forms of degree n, show that the semistable (resp., stable) locus consists of binary forms $f(x, y)$ such that every linear factor has multiplicity less than or equal to (resp., less than) $n/2$.

(b) Under the SL_2-linearization $\mathcal{O}(a_1) \boxtimes \cdots \boxtimes \mathcal{O}(a_n)$ on $(\mathbb{P}^1)^n$ with each $a_i > 0$, show that the semistable (resp., stable) locus consists of tuples $(p_1, \ldots, p_n)$ such that for all $q \in \mathbb{P}^1(\Bbbk)$,

$$\sum_{p_i = q} a_i \leq \Big(\sum_{i=1}^{n} a_i\Big)/2 \qquad (\text{resp., } <).$$

(c) Under the SL_{r+1} action on $(\mathbb{P}^r)^n$ and the SL_{r+1}-linearization $\mathcal{O}(a_1) \boxtimes \cdots \boxtimes \mathcal{O}(a_n)$ with each $a_i > 0$, show that the semistable (resp., stable) locus consists of tuples $(p_1, \ldots, p_n)$ such that for every linear subspace $W \subsetneq \mathbb{P}^r$

$$\sum_{p_i \in W} a_i \leq \frac{\dim W + 1}{r+1} \Big(\sum_{i=1}^{n} a_i\Big) \qquad (\text{resp., } <).$$

7.5.2 Stability of hypersurfaces

Exercise 7.5.3 (Cubic curves)**.** Consider the action of SL_3 on the projective space $\mathbb{P}(\mathrm{H}^0(\mathbb{P}^2, \mathcal{O}(3)))$ of cubic curves in $\mathbb{P}^2$. Show that the stable (resp., semistable) locus consists of smooth (resp., at worst nodal) curves. See also [Mum77, §1.11], and [New78, §4.4], and [HM98, pp.202-6]

Exercise 7.5.4 (Quartic curves, hard)**.** A more involved calculation shows that under the SL_3 action on $\mathbb{P}(\mathrm{H}^0(\mathbb{P}^2, \mathcal{O}(4)))$, a quartic curve is semistable if and only if it doesn't contain a triple point and is not the union of a cubic curve and an inflection tangent line, and is stable if and only if it has at worst nodal and cuspidal singularities. See also [Mum77, §1.12-13]. See also [HM98, §4A]

Remark 7.5.5 (Cubic surfaces). Under the action of SL_4 on $\mathbb{P}(\mathrm{H}^0(\mathbb{P}^3, \mathcal{O}(3)))$, a cubic surface is stable (resp., semistable) if and only if it has finitely many singular points and the singularities are ordinary double points (resp., ordinary double points or rank two double points whose axes are not contained in the surface). See [Muk03, Thm. 7.14].

7.5.3 Quiver and toric GIT

Exercise 7.5.6 (Quiver GIT)**.** A *quiver* $Q = (Q_0, Q_1)$ is a directed graph, where Q_0 is a finite set of vertices and Q_1 is a finite set of arrows; there are source and target maps $s, t \colon Q_1 \to Q_0$. A *quiver representation* of Q consists of a vector space V_i for every $i \in Q_0$ together with linear maps $L_\alpha \colon V_i \to V_j$ for every arrow $\alpha \colon i \to j$. If each V_i is finite dimensional with $d_i = \dim V_i$, we say that $d = (d_i)$ is the *dimension vector* of V.

Fix a dimension vector $d = (d_i) \in \mathbb{Z}^{Q_0}$ and consider the space

$$R(Q, d) = \prod_{\alpha \in Q_1} \operatorname{Hom}(\Bbbk^{d_{s(\alpha)}}, \Bbbk^{d_{t(\alpha)}})$$

of representations with dimension vector d. This inherits an action of $\prod_i \mathrm{GL}_{d_i}$ via $(g_i) \cdot (L_\alpha) = (g_{t(\alpha)} L_\alpha g_{s(\alpha)}^{-1})$. The diagonal subgroup $\mathbb{G}_m \subseteq \prod_i \mathrm{GL}_{d_i}$ consisting of tuples $(t \operatorname{id}_{\Bbbk^{d_i}})$ of scalar matrices for $t \in \mathbb{G}_m$ is normal and acts trivially. Therefore the quotient $G := (\prod_i \mathrm{GL}_{d_i})/\mathbb{G}_m$ also acts on $R(Q, d)$.

For any tuple $a = (a_i)_{i \in Q_0}$ of integers such that $\sum_i a_i d_i = 0$, consider the character

$$\chi_a \colon G \to \mathbb{G}_m, \quad (g_i) \mapsto \prod_i \det(g_i)^{a_i}.$$

Use the Affine Hilbert–Mumford Criterion (7.4.7) to show that a representation $V \in R(Q, d)$ is semistable (resp., stable) with respect to χ if and only if for every subrepresentation $W \subseteq V$ (i.e., subspaces $W_i \subseteq V_i$ such that $L_\alpha(W_{s(\alpha)}) \subseteq W_{t(\alpha)}$),

$$\sum_i a_i \dim W_i \geq 0 \qquad (\text{resp.,} >).$$

See also [Kin94, Prop. 3.1].

Remark 7.5.7 (Cox construction of toric varieties)*.* Let $X = X(\Sigma)$ be a proper toric variety with fan $\Sigma \subseteq N_{\mathbb{R}}$ and torus T_N, where N is a lattice with dual M. Letting $\Sigma(1)$ denote the rays of the fan, the divisors D_ρ associated to $\rho \in \Sigma(1)$ generate the class group. There is a short exact sequence

$$0 \to M \to \mathbb{Z}^{\Sigma(1)} \to \mathrm{Cl}(X) \to 0.$$

The algebraic group $G := \operatorname{Hom}(\mathrm{Cl}(X), \mathbb{G}_m)$ is diagonalizable (hence linearly reductive) and sits in a short exact sequence

$$1 \to G \to \mathbb{G}_m^{\Sigma(1)} \to T_N \to 1$$

obtained by applying $\operatorname{Hom}(-, \mathbb{G}_m)$ to the above sequence. The group G acts naturally on $\mathbb{A}^{\Sigma(1)}$.

For a cone $\sigma \in \Sigma$, let $x^\sigma := \prod_{\rho \in \sigma(1)} x_\rho$. Define the closed G-invariant subscheme $Z \subseteq \mathbb{A}^{\Sigma(1)}$ by the vanishing of the ideal generated by the monomials x^σ as σ varies over maximal dimensional cones; this set can also be described as the union $\bigcup_C V(x_\rho \mid \rho \in C)$ where the union runs over primitive collections $C \subseteq \Sigma(1)$, i.e., subsets C such that C is not contained in $\sigma(1)$ for any $\sigma \in \Sigma$ and such that for any $C' \subsetneq C$, there exists $\sigma \in \Sigma$ with $C' \subseteq \sigma(1)$.

The main theorem here is that X is isomorphic to the good quotient $(\mathbb{A}^{\Sigma(1)} \smallsetminus Z)/\!\!/G$. This is the so-called *Cox construction of* X, and it gives X homogeneous coordinates in a similar fashion to how $\mathbb{A}^{n+1}$ gives homogeneous coordinates for

$\mathbb{P}^n = (\mathbb{A}^{n+1} \smallsetminus 0)/\mathbb{G}_m$. When Σ is a simplicial fan, X is a geometric quotient $(\mathbb{A}^{\Sigma(1)} \smallsetminus Z)/G$. Moreover, the class group $\mathrm{Cl}(X)$ is identified with group of character $\mathbb{X}^*(G)$, and if L is an ample line bundle on X corresponding to a character χ, then $\mathbb{A}^{\Sigma(1)} \smallsetminus Z$ is the semistable locus for the action of G on $\mathbb{A}^{\Sigma(1)}$ with respect to the character χ. See [Cox95] and [CLS11, §5].

7.5.4 GIT and birational geometry

Example 7.5.8 (Variation of GIT for $\mathbb{G}_m$-actions)**.** Consider a $\mathbb{G}_m$-action on an affine scheme $X = \operatorname{Spec} A$ of finite type over $\Bbbk$. In this example, we will consider how the GIT quotients vary as we vary the character of $\mathbb{G}_m$; see Exercise 7.2.10 for affine GIT with respect to a character. There is a bijection $\mathrm{Hom}(\mathbb{G}_m, \mathbb{G}_m) \cong \mathbb{Z}$ and we write $\chi_d(t) = t^d$ as the character corresponding to $d \in \mathbb{Z}$. Let $A = \bigoplus_{n \in \mathbb{Z}} A_n$ be the induced grading. There are three cases for the semistable locus $X^{\mathrm{ss}}_{\chi_d}$ with respect to the character χ_d:

(i) $d = 0$: $X^{\mathrm{ss}}(0) := X^{\mathrm{ss}}_{\chi_0} = X$ and $X^{\mathrm{ss}}_{\chi_0} /\!\!/ \mathbb{G}_m = \operatorname{Spec} A_0$.

(ii) $d > 0$: $X^{\mathrm{ss}}(+) := X^{\mathrm{ss}}_{\chi_d} = X \smallsetminus V(\sum_{n>0} A_n)$ and $X^{\mathrm{ss}}_{\chi_d} = \operatorname{Proj} \bigoplus_{n \geq 0} A_{nd}$ is independent of d; moreover $X^{\mathrm{ss}}(0)$ is identified with $X^+_{\chi_d}$ with respect to the one-parameter subgroup χ_d (Exercise 6.8.7).

(iii) $d < 0$: $X^{\mathrm{ss}}(-) := X^{\mathrm{ss}}_{\chi_d} = X \smallsetminus V(\sum_{n<0} A_n) = X^+_{\chi_d}$ and $X^{\mathrm{ss}}_{\chi_d} = \operatorname{Proj} \bigoplus_{n \geq 0} A_{-nd}$ is independent of d.

There is a commutative diagram

$$\begin{tikzcd}
X^{\mathrm{ss}}(+) \arrow[r, hook] \arrow[d] & X \arrow[d] & X^{\mathrm{ss}}(-) \arrow[l, hook'] \arrow[d] \\
X^{\mathrm{ss}}(+) /\!\!/ \mathbb{G}_m \arrow[r] & X /\!\!/ \mathbb{G}_m & X^{\mathrm{ss}}(-) /\!\!/ \mathbb{G}_m \arrow[l]
\end{tikzcd}$$

where the vertical maps are good quotients, the top maps are open immersions, and the bottom maps are projective. The Affine Hilbert–Mumford Criterion (7.4.7) implies that there are identifications of the stable loci with respect to χ_0, χ_1, and χ_{-1}: $X^{\mathrm{s}}(0) = X \smallsetminus (X^{\mathrm{ss}}(+) \cap X^{\mathrm{ss}}(-))$, $X^{\mathrm{s}}(+) = X^{\mathrm{ss}}(+) = X \smallsetminus X^{\mathrm{ss}}(-)$, and $X^{\mathrm{s}}(-) = X^{\mathrm{ss}}(-) = X \smallsetminus X^{\mathrm{ss}}(+)$. Therefore, we see that if both $X^{\mathrm{ss}}(+)$ and $X^{\mathrm{ss}}(-)$ are nonempty, then $X^{\mathrm{ss}}(+)/\mathbb{G}_m \to X/\!\!/\mathbb{G}_m$ and $X^{\mathrm{ss}}(-)/\mathbb{G}_m \to X/\!\!/\mathbb{G}_m$ are isomorphisms over $X^{\mathrm{s}}(0)/\mathbb{G}_m$, and in particular birational. We also see that if the complements of $X^{\mathrm{ss}}(+)$ and $X^{\mathrm{ss}}(-)$ in X each have codimension at least two, then the birational map $X^{\mathrm{ss}}(+)/\!\!/\mathbb{G}_m \dashrightarrow X^{\mathrm{ss}}(-)/\!\!/\mathbb{G}_m$ is an isomorphism in codimension 2 such that the divisor $\mathcal{O}(1)$ (which is relatively ample over $X/\!\!/\mathbb{G}_m$) pushes forward to a divisor on $X^{\mathrm{ss}}(-)/\!\!/\mathbb{G}_m$ whose dual is relatively ample, i.e., $X^{\mathrm{ss}}(+)/\!\!/\mathbb{G}_m \dashrightarrow X^{\mathrm{ss}}(-)/\!\!/\mathbb{G}_m$ is a *flip with respect to* $\mathcal{O}(1)$.

Remark 7.5.9 (Variation of GIT). The previous example can be generalized to actions of a linearly reductive group G. Extending the notion of algebraic independence, two line bundles with G-actions L_1 and L_2 on a projective scheme X are *G-algebraically equivalent* if there is a connected variety T, points $t_1, t_2 \in T(\Bbbk)$, and a line bundle with a G-action $\mathcal{L}$ on $X \times T$ such that $L_i \cong \mathcal{L}|_{X \times \{t_i\}}$. The *$G$-equivariant Neron–Severi* group $\mathrm{NS}^G(X)$ of line bundles with a G-action on X up to G-algebraic equivalence is finitely generated. The kernel of $\mathrm{NS}^G(X)_{\mathbb{R}} \to \mathrm{NS}(X)$ is identified with the rational character group $\mathbb{X}^*(G)_{\mathbb{R}}$. We let $\mathrm{Eff}^G(X) \subseteq \mathrm{NS}^G(X)_{\mathbb{R}}$ be the cone of G-effective linearizations, i.e., G-linearizations L such that there is a nonzero invariant section of $L^{\otimes d}$ for some $d > 0$, or in other words such that $X_L^{\mathrm{ss}} \neq \emptyset$. We also let $\mathrm{Amp}^G(X) \subseteq \mathrm{NS}^G(X)_{\mathbb{R}}$ be the cone of ample G-linearizations.

The main results of *Variation of GIT* can be formulated as follows. The semistable locus X_L^{ss} only depends on the G-algebraic equivalence class of L. There is a polyhedral decomposition of the cone $\mathrm{Amp}^G(X) \cap \mathrm{Eff}^G(X)$ defined by codimension 1 walls such that the semistable locus is constant in any open chamber. If L_0 is on a wall while L_+ and L_- are on opposite adjacent chambers, then there is a commutative diagram

$$\begin{tikzcd}
X_{L_+}^{\mathrm{ss}} \arrow[r, hook] \arrow[d] & X_{L_0}^{\mathrm{ss}} \arrow[d] & X_{L_-}^{\mathrm{ss}} \arrow[l, hook'] \arrow[d] \\
X_{L_+}^{\mathrm{ss}} /\!\!/ G \arrow[r] & X_{L_0}^{\mathrm{ss}} /\!\!/ G & X_{L_-}^{\mathrm{ss}} /\!\!/ G \arrow[l]
\end{tikzcd}$$

where the vertical maps are good quotients, the top maps are open immersions, and the bottom maps are projective. If $X_{L_+}^{\mathrm{ss}}$ and $X_{L_-}^{\mathrm{ss}}$ are non-empty, the bottom maps are birational; when the bottom maps are isomorphisms in codimension 2, then $X_{L_+}^{\mathrm{ss}} /\!\!/ G \dashrightarrow X_{L_-}^{\mathrm{ss}} /\!\!/ G$ is a flip with respect to the line bundle $\mathcal{O}(1)$ on $X_{L_+}^{\mathrm{ss}} /\!\!/ G$, which is relatively ample over $X_{L_0}^{\mathrm{ss}} /\!\!/ G$. See [Tha96] and [DH98].

Remark 7.5.10 (Mori Dream Spaces). There is an interesting connection between the Mori program and variation of GIT. A normal $\mathbb{Q}$-factorial projective variety X is a *Mori dream space* if (1) $\mathrm{Pic}(X)_{\mathbb{Q}} = \mathrm{NS}(X)_{\mathbb{Q}}$, (2) the cone $\mathrm{Nef}(X)$ of nef line bundles is the affine hull of finitely many semiample line bundles, and (3) there are finitely many birational maps $f_i \colon X \dashrightarrow X_i$, which are isomorphisms in codimension 1, to a $\mathbb{Q}$-factorial normal projective variety X_i such that the movable cone $\mathrm{Mov}(X)$ is the union of $f_i^{-1}(\mathrm{Amp}(X_i)_{\mathbb{Q}})$ (where a line bundle is called *movable* if its stable base locus has codimension at least 2). In other words, X is a Mori dream space if $\mathrm{Mov}(X)$ has a finite wall and chamber decomposition such that the projective variety determined by the line bundle is constant within an open chamber. Equivalently, X is a Mori dream space if $\mathrm{Pic}(X)_{\mathbb{Q}} = \mathrm{NS}(X)_{\mathbb{Q}}$ and the *Cox ring*

$$\mathrm{Cox}(X) := \bigoplus_{(d_1, \ldots, d_n) \in \mathbb{N}^n} \Gamma(X, L_1^{d_1} \otimes \cdots \otimes L_n^{d_n})$$

is finitely generated, where $L_1, \ldots, L_n$ is a basis for $\mathrm{Pic}(X)_{\mathbb{Q}}$ such that their affine hull contains $\mathrm{Eff}(X)_{\mathbb{Q}}$. If X is a Mori dream space, then X along with each birational model X_i is a GIT quotient of the semistable locus of $\mathrm{Spec}(\mathrm{Cox}(X))$ by the torus $\mathbb{G}_m^n$ with respect to some character. Moreover, there is an identification of the Mori chambers of $\mathrm{Mov}(X)$ with the variation of GIT chambers for the action of $\mathbb{G}_m^n$ on $\mathrm{Spec}(\mathrm{Cox}(X))$. See [HK00].

Example 7.5.11 (Partial desingularization)**.** If U is a smooth variety and $U \to X$ is a geometric quotient by a linearly reductive group, then X necessarily has finite quotient singularities; this is a consequence of the Local Structure Theorem (4.4.13). On the other hand, if $U \to X$ is a good quotient, then X can have worse singularities (Section 6.5.11). Nevertheless, in the case that there is an open subset $X' \subseteq X$ such that $\pi_0(X') \to X'$ is a geometric quotient (e.g., $U = V^{\mathrm{ss}}$ is the semistable locus with respect to a G-equivariant embedding $V \hookrightarrow \mathbb{P}^n$ and the stable locus V^{s} is nonempty), there is a canonical procedure to partially resolve the singularities of X so that they become finite quotient singularities. Namely, there exists a commutative diagram

$$\begin{tikzcd} U_n \arrow[r] \arrow[d, "\pi_n"] & U_{n-1} \arrow[r] \arrow[d, "\pi_{n-1}"] & \cdots \arrow[r] & U = U_0 \arrow[d, "\pi_0"] \\ X_n \arrow[r] & X_{n-1} \arrow[r] & \cdots \arrow[r] & X = X_0 \end{tikzcd}$$

such that:

- Each U_{i+1} is a G-invariant open subscheme of a blowup $\mathrm{Bl}_{Z_i} U_i$, where Z_i is a G-invariant smooth closed subscheme whose stabilizers are of maximal dimension, and $U_{i+1} \subseteq \mathrm{Bl}_{Z_i} U_i$ is the complement of the strict transform of $\pi_i^{-1}(\pi_i(Z_i))$. If $U = V^{\mathrm{ss}}$ is the semistable locus of a projective variety with respect to a G-linearization L, then U_{i+1} is the semistable locus with respect to $(q^*L)^{\otimes n} \otimes \mathcal{O}(-E)$ for $n \gg 0$, where $q\colon \mathrm{Bl}_{Z_i} U_i \to U_i$ and E denotes the exceptional divisor.
- The maps $X_{i+1} \to X_i$ are projective and birational.
- The maps $\pi_i\colon U_i \to X_i$ are good quotients by G, and the map $\pi_n\colon U_n \to X_n$ is a geometric quotient. In particular, X_n has finite quotient singularities.

See [Kir85], [Rei89], and [ER21].

For a simple example of this procedure in action, consider the $\mathbb{G}_m$-action on $\mathbb{A}^2$ with weights 1 and -1. In this case, the quotient $\mathbb{A}^2/\!\!/\mathbb{G}_m \cong \mathbb{A}^1$ is smooth but not a geometric quotient. The procedure tells us to take the blowup $\mathrm{Bl}_0 \mathbb{A}^2$ at the origin and remove of the strict transform of $V(xy)$. Then $\mathbb{G}_m$-acts acts on the complement U_1 with finite stabilizers, and $U_1 \to \mathbb{A}^2$ is $\mathbb{G}_m$-invariant birational (but neither proper nor surjective) map inducing an isomorphism $U_1/\mathbb{G}_m \to \mathbb{A}^2/\!\!/\mathbb{G}_m$ on quotients.

7.6 Kempf's Optimal Destabilization Theorem

By the Hilbert–Mumford Criterion (7.4.4), for every non-semistable point $x \in X$, there exists a one-parameter subgroup λ such that the Hilbert–Mumford index $\mu(x, \lambda) < 0$. Kempf's Optimal Destabilization Theorem (7.6.6) asserts that there is a canonical one-parameter subgroup λ minimizing the *normalized Hilbert–Mumford index* $\mu(x, \lambda)/\|\lambda\|$. In other words, there is a 'worst' one-parameter subgroup that is most responsible for the instability of x.

7.6.1 Measuring instability

Given an algebraic group G over an algebraically closed field $\Bbbk$, we denote by $\mathbb{X}_*(G)$ the set of one-parameter subgroups $\mathbb{G}_m \to G$. Recall that for a torus $T \cong \mathbb{G}_m^n$, $\mathbb{X}_*(T) \cong \mathbb{Z}^n$ (see Example B.1.22).

Definition 7.6.1. A *length* $\|-\|$ *on* $\mathbb{X}_*(G)$ is a nonnegative real-valued function on $\mathbb{X}_*(G)$ which is conjugation invariant, i.e., $\|g\lambda g^{-1}\| = \|\lambda\|$ for $\lambda \in \mathbb{X}_*(G)$ and $g \in G(\Bbbk)$, and such that for every maximal torus $T \subseteq G$, there is a positive definite integral-valued bilinear form $(-,-)$ on $\mathbb{X}_*(T)$ with $(\lambda, \lambda) = \|\lambda\|^2$ for $\lambda \in \mathbb{X}_*(T)$.

Example 7.6.2. If $G = \mathrm{GL}_n$, then any one-parameter subgroup λ is conjugate to a one-parameter subgroup of the form $t \mapsto \mathrm{diag}(t^{d_1}, \ldots, t^{d_n})$ and we can define $\|\lambda\| = \sqrt{d_1^2 + \cdots + d_n^2}$.

Example 7.6.3. For every reductive group G, there exists a length $\|-\|$ on $\mathbb{X}_*(G)$. To see this, let $T \subseteq G$ be a maximal torus and choose a positive definite integral-valued bilinear form $(-,-)$ on $\mathbb{X}_*(T)$, which is invariant under the conjugation action of the Weyl group $W := N(T)/T$. There is a bijection $\mathbb{X}_*(G)/G \cong \mathbb{X}_*(T)/W$ between conjugacy classes of $\mathbb{X}_*(G)$ under G and conjugacy classes of $\mathbb{X}_*(T)$ under W. In other words, for every $\lambda \in \mathbb{X}_*(G)$ there exists $g \in G(\Bbbk)$ such that $g\lambda g^{-1} \in \mathbb{X}_*(T)$, and moreover for any other element $g' \in G(\Bbbk)$ such that $g'\lambda g'^{-1} \in \mathbb{X}_*(T)$, then $g\lambda g^{-1}$ and $g'\lambda g'^{-1}$ are conjugate under W. It follows that $\|\lambda\|^2 := (g\lambda g^{-1}, g\lambda g^{-1})$ is well-defined.

Let $X = \operatorname{Spec} A$ be an affine $\Bbbk$-scheme with the action of G and let $x_0 \in X(\Bbbk)$ be a point with closed orbit. For every point $x \in X(\Bbbk)$ with $Gx_0 \subseteq \overline{Gx}$ and a one-parameter subgroup $\lambda\colon \mathbb{G}_m \to G$ such that $\lim_{t\to 0} \lambda(t)\cdot x$ exists, we define the *Hilbert–Mumford index of x with respect to λ* as

$$\mu(x,\lambda) = -\deg f_{x,\lambda}^{-1}(Gx_0). \tag{7.6.4}$$

where $f_{x,\lambda}\colon \mathbb{A}^1 \to X$ is the map extending $\mathbb{G}_m \to X$, $t \mapsto \lambda(t)\cdot x$. Note that if $\lim_{t\to 0}\lambda(t)\cdot x \notin Gx_0$, then $\mu(x,\lambda) = 0$.

In the projective case of a G-equivariant embedding $X \hookrightarrow \mathbb{P}(V)$, we have already defined the Hilbert–Mumford index $\mu(x,\lambda)$ in (7.4.2) as follows: choosing a basis of V such that $\mathbb{G}_m$ acts on $\mathbb{A}(V) = \mathbb{A}^n$ with weights $d_1,\dots,d_n$ and a lift $\widehat{x} = (u_1,\dots,u_n) \in \mathbb{A}(V)$ of x, then $-\mu(x,\lambda)$ is defined as the smallest d_i with $u_i \neq 0$. If $\lim_{t\to 0}\lambda(t)\cdot\widehat{x}$ exists, then this agrees with the definition in (7.6.4). To see this, observe that the extension $f_{\widehat{x},\lambda}\colon \mathbb{A}^1 \to \mathbb{A}^n$ of the map $t \mapsto \lambda(t)\cdot\widehat{x}$ is the map $t \mapsto (t^{d_i}u_i)$ and $f_{\widehat{x},\lambda}^{-1}(0) = \operatorname{Spec}\Bbbk[t]/(t^d)$ where d is the smallest d_i with $u_i \neq 0$.

Since $\mu(x,\lambda^n) = n\cdot\mu(x,\lambda)$, it natural to consider the *normalized Hilbert–Mumford index*

$$\frac{\mu(x,\lambda)}{\|\lambda\|}$$

as a measure of how quickly $\lambda(t)\cdot x$ approaches the closed orbit Gx_0. The more negative the normalized Hilbert–Mumford index is, the faster $\lambda(t)\cdot x$ approaches Gx_0. Kempf proved that there is a one-parameter subgroup minimizing this index and that it is unique up to conjugation.

7.6.2 Statements of Kempf's theorem

Given a one-parameter subgroup $\lambda\colon \mathbb{G}_m \to G$, recall from §B.1.4 that the *centralizer* C_λ is the subgroup of elements commuting with λ, the *parabolic* subgroup is $P_\lambda = \{g \in G \mid \lim_{t\to 0}\lambda_0(t)g\lambda_0(t)^{-1} \text{ exists}\}$ and the *unipotent* subgroup is $U_{\lambda_0} = \{g \in G \mid \lim_{t\to 0}\lambda_0(t)g\lambda_0(t)^{-1} = 1\}$. There is a short exact sequence $1 \to U_\lambda \to P_\lambda \to C_\lambda \to 1$, identifying U_λ as the unipotent radical of P_λ.

Theorem 7.6.5 (Kempf's Optimal Destabilization Theorem—affine version)**.** *Let G be a reductive group over an algebraically closed field $\Bbbk$ with a length $\|-\|$ on $\mathbb{X}_*(G)$. Let $X = \operatorname{Spec} A$ be an affine scheme of finite type over $\Bbbk$ with an action of G. Let $x_0 \in X(\Bbbk)$ be a point with a closed orbit. For every point $x \in X(\Bbbk)$ with $Gx_0 \subseteq \overline{Gx}$, there exists a one-parameter subgroup $\lambda_0\colon \mathbb{G}_m \to G$ such that $\mu(x,\lambda_0)/\|\lambda_0\|$ achieves the minimal value*

$$M(x) := \inf_{\lambda \in \mathbb{X}_*(G)} \left\{ \frac{\mu(x,\lambda)}{\|\lambda\|} \;\middle|\; \lim_{t\to 0}\lambda(t)\cdot x \in Gx_0 \right\}$$

If λ_0' is another such one-parameter subgroup, then $P_{\lambda_0} = P_{\lambda_0'}$ and $\lambda_0' = u\lambda_0 u^{-1}$ for a unique element $u \in U_{\lambda_0}$. Every maximal torus $T \subseteq P_{\lambda_0}$ contains a unique indivisible element achieving this minimum value.

The projective version below follows from applying the affine version (Theorem 7.6.5) to a lift $\widehat{x} \in \mathbb{A}(V)$ of a non-semistable point $x \in \mathbb{P}(V)$, in which case the closed orbit in $\overline{G\widehat{x}}$ is the fixed point 0. The projective version also holds for reductive groups, but we restrict to linearly reductive groups as we've only discussed semistability in that context.

Theorem 7.6.6 (Kempf's Optimal Destabilization Theorem—projective version)**.** *Let G be a reductive group over an algebraically closed field $\Bbbk$ with a length $\|-\|$ on $\mathbb{X}_*(G)$. Let $X \subseteq \mathbb{P}(V)$ be a G-equivariant closed subscheme where V is a finite dimensional G-representation. For every non-semistable point $x \in X(\Bbbk)$, there exists a one-parameter subgroup $\lambda_0 \colon \mathbb{G}_m \to G$ such that $\mu(x, \lambda_0)/\|\lambda_0\|$ achieves the minimal value*

$$M(x) := \inf_{\lambda \in \mathbb{X}_*(G)} \left\{ \frac{\mu(x,\lambda)}{\|\lambda\|} \right\}$$

If λ_0' is another such one-parameter subgroup, then $P_{\lambda_0} = P_{\lambda_0'}$ and $\lambda_0' = u\lambda_0 u^{-1}$ for a unique element $u \in U_{\lambda_0}$. Every maximal torus $T \subseteq P_{\lambda_0}$ contains a unique indivisible element achieving this minimum value. □

Definition 7.6.7. We call any λ_0 satisfying Theorem 7.6.5 or Theorem 7.6.6 an *optimal destabilizing one-parameter subgroup for x*, and we call $M(x) < 0$ the *optimal normalized Hilbert–Mumford index* for x.

7.6.3 Proof of Kempf's theorem

The proof below is simpler when $x_0 \in \overline{Gx}$ is a fixed point, such as in the projective version when the closed orbit is $0 \in \mathbb{A}(V)$. The reader may want to keep this case in mind.

Proof of Theorem 7.6.5. By Proposition B.1.18, we may choose finite dimensional G-representations V and W along with G-equivariant maps

$$\begin{tikzcd} & \mathbb{A}(V) \\ X \arrow[ur, hook, "i"] \arrow[dr, "f"] & \\ & \mathbb{A}(W), \end{tikzcd} \tag{7.6.8}$$

where $i \colon X \hookrightarrow \mathbb{A}(V)$ is a closed immersion with $i(x_0) = 0$ and $f \colon X \to \mathbb{A}(W)$ is a morphism with $f^{-1}(0) = Gx_0$. (When x_0 is a fixed point, we can take $f = i$ in (7.6.8).) A one-parameter subgroup $\lambda \colon \mathbb{G}_m \to G$ induces $\mathbb{G}_m$-actions on V and W, and thus gradings $V = \bigoplus_{d\in\mathbb{Z}} V_d$ and $W = \bigoplus_{d\in\mathbb{Z}} W_d$. We define

$$\begin{aligned} m(i(x),\lambda) &= \min\{d \mid \text{the projection of } i(x) \text{ to } V_d \text{ is nonzero}\}, \\ m(f(x),\lambda) &= \min\{d \mid \text{the projection of } f(x) \text{ to } W_d \text{ is nonzero}\}. \end{aligned}$$

For any $g \in G$, we have the identities $m(i(x),\lambda) = m(i(g\cdot x), g\lambda g^{-1})$ and $m(f(x),\lambda) = m(f(g\cdot x), g\lambda g^{-1})$. It is easy to see that if $\lim_{t\to 0}\lambda(t)\cdot x$ exists, then $\mu(x,\lambda) =$

$-m(f(x),\lambda)$, and that

$$\begin{aligned}\lim_{t\to 0}\lambda(t)\cdot x \text{ exists} &\iff m(i(x),\lambda)\geq 0,\\ \lim_{t\to 0}\lambda(t)\cdot x\in Gx_0 &\iff m(i(x),\lambda)\geq 0 \text{ and } m(f(x),\lambda)>0.\end{aligned}$$

By the Destabilization Theorem (7.3.6), there exists $\lambda_x\in\mathbb{X}_*(G)$ such that $m(i(x),\lambda_x)\geq 0$ and $m(f(x),\lambda_x)>0$.

Case of a torus: Let $T\subseteq G$ be a maximal torus containing λ_x. We can decompose $V=\bigoplus_{\chi\in\mathbb{X}^*(T)}V_\chi$ as a T-representation, where $\mathbb{X}^*(T)$ denotes the set of characters of T. We define *the state of* $i(x)\in V$ *with respect to* T as the set

$$\mathrm{State}_T(i(x)):=\{\chi\in\mathbb{X}^*(T)\mid \text{the projection of } i(x) \text{ to } V_\chi \text{ is nonzero}\}.$$

Likewise, we have the state $\mathrm{State}_T(f(x))\subseteq\mathbb{X}^*(T)$ of $f(x)\in W$ with respect to T. Let $\langle -,-\rangle$ be the natural pairing $\mathbb{X}^*(T)\times\mathbb{X}_*(T)\to\mathbb{Z}$. For a one-parameter subgroup $\lambda\in\mathbb{X}_*(T)$, we have identifications

$$m(i(x),\lambda)=\min_{\chi\in\mathrm{State}_T(i(x))}\langle\chi,\lambda\rangle \quad\text{and}\quad m(f(x),\lambda)=\min_{\chi\in\mathrm{State}_T(f(x))}\langle\chi,\lambda\rangle.$$

We claim that the function $\lambda\mapsto m(f(x),\lambda)/\|\lambda\|$ achieves a maximum value on the set $\{\lambda\neq 0\in\mathbb{X}_*(T)\mid m(i(x),\lambda_T)\geq 0\}$ at a one-parameter subgroup λ_T, and that any other one-parameter subgroup achieving this minimum is a positive multiple of λ_T. This is precisely the conclusion of Lemma 7.6.9 below applied to the lattice $L=\mathbb{X}_*(T)\cong\mathbb{Z}^r$ and the subsets of $\mathbb{X}^*(T)\cong\mathrm{Hom}_{\mathbb{Z}}(L,\mathbb{Z})$ given by $F:=\mathrm{State}_T(i(x))$ and $G:=\mathrm{State}_T(f(x))$.

General case: For each maximal torus $T\subseteq G$ and each $g\in G(\Bbbk)$, there is an identification $\mathbb{X}^*(T)\cong\mathbb{X}^*(gTg^{-1})$, given by identifying $\chi\leftrightarrow g\chi g^{-1}$, under which $\mathrm{State}_T(i(x))=\mathrm{State}_{gTg^{-1}}(i(gx))$. Given a one-parameter subgroup $\lambda\in\mathbb{X}_*(G)$, we have seen that $m(f(x),\lambda)=m(f(gx),g\lambda g^{-1})$ for $g\in G(\Bbbk)$. We claim that, in fact, $m(f(x),\lambda)=m(f(x),p\lambda p^{-1})$ for $p\in P_\lambda$. By symmetry, it suffices to show that $m(f(x),\lambda)\leq m(f(x),p\lambda p^{-1})$. Interpreting $-m(f(x),\lambda)$ as the smallest integer d such that $\lim_{t\to 0}t^d\lambda(t)\cdot f(x)\in\mathbb{A}(W)$ exists, we need to show that $\lim_{t\to 0}t^dp\lambda(t)p^{-1}\cdot f(x)\in\mathbb{A}(W)$ exists. This follows from the computation

$$\begin{aligned}\lim_{t\to 0}\left(t^dp\lambda(t)p^{-1}\cdot f(x)\right)&=\lim_{t\to 0}\Big(p\cdot\left(\lambda(t)p^{-1}\lambda(t)^{-1}\right)\cdot\left(t^d\lambda(t)f(x)\right)\Big)\\ &=p\cdot\big(\lim_{t\to 0}\lambda(t)p^{-1}\lambda(t)^{-1}\big)\cdot\big(\lim_{t\to 0}t^d\lambda(t)f(x)\big).\end{aligned}$$

We now show that the function $\lambda\mapsto m(f(x),\lambda)/\|\lambda\|$ achieves a minimum value on

$$\Lambda:=\{\lambda\in\mathbb{X}_*(G)\mid m(i(x),\lambda)\geq 0\}.$$

By the torus case, we know that for every $g\in G(\Bbbk)$ there is a minimum value on each *non-empty* set $\mathbb{X}_*(gTg^{-1})\cap\Lambda$, and that the minimum is determined by the subsets of $\mathbb{X}_*(T)$ given by $\mathrm{State}_{gTg^{-1}}(i(x))\cong\mathrm{State}_T(i(g^{-1}u))$ and $\mathrm{State}_{gTg^{-1}}(f(x))\cong\mathrm{State}_T(f(g^{-1}u))$. Since these subsets are contained in the finite set of characters χ with $V_\chi\neq 0$ (resp., $W_\chi\neq 0$), there are only finitely many minimum values as g ranges over $G(\Bbbk)$. Since the image of any $\lambda\in\mathbb{X}_*(G)$ is contained in gTg^{-1} for some $g\in G(\Bbbk)$, it follows that there is a global minimum value achieved by a one-parameter subgroup $\lambda_0\in\Lambda$. We may assume that λ_0 is indivisible, i.e., λ_0 cannot be written as a positive multiple of another one-parameter subgroup.

To establish the uniqueness, choose a maximal torus $T_0 \subseteq G$ containing λ_0. By the torus case, $\lambda_0 \in \mathbb{X}_*(T_0) \cap \Lambda$ is the unique indivisible one-parameter subgroup achieving the minimal value. For $p \in P_{\lambda_0}$, the conjugate one-parameter subgroup $p\lambda_0 p^{-1}$ also achieves this minimal value. Since any other maximal torus $T' \subseteq P_{\lambda_0}$ is pT_0p^{-1} for some $p \in P_{\lambda_0}$, we see that $\mathbb{X}_*(T') \cap \Lambda$ also contains a unique indivisible element achieving the minimum value. Finally, let $\lambda_1 \in \mathbb{X}_*(G)$ be another indivisible element achieving the minimum value. The intersection $P_{\lambda_0} \cap P_{\lambda_1}$ contains a maximal torus T of G (Proposition B.1.26(d)), and we can write $\lambda_T = p_0\lambda_0 p_0^{-1} = p_1\lambda_1 p_1^{-1}$ for $p_0, p_1 \in P_{\lambda_T}$. It follows that $P_{\lambda_0} = P_{\lambda_T} = P_{\lambda_1}$, and that λ_0 and λ_1 are conjugate by a unique element element of U_{λ_T} (Proposition B.1.26(c)). See also [Kem78, Thm. 3.4]. □

The argument above used the following lemma in convex geometry.

Lemma 7.6.9. *Let Λ be a finite dimensional lattice, and let F and G be non-empty finite subsets of $\Lambda^\vee = \operatorname{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{Z})$. Assume that $\Lambda_{\mathbb{R}} = \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$ has a positive definite inner product which is integral valued on Λ. Define*

$$f_{\min}\colon \Lambda_{\mathbb{R}} \to \mathbb{R},\ \lambda \mapsto \min_{f \in F} f(\lambda) \quad \text{and} \quad g_{\min}\colon \Lambda_{\mathbb{R}} \to \mathbb{R},\ \lambda \mapsto \min_{g \in G} g(\lambda).$$

Suppose that there exists $\lambda \in \Lambda_{\mathbb{R}}$ such that $f_{\min}(\lambda) \geq 0$ and $g_{\min}(\lambda) > 0$. Then the function

$$\begin{aligned} C_F := \{\lambda \neq 0 \in \Lambda_{\mathbb{R}} \mid f_{\min}(\lambda) \geq 0\} &\to \mathbb{R} \\ \lambda &\mapsto g_{\min}(\lambda)/\left\|\lambda\right\|, \end{aligned}$$

obtains a maximum value M. There exists a unique element $\lambda_0 \in C_F \cap \Lambda$ such that $M = g_{\min}(\lambda_0)/\left\|\lambda_0\right\|$ and such that any other element $\lambda \in C_F \cap \Lambda$ with $M = g_{\min}(\lambda)/\left\|\lambda\right\|$ is an integral multiple of λ_0.

Proof. The set $\{\lambda \in C_F \mid g_{\min}(\lambda) \geq 1\}$ is closed and convex, and therefore contains a unique point λ' closest to the origin. Since $g_{\min}(\alpha\lambda') = \alpha g_{\min}(\lambda')$ for $\alpha \in \mathbb{R}$, we must have that $g_{\min}(\lambda') = 1$ and that $\lambda' \in C_F$ is the unique point with $g_{\min}(\lambda') = 1$ and $g_{\min}(\lambda')/\left\|\lambda'\right\| = M$.

We now argue that the ray spanned by λ' contains an integral point. If λ' is in the interior of $\{\lambda \in C_F \mid g_{\min}(\lambda) \geq 1\}$, i.e., $f(\lambda') > 0$ for all $f \in F$ and there is a unique $g \in G$ with $g(\lambda') = 1$, then λ' is the closest point to the origin on the affine plane defined by $g = 1$. We claim that $\lambda' = g^*/\langle g^*, g^*\rangle$ where $g^* \in \Lambda_{\mathbb{R}}$ is the unique point such that $\langle g^*, \lambda\rangle = g(\lambda)$ for all $\lambda \in \Lambda_{\mathbb{R}}$. Indeed, the point λ' is contained in the plane $g = 1$, and for any other point λ on this plane, we have that $\langle \lambda', \lambda\rangle = 1/\langle g^*, g^*\rangle = \langle \lambda', \lambda'\rangle$. The Cauchy–Schwarz inequality implies that $\langle \lambda', \lambda'\rangle^2 = \langle \lambda', \lambda\rangle^2 \leq \langle \lambda', \lambda'\rangle\langle \lambda, \lambda\rangle$ so that $\langle \lambda', \lambda'\rangle \leq \langle \lambda, \lambda\rangle$. Since the inner product and g take integral values, $g^* \in \Lambda$. We then take λ_0 to be the unique indivisible element in the ray spanned by g^*.

To reduce to this case, let $f_1, \ldots, f_t \in F$ be the functions satisfying $f_i(\lambda') = 0$, and let $g_1, \ldots, g_s \in G$ be the functions satisfying $g_i(\lambda') = g_{\min}(\lambda')$. Since each f_i and g_j take integral values, we may restrict to the subspace

$$W := \left\{ \lambda \in \Lambda_{\mathbb{R}} \;\middle|\; \begin{array}{l} f_1(\lambda) = \cdots = f_t(\lambda) = 0 \\ g_1(\lambda) = \cdots = g_s(\lambda) \end{array} \right\},$$

and the lattice $W \cap \Lambda$. Then λ' is in the interior of $\{\lambda \in C_F \cap W \mid g_{\min}(\lambda) \geq 1\}$ and thus is the closest point to the origin contained in the affine plane define by $g_1 = 1$. □

Corollary 7.6.10. *In the setting of Theorem 7.6.5 or Theorem 7.6.6, there is a unique morphism $f\colon [\mathbb{A}^1/\mathbb{G}_m] \to [X/G]$ with $f(1) \simeq x$ and $f(0) \simeq x_0$.*

Proof. By Proposition 6.9.1, a morphism $[\mathbb{A}^1/\mathbb{G}_m] \to [X/G]$ is determined by a one-parameter subgroup λ such that $\lim_{t\to 0} \lambda(t)x \in Gx_0$, and that λ is unique up to conjugation by P_λ. Since any two optimal destabilizing one-parameter subgroups are conjugate under U_λ (and thus P_λ), the statement follows interpretation of maps from $[\mathbb{A}^1/\mathbb{G}_m]$ into quotient stacks (Proposition 6.9.1). $\square$

7.6.4 Examples and extensions

Example 7.6.11. We revisit the SL_2-action on $(\mathbb{P}^1)^n$ with the linearization given by the Segre embedding $(\mathbb{P}^1)^n \hookrightarrow \mathbb{P}^{2^n-1}$ (Example 7.5.1). The non-semistable locus consists of tuples $x = (p_1, \ldots, p_n)$ where more than $n/2$ points are equal. Suppose that precisely $k > n/2$ points are equal. Since the Hilbert–Mumford index is symmetric, we can assume that the first k are equal. If $\lambda\colon \mathbb{G}_m \to \mathrm{SL}_2$ is a one-parameter subgroup, we can choose $g \in \mathrm{SL}_2(\Bbbk)$ with $g\lambda g^{-1} = \lambda_0^d$ where $d \in \mathbb{Z}$ and $\lambda_0(t) = \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix}$. After rescaling the norm, we can assume that $\|\lambda_0\| = 1$. We also assume that $d \geq 0$ as the $d < 0$ case can be handled similarly. Then

$$\frac{\mu(x,\lambda)}{\|\lambda\|} = \frac{\mu(gx, g\lambda g^{-1})}{\|g\lambda g^{-1}\|} = \mu(gx, \lambda_0)$$

This index is negative if and only if $gx = \{0, \ldots, 0, p_{k+1}, \ldots, p_n\}$ in which case $\mu(gx, \lambda_0) = n - 2k$. It follows that λ_0 (resp., $g^{-1}\lambda_0 g$) is an optimal destabilizing one parameter subgroup for gx (resp x). Observe that the parabolic $P_{\lambda_0} \subseteq \mathrm{SL}_2$ of lower triangular matrices is also the stabilizer of $0 \in \mathbb{P}^1$, and thus $G_{gx} \subseteq P_{\lambda_0}$.

Exercise 7.6.12. Under the action of SL_3 on the space $\mathbb{P}(\Gamma(\mathbb{P}^2, \mathcal{O}(3)))$ on ternary cubics, the cuspidal cubic $y^2z - x^3$ is not semistable (see Exercise 7.5.3). Find the optimal destabilizing one-parameter subgroup.

Exercise 7.6.13. Let G be a reductive group over an algebraically closed field $\Bbbk$ with a length $\|-\|$ on $\mathbb{X}_*(G)$. Let $X = \operatorname{Spec} A$ be an affine scheme of finite type over $\Bbbk$ with an action of G. Let $x_0 \in X(\Bbbk)$ have closed G-orbit. Let $x \in X(\Bbbk)$ be a point such that $Gx_0 \subseteq \overline{Gx}$, and let P_x be the parabolic determined by Kempf's Optimal Destabilization Theorem (7.6.5).

(a) Show that for all $g \in G(\Bbbk)$ that $gP_xg^{-1} = P_{gx}$.

Hint: Show that if $P_x = P_\lambda$ for a one-parameter subgroup λ, then $P_{gx} = P_{g\lambda g^{-1}}$.

(b) Show that $G_x \subseteq P_x$.

Hint: Use that for a parabolic P, $N_G(P) = P$ (Proposition B.1.26).

The following criterion can sometimes be used to check stability/semistability by computing Hilbert–Mumford indices only for one-parameter subgroups in a fixed maximal torus.

Exercise 7.6.14 (Kempf–Morrison Criterion)**.** Let $G = \mathrm{GL}(W)$ or $\mathrm{SL}(W)$, where W is finite dimensional vector space over an algebraically closed field $\Bbbk$ of characteristic 0. Let $X \subseteq \mathbb{P}(V)$ be a G-invariant closed subscheme, where V is a finite dimensional G-representation. Let $x \in X(\Bbbk)$. Assume that there is a linearly reductive subgroup

$H \subseteq G_x$ such that W decomposes as a direct sum of distinct H-representations. Let $T \subseteq G$ be a maximal torus compatible with this decomposition. Show that

$$x \in X^{\mathrm{ss}} \iff \mu(x, \lambda) \leq 0 \text{ for all } \lambda \colon \mathbb{G}_m \to T,$$
$$x \in X^{\mathrm{s}} \iff \mu(x, \lambda) < 0 \text{ for all } \lambda \colon \mathbb{G}_m \to T.$$

Hint: If $x \notin X^{\mathrm{ss}}$, let $\lambda_0 \colon \mathbb{G}_m \to G$ be an optimal destabilizing one-parameter subgroup and $0 = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_k = V$ be the filtration induced by the parabolic P_{λ_0}. Use Exercise 7.6.13 to conclude that each V_i is H-invariant, and use the hypothesis on the H-representation V to show that each V_i is T-invariant; thus $T \subseteq P_{\lambda_0}$. Apply Kempf's Optimal Destabilization Theorem again to find λ in T with $\mu(x, \lambda) < 0$. If $x \notin X^{\mathrm{s}}$, letting $\widehat{x} \in \mathbb{A}(V)$ be a lift of x and $\widehat{x}_0 \in \overline{G\widehat{x}}$ be a point with closed orbit, repeat the above argument using the affine version of Kempf's Optimal Destabilization Theorem.

Exercise 7.6.15 (Existence of destabilizing one-parameter subgroups over a perfect field)**.** Let X be an affine scheme of finite type over a perfect field $\Bbbk$, and let G be a reductive group over $\Bbbk$ acting on X. This exercise will show that for every point $x \in X(\Bbbk)$, there exists a one-parameter subgroup $\lambda \colon \mathbb{G}_m \to G$ *defined over* $\Bbbk$ such that $\lim_{t \to 0} \lambda(t) \cdot x$ has closed G-orbit. See also [Kem78, §4].

(1) Show that if $\mathrm{Gal} := \mathrm{Gal}(\overline{\Bbbk}/\Bbbk)$ is the geometric Galois group, then Gal acts on the set $\mathbb{X}_*(G_{\overline{\Bbbk}})$ of one-parameters subgroups such that $\mathbb{X}_*(G) = \mathbb{X}_*(G_{\overline{\Bbbk}})^{\mathrm{Gal}}$.

(2) Show that there exists a length $\|-\|$ on $\mathbb{X}_*(G_{\overline{\Bbbk}})$ which is invariant under the action of Gal.

(3) Show that the subsets $\{\lambda \in \mathbb{X}_*(G_{\overline{\Bbbk}}) \mid \lim_{t \to 0} \lambda(t) \cdot x \in X(\overline{\Bbbk}) \text{ exists}\}$ and $\{\lambda \in \mathbb{X}_*(G_{\overline{\Bbbk}}) \mid \lim_{t \to 0} \lambda(t) \cdot x \in G_{\overline{\Bbbk}} x_0\}$ are Gal-invariant where $G_{\overline{\Bbbk}} x_0$ is the unique closed orbit in $\overline{G_{\overline{\Bbbk}} x}$. Moreover, show that if V and W are G-representations as in (7.6.8), then the functions $m(i(x), \lambda)$ and $m(f(x), \lambda)$ are Gal-invariant.

(4) Generalize Theorem 7.6.5 and Theorem 7.6.6 to the case when $\Bbbk$ is a perfect field and $x \in X(\Bbbk)$.

In particular, if G has no non-trivial one-parameter subgroups defined over $\Bbbk$, then the G-orbit of any $\Bbbk$-point is closed.

Finally, we record the following consequence of *the proof* of Kempf's Optimal Destabilization Theorem (7.6.5). This will play a key role in the proof of the HKKN Stratification (7.7.1).

Proposition 7.6.16. *Let G be a reductive group over an algebraically closed field $\Bbbk$ with a length $\|-\|$ on $\mathbb{X}_*(G)$. Let $X = \operatorname{Spec} A$ be an affine scheme of finite type over $\Bbbk$ with an action of G with a unique closed orbit Gx_0. Fix a maximal torus $T \subseteq G$. There are finitely many one-parameter subgroups $\lambda_1, \ldots, \lambda_n \in \mathbb{X}_*(T)$ and numbers $M_1, \ldots, M_n \in \mathbb{R}_{<0}$ such that for every point $x \in X(\Bbbk)$, there exists a unique $i = 1, \ldots, n$ such that λ_i is an optimal destabilizing one-parameter subgroup for gx for some $g \in G$, and such that $M_i = \mu(x, \lambda_i)/\|\lambda_i\|$.*

Proof. We will use the notation of the proof of Theorem 7.6.5. For $x \in X(\Bbbk)$, the unique parabolic subgroup of a optimal destabilization one-parameter subgroup is determined by the subsets $\mathrm{State}_{gTg^{-1}}(i(x)) \cong \mathrm{State}_T(i(gx)) \subseteq \mathbb{X}_*(T)$ and $\mathrm{State}_{gTg^{-1}}(f(x)) \cong \mathrm{State}_T(f(gx)) \subseteq \mathbb{X}_*(T)$ as g ranges over $G(\Bbbk)$. These subsets are contained in the finite subset of characters $\chi \in \mathbb{X}_*(T)$ with $V_\chi \neq 0$ or $W_\chi \neq 0$. Thus there are only finitely many possibilities for an optimal destabilizing subgroup of T. $\square$

7.7 The HKKN stratification of the unstable locus

For an action of a linearly reductive group G on a projective variety $X \subseteq \mathbb{P}^n$, we show that the non-semistable locus $X \setminus X^{\mathrm{ss}}$ admits a stratification, called the *Hesselink–Kempf–Kirwan–Ness Stratification* or simply the *HKKN stratification*, into locally closed subschemes according to the function

$$M(x) := \inf_{\lambda \in \mathbb{X}_*(G)} \frac{\mu(x, \lambda)}{\|\lambda\|} \in \mathbb{R}_{<0}.$$

By Kempf's Optimal Destabilization Theorem (7.6.6), for each $x \in X \setminus X^{\mathrm{ss}}$, the infimum is obtained by an optimal destabilizing one-parameter subgroup λ. The more negative the index $M(x)$ is, the more non-semistable or 'unstable' the point x is. The strata will be indexed by pairs (λ, M) where $\lambda \in \mathbb{X}_*(G)$ and $M \in \mathbb{R}_{<0}$. Like the Białynicki-Birula Stratification (6.8.14), the HKKN stratification is a powerful tool to compute equivariant cohomology.

7.7.1 The Hesselink–Kempf–Kirwan–Ness Stratification

For a one-parameter subgroup $\lambda \colon \mathbb{G}_m \to G$, recall that $X^\lambda := \mathrm{Mor}^{\mathbb{G}_m}(\operatorname{Spec} \Bbbk, X)$ is the λ-*fixed locus* (Definition 6.8.1) and that $X_\lambda^+ := \mathrm{Mor}^{\mathbb{G}_m}(\mathbb{A}^1, X)$ is the *the attractor locus* parameterizing points x such that $\lim_{t \to 0} \lambda(t) \cdot x$ exists (Definition 6.8.6). We denote by $\mathrm{ev}_0 \colon X_\lambda^+ \to X^\lambda$ the *evaluation map at 0* defined by $x \mapsto \lim_{t\to 0} \lambda(t) \cdot x$. If X is projective, the Białynicki-Birula Stratification (6.8.14) implies that X_λ^+ is a disjoint union of locally closed subschemes. Recall also that C_λ and P_λ denote the centralizer and parabolic subgroups associated to λ (Definition B.1.24). The HKKN Stratification also holds for reductive groups, but as usual we restrict to linearly reductive groups as we've only developed GIT in that setting.

Theorem 7.7.1 (The HKKN Statification)**.** *Let G be a linearly reductive group over an algebraically closed field $\Bbbk$ and a length $\|-\|$ on $\mathbb{X}_*(G)$. Let $X \subseteq \mathbb{P}(V)$ be a G-equivariant closed subscheme where V is a finite dimensional G-representation. For a maximal torus $T \subseteq G$, there is a finite subset $\Sigma \subseteq \mathbb{X}_*(T) \times \mathbb{R}_{<0}$ and a stratification of the non-semistable locus into G-invariant locally closed subschemes*

$$X \setminus X^{\mathrm{ss}} = \coprod_{(\lambda, M) \in \Sigma} S_{\lambda, M}$$

such that for each $(\lambda, M) \in \Sigma$,

(1) The locus

$$Y_{\lambda,M} := \left\{ x \in X_\lambda^+ \;\middle|\; M = M(x) = \mu(x,\lambda)/\|\lambda\| \right\},$$

is a P_λ-invariant locally closed subscheme of X and $S_{\lambda,M} := G \cdot Y_{\lambda,M}$ is a G-invariant locally closed subscheme of X;

(2) a point $x \in Y_{\lambda,M}$ if and only if $\mathrm{ev}_0(x) = \lim_{t \to 0} \lambda(t) \cdot x \in Y_{\lambda,M} \cap X^\lambda$; thus

$$Z_{\lambda,M} := \left\{ x \in X^\lambda \;\middle|\; M(x) = M = \mu(x,\lambda)/\|\lambda\| \right\}$$

is a C_λ-invariant closed subscheme of $Y_{\lambda,M}$ such that $Y_{\lambda,M} = \mathrm{ev}_0^{-1}(Z_{\lambda,M})$.

(3) the natural map $G \times^{P_\lambda} Y_{\lambda,M} \to S_{\lambda,M}$ is finite, surjective, and universally injective; if $\mathrm{char}(\Bbbk) = 0$, then $G \times^{P_\lambda} Y_{\lambda,M} \to S_{\lambda,M}$ is an isomorphism.

(4) the locus

$$\bigcup_{(\lambda',M')\in\Sigma,\, M'\leq M} S_{\lambda',M'}$$

is closed and in particular contains $\overline{S}_{\lambda,M}$*;*

(5) if X *is smooth, then so is each* $Y_{\lambda,M}$*; in* $\mathrm{char}(\Bbbk)=0$*, the strata* $S_{\lambda,M}$ *are also smooth.*

Remark 7.7.2 (Strata, blades, and centers). The locus $S_{\lambda,M}$ is called a *stratum*, Y_λ a *blade*, and $Z_{\lambda,M}$ a *center* (or sometimes a *limit set*). Stack-theoretically, there is a stratification

$$[X/G] = [X^{\mathrm{ss}}/G] \amalg \coprod_{(\lambda,M)\in\Sigma} [S_{\lambda,M}/G],$$

and for each (λ, M), there is a diagram

$$\begin{tikzcd} {[Z_{\lambda,M}/C_\lambda]} \arrow[r, "i"'] & {[Y_{\lambda,M}/P_\lambda]} \arrow[l, bend right, "\mathrm{ev}_0"'] \arrow[r] & {[X/G]} \end{tikzcd} \tag{7.7.3}$$

such that $\mathrm{ev}_0 \circ i = \mathrm{id}$. In characteristic 0, there is an equivalence $[Y_{\lambda,M}/P_\lambda] \cong [S_{\lambda,M}/G]$ and the map $[Y_{\lambda,M}/P_\lambda] \to [X/G]$ is a locally closed immersion.

Proof. Let $\widehat{X} \subseteq \mathbb{A}(V)$ be the affine cone of X, and let $\widehat{N} \subseteq \widehat{X}$ be the nullcone, i.e., the affine cone of $X \smallsetminus X^{\mathrm{ss}}$, which has $0 \in \widehat{N}$ as the unique closed G-orbit. Applying Proposition 7.6.16 to the nullcone $\widehat{N} \subseteq \mathbb{A}(V)$, there is a finite subset $\Sigma \subseteq \mathbb{X}_*(T) \times \mathbb{R}_{<0}$ such that for every point $\widehat{x} \in \widehat{N} \smallsetminus 0$, there is a unique $(\lambda, M) \in \Sigma$ such that λ is an optimal destabilizing one-parameter subgroup for $\widehat{x}$ with $M = \mu(\widehat{x}, \lambda)/\left\|\lambda\right\|$. Since $\widehat{N}$ is affine, the locus $\widehat{N}^+_\lambda \subseteq \widehat{N}$ is a closed subscheme for each $(\lambda, M) \in \Sigma$ (Exercise 6.8.7). As $\widehat{N}^+_\lambda$ is invariant under scaling, it corresponds to a closed subscheme $Y_\lambda \subseteq X$. We consider the loci

$$Y_{\lambda,M} \subseteq \widetilde{Y}_{\lambda,M} \subseteq Y_\lambda,$$

where $\widetilde{Y}_\lambda = \{x \in Y_\lambda \mid M = \mu(x,\lambda)/\left\|\lambda\right\|\}$ and $Y_{\lambda,M} := \{x \in Y_\lambda \,|\, M = M(x) = \mu(x,\lambda)/\left\|\lambda\right\|\}$. Since the function $x \in Y_\lambda \mapsto \mu(x,\lambda)$ is locally constant, $\widetilde{Y}_\lambda \subseteq Y_\lambda$ is open and closed. Using the identification

$$Y_{\lambda,M} = \widetilde{Y}_{\lambda,M} \smallsetminus \bigcup_{(\lambda',M')\in\Sigma,\,,M'<M} \widetilde{Y}_{\lambda',M'},$$

we conclude that $Y_{\lambda,M} \subseteq \widetilde{Y}_{\lambda,M}$ is open. Since G is reductive, $P_\lambda \subseteq G$ is parabolic and

$$[Y_\lambda/P_\lambda] \cong [G \times^{P_\lambda} Y_\lambda/G] \to [X/G]$$

is projective, and thus the image $\widetilde{S}_{\lambda,M} = G \cdot \widetilde{Y}_{\lambda,m} \subseteq X$ of $\widetilde{Y}_{\lambda,M}$ is closed, and the image $S_{\lambda,M} = G \cdot \widetilde{S}_{\lambda,M} \subseteq \widetilde{S}_{\lambda,M}$ of $Y_{\lambda,M}$ is open. This implies both (1) and (4).

For (2), if $x \in X \subseteq \mathbb{P}(V)$, then the limit $x_0 = \lim_{t\to 0} \lambda(t)\cdot x$ is the projection onto the subspace $W = \oplus V_\chi$ ranging over characters $\chi \in \mathbb{X}^*(T)$ such that the projection $\mathrm{proj}_\chi(x)$ of x to V_χ is nonzero and $\langle \chi, \lambda\rangle = -\mu(x,\lambda)$. By Lemma 7.6.9, λ lies on the ray spanned by the unique point closest to the origin in the closed convex set of $C_x = \{\lambda \in \mathbb{X}_*(T)_{\mathbb{R}} \mid \langle \chi, \lambda\rangle \geq 1, \mathrm{proj}_\chi(x) \neq 0\}$. It follows that λ is also the closest point to the origin in the analogously defined set C_{x_0}. Alternatively, one can check that if $\lambda_0 \in \mathbb{X}_*(T)$ is a optimal destabilizing one-parameter subgroup for x_0, then

$\mu(x_0, \lambda_0)/\|\lambda_0\| \leq \mu(x, \lambda)/\|\lambda\|$ (giving the implication $x_0 \in Y_{\lambda,M} \Rightarrow x \in Y_{\lambda,M}$) and $\mu(x, \lambda^N \lambda_0)/\|\lambda^N \lambda_0\| \leq \mu(x, \lambda)/\|\lambda\|$ for $N \gg 0$ (giving the implication $x \in Y_{\lambda,M} \Rightarrow x_0 \in Y_{\lambda,M}$).

For (3), for $x \in Y_{\lambda,M}$ we claim that

$$P_\lambda = \{g \in G(\Bbbk) \mid gx \in Y_{\lambda,M}\}. \tag{7.7.4}$$

Since $Y_{\lambda,M}$ is P_λ-invariant, we have the inclusion '$\subseteq$'. Conversely, if $gx \in Y_{\lambda,M}$, then both λ and $g\lambda g^{-1}$ are optimal destabilization one-parameter subgroups for x. By Kempf's Optimal Destabilization Theorem (7.6.5), the parabolics P_λ and $P_{g\lambda g^{-1}} = gP_\lambda g^{-1}$ are equal. Since $N_G(P_\lambda) = P_\lambda$ (Proposition B.1.26), we conclude that $g \in P_\lambda$. Since $[X_\lambda^+/P_\lambda] \to [X/G]$ is proper, so is $[Y_{\lambda,M}/P_\lambda] \to [S_{\lambda,M}/G]$. The map $[Y_{\lambda,M}/P_\lambda] \to [S_{\lambda,M}/G]$ is surjective by construction, and injective on $\Bbbk$-points by (7.7.4); it is thus finite, surjective, and universally injective. In $\mathrm{char}(\Bbbk) = 0$, it is an isomorphism.

For (5), if X is smooth, by Theorem 6.8.9, so is each Y_λ. Since $Y_{\lambda,M} \subseteq Y_\lambda$ is open, $Y_{\lambda,M}$ is also smooth. In $\mathrm{char}(\Bbbk) = 0$, by (3), $S_{\lambda,M} = G \times^{P_\lambda} Y_{\lambda,M}$ and thus also smooth. See also [Hes81, §3], [Hes79, §4], and [Kir84, Thms. 12.26 and 13.5]. □

7.7.2 Examples and related stratifications

Example 7.7.5. Let $\mathbb{G}_m$ act linearly on $X = \mathbb{P}^2$ with weights $-1, 2, 3$. Letting $\lambda = \mathrm{id}$ be the identity one-parameter subgroup, the non-semistable locus is $V(x^2y, x^3z)$ has the stratification $S_{\lambda^{-1},-1} \cup S_{\lambda,-2} \cup S_{\lambda,-3}$ where $S_{\lambda^{-1},-1} = \{[1:0:0]\}$, $S_{\lambda,-2} = \{[0 : y : z] \mid y \neq 0\}$, and $S_{\lambda,-3} = \{[0:0:1]\}$.

Example 7.7.6. Revisiting the action of SL_2 on $X = (\mathbb{P}^1)^n$ with the Segre linearization (Example 7.6.11), let $\lambda_0 \colon \mathbb{G}_m \to \mathrm{SL}_2$ be the one-parameter subgroup defined by $\lambda_0(t) = \mathrm{diag}(t^{-1}, t)$. The strata are indexed by $(\lambda_0, -1), (\lambda_0, -3), \ldots, (\lambda_0, -n)$ if n is odd and by $(\lambda_0, -2), (\lambda_0, -4), \ldots, (\lambda_0, -n)$ if n is even. The strata $S_{\lambda_0, n-2k}$ consists of tuples with precisely $k > n/2$ points in common and has codimension $k-1$. The blade $Y^+_{\lambda_0, n-2k}$ consists of tuples where precisely k points are 0 while the center $Z_{\lambda_0, n-2k}$ is the set of $\mathbb{G}_m$-fixed points where k points are 0 and $n-k$ points are ∞.

Remark 7.7.7 (Morse stratifications). When X is a smooth projective variety over $\mathbb{C}$, the HKKN stratification coincides with the Morse stratification of the square-norm of the moment map $\|-\|^2 : X \to \mathbb{R}$. Given $x \in X$, the optimal destabilizing one-parameter subgroup corresponds to the path of steepest descent starting from x. The centers $Z_{\lambda,M}$ correspond to the set of critical values of $\|-\|^2$, while the strata $S_{\lambda,M}$ are the locally closed submanifolds consisting of points which flow to $Z_{\lambda,M}$. See [Kir84, §6] and [Nes84].

Remark 7.7.8 (Θ-stratifications). As indicated in Remark 6.9.5, there is an identification

$$\underline{\mathrm{Mor}}([\mathbb{A}^1/\mathbb{G}_m], [X/G]) = \coprod_{\lambda \in \mathbb{X}_*(G)/\sim} [X_\lambda^+/P_\lambda],$$

where $\mathbb{X}_*(G)/\sim$ represents the set of one-parameter subgroups up to conjugation. A Θ-*stratification* of an algebraic stack $\mathcal{X}$ locally of finite type over $\Bbbk$ is the data of a totally ordered set Σ with a minimal element $0 \in \Sigma$ and a stratification into locally closed substacks

$$\mathcal{X} = \coprod_{\lambda \in \Sigma} \mathcal{S}_\lambda$$

such that:

(1) for each $\lambda \in \Sigma$, $\mathcal{X}_{\leq \lambda} := \bigcup_{\rho \leq \lambda} \mathcal{S}_\rho$ is an open substack of $\mathcal{X}$,

(2) for each $\lambda \in \Sigma$, there is a union of connected components (called a Θ-*stratum* of $\mathcal{X}_{\leq \lambda}$)

$$\mathcal{S}'_\lambda \subseteq \underline{\mathrm{Mor}}([\mathbb{A}^1/\mathbb{G}_m], \mathcal{X}_{\leq \lambda})$$

such that $\mathrm{ev}_1 \colon \mathcal{S}'_\lambda \to \mathcal{X}_{\leq \lambda}$ is a closed immersion mapping isomorphically onto $\mathcal{S}_\lambda$, and

(3) for every $x \in |\mathcal{X}|$, the set $\{\lambda \in \Sigma \mid x \in |\mathcal{X}_{\leq \lambda}|\}$ has a minimal element.

See [HL14]. The semistable locus $\mathcal{X}^{\mathrm{ss}}$ is by definition the open substack $\mathcal{X}_{\leq 0} = \mathcal{S}_0$. Let $\mathcal{Z}'_\lambda$ be the preimage of $\mathcal{S}'_\lambda$ under the map

$$i \colon \underline{\mathrm{Mor}}(B\mathbb{G}_m, \mathcal{X}) \to \underline{\mathrm{Mor}}([\mathbb{A}^1/\mathbb{G}_m], \mathcal{X}).$$

The map $\mathrm{ev}_0 \colon \underline{\mathrm{Mor}}([\mathbb{A}^1/\mathbb{G}_m], \mathcal{X}) \to \underline{\mathrm{Mor}}(B\mathbb{G}_m, \mathcal{X})$ obtained by restricting to 0 is a section of i, and there is a diagram analogous to (7.7.3)

$$\begin{tikzcd} \mathcal{Z}'_\lambda \arrow[r, "i"] & \mathcal{S}'_\lambda \arrow[l, bend right, "\mathrm{ev}_0"'] \arrow[r, hook] & \mathcal{X}. \end{tikzcd}$$

In characteristic 0, the HKKN stratification is an example of a Θ-stratification, where one orders the indices (λ, M) first by $-M$ and then arbitrarily by λ. In the next chapter, we will see that the moduli stack $\mathcal{B}un_{r,d}(C)$ has a Θ-stratification called the Harder–Narasimhan–Shatz stratification.

7.7.3 Applications to cohomology: Kirwan surjectivity

Kirwan Surjectivity (7.7.13) provides a formula for the equivariant Chow groups of X in terms of the equivariant Chow groups of X^{ss} and the centers $Z_{\lambda,M}$ of the HKKN stratification. To prove this, we will need some preparatory lemmas.

Lemma 7.7.9. *Let X be a smooth irreducible scheme over an algebraically closed $\Bbbk$ with an action of a smooth affine algebraic group G. Let $S_1, \ldots, S_r \subseteq X$ be nonempty, disjoint, smooth, irreducible, and locally closed G-invariant subschemes such that $X = \coprod_i S_i$ and such that $S_{\geq i} := \bigcup_{j \geq i} S_j$ is closed for each i. Let d_i be the codimension of S_i in X. If the top Chern class $c^G_{d_i}(N_{S_i/X}) \in \mathrm{CH}^*_G(S_i)_{\mathbb{Q}}$ is a nonzerodivisor for each i, then*

$$\dim \mathrm{CH}^k_G(X)_{\mathbb{Q}} = \sum_{i=1}^{r} \dim \mathrm{CH}^{k-d_i}_G(S_i)_{\mathbb{Q}}$$

for each k.

Proof. By assumption, $S_{\leq i} = \bigcup_{j \leq i} S_j$ is open for each i, and $S_i \subseteq S_{\leq i}$ is a closed subscheme with open complement $S_{<i}$. We have a commutative diagram

$$\begin{tikzcd} \mathrm{CH}^{k-d_i}_G(S_i) \arrow[r] \arrow[dr] & \mathrm{CH}^k_G(S_{\leq i}) \arrow[r] \arrow[d] & \mathrm{CH}^k_G(S_{<i}) \arrow[r] & 0 \\ & \mathrm{CH}^k_G(S_i) & & \end{tikzcd}$$

where the top row is the right exact excision sequence (6.1.36(3)) and the vertical downward arrow is given by intersecting with S_i. By the self-intersection formula

(6.1.36(6)), the composition $\mathrm{CH}_G^{k-d_i}(S_i) \to \mathrm{CH}_G^k(S_i)$ is multiplication by $c_{d_i}^G(N_{S_i/X})$. By hypothesis, this map is injective after tensoring with $\mathbb{Q}$. It follows that the top row is an exact sequence after tensoring with $\mathbb{Q}$, and that

$$\dim \mathrm{CH}_G^k(S_{\leq i})_{\mathbb{Q}} = \dim \mathrm{CH}_G^{k-d_i}(S_i)_{\mathbb{Q}} + \dim \mathrm{CH}_G^k(S_{<i})_{\mathbb{Q}}.$$

The formula follows from induction. See also [AB83, Prop. 1.9]. □

Remark 7.7.10. If $[S_i/G]$ is Deligne–Mumford, then $\mathrm{CH}_G^k(S_i)$ vanishes for $k \gg 0$ and $c_{d_i}^G(N_{S_i/X})$ is a zero divisor.

There is a convenient criterion for the top Chern class to be a nonzerodivisor.

Lemma 7.7.11. *Let X be a smooth irreducible scheme over an algebraically closed $\Bbbk$ with an action of a connected, smooth, and affine algebraic group G, and let N be a G-equivariant vector bundle of rank d on X. Suppose that there is a subgroup $\mathbb{G}_m \subseteq G$ acting trivially on X and a point $x \in X(\Bbbk)$ such that $N \otimes \kappa(x)$ contains no $\mathbb{G}_m$-invariant vectors. Then $c_d^G(N) \in \mathrm{CH}_G^*(X)_{\mathbb{Q}}$ is a nonzerodivisor.*

Proof. Choose a maximal torus T containing $\mathbb{G}_m$ and a character $T \to \mathbb{G}_m$ such that the composition $\mathbb{G}_m \hookrightarrow T \to \mathbb{G}_m$ is given by $t \mapsto t^d$ for $d > 0$. By (6.1.38(b)), $\mathrm{CH}_G^*(X)_{\mathbb{Q}} = \mathrm{CH}_T^*(X)_{\mathbb{Q}}^W$ where W is the Weyl group. Since $\mathrm{CH}_G^*(X)_{\mathbb{Q}}$ is a subring of $\mathrm{CH}_T^*(X)_{\mathbb{Q}}$, we are reduced to show that $c_d^T(N) \in \mathrm{CH}_T^*(X)_{\mathbb{Q}}$ is a nonzerodivisor. If we write T as the product of the given $\mathbb{G}_m$ and a subtorus T', then

$$\mathrm{CH}_T^*(X) \cong \mathrm{CH}_{T'}^*(X) \otimes \mathrm{CH}^*(B\mathbb{G}_m) \cong \mathrm{CH}_{T'}^*(X)[z]$$

by (6.1.38(a)). For $x \in X(\Bbbk)$, we can write

$$\begin{aligned} c_d^T(N) &= \sum c_i^{T'}(N) \otimes c_{d-i}^{\mathbb{G}_m}(N \otimes \kappa(x)) \\ &= 1 \otimes c_d^{\mathbb{G}_m}(N \otimes \kappa(x)) + \text{higher degree terms.} \end{aligned}$$

If $a_1, \ldots, a_d$ denote the $\mathbb{G}_m$-weights of $N \otimes \kappa(x)$, then by hypothesis each $a_i \neq 0$ and

$$c_d^{\mathbb{G}_m}(N \otimes \kappa(x)) = \big(\prod_i a_i\big) z^d \in \mathrm{CH}^*(B\mathbb{G}_m)_{\mathbb{Q}} \cong \mathbb{Q}[z]$$

is a nonzerodivisor, and therefore $c_d^T(N)$ is also a nonzerodivisor. See also [AB83, Prop. 13.4] and [Bri97, §3.2]. □

We define the *G-equivariant Chow–Poincaré polynomial* of a G-equivariant scheme X as

$$p_G(X, t) = \sum_{d=0}^{\infty} \big(\dim \mathrm{CH}_G^d(X)_{\mathbb{Q}}\big) t^d.$$

We also denote $p(X,t) = \sum_{d=0}^{\infty} \big(\dim \mathrm{CH}^d(X)_{\mathbb{Q}}\big) t^d$ as the (non-equivariant) *Chow–Poincaré polynomial.*

Exercise 7.7.12. Let X be a smooth, irreducible, and quasi-projective scheme over an algebraically closed field $\Bbbk$ with an action of $\mathbb{G}_m$ such that $X^+ \to X$ is surjective (i.e., X is projective). Let $X = \coprod_{i=1}^r X_i$ and $X^{\mathbb{G}_m} = \coprod_{i=1}^r F_i$ be the Białynicki-Birula Stratification (6.8.14), and let d_i be the codimension of X_i in X. Show that

$$p_{\mathbb{G}_m}(X, t) = \sum_{i=1}^r p(F_i, t) \cdot t^{d_i} (1-t)^{-1}.$$

Proposition 7.7.13 (Kirwan Surjectivity)**.** *Under the hypotheses of the HKKN Stratification (7.7.1), assume further assume that X is smooth and irreducible, and that $\operatorname{char}(\Bbbk) = 0$. Suppose that for all (λ, M), the stratum $S_{\lambda,M}$ is equidimensional of codimension $d_{\lambda,M}$. Then*

$$\dim \mathrm{CH}^k_G(X)_{\mathbb{Q}} = \dim \mathrm{CH}^k_G(X^{\mathrm{ss}})_{\mathbb{Q}} + \sum_{(\lambda,M)} \dim \mathrm{CH}^{k-d_{\lambda,M}}_{C_\lambda}(Z_{\lambda,M})_{\mathbb{Q}}$$

and

$$p_G(X,t) = p_G(X^{\mathrm{ss}},t) + \sum_{(\lambda,M)} p_{C_\lambda}(Z_{\lambda,M},t)t^{d_{\lambda,M}}.$$

Proof. From the HKKN Stratification (7.7.1), $[S_{\lambda,M}/G] \cong [Y_{\lambda,M}/P_\lambda]$, and from the Białynicki-Birula Stratification (6.8.14), $\mathrm{ev}_0 \colon Y_{\lambda,M} \to Z_{\lambda,M}$ is a Zariski-local affine fibration and equivariant with respect to $P_\lambda \to C_\lambda$. We claim that $[Y_{\lambda,M}/P_\lambda] \to [Z_{\lambda,M}/C_\lambda]$ induces an isomorphism

$$\mathrm{CH}^*_{C_\lambda}(Z_{\lambda,M}) \to \mathrm{CH}^*_{P_\lambda}(Y_{\lambda,M}). \tag{7.7.14}$$

By the definition of the equivariant Chow groups (§6.1.6), $\mathrm{CH}^i_{P_\lambda}(Y_{\lambda,M})$ is identified by $\mathrm{CH}^i(Y_{\lambda,M} \times^{P_\lambda} V)$, where V is an open subspace $\mathbb{A}(W)$ of a P_λ-representation W such that P_λ acts freely on V and $\mathbb{A}(W) \smallsetminus V$ has sufficiently high codimension. Similarly, $\mathrm{CH}^i_{C_\lambda}(Z_{\lambda,M})$ is identified with $\mathrm{CH}^i(Z_{\lambda,M} \times^{C_\lambda} V)$. The map (7.7.14) corresponds to the pullback map on Chow induced from the composition

$$Y_{\lambda,M} \times^{P_\lambda} V \to Z_{\lambda,M} \times^{P_\lambda} V \to Z_{\lambda,M} \times^{C_\lambda} V.$$

The first map is a Zariski local affine fibration and the second map is a principal bundle under $U_\lambda = \ker(P_\lambda \to C_\lambda)$. Since U_λ is unipotent, U_λ is isomorphic to affine space (see Example B.1.20) and principal U_λ-bundles are locally trivial in the Zariski topology (see Example B.1.59). From Properties 6.1.36(2), we conclude that (7.7.14) is an isomorphism.

We also claim that $c_{d_{\lambda,M}}(N_{S_{\lambda,M}/X}) \in \mathrm{CH}^*_G(S_{\lambda,M})$ is a nonzerodivisor. Since $(N_{S_{\lambda,M}/X})|_{Z_{\lambda,M}}$ is identified with $N_{S_{\lambda,M}/X}$ under $\mathrm{CH}^*_{C_\lambda}(Z_{\lambda,M}) \cong \mathrm{CH}^*_G(S_{\lambda,M})$, it suffices to show that $c_{d_{\lambda,M}}((N_{S_{\lambda,M}/X})|_{Z_{\lambda,M}}) \in \mathrm{CH}^*_{C_\lambda}(Z_{\lambda,M})_{\mathbb{Q}}$ is a nonzerodivisor. By the Białynicki-Birula Stratification (6.8.14), λ acts on a fiber of the normal bundle with nonzero weights. Thus Lemma 7.7.11 implies that $c_{d_{\lambda,M}}((N_{S_{\lambda,M}/X})|_{Z_{\lambda,M}})$ is a nonzerodivisor.

We therefore can apply Lemma 7.7.9 with the strata $S_{\lambda,M}$ ordered first by $-M$ and then with any ordering of the λ's; the semistable locus U^{ss} is viewed as a stratum with the smallest index. This yields

$$\begin{aligned}\dim \mathrm{CH}^k_G(X)_{\mathbb{Q}} &= \dim \mathrm{CH}^k_G(X^{\mathrm{ss}})_{\mathbb{Q}} + \sum_{(\lambda,M)} \dim \mathrm{CH}^{k-d_{\lambda,M}}_G(S_{\lambda,M})_{\mathbb{Q}} \\ &= \dim \mathrm{CH}^k_G(X^{\mathrm{ss}})_{\mathbb{Q}} + \sum_{(\lambda,M)} \dim \mathrm{CH}^{k-d_{\lambda,M}}_{C_{\lambda,M}}(Z_{\lambda,M})_{\mathbb{Q}},\end{aligned}$$

and the formula for the Chow–Poincaré polynomial also follows. □

Remark 7.7.15. This formula was established for de Rham cohomology in [Kir84, Thm. 5.4]. Instead of the excision sequence

$$\mathrm{CH}^{k-d_{\lambda,M}}_G(S_{\lambda,M}) \to \mathrm{CH}^k_G(S_{\leq(\lambda,M)}) \to \mathrm{CH}^k_G(S_{<(\lambda,M)}) \to 0,$$

one uses the Thom–Gysin long exact sequence

$$\cdots \to \mathrm{H}_G^{k-d_{\lambda,M}}(S_{\lambda,M}) \to \mathrm{H}_G^k(S_{\leq(\lambda,M)}) \to \mathrm{H}_G^k(S_{<(\lambda,M)}) \to \cdots .$$

Kirwan proved that the long exact sequence splits into short exact sequences by showing that each map on the right is surjective, which explains why the result is referred to *Kirwan Surjectivity.* To see this, observe that the surjectivity of the right map for all (λ, M) is equivalent to the injectivity of the left map for all (λ, M), and that the latter condition is verified by arguing as above that the top Chern class of the normal bundle is a nonzerodivisor.

Example 7.7.16. We compute the dimension of the rational Chow groups of $[(\mathbb{P}^1)^{n,\mathrm{ss}}/\operatorname{SL}_2]$ using the HKKN Stratification, as determined in Example 7.7.6. When n is odd, this also gives the dimension of the rational Chow groups of the GIT quotient $(\mathbb{P}^1)^{n,\mathrm{ss}}/\operatorname{SL}_2$ by Properties 6.1.36(4).

Since $[(\mathbb{P}^1)^n/\operatorname{SL}_2] \to B\operatorname{SL}_2$ is an iterated $\mathbb{P}^1$-bundle and $\mathrm{CH}^*(\mathrm{SL}_2) \cong \mathbb{Z}[T]$ generated in degree 2,

$$\begin{aligned}\mathrm{CH}^*([(\mathbb{P}^1)^n/\operatorname{SL}_2]) &\cong \mathrm{CH}^*((\mathbb{P}^1)^n) \otimes \mathrm{CH}^*(B\operatorname{SL}_2) \\ &\cong \mathbb{Z}[H_1,\dots,H_n]/(H_1,\dots,H_n)^2 \otimes \mathbb{Z}[T],\end{aligned}$$

and the Chow–Poincaré polynomial is $p_{\mathrm{SL}_2}((\mathbb{P}^1)^n, t) = (1+t)^n(1-t^2)^{-1}$. On the other hand, the strata $S_{\lambda,n-2k}$, where precisely k points are the same, has codimension $k-1$ and its center $Z_{\lambda,n-2k}$ consists of $\binom{n}{k}$ $\mathbb{G}_m$-fixed points. Thus $p_{\mathbb{G}_m}(Z_{\lambda,n-2k}, t) = \binom{n}{k}(1-t)^{-1}$ and

$$\begin{aligned}p_G((\mathbb{P}^1)^{\mathrm{ss}}, t) &= (1+t)^n(1-t)^{-1} - \sum_{k>n/2}\binom{n}{k} t^{k-1}(1-t)^{-1} \\ &= 1 + nt + \cdots + \left(1 + (n-1) + \binom{n-1}{2} + \cdots + \binom{n-1}{\min(d, n-3-d)}\right)t^d \\ &+ \cdots + nt^{n-4} + t^{n-3}.\end{aligned}$$

See also [Kir84, §16.1].

Chapter 8

Moduli of semistable vector bundles

Vector bundles and their associated moduli spaces are of fundamental importance in algebraic geometry. In recent decades the richness of the subject has been enhanced by its relationship to other areas of mathematics, including differential geometry (which has always played a fundamental role), topology (where the moduli spaces not only present interesting topological features but turn out to be useful in settling topo- logical problems) and, perhaps most surprising of all, theoretical physics, in particular gauge theory, quantum field theory and string theory.

Editors [BBPGPR09, preface]

This chapter proves Theorem B: for a smooth, connected, and projective curve C of genus g over an algebraically closed field $\Bbbk$ of characteristic 0, the stack $\mathcal{B}un^{\mathrm{ss}}_{r,d}(C)$ parameterizing semistable vector bundles of rank r and degree d over is a smooth, universally closed, and irreducible algebraic stack of dimension $r^2(g-1)$ which admits a projective good moduli space $M^{\mathrm{ps}}_{r,d}(C)$. We follow the six-step strategy to construct projective moduli spaces outlined in §0.7.2:

① (Algebraicity) We express $\mathcal{B}un^{\mathrm{ss}}_{r,d}(C)$ as a substack of $\underline{\mathrm{Coh}}(C)$, parameterizing all coherent sheaves on C, which is an algebraic stack locally of finite type (Theorem 3.1.21).

② (Openness of Semistability) $\mathcal{B}un^{\mathrm{ss}}_{r,d}(C) \subseteq \underline{\mathrm{Coh}}(C)$ is an open substack (Theorem 8.2.4).

③ (Boundedness of Semistability) $\mathcal{B}un^{\mathrm{ss}}_{r,d}(C)$ is of finite type (Theorem 8.2.8).

④ (Semistable reduction) $\mathcal{B}un^{\mathrm{ss}}_{r,d}(C)$ is universally closed by Langton's Theorem (8.3.1).

⑤ (Existence of a moduli space) Since $\mathcal{B}un^{\mathrm{ss}}_{r,d}(C)$ is Θ- and S-complete (Proposition 8.3.8), the Existence Theorem for Good Moduli Spaces (6.10.1) implies that exists a good moduli space $\mathcal{B}un^{\mathrm{ss}}_{r,d}(C) \to M^{\mathrm{ps}}_{r,d}(C)$, with $M^{\mathrm{ps}}_{r,d}(C)$ a proper algebraic space of $\dim M^{\mathrm{ps}}_{r,d}(C) = r^2(g-1)+1$ (Theorem 8.3.10).

⑥ (Projectivity) We follow Faltings's approach to verify that $M^{\mathrm{ps}}_{r,d}(C)$ is projective.

We also provide a GIT construction of $M^{\mathrm{ps}}_{r,d}(C)$. In the rank $r=1$ case, $\mathcal{B}un_{1,d}(C)$ is identified with the Picard stack $\underline{\mathcal{P}\mathrm{ic}}^{\,d}_C$ of degree d line bundles on C. We showed in Theorem 6.4.51 that there is a banded $\mathbb{G}_m$-gerbe $\underline{\mathcal{P}ic}_C^d \to \underline{\mathrm{Pic}}^d(C)$ (which in particular is a coarse moduli space), where $\underline{\mathrm{Pic}}^d(C)$ is a projective scheme. If in addition $d = 0$, then $\underline{\mathrm{Pic}}^0(C)$ is an abelian variety with the group law given by tensor product.

Notation: Throughout this chapter, C denotes a smooth, geometrically connected, and projective curve over a field $\Bbbk$. We will often further impose that $\Bbbk$ is algebraically closed and/or characteristic 0.[1] As C is fixed, we will drop it from the notation of the moduli stacks and spaces, e.g., $\mathcal{B}un_{r,d}^{\mathrm{ss}} := \mathcal{B}un_{r,d}^{\mathrm{ss}}(C)$ and $M_{r,d}^{\mathrm{ps}} := M_{r,d}^{\mathrm{ps}}(C)$.

8.1 Semistable vector bundles

We introduce the fundamental notion of *semistability* for vector bundles on a smooth curve. At first glance, the definition may appear unmotivated, but we will show that it enjoys several desirable properties:

- Every vector bundle has a unique filtration—the *Harder–Narasimhan filtration*—with semistable factors (Theorem 8.1.21), while every semistable vector bundle has a (non-canonical) filtration—the *Jordan–Hölder filtration*—with stable factors (Proposition 8.1.32).
- Semistability is an open and bounded condition (Theorems 8.2.4 and 8.2.8).
- The algebraic stack of semistable vector bundles $\mathcal{B}un_{r,d}^{\mathrm{ss}}$ is universally closed (Theorem 8.3.1).
- There is a projective good moduli space $\mathcal{B}un_{r,d}^{\mathrm{ss}} \to M_{r,d}^{\mathrm{ps}}$ identifying S-equivalent vector bundles and inducing a bijection between the polystable vector bundles (which are identified with the closed points of $\mathcal{B}un_{r,d}^{\mathrm{ss}}$) and $M_{r,d}^{\mathrm{ps}}(\Bbbk)$ (Theorem B and Proposition 8.1.35).
- Semistable vector bundles exist for every $r > 0$ and d (Theorem 8.1.36). In fact, stable vector bundles also exist (Theorem 8.2.14).

In §8.1.6, we provide a case study of vector bundles on elliptic curves.

8.1.1 Review of coherent sheaves on a smooth curve

Let C be a smooth, geometrically connected, and projective curve C over an arbitrary field $\Bbbk$. Any coherent sheaf F on C has a torsion subsheaf F_{tors} whose sections on an open subset $U \subseteq C$ is the subset $F(U)_{\mathrm{tors}} \subseteq F(U)$ of torsion elements. The quotient F/F_{tors} is a vector bundle and since $\mathrm{Ext}^1_{\mathcal{O}_C}(F/F_{\mathrm{tors}}, F_{\mathrm{tors}}) = \mathrm{H}^1(C, F_{\mathrm{tors}} \otimes (F/F_{\mathrm{tors}})^\vee) = 0$ (as $\dim(\mathrm{Supp}\, F_{\mathrm{tors}}) = 0$), the short exact sequence $0 \to F_{\mathrm{tors}} \to F \to F/F_{\mathrm{tors}} \to 0$ splits and $F \cong F_{\mathrm{tors}} \oplus F/F_{\mathrm{tors}}$.

If F is a vector bundle of rank $\mathrm{rk}\, F = r$, we define its *determinant* as the line bundle $\det F := \bigwedge^r F$ and its *degree* as $\deg F := \deg(\det F)$. For a coherent sheaf $F \cong F_{\mathrm{tors}} \oplus F/F_{\mathrm{tors}}$, the *degree* of F is $\deg F := \mathrm{h}^0(C, F_{\mathrm{tors}}) + \deg(F/F_{\mathrm{tors}})$. We define the *rank* of a coherent sheaf F as $\mathrm{rk}\, F := \dim_{\kappa(\xi)} F \otimes \kappa(\xi)$ where $\xi \in C$ is the generic point. Note that the rank of a torsion sheaf is 0. If $0 \to F' \to F \to F'' \to 0$ is a short exact sequence, then $\deg F = \deg F' + \deg F''$ and $\mathrm{rk}\, F = \mathrm{rk}\, F' + \mathrm{rk}\, F''$. In other words, the degree and rank extend to group homomorphisms

$$\deg, \mathrm{rk}\colon K(C) \to \mathbb{Z}$$

[1] Most results in this chapter hold (with the same proof) when $\Bbbk$ is an arbitrary field of characteristic 0. The only place where the characteristic 0 hypothesis is essential is in the application of the Existence Theorem for Good Moduli Spaces (Theorem 6.10.1).

from the Grothendieck group.

Exercise 8.1.1 (easy)**.** If F_1 and F_2 are vector bundles on C, show that $\deg(F_1 \otimes F_2) = \deg F_1 \operatorname{rk} F_2 + \operatorname{rk} F_1 \deg F_2$.

Riemann–Roch (5.1.2) for line bundles extends to coherent sheaves.

Theorem 8.1.2 (Riemann–Roch for Coherent Sheaves)**.** *Let C be a smooth, connected, and projective curve over an algebraically closed field $\Bbbk$. If F is a coherent sheaf on C, then*

$$\chi(C, F) = \deg(F) + \operatorname{rk}(F)(1 - g).$$

Proof. We will prove this by induction on the rank using the observation that the Euler characteristic, rank, and degree are each additive in short exact sequences. If F is torsion, then $\chi(C, F) = \deg F$ by the definition of degree. If $\operatorname{rk}(F) > 0$, then there is a line bundle subsheaf $L \subseteq F$ with $\operatorname{rk}(F/L) = \operatorname{rk}(F) - 1$. Since the statement holds for F/L by induction and for L by Riemann–Roch (5.1.2), it also holds for F. □

Exercise 8.1.3 (easy)**.** Show that a vector bundle F on C of rank r and degree d has Hilbert polynomial $P(t) := rt + (d + r(1 - g))$.

Exercise 8.1.4 (easy)**.** If F_1 and F_2 are vector bundles on C of degree d_i and rank r_i, show that

$$\dim \operatorname{Ext}^1_{\mathcal{O}_C}(F_1, F_2) - \dim \operatorname{Ext}^0_{\mathcal{O}_C}(F_1, F_2) = r_1 r_2 (g - 1) + (d_1 r_2 - d_2 r_1).$$

In particular, if F is a vector bundle of r and degree d, then $\dim \operatorname{Ext}^1_{\mathcal{O}_C}(F, F) - \dim \operatorname{Ext}^0_{\mathcal{O}_C}(F, F) = r^2(g - 1)$.

We also recall Serre–Duality (5.1.3): for a coherent sheaf F on C, there is an natural isomorphism

$$\mathrm{H}^1(C, F) \cong \operatorname{Hom}_{\mathcal{O}_C}(F, \Omega_C)^\vee.$$

Example: vector bundles on $\mathbb{P}^1$. It is a classical theorem that vector bundles on $\mathbb{P}^1$ split as direct sums of line bundles.

Theorem 8.1.5 (Birkhoff–Grothendieck Theorem)**.** *If E is a vector bundle on $\mathbb{P}^1$ over an arbitrary field $\Bbbk$, there exists unique integers $d_1 \geq \cdots \geq d_r$ such that $E \cong \bigoplus_{i=1}^r \mathcal{O}_{\mathbb{P}^1}(d_i)$.*

Proof. We will argue by induction on the rank r. If $r = 1$, every line bundle is isomorphic to $\mathcal{O}(d)$ [Har77, Prop. II.6.4]. If E is a vector bundle of rank $r > 0$, we let d_1 be the largest integer such that $\mathrm{H}^0(\mathbb{P}^1, E(-d_1)) \neq 0$. A nonzero section of $E(-d_1)$ defines an injection $\mathcal{O}(d_1) \to E$ and a short exact sequence

$$0 \to \mathcal{O}(d_1) \to E \to E' \to 0. \tag{8.1.6}$$

We claim that E' is a vector bundle: otherwise, the kernel $K := \ker(E \to E'/E'_{\text{tors}})$ is a line bundle satisfying $\mathcal{O}(d_1) \subsetneq K$ so that $K \cong \mathcal{O}(d)$ with $d > d_1$, contradicting the choice of d_1. By induction, we can write $E' = \bigoplus_{i=2}^r \mathcal{O}(d_i)$ with $d_2 \geq \cdots \geq d_r$. Twisting the above sequence gives a short exact sequence $0 \to \mathcal{O}(-1) \to E(-d_1 - 1) \to \bigoplus_{i=2}^r \mathcal{O}(d_i - d_1 - 1) \to 0$. Since $\mathrm{H}^0(\mathbb{P}^1, E(-d_1 - 1)) = 0$ and $\mathrm{H}^0(\mathbb{P}^1, \mathcal{O}(-1)) = \mathrm{H}^1(\mathbb{P}^1, \mathcal{O}(-1)) = 0$, we have that $\mathrm{H}^0(\mathbb{P}^1, \mathcal{O}(d_i - d_1 - 1)) = 0$ for all $i \geq 2$. Therefore, $d_i - d_1 - 1 < 0$ or equivalently $d_i \leq d_1$. To show that (8.1.6) splits, we apply $\operatorname{Hom}_{\mathcal{O}_{\mathbb{P}^1}}(-, \mathcal{O}(d_1))$ and compute that

$$\operatorname{Ext}^1_{\mathcal{O}_{\mathbb{P}^1}}(E', \mathcal{O}(d_1)) = \bigoplus_{i \geq 2} \mathrm{H}^1(\mathbb{P}^1, \mathcal{O}(d_1 - d_i)) = 0$$

because $d_1 - d_i \geq 0$. We leave the uniqueness to the reader. See also [LP97, Lem. 4.4.1] and [HL10, Thm. 1.3.1].

Alternatively, this can be proved using matrix factorizations, which is a similar spirit to Birkhoff's argument [Bir09]. As every vector bundle on $\mathbb{A}^1$ is trivial, a vector bundle E on $\mathbb{P}^1$ can be specified by a transition function $g \in \mathrm{GL}_r(\Bbbk[x]_x)$ which is unique up to automorphisms of $E|_{\mathbb{P}^1 \smallsetminus 0}$ and $E|_{\mathbb{P}^1 \smallsetminus \infty}$, i.e., there is a bijection

$$\left\{\text{vector bundles on } \mathbb{P}^1\right\}/\!\sim \longleftrightarrow \mathrm{GL}_r(\Bbbk[x]) \backslash \mathrm{GL}_r(\Bbbk[x]_x) / \mathrm{GL}_r(\Bbbk[1/x]).$$

It suffices to show that for any $g \in \mathrm{GL}_r(\Bbbk[x]_x)$, there exist $A \in \mathrm{GL}_r(\Bbbk[x])$ and $B \in \mathrm{GL}_r(\Bbbk[1/x])$ such that $AgB = \operatorname{diag}(x^{d_1}, \ldots, x^{d_r})$ for integers $d_1 \geq \cdots \geq d_r$. We leave this linear algebra fact as an exercise. □

Remark 8.1.7. Grothendieck proved an analogous statement for G-bundles on $\mathbb{P}^1$ for a reductive group G [Gro57a], which recovers the above statement with $G = \mathrm{GL}_r$.

8.1.2 Definition of semistability and first properties

Definition 8.1.8 (Slope)**.** The *slope* of a coherent sheaf F on a smooth, geometrically connected, and projective curve C over an arbitrary field $\Bbbk$ is defined as

$$\mu(F) := \frac{\deg F}{\operatorname{rk} F} \in \mathbb{Q} \cup \{\infty\}$$

with the convention that $\mu(F) = \infty$ if F is torsion.

Remark 8.1.9 (Visualizing the slope). We may plot a coherent sheaf F in the complex plane as the point $Z(F) := -\deg(F) + \operatorname{rk}(F)i$.[2] With this convention, E has larger slope than F if and only if the complex argument of $Z(E)$ is larger than $Z(F)$. The function $Z(F)$, sometimes called the *charge*, is additive in short exact sequences. This perspective can be useful to visualize basic properties, e.g., for a subsheaf $E \subseteq F$, $\mu(E) \leq \mu(F)$ if and only if $\mu(F/E) \geq \mu(F)$.

[2] The reader may find it more natural to plot instead the point $(\operatorname{rk} F, \deg F)$. Our choice of writing the charge as $Z(F) = -\deg(F) + \operatorname{rk}(F)i \in \mathbb{C}$ is motivated by Bridgeland stability: $(\operatorname{Coh}(C), Z)$ defines a stability condition on $D^b(C)$.

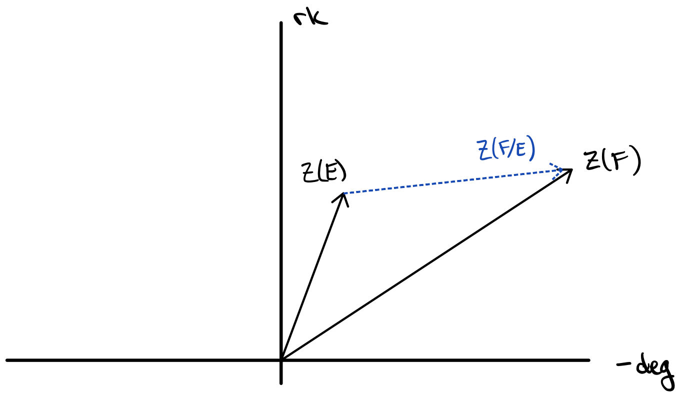


Figure 8.1.10: Visualizing the slope: for a subsheaf $E \subseteq F$, $Z(F) = Z(E) + Z(F/E)$.

Definition 8.1.11 (Semistability/stability/polystability)**.** A nonzero coherent sheaf F on a smooth, geometrically connected, and projective curve C over a field $\Bbbk$ is defined to be

(1) *semistable* if $\mu(E) \leq \mu(F)$ for every nonzero subsheaf $0 \neq E \subseteq F$,

(2) *stable* if $\mu(E) < \mu(F)$ for every nonzero proper subsheaf $0 \neq E \subsetneq F$, and

(3) *polystable* if $F \cong \bigoplus_i F_i$ where each F_i is stable with $\mu(F_i) = \mu(F)$.

If F is not semistable, a subsheaf $E \subseteq F$ with $\mu(E) > \mu(F)$ is called a *destabilizing subsheaf*. If F is semistable but not stable, we say that F is *stricltly semistable*.

Example 8.1.12 ($C = \mathbb{P}^1$)**.** The only stable vector bundles on $\mathbb{P}^1$ are line bundles. A vector bundle $E \cong \mathcal{O}(d_i)$ is semistable if and only if it is polystable if and only if all the d_i are equal.

Lemma 8.1.13.

(1) If F is a semistable vector bundle of rank r and degree d with $\gcd(r, d) = 1$, then F is stable.

(2) A semistable coherent sheaf is a vector bundle.

(3) A vector bundle F is semistable (resp., stable) if and only if $\mu(F') \leq \mu(F)$ (resp., $<$) for every nonzero proper subbundle $F' \subsetneq F$ (i.e., F' is a subsheaf such that F/F' is a vector bundle).

(4) For a line bundle L on C, a vector bundle F is semistable if and only if $F \otimes L$ is.

Proof. For (1), if $0 \neq E \subseteq F$ is a subbundle with $\mu(E) = \mu(F)$, then there exists an integer $k \geq 1$ with $\operatorname{rk} F = k \operatorname{rk} E$ and $\deg F = k \deg E$. Since $\gcd(r, d) = 1$, $k = 1$ and $F = E$. For (2), if F is not a vector bundle, then the torsion subsheaf $F_{\mathrm{tors}} \subseteq F$ is a destabilizing subsheaf. For (3), if $E \subseteq F$ is a destabilizing subsheaf with quotient $Q = F/E$, then $E' := \ker(F \to Q \to Q/Q_{\mathrm{tors}})$ is a subbundle of F containing E such that $\operatorname{rk}(E') = \operatorname{rk}(E)$ and $\deg(E') \geq \deg(E)$. Since $\mu(E') \geq \mu(E)$, we see that $E' \subseteq F$ is a destabilizing subbundle. For (4), observe that $\deg(F \otimes L) = \deg(F) + \operatorname{rk}(F) \cdot \deg(L)$ and $\mu(F \otimes L) = \mu(F) + \deg(L)$. Hence, for any subsheaf $E \subseteq F$, we have that

$$\mu(F) - \mu(E) = \mu(F \otimes L) - \mu(E \otimes L),$$

which implies that F is semistable if and only if $F \otimes L$ is. $\square$

Exercise 8.1.14 (easy)**.**

(a) Show that a coherent sheaf F is semistable if and only if $\mu(Q) \geq \mu(F)$ for every quotient coherent sheaf $F \twoheadrightarrow Q$.

(b) Show that a vector bundle F is semistable if and only if $F^\vee$ is.

Lemma 8.1.15 (easy)**.**

(1) If E and F are semistable vector bundles with $\mu(E) > \mu(F)$, then $\mathrm{Hom}_{\mathcal{O}_C}(E, F) = 0$.

(2) If E and F are stable vector bundles with $\mu(E) = \mu(F)$, then a nonzero homomorphism $E \to F$ is an isomorphism.

Proof. For (1), suppose that $E \to F$ is a nonzero map, and let K and $I = E/K$ denote its kernel and image respectively. Since E is semistable, $\mu(K) \leq \mu(E)$. Hence,

$$\begin{aligned} \deg(I) = \mu(I)\,\mathrm{rk}(I) &= \mu(E)\,\mathrm{rk}(E) - \mu(K)\,\mathrm{rk}(K) \\ &\geq \mu(E)\,\mathrm{rk}(E) - \mu(E)\,\mathrm{rk}(K) \\ &= \mu(E)\,\mathrm{rk}(I) \end{aligned}$$

so $\mu(I) \geq \mu(E) > \mu(F)$. This contradicts the semistability of F unless $I = 0$. For (2), if $E \to F$ is a nonzero map of stables sheaves of slope $\alpha = \mu(E) = \mu(F)$, the kernel K also has slope α. Since E is stable and $K \subsetneq E$, $K = 0$. Similarly the image I has slope α and since F is stable, $I = F$. Thus $E \to F$ is an isomorphism. □

Corollary 8.1.16. *Assume $\Bbbk$ is algebraically closed. If F is a stable vector bundle, then $\underline{\mathrm{Aut}}(F) \cong \mathbb{G}_m$. More generally, if $F = \bigoplus F_i^{\oplus r_i}$ is a polystable vector bundle with each F_i stable of slope $\mu(F)$ such that $F_i \ncong F_j$ for $i \neq j$, then $\underline{\mathrm{Aut}}(F) \cong \prod_i \mathrm{GL}_{r_i}$.*

Proof. If a stable bundle F, $\mathrm{Aut}(F)$ is a division algebra by part (2). Because $\Bbbk$ is algebraically closed, $\mathrm{Aut}(F) \cong \Bbbk^*$. The isomorphism $\underline{\mathrm{Aut}}(F) \cong \mathbb{G}_m$ of group schemes follows from a simple formal argument (see Proposition 8.2.17(2)). For the final statement, since $\mathrm{Hom}_{\mathcal{O}_C}(F_i, F_j) = 0$ for $i \neq j$, we have that $\mathrm{Aut}(F) \cong \prod_i \mathrm{Aut}(F_i^{\oplus r_i}) = \prod_i \mathrm{GL}_{r_i}$. □

Exercise 8.1.17 (good practice, easy)**.** Show that category of semistable vector bundles of fixed slope is an abelian category closed under extension.

A polystable vector bundle has a reductive automorphism group (Corollary 8.1.16), but this does not hold for every semistable vector bundle.

Exercise 8.1.18 (moderate)**.** Show that a strictly semistable vector bundle may have a non-reductive automorphism group.

Remark 8.1.19 (Narasimhan–Seshadri's Theorem)**.** Let C be a smooth, connected, and projective curve over $\mathbb{C}$, viewed as a Riemann surface. Let $\widetilde{C} \to C$ be the universal cover. If $\pi\colon \pi_1(C) \to \mathrm{GL}_n(\mathbb{C})$ is a representation, then $\pi_1(C)$ acts diagonally on $\widetilde{C} \times \mathbb{C}^n$ and the quotient $(\widetilde{C} \times \mathbb{C}^n)/\pi_1(C)$ is the total space of a vector bundle on C. Narasimhan and Seshadri proved that this correspondence induces a bijection between *irreducible unitary representations of* $\pi_1(C)$ and *stable vector bundles on* C [NS65].

Remark 8.1.20 (Semistability under tensor products)**.** Narasimhan and Seshadri applied their correspondence between stability and irreducible unitary representations to prove that over $\mathbb{C}$ the tensor product of two semistable vector bundles is semistable. This property fails in positive characteristic.

8.1.3 Harder–Narasimhan filtrations

Every vector bundle is built from semistable vector bundles in a *unique* way: this is called the *Harder–Narasimhan filtration*, sometimes abbreviated as the *HN filtration*.

Theorem 8.1.21 (Harder–Narasimhan Filtration)**.** *A vector bundle on a smooth, geometrically connected, and projective curve C over a field $\Bbbk$ admits a unique filtration*

$$F_\bullet : 0 = F_0 \subseteq F_1 \subseteq F_2 \subseteq \cdots \subseteq F_n = F$$

such that each factor F_i/F_{i-1} is semistable for each i and

$$\mu(F_1/F_0) > \mu(F_2/F_1) > \cdots > \mu(F_n/F_{n-1}).$$

If $\Bbbk'/\Bbbk$ is a field extension, then $0 \subseteq F_1 \otimes_\Bbbk \Bbbk' \subseteq \cdots \subseteq F_n \otimes_\Bbbk \Bbbk'$ is the Harder–Narasimhan filtration of $F \otimes_\Bbbk \Bbbk'$.

Remark 8.1.22. The subsheaf $F_1 \subseteq F$ is called the *maximal destabilizing subsheaf*: it is semistable and characterized uniquely by the property that for every subsheaf $E \subseteq F$, $\mu(F_1) \geq \mu(E)$ with equality if and only if $E \subseteq F_1$.

Remark 8.1.23. If $C = \mathbb{P}^1$ and $F = \bigoplus_{i=1}^n \mathcal{O}(d_i)^{\oplus r_i}$ with $d_1 > \cdots > d_n$, then the Harder–Narasimhan filtration is given by $F_j = \bigoplus_{i \leq j} \mathcal{O}(d_i)^{\oplus r_i}$ such that the jth factor $F_j/F_{j-1} \cong \mathcal{O}(d_j)^{\oplus r_j}$ has slope d_j.

Remark 8.1.24. Using the correspondence between maps $[\mathbb{A}^1/\mathbb{G}_m] \to \mathcal{B}un_{r,d}$ and filtrations of vector bundles (Proposition 6.9.1), the Harder–Narasimhan filtration of a vector bundle F corresponds to a unique map $[\mathbb{A}^1/\mathbb{G}_m] \to \mathcal{B}un_{r,d}$ such that 1 maps to F.

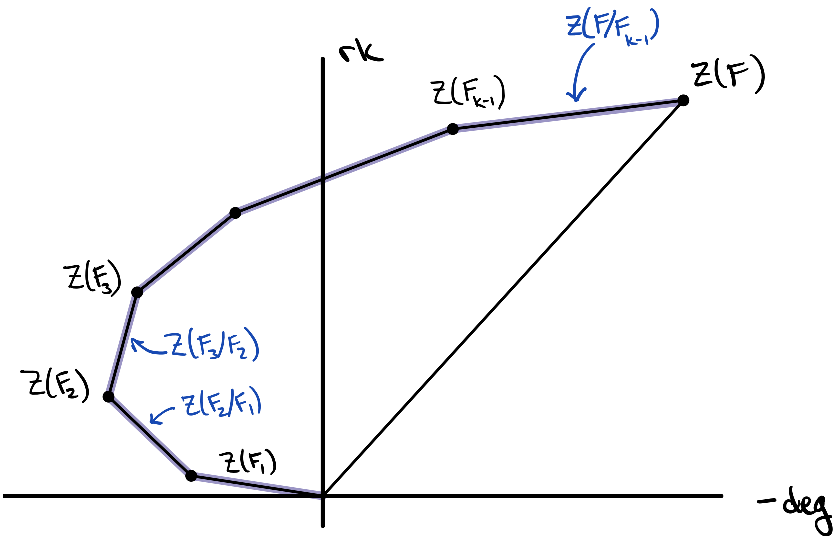


Figure 8.1.25: Using $Z(F) = -\deg(F) + \mathrm{rk}(F)i$ to visualize the slope, the Harder–Narasimhan filtration is determined by the Harder–Narasimhan polygon.

Proof. As in Figure 8.1.25, consider the *Harder–Narasimhan polygon of F*

$$HNP(F) := \text{convex hull of } \{Z(E) \mid E \subseteq F \text{ with } \mu(E) \geq \mu(F)\}.$$

By Lemma 8.1.27, there exists an integer D such that $\deg(E) \leq D$ for all $E \subseteq F$, and thus $HNP(F)$ is bounded from below, i.e., appears to the right of the vertical line $x = -D$. It is also bounded above by the horizontal line $y = \mathrm{rk}(E)$. Since $Z(E) \in \mathbb{Z} + i\mathbb{Z}$, there are only finitely many possibilities for $Z(E)$. Thus, there are finitely many extremal points $z_i \in HNP(F)$, each that can be written as $z_i = Z(F_i)$ for a subsheaf $F_i \subseteq F$. We may order the F_i so that $\mu(F_1) > \cdots > \mu(F_k) = \mu(F)$, and we set $F_0 = 0$. We claim that

(a) $F_{i-1} \subsetneq F_i$,

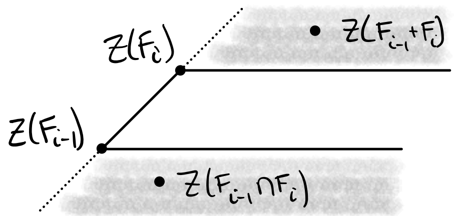
(b) if $F_i \subseteq F$ is the unique subsheaf with $z_i = Z(F_i)$, and

(c) F_i/F_{i-1} is semistable with decreasing slopes $\mu(F_1/F_0) > \cdots > \mu(F_k/F_{k-1})$.

To see (a), consider the subsheaves $F_{i-1} \cap F_i$ and $F_{i-1} + F_i$ of F. The short exact sequence

$$0 \to F_{i-1} \cap F_i \to F_{i-1} \oplus F_i \to F_{i-1} + F_i \to 0,$$

implies that $Z(F_{i-1} \cap F_i) + Z(F_{i-1} + F_i) = Z(F_{i-1}) + Z(F_i)$.



Since F_{i-1} and F_i are extremal points, $Z(F_{i-1} \cap F_i)$ lies to the right and below $Z(F_{i-1})$, while $Z(F_{i-1} + F_i)$ lies to the right and above $Z(F_i)$. Since F_{i-1} and F_i are adjacent extremal points, this can only happen if $Z(F_{i-1} \cap F_i) = Z(F_{i-1})$ and $Z(F_{i-1} + F_i) = Z(F_i)$. This implies that $F_{i-1} \cap F_i = F_{i-1}$, i.e., $F_{i-1} \subseteq F_i$. The same argument implies (b): if $F_i' \subseteq F$ is a subsheaf with $Z(F_i') = Z(F_i)$, then $F_i' = F_i$. For (c), the semistability of F_i/F_{i-1} follows from the convexity of $HNP(F)$: if $E \subseteq F_i/F_{i-1}$ is a destabilizing subsheaf, then its preimage $\widetilde{E} \subseteq F_i$ would appear left to the line segment connecting $Z(F_{i-1})$ and $Z(F_i)$. Because $\mu(F_{i-1}) > \mu(F_i)$, $\mu(F_i/F_{i-1}) > \mu(F_{i+1}/F_i)$.

The uniqueness follows formally from the uniqueness of the extremal points of the convex hull, and this implies that filtration is stable under field extensions. See also [HN75, Lem. 1.3.7], [LP97, Prop. 5.4.2], [HL10, Thm. 1.3.4], and [Bay11, §2]. $\square$

Corollary 8.1.26 (of the proof)**.** *If $E \subseteq F$ is a subsheaf with $\mu(E) \geq \mu(F)$, then $Z(E) \in HNP(F)$.* $\square$

Lemma 8.1.27. *If F is a vector bundle on C, there exists an integer D such that $\deg E \leq D$ for all subsheaves $E \subseteq F$.*

Proof. If $C = \mathbb{P}^1$, this follows from the Birkhoff–Grothendieck Theorem (8.1.5) classifying vector bundles. Otherwise, we choose a finite morphism $f\colon C \to \mathbb{P}^1$. Since f is flat, f_*F is a vector bundle. If $E \subseteq F$ is a subsheaf, then $f_*E \subseteq f_*F$ is a subsheaf. By Riemann–Roch (8.1.2), $\chi(E) = \deg(E) + \mathrm{rk}(E)(1-g)$ and it thus suffices to show that $\chi(E)$ is bounded above, but this follows from the case of $\mathbb{P}^1$ as $\chi(C, E) = \chi(\mathbb{P}^1, f_*E) = \deg(f_*E) + \mathrm{rk}(f_*E)$. $\square$

Exercise 8.1.28 (easy, needed for later)**.** Establish the following relative version of the above lemma: if S is a noetherian $\Bbbk$-scheme and F is a family of vector bundles on $C \times S$ of rank r and degree d, show that there exists an integer D such that for every $s \in S$ and every subsheaf $E \subseteq F_s$, $\deg E \leq D$.

Since the Harder–Narasimhan filtration is unique and stable under field extensions, we obtain:

Corollary 8.1.29. *For a field extension $\Bbbk'/\Bbbk$, a vector bundle F on C is semistable if and only if $F \otimes_{\Bbbk} \Bbbk'$ is semistable on $C_{\Bbbk'}$.* □

Exercise 8.1.30 (moderate)**.** Show that the analogous fact is not true for stability.

Exercise 8.1.31. For a vector bundle E on C, denote by $\mu_{\max}(E)$ and $\mu_{\min}(E)$ the maximum and minimum slope of the Harder–Narasimhan factors of E. Extending Lemma 8.1.15(1), show that if E and F be vector bundles on C with $\mu_{\min}(E) > \mu_{\max}(F)$, then $\mathrm{Hom}_{\mathcal{O}_C}(E, F) = 0$.

8.1.4 Jordan–Hölder filtrations and S-equivalence

Every semistable vector bundle F on C admits a *Jordan–Hölder filtration* $F_\bullet$ where the factors $\mathrm{gr}_i := F_i/F_{i-1}$ are *stable* vector bundles with the same slope as F. While this filtration is not unique, the factors are unique up to permutation. By combining this with the Harder–Narasimhan filtration, we can filter every vector bundle by stable vector bundles.

Proposition 8.1.32 (Jordan–Hölder Filtration)**.** *Let F be a semistable vector bundle on C. Suppose that*

$$0 = F_0 \subseteq F_1 \subseteq \cdots \subseteq F_k = F \quad \textit{and} \quad 0 = F'_0 \subseteq F'_1 \subseteq \cdots \subseteq F'_\ell = F$$

are filtrations such that the factors $\mathrm{gr}_i = F_i/F_{i-1}$ and $\mathrm{gr}'_i = F'_i/F'_{i-1}$ are stable vector bundles with $\mu(F) = \mu(\mathrm{gr}_i) = \mu(\mathrm{gr}'_i)$. Then $\ell = k$ and there exists a permutation $\sigma \in \mathrm{Sym}_k$ such that $\mathrm{gr}_i \cong \mathrm{gr}'_{\sigma(i)}$.

Proof. Semistable vector bundles of a given slope form an abelian category (Exercise 8.1.17), whose simple objects are the stable vector bundles. This proposition holds for any artinian and noetherian object in an abelian category [SP, Tag 0FCK]. The proof is an elementary induction argument analogous to the existence of compositions series for finite groups. □

As a result, the following definitions are well-defined.

Definition 8.1.33 (Associated graded)**.** The *associated graded* of a semistable bundle F is

$$\mathrm{gr}\, F \cong \bigoplus_i \mathrm{gr}_i,$$

where $\mathrm{gr}_i = F_i/F_{i-1}$ are the factors with respect to any Jordan–Hölder filtration $0 = F_0 \subseteq F_1 \subseteq \cdots \subseteq F_k$.

Definition 8.1.34 (S-equivalence)**.** Two semistable vector bundles E and F' of the same rank and degree are called *S-equivalent*[3] if $\mathrm{gr}\, E \cong \mathrm{gr}\, F$.

[3]The 'S' in 'S-equivalence' refers to Seshadri.

By the Jordan–Hölder Filtration (8.1.32), every semistable vector bundle is S-equivalent to its associated graded. This filtration corresponds via Proposition 6.9.1 to a map $[\mathbb{A}^1/\mathbb{G}_m] \to \mathcal{B}un^{\mathrm{ss}}_{r,d}$ taking 1 to $[F]$ and 0 to the associated graded $\operatorname{gr} F = \bigoplus_i F_i/F_{i-1}$, which is polystable. While the polystable limit $\operatorname{gr} F$ is unique, the map $[\mathbb{A}^1/\mathbb{G}_m] \to \mathcal{B}un^{\mathrm{ss}}_{r,d}$ is not canonical.

For the following statement and proof, we will appeal to the fact (proven later but independently) that $\mathcal{M}^{\mathrm{ss}}_{r,d}$ is an algebraic stack of finite type (Theorem 8.2.8).

Proposition 8.1.35. *Let $\Bbbk$ be an algebraically closed field of characteristic 0.*

(1) A vector bundle F on C of rank r and degree d is polystable if and only if $[F] \in \mathcal{B}un^{\mathrm{ss}}_{r,d}$ is a closed point.

(2) Two semistable vector bundles F_1, F_2 on C of rank r and degree d are S-equivalent if and only if $\overline{\{[F_1]\}} \cap \overline{\{[F_2]\}} \neq \emptyset$ in $|\mathcal{B}un^{\mathrm{ss}}_{r,d}|$.

Proof. For (1), if F is not polystable, then the Jordan–Hölder filtration $F_\bullet$ defines a map $[\mathbb{A}^1/\mathbb{G}_m] \to \mathcal{B}un^{\mathrm{ss}}_{r,d}$ taking 1 to F and 0 to $\operatorname{gr} F$, which shows that $[F]$ is not a closed point. On the other hand, if F is polystable and $[F]$ is not a closed point, there exists a closed point $[G] \in \mathcal{M}^{\mathrm{ss}}_{r,d}$ in the closure of $[F]$ (by the quasi-compactness of $\mathcal{M}^{\mathrm{ss}}_{r,d}$) with $G \ncong F$. By above, G is necessarily also polystable and, in particular, $\underline{\mathrm{Aut}}(G)$ is reductive by Corollary 8.1.16, hence linearly reductive (because $\mathrm{char}(\Bbbk) = 0$!). By the Destabilization Theorem (7.3.8), there exists a morphism $[\mathbb{A}^1/\mathbb{G}_m] \to \mathcal{B}un_{r,d}$ taking 1 to F and 0 to G. By Proposition 6.9.6, this corresponds to filtration $0 = F_0 \subseteq \cdots \subseteq F_n = F$ with associated graded G. Since $F = \bigoplus_{i \in I} S_i$ is direct sum of stable vector bundles of the same slope of F, each F_i must be a direct sum of a subset of the S_i. This implies that the associated graded bundle G is isomorphic to F, a contradiction.

For (2), if F_1 and F_2 are S-equivalent, then there exist Jordan–Hölder filtrations $F_{1,\bullet}$ and $F_{2,\bullet}$ with isomorphic associated graded bundles. These correspond to maps $f_1, f_2 \colon [\mathbb{A}^1/\mathbb{G}_m] \to \mathcal{B}un^{\mathrm{ss}}_{r,d}$ with $f_i(1) \simeq F_i$ and $f_1(0) \simeq f_2(0)$. Hence, the closures of $[F_1]$ and $[F_2]$ in $\mathcal{B}un^{\mathrm{ss}}_{r,d}$ have nonempty intersection. Conversely, if $\overline{\{[F_1]\}} \cap \overline{\{[F_2]\}}$ is nonempty, then it contains a closed point $[G] \in |\mathcal{B}un^{\mathrm{ss}}_{r,d}|$ in the closure of both $[F_1]$ and $[F_2]$. By the Destabilization Theorem (7.3.8), there exists maps $f_1, f_2 \colon [\mathbb{A}^1/\mathbb{G}_m] \to \mathcal{B}un^{\mathrm{ss}}$ with $f_i(1) \simeq F_i$ and $f_1(0) \simeq f_2(0) \simeq G$. Since G is a closed point, G is polystable and the filtrations corresponding to f_1 and f_2 are Jordan–Hölder filtrations with isomorphic associated graded bundles. Hence, F_1 and F_2 are S-equivalent. $\square$

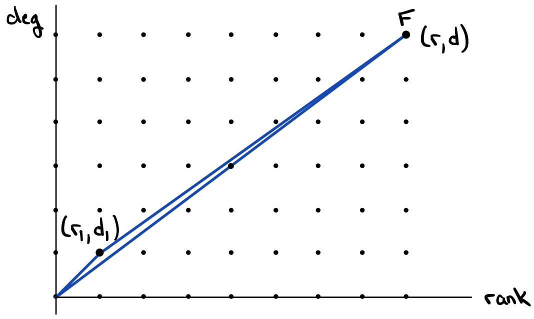
Once we have established the existence of a good moduli space $\pi \colon \mathcal{B}un^{\mathrm{ss}}_{r,d} \to M^{\mathrm{ps}}_{r,d}$ (Theorem 8.3.10), part (2) above implies that two vector bundles F_1 and F_2 are S-equivalent if and only if $\pi([F_1]) = \pi([F_2])$.

8.1.5 Existence of semistable bundles

Theorem 8.1.36 (Existence of Semistable Bundles)**.** *Let C be a smooth, connected, and projective curve over an algebraically closed field $\Bbbk$ of genus $g \geq 2$. Then for every degree d and rank $r > 0$, there exists a semistable vector bundle on C of rank r and degree d.*

Remark 8.1.37. When $\gcd(r, d) = 1$, this theorem implies the existence of a stable vector bundle. In a moment, we will show that there exists a *stable* vector bundle of every rank and degree (Theorem 8.2.14).

Proof. We will use induction on the rank r. When $r = 1$, the statement is clear as there exist line bundles of every degree and every line bundle is stable. If $r > 1$, choose (r_1, d_1) such that $d_1/r_1 > d/r$ and the triangle with vertices $(0,0)$, (r_1, d_1), (r, d) has no integral points in the interior nor integral points on the boundary line segments $(0,0) \leftrightarrow (r_1, d_1)$ and $(r_1, d_1) \leftrightarrow (r, d)$. This can be arranged by choosing the closest integral point to the left of the line segment $(0,0) \leftrightarrow (r, d)$.



Set $r_2 = r - r_1$ and $d_2 = d - d_2$. By induction, we may choose semistable vector bundles F_1 and F_2 with $\operatorname{rk} F_i = r_i$ and $\deg F_i = d_i$. Since $\gcd(r_1, d_1) = \gcd(r_2, d_2) = 1$, both F_1 and F_2 are stable. By Riemann–Roch (Exercise 8.1.4),

$$\dim \operatorname{Ext}^1_{\mathcal{O}_C}(F_1, F_2) \geq r_1 r_2 (g-1) + (d_1 r_2 - d_2 r_1) > 0$$

is strictly positive as $g > 1$ and $d_1/r_1 > d/r > d_2/r_2$. Choose a non-trivial extension

$$0 \to F_2 \to F \to F_1 \to 0. \tag{8.1.39}$$

We will show that any destabilizing subsheaf of F is necessarily isomorphic to F_1 and splits the sequence. Namely, if $0 \neq E \subseteq F$ is a subsheaf with $\mu(E) > \mu(F) = d/r$. Let $E_2 := E \cap F_2 \subseteq F_2$ and $E_1 = E/E_2 \subseteq F_1$. As F_1 and F_2 are stable, $\mu(E_i) \leq d_i/r_i$ as long as $E_i \neq 0$. It follows that $(\operatorname{rk} E, \deg E)$ is contained in the triangle with vertices $(0,0)$, (r_1, d_1), (r, d). Since the line segments $(0,0) \leftrightarrow (r_1, d_1) \leftrightarrow (r, d)$ have no integral points, we must have that $\operatorname{rk} E = r_1$ and $\deg E = d_1$. This implies that $E_2 = 0$, $E_1 = F_1$, and $E \xrightarrow{\sim} E_1$ is an isomorphism. Thus $F_1 = E_1 \xrightarrow{\sim} E \hookrightarrow F$ splits the exact sequence (8.1.39), contradicting that F is a non-trivial extension. See also [Ari15]. □

Exercise 8.1.40.

(a) If $\Bbbk$ not algebraically closed, show that the above argument is valid as long as $C(\Bbbk) \neq \emptyset$.

(b) Show that semistable bundles exist if $\Bbbk$ is a finite field.

Example 8.1.41 (Examples from geometry)**.** Let $C \subset \mathbb{P}^{g-1}$ be a general canonical curve of genus $g \geq 3$. The restricted tangent bundle $T_{\mathbb{P}^{g-1}}|_C$ is always stable, and if $g \neq 4, 6$ (resp., $g \equiv 1$ or $3 \pmod 6$), then the normal bundle $N_{C/\mathbb{P}^{g-1}}$ is semistable (resp., stable); see [CLV23].

8.1.6 Case study: elliptic curves

Let (C, p) be an elliptic curve over an algebraically closed field $\Bbbk$. Then $\omega_C \cong \mathcal{O}_C$, and for a vector bundle F, Riemann–Roch (8.1.2) asserts that $\chi(C, F) = \deg F$, while Serre–Duality states that $\mathrm{H}^1(C, F) = \mathrm{H}^0(C, F^\vee)^\vee$.

Exercise 8.1.42 (Line bundles)**.**

(a) For a line bundle L on C of degree 1, show that $\mathrm{h}^0(C, L) = \chi(C, L) = 1$ and a nonzero section vanishes at a unique point $x \in C(\Bbbk)$.

(b) Conclude the map $x \mapsto [\mathcal{O}_C(x)]$ defines an isomorphism $C \xrightarrow{\sim} \underline{\mathrm{Pic}}^1(C)$ from the Picard scheme (see Theorem 6.4.51) and extend this to an isomorphism $C \xrightarrow{\sim} \underline{\mathrm{Pic}}^d(C)$ for all d.

Atiyah's classification of indecomposable bundles. A vector bundle F is *indecomposable* if it cannot be written as $F_1 \oplus F_2$ for nonzero vector bundles F_1 and F_2. By the Krull–Schmidt Theorem (e.g., [Ati56]), every vector bundle F can be uniquely written as $F = \bigoplus_i F_i$ for indecomposable vector bundles F_i. This holds in any genus, but what is unique to elliptic curves is the relationship between indecomposability and stability (Exercise 8.1.46). Let $\mathrm{Bun}^{\mathrm{ind}}_{r,d}$ denote the set of isomorphism classes of indecomposable bundles of rank r and degree d.

Exercise 8.1.43 (moderate)**.**

(a) Show that there is an isomorphism of stacks

$$\mathcal{B}un_{r,d} \xrightarrow{\sim} \mathcal{B}un_{r,d+r}, \qquad [F] \mapsto [F \otimes \mathcal{O}_C(p)],$$

which restricts to a bijection $\mathrm{Bun}^{\mathrm{ind}}_{r,d} \to \mathrm{Bun}^{\mathrm{ind}}_{r,d+r}$.

(b) If F is a vector bundle of rank r and degree $0 \leq d < r$, show that there is bijection $\mathrm{Ext}^1_{\mathcal{O}_C}(F, \mathrm{H}^0(C, F) \otimes \mathcal{O}_C) \cong \mathrm{End}_{\mathcal{O}_C}(F)$, under which the identity map id_F corresponds to an extension

$$0 \to \mathrm{H}^0(C, F) \otimes \mathcal{O}_C \to E \to F \to 0$$

such that the boundary map $\mathrm{H}^0(C, F) \to \mathrm{H}^1(C, \mathrm{H}^0(C, F) \otimes \mathcal{O}_C)$ is an isomorphism. Show that F is indecomposable if and only if E is, and that the map $[F] \mapsto [E]$ defines a bijection $\mathrm{Bun}^{\mathrm{ind}}_{r,d} \to \mathrm{Bun}^{\mathrm{ind}}_{r+d,d}$.

(c) Letting $h = \gcd(r, d)$, show that there is a bijection $\alpha_{r,d} \colon \mathrm{Bun}^{\mathrm{ind}}_{h,0} \xrightarrow{\sim} \mathrm{Bun}^{\mathrm{ind}}_{r,d}$ such that $\det \alpha_{r,d}(F) = \det(F) \otimes \mathcal{O}_C(dp)$.

See also [Ati57, Lem. 10, Thm. 6].

Exercise 8.1.44 (hard)**.** Let $r > 0$.

(a) Show that there exists a unique indecomposable vector bundle $\mathcal{F}_r$ of rank r and degree 0 with $\mathrm{H}^0(C, \mathcal{F}_r) \neq 0$. Moreover, show that $\mathrm{H}^0(C, \mathcal{F}_r) = 1$, $\mathcal{F}_r^\vee \cong \mathcal{F}_r$, and that $\mathcal{F}_r$ is an extension $0 \to \mathcal{O}_C \to \mathcal{F}_r \to \mathcal{F}_{r-1} \to 0$.

(b) Show that every $[E] \in \mathrm{Bun}^{\mathrm{ind}}_{r,0}$ can be expressed as $E \cong \mathcal{F}_r \otimes L$ for a unique line bundle L of degree 0. Conclude that there is a bijection $C(\Bbbk) \cong \mathrm{Bun}^{\mathrm{ind}}_{r,0}$.

(c) For every d, show that there is a canonical vector bundle $[\mathcal{F}_{r,d}] \in \mathrm{Bun}^{\mathrm{ind}}_{r,d}$ of rank r and degree d such that every $[E] \in \mathrm{Bun}^{\mathrm{ind}}_{r,d}$ can be expressed as $E \cong \mathcal{F}_{r,d} \otimes L$ for a line bundle L of degree 0. If $\mathcal{F}_{r,d} \otimes L \cong \mathcal{F}_{r,d} \otimes L'$, show that $(L' \otimes L^\vee)^{r/\gcd(r,d)} \cong \mathcal{O}_C$.

(d) Conclude that $C(\Bbbk) \cong \mathrm{Bun}^{\mathrm{ind}}_{r,d}$ for every $r > 0$ and d.

See also [Ati57, Thm. 10].

Semistable vector bundles and their moduli.

Exercise 8.1.45 (Unstable bundles, easy)**.**

(a) If F_1 and F_2 are semistable vector bundles on C with $\mu(F_1) < \mu(F_2)$, show that $\mathrm{Ext}^1_{\mathcal{O}_C}(F_1, F_2) = 0$.

(b) Conclude that an unstable vector bundle is a direct sum of semistable vector bundles, not all of the same slope.

Exercise 8.1.46 (Stable vs. indecomposable bundles)**.**

(a) (easy) Prove that a stable vector bundle is indecomposable. (This holds in any genus.)

(b) (hard) Show that an indecomposable vector bundle F on C is semistable.

(c) (easy) Conclude that if $\gcd(r, d) = 1$, a vector bundle of rank r and degree d is stable if and only if it is indecomposable.

(d) (moderate) If $\gcd(r, d) \neq 1$, show that there are no stable vector bundles of rank r and degree d.

See also [Tu93, App. A].

Exercise 8.1.47 (hard)**.** For $r > 0$ and d, set $h = \gcd(r, d)$, $r' = r/h$, and $d' = d/h$.

(a) Show that every semistable vector bundle F of rank r and degree d can be written as

$$F \cong \mathcal{F}_{r',d'} \otimes \bigoplus_{i=1}^{h} L_i,$$

where $\mathcal{F}_{r',d'}$ is the vector bundle introduced in Exercise 8.1.44 and each L_i is a line bundle of degree 0. Show that the line bundles L_i are unique up to permutation and multiplication by an r'-torsion element of $\mathrm{Pic}^0(C)$.

(b) Assume the existence of a proper good moduli space $\mathcal{B}un^{\mathrm{ss}}_{r,d} \to M^{\mathrm{ss}}_{r,d}$ (Theorem 8.3.10). Using the isomorphism $\underline{\mathrm{Pic}}^0(C) \xrightarrow{\sim} C$ (see Exercise 8.1.42), show that there is an isomorphism

$$M^{\mathrm{ps}}_{r,d} \xrightarrow{\sim} \mathrm{Sym}^h C, \qquad \mathcal{F}_{r',d'} \otimes \bigoplus_{i=1}^{h} L_i \mapsto \sum_{i=1}^{h} L_i(p).$$

See also [Tu93, Thms. 1, 2, and 16].

8.2 Moduli stack of semistable vector bundles

It then becomes apparent that the successes arising from the theory of "twisted sheaves" are primarily psychological: by viewing the moduli stack as its own object (rather than something to be compressed into a scheme), one more easily retains information that can be flowed back to the land of varieties.

Max Lieblich [Lie18]

We cover properties of the moduli stack $\mathcal{B}un^{\mathrm{ss}}_{r,d}$ of semistable vector bundles. Highlights include Openness of Semistability (8.2.6), Boundedness of Semistability (8.2.8), and the non-emptiness of the stable locus $\mathcal{B}un^{s}_{r,d}$ (Theorem 8.2.14). This latter property is proved inductively using stacks of extensions (§8.2.4). We also demonstrate that $\mathcal{B}un^{\mathrm{s}}_{r,d}$ is a banded $\mathbb{G}_m$-gerbe over an algebraic space $M^{\mathrm{s}}_{r,d}$ (Corollary 8.2.23). Finally, we introduce the stack of vector bundles with fixed determinant (§8.2.7).

8.2.1 Review of the stack of vector bundles over a curve

Let C be a smooth, connected, and projective curve over an algebraically closed field $\Bbbk$ of genus $g \geq 2$. We have already established a handful of properties of the stack $\underline{\mathrm{Coh}} := \underline{\mathrm{Coh}}(C)$ parameterizing coherent sheaves on C, which we recall now. First, $\underline{\mathrm{Coh}}$ is an algebraic stack (Theorem 3.1.21), which is locally of finite type (Example 3.3.3) and has affine diagonal (Example 3.3.15). Theorem 3.1.21 also implies that the substack $\underline{\mathrm{Coh}}_{r,d} \subseteq \underline{\mathrm{Coh}}$, parameterizing coherent sheaves of rank r and degree d, is open and closed, and furthermore that the substacks $\mathcal{B}un \subseteq \underline{\mathrm{Coh}}$ and $\mathcal{B}un_{r,d} \subseteq \underline{\mathrm{Coh}}_{r,d}$ of vector bundles are open substacks. Every quasi-compact open substack of $\underline{\mathrm{Coh}}_{r,d}$ is isomorphic to a quotient stack $[Q'_N/\operatorname{GL}_{P(N)}]$ for some N, where $Q'_N \subset \operatorname{Quot}^P(\mathcal{O}_C(-N)^{\oplus P(N)}/C/\Bbbk)$ is an open subscheme of the projective Quot scheme and P is the Hilbert polynomial of any coherent sheaf of rank r and degree d.

Using the Deformation Theory (C.2.11), we showed in Proposition 3.7.9 that $\underline{\mathrm{Coh}}_{r,d}$ is smooth over $\operatorname{Spec}\Bbbk$ of dimension $r^2(g-1)$. Indeed, if $[F] \in \underline{\mathrm{Coh}}_{r,d}(\Bbbk)$, then $T_{\underline{\mathrm{Coh}}_{r,d},[F]} \cong \operatorname{Ext}^1(F,F)$, which is computed by Riemann-Roch to have dimension $r^2(g-1)$ (see Exercise 8.1.4). The stacks $\underline{\mathrm{Coh}}_{r,d}$ and $\mathcal{B}un_{r,d}$ satisfy the valuative criterion for universally closed (Exercise 3.8.13) but are not quasi-compact (Exercise 8.2.3).

The stacks $\underline{\mathrm{Coh}}_{r,d}$ and $\mathcal{B}un_{r,d}$ are not Deligne–Mumford as $\underline{\mathrm{Aut}}(F)$ always contains a copy of $\mathbb{G}_m$ (as long as $(r,d) \neq (0,0)$). It follows that $\underline{\mathrm{Coh}}_{r,d}$ and $\mathcal{B}un_{r,d}$ are not separated: if they were separated, then each automorphism group would be proper and affine, hence finite. On the other hand, we know that $\underline{\mathrm{Coh}}_{r,d}$ is both Θ- and S-complete (Proposition 6.9.30), which will be useful in establishing the existence of a good moduli space in §8.3.2.

Exercise 8.2.1. Let F be a coherent sheaf of rank r and degree d on C, and let N be an integer such that $\mathrm{H}^1(C, F(N)) = 0$ and $F(N)$ is globally generated, i.e., such that F is expressed as a quotient $[\mathcal{O}_C(-N) \otimes V \twoheadrightarrow F] \in Q'_N$, where $V \cong \mathrm{H}^0(C, F(N))$ is a vector space of dimension $P(N)$. Show that there is an exact sequence where each term has the displayed identification

$$0 \to \underbrace{\operatorname{Hom}_{\mathcal{O}_C}(F,F)}_{\operatorname{End}(F)} \to \underbrace{\operatorname{Hom}_{\mathcal{O}_C}(\mathcal{O}_C(-N)\otimes V, F)}_{\operatorname{End}(V)} \to \underbrace{\operatorname{Hom}_{\mathcal{O}_C}(K,F)}_{T_{Q'_N,[\mathcal{O}_C(-N)\otimes V \to F]}}$$
$$\to \underbrace{\operatorname{Ext}^1_{\mathcal{O}_C}(F,F)}_{T_{\underline{\mathrm{Coh}}_{r,d},[F]}} \to \underbrace{\operatorname{Ext}^1_{\mathcal{O}_C}(\mathcal{O}_C(-N)\otimes V, F)}_{0}.$$

Remark 8.2.2 (The stack $\mathcal{B}un_{r,d}(\mathbb{P}^1)$). Despite that the Birkhoff–Grothendieck Theorem (8.1.5) provides a simple classification of isomorphism classes of vector bundles on $\mathbb{P}^1$, the stack $\mathcal{B}un_{r,d}(\mathbb{P}^1)$ of vector bundles is more complicated. It is *not* true that $\mathcal{B}un_{r,d}(\mathbb{P}^1)$ is a disjoint union over the isomorphism classes of vector bundles. For instance, consider the short exact sequence $0 \to \mathcal{O}(-1) \to \mathcal{O}^{\oplus 2} \to \mathcal{O}(1) \to 0$ defined by a non-zero element of $\operatorname{Ext}^1_{\mathcal{O}_{\mathbb{P}^1}}(\mathcal{O}(1), \mathcal{O}(-1)) \cong \Bbbk$. The universal extension defines a $\mathbb{G}_m$-equivariant map $\mathbb{A}^1 \to \mathcal{B}un_{2,0}(\mathbb{P}^1)$ taking 1 to $[\mathcal{O}^{\oplus 2}]$ and 0 to $[\mathcal{O}(-1) \oplus \mathcal{O}(1)]$. Equivalently, we can view this as a map $[\mathbb{A}^1/\mathbb{G}_m] \to \mathcal{B}un_{2,0}(\mathbb{P}^1)$, whose existence also follows from the characterization of maps from $[\mathbb{A}^1/\mathbb{G}_m]$ in Proposition 6.9.6. This implies that $[\mathcal{O}(-1) \oplus \mathcal{O}(1)] \in |\mathcal{B}un_{2,0}(\mathbb{P}^1)|$ is in the closure of $[\mathcal{O}^{\oplus 2}]$. Moreover, since $\mathcal{B}un_{2,0}(\mathbb{P}^1)$ contains $\mathcal{O}(d) \oplus \mathcal{O}(-d)$ for every $d > 0$, it follows that $\mathcal{B}un_{2,0}(\mathbb{P}^1)$ is not bounded, i.e., not of finite type.

Exercise 8.2.3. Show that the algebraic stack $\mathcal{B}un_{r,d}(C)$ is not bounded.

8.2.2 Openness of semistability

We prove that stability and semistability are open properties, and conclude that the stacks $\mathcal{B}un^{\mathrm{s}}_{r,d}$ and $\mathcal{B}un^{\mathrm{ss}}_{r,d}$ are open substacks of $\mathcal{B}un_{r,d}$.

Theorem 8.2.4 (Openness of Stablity/Semistability)**.** *Let C be a smooth, connected, and projective curve over an algebraically closed field $\Bbbk$ of genus $g \geq 2$. Let S be a $\Bbbk$-scheme and F be a family of vector bundles of rank r and degree d on C_S. The locus of points $s \in S$ such that $F_s := F|_{C_s}$ is semistable is open. Likewise, the locus of points $s \in S$ such that $F_{\overline{s}} := F|_{C \times \operatorname{Spec} \overline{\kappa(s)}}$ is stable is open.*

Remark 8.2.5. The difference in the two formulations above is due to the fact that unlike semistability, stability is not always preserved under field extensions (Exercise 8.1.30).

Proof. By the Limit Methods of §B.3, we can assume that S is of finite type over $\Bbbk$. We may also assume that S is connected. We will express the locus of points $s \in S$ such that F_s is *not* semistable as a finite union of images of proper morphisms. Let $P(t) := rt + (d + r(1-g))$ be the Hilbert polynomial of a rank r and degree d vector bundle. By Exercise 8.1.28, there exists an integer D such that every subsheaf $E \subseteq F_s$ of a fiber has $\deg E \leq D$. By definition, F_s is not semistable if and only if there exists a subsheaf $E \subseteq F_s$ with $\mu(F) > d/r$. Since $\operatorname{rk} E \leq r$ and $\deg E \leq D$, there are only finitely many possibilities of $(\operatorname{rk} E, \deg E)$ for a destabilizing subsheaf E of F_s. For each possible rank and degree (r', d'), let $P'(t) = r't + (d' + r'(1-g))$ be the Hilbert polynomial of a rank r' and degree d' vector bundle. Let $Q(t) = P(t) - P'(t)$ be the Hilbert polynomial of a quotient F_s/E. By Projectivity of the Quot scheme (1.1.3), there is a scheme $\operatorname{Quot}^Q(F/C_S/S)$, parameterizing quotients of F with Hilbert polynomial Q, which is projective over S. The image of $\operatorname{Quot}^Q(F/C_S/S) \to S$ is closed and identified with the locus of points $s \in S$ such that F_s has a destabilizing subsheaf of rank r' and degree d'. Therefore, the non-semistable locus of S is closed.

Similarly, if $F_{\overline{s}}$ is semistable but not stable, then letting $h = \gcd(r, d)$, there exists a subsheaf $0 \neq E \subsetneq F_{\overline{s}}$ of rank kr/h and degree kd/h for some integer $0 < k < \gcd(r, d)$. The above argument extends to show that the locus of points $s \in S$ such that $F_{\overline{s}}$ is *not* stable is closed. $\square$

Corollary 8.2.6. *The stacks $\mathcal{B}un^{\mathrm{s}}_{r,d}$ and $\mathcal{B}un^{\mathrm{ss}}_{r,d}$ are open substacks of $\mathcal{B}un_{r,d}$ and, in particular, algebraic stacks locally of finite type over $\Bbbk$.* $\square$

8.2.3 Boundedness of semistability

The crucial input to the boundedness of semistable vector bundles on a curve is the following lemma.

Lemma 8.2.7. *If E is a semistable vector bundle on C and $\mu(E) > 2g - 1$, then E is globally generated and $\mathrm{H}^1(C, E) = 0$.*

Proof. By Serre Duality (5.1.3), $\mathrm{H}^1(C, E) \cong \operatorname{Hom}_C(E, \Omega_C)^\vee$. The line bundle Ω_C is semistable of slope $2g - 2$. Since E is also semistable and $\mu(E) > 2g - 2$, $\operatorname{Hom}_C(E, \Omega_C) = 0$ by Lemma 8.1.15(1). Thus $\mathrm{H}^1(C, E) = 0$. The same argument shows that $\mathrm{H}^1(C, E \otimes \mathcal{O}_C(-p)) = 0$ for any point $p \in C(\Bbbk)$ because $E \otimes \mathcal{O}_C(-p)$ is a vector bundle which is semistable (Lemma 8.1.13(4)) of slope $\mu(E) - 1 > 2g - 2$. Applying this vanishing to the short exact sequence $0 \to E \otimes \mathcal{O}_C(-p) \to E \to E \otimes \kappa(p) \to 0$, we obtain that $\mathrm{H}^0(C, E) \to E \otimes \kappa(p)$ is surjective for every $p \in C$. $\square$

Theorem 8.2.8 (Boundedness of Semistability)**.** *Let C be a smooth, connected, and projective curve over an algebraically closed field $\Bbbk$ of genus $g \geq 2$. For every pair of integers $r > 0$ and d, the algebraic stack $\mathcal{B}un^{\mathrm{ss}}_{r,d}$ is of finite type over $\Bbbk$.*

Proof. Choose a point $p \in C(\Bbbk)$ and an integer N such that $d/r + N > 2g-1$. For every semistable vector bundle E of rank r and degree d, the vector bundle $E \otimes \mathcal{O}_C(Np)$ is semistable by (Lemma 8.1.13(4)) and has slope $\mu(E \otimes \mathcal{O}_C(Np)) = \mu(E) + N > 2g-1$. By Lemma 8.2.7, $E \otimes \mathcal{O}_C(Np)$ is globally generated and $\mathrm{H}^1(C, E \otimes \mathcal{O}_C(Np)) = 0$. Thus, $E \otimes \mathcal{O}_C(Np)$ can be expressed as a quotient of $V \otimes \mathcal{O}_C$ where V is a vector space of fixed dimension $\mathrm{h}^0(C, E \otimes \mathcal{O}_C(Np)) = \chi(C, E \otimes \mathcal{O}_C(Np))$ (which is independent of E), and thus E can be expressed as a quotient

$$V \otimes \mathcal{O}_C(-Np) \to E.$$

Letting $P(t) = rt + (d + r(1-g))$, the Quot scheme $\mathrm{Quot}^P(V \otimes \mathcal{O}_C(-Np)/C)$ is of finite type over $\Bbbk$ by Theorem 1.1.3. By Openness of Semistability (8.2.4), there is an open subscheme $Q' \subseteq \mathrm{Quot}^P(V \otimes \mathcal{O}_C(-Np)/C)$ parameterizing semistable quotients. The universal quotient on Q' defines a surjective morphism $Q' \twoheadrightarrow \mathcal{B}un^{\mathrm{ss}}_{r,d}$ (and in fact expresses $\mathcal{B}un^{\mathrm{ss}}_{r,d}$ as a quotient stack $[Q'/\mathrm{GL}_{P(N)}]$). Since Q' is quasi-compact, so is $\mathcal{B}un^{\mathrm{ss}}_{r,d}$. $\square$

8.2.4 Stacks of extensions

We discuss the algebraicity of moduli functors and stacks parameterizing extensions of two given vector bundles. As a consequence, we will be able construct a stack of extensions over $\mathcal{B}un_{r_1,d_1} \times \mathcal{B}un_{r_2,d_2}$ which admits a morphism to $\mathcal{B}un_{r_1+r_2,d_1+d_2}$. This can be viewed as a analogous construction to the Gluing Morphisms (5.6.16) $\overline{\mathcal{M}}_{g_1,n_1} \times \overline{\mathcal{M}}_{g_2,n_2} \to \overline{\mathcal{M}}_{g_1+g_2,n_1+n_2-2}$. As in the case of stable curves, this provides an inductive strategy to establish properties of $\mathcal{B}un_{r,d}$. We will use this strategy in the next subsection to show the nonemptiness of the stable locus $\mathcal{B}un^{\mathrm{s}}_{r,d}$.

Extensions. If F_1 and F_2 are vector bundles on a scheme X, let $\mathrm{EXT}(F_1, F_2)$ denote the groupoid of extensions $0 \to F_2 \xrightarrow{a} E \xrightarrow{b} F_1 \to 0$, where a morphism of extensions is an isomorphism $\alpha\colon E \to E$ fitting in a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & F_2 \arrow[r, "a"] \arrow[d, "\mathrm{id}"] & E \arrow[r, "b"] \arrow[d, "\alpha"] & F_1 \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\
0 \arrow[r] & F_2 \arrow[r, "a"] & E' \arrow[r, "b"] & F_1 \arrow[r] & 0.
\end{tikzcd}$$

Exercise 8.2.9 (easy)**.** Show that $\mathrm{Aut}_{\mathrm{EXT}(F_1,F_2)}([E]) \cong \mathrm{Hom}_{\mathcal{O}_X}(F_1, F_2)$, where $\gamma \in \mathrm{Hom}_{\mathcal{O}_X}(F_1, F_2)$ corresponds to the automorphism $\mathrm{id} + a \circ \gamma \circ b$ of the extension E.

Functor of extensions. Before discussing the *stack* of extensions, we first describe properties of the *functor* of extensions. Recall that the group $\mathrm{Ext}^1_{\mathcal{O}_X}(F_1, F_2)$ classifies isomorphism classes of $\mathrm{EXT}(F_1, F_2)$. If X is defined over a field $\Bbbk$, then the functor $T \mapsto \mathrm{Ext}^1_{\mathcal{O}_{X \times T}}(p_1^* F_1, p_1^* F_2)$ from the category of *affine* $\Bbbk$-schemes, is representable by $\mathbb{A}(\mathrm{Ext}^1_{\mathcal{O}_X}(F_1, F_2)^\vee)$ (Exercise 0.3.21). The group $\underline{\mathrm{Aut}}(F_1) \times \underline{\mathrm{Aut}}(F_2)$ acts on $\mathbb{A}(\mathrm{Ext}^1_{\mathcal{O}_X}(F_1, F_2)^\vee)$ via

$$(\beta_1, \beta_2) \cdot [0 \to F_2 \xrightarrow{a} E \xrightarrow{b} F_1 \to 0] = [0 \to F_2 \xrightarrow{a \circ \beta_2^{-1}} E \xrightarrow{\beta_1 \circ b} F_1 \to 0].$$

The anti-diagonal subgroup $\mathbb{G}_m = \{(t^{-1}, t)\} \subseteq \underline{\mathrm{Aut}}(F_1) \times \underline{\mathrm{Aut}}(F_2)$ corresponds to the scaling automorphisms on E and acts trivially on $\mathbb{A}(\mathrm{Ext}^1_{\mathcal{O}_X}(F_1, F_2)^\vee)$. On the other hand, $\mathbb{G}_m = \{(t, 1)\} \subseteq \underline{\mathrm{Aut}}(F_1) \times \underline{\mathrm{Aut}}(F_2)$ acts non-trivially via the scaling action on $\mathbb{A}(\mathrm{Ext}^1_{\mathcal{O}_X}(F_1, F_2)^\vee)$. Under this latter action, the universal extension on $X \times \mathbb{A}(\mathrm{Ext}^1_{\mathcal{O}_X}(F_1, F_2)^\vee)$ is $\mathbb{G}_m$-equivariant and defines a morphism

$$[\mathbb{A}(\mathrm{Ext}^1_{\mathcal{O}_X}(F_1, F_2)^\vee)/\mathbb{G}_m] \to \mathcal{B}un(X),$$

restricting to a morphism $\mathbb{P}(\mathrm{Ext}^1_{\mathcal{O}_X}(F_1, F_2)^\vee) \to \mathcal{B}un(X)$ on the locus of non-split extensions.

In the relative setting, we have:

Exercise 8.2.10. Let $f \colon X \to S$ be a proper flat morphism of noetherian schemes, e.g., $X = C \times S \to S$ for a curve C. Let F_1 and F_2 be vector bundles on X such that $f_* \mathscr{E}xt^1_{\mathcal{O}_X}(F_1, F_2)$ is a vector bundle on S whose construction commutes with base change. Show that the functor

$$\begin{aligned} \mathrm{Sch}/S &\to \mathrm{Sets} \\ (T \to S) &\mapsto \mathrm{H}^0(X_T, \mathscr{E}xt^1_{\mathcal{O}_{X_T}}(F_{1,T}, F_{2,T})), \end{aligned}$$

is representable by $\mathbb{A}((f_* \mathscr{E}xt^1_{\mathcal{O}_X}(F_1, F_2))^\vee) \to S$. See also [Lan83].

If T is affine, observe that $\mathrm{H}^0(X_T, \mathscr{E}xt^1_{\mathcal{O}_{X_T}}(F_{1,T}, F_{2,T})) = \mathrm{Ext}^1_{\mathcal{O}_{X_T}}(F_{1,T}, F_{2,T})$. In particular, $\mathbb{A}((p_* \mathscr{E}xt^1_{\mathcal{O}_X}(F_1, F_2))^\vee) \to S$ represents the functor $(T \to S) \mapsto \mathrm{Ext}^1_{\mathcal{O}_{X_T}}(F_{1,T}, F_{2,T})$ on the category of *affine* schemes over S. For non-affine schemes T, these groups are connected via the local-to-global spectral sequence $\mathrm{H}^p(T, f_{T,*} \mathscr{E}xt^q_{\mathcal{O}_{X_T}}(F_{1,T}, F_{2,T})) \Rightarrow \mathrm{Ext}^{p+q}_{\mathcal{O}_{X_T}}(F_{1,T}, F_{2,T})$, which on low degree terms yields an exact sequence

$$\begin{aligned} 0 \to \mathrm{H}^1(T, f_{T,*} \mathscr{H}om_{\mathcal{O}_{X_T}}(F_{1,T}, F_{2,T})) \to \mathrm{Ext}^1_{\mathcal{O}_{X_T}}(F_{1,T}, F_{2,T}) \to \\ \mathrm{H}^0(T, f_{T,*} \mathscr{E}xt^1_{\mathcal{O}_{X_T}}(F_{1,T}, F_{2,T})) \to \mathrm{H}^1(T, f_{T,*} \mathscr{H}om_{\mathcal{O}_{X_T}}(F_{1,T}, F_{2,T})). \end{aligned}$$

When T is affine, the left and right terms are 0, and the two middle terms are identified.

Stack of extensions. Due to the presence of automorphisms of extensions, it is more natural to describe the moduli of extensions as a *stack*. This allows us to conveniently handle the case that $f_* \mathscr{E}xt^1_{\mathcal{O}_X}(F_1, F_2)$ is not a vector bundle, e.g., when the dimension of $\mathrm{Ext}^1_{\mathcal{O}_{X_s}}(F_{1,s}, F_{2,s})$ depends on the point $s \in S$.

Define the stack

$$\underline{\mathrm{EXT}}(F_1, F_2)$$

over $(\mathrm{Sch}/S)_{\text{ét}}$, whose category over $T \to S$ is the groupoid $\mathrm{EXT}(F_{1,T}, F_{2,T})$ of extensions of the base changes $F_{1,T}$ and $F_{2,T}$ to X_T.

Proposition 8.2.11. *If $f \colon X \to S$ is a family of curves and F_1 and F_2 are vector bundles on X, then $\underline{\mathrm{EXT}}(F_1, F_2)$ is an algebraic stack isomorphic to the vector bundle stack $\mathfrak{C}(\mathrm{R}f_* \mathscr{H}om_{\mathcal{O}_X}(F_1, F_2))$ over S defined in Example 3.9.32. In particular, $\underline{\mathrm{EXT}}(F_1, F_2) \to S$ is smooth and quasi-compact, whose relative dimension is computed as*

$$\dim \mathrm{Ext}^1_{\mathcal{O}_{X_s}}(F_{1,s}, F_{2,s}) - \dim \mathrm{Ext}^0_{\mathcal{O}_{X_s}}(F_{1,s}, F_{2,s})$$

for any point $s \in S$.

Proof. Let $K^\bullet\colon K^0 \xrightarrow{d} K^1$ be a complex of vector bundles computing $\mathrm{R}f_*\mathscr{H}om_{\mathcal{O}_X}(F_1,F_2)$. Then

$$\mathfrak{C}(\mathrm{R}f_*\mathscr{H}om_{\mathcal{O}_X}(F_1,F_2)) \cong \mathfrak{C}(K^\bullet) \cong [\mathbb{A}(K^{1,\vee})/\mathbb{A}(K^{0,\vee})],$$

where the additive group $\mathbb{A}(K^{0,\vee})$ acts on $\mathbb{A}(K^{1,\vee})$ via $v_0 \cdot v_1 = v_1 + d(v_0)$. If $g\colon T \to S$ is a morphism from an *affine* scheme, then all principal $\mathbb{A}(K^{0,\vee})$-bundles are trivial, so that

$$\begin{aligned}
\mathfrak{C}(K^\bullet)(T) &\cong [\mathbb{A}(K^{1,\vee})(T)/\mathbb{A}(K^{0,\vee})(T)] \\
&\cong [\mathrm{H}^0(T,g^*K^1)/\mathrm{H}^0(T,g^*K^0)] && \text{(see (3.9.33))} \\
&\cong \mathrm{EXT}(F_{1,T},F_{2,T}), && \text{(as } \mathrm{Ext}^i_{\mathcal{O}_{X_T}}(F_{1,T},F_{2,T}) \cong \mathrm{H}^i(g^*K^\bullet)).
\end{aligned}$$

□

Remark 8.2.12. If $\mathrm{Hom}(F_{1,s},F_{2,s}) = 0$ for every $s \in S$, then $f_*\mathscr{E}xt^1_{\mathcal{O}_X}(F_1,F_2)$ is a vector bundle on S by Cohomology and Base Change (A.6.8). In this case, we recover Exercise 8.2.10: $\underline{\mathrm{EXT}}(F_1,F_2)$ is representable over S by $\mathbb{A}(f_*\mathscr{E}xt^1_{\mathcal{O}_X}(F_1,F_2)^\vee)$. More generally, if $f_*\mathscr{E}xt^i_{\mathcal{O}_X}(F_1,F_2)$ are vector bundles for $i=0,1$, then $\underline{\mathrm{EXT}}(F_1,F_2) \to S$ is a gerbe banded by $f_*\mathscr{H}om_{\mathcal{O}_X}(F_1,F_2)$.

Remark 8.2.13 (The universal extension). For two rank-degree pairs (r_1,d_1) and (r_2,d_2), we can apply this construction to the morphism

$$C \times \mathcal{B}un_{r_1,d_1} \times \mathcal{B}un_{r_2,d_2} \to \mathcal{B}un_{r_1,d_1} \times \mathcal{B}un_{r_2,d_2}$$

and the vector bundles $p_{12}^*\mathcal{F}_{1,\mathrm{univ}}$ and $p_{13}^*\mathcal{F}_{2,\mathrm{univ}}$, where $\mathcal{F}_{i,\mathrm{univ}}$ is the universal vector bundle on $C \times \mathcal{B}un_{r_i,d_i}$. We have the stack of extensions

$$q\colon \underline{\mathrm{EXT}}(p_{12}^*\mathcal{F}_{1,\mathrm{univ}}, p_{13}^*\mathcal{F}_{2,\mathrm{univ}}) \to \mathcal{B}un_{r_1,d_1} \times \mathcal{B}un_{r_2,d_2},$$

where an object over T is that data of vector bundles F_1 and F_2 on C_T with $\operatorname{rk} F_i = r_i$ and $\deg F_i = d_i$ together with an extension $0 \to F_2 \to E \to F_1 \to 0$, while a morphism over the identity on T is the data of a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & F_2 \arrow[r, "a"] \arrow[d, "\beta_2"] & E \arrow[r, "b"] \arrow[d, "\alpha"] & F_1 \arrow[r] \arrow[d, "\beta_1"] & 0 \\
0 \arrow[r] & F_2' \arrow[r, "a'"] & E' \arrow[r, "b'"] & F_1' \arrow[r] & 0,
\end{tikzcd}$$

where α, β_1, and β_2 are isomorphisms.

There is a universal extension $\mathcal{E}_{\mathrm{univ}}$ of rank $r = r_1 + r_2$ and degree $d = d_1 + d_2$

$$0 \to (\mathrm{id} \times q)^* p_{13}^*\mathcal{F}_{2,\mathrm{univ}} \to \mathcal{E}_{\mathrm{univ}} \to (\mathrm{id} \times q)^* p_{12}^*\mathcal{F}_{1,\mathrm{univ}} \to 0$$

on $C \times \underline{\mathrm{EXT}}(p_{12}^*\mathcal{F}_{1,\mathrm{univ}}, p_{13}^*\mathcal{F}_{2,\mathrm{univ}})$. This defines a *representable* morphism $\underline{\mathrm{EXT}}(p_{12}^*\mathcal{F}_{1,\mathrm{univ}}, p_{13}^*\mathcal{F}_{2,\mathrm{univ}}) \to \mathcal{B}un_{r,d}$ fitting in a diagram

$$\begin{tikzcd}
\underline{\mathrm{EXT}}(p_{12}^*\mathcal{F}_{1,\mathrm{univ}}, p_{13}^*\mathcal{F}_{2,\mathrm{univ}}) \arrow[r, "{[\mathcal{E}_{\mathrm{univ}}]}"] \arrow[d, "q"] & \mathcal{B}un_{r,d} \\
\mathcal{B}un_{r_1,d_1} \times \mathcal{B}un_{r_2,d_2} &
\end{tikzcd}
\qquad
\begin{tikzcd}
{[0 \to F_2 \to E \to F_1 \to 0]} \arrow[r, mapsto] \arrow[d, mapsto] & {[E]} \\
{([F_1],[F_2]).} &
\end{tikzcd}$$

8.2.5 Existence of stable bundles

We prove the Existence of Stable Bundles (8.2.14), extending a prior result on the Existence of Semistable Bundles (8.1.36). Our argument is moduli-theoretic relying on the algebraicity of $\mathcal{B}un^{\mathrm{s}}_{r,d}$, $\mathcal{B}un^{\mathrm{ss}}_{r,d}$, and the stack of extensions.

Theorem 8.2.14 (Existence of Stable Bundles)**.** *Let C be a smooth, connected, and projective curve over an algebraically closed field $\Bbbk$ of genus $g \geq 2$. Then $\mathcal{B}un^{\mathrm{s}}_{r,d}$ is nonempty and dense in $\mathcal{B}un^{\mathrm{ss}}_{r,d}$.*

Proof. Theorem 8.1.36 gives the existence of a semistable vector bundle, and hence a stable vector bundle if $\gcd(r,d) = 1$. We will prove the theorem by induction on the rank by showing that $\dim \mathcal{B}un^{\mathrm{ss}}_{r,d} \smallsetminus \mathcal{B}un^{\mathrm{s}}_{r,d} < \dim \mathcal{B}un^{\mathrm{ss}}_{r,d}$. If $\gcd(r,d) > 1$, then a vector bundle F of rank r and degree d is strictly semistable if and only if if it can be written as an extension $0 \to F_2 \to F \to F_1 \to 0$ of semistable vector bundles with $\mathrm{rk}(F_i) = r_i < r$ and $\mu(F_i) = \mu(F)$. Letting $\underline{\mathrm{EXT}}(p_{12}^*\mathcal{F}_{1,\mathrm{univ}}, p_{13}^*\mathcal{F}_{2,\mathrm{univ}})$ be the stack of extensions over $\mathcal{B}un^{\mathrm{ss}}_{r_1,d_1} \times \mathcal{B}un^{\mathrm{ss}}_{r_2,d_2}$ (Proposition 8.2.11), the universal extension $\mathcal{E}_{\mathrm{univ}}$ (Remark 8.2.13) defines a representable morphism

$$\underline{\mathrm{EXT}}(p_{12}^*\mathcal{F}_{1,\mathrm{univ}}, p_{13}^*\mathcal{F}_{2,\mathrm{univ}}) \to \mathcal{B}un^{\mathrm{ss}}_{r,d},$$

whose image consists of strictly semistable vector bundles E containing a subbundle of rank r_2 and slope d/r. The theorem follows from the calculation

$$\begin{aligned}
\dim \underline{\mathrm{EXT}}(p_{12}^*\mathcal{F}_{1,\mathrm{univ}}, p_{13}^*\mathcal{F}_{2,\mathrm{univ}}) &= \dim \mathcal{B}un^{\mathrm{ss}}_{r_1,d_1} + \dim \mathcal{B}un^{\mathrm{ss}}_{r_2,d_2} \\
&\quad + \dim \mathrm{Ext}^1_{\mathcal{O}_C}(F_1,F_2) - \dim \mathrm{Ext}^0_{\mathcal{O}_C}(F_1,F_2) && (\text{for any } F_i \in \mathcal{B}un^{\mathrm{ss}}_{r_i,d_i}) \\
&= (r_1^2 + r_2^2 + r_1 r_2)(g-1) && (\text{by Riemann-Roch } (8.1.4)) \\
&< (r_1+r_2)^2(g-1) = \dim \mathcal{B}un^{\mathrm{ss}}_{r,d}.
\end{aligned}$$

□

Exercise 8.2.15. Provide an alternative argument for the Existence of Semistable Vector Bundles (8.2.14).

Hint: Follow the proof above using the stack of extensions and the observation that there are only finitely many ranks and degrees of destabilizing subbundles. Instead of using the stack of extensions, one can also argue using dimensions of Quot *schemes as in [NR69, Lemma 4.3] and [LP97, §8.6].*

8.2.6 Simple vector bundles and their rigidification

Definition 8.2.16 (Simple vector bundles)**.** A vector bundle E on C is *simple* if $\mathrm{Aut}(C) \cong \Bbbk^*$. For a $\Bbbk$-scheme S, a family of vector bundles $\mathcal{E}$ on C_S over S is *simple* if for every $s \in S$, the geometric fiber $\mathcal{E}_{\overline{s}} := \mathcal{E}|_{C \times \operatorname{Spec} \overline{\kappa(s)}}$ is simple.

Every stable vector bundle is simple by Lemma 8.1.15.

Proposition 8.2.17. *Let $\mathcal{E}$ be a family of vector bundles on C_S over a $\Bbbk$-scheme S.*

(1) There is a closed immersion $i_{\mathcal{E}} \colon \mathbb{G}_{m,S} \hookrightarrow \underline{\mathrm{Aut}}(\mathcal{E})$ of group schemes which is compatible under base change: if $S' \to S$ is a map of schemes and $\mathcal{E}'$ denotes the pullback of $\mathcal{E}$ to $C_{S'}$, then $i_{\mathcal{E}'}$ is identified with the base changes of $i_{\mathcal{E}}$ under $S' \to S$ under the natural isomorphism $\underline{\mathrm{Aut}}(\mathcal{E}) \times_S S' \cong \underline{\mathrm{Aut}}(\mathcal{E}')$.

(2) If $\mathcal{E}$ is simple, then $i_{\mathcal{E}} \colon \mathbb{G}_{m,S} \xrightarrow{\sim} \underline{\mathrm{Aut}}(\mathcal{E})$ is an isomorphism.

Proof. We define $i_{\mathcal{E}}$ functorially: for an S-scheme S', let $i_{\mathcal{E}}(S') \colon \mathbb{G}_{m,S}(S') \to \operatorname{Aut}(\mathcal{E}')$ be the map taking a unit $f \in \Gamma(S', \mathcal{O}_{S'})^*$ to the isomorphism $\mathcal{E}' \to \mathcal{E}'$ given by multiplication by the unit $p_2^* f \in \Gamma(C_{S'}, \mathcal{O}_{C_{S'}})^*$. Since $i_{\mathcal{E}}(S')$ is injective, $i_{\mathcal{E}} \colon \mathbb{G}_{m,S} \to \underline{\operatorname{Aut}}(\mathcal{E})$ is a monomorphism of group schemes. To see that $i_{\mathcal{E}}$ is a closed immersion, we first claim that the algebraic space quotient $X := \underline{\operatorname{Aut}}(\mathcal{E})/\mathbb{G}_{m,S}$ is affine over S. Indeed, since $p \colon \underline{\operatorname{Aut}}(\mathcal{E}) \to S$ is affine and $\mathbb{G}_m$ is linearly reductive, the natural map $X \to \mathcal{S}\mathrm{pec}_S(p_* \mathcal{O}_X)^{\mathbb{G}_{m,S}}$ is a good moduli space. By Theorem 6.5.6, this map is universal for maps to algebraic spaces and therefore an isomorphism. The inclusion $\mathbb{G}_{m,S} \to \underline{\operatorname{Aut}}(\mathcal{E})$ is obtained by base changing $X \to X \times X$ by $\underline{\operatorname{Aut}}(\mathcal{E}) \to X \times X$. As X is separated, $\mathbb{G}_{m,S} \to \underline{\operatorname{Aut}}(\mathcal{E})$ is a closed immersion.

For (2), we first observe that the Fibral Flatness Criterion (A.2.10) implies that $i_{\mathcal{E}}$ is flat. As $i_{\mathcal{E}}$ is a flat surjective monomorphism of finite type, it is an isomorphism (Proposition A.2.15). □

Exercise 8.2.18. For a vector bundle E on C, show that the following are equivalent:

$$E \text{ is simple} \iff \underline{\operatorname{Aut}}(E) \cong \mathbb{G}_m \iff \dim \underline{\operatorname{Aut}}(E) = 1.$$

Definition 8.2.19. The *stack of simple vector bundles of rank r and degree d* is the substack

$$\mathcal{B}un_{r,d}^{\mathrm{simple}} \subseteq \mathcal{B}un_{r,d}$$

whose objects over a $\Bbbk$-scheme S are families of simple vector bundles of rank r and degree d.

Proposition 8.2.20. *The substack $\mathcal{B}un_{r,d}^{\mathrm{simple}} \subseteq \mathcal{B}un_{r,d}$ is open and is a banded $\mathbb{G}_m$-gerbe over the algebraic space $M_{r,d}^{\mathrm{simple}} := \mathcal{B}un_{r,d}^{\mathrm{simple}} /\!\!/ \mathbb{G}_m$.*

Proof. For openness, we need to show that if $\mathcal{E}$ is a family of vector bundles over a $\Bbbk$-scheme S, then the locus

$$\{s \in S \mid \text{the geometric fiber } \mathcal{E}_{\overline{s}} \text{ is simple}\}$$

is an open subset of S. Indeed, by Exercise 8.2.18 this locus is the set of points $s \in S$ where the group scheme $\operatorname{Aut}(\mathcal{E}) \to S$ has fiber dimension 1, and this is open by the upper semicontinuity of the fiber dimension of a group scheme (Proposition B.1.8). The canonical isomorphism $i_{\mathcal{E}} \colon \mathbb{G}_{m,S} \xrightarrow{\sim} \underline{\operatorname{Aut}}(\mathcal{E})$ from Proposition 8.2.17(2) induces an isomorphism between the inertial stack of $\mathcal{B}un_{r,d}^{\mathrm{simple}}$ and $\mathbb{G}_m \times \mathcal{B}un_{r,d}^{\mathrm{simple}}$. The statement therefore follows by rigidifying the $\mathbb{G}_m$ using Proposition 6.4.18 (or using Proposition 6.4.41). □

Remark 8.2.21 (Ridigification of the stack of vector bundles)*.* We may apply Proposition 6.4.41 to construction the rigidification $\mathcal{B}un_{r,d} /\!\!/ \mathbb{G}_m$ of the stack of all vector bundles using the fppf closed subgroup scheme $\mathbb{G}_m := \mathbb{G}_{m,\mathcal{B}un_{r,d}}$ of the inertia stack $I_{\mathcal{B}un_{r,d}}$ defined by the closed immersions $i_{\mathcal{E}} \colon \mathbb{G}_{m,S} \hookrightarrow \underline{\operatorname{Aut}}(\mathcal{E})$ of Proposition 8.2.17(1). Alternatively, by viewing $\mathcal{B}un_{r,d} = \bigcup_N [Q'_N / \operatorname{GL}_{P(N)}]$ as a union of quotient stacks (Theorem 3.1.21), where $Q'_N \subseteq \operatorname{Quot}^P(\mathcal{O}_C(-N)^{\oplus P(N)}/C)$ are open subschemes, then $\mathcal{B}un_{r,d} /\!\!/ \mathbb{G}_m = \bigcup_N [Q'_N / \operatorname{PGL}_{P(N)}]$. The stack $\mathcal{B}un_{r,d} /\!\!/ \mathbb{G}_m$ contains the algebraic space $M_{r,d}^{\mathrm{simple}}$ as an open substack.

Exercise 8.2.22 (moderate)**.** For each pair of integers $r > 0$ and d, show that $M_{r,d}^{\mathrm{simple}}$ is not separated and not of finite type over $\Bbbk$.

Application to stable bundles. Since $\mathcal{B}un^{\mathrm{s}}_{r,d} \subseteq \mathcal{B}un^{\mathrm{simple}}_{r,d}$ is an open substack, we may apply Proposition 8.2.20 to conclude:

Corollary 8.2.23. *The stack $\mathcal{B}un^{\mathrm{s}}_{r,d}$ of stable bundles is a banded $\mathbb{G}_m$-gerbe over the algebraic space $M^{\mathrm{s}}_{r,d} := \mathcal{B}un^{\mathrm{s}}_{r,d} /\!\!/ \mathbb{G}_m$.* □

8.2.7 Vector bundles with fixed determinant

Definition 8.2.24. For a line bundle $\xi \in \mathrm{Pic}(C)$, define the stack $\mathcal{B}un_{r,\xi}$ over $(\mathrm{Sch}/\Bbbk)_{\text{ét}}$ whose objects over a $\Bbbk$-scheme S are pairs (E, φ) where E is a vector bundle on C_S and $\varphi\colon \det E \xrightarrow{\sim} p_1^* \xi$ is an isomorphism. A morphism $(E, \varphi) \to (E', \varphi')$ is an isomorphism $\alpha\colon E \xrightarrow{\sim} E'$ compatible with the isomorphisms φ and φ'.

The stack $\mathcal{B}un_{r,\xi}$ of vector bundles with determinant ξ fits in a cartesian diagram

$$\begin{tikzcd}
\mathcal{B}un_{r,\xi} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \operatorname{Spec} \Bbbk \arrow[d, "{[\xi]}"] \\
\mathcal{B}un_{r,d} \arrow[r, "\det"] & \underline{\mathcal{P}\mathrm{ic}}^{\,d},
\end{tikzcd} \tag{8.2.25}$$

where $\underline{\mathrm{Pic}}^d = \mathcal{B}un_{1,d}$ is the Picard stack of line bundles of degree d (see also the discussion in §6.4.7), and $\det\colon \mathcal{B}un_{r,d} \to \underline{\mathcal{P}\mathrm{ic}}^{\,d}$ is the morphism defined by $E \mapsto \det E$. There are open substacks

$$\mathcal{B}un^{\mathrm{s}}_{r,\xi} \subseteq \mathcal{B}un^{\mathrm{ss}}_{r,\xi} \subseteq \mathcal{B}un_{r,\xi}$$

consisting of stable and semistable vector bundles with determinant ξ. For a stable bundle E of rank r, there is a canonical isomorphism $\underline{\mathrm{Aut}}(E) \cong \boldsymbol{\mu}_r$. In particular, $\mathcal{B}un^{\mathrm{s}}_{r,\xi}$ is a Deligne–Mumford stack. Moreover, the same argument as Proposition 8.2.20 shows that $\mathcal{B}un^{\mathrm{s}}_{r,\xi}$ is a $\boldsymbol{\mu}_r$-gerbe over $M^{\mathrm{s}}_{r,\xi} := \mathcal{B}un^{\mathrm{s}}_{r,\xi} /\!\!/ \boldsymbol{\mu}_r$.

Proposition 8.2.26. *For every line bundle $\xi \in \mathrm{Pic}(C)$ and rank $r > 0$, there exists a stable vector bundle E on C of rank r with $\det(E) \cong \xi$.*

Proof. By the Existence of Stable Bundles (8.2.14), there exists a stable bundle E of rank r and degree $d = \deg \xi$. Then $\xi \otimes (\det E)^{-1}$ is a line bundle of degree 0. Since $\mathrm{Pic}^0(C)$ is an abelian variety, it is a divisible group, and hence there is a line bundle L of degree 0 with $L^{\otimes r} \cong \xi \otimes (\det E)^{-1}$. The vector bundle $E \otimes L$ is semistable (Lemma 8.1.13(4)) of rank r with $\det(E \otimes L) \cong (\det E) \otimes L^{\otimes r} \cong \xi$. □

Exercise 8.2.27 (hard)**.** If (C, p) is an elliptic curve and ξ is a line bundle of degree d, show that $M^{\mathrm{ps}}_{r,\xi} \cong \mathbb{P}^{\gcd(r,d)-1}$. (This extends the classification of Exercise 8.1.47 to bundles with fixed determinant. See also [Tu93, Thm. 3].)

Remark 8.2.28. Instead of using the fiber diagram (8.2.25) to define a stack of vector bundles with fixed determinant, one could use the residual gerbe instead: if $B\mathbb{G}_m \hookrightarrow \underline{\mathcal{P}\mathrm{ic}}^{\,d}$ is the inclusion of the residual gerbe of ξ, then the fiber product $\mathcal{X} := \mathcal{B}un_{r,d} \times_{\underline{\mathcal{P}\mathrm{ic}}^{\,d}} B\mathbb{G}_m$ parameterizes vector bundles E of rank r such that there *exists* an isomorphism $\det E \xrightarrow{\sim} \xi$. The automorphism of a stable vector bundle $E \in \mathcal{X}(\Bbbk)$ is $\mathbb{G}_m$ and the stable locus $\mathcal{X}^{\mathrm{s}} := \mathcal{B}un^{\mathrm{s}}_{r,d} \times_{\underline{\mathcal{P}\mathrm{ic}}^{\,d}} B\mathbb{G}_m$ is a $\mathbb{G}_m$-gerbe over

$$\mathcal{X}^{\mathrm{s}} /\!\!/ \mathbb{G}_m \cong M^{\mathrm{s}}_{r,\xi} \cong \mathcal{B}un^{\mathrm{s}}_{r,\xi} /\!\!/ \boldsymbol{\mu}_r.$$

8.3 Semistable reduction, moduli spaces, and stratifications

Algebra is the offer made by the devil to the mathematician. The devil says: "I will give you this powerful machine; it will answer any question you like. All you need to do is give me your soul: give up geometry and you will have this marvelous machine."

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We prove further properties of the moduli stack of vector bundles over a curve: the semistable locus $\mathcal{B}un_{r,d}^{ss}$ is universally closed (Theorem 8.3.1) and admits a proper moduli space (Theorem 8.3.10), while the unstable locus $\mathcal{B}un_{r,d} \setminus \mathcal{B}un_{r,d}^{ss}$ admits a stratification indexed by the Harder–Narasimhan polygon (Theorem 8.3.13).

8.3.1 Semistable reduction: Langton's Theorem

Theorem 8.3.1 (Langton's Theorem). *Let C be a smooth, connected, and projective curve of genus $g \geq 2$ over an algebraically closed field $\Bbbk$. Let R be a DVR over $\Bbbk$ with fraction field K. If E^* is a semistable vector bundle on C_K, then there exists a family E of semistable vector bundles on C_R such that $E_K \cong E^*$.*

Remark 8.3.2. Unlike the general case of algebraic stacks where base changes of R are necessary for the existence part of the valuative criterion (see the examples 3.8.16–3.8.18 and 5.5.11), no base change is necessary here.

Proof. We may assume that $R/\pi \cong \Bbbk$,[4] where π is a uniformizer. Let $r = \operatorname{rk}(E^*)$ and $d = \deg(E^*)$. By the valuative criterion for $\mathcal{B}un_{r,d}$ (Exercise 3.8.13), there exists a vector bundle E on C_R with $E_K \cong E^*$. Our strategy will be to modify E by 'elementary modifications' so that $E_\Bbbk$ becomes semistable. Namely, if $E_\Bbbk$ is not semistable, then we may choose a short exact sequence

$$0 \to D \to E_\Bbbk \to F \to 0 \qquad (\star)$$

such that F is maximal among semistable quotients of F of smallest slope, i.e., D is the last term of the Harder–Narasimhan filtration of $E_\Bbbk$; see Figure 8.3.3.[5] Define the *elementary modification*

$$E' := \ker(E \twoheadrightarrow E_\Bbbk \twoheadrightarrow F) \subseteq E.$$

By construction, $E'_K \cong E_K \cong E^*$ and $E/E' \cong F$. The containments $\pi E \subseteq E' \subseteq E$ imply that $E'/\pi E \cong D$ by using the identifications in the short exact sequence $(\star)$:

$$0 \to \underbrace{E'/\pi E}_{D} \to \underbrace{E/\pi E}_{E_\Bbbk} \to \underbrace{E/E'}_{F} \to 0.$$

We have a short exact sequence

$$0 \to F \to E'_\Bbbk \to D \to 0 \qquad (\star\star)$$

[4] In fact, the theorem holds with the same proof when $\Bbbk$ is an arbitrary field.

[5] As in Langton's proof, one can modify the argument to take instead $D \subseteq E_\Bbbk$ to be the maximal destabilizing subsheaf, i.e., the first term of the Harder–Narasimhan filtration of $E_\Bbbk$.

where the roles of D and F are flipped in comparison to ($\star$). This follows from the surjection $E'_{\Bbbk} \cong E'/\pi E' \twoheadrightarrow E'/\pi E \cong D$ with kernel $F \cong \pi E/\pi E'$.[6]

Let $HNP(E_{\Bbbk}) \subseteq \mathbb{C}$ be the Harder–Narasimhan polygon of $E_{\Bbbk}$, i.e., the convex hull of the points $Z(E') = -\deg(E') + \mathrm{rk}(E')i \in \mathbb{C}$ as $E' \subseteq E_{\Bbbk}$ ranges over all subsheaves with $\mu(E') > d/r$.

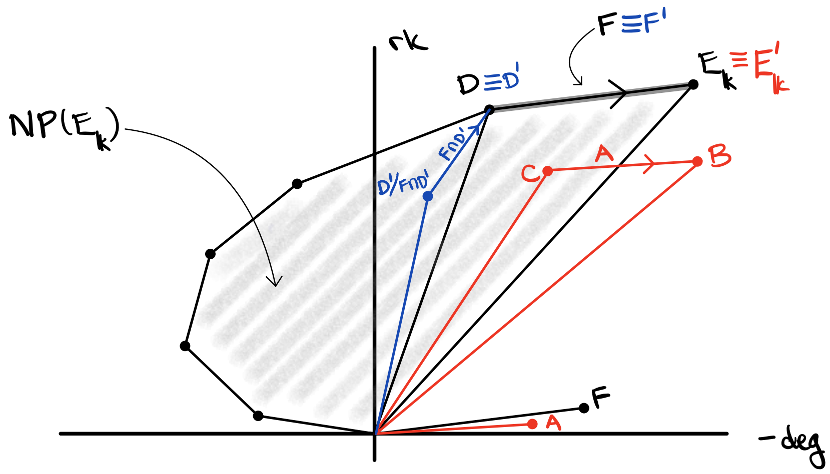

Figure 8.3.3: The Harder–Narasimhan polygon $HNP(E_{\Bbbk})$ of $E_{\Bbbk}$ is featured in grey. The red (resp., blue) vectors are drawn to help illustrate the proof of the inclusion $HNP(E'_{\Bbbk}) \subseteq HNP(E_{\Bbbk})$ (resp., the splitting of ($\star\star$) in the case of equality).

Mirroring the construction of ($\star$) for $E_{\Bbbk}$, let $0 \to D' \to E'_{\Bbbk} \to F' \to 0$ be the short exact sequence obtained by taking D' to be the last term in the Harder–Narasimhan filtration of $E'_{\Bbbk}$. We claim that:

(i) $HNP(E'_{\Bbbk}) \subseteq HNP(E_{\Bbbk})$, and

(ii) if $HNP(E'_{\Bbbk}) = HNP(E_{\Bbbk})$, then the compositions $F \to E'_{\Bbbk} \to F'$ and $D' \to E'_{\Bbbk} \to D$ are isomorphisms. In particular, ($\star\star$) is split.

For (i), a subsheaf $B \subseteq E'_{\Bbbk}$ induces a commutative diagram of short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, hook] & B \arrow[r] \arrow[d, hook] & C \arrow[r] \arrow[d, hook] & 0 \\ 0 \arrow[r] & F \arrow[r] & E'_{\Bbbk} \arrow[r] & D \arrow[r] & 0. \end{tikzcd}$$

[6]We provide another way to think about this setup. Recall from §6.9.2 that $\phi_R := [\mathrm{Spec}\,(R[s,t]/(st-\pi))/\mathbb{G}_m]$ and that $\phi_R \smallsetminus 0 \cong \mathrm{Spec}\,R \bigcup_{\mathrm{Spec}\,K} \mathrm{Spec}\,R$ is the non-separated DVR. Since $E \bigcup_{E^*} E'$ defines family over $C \times (\phi_R \smallsetminus 0)$ and since $\underline{\mathrm{Coh}}(C)$ is S-complete (Proposition 6.9.30), we know that this family extends to a family $\mathcal{F}$ over ϕ_R, but in this case we can describe it explicitly using the description from Exercise 6.9.16 as a $\mathbb{Z}$-graded sheaf $\mathcal{F} = \bigoplus_{n \in \mathbb{Z}} \mathcal{F}_n$ on C_R with maps s and t in a diagram:

$$\begin{tikzcd} \cdots \arrow[r, bend left, "s"] & \mathcal{F}_{-2} \arrow[l, bend left, "t"] \arrow[r, bend left, "s=\pi"] \arrow[d, equal] & \mathcal{F}_{-1} \arrow[l, bend left, "t=1"] \arrow[r, bend left, "s=1"] \arrow[d, equal] & \mathcal{F}_0 \arrow[l, bend left, "t=\pi"] \arrow[r, bend left, "s=1"] \arrow[d, equal] & \mathcal{F}_1 \arrow[l, bend left, "t=\pi"] \arrow[r, bend left, "s"] \arrow[d, equal] & \cdots . \arrow[l, bend left, "t"] \\ & E' & E' & E & E & \end{tikzcd}$$

The exact sequences ($\star$) and ($\star\star$) then correspond to the filtrations obtained by restricting to $\Theta_{\Bbbk}$ along $t = 0$ and $s = 0$, respectively.

Since $C \subseteq D \subseteq E_{\Bbbk}$, we have that $\mu(C) \le d/r$ or $Z(C) \in HNP(E_{\Bbbk})$. Since F is semistable, $\mu(A) \le \mu(F)$ (if $A \neq 0$). Using that $Z(B) = Z(A) + Z(C)$, these two facts implies that either $\mu(B) \le d/r$ or that $Z(B) \in HNP(E_{\Bbbk})$.

For (ii), we have a short exact sequence $0 \to D' \to E'_{\Bbbk} \to F' \to 0$ with $Z(F') = Z(F)$. The kernel of the composition $F \to E'_{\Bbbk} \to F'$ is $D' \cap F$ which sits in a short exact sequence

$$0 \to F \cap D' \to D' \to D'/F \cap D' \to 0.$$

Assuming that $F \cap D' \neq 0$, the semistability of F and the constructions of $F = E_{\Bbbk}/D$ and $D' \subseteq E'_{\Bbbk}$ yield inequalities

$$\mu(F \cap D') \le \mu(F) < \mu(E_{\Bbbk}) = \mu(E'_{\Bbbk}) < \mu(D'),$$

which in turn implies that $\mu(D'/F \cap D') > \mu(D')$. Since $D'/F \cap D' \subseteq D \subseteq E_{\Bbbk}$, $Z(D'/F \cap D') \in HNP(E_{\Bbbk})$ lies to the left of $Z(D')$. But this implies that $\mu(F \cap D') > \mu(F)$, contradicting the semistability of F. Thus $F \cap D' = 0$ and $F \to E'_{\Bbbk} \to F'$ is injective. Since F and F' have the same degree and rank, $F \to F'$ is an isomorphism splitting ($\star\star$).

Since there are only finitely many Harder–Narasimhan polygons contained in $HNP(E_{\Bbbk})$, by inductively replacing E with E' and applying claim (i), we either obtain a family of semistable vector bundles extending E^* or we obtain an infinite family $\cdots \subseteq E^{(2)} \subseteq E^{(1)} \subseteq E^{(0)}$ such that each $E^{(n)}$ has the same Harder–Narasimhan polygon. We will show that the latter case leads to a contradiction. By claim (ii), we know that each $E^{(n)}_{\Bbbk} \cong D \oplus F$ is split. After replacing E with $E^{(0)}$, the reader is left to check that the sheaf $E/E^{(n)}$ on $C_{R/\mathfrak{m}^n}$ is flat over $R/\mathfrak{m}^n$ with $(E/E^{(n+1)}) \otimes_{R/\mathfrak{m}^{n+1}} R/\mathfrak{m}^n \cong E/E^{(n)}$ (Exercise 8.3.4). This implies that we have compatible lifts

$$\begin{tikzcd}
\operatorname{Spec} \Bbbk \arrow[r, "{[E_{\Bbbk} \twoheadrightarrow F]}"] \arrow[d, hook] & \operatorname{Quot}^{P_F}(E/C_R/R) \arrow[d] \\
\operatorname{Spec} R/\mathfrak{m}^n \arrow[r, hook] \arrow[ur, dashed, "{[E_{R/\mathfrak{m}^n} \twoheadrightarrow E/E^{(n)}]}"] & \operatorname{Spec} R,
\end{tikzcd}$$

where P_F is the Hilbert polynomial of F. By properness of the Quot scheme (Proposition 1.4.2), the map $\operatorname{Quot}^{P_F}(E/C_R/R) \to \operatorname{Spec} R$ has closed image and is thus surjective. Therefore, there exists an extension $R \to R'$ and a section $[E_{R'} \twoheadrightarrow \widetilde{F}]\colon \operatorname{Spec} R' \to \operatorname{Quot}^{P_F}(E/C_R/R)$. (If R were complete, Grothendieck's Existence Theorem (C.5.3) would imply the existence of a section without an extension.) Restricting to the fraction field K' of R', we have a quotient $E_{K'} \twoheadrightarrow \widetilde{F}_{K'}$ with $\mu(\widetilde{F}_{K'}) > \mu(E_{K'})$. This implies that $E_{K'}$ is not semistable. Since semistability is insensitive to field extension (Corollary 8.1.29), this contradicts the semistability of E_K. See also [Lan75], [HL10, Thm. 2.B.1], and [Mar78, Thm. 5.7]. □

Exercise 8.3.4. Verify the missing details in the proof above: $(E/E^{(n+1)}) \otimes_{R/\mathfrak{m}^{n+1}} R/\mathfrak{m}^n \cong E/E^{(n)}$ and $E/E^{(n)}$ is flat over $R/\mathfrak{m}^n$.

Hint: First show that $(E/E^{(n)})_{\Bbbk} \cong F$. Then compare the short exact sequence $0 \to E^{(n)}/E^{(n+1)} \to E/E^{(n+1)} \to E/E^{(n)} \to 0$ with the right exact sequence

$$E/E^{(n+1)} \otimes_{R/\mathfrak{m}^{n+1}} \mathfrak{m}^n \to E/E^{(n+1)} \to E/E^{(n+1)} \otimes_{R/\mathfrak{m}^{n+1}} R/\mathfrak{m}^n \to 0$$

using the identifications $E^{(n)}/E^{(n+1)} \cong F \cong (E/E^{(n+1)})_{\Bbbk} \cong E/E^{(n+1)} \otimes_{R/\mathfrak{m}^{n+1}} \mathfrak{m}^n$. Establish the flatness of $E/E^{(n+1)}$ over $R/\mathfrak{m}^{n+1}$ by using induction on n and the Local Criterion for Flatness (A.2.6) with respect to $R/\mathfrak{m}^{n+1} \to R/\mathfrak{m}^n$.

Corollary 8.3.5. *The algebraic stack $\mathcal{B}un^{\mathrm{ss}}_{r,d}(C)$ is universally closed over $\Bbbk$.*

Proof. By Boundedness of Semistability (8.2.8), $\mathcal{B}un^{\mathrm{ss}}_{r,d}(C)$ is of finite type and, in particular, quasi-compact. We may therefore apply the Valuative Criteria for Universal Closedness (3.8.7). $\square$

8.3.2 Existence of a proper moduli space

Recall that Θ- and S-completeness are valuative criteria (Definitions 6.9.10 and 6.9.14) which guarantee the existence of a separated good moduli space (Theorem 6.10.1). We will verify that $\mathcal{B}un^{\mathrm{ss}}_{r,d}$ is both Θ- and S-complete by interpreting maps from Θ as filtrations of coherent sheaves (Proposition 6.9.6) and exploiting that the stack $\underline{\mathrm{Coh}}(X)$ parameterizing coherent sheaves is Θ- and S-complete (Proposition 6.9.30), a fact that holds for any projective scheme X. To this end, we provide a general criterion for when an open substack of $\underline{\mathrm{Coh}}(X)$ is Θ- and S-complete. To formulate the S-completeness criterion, we will need the following notion: two $\mathbb{Z}$-graded filtrations

$$E_\bullet : 0 \subseteq \cdots \subseteq E_{i-1} \subseteq E_i \subseteq E_{i+1} \subseteq \cdots \subseteq E$$

and

$$F^\bullet : F \supseteq \cdots \supseteq F^{i-1} \supseteq F^i \supseteq F^{i+1} \supseteq \cdots \supseteq 0$$

are called *opposite* if $E_i/E_{i-1} \cong F^i/F^{i+1}$ for all i. Extending the correspondence between filtrations and maps from Θ, the data of opposite filtrations $E_\bullet$ and $F^\bullet$ such that $E_i = 0$ and $F^i = F$ for $i \ll 0$, and $E_i = E$ and $F^i = 0$ for $i \gg 0$ is equivalent to giving a map

$$[(\operatorname{Spec} \Bbbk[x,y]/xy)/\mathbb{G}_m] \to \underline{\mathrm{Coh}}(X),$$

where $t \cdot (x,y) = (tx, t^{-1}y)$. In this case, this map takes $(1,0) \mapsto E$, $(0,1) \mapsto F$, and $(0,0) \mapsto \operatorname{gr} E_\bullet \cong \operatorname{gr} F^\bullet$.

Proposition 8.3.6. *Let X be a projective scheme over an algebraically closed field $\Bbbk$, and let $\mathcal{U} \subseteq \underline{\mathrm{Coh}}(X)$ be an open substack.*

(1) The substack $\mathcal{U}$ is Θ-complete if and only if for every DVR R (with fraction field K and residue field κ), for every coherent sheaf $\mathcal{E}$ on X_R flat over R, and for every $\mathbb{Z}$-graded filtration $\mathcal{E}_\bullet$ satisfying $\mathcal{E}_i = 0$ for $i \ll 0$, $\mathcal{E}_i = \mathcal{E}$ for $i \gg 0$, and with each $\mathcal{E}_i/\mathcal{E}_{i-1}$ flat over R, then if $\mathcal{E}$ and $\operatorname{gr}(\mathcal{E}_\bullet|_K)$ are in $\mathcal{U}$, then $\operatorname{gr}(\mathcal{E}_\bullet|_\kappa)$ must be in $\mathcal{U}$ as well.

(2) The substack is S-complete if for every pair of opposite filtrations $E_\bullet$, $F^\bullet$ of $E, F \in \mathcal{U}(\kappa)$ for a field extension $\kappa/\Bbbk$, the associated graded $\operatorname{gr} E_\bullet$ is in $\mathcal{U}(\kappa)$.

Proof. Since we already know that $\underline{\mathrm{Coh}}(X)$ is Θ- and S-complete (Proposition 6.9.30), the valuative criteria for $\mathcal{U}$ are equivalent to the existence of lifts for all commutative diagrams

$$\begin{tikzcd} \Theta_R \setminus 0 \arrow[r] \arrow[d, hook] & \mathcal{U} \arrow[d, hook] \\ \Theta_R \arrow[r] \arrow[ur, dashed] & \underline{\mathrm{Coh}}(X) \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \phi_R \setminus 0 \arrow[r] \arrow[d, hook] & \mathcal{U} \arrow[d, hook] \\ \phi_R \arrow[r] \arrow[ur, dashed] & \underline{\mathrm{Coh}}(X), \end{tikzcd}$$

where R is a DVR. In other words, we need to show that the images of 0 under the unique fillings $\Theta_R \to \underline{\mathrm{Coh}}(X)$ and $\phi_R \to \underline{\mathrm{Coh}}(X)$ are contained in $\mathcal{U}$. By Proposition 6.9.6, the map $\Theta_R \to \underline{\mathrm{Coh}}(X)$ is classified by a coherent sheaf $\mathcal{E}$ on C_R flat over R and a $\mathbb{Z}$-graded filtration $\mathcal{E}_\bullet$ satisfying $\mathcal{E}_i = 0$ for $i \ll 0$, $\mathcal{E}_i = \mathcal{E}$ for $i \gg 0$, and with each $\mathcal{E}_i/\mathcal{E}_{i-1}$ flat over R. Moreover, the restriction $\Theta_R \smallsetminus 0 \to \underline{\mathrm{Coh}}(X)$ factors through $\mathcal{U}$ if and only if $\mathcal{E}$ and $\mathrm{gr}(\mathcal{E}_\bullet|_K)$ are in $\mathcal{U}$. As the image of 0 under $\Theta_R \to \underline{\mathrm{Coh}}(X)$ is $\mathrm{gr}(\mathcal{E}_\bullet|_\kappa)$, we see that part (1) holds.

For (2), the restriction of $\phi_R \to \underline{\mathrm{Coh}}(X)$ along $\pi = 0$ yields a map $[\mathrm{Spec}(\kappa[x,y]/xy)/\mathbb{G}_m] \to \underline{\mathrm{Coh}}(X)$ corresponding to opposite filtrations $E_\bullet$ and $F^\bullet$. If $\mathrm{gr}\, E_\bullet$ is in $\mathcal{U}$, then the image of $\phi_R \to \underline{\mathrm{Coh}}(X)$ is contained in $\mathcal{U}$. □

While not needed for later, the following exercise is conceptually helpful.

Exercise 8.3.7. Show that the converse of Proposition 8.3.6(2) holds when $X = C$ is a curve.

Hint: Letting $\mathcal{X}_0 := [\mathrm{Spec}(\Bbbk[x,y]/xy)/\mathbb{G}_m] \subset \phi_R$ and $\mathcal{X}_n$ be its nilpotent thickenings, use deformation theory to extend a map $\mathcal{X}_0 \to \underline{\mathrm{Coh}}(C)$ to compatible maps $\mathcal{X}_n \to \underline{\mathrm{Coh}}(C)$ and use Coherent Tannaka Duality (6.6.9) to extend further to a map $\phi_R \to \underline{\mathrm{Coh}}(C)$.

Proposition 8.3.8. *For a smooth, connected, and projective curve C over an algebraically closed field $\Bbbk$, the algebraic stack $\mathcal{B}un^{\mathrm{ss}}_{r,d}$ is Θ- and S-complete.*

Proof. We will apply Proposition 8.3.6 to establish each property. For Θ-completeness, let $\mathcal{E}$ be a coherent sheaf on C_R flat over R and $\mathcal{E}_\bullet$ be a $\mathbb{Z}$-graded filtration satisfying $\mathcal{E}_i = 0$ for $i \ll 0$, $\mathcal{E}_i = \mathcal{E}$ for $i \gg 0$, and with each $\mathcal{E}_i/\mathcal{E}_{i-1}$ flat over R. Suppose that $\mathcal{E}$ and $\mathrm{gr}(\mathcal{E}_\bullet|_K)$ are in $\mathcal{B}un^{\mathrm{ss}}_{r,d}$. Since the degree and rank are constant in flat families, $\mathrm{gr}(\mathcal{E}_\bullet|_\kappa) \in \underline{\mathrm{Coh}}_{r,d}$. To show that $\mathrm{gr}(\mathcal{E}_\bullet|_\kappa)$ is semistable, it suffices to show that each factor $(\mathcal{E}_i/\mathcal{E}_{i-1})_\kappa$ is semistable of slope d/r. Since $(\mathcal{E}_i)_\kappa \subseteq \mathcal{E}_\kappa$ is a subsheaf of a semistable sheaf of the same slope, each $(\mathcal{E}_i)_\kappa$ is semistable. As semistable sheaves of a fixed slope form an abelian category (Exercise 8.1.17), $(\mathcal{E}_i/\mathcal{E}_{i-1})_\kappa = (\mathcal{E}_i)_\kappa/(\mathcal{E}_{i-1})_\kappa$ is semistable of slope d/r.

For S-completeness, suppose that

$$E_\bullet : \cdots \subseteq E_{i-1} \subseteq E_i \subseteq \cdots \subseteq E \qquad \text{and} \qquad F^\bullet : F \supseteq \cdots \supseteq F^i \supseteq F^{i+1} \supseteq \cdots$$

are opposite filtrations of semistable vector bundles E and F on C_κ for a field extension $\kappa/\Bbbk$. This means that $E_i/E_{i-1} \cong F^i/F^{i+1}$. We will prove by induction that each E_i and F^i is semistable of slope d/r. This establishes the desired semistability of $\mathrm{gr}\, E_\bullet = \bigoplus E_i/E_{i-1}$ by Exercise 8.1.17. Since $E_i = 0$ and $F_i = F$ for $i \ll 0$ and F is semistable, E_j and F_j are semistable for $i \ll 0$. Let's suppose that E_j, F_j are semistable for $j \leq i$. Since $F_{i+1} \cong \ker(F_i \to E_i/E_{i-1})$ is the kernel of semistable vector bundles of the same slope, F_{i+1} is semistable of slope d/r. On the other hand, we can express $E_{i+1}/E_i \cong F_{i+1}/F_{i+2}$ both as a subsheaf of the semistable vector bundle E/E_i (using the semistability of E and the inductive hypothesis) and as a quotient of the semistable vector bundle F_{i+1}. It follows that $\mu(E_{i+1}/E_i) = d/r$, and hence E_{i+1} is semistable of slope d/r. □

The stack $\mathcal{B}un_{r,d}$ of vector bundles, on the other hand, is not Θ-complete nor S-complete.

Example 8.3.9. Suppose that $g \geq 2$. Let $p \in C(\Bbbk)$ be a point defined by the vanishing of a section $s \in \Gamma(C, \mathcal{O}(p))$, and let $I \subseteq \mathcal{O}_{C_R}$ be the ideal sheaf of

$(p,0) \in C \times \operatorname{Spec} R$. The injection $(s,-\pi)\colon \mathcal{O}_{C_R}(-p) \hookrightarrow \mathcal{O}_{C_R} \oplus \mathcal{O}_{C_R}(-p)$ has quotient I, which is torsion free, hence flat over R, but is not a vector bundle. By Proposition 8.3.6, $\mathcal{B}un_{2,-1}$ is not Θ-complete.

Let L and M be line bundles on C, and let $p \in C(\Bbbk)$ be a point such that $\operatorname{Ext}^1_{\mathcal{O}_C}(M, L(p))$ and $\operatorname{Ext}^1_{\mathcal{O}_C}(L, M(p))$ are nonzero; if L and M have the same degree, then a Riemann–Roch calculation shows that both Ext^1 groups are nonzero. Let Q (resp., Q') be a non-trivial extension of M by $L(p)$ (resp., L by $M(p)$). Then

$$E_\bullet : 0 \subseteq L \subseteq L(p) \subseteq Q \quad \text{and} \quad F^\bullet : Q' \supseteq M(p) \supseteq M \supseteq 0$$

define opposite filtrations where $E_0 = L$ and $F^0 = Q'$. The associated graded $\operatorname{gr} E_\bullet = L \oplus \kappa(p) \oplus M$ is not a vector bundle. By Proposition 8.3.6, $\mathcal{B}un_{1,d}$ is not S-complete where $d = \deg L + \deg M + 1$.

Theorem 8.3.10. *Let C be a smooth, connected, and projective curve over an algebraically closed field $\Bbbk$ of characteristic 0. For each $r > 0$ and d, there exists a cartesian diagram*

$$\begin{tikzcd} \mathcal{B}un^{\mathrm{s}}_{r,d} \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{B}un^{\mathrm{ss}}_{r,d} \arrow[d] \\ M^{\mathrm{s}}_{r,d} \arrow[r, hook] & M^{\mathrm{ps}}_{r,d}, \end{tikzcd}$$

where $M^{\mathrm{ps}}_{r,d}$ is a proper algebraic space over $\Bbbk$, $\mathcal{B}un^{\mathrm{ss}}_{r,d} \to M^{\mathrm{ps}}_{r,d}$ is a good moduli space, and $\mathcal{B}un^{\mathrm{s}}_{r,d} \to M^{\mathrm{s}}_{r,d}$ is a banded $\mathbb{G}_m$-gerbe. Moreover,

$$\dim M^{\mathrm{ps}}_{r,d} = \dim \mathcal{B}un^{\mathrm{ss}}_{r,d} + 1 = r^2(g-1) + 1.$$

Proof. The existence of a separated algebraic space $\pi\colon \mathcal{B}un^{\mathrm{ss}}_{r,d} \to M^{\mathrm{ps}}_{r,d}$ follows directly from the Existence Theorem of Good Moduli Spaces (6.10.1) using that $M^{\mathrm{ps}}_{r,d}$ is of finite type by Boundedness of Semistability (8.2.8) and both Θ- and S-complete (Proposition 8.3.8). Since $\mathcal{B}un^{\mathrm{ss}}_{r,d}$ is universally closed by Langton's Theorem (8.3.5), $M^{\mathrm{ps}}_{r,d}$ is proper.

Since every stable bundle is polystable, hence a closed point of $\mathcal{B}un^{\mathrm{ss}}_{r,d}$ by Proposition 8.1.35, it follows that $\pi^{-1}(\pi(\mathcal{B}un^{\mathrm{s}}_{r,d})) = \mathcal{B}un^{\mathrm{s}}_{r,d}$. Since π is universally closed, $\pi(\mathcal{B}un^{\mathrm{s}}_{r,d}) \subseteq M^{\mathrm{ps}}_{r,d}$ is open. On the hand, Corollary 8.2.23 gives the existence of a banded $\mathbb{G}_m$-gerbe $\mathcal{B}un^{\mathrm{s}}_{r,d} \to M^{\mathrm{s}}_{r,d}$. Since both good moduli spaces and banded $\mathbb{G}_m$-gerbes are universal for maps to algebraic spaces, $M^{\mathrm{s}}_{r,d} \cong \pi(\mathcal{B}un^{\mathrm{s}}_{r,d})$.

To compute the dimension, we will use that $M^{\mathrm{s}}_{r,d} \subseteq M^{\mathrm{ps}}_{r,d}$ is open and dense; this follows from the fact $\mathcal{B}un^{\mathrm{s}}_{r,d} \subseteq \mathcal{B}un_{r,d}$ is open and dense (Theorem 8.2.14). Since $\mathcal{B}un^{\mathrm{s}} \to M^{\mathrm{s}}_{r,d}$ is a banded $\mathbb{G}_m$-gerbe and $\dim B\mathbb{G}_m = -1$, we conclude that

$$\dim M^{\mathrm{ps}}_{r,d} = \dim M^{\mathrm{s}}_{r,d} = \dim \mathcal{B}un^{\mathrm{s}}_{r,d} + 1 = r^2(g-1) + 1,$$

using the calculation of the dimension of $\mathcal{B}un_{r,d}$ from Proposition 3.7.9. □

Exercise 8.3.11. For every $\xi \in \operatorname{Pic}(C)$, denote by $\mathcal{B}un^{\mathrm{ss}}_{r,\xi}$ (resp., $\mathcal{B}un^{\mathrm{s}}_{r,\xi}$) the substack of the moduli stack $\mathcal{B}un_{r,\xi}$ parameterizing semistable (resp., stable) vector bundles with determinant ξ (Definition 8.2.24). Show that $\mathcal{B}un^{\mathrm{ss}}_{r,\xi}$ is Θ- and S-complete. Conclude that in characteristic 0, there exists a proper good moduli space $\mathcal{B}un^{\mathrm{ss}}_{r,\xi} \to M^{\mathrm{ps}}_{r,\xi}$ restricting to a $\boldsymbol{\mu}_r$-gerbe $\mathcal{B}un^{\mathrm{s}}_{r,\xi} \to M^{\mathrm{s}}_{r,\xi}$ with $M^{\mathrm{s}}_{r,\xi} \subseteq M^{ps}_{r,\xi}$ an open subspace.

Hint: Use that $\mathcal{B}un^{\mathrm{ss}}_{r,\xi} \to \mathcal{B}un^{\mathrm{ss}}_{r,\deg L}$ is affine.

Note that every closed point of $\mathcal{B}un^{\mathrm{ss}}_{r,d}(C)$ (and similarly for $\mathcal{B}un^{\mathrm{ss}}_{r,\xi}(C)$) has reductive automorphism group; this follows either from the S-completeness of $\mathcal{B}un^{\mathrm{ss}}_{r,d}(C)$ using Proposition 6.10.30, or in characteristic 0 by the existence of a good moduli space using Exercise 6.5.25. Since the closed points of $\mathcal{B}un^{\mathrm{ss}}_{r,d}(C)$ correspond to polystable vector bundles (a fact that holds generally but that we only proved in Proposition 8.1.35 in characteristic 0), we recover the fact that a polystable vector bundle has a reductive automorphism group, which we established directly in Corollary 8.1.16. A strictly semistable but non polystable vector bundle, by contrast, may have a non-reductive automorphism group.

Exercise 8.3.12. Construct a semistable vector bundle with non-reductive automorphism group.

Hint: Consider a non-split extension of stable vector bundles.

8.3.3 A stratification of the unstable locus

The moduli stack $\mathcal{B}un_{r,d}$ of vector bundles admits a stratification into locally closed substacks indexed by the Harder–Narasimhan polygon. This stratification is called the *Harder–Narasimhan* or *Shatz stratification*. It was first proven in [Sha77], and as with most other results in this chapter (e.g., the Harder–Narasimhan Filtration and Langton's Theorem), it generalizes to torsion free sheaves on projective varieties of arbitrary dimension.

Recall that if F is a vector bundle on C, the *Harder–Narasimhan polygon* $HNP(F) \subseteq \mathbb{C}$ is the convex hull of $Z(E) = -\deg(E) + \mathrm{rk}(E)i$ as $E \subseteq F$ ranges over all destabilizing subsheaves, i.e., $\mu(E) \geq \mu(F)$ or equivalently $\arg Z(E) \geq \arg Z(F)$; see Figure 8.1.25. If $0 = F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_l = F$ is the Harder–Narasimhan Filtration (8.1.21), then $Z(F_i)$ are the extremal points of $HNP(F)$ (Corollary 8.1.26). We will use the partial order on Harder–Narasimhan polygons given by containment. We also note that if F is semistable, then $HNP(F)$ is the straight line from 0 to $Z(F)$.

Theorem 8.3.13 (The Harder–Narasimhan/Shatz Stratification of $\mathcal{B}un_{r,d}$)**.** *Let C be a smooth, connected, and projective curve of genus $g \geq 2$ over an algebraically closed field $\Bbbk$. There is a stratification*

$$\mathcal{B}un_{r,d} = \coprod_{\textit{convex polygons } \mathcal{P}} \mathcal{M}_{\mathcal{P}},$$

where $\mathcal{M}_{\mathcal{P}} \subseteq \mathcal{B}un_{r,d}$ is a locally closed substack such that for a vector bundle E on C of rank r and degree d, then

$$E \in \mathcal{M}_{\mathcal{P}} \iff HNP(E) = \mathcal{P}.$$

Moreover, for each convex polygon $\mathcal{P}$:

(1) Let S be a $\Bbbk$-scheme and $\mathcal{F}$ be a family of vector bundle on C_S of rank r and degree d.

(a) The map $S \to \mathcal{B}un_{r,d}$ classifying $\mathcal{F}$ factors through $\mathcal{M}_{\mathcal{P}}$ if and only if there exists a filtration

$$0 = \mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \cdots \subseteq \mathcal{F}_l = \mathcal{F}$$

of vector bundles with vector bundle factors $\mathcal{F}_i/\mathcal{F}_{i-1}$ such that for every $s \in S$, $HNP(\mathcal{F}_s) = \mathcal{P}$ and $0 = \mathcal{F}_{0,s} \subsetneq \mathcal{F}_{1,s} \subsetneq \cdots \subsetneq \mathcal{F}_{l,s} = \mathcal{F}_s$ is the Harder–Narasimhan filtration of $\mathcal{F}_s$.

(b) If $s \rightsquigarrow s_0$ is a specialization, then $HNP(\mathcal{F}_s) \subseteq HNP(\mathcal{F}_{s_0})$.

(2) the union

$$\mathcal{M}_{\leq \mathcal{P}} := \bigcup_{\mathcal{P}' \subseteq \mathcal{P}} \mathcal{M}_{\mathcal{P}'}$$

is an open substack of $\mathcal{B}un_{r,d}$, while $\mathcal{M}_{\mathcal{P}} \subseteq \mathcal{M}_{\leq \mathcal{P}}$ is a closed substack.

(3) Letting $e_1, \ldots, e_l \in \mathbb{C}$ denote the extremal points of $\mathcal{P}$ with increasing imaginary parts and $P_1, \ldots, P_l$ denote the Hilbert polynomials corresponding to vector bundles G_i with $Z(G_i) = e_i - e_{i-1}$, there is an identification

$$\mathcal{M}_{\mathcal{P}} \cong \mathrm{Flag}^{P_1,\ldots,P_l}(\mathcal{E}_{\mathrm{univ}}/C \times \mathcal{M}_{\leq \mathcal{P}}/\mathcal{M}_{\leq \mathcal{P}})$$

with the Flag scheme (Exercise 1.4.7) over $\mathcal{M}_{\leq \mathcal{P}}$ parameterizing filtrations $\mathcal{F}_\bullet$ of the universal bundle $\mathcal{E}_{\mathrm{univ}}$ whose ith factor $\mathcal{F}_i/\mathcal{F}_{i-1}$ has Hilbert polynomial P_i.

Remark 8.3.14. When $\mathcal{P}$ corresponds to the straight line from 0 to $Z(F)$, $\mathcal{M}_{\mathcal{P}} = \mathcal{B}un^{\mathrm{ss}}_{r,d}$ is the semistable locus.

If S is a noetherian $\Bbbk$-scheme and $\mathcal{F}$ is a family of vector bundles on C_S, there is a *finite* stratification $S = \coprod S_{\mathcal{P}}$ with each $S_{\mathcal{P}} \subseteq S$ locally closed and stable under base change (i.e., for $T \to S$, $T_{\mathcal{P}} = T \times_S S_{\mathcal{P}}$).

Proof. We begin by showing that if $\mathcal{F}$ is a vector bundle on C_S over a noetherian scheme S, then there is a finite stratification $S = \coprod_{\mathcal{P}} S_{\mathcal{P}}$ into locally closed subschemes. First, there are only finitely many polygons $\mathcal{P}$ that can arise as the Harder–Narasimhan polygon of a fiber $\mathcal{F}_s$; this is because the degree of a subsheaf of a fiber is bounded from above (Exercise 8.1.28). Second, for each such polygon $\mathcal{P}$ whose extremal vertices have Hilbert polynomials $P_1, \ldots, P_l$ (following the notation of (3)), there is a Flag scheme $\mathrm{Flag}^{P_1,\ldots,P_l}(\mathcal{F}/C_S/S)$ proper over S. Denote the *closed* image as

$$S_{\geq \mathcal{P}} := \mathrm{im}(\mathrm{Flag}^{P_1,\ldots,P_l}(\mathcal{F}/C_S/S) \to S).$$

A point $s \in S$ lies in $S_{\geq \mathcal{P}}$ if and only if $\mathcal{F}_s$ admits a filtration whose associated polygon is $\mathcal{P}$. Then

$$S_{\mathcal{P}} := S_{\geq \mathcal{P}} \smallsetminus \bigcup_{\mathcal{P}' \supsetneq \mathcal{P}} S_{\geq \mathcal{P}'}$$

is an open substack of $S_{\geq \mathcal{P}}$. Since the Harder–Narasimhan polygon contains every destabilizing subsheaf (Corollary 8.1.26), it follows that $s \in S$ lies in $S_{\mathcal{P}}$ if and only if $HNP(\mathcal{F}_s) = \mathcal{P}$.

To see (1)(a) let $\operatorname{Spec} R \to S$ be a map from a DVR representing the specialization. Since the Harder–Narasimhan filtration is stable under field extensions, it suffices to show that $HNP(\mathcal{F}_K) \subseteq HNP(\mathcal{F}_\kappa)$, where K and κ are the fraction and residue field of R. By the valuative criterion for Quot schemes (Proposition 1.4.2), the Harder–Narasimhan filtration of $\mathcal{F}_K$ extends to a filtration of $\mathcal{F}_R$, which in turn restricts to a filtration of $\mathcal{F}_\kappa$. Since each term of this filtration of $\mathcal{F}_\kappa$ lies in $HNP(\mathcal{F}_\kappa)$, we have that $HNP(\mathcal{F}_K) \subseteq HNP(\mathcal{F}_\kappa)$. Part (1)(a) implies part $S_{\leq \mathcal{P}} := \bigcup_{\mathcal{P}' \subseteq \mathcal{P}} S_{\mathcal{P}'}$ is open: it is constructible and stable under generization.

It remains to show that the scheme structures $S_{\mathcal{P}} \subseteq S$ are functorial in S, which will follow if we can establish that $S_{\mathcal{P}} \cong \mathrm{Flag}^{P_1,\ldots,P_l}(\mathcal{F}/C \times S_{\leq \mathcal{P}}/S_{\leq \mathcal{P}})$. In other words, we need to show that the proper morphism

$$\mathrm{Flag}^{P_1,\ldots,P_l}(\mathcal{F}/C \times S_{\leq \mathcal{P}}/S_{\leq \mathcal{P}}) \to S_{\leq \mathcal{P}}$$

is a closed immersion mapping isomorphically onto $S_{\mathcal{P}}$. To see this, it suffices to show that it is injective on tangent spaces or, in other words, that the relative tangent spaces are zero. We can factor the above map as a composition

$$\begin{aligned}\operatorname{Flag}^{P_1,\dots,P_l}(\mathcal{F}/C\times S_{\leq\mathcal{P}}/S_{\leq\mathcal{P}})\to\operatorname{Flag}^{P_2,\dots,P_l}(\mathcal{F}/C\times S_{\leq\mathcal{P}}/S_{\leq\mathcal{P}})\to\cdots\\ \to\operatorname{Quot}^{P_l}(\mathcal{F}/C\times S_{\leq\mathcal{P}}/S_{\leq\mathcal{P}})\to S_{\leq\mathcal{P}},\end{aligned}$$

where each morphism is identified by a relative Quot scheme (see Exercise 1.4.7). By analyzing the relative tangent space of each map above, we can reduce the injectivity claim to the following statement: if $0=\mathcal{F}_{s,0}\subseteq\mathcal{F}_{s,1}\subseteq\cdots\subseteq\mathcal{F}_{s,l}=\mathcal{F}_s$ is the Harder–Narasimhan filtration of a fiber $\mathcal{F}_s$ over $s\in S_{\mathcal{P}}$ and $q\colon\mathcal{F}_{s,i}\twoheadrightarrow\mathcal{F}_{s,i}/\mathcal{F}_{s,i-1}$ denotes the ith quotient, then $T_q\operatorname{Quot}^{P_i}(\mathcal{F}_{s,i}/C_s/\kappa(s))=0$. By Exercise 1.5.6, this tangent space is identified with $\operatorname{Hom}_{\mathcal{O}_{C_s}}(\mathcal{F}_{s,i},\mathcal{F}_{s,i}/\mathcal{F}_{s,i-1})$. Since the slope of every factor in the Harder–Narasimhan filtration of $\mathcal{F}_{s,i}$ is strictly greater than the slope of the semistable sheaf $\mathcal{F}_{s,i}/\mathcal{F}_{s,i-1}$, there are no nontrivial homomorphisms (Exercise 8.1.31). See also [Sha77] and [Nit11]. □

Remark 8.3.15 (Relative Harder–Narasimhan Filtrations). As a consequence of the theorem, for a vector bundle $\mathcal{F}$ on C_S, the restriction $\mathcal{F}_{S_{\mathcal{P}}}$ of a vector bundle $\mathcal{F}$ admits a filtration $0=\mathcal{F}_0\subseteq\mathcal{F}_1\subseteq\cdots\subseteq\mathcal{F}_l=\mathcal{F}_{S_{\mathcal{P}}}$ of vector bundles with vector bundle factors $\mathcal{F}_i/\mathcal{F}_{i-1}$. This filtration restricts to the Harder–Narasimhan filtration of $\mathcal{F}_s$ on C_s for each $s\in S_{\mathcal{P}}$, and for this reason is called the *relative Harder–Narasimhan filtration*. By taking $\mathcal{P}$ to be minimal among Harder–Narasimhan polygons of the fibers $\mathcal{F}_s$, we see that there is an open subscheme $U:=S_{\mathcal{P}}\subseteq S$ is an open subscheme over which there exists a relative Harder–Narasimhan filtration. While this filtration may not extend over S in general, it does extend after a blow-up [HL10, Thm. 2.3.2].

Remark 8.3.16 (Θ-stratification). The Harder–Narasimhan/Shatz stratification of $\mathcal{B}un_{r,d}$ is an example of a Θ-*stratification* as defined in Remark 7.7.8. For each convex polygon $\mathcal{P}$, $\mathcal{M}_{\leq\mathcal{P}}:=\bigcup_{\mathcal{P}'\leq\mathcal{P}}\mathcal{M}_{\mathcal{P}'}\subseteq\mathcal{B}un_{r,d}$ is an open substack and there is an identification

$$\underline{\operatorname{Mor}}([\mathbb{A}^1/\mathbb{G}_m],\mathcal{M}_{\leq\mathcal{P}})\cong\coprod_{\mathcal{P}'}\operatorname{Flag}^{\mathcal{P}'}(\mathcal{E}_{\mathrm{univ}}/C\times\mathcal{M}_{\leq\mathcal{P}}/\mathcal{M}_{\leq\mathcal{P}}),$$

where $\operatorname{Flag}^{\mathcal{P}'}$ is shorthand notation for $\operatorname{Flag}^{P'_1,\dots,P'_l}$ as in part (3) above; this fact is an enhancement of the correspondence between maps from $[\mathbb{A}^1/\mathbb{G}_m]$ and filtrations (Proposition 6.9.6). Taking $\mathcal{P}'=\mathcal{P}$, the Flag scheme $\operatorname{Flag}^{\mathcal{P}}(\mathcal{E}_{\mathrm{univ}}/C\times\mathcal{M}_{\leq\mathcal{P}}/\mathcal{M}_{\leq\mathcal{P}})\subseteq\underline{\operatorname{Mor}}([\mathbb{A}^1/\mathbb{G}_m],\mathcal{M}_{\leq\mathcal{P}})$ is a union of connected components and by part (3) defines a Θ-stratum, i.e., that the induced map $\operatorname{Flag}^{\mathcal{P}}(\mathcal{E}_{\mathrm{univ}}/C\times\mathcal{M}_{\leq\mathcal{P}}/\mathcal{M}_{\leq\mathcal{P}})\to\mathcal{M}_{\leq\mathcal{P}}$ is a closed immersion mapping isomorphically onto $\mathcal{M}_{\mathcal{P}}$.

Appendix A

Morphisms of schemes

The theory of schemes is widely regarded as a horribly abstract algebraic tool that hides the appeal of geometry to promote and overwhelming and often unnecessary generality. By contrast, experts know that schemes make things simpler.

David Eisenbud and Joe Harris [EH00]

We recall definitions and properties of morphisms of schemes—locally of finite presentation, flat, smooth, étale, unramified, quasi-finite, and proper. As we intend to highlight results important in moduli theory, we pay close attention to *functorial* properties, i.e., properties of schemes and their morphisms characterized by their functors.

While first courses in algebraic geometry (e.g., [Har77]) often impose noetherian hypotheses, we try to state results in the non-noetherian setting when possible. Since we define moduli functors and stacks on the entire category of schemes, it is essential to work with non-noetherian schemes. Limit Methods (B.3) allow one to reduce properties of general schemes to noetherian schemes.

A.1 Morphisms locally of finite presentation

A morphism of schemes $f: X \to Y$ is *locally of finite type* (resp., *locally of finite presentation*) if for all affine open subschemes $\operatorname{Spec} B \subseteq Y$ and $\operatorname{Spec} A \subseteq f^{-1}(\operatorname{Spec} B)$, there is a surjection $B[x_1, \ldots, x_n] \to A$ of B-algebras (resp., a surjection $B[x_1, \ldots, x_n] \to A$ with finitely generated kernel). If in addition f is quasi-compact (resp., quasi-compact and quasi-separated), we say that f is *of finite type* (resp., *of finite presentation*). When Y is locally noetherian, being locally of finite type (resp., finite type) is equivalent to being locally of finite presentation (resp., finite presentation). In the non-noetherian setting, even closed immersions may not be locally of finite presentation, e.g., $\operatorname{Spec} \mathbb{C} \hookrightarrow \operatorname{Spec} \mathbb{C}[x_1, x_2, \ldots]$. Morphisms of finite presentation are better behaved than morphisms of finite type. Many standard results (e.g., Semicontinuity (A.6.4)) for proper flat morphisms do not hold in the non-noetherian setting without a finite presentation condition; see [Vak17, §28.2.11] and [SP, Tag 05LB]. Therefore, in this text, when we define for instance a family of stable curves $\pi: \mathcal{C} \to S$ (Definition 5.3.19), we require not only that π is proper and flat but also of finite presentation.

The functorial characterization of locally of finite presentation morphisms uses the notion of an *inverse system* in a category $\mathcal{C}$: a partially ordered set $(I, \geq)$ which is directed, i.e., for every $i, j \in I$ there exists $k \in I$ such that $k \geq i$ and $k \geq j$, together with a contravariant functor $I \to \mathcal{C}$.

Proposition A.1.1. *A morphism $f\colon X \to Y$ of schemes is locally of finite presentation if and only if for every inverse system $\{\operatorname{Spec} A_\lambda\}_{\lambda \in I}$ of affine schemes over Y, the natural map*

$$\operatorname{colim}_\lambda \operatorname{Mor}_Y(\operatorname{Spec} A_\lambda, X) \to \operatorname{Mor}_Y(\operatorname{Spec}(\operatorname{colim}_\lambda A_\lambda), X) \tag{A.1.2}$$

is bijective.

Proof. See [EGA, IV.8.14.2] or [SP, Tag 01ZC], where (A.1.2) is also shown to be bijective for inverse systems of quasi-compact and quasi-separated schemes with affine transition maps. □

This is not a deep result: requiring that every map $\operatorname{Spec} A \to X$ over Y factors through some $\operatorname{Spec} A_\lambda \to \operatorname{Spec} A$ is essentially the condition that $\operatorname{Spec} A \to X$ depends on only finite data, i.e., there are only finitely many generators and relations for the ring maps locally defining $X \to Y$.

Exercise A.1.3. Verify Proposition A.1.1 in the case of a morphism $\operatorname{Spec} A \to \operatorname{Spec} B$ of affine schemes.

Proposition A.1.1 is a *functorial* condition as it only depends on the functor $\operatorname{Mor}_Y(-, X)$, and therefore we can extend the definition of locally of finite presentation to functors.

Definition A.1.4. Let Y be a scheme. A contravariant $F\colon \mathrm{Sch}/Y \to \mathrm{Sets}$ is *locally of finite presentation* (or *limit preserving*) if for every inverse system $\{\operatorname{Spec} A_\lambda\}_{\lambda \in I}$ of affine schemes over Y, the natural map

$$\operatorname{colim}_\lambda F(A_\lambda) \to F(\operatorname{colim}_\lambda A_\lambda)$$

is bijective.

By Proposition A.1.1, a scheme X is locally of finite presentation over Y if and only if the functor $\operatorname{Mor}_Y(-, X)$ is locally of finite presentation.

A.2 Flatness

Art is fire plus algebra.

Jorge Luis Borges

You cannot get very far in moduli theory without flatness. While its definition is seemingly abstract and algebraic, it is a magical geometric property of a morphism $X \to Y$ that ensures that fibers X_y 'vary nicely' as $y \in Y$ varies. This principle is nicely evidenced by Flatness via the Hilbert Polynomial (A.2.4). It is the reason why we define objects of our moduli stacks as flat families.

A.2.1 Flatness criteria

A module M over a ring A is *flat* if the functor

$$- \otimes_A M \colon \operatorname{Mod}(A) \to \operatorname{Mod}(A)$$

is exact. We recall the following criteria:

(1) (Stalk Criterion) M is flat over A if and only if $M_{\mathfrak{p}}$ is flat over $A_{\mathfrak{p}}$ for every prime (equivalently maximal) ideal $\mathfrak{p}$. More generally, if $A \to B$ is a ring map, a B-module N is flat if and only if for every prime $\mathfrak{q} \subseteq B$ with preimage $\mathfrak{p} \subseteq A$, $N_{\mathfrak{q}}$ is flat over $A_{\mathfrak{p}}$.

(2) (Ideal Criterion) M is flat if and only if for every finitely generated ideal $I \subseteq A$, the map $I \otimes_A M \to M$ is injective [Eis95, Prop. 6.1]. (When A is a PID, this implies that M is flat if and only if M is torsion free.)

(3) (Tor Criterion) M is flat if and only if $\operatorname{Tor}_1^A(A/I, M) = 0$ for all finitely generated ideals $I \subseteq A$ [Eis95, Prop. 6.1].

(4) (Finitely Presented Criterion) M is finitely presented and flat over A if and only if M is finite and projective if and only if M is finite and locally free (i.e., $M_{\mathfrak{p}}$ is finite and free for all prime—or equivalently maximal—ideals $\mathfrak{p}$); see [SP, Tag 00NX]. (Without the finitely presented hypothesis, Lazard's Theorem states that M is flat over A if and only if M can be written as a directed colimit $\operatorname{colim}_{i \in I} M_i$ of free finite A-modules M_i; see [Eis95, A6.6] or [SP, Tag 058G].)

(5) (Equational Criterion) M is flat if and only if for every relation $\sum_{i=1}^n a_i m_i = 0$ with $a_i \in A$ and $m_i \in M$, there exists $m'_j \in M$ for $j = 1, \ldots, r$ and $a'_{ij} \in A$ such that $\sum_{j=1}^r a'_{ij} m'_j = m_i$ for all i and $\sum_{i=1}^n a'_{ij} a_i = 0$ for all j [Eis95, Cor. 6.5].

If $f \colon X \to Y$ is a morphism of schemes, then a quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ is *flat* if for all affine opens $\operatorname{Spec} B \subseteq Y$ and $\operatorname{Spec} A \subseteq f^{-1}(\operatorname{Spec} B)$, the B-module $\Gamma(\operatorname{Spec} A, \mathcal{F})$ is a flat.

Proposition A.2.1 (Flat Equivalences)**.** *Let $f \colon X \to Y$ be a morphism of schemes and $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. The following are equivalent:*

(1) $\mathcal{F}$ is flat over Y;

(2) There exists a Zariski-cover $\{\operatorname{Spec} B_i\}$ of Y and $\{\operatorname{Spec} A_{ij}\}$ of $f^{-1}(\operatorname{Spec} B_i)$ such that $\Gamma(\operatorname{Spec} A_{ij}, \mathcal{F})$ is flat as an B_i-module under the ring map $B_i \to A_{ij}$;

(3) For all $x \in X$, the $\mathcal{O}_{X,x}$-module $\mathcal{F}_x$ is flat as an $\mathcal{O}_{Y,y}$-module.

(4) The functor

$$\operatorname{QCoh}(Y) \to \operatorname{QCoh}(X), \qquad \mathcal{G} \mapsto f^*\mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{F}$$

is exact.

Proof. See [Har77, §III.9] or [SP, Tag 01U2]. □

We say that a morphism $f \colon X \to Y$ of schemes is *flat at $x \in X$* (resp., a quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ is *flat at $x \in X$*) if there exists a Zariski open neighborhood $U \subseteq X$ containing x such that $f|_U$ (resp., $\mathcal{F}|_U$) is flat over Y. This is equivalent to the flatness of $\mathcal{O}_{X,x}$ (resp., $\mathcal{F}_x$) as an $\mathcal{O}_{Y,y}$-module.

Proposition A.2.2 (Flatness Criterion over Smooth Curves)**.** *Let C be an integral and regular scheme of dimension 1 (e.g., the spectrum of a DVR or a smooth* *connected curve over a field), and let $X \to C$ be a morphism of schemes. A quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ is flat over C if and only if every associated point of $\mathcal{F}$ maps to the generic point of C.*

Proof. A short argument shows that this follows from the fact that a module over a DVR is flat if and only if it is torsion free; see [Har77, III.9.7]. $\square$

Over higher dimensional bases, it is sometimes possible to check flatness by reducing to the above criterion over a smooth curve. This is called the valuative criterion for flatness: if $f\colon X \to S$ is a finite type morphism of noetherian schemes over a reduced scheme S and $\mathcal{F}$ is a coherent $\mathcal{O}_X$-module, then $\mathcal{F}$ is flat at $x \in X$ if and only if for every map $(\operatorname{Spec} R, 0) \to (S, f(x))$ from a DVR, the restriction $\mathcal{F}|_{X_R}$ is flat over R at all points in $X_R := X \times_S \operatorname{Spec} R$ over 0 and x [EGA, IV.11.8.1]. Despite providing a conceptual geometric criterion for flatness, it is surprisingly rarely used in moduli theory.

Proposition A.2.3 (Flatness Criterion over Artinian Rings)**.** *A module over an artinian ring is flat if and only if it is free if and only if it is projective.*

Proof. See [SP, Tag 051E]. $\square$

Recall that if $X \subseteq \mathbb{P}^n_K$ is a subscheme and $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_X$-module, the *Hilbert polynomial* of $\mathcal{F}$ is $P_{\mathcal{F}}(z) = \chi(X, \mathcal{F}(z)) \in \mathbb{Q}[z]$.

Proposition A.2.4 (Flatness via the Hilbert Polynomial)**.** *Let S be a reduced, connected, and noetherian scheme, and let $X \subseteq \mathbb{P}^n_S$ be a closed subscheme. A coherent $\mathcal{O}_X$-module $\mathcal{F}$ is flat over S if and only if the function*

$$S \to \mathbb{Q}[z], \qquad s \mapsto P_{\mathcal{F}|_{X_s}}$$

assigning a point $s \in S$ to the Hilbert polynomial of the restriction $\mathcal{F}|_{X_s}$ to the fiber $X_s \subseteq \mathbb{P}^n_{\kappa(s)}$ is constant.

Proof. See [Har77, Thm. 9.9]. $\square$

Theorem A.2.5 (Local and Infinitesimal Criteria for Flatness)**.** *Let $A \to B$ be a local homomorphism of noetherian local rings, and let M be a finite B-module. The following are equivalent:*

(1) M is flat over A,

(2) (Local Criterion) $\operatorname{Tor}_1^A(A/\mathfrak{m}_A, M) = 0$, and

(3) (Infinitesimal Criterion) $M/\mathfrak{m}_A^n M$ is flat over $A/\mathfrak{m}_A^n$ for every $n \geq 1$.

Proof. See [Eis95, Thm. 6.8, Exc. 6.5] or [SP, Tag 00MK]. $\square$

The following consequence of the Local Criterion for Flatness is particularly useful in deformation theory.

Corollary A.2.6. *Let $A \twoheadrightarrow A_0$ be a surjective homomorphism of noetherian rings with kernel I such that $I^2 = 0$. An A-module M is flat over A if and only if*

(1) $M_0 := M \otimes_A A_0$ is flat over A_0, and

(2) the map $M_0 \otimes_{A_0} I \to M$ is injective.

Proof. For ($\Rightarrow$), condition (1) holds by base change and condition (2) holds by tensoring the exact sequence $0 \to I \to A \to A_0 \to 0$ with M and using the identification $M \otimes_A I \cong M_0 \otimes_{A_0} I$. For ($\Leftarrow$), by the Local Criterion for Flatness (A.2.5) it suffices to show that $\mathrm{Tor}_1^A(A/\mathfrak{p}, M) = 0$ for all prime ideals $\mathfrak{p} \subseteq A$. Let $\mathfrak{p}_0 := \mathfrak{p}/I \subseteq A$. Consider the following diagram which is obtained by tensoring the exact sequences $0 \to I \to \mathfrak{p} \to \mathfrak{p}_0 \to 0$ and $0 \to I \to A \to A_0 \to 0$ with M:

$$\begin{tikzcd}
 & & \mathrm{Tor}_1^A(M, A/\mathfrak{p}) \arrow[r] \arrow[d] & \mathrm{Tor}_1^{A_0}(M_0, A_0/\mathfrak{p}_0) \arrow[d] & \\
0 \arrow[r] & M \otimes_A I \arrow[r] \arrow[d, equal] & M \otimes_A \mathfrak{p} \arrow[r] \arrow[d] & M_0 \otimes_{A_0} \mathfrak{p}_0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M \otimes_A I \arrow[r] & M \arrow[r] \arrow[d, two heads] & M_0 \arrow[r] \arrow[d, two heads] & 0 \\
 & & M \otimes_A A/\mathfrak{p} \arrow[r] & M_0 \otimes_{A_0} A_0/\mathfrak{p}_0 &
\end{tikzcd}$$

Condition (2) implies that the second row is exact, and it follows that the first row is also exact, where we have used the identification $M \otimes_A \mathfrak{p}_0 \cong M_0 \otimes_{A_0} \mathfrak{p}_0$. Condition (1) implies that $\mathrm{Tor}_1^{A_0}(M_0, A_0/\mathfrak{p}_0) = 0$ and it follows from the snake lemma that $\mathrm{Tor}_1^A(M, A/\mathfrak{p}) = 0$. See also [Har10, Prop. 2.2]. $\square$

Remark A.2.7. Applying this with $A = \Bbbk[\epsilon]/(\epsilon^2)$ being the dual numbers and $A' = \Bbbk$, we recover the fact that an A-module M is flat if and only if $M \otimes_{\Bbbk[\epsilon]/(\epsilon^2)} \Bbbk \xrightarrow{\epsilon} M$ is injective. This also follows from the fact that a module N over a ring B is flat if and only if for every ideal $I \subseteq B$, the map $I \otimes_B M \to M$ is injective, and using that the only ideal in $\Bbbk[\epsilon]/(\epsilon^2)$ is (ϵ).

The following convenient facts are closely related to the Local Criterion of Flatness (A.2.5).

Lemma A.2.8. *Let $(A, \mathfrak{m}_A) \to (B, \mathfrak{m}_B)$ be a local ring homomorphism of noetherian local rings.*

(1) Let M be a flat A-module and N be a finitely generated B-module. If $\phi\colon N \to M$ is a morphism of R-modules such that $N/\mathfrak{m}N \to M/\mathfrak{m}M$ is injective, then $\phi\colon N \to M$ is injective and $M/\phi(N)$ is flat over A.

(2) If in addition $A \to B$ is flat and $f \in \mathfrak{m}_B$ is a nonzerodivisor in $B \otimes_A A/\mathfrak{m}_A$, then $A \to B/(f)$ is flat.

Proof. The proofs are elementary (see [Mat89, Thm. 22.5] or [SP, Tag 00ME]). Note that Part (2) follows directly from (1). $\square$

Part (2) can be viewed as a 'slicing criterion for flatness' and is often applied inductively to regular sequences. It has the following geometric interpretation.

Corollary A.2.9 (Slicing Criterion for Flatness)**.** *Let $f\colon X \to S$ be a morphism locally of finite presentation, and let $x \in X$ be a point with image $s \in S$. If f is flat at x and the image of $h \in \mathfrak{m}_x \subseteq \mathcal{O}_{X,x}$ in the local ring $\mathcal{O}_{X_s,x}$ of the fiber is a nonzerodivisor, then there exists an open neighborhood $U \subseteq X$ of x such that h extends to a global function on U and the composition $V(h) \hookrightarrow U \to S$ is locally of finite presentation and flat at x.*

Proof. The noetherian case is a direct consequence of Lemma A.2.8(2), and the general case can be reduced to the noetherian case using the limit methods of §B.3. See also [SP, Tag 056X]. ◻

Theorem A.2.10 (Fibral Flatness Criterion). *Consider a commutative diagram*

$$\begin{tikzcd} X \arrow[rr] \arrow[dr] & & Y \arrow[dl] \\ & S & \end{tikzcd}$$

of schemes, and let F be a quasi-coherent $\mathcal{O}_X$-module of finite presentation. Assume that $X \to S$ is locally of finite presentation and $Y \to S$ is locally of finite type. Let $x \in X$ with images $y \in Y$ and $s \in S$. If the stalk $\mathcal{F}_x$ is nonzero, then the following are equivalent:

(1) F is flat over S at x, and $\mathcal{F}_s := \mathcal{F}|_{X_s}$ is flat over Y_s at x, and

(2) Y is flat over S at y and $\mathcal{F}$ is flat over Y at x.

Proof. See [SP, Tag 039A]. ◻

If $A \to B$ is a local ring map of noetherian local rings, then $\dim B = \dim A + \dim B/\mathfrak{m}_A B$. The following is a partial converse.

Theorem A.2.11 (Miracle Flatness). *Let $A \to B$ be a local homomorphism of noetherian local rings. Assume that*

1. *A is regular,*
2. *B is Cohen–Macaulay, and*
3. *$\dim B = \dim A + \dim B/\mathfrak{m}_A B$.*

Then $A \to B$ is flat.

Proof. See [Nag62, Thm. 25.16] or [SP, Tag 00R4]. ◻

A.2.2 Properties of flatness

Proposition A.2.12 (Flat Base Change). *Consider a cartesian diagram*

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"] \arrow[dr, phantom, "\square"] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

of schemes, and let F be a quasi-coherent sheaf on X. If $g\colon Y' \to Y$ is flat and $f\colon X \to Y$ is quasi-compact and quasi-separated, the natural adjunction map

$$g^* \mathrm{R}^i f_* F \to \mathrm{R}^i f'_* g'^* F$$

is an isomorphism for all $i \geq 0$.

Proof. See [Har77, Prop. III.9.3] or [SP, Tag 02KH]. ◻

Theorem A.2.13 (Generic Flatness). *Let $f\colon X \to S$ be a finite type morphism of schemes and $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_X$-module. If S is reduced, there exists a dense open subscheme $U \subseteq S$ such that $X_U \to U$ is flat and of presentation and such that $\mathcal{F}|_{X_U}$ is flat over U and of finite presentation as on $\mathcal{O}_{X_U}$-module.*

Proof. See [SP, Tag 052B]. □

Proposition A.2.14 (Fppf Morphisms are Open)**.** *Let $f\colon X \to Y$ be a morphism of schemes. If f is flat and locally of finite presentation, then for every open subset $U \subseteq X$, the image $f(U) \subseteq Y$ is open.*

Proof. See [SP, Tag 01UA]. □

Proposition A.2.15. *A flat monomorphism locally of finite presentation (e.g., an étale monomorphism) is an open immersion.*

Proof. This is a nice application of descent theory. By the previous result, it suffices to assume that $f\colon X \to Y$ is surjective, hence fppf. Since $X \to Y$ is a monomorphism, the base change $X \times_Y X \to Y$ is an isomorphism. Since being an isomorphism is an Fpqc Local Property on the Target (2.1.26), $X \to Y$ is an isomorphism. □

The following theorem is the an essential ingredient in Projectivity of the Quot Scheme (1.1.3).

Theorem A.2.16 (Existence of Flattening Stratifications)**.** *Let $X \to S$ be a projective morphism of noetherian schemes, $\mathcal{O}_X(1)$ be a relatively ample line bundle, and $\mathcal{F}$ be a coherent sheaf on X. For each polynomial $P \in \mathbb{Q}[z]$, there exists a locally closed subscheme $S_P \hookrightarrow S$ such that a morphism $T \to S$ factors through S_P if and only if the pullback $\mathcal{F}_T$ of $\mathcal{F}$ to X_T is flat over T and for every $t \in T$, the pullback $\mathcal{F}_{\kappa(t)}$ to $X_{\kappa(t)}$ has Hilbert polynomial P.*

Moreover, there exists a finite indexing set I of polynomials such that $S = \coprod_{P \in I} S_P$ set-theoretically. The closure of S_P in S is contained set-theoretically in the union $\bigcup_{P \le Q} S_Q$, where $P \le Q$ if and only if $P(z) \le Q(z)$ for $z \gg 0$.

Proof. See [FGAIV, Lem. 3.3] or [Mum66a, §8]. The central idea is to reduce to $X = S$, in which case the closed loci $\overline{S_P}$ are defined by fitting ideals of $\mathcal{F}$ [SP, Tag 05P9]. □

Remark A.2.17. When $X \to S$ is only proper, there is a *universal flattening*, i.e., an algebraic space S' and a morphism $S' \to S$ such that a map $T \to S$ factors through $S' \to S$ if and only if the pullback $\mathcal{F}|_{X_T}$ to $X_T := X \times_S T$ is flat over T [SP, Tag 05UG]. In general, S' may not be a disjoint union of locally closed subschemes of S; see [Kre13].

Theorem A.2.18 (Raynaud-Gruson Flatification)**.** *Let Y be a quasi-compact and quasi-separated scheme and $X \to Y$ be a finitely presented morphism which is flat over a quasi-compact open subscheme $U \subseteq Y$. Then there is a commutative diagram*

$$\begin{tikzcd} \widetilde{X} \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ Y' \arrow[r, "p"] & Y \end{tikzcd}$$

where $p\colon Y' \to Y$ is a blowup of a finitely presented closed subscheme $Z \subseteq Y$ disjoint from U and the strict transform $\widetilde{X}$ of X is flat over Y'.

The *strict transform* $\widetilde{X}$ above is by definition the closure of $(Y' \smallsetminus p^{-1}(Z)) \times_Y X$ in the base change $Y' \times_Y X$.

Proof. See [GR71, Thm. I.5.2.2] or [SP, Tag 0815]. □

A.2.3 Faithful flatness

A module M over a ring A is *faithfully flat* if the functor $-\otimes_A M\colon \operatorname{Mod}(A) \to \operatorname{Mod}(A)$ is faithfully exact, i.e., a sequence $N' \to N \to N''$ of A-modules is exact if and only if $N' \otimes_A M \to N \otimes_A M \to N'' \otimes_A M$ is exact

for every nonzero map $\phi\colon N \to N'$ of A-modules, the induced map $\phi \otimes_A M\colon N \otimes_A M \to N' \otimes_A M$ is also nonzero.

Proposition A.2.19 (Faithfully Flat Equivalences)**.** *Let A be a ring and M be a* flat *A-module. The following are equivalent:*

(1) M is faithfully flat;

(2) for every nonzero map $\phi\colon N \to N'$ of A-modules, the induced map $\phi \otimes_A M\colon N \otimes_A M \to N' \otimes_A M$ is also nonzero;

(3) for every nonzero A-module N, the tensor product $N \otimes_A M$ is nonzero;

(4) for every prime ideal $\mathfrak{p} \subseteq A$, the tensor product $M \otimes_A \kappa(\mathfrak{p})$ is nonzero; and

(5) for every maximal ideal $\mathfrak{m} \subseteq A$, the tensor product $M \otimes_A \kappa(\mathfrak{m}) \cong M/\mathfrak{m}M$ is nonzero.

Proof. See [SP, Tag 00H9]. □

When $M = B$ is an A-algebra, then by (4) a flat ring map $A \to B$ is faithfully flat if $\operatorname{Spec} B \to \operatorname{Spec} A$ is surjective, or equivalently by (5) every maximal ideal of A is in the image of $\operatorname{Spec} B \to \operatorname{Spec} A$. The latter equivalence implies that any flat local ring map is faithfully flat.

A morphism $f\colon X \to Y$ of schemes is *faithfully flat* if f is flat and surjective. This is equivalent to the condition that $f^*\colon \operatorname{QCoh}(Y) \to \operatorname{QCoh}(X)$ is faithfully exact. It is also equivalent to the condition that a quasi-coherent $\mathcal{O}_Y$-module (resp., a morphism of quasi-coherent $\mathcal{O}_Y$-modules) is zero if and only if its pullback is. Faithfully flat morphisms play an important role in descent theory; see §2.1.

A.2.4 Fppf and fpqc morphisms

Fppf and fpqc morphisms are acronyms for 'fidèlement plate de présentation finie' and 'fidèlement plat et quasi-compact,' respectively. Despite this terminology, an fpqc morphism is more general than a faithfully flat and quasi-compact map.

Definition A.2.20. A morphism $f\colon X \to Y$ of schemes is:

(1) *fppf* if f is faithfully flat and locally of finite presentation, and

(2) *fpqc* if f is faithfully flat and every quasi-compact open subset of Y is the image of a quasi-compact open subset of X.

Remark A.2.21. A quasi-compact and faithfully flat morphism is fpqc. In addition, an open and faithfully flat morphism is fpqc: for a quasi-compact open subset $V \subseteq Y$, we can write $f^{-1}(V) = \bigcup_i U_i$ as a union of affines, and since each $f(U_i) \subseteq V$ is open and V is quasi-compact, we see that V is the image of finitely many of the U_i's. Fppf Morphisms are Open (A.2.14) gives the implication: fppf $\Rightarrow$ fpqc.

An fpqc morphism $f\colon X \to Y$ can be equivalently characterized by either requiring that there exists an affine covering $\{Y_i\}$ of Y such that each Y_i is the image of quasi-compact open subset of X, or by requiring that every point $x \in X$ has an open (resp., quasi-compact open) neighborhood U such that $f(U)$ is open and $U \to f(U)$ is quasi-compact; see [Nit05, Prop. 2.33].

An fppf (resp., fpqc) cover $\{X_i \to X\}$ is a collection of morphisms such that $\coprod_i X_i \to X$ is fppf (resp., fpqc).

A.2.5 Universally injective homomorphisms

The defining characteristic of a flat module is that it preserves every injection under tensoring. The dual notion of an injection of modules, which is preserved under tensoring by every module, is also a very useful property.

Definition A.2.22. A homomorphism $M \to N$ of A-modules is *universally injective* if for every A-module P, the map $M \otimes_A P \to N \otimes_A P$ is injective. A ring homomorphism $A \to B$ is *universally injective* if it is as a map of A-modules.

Remark A.2.23. This should not be confused with a *universally injective* or *radiciel* morphism of schemes $X \to Y$, i.e., an injective map that remains injective after any base change; see [SP, Tag 01S2].

We will use this notion in a fundamental way in our proof of Coherent Tannaka Duality (Theorem 6.6.1). To this end, the following properties will be used:

Proposition A.2.24.

(1) A faithfully flat ring homomorphism $A \to B$ is universally injective.

(2) A split injective $M \to N$ of A-modules is universally injective. The converse is true if N/M is finitely presented.

(3) If $A \to A'$ is faithfully flat, then a map $M \to N$ of A-modules is universally injective if and only if $M \otimes_A A' \to N \otimes_A A'$ is.

(4) If $A \to B$ is universally injective and $B \to B \otimes_A B$, defined by $b \mapsto b \otimes 1$, is faithfully flat, then $A \to B$ is faithfully flat.

Proof. For (1), (2), and (4), see [SP, Tags 08WP, 058L, and 08XD]. Part (3) follows directly from the faithful exactness of $- \otimes_A A'$. See also [Laz69] or [Lam99, §4J]. $\square$

Remarkably universally injective ring maps are precisely those maps that satisfy effective descent for modules; see Remark 2.1.6.

A.3 Étale, smooth, and unramified morphisms

A morphism $f\colon X \to Y$ of schemes is

- *smooth* if f is locally of finite presentation, flat, and for every $y \in Y$, the geometric fiber $X_{\overline{\kappa(y)}} = X \times_Y \operatorname{Spec} \overline{\kappa(y)}$ is regular,
- *étale* if f is smooth of relative dimension 0, i.e., $\dim X_y = 0$ for all $y \in Y$, and
- *unramified* if f is locally of finite type[1] and every geometric fiber is discrete and reduced, i.e., for all $y \in Y$, the $X_y \cong \coprod_i \operatorname{Spec} K_i$ where each K_i is a separable field extension of $\kappa(y)$.

We say that a morphism $f\colon X \to Y$ of schemes is *smooth (resp., étale, unramified) at* $x \in X$ if there exists an open neighborhood $U \subseteq X$ of x such that $f|_U\colon U \to Y$ is smooth (resp., étale, unramified). In §A.3.5, we discuss local complete intersections and syntomic morphisms. There are the following implications:

$$\text{unramified} \iff \text{étale} \Rightarrow \text{smooth} \Rightarrow \text{syntomic} \Rightarrow \text{fppf} \Rightarrow \text{fpqc}.$$

[1]We are following the conventions of [GR71] and [SP] rather than [EGA] as we only require that f is locally of finite type rather than locally of finite presentation.

A.3.1 Equivalences

Smooth, étale, and unramified morphisms have many equivalent characterizations. These equivalences take considerable work to establish, and we recommend [Mil80, Ch. 1] and [Liu02, §4.3] for accessible accounts.

Theorem A.3.1 (Smooth Equivalences)**.** *Let $f\colon X \to Y$ be a locally of finite presentation morphism of schemes (resp., locally noetherian schemes). The following are equivalent:*

(1) f is smooth;

(2) (Differential Criterion) *for every point $x \in X$, f is flat at x and the $\mathcal{O}_{X,x}$-module $\Omega_{X/S,x}$ can be generated by at most $\dim_x X_{f(x)}$ elements (equivalently is free of rank $\dim_x X_{f(x)}$);*

(3) (Infinitesimal Lifting Criterion) *for every surjection $A \to A_0$ of rings with nilpotent kernel (resp., surjection of local artinian rings with $\ker(A \to A_0) \cong A/\mathfrak{m}_A$) and every commutative diagram*

$$\begin{tikzcd} \operatorname{Spec} A_0 \arrow[r] \arrow[d, hook] & X \arrow[d, "f"] \\ \operatorname{Spec} A \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

of solid arrows, there exists a dotted arrow filling in the diagram;

(4) (Jacobian Criterion) *for every point $x \in X$, there exists affine open neighborhoods $\operatorname{Spec} B$ of $f(x)$ and $\operatorname{Spec} A \subseteq f^{-1}(\operatorname{Spec} B)$ of x and an A-algebra isomorphism*

$$B \cong A[x_1, \ldots, x_n]/(h_1, \ldots, h_r)$$

for some $h_1, \ldots, h_r \in A[x_1, \ldots, x_n]$ with $r \le n$ such that the determinant $\det(\frac{\delta h_j}{\delta x_i})_{1 \le i,j \le r} \in B$ of the Jacobi matrix, defined by the partial derivatives with respect to the first r x_i's, is a unit. (The map $\operatorname{Spec} A \to \operatorname{Spec} B$ is called a standard smooth morphism*.).*

If in addition X and Y are locally noetherian and $x \in X$ has image $y \in Y$ with $\kappa(x) = \kappa(y)$, then $f\colon X \to Y$ is smooth at x if and only if

(4) there is an isomorphism $\widehat{\mathcal{O}}_{X,x} \cong \widehat{\mathcal{O}}_{Y,y}[\![x_1, \ldots, x_r]\!]$ of $\widehat{\mathcal{O}}_{Y,y}$-algebras.

If in addition X and Y are smooth over an algebraically closed field $\Bbbk$, then f is smooth at $x \in X(\Bbbk)$ if and only if

(5) the induced map $T_{X,x} \to T_{Y,y}$ on tangent spaces is surjective.

Proof. See [Har77, Exc. II.8.6, Prop. III.10.4], [EGA, 0.22.6.1, IV$_4$.17.5,14], and [SP, Tags 01V9, 02H6, and 02HX]. □

We say that $f\colon X \to Y$ is *smooth of relative dimension n* if f is smooth and every fiber is equidimensional of dimension n, or equivalently if f is fppf, all fibers are equidimensional of dimension n, and $\Omega_{X/S}$ is locally free of rank n. If f is only fppf and $\Omega_{X/S}$ is locally free of dimension d, it is not necessarily true that f is smooth of relative dimension d, e.g., $\operatorname{Spec} \mathbb{F}_p[x] \to \operatorname{Spec} \mathbb{F}_p[x^p]$.

Theorem A.3.2 (Étale Equivalences)**.** *Let $f\colon X \to Y$ be a locally of finite presentation morphism of schemes (resp., locally noetherian schemes). The following are equivalent:*

(1) f is étale;

(2) f is smooth and $\Omega_{X/Y} = 0$;

(3) f is smooth and unramified;

(4) f is flat and unramified;

(5) (Infinitesimal Lifting Criterion) *for every surjection $A \to A_0$ of rings with nilpotent kernel (resp., surjection of local artinian rings with $\ker(A \to A_0) \cong A/\mathfrak{m}_A$) and every commutative diagram*

$$\begin{tikzcd} \operatorname{Spec} A_0 \arrow[r] \arrow[d, hook] & X \arrow[d, "f"] \\ \operatorname{Spec} A \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

of solid arrows, there exists a unique dotted arrow filling in the diagram; and

(6) (Jacobi Criterion) *for every point $x \in X$, there exists affine open neighborhoods $\operatorname{Spec} B$ of $f(x)$ and $\operatorname{Spec} A \subset f^{-1}(\operatorname{Spec} B)$ of x and an A-algebra isomorphism*

$$B \cong A[x_1, \ldots, x_n]/(h_1, \ldots, h_n)$$

for some $h_1, \ldots, h_n \in A[x_1, \ldots, x_n]$ such that the determinant $\det(\frac{\delta h_j}{\delta x_i})_{1 \leq i,j \leq n} \in B$ is a unit. (The map $\operatorname{Spec} A \to \operatorname{Spec} B$ is called a standard étale morphism.*)*

If in addition X and Y are locally noetherian and $x \in X$ has image $y \in Y$ with $\kappa(x) = \kappa(y)$, then $f \colon X \to Y$ is smooth at x if and only if

(6) $\widehat{\mathcal{O}}_{X,x} \cong \widehat{\mathcal{O}}_{Y,y}$.

If in addition X and Y are smooth over an algebraically closed field $\Bbbk$, then f is étale at $x \in X(\Bbbk)$ if and only if

(7) the induced map $T_{X,x} \to T_{Y,y}$ on tangent spaces is an isomorphism.

Proof. See [Har77, Exc. III.10.3], [EGA, IV_4.17.14.1-2, IV_4.17.6.3], and [SP, Tags 02GH and 02HF]. □

Theorem A.3.3 (Unramified Equivalences)**.** *Let $f \colon X \to Y$ be morphism of schemes locally of finite type. The following are equivalent:*

(1) f is unramified;

(2) $\Omega_{X/Y} = 0$;

(3) the diagonal $\Delta_f \colon X \to X \times_Y X$ is an open immersion;

(4) (Infinitesimal Lifting Criterion for Unramifiedness) *for every surjection $A \to A_0$ of rings with nilpotent kernel (resp., surjection of local artinian rings with $\ker(A \to A_0) \cong A/\mathfrak{m}_A$) and every commutative diagram*

$$\begin{tikzcd} \operatorname{Spec} A_0 \arrow[r] \arrow[d, hook] & X \arrow[d, "f"] \\ \operatorname{Spec} A \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

of solid arrows, there exists at most one dotted arrow filling in the diagram.

If in addition X and Y are locally noetherian and $x \in X$ has image $y \in Y$ with $\kappa(x) = \kappa(y)$, then $f \colon X \to Y$ is smooth at x if and only if

(4) $\widehat{\mathcal{O}}_{Y,y} \to \widehat{\mathcal{O}}_{X,x}$ *is surjective.*

Proof. See [EGA, IV$_4$.17.14.1-2, IV$_4$.17.6.3], and [SP, Tags 02G3, 02H7, and 02GE]. $\square$

A.3.2 Étale-local structure of smooth, étale, and unramified morphisms

Every smooth morphism is étale-locally relative affine space.

Proposition A.3.4 (Local Structure of Smooth Morphisms)**.** *A morphism* $X \to Y$ *of schemes is smooth at* $x \in X$ *if and only if there exists affine open subschemes* $\operatorname{Spec} A \subseteq X$ *and* $\operatorname{Spec} B \subseteq Y$ *with* $x \in \operatorname{Spec} A$*, and a commutative diagram*

$$\begin{tikzcd} X \arrow[d] & \operatorname{Spec} A \arrow[l, hook', "\mathrm{op}"'] \arrow[r, "\text{ét}"] \arrow[d] & \mathbb{A}^n_B \arrow[dl] \\ Y & \operatorname{Spec} B \arrow[l, hook', "\mathrm{op}"'] & \end{tikzcd}$$

where $\operatorname{Spec} A \to \mathbb{A}^n_B$ *is étale.*

Proof. See [SP, Tag 039P] and [EGA, IV$_4$.17.11.4]. $\square$

An important consequence is that smooth morphisms have sections étale locally.

Corollary A.3.5. *Let* $f\colon X \to Y$ *be a morphism of schemes smooth at* $x \in X$*. Then there exists an étale neighborhood* $Y' \to Y$ *of* $f(x)$ *such that* $X \times_Y Y' \to Y'$ *has a section.*

Proof. After applying the proposition, observe that the morphism $\mathbb{A}^n_B \to \operatorname{Spec} B$ admits the zero section $\operatorname{Spec} B \to \mathbb{A}^n_B$. The scheme $Y' := \operatorname{Spec} B \times_{\mathbb{A}^n_B} \operatorname{Spec} A$ is étale over Y, and the composition $Y' \to \operatorname{Spec} A \hookrightarrow X$ defines a section $Y' \to X \times_Y Y'$ of $X \times_Y Y' \to Y'$. $\square$

Every étale (resp., unramified) morphism is étale-locally an isomorphism (resp., closed immersion).

Proposition A.3.6. *Let* $f\colon X \to S$ *be a separated morphism of schemes étale (resp., unramified) at* $x \in X$*. Then there exists an étale neighborhood* $(U, u) \to (S, f(x))$ *and a finite disjoint union decomposition*

$$X_U = W \amalg \coprod_i V_i$$

such that each $V_i \to U$ *is an isomorphism (resp., closed immersion) and the fiber* W_u *contains no point over* x*.*

Proof. See [SP, Tags 04HM and 04HG]. $\square$

It is sometimes convenient to know that étale and smooth morphisms of affine schemes can be lifted along closed immersions. It is also holds for syntomic morphisms (see Definition A.3.17).

Proposition A.3.7. *Consider a diagram*

$$\begin{tikzcd}
\operatorname{Spec} A_0 \arrow[r, hook, dashed] \arrow[d] \arrow[dr, phantom, "\square"] & \operatorname{Spec} A \arrow[d, dashed] \\
\operatorname{Spec} B_0 \arrow[r, hook] & \operatorname{Spec} B
\end{tikzcd}$$

of solid arrows where $\operatorname{Spec} B \hookrightarrow \operatorname{Spec} B_0$ *is a closed immersion. If* $\operatorname{Spec} A_0 \to \operatorname{Spec} B_0$ *is étale (resp., smooth, syntomic), then there exists an étale (resp., smooth, syntomic) morphism* $\operatorname{Spec} A \to \operatorname{Spec} B$ *making the above diagram cartesian.*

Proof. See [SP, Tags 04D1 and 07M8]. □

A.3.3 Further properties

Proposition A.3.8 (Fibral Étaleness/Smoothness/Unramifiedness Criteria)**.** *Consider a diagram*

$$\begin{tikzcd}
X \arrow[rr] \arrow[dr] & & Y \arrow[dl] \\
& S &
\end{tikzcd}$$

of schemes where $X \to S$ *and* $Y \to S$ *are locally of finite presentation. Let* $x \in X$ *with image* $s \in S$*. Then*

(1) $X \to Y$ *is unramified at* x *if and only if* $X_s \to Y_s$ *is unramified at* x*, and*

(2) *if* $X \to S$ *is flat at* x*, then* $X \to Y$ *is étale (resp., smooth) at* x *if and only if* $X_s \to Y_s$ *is étale (resp., smooth) at* x*.*

Proof. Let $y \in Y$ be the image of $x \in X$. Part (1) holds since unramified is defined as a condition on the fiber and the fiber X_y is identified with the fiber of $X_s \to Y_s$ over $y \in Y_s$. For the non-trivial direction $(\Leftarrow)$ of (2), the Fibral Flatness Criterion (A.2.10) implies that $X \to Y$ is flat at x. Therefore, the smoothness (resp., étaleness) of $X \to Y$ at x is equivalent to the smoothness (resp., étaleness) of $X_y \to \operatorname{Spec} \kappa(y)$ at x. The latter condition holds by the smoothness (resp., étaleness) of $X_s \to Y_s$ at x. □

While smoothness is clearly an open condition on the *source*, it is also an open condition on the *target* when the morphism is proper.

Corollary A.3.9. *If* $f \colon X \to Y$ *is a proper, flat, and locally of finite presentation morphism, then the set of points* $y \in Y$ *where* $X_y \to \operatorname{Spec} \kappa(y)$ *is smooth is open.*

Proof. If $y \in Y$ is a point such that $X_y \to \operatorname{Spec} \kappa(y)$ is smooth, then $f \colon X \to Y$ is smooth in an open neighborhood of X_y. If $Z \subset X$ is the closed locus where $f \colon X \to Y$ is not smooth, then $f(Z) \subseteq Y$ is precisely the locus where the fibers of f are not smooth. Since f is proper, $f(Z)$ is closed. □

Proposition A.3.10. *Let* $X \to Y$ *be a smooth morphism of noetherian schemes. For every point* $x \in X$ *with image* $y \in Y$*,*

$$\dim_x(X) = \dim_y(Y) + \dim_x(X_y).$$

Proof. See [SP, Tag 0AFF]. □

Proposition A.3.11. *If $X \to Y$ is a finite étale morphism, there exists a finite étale cover $Y' \to Y$ such that $X \times_Y Y' \to Y'$ is a trivial covering, i.e., $X \times_Y Y'$ is isomorphism to $\coprod_i Y'$ over Y'.*

Proof. We may assume that the degree d of $X \to Y$ is constant. The scheme

$$(X/Y)^d = \underbrace{X \times_Y \cdots \times_Y X}_{d}$$

represents the functor $\mathrm{Sch}/Y \to \mathrm{Sets}$ assigning a Y-scheme T to the set of d sections of $X \times_Y T \to T$. Each pairwise diagonal $(X/Y)^{d-1} \to (X/Y)^d$ is an open and closed immersion, and we set $(X/Y)^d_0 \subseteq (X/Y)^d$ to be the complement of all pairwise diagonals. The projection morphism $(X/Y)^d_0 \to Y$ is finite étale and the functorial description gives d disjoint sections of $X \times_Y (X/Y)^d_0 \to (X/Y)^d_0$. □

Proposition A.3.12. *A dominant unramified morphism $X \to Y$ of schemes with Y normal and X connected is étale.*

Proof. See [SGA1, Cor. I.9.11]. □

The following result is often called 'Nagata–Zariski Purity'.

Proposition A.3.13 (Purity of the Branch Locus)**.** *Let $f\colon X \to Y$ be a quasi-finite morphism of integral noetherian schemes such that X is normal and Y is regular. Then the locus of points in X where f is not étale is either empty or codimension 1.*

Proof. See [Zar58], [Nag59], [SGA1, Thm. X.3.1], and [SP, Tag 0BMB]. □

A.3.4 Fitting ideals and the singular locus

Fitting ideals allows for a scheme-theoretic description of the singular locus of a scheme. We use fitting ideals in the Characterization of Nodes (5.2.4). For background references on fitting ideals, we recommend [SP, Tag 07Z6] and [Eis95, §20].

If R is a ring and M is a finitely generated R-module, the *kth fitting ideal* $\mathrm{Fit}_k(M)$ *of M* is the ideal generated by the $(n-k) \times (n-k)$ minors of a matrix A defining a presentation

$$\bigoplus_{i \in I} R \xrightarrow{A} R^n \to M \to 0.$$

Of course, when M is finitely presented (e.g., R is noetherian), then the left-hand term can be assumed to be a finite free module R^m, in which case A is an $m \times n$ matrix and $\mathrm{Fit}_k(M)$ is a finitely generated ideal. The fitting ideal is independent of the choice of presentation, and defines an increasing sequence of ideals

$$0 = \mathrm{Fit}_{-1}(M) \subseteq \mathrm{Fit}_0(M) \subseteq \mathrm{Fit}_1(M) \subseteq \cdots R$$

such that $\mathrm{Fit}_k(M) = R$ if M can be generated by k elements. The R-module M is locally free of rank r if and only if $\mathrm{Fit}_{r-1}(M) = 0$ and $\mathrm{Fit}_r(M) = R$, and in this case $\mathrm{Fit}_k(M) = 0$ for all $k < r$. There is an identification $\mathrm{Fit}_k(M \otimes_R S) = \mathrm{Fit}_k(M)S$ for a ring map $R \to S$. In particular, $\mathrm{Fit}_k(M_f) = \mathrm{Fit}_k(M)_f$ for $f \in R$, $\mathrm{Fit}_k(M_{\mathfrak{p}}) = \mathrm{Fit}_k(M)_{\mathfrak{p}}$ for a prime ideal $\mathfrak{p} \subseteq R$, and $\mathrm{Fit}_k(M) \otimes_R \widehat{R} = \mathrm{Fit}_k(\widehat{M})$ if R is a noetherian local ring.

If X is a scheme and F is a finite type quasi-coherent sheaf on X, the *kth fitting ideal sheaf of F* is the quasi-coherent sheaf of ideals $\mathrm{Fit}_k(F) \subseteq \mathcal{O}_X$ defined by $\Gamma(U, \mathrm{Fit}_k(F)) = \mathrm{Fit}_k(\Gamma(F, U))$ for affine open subsets $U \subseteq X$. Fitting ideal sheaves give a scheme structure to the singular locus.

Definition A.3.14. If X is a noetherian scheme of pure dimension d over a field $\Bbbk$, we define the *singular locus of* X as the subscheme $\operatorname{Sing}(X) := V(\operatorname{Fit}_d(\Omega_{X/\Bbbk}))$ defined by the dth fitting ideal of of $\Omega_{X/\Bbbk}$. More generally, if $X \to S$ is an fppf morphism such that every fiber has pure dimension d, we define the *relative singular locus* as the subscheme $\operatorname{Sing}(X/S) := V(\operatorname{Fit}_d(\Omega_{X/S}))$.

For example, if $X = \operatorname{Spec} \Bbbk[x_1, \dots, x_n]/I$ with $I = (f_1, \dots, f_m)$, the exact sequence $I/I^2 \to \Omega_{\mathbb{A}^n/\Bbbk}|_X \to \Omega_{X/\Bbbk} \to 0$ induces a resolution

$$\mathcal{O}_X^m \xrightarrow{J} \mathcal{O}_X^n \to \Omega_{X/\Bbbk} \to 0 \quad \text{with } J = \left(\frac{\partial f_j}{\partial x_i}\right),$$

and $\operatorname{Sing}(X)$ is defined by all $(n-d) \times (n-d)$ minors of J.

A.3.5 Local complete intersections and syntomic morphisms

Definition A.3.15. A scheme X locally of finite type over a field $\Bbbk$ is a *local complete intersection at* $p \in X$ (or *lci* at p) if there exists an affine open neighborhood $p \in \operatorname{Spec} A \subseteq X$ such that A is a global complete intersection over $\Bbbk$, i.e., $A \cong \Bbbk[x_1, \dots, x_n]/(f_1, \dots, f_c)$ with $\dim A = n - c$. The scheme X is a *local complete intersection* if it is at every point.

Proposition A.3.16. *For a scheme X locally of finite type over a field $\Bbbk$ and a point $p \in X$, the following are equivalent:*

(1) X is a local complete intersection at p,

(2) the local ring $\mathcal{O}_{X,x} \cong R/(f_1, \dots, f_c)$ where R is a regular local ring and $f_1, \dots, f_c \in R$ is a regular sequence, and

(3) the completion $\widehat{\mathcal{O}}_{X,x} \cong R/(f_1, \dots, f_c)$ where R is a regular complete local ring and $f_1, \dots, f_c \in R$ is a regular sequence.

Proof. See [SP, Tags 00S8 and 09PY]. □

For a scheme locally of finite type over a field $\Bbbk$, there are implications:

smooth ⇒ local complete intersection ⇒ Gorenstein ⇒ Cohen–Macaulay.

Here is the relative notion:

Definition A.3.17. A morphism of schemes $f\colon X \to S$ is *syntomic* (or a *flat local complete intersection morphism*) if f is fppf and every fiber is a local complete intersection. We say that $f\colon X \to S$ is *syntomic at* $x \in X$ if there is an open neighborhood U of x such that $f|_U$ is syntomic.

Syntomic morphisms have a local structure analogous to local complete intersections.

Proposition A.3.18. *A morphism $f\colon X \to S$ is syntomic at $x \in X$ if and only if there are affine open neighborhood $x \in \operatorname{Spec} A \subseteq X$ and $\operatorname{Spec} B \subseteq Y$ with $f(\operatorname{Spec} A) \subseteq \operatorname{Spec} B$ such that $A \cong B[x_1, \dots, x_n]/(f_1, \dots, f_c)$ and every nonempty fiber of $\operatorname{Spec} A \to \operatorname{Spec} B$ has dimension $n - c$.*

Proof. See [EGA, IV_4 §19.3], [SGA6, VIII §1] and [SP, Tag 01UB]. □

A.4 Properness and the Valuative Criterion

Each day learn something new, and just as important, relearn something old

ROBERT BRAULT

Properness, separatedness, and universal closedness can be verified using the Valuative Criteria (3.8.7). While the importance of valuative criteria may not be apparent after a first course in algebraic geometry, they becomes indispensable in moduli theory, as it provides a geometric strategy to verify separated, properness, and universal closedness. In this text, we apply the Valuative Criteria to show that $\overline{\mathcal{M}}_g$ is proper (Theorem 5.5.23) and $\mathcal{B}un^{\mathrm{ss}}_{r,d}(C)$ is universally closed.

A.4.1 The Valuative Criteria

As we generalize the criteria to algebraic stacks in Theorem 3.8.7, we quickly recap how the Valuative Criteria (A.4.5) are established for schemes. The starting point of the proof of is the following lifting criterion for quasi-compact morphisms to be closed.

Lemma A.4.1. *A quasi-compact morphism $f\colon X \to Y$ of schemes is closed if and only if for every point $x \in X$, every specialization $f(x) \rightsquigarrow y_0$ in Y lifts to a specialization $x \rightsquigarrow x_0$ in X:*

$$\begin{tikzcd} X \arrow[d, "f"] & x \arrow[r, squiggly, dashed] \arrow[d, mapsto] & x_0 \arrow[d, mapsto, dashed] \\ Y & f(x) \arrow[r, squiggly] & y_0. \end{tikzcd}$$

Proof. The implication ($\Rightarrow$) is clear as $f(\overline{\{x\}}) \subseteq Y$ is closed. For the converse, after replacing X with a closed subscheme, it suffices to show that $f(X)$ is closed. We can assume that $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$ are affine (since f is quasi-compact) and reduced (since the question is topological). The scheme-theoretic image of $\operatorname{Spec} A \to \operatorname{Spec} B$ is defined by $I := \ker(B \to A)$. By replacing B with B/I, we can assume that $B \to A$ is injective. For every minimal prime $\mathfrak{p} \in \operatorname{Spec} B$, the localization $B_{\mathfrak{p}}$ is a field and the map $B_{\mathfrak{p}} \to A_{\mathfrak{p}}$ is injective. Thus, $A_{\mathfrak{p}} \neq 0$ and the fiber $f^{-1}(\mathfrak{p}) = \operatorname{Spec} A_{\mathfrak{p}}$ is non-empty. Since $f(X)$ contains all the minimal primes and is closed under specialization, $f(X) = Y$ is closed. □

The noetherian valuative criterion depends on the following algebraic fact:

Proposition A.4.2. *Let $(A, \mathfrak{m}_A)$ be a noetherian local domain with fraction field K such that A is not a field. If $K \to L$ is a finitely generated field extension, then there exists a DVR R with fraction field L dominating A (i.e., $A \subseteq R$ and $\mathfrak{m}_A \cap K = \mathfrak{m}_R$).*

Proof. We reduce to the case that $K \to L$ is a finite field extension by choosing a transcendence basis $x_1, \ldots, x_n \in L$ over K and replacing A with $A[x_1, \ldots, x_n]_{\mathfrak{n}}$ where $\mathfrak{n} = \mathfrak{m}_A A[x_1, \ldots, x_n] + (x_1, \ldots, x_n)$. Let B be the blow of $\operatorname{Spec} A$ at $\mathfrak{m}_A$ and let $E \subseteq B$ be the exceptional divisor. If $\xi \in E$ is a generic point, then $\mathcal{O}_{B,\xi}$ is a noetherian domain of dimension 1 (by Krull's Hauptidealsatz) with fraction field K. We now let $R \subseteq L$ be the integral closure of $\mathcal{O}_{B,\xi}$ in L. By Krull–Akizuki (A.4.3), R is noetherian. Since R is also normal of dimension 1, it is a DVR. □

Proposition A.4.3 (Krull–Akizuki)**.** *Let R be a noetherian domain of dimension 1 with fraction field K. If $K \to L$ is a finite extension of fields, then every ring A with $R \subseteq A \subseteq L$ is noetherian.*

Proof. See [Nag62, p. 115] or [SP, Tag 00PG]. $\square$

Proposition A.4.2 and Krull–Akizuki have the following geometric implication.

Proposition A.4.4. *If $f\colon X \to Y$ is a finite type morphism of noetherian schemes, $x \in X$, and $f(x) \rightsquigarrow y_0$ is a specialization, there exists a commutative diagram*

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & X \arrow[d, "f"] & x \arrow[d, mapsto] \\ \operatorname{Spec} R \arrow[r] & Y & f(x) \arrow[r, rightsquigarrow] & y_0. \end{tikzcd}$$

where R is a DVR with fraction field K, the image of $\operatorname{Spec} K \to X$ is x and $\operatorname{Spec} R \to Y$ realizes the specialization $f(x) \rightsquigarrow y_0$. In particular, every specialization $x \rightsquigarrow x_0$ in a noetherian scheme is realized by a map $\operatorname{Spec} R \to X$ from a DVR.

Proof. After replacing X with $\overline{\{f(x)\}}$ and Y with $\overline{\{x\}}$, we may assume that X and Y are integral with generic points x and $f(x)$. Then $\mathcal{O}_{Y,y_0}$ is a noetherian local domain with fraction field $\kappa(f(x))$. By applying Proposition A.4.2 to the field extension $\kappa(f(x)) \to \kappa(x)$, we obtain a DVR R with fraction field $\kappa(x)$ dominating $\mathcal{O}_{Y,y_0}$, yielding the desired diagram. $\square$

We only state a noetherian version of the Valuative Criteria.

Theorem A.4.5 (Valuative Criteria for Proper/Separated/Universally Closed Morphisms)**.** *Let $f\colon X \to Y$ be a quasi-compact morphism of noetherian schemes. Consider a commutative diagram*

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \operatorname{Spec} R \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd} \tag{A.4.6}$$

of solid arrows where R is a DVR with fraction field K. Then

(1) f is proper if and only if f is of finite type and for every diagram (A.4.6), there exists a unique lift,

(2) f is separated if and only if for every diagram (A.4.6), any two lifts are equal, and

(3) f is universally closed if and only if for every diagram (A.4.6), there exists a lift.

Proof. We first claim that it suffices to handle the universally closed case. Indeed, a morphism $X \to Y$ is separated if and only if the diagonal $X \to X \times_Y X$ is universally closed, and the equality of two lifts in the valuative criterion for $X \to Y$ corresponds to the existence of a lift in the valuative criterion for $X \to X \times_Y X$.

Suppose that $X \to Y$ satisfies the valuative criterion for universal closedness. To show that $X \to Y$ is universally closed, we claim that it suffices to check that the base change $X_T \to T$ is closed for every *finite type* morphism $T \to Y$. Indeed, suppose that $f_T\colon X_T \to T$ is not closed for some map $T \to Y$. By Lemma A.4.1, there exists

$x \in X_T$ and a specialization $f_T(x) \rightsquigarrow t_0$ which doesn't lift to a specialization $x \rightsquigarrow x_0$. This implies that $Z = \overline{\{x\}} \subseteq X_T$ has trivial intersection with the fiber $(X_T)_{t_0}$. Applying Lemma A.4.8 shows that, after replacing T with an open neighborhood of t_0, there is a commutative diagram

$$\begin{tikzcd} x \arrow[d, mapsto] \\ f_T(x) \arrow[r, rightsquigarrow] & t_0 \end{tikzcd} \qquad \begin{tikzcd} X_T \arrow[r] \arrow[d, "f_T"] & X_{T'} \arrow[r] \arrow[d, "f_{T'}"] & X \arrow[d, "f"] \\ T \arrow[r, "g"] & T' \arrow[r] & Y \end{tikzcd}$$

where $T' \to Y$ is finite type and a closed subscheme $Z' \subseteq X_{T'}$ such that $f_{T'}(Z')$ contains $g(f_T(x))$ but not $g(t_0)$. This shows that $f_{T'} : X_{T'} \to T'$ is not closed.

Since the valuative criterion holds for $X \to Y$, it also holds for the morphism $X_T \to T$ of *noetherian* schemes. It therefore suffices to show that $X \to Y$ is closed. By Lemma A.4.1, it suffices to show that given $x \in X$, every specialization $f(x) \rightsquigarrow y_0$ lifts to a specialization $x \rightsquigarrow x_0$. By Proposition A.4.4, there exists a diagram (A.4.6) such that $\operatorname{Spec} R \to Y$ realizes $f(x) \rightsquigarrow y_0$ with a lift $\operatorname{Spec} K \to X$ whose image is x. The valuative criterion implies the existence of a lift $\operatorname{Spec} R \to X$, which in turn yields a specialization $x \rightsquigarrow x_0$ lifting $f(x) \rightsquigarrow y_0$.

Conversely, assume that $f : X \to Y$ is universally closed and that we are given a diagram (A.4.6). By replacing Y with $\operatorname{Spec} R$ and X with $X \times_Y \operatorname{Spec} R$, we may assume that $Y = \operatorname{Spec} R$ and that we have a diagram

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r, "x"] \arrow[d] & X \arrow[dl] \\ \operatorname{Spec} R \end{tikzcd}$$

By replacing X with $\overline{\{x\}}$, we may assume that X is integral with generic point x. Since $X \to \operatorname{Spec} R$ is closed, there exists a specialization $x \rightsquigarrow x_0$ in X over the specialization of the generic point to the closed point in $\operatorname{Spec} R$. This gives an inclusion of local rings $R \hookrightarrow \mathcal{O}_{X,x_0}$ in K. Since R is a valuation ring with fraction field K (i.e., is maximal among local rings properly contained in K), we see that $R = \mathcal{O}_{X,x_0}$ and the inclusion $\operatorname{Spec} \mathcal{O}_{X,x_0} \to X$ gives the desired lift.

See also [Har77, Thm. 4.7, Exc. II.4.11], [EGA, §II.7], and [SP, Tags 0BX4 and 0CM1] □

Remark A.4.7. The quasi-compactness of f (resp., Δ_f, Δ_{Δ_f}) are essential in the valuative criterion of universal closedness (resp., separatedness, separated diagonal). In fact, universally closed morphisms are necessarily quasi-compact [SP, Tag 04XU].

The lemma below was used in the proof and is also used in the proof of the Valuative Criteria (3.8.7) for algebraic stacks.

Lemma A.4.8. *Let $f : X \to Y$ be a quasi-compact morphism of schemes. Let $T \to Y$ be a morphism of schemes, $t_0 \in T$ be a point, and $Z \subseteq X_T$ a closed subscheme such that $Z \cap (X_T)_{t_0} = \emptyset$. Then after replacing T with an open neighborhood of t_0, there exists a finite type morphism $T' \to Y$ of schemes with a factorization $T \xrightarrow{g} T' \to Y$ and a closed subscheme $Z' \subseteq X_{T'}$ such that $Z' \cap (X_{T'})_{g(t_0)} = \emptyset$ and $\operatorname{im}(Z \hookrightarrow X_T \to X_{T'}) \subseteq Z'$.*

Proof. After reducing to the affine case $T = \operatorname{Spec} B \to Y = \operatorname{Spec} A$, we write B as a colimit of finite type A-algebras B_λ. Using techniques analogous to Limits of

Schemes (§B.3), one shows that for $\lambda \gg 0$ there exists a subscheme $Z_\lambda \subseteq X_{B_\lambda}$ with the desired properties. The details are not hard, but also not inspiring. See [SP, Tag 05BD]. $\square$

A.4.2 Universally submersive morphisms

A morphism $f\colon X \to Y$ of schemes is *submersive* if f is surjective and Y has the quotient topology, i.e., a subset $U \subseteq Y$ is open if and only if $f^{-1}(U)$ is open, and $f\colon X \to Y$ is *universally submersive* if for every map $Y' \to Y$, the base change $X \times_Y Y' \to Y'$ is submersive.

Exercise A.4.9.

(1) Show that a morphism $f\colon X \to Y$ of noetherian schemes is universally submersive if and only if every map $\operatorname{Spec} R \to Y$ from a DVR has a lift

$$\begin{tikzcd} \operatorname{Spec} R' \arrow[r, dashed] \arrow[d, dashed] & X \arrow[d, "f"] \\ \operatorname{Spec} R \arrow[r] & Y, \end{tikzcd}$$

where $R \to R'$ is a local homomorphism of DVRs.

(2) Show that universally closed morphism of noetherian schemes is universally submersive.

(3) Show that every fpqc morphism of schemes is universally submersive.

A.5 Dévissage and finiteness of cohomology

MacPherson told me that my theorem can be viewed as blah blah blah Grothendieck blah blah blah, which makes it much more respectable.

Jim Propp

Dévissage, or 'unscrewing' in French, is a specific type of noetherian induction designed to verify properties of coherent sheaves. We apply it to extend the theorem of Finiteness of Cohomology (A.5.3) from projective morphisms to proper morphisms.

A.5.1 Dévissage

Proposition A.5.1 (Dévissage)**.** *Let X be a noetherian scheme. Let $\mathcal{P}$ be a property of coherent sheaves on X satisfying*

(a) if $0 \to F' \to F \to F'' \to 0$ is a short exact sequence of coherent sheaves on X and two out of the three satisfy $\mathcal{P}$, then the third satisfies $\mathcal{P}$, and

(b) for every integral closed subscheme $Z \subseteq X$ with generic point ξ, there exists a coherent sheaf G satisfying $\mathcal{P}$ with $\mathfrak{m}_\xi G_\xi = 0$ and $\dim_{\kappa(\xi)} G_\xi = 1$ (so that $\operatorname{Supp}(G) = Z$).

Then every coherent sheaf on X satisfies $\mathcal{P}$.

Proof. We say that the property $\mathcal{P}_Y$ holds for a closed subset $Y \subseteq X$ if every coherent sheaf F on X with $\operatorname{Supp}(F) \subseteq Y$ satisfies $\mathcal{P}$. We prove the proposition by noetherian induction: we need to show that if $\mathcal{P}_{Y'}$ holds for every closed subset

$Y' \subsetneq Y$, then $\mathcal{P}_Y$ also holds. Let F be a coherent sheaf on X with $\operatorname{Supp}(F) \subseteq Y$. Giving $Y \subseteq X$ the reduced scheme structure, let $I \subseteq \mathcal{O}_X$ be its ideal sheaf. To show that F satisfies $\mathcal{P}$, we first claim that it suffices to assume that $IF = 0$. Since $I^n F = 0$ for some $n > 0$, we have a filtration $0 = I^n F \subseteq I^{n-1} F \subseteq \cdots \subseteq F$ and short exact sequences

$$0 \to I^{k-1}F/I^kF \to F/I^kF \to F/I^{k-1}F \to 0.$$

By induction and property (a), it suffices to show that $\mathcal{P}$ holds for $I^{k-1}F/I^kF$, which is annihilated by I. Second, we claim that we can assume that Y is irreducible, hence integral. Supposing that $Y = Y_1 \cup Y_2$ with $F_1 = F|_{Y_1}$ and $F_2 = F|_{Y_2}$, the map $\phi\colon F \to F_1 \oplus F_2$ has kernel and cokernel supported on $Y_1 \cap Y_2$. Applying property (a) to the exact sequence $0 \to \operatorname{im}\phi \to F_1 \oplus F_2 \to \operatorname{coker}\phi \to 0$ and then to $0 \to \ker\phi \to F \to \operatorname{im}\phi \to 0$, shows that $\mathcal{P}$ also holds for F. Finally, letting $\xi \in Y$ be the generic point, property (b) gives a coherent sheaf G satisfying $\mathcal{P}$ with $\operatorname{Supp}(G) \subseteq Y$ and $\dim_{\kappa(\xi)} G_\xi = 1$. Setting $d = \dim_{\kappa(\xi)} F_\xi$, since F_ξ and $G_\xi^{\oplus d}$ are isomorphic, there is an open subscheme $U \subseteq Y$ and an isomorphism $F|_U \to G^{\oplus d}|_U$. Let H be the image of the graph $F|_U \to F|_U \oplus G^{\oplus d}|_U$, and $\widetilde{H} \subseteq F \oplus G^{\oplus d}$ be a subsheaf on Y extending H [Har77, Exc. II.5.15]. The projections $H \to G^{\oplus d}$ and $H \to F$ induce isomorphisms over U and hence have kernels and cokernels supported on a closed subscheme $Y' \subsetneq Y$. Since $\mathcal{P}$ holds for G, $\mathcal{P}$ also holds for $G^{\oplus d}$ by property (a). It follows that $\mathcal{P}$ holds for H and thus also F. See also [EGA, III_1.3.1.2] and [SP, Tag 01YI]. ☐

Remark A.5.2. The same proof also establishes some useful variants. First, if we assumed that $\mathcal{F}^{\oplus n} \in \mathcal{P}$ implies that $\mathcal{F} \in \mathcal{P}$, then condition (b) can be weakened to exhibiting a coherent sheaf G satisfying $\mathcal{P}$ with $\operatorname{Supp}(G) \subseteq Z$ and $G_\xi \neq 0$. Alternatively, (b) can be replaced with the condition that for every integral closed subscheme $Z \subseteq X$ with ideal sheaf I_Z and every coherent sheaf F on X with $I_Z F = 0$, there exists a coherent sheaf G on X with $I_Z G = 0$ satisfying $\mathcal{P}$ and a morphism $F \to G$ which is an isomorphism on a non-empty open subset of Z.

A.5.2 Dévissage

Theorem A.5.3 (Finiteness of Cohomology)**.** *Let $f\colon X \to Y$ be a proper morphism of noetherian schemes. For any coherent sheaf F on X and any $i \geq 0$, $\mathrm{R}^i f_* F$ is coherent.*

Proof. By Flat Base Change (A.2.12), we can assume that $Y = \operatorname{Spec} A$ is the spectrum of a noetherian ring. We need to show that for every coherent sheaf F on X and any $i \geq 0$, $\mathrm{H}^i(X, F)$ is a finite A-module. When $X \to \operatorname{Spec} A$ is projective, this is [Har77, Thm. III.5.2]—this is a really nice argument exhibiting the power of cohomology, even if you are only interested in the H^0 case. In the proof, one quickly reduces to the case that $X = \mathbb{P}^n_A$. Choosing an exact sequence $0 \to K \to \mathcal{O}_{\mathbb{P}^n_A}(-m)^{\oplus d} \to F \to 0$, the statement holds for the middle term by a Céch cohomology computation, and the vanishing of cohomology in sufficiently high degree and a descending induction argument shows that it holds for F.

To apply Dévissage (A.5.1), we let $\mathcal{P}$ be the property of a coherent sheaf F on X that $\mathrm{H}^i(X, F)$ is a finite A-module for all i. This satisfies the two-out-of-three condition (a) of (A.5.1). To see that (b) holds, let $Z \subseteq X$ be an integral closed subscheme with generic point ξ. By Chow's Lemma [Har77, Exc. II.4.10], there exists a projective birational morphism $g\colon Z' \to Z$ such that $Z' \to Z \to Y$ is also projective. The ample sheaf $\mathcal{O}_{Z'}(1)$ is relatively ample over Z and thus for $d \gg 0$ and $i > 0$, $\mathrm{R}^i g_*(\mathcal{O}_{Z'}(d)) = 0$ and $\mathrm{H}^i(Z', \mathcal{O}_{Z'}(d)) = 0$. Taking $G := g_*\mathcal{O}_{Z'}(d)$ for $d \gg 0$, $\dim_{\kappa(\xi)} G_\xi = 1$. Using the vanishing of $\mathrm{R}^i g_*(\mathcal{O}_{Z'}(d))$, the Leray spectral sequence $\mathrm{H}^p(Z, \mathrm{R}^q g_*\mathcal{O}_{Z'}(d)) \Rightarrow \mathrm{H}^{p+q}(Z', \mathcal{O}_{Z'}(d))$ implies that $\mathrm{H}^i(Z, G) = \mathrm{H}^i(Z', \mathcal{O}_{Z'}(d))$ is a finite A-module (and in fact 0 for $i > 0$). See also [EGA, III_1.3.2.1] and [SP, Tag 02O5].

This argument can also be formulated using derived categories. Chow's Lemma gives a projective birational morphism $g \colon X' \to X$ with X' projective over $Y = \operatorname{Spec} A$, and consider the exact triangle $F \to \mathrm{R}g_*g^*F \to C$. Since g is projective, $\mathrm{R}g_*g^*F \in D^b_{\mathrm{Coh}}(X)$ and thus $C \in D^b_{\mathrm{Coh}}(X)$. Since g is birational $F \to \mathrm{R}g_*g^*F$ is an isomorphism over a dense open. Using the exact triangles $C^n \to C \to \tau_{>n}C$ arising from truncation and the fact that $C^n = 0$ for $n \ll 0$, an induction argument shows that $\mathrm{R}\Gamma(X, C) \in D^b_{\mathrm{Coh}}(Y)$. Since $\mathrm{R}\Gamma(X, g_*g^*F) \in D^b_{\mathrm{Coh}}(Y)$, we conclude that $\mathrm{R}\Gamma(X, F) \in D^b_{\mathrm{Coh}}(Y)$. Formalizing this argument leads to version of dévissage for derived categories (e.g., [LMB00, Lem. 15.7]). □

The following version of Formal Functions is often applied over a complete *local* ring $(A, \mathfrak{m})$, but the non-local case is sometimes useful.

Theorem A.5.4 (Formal Functions)**.** *Let X be a scheme proper over a noetherian ring A which is complete with respect to an ideal $I \subseteq A$. Let $X_n = X \times_A A/I^{n+1}$. If F is a coherent sheaf on X, there is a natural isomorphism*

$$\mathrm{H}^i(X, F) \xrightarrow{\sim} \varprojlim_n \mathrm{H}^i(X_n, F|_{X_n})$$

for every $i \geq 0$.

Proof. See [Har77, Thm. III.11.1] (projective over complete local), [Vak17, Thm. 30.8.1], [Ill05, Cor. 8.2.4], [EGA, III_1.4.1.7], and [SP, Tag 02OC]. □

Exercise A.5.5. Show more generally that for coherent sheaves F and G on X, there is a natural isomorphism

$$\mathrm{Ext}^i_{\mathcal{O}_X}(F, G) \xrightarrow{\sim} \varprojlim_n \mathrm{Ext}^i_{\mathcal{O}_{X_n}}(F|_{X_n}, G|_{X_n})$$

for every $i \geq 0$.

Exercise A.5.6. Use dévissage to reduce the proper case of Formal Functions (A.5.4) to the projective case.

Hint: Reduce to showing that there is a universal d such that $\mathrm{R}^i g_*\mathcal{O}_{Z'_n}(d) = 0$ for all $n, i \geq 0$, where Z' is as in the proof of Finiteness of Cohomology (A.5.3) and $Z'_n = Z' \times_A A/\mathfrak{m}^{n+1}$. To show this, apply Serre's Vanishing Theorem [Har77, Thm. II.5.2] to the projective morphism $\mathcal{S}\mathrm{pec}_{Z'} \bigoplus_{i \geq 0} \mathfrak{m}^i\mathcal{O}_{Z'} \to \operatorname{Spec} \bigoplus_{i \geq 0} \mathfrak{m}^i$.

A.6 Cohomology and Base Change

There is hardly any theory which is more elementary [than linear algebra], in spite of the fact that generations of professors and textbook writers have obscured its simplicity by preposterous calculations with matrices.

Jean Dieudonné

If $f\colon X \to Y$ is a proper morphism of noetherian schemes and F is a coherent sheaf on X, then Finiteness of Cohomology (A.5.3) implies that $\mathrm{R}^i f_* F$ is coherent. We often want to know more:

(a) When is $\mathrm{R}^i f_* F$ a vector bundle on Y?

(b) When does the construction of $\mathrm{R}^i f_* F$ commute with base change, i.e., for a map $g\colon Y' \to Y$ of schemes inducing a cartesian diagram

$$\begin{tikzcd} X_{Y'} \arrow[r, "g'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y, \end{tikzcd}$$

when is the comparison map

$$\phi^i_{Y'}\colon g^* \mathrm{R}^i f_* F \to \mathrm{R}^i f'_* g'^* F \tag{A.6.1}$$

an isomorphism?

When $f\colon X \to Y$ is flat, Flat Base Change (A.2.12) tells us that (A.6.1) is always an isomorphism. Cohomology and Base Change (A.6.8) provides an answer when F is flat over Y.

Cohomology and Base Change is an essential tool in moduli theory. It can be applied to verify properties of families of objects and construct vector bundles on moduli spaces. For instance, for a family $\pi\colon \mathcal{C} \to S$ of smooth curves, we can verify that $\pi_*(\Omega^{\otimes k}_{\mathcal{C}/S})$ is a vector bundle for $k > 0$ whose construction commutes with base change on S, and show that $\mathcal{C}$ embeds canonically into $\mathbb{P}(\pi_*(\Omega^{\otimes k}_{\mathcal{C}/S}))$ for $k \geq 3$ (Proposition 5.1.16). We apply Cohomology and Base Change in our study of $\mathcal{M}_g$. It is used to verify that it is an algebraic stack (Theorem 3.1.17) and to establish various geometric properties. Applied to the universal family $\pi\colon \mathcal{U}_g \to \mathcal{M}_g$, Cohomology and Base Change shows that $\pi_*(\Omega_{\mathcal{U}_g/\mathcal{M}_g})$ is a vector bundle of rank g on $\mathcal{M}_g$, called the *Hodge bundle* (Example 4.1.4).

A.6.1 Formulations of Cohomology and Base Change

We begin with the key algebraic version of Cohomology and Base Change, which is used to establish the other versions.

Theorem A.6.2 (Cohomology and Base Change I)**.** *Let $X \to \operatorname{Spec} A$ be a proper morphism of noetherian schemes and F be a coherent sheaf on X which is flat over A. There is a complex*

$$K^\bullet\colon 0 \to K^0 \to K^1 \to \cdots \to K^n \to 0$$

of finite and locally free A-modules such that $\mathrm{H}^i(X, F) = \mathrm{H}^i(K^\bullet)$ for all i. Moreover, for every A-module M, $\mathrm{H}^i(X, F \otimes_A M) = H^i(K^\bullet \otimes_A M)$. In particular, for a morphism $\operatorname{Spec} B \to \operatorname{Spec} A$ of schemes, $\mathrm{H}^i(X_B, F_B) = \mathrm{H}^i(K^\bullet \otimes_A B)$ where $X_B := X \times_{\operatorname{Spec} A} \operatorname{Spec} B$ and F_B is the pullback of F to X_B.

Proof. This is established by choosing a finite affine cover $\{U_i\}$ of X and considering the corresponding alternating Čech complex $C^\bullet$ on $\{U_i\}$ with coefficients in F. Then $C^\bullet$ is a finite complex of flat (but not finitely generated) A-modules and $\mathrm{H}^i(X, F) = \mathrm{H}^i(C^\bullet)$. The result is then obtained by inductively refining $C^\bullet$ to build a finite complex $K^\bullet$ of finite and flat A-modules which is quasi-isomorphic to $C^\bullet$. See [Mum70a, Thm. p.46], [SP, Tag 07VK], and [Vak17, 28.2.1]. □

Remark A.6.3 (Perfect complexes). A bounded complex $K^\bullet$ of coherent sheaves on a noetherian scheme X is *perfect* if there is an affine cover $X = \bigcup_i U_i$ such that each $K^\bullet|_{U_i}$ is quasi-isomorphic to a bounded complex of vector bundles on U_i. If X is affine (or more generally has the resolution property, i.e., every coherent sheaf is the quotient of a vector bundle), then $K^\bullet$ is perfect if and only if it is quasi-isomorphic to a bounded complex of vector bundles on X [SP, Tag 0F8F]. Moreover, the compact objects in $D_{\mathrm{QCoh}}(X)$ are precisely the perfect complexes [SP, Tag 09M8].

With this terminology in place, Theorem A.6.2 has the following translation: $\mathrm{R}f_*F \in D^b_{\mathrm{Coh}}(\operatorname{Spec} A)$ is perfect. More generally, if $F^\bullet$ is a perfect complex on X, then $\mathrm{R}f_*F^\bullet$ is also perfect [SP, Tag 0A1H].

Theorem A.6.2 tells us that the cohomology $\mathrm{H}^i(X, F)$ can be computed as the cohomology of a bounded complex $K^\bullet$ of vector bundles on $\operatorname{Spec} A$, and thus reduces cohomological questions to linear algebra.

Theorem A.6.4 (Semicontinuity Theorem)**.** *Let $X \to Y$ be a proper morphism of noetherian schemes and F be a coherent sheaf on X which is flat over Y.*

(1) For each $i \geq 0$, the function

$$Y \to \mathbb{Z}, \quad y \mapsto \mathrm{h}^i(X_y, F_y)$$

is upper semicontinuous.

(2) The function

$$Y \to \mathbb{Z}, \quad y \mapsto \chi(X_y, F_y) = \sum_{i=0}^{\infty} (-1)^i \mathrm{h}^i(X_y, F_y)$$

is locally constant.

Proof. We may assume that $Y = \operatorname{Spec} A$ so that Theorem A.6.2 applies: there is bounded complex $K^\bullet\colon \cdots \to K^{i-1} \xrightarrow{d^{i-1}} K^i \xrightarrow{d^i} K^{i+1} \to \cdots$ of finite and locally free A-modules such that $\mathrm{H}^i(X_y, F_y) = H^i(K^\bullet \otimes_A \kappa(y))$ for all $y \in Y$. Using that $\operatorname{im}(d^i \otimes \kappa(y)) = (K^i \otimes_A \kappa(y)) / \ker(d^i \otimes \kappa(y))$, we have

$$\begin{aligned} \mathrm{h}^i(X_y, F_y) &= \dim_{\kappa(y)} \ker(d^i \otimes \kappa(y)) - \dim_{\kappa(y)} \operatorname{im}(d^{i-1} \otimes \kappa(y)) \\ &= \dim_{\kappa(y)} K^i \otimes \kappa(y) - \dim_{\kappa(y)} \operatorname{im}(d^i \otimes \kappa(y)) - \dim_{\kappa(y)} \operatorname{im}(d^{i-1} \otimes \kappa(y)). \end{aligned} \tag{A.6.5}$$

The statement follows as both $\dim_{\kappa(y)} \operatorname{im}(d^i \otimes \kappa(y))$ and $\dim_{\kappa(y)} \operatorname{im}(d^i \otimes \kappa(y))$ are lower semicontinuous. See also [Mum70a, p. 47], [Har77, Thm. 12.8], or [Vak17, Thm. 25.1.1]. $\square$

To show more powerful results, we will need more sophisticated linear algebra. We follow an argument of Eric Larson, as also described in [Vak17, §25.2].

Proposition A.6.6. *Let X be a scheme and $\phi\colon E \to F$ be a map of vector bundles on X of rank e and f. For every point $x \in X$ and an integer $r \leq \min(e, f)$, the following are equivalent:*

(1) $\operatorname{coker}(\phi)$ is a vector bundle of rank $f - r$ in an open neighborhood of x,

(2) there is an open neighborhood U of x and identifications $E|_U \cong \mathcal{O}_U^{\oplus e}$ and $F|_U \cong \mathcal{O}_U^{\oplus f}$ such that $\phi|_U$ corresponds to the composition of the projection $\mathcal{O}_U^{\oplus e} \twoheadrightarrow \mathcal{O}_U^{\oplus r}$ and inclusion $\mathcal{O}_U^{\oplus r} \hookrightarrow \mathcal{O}_U^{\oplus f}$ of the first r summands, and

(3) $\ker(\phi) \otimes \kappa(x) \to \ker(\phi \otimes \kappa(x))$ is surjective.

If in addition X is reduced, the above are equivalent to

(4) *there is an open neighborhood U of x such that $\phi \otimes \kappa(u)$ has rank r for all $u \in U$.*

If these hold, then $\operatorname{coker}(\phi)$*,* $\ker(\phi)$*, and* $\operatorname{im}(\phi)$ *are vector bundles, and their construction commutes with base change.*

Proof. This is an exercise in linear algebra. □

If the equivalent conditions above hold, we say that $\phi\colon E \to F$ is a *map of vector bundles (of rank r)*. We can now provide a quick proof of Grauert's Theorem.

Theorem A.6.7 (Grauert's Theorem). *Let $f\colon X \to Y$ be a proper morphism of noetherian schemes such that Y is reduced and connected. Let F be a coherent sheaf on X flat over Y. For each integer i, the following are equivalent:*

(1) *the function $y \mapsto \mathrm{h}^i(X_y, F_y)$ is constant; and*

(2) $\mathrm{R}^i f_* F$ *is a vector bundle and the comparison map*

$$\phi^i_y\colon \mathrm{R}^i f_* F \otimes \kappa(y) \to \mathrm{H}^i(X_y, F_y)$$

is an isomorphism for all $y \in Y$.

If these hold, then the construction of $\mathrm{R}^i f_* F$ commutes with base change by an arbitrary map $T \to Y$.

Proof. The direction (1) ⇒ (2) is clear. For the converse, we can reduce to the case that $Y = \operatorname{Spec} A$ as the question is Zariski local. Let

$$K^\bullet\colon \cdots \to K^{i-1} \xrightarrow{d^{i-1}} K^i \xrightarrow{d^i} K^{i+1} \to \cdots$$

be the complex of vector bundles on Y produced by Theorem A.6.2. As $y \mapsto \mathrm{h}^i(X_y, F_y)$ is constant, the identity (A.6.5) implies that $y \mapsto \dim_{\kappa(y)} \operatorname{im}(d^{i-1} \otimes \kappa(y))$ and $y \mapsto \dim_{\kappa(y)} \operatorname{im}(d^i \otimes \kappa(y))$ are also constant. As Y is reduced, Proposition A.6.6 implies that d^{i-1} and d^i are maps of vector bundles, that $\operatorname{im}(d^i)$ and $\operatorname{coker}(d^{i-1})$ are vector bundles, and that $\ker(d^i)$ commutes with base change. The cohomology $\mathrm{H}^i(K^\bullet) = \operatorname{im}(d^{i-1})/\ker(d^i)$ sits in a short exact sequence

$$0 \to \mathrm{H}^i(K^\bullet) \to \underbrace{K^i/\ker(d^i)}_{\operatorname{im}(d^i)} \to \operatorname{coker}(d^{i-1}) \to 0,$$

and thus $\mathrm{H}^i(K^\bullet)$ is also a vector bundle. As cokernels always compute with base change, so does $\mathrm{H}^i(K^\bullet) = \operatorname{coker}(K^{i-1} \to \ker(d^i))$. See also [Mum70a, Cor. 2, p.48], [Har77, Cor. 12.9], or [Vak17, 28.1.5]. □

The reducedness hypothesis in Grauert's Theorem is quite restrictive in applications to moduli theory, where we often need to establish properties of families of objects over an arbitrary base. Fortunately, with a little more linear algebra, we can establish the following criterion which holds over any base.

Theorem A.6.8 (Cohomology and Base Change II). *Let $f\colon X \to Y$ be a proper and finitely presented morphism of schemes, and let F be a finitely presented quasi-coherent sheaf on X flat over Y. Suppose that for a point $y \in Y$ and integer i, the comparison map $\phi^i_y\colon \mathrm{R}^i f_* F \otimes \kappa(y) \to \mathrm{H}^i(X_y, F_y)$ is surjective. Then the following hold:*

(a) *There is an open neighborhood $V \subseteq Y$ of y such that for every morphism $Y' \to V$ of schemes, the comparison map $\phi^i_{Y'} \colon g^*\mathrm{R}^i f_* F \to \mathrm{R}^i f'_* g'^* F$ is an isomorphism. In particular, ϕ^i_y is an isomorphism.*

(b) *ϕ^{i-1}_y is surjective if and only if $\mathrm{R}^i f_* F$ is a vector bundle in an open neighborhood of y.*

Proof. Assuming that Y is noetherian, we reduce to the case that $Y = \operatorname{Spec} A$ is affine. Theorem A.6.2 constructs a complex $K^\bullet$ such that for each $y \in Y$, there is a morphism of complexes

$$\begin{tikzcd} K^{i-1} \arrow[r, "d^{i-1}"] \arrow[d] & K^i \arrow[r, "d^i"] \arrow[d] & K^{i+1} \arrow[d] & \text{computing } \mathrm{H}^i(X,F) \\ K^{i-1} \otimes_A \kappa(y) \arrow[r, "d^{i-1}\otimes\kappa(y)"] & K^i \otimes_A \kappa(y) \arrow[r, "d^i\otimes\kappa(y)"] & K^{i+1} \otimes_A \kappa(y) & \text{computing } \mathrm{H}^i(X_y,F_y) \end{tikzcd}$$

The following claim will be used twice in the proof: the surjectivity of $\mathrm{H}^i(X,F) \to \mathrm{H}^i(X_y,F_y)$ is equivalent to the surjectivity of $\ker(d^i) \to \ker(d^i \otimes \kappa(y))$. Since $K^{i-1} \to K^{i-1} \otimes_A \kappa(y)$ is surjective, so is $\operatorname{im}(d^{i-1}) \to \operatorname{im}(d^{i-1} \otimes \kappa(y))$. The snake lemma applied to

$$\begin{tikzcd} 0 \arrow[r] & \operatorname{im}(d^{i-1}) \arrow[r] \arrow[d, two heads] & \ker(d^i) \arrow[r] \arrow[d] & \mathrm{H}^i(X,F) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \operatorname{im}(d^{i-1} \otimes \kappa(y)) \arrow[r] & \ker(d^i \otimes \kappa(y)) \arrow[r] & \mathrm{H}^i(X_y,F_y) \arrow[r] & 0 \end{tikzcd}$$

establishes the claim.

For (a), our hypothesis is that $\mathrm{H}^i(X,F) \to \mathrm{H}^i(X_y,F_y)$ is surjective, which by the claim implies that $\ker(d^i) \to \ker(d^i \otimes \kappa(y))$ is surjective. Proposition A.6.6 implies that $d^i \colon K^i \to K^{i+1}$ is a map of vector bundles, and that after replacing Y with an open neighborhood of y, $\ker(d^i)$ is a vector bundle whose construction commutes with base change. Thus $\mathrm{H}^i(X,F) = \operatorname{coker}(K^{i-1} \to \ker(d^i))$ also commutes with base change. For (b), we use the equivalences:

$$\begin{aligned} \mathrm{H}^i(X,F) \text{ is a vector bundle} &\overset{A.6.6}{\iff} K^{i-1} \to \ker(d^i) \text{ is a map of vector bundles} \\ &\iff K^{i-1} \to K^i \text{ is a map of vector bundles} \\ &\overset{A.6.6}{\iff} \ker(d^{i-1}) \otimes \kappa(y) \to \ker(d^{i-1} \otimes \kappa(y)) \text{ is surjective} \\ &\overset{\text{claim}}{\iff} \mathrm{H}^{i-1}(X,F) \to \mathrm{H}^{i-1}(X_y,F_y) \text{ is surjective.} \end{aligned}$$

The first equivalence follows from Proposition A.6.6 as $\mathrm{H}^i(X,F)$ is the cokernel of $K^{i-1} \to \ker(d^i)$, the second follows from the observation that since d^i is a map of vector bundles, $\ker(d^i)$ and $\operatorname{im}(d^i)$ are vector bundles, and the map $\ker(d^i) \to K^i$ (whose cokernel is $\operatorname{im}(d^i)$) is also a map of vector bundles, the third also follows from Proposition A.6.6, and the fourth is the claim at the beginning of the proof.

Using the methods of Limits of Schemes (§B.3), it is not hard to see how the general statement follows from the noetherian version. Assuming Y is affine, write $Y = \lim_{\lambda \in \Lambda} Y_\lambda$ as a limit of affine schemes of finite type over $\mathbb{Z}$. Since $X \to Y$ is finitely presented, there exists an index $0 \in \Lambda$ and a finitely presented morphism $X_0 \to Y_0$ such that $X \cong X_0 \times_{Y_0} Y$ (Proposition B.3.3). For each $\lambda > 0$, we can define $X_\lambda = X_0 \times_{Y_0} Y_\lambda$ and we have $X \cong X_\lambda \times_{Y_\lambda} Y$. By Proposition B.3.7, $X_\lambda \to Y_\lambda$

is proper for $\lambda \gg 0$. By Proposition B.3.4(1), there exists an index $\mu \in \Lambda$ and a coherent sheaf F_μ on X_μ that pulls back to F under $X \to X_\mu$. For $\lambda > \mu$, set F_λ to be the pullback of F_μ under $X_\lambda \to X_\mu$. By Proposition B.3.4(3), F_λ is flat over Y_λ for $\lambda \gg 0$. We may now apply noetherian Cohomology and Base Change to the data of $X_\lambda \to Y_\lambda$ and F_λ for $\lambda \gg 0$, and we may deduce the same properties for $X \to Y$ and F under the base change $Y \to Y_\lambda$. See also [EGA, III_2.7.7.5, III_2.7.7.10, III_2.7.8.4], [Har77, Thm. 12.11], and [Vak17, Thm. 25.1.6]. □

The following exercise will give you some practice applying Cohomology and Base Change.

Exercise A.6.9. Let $f\colon X \to Y$ be a proper morphism of noetherian schemes. For a coherent sheaf F flat over Y, the following are equivalent:

(1) $\mathrm{H}^i(X_y, F_y) = 0$ for all $y \in Y$ and $i > 0$; and

(2) $\mathrm{R}^i f_* F = 0$ for all $i > 0$, and $f_* F$ is a vector bundle whose construction commutes with base change on Y.

A.6.2 Applications to moduli theory

Here is a typical moduli-theoretic application of Cohomology and Base Change establishing properties of smooth families of curves (Proposition 5.1.16), which is applied for instance in the Algebraicity of $\mathcal{M}_g$ (3.1.17). The argument below applies equally to families of stable curves (Proposition 5.3.23).

Proposition A.6.10. *Let $\pi\colon \mathcal{C} \to S$ be a family of smooth curves of genus $g \geq 2$ (i.e., $\mathcal{C} \to S$ is a smooth, proper morphism of schemes such that every geometric fiber is a connected curve of genus g). Then*

(1) $\pi_* \mathcal{O}_{\mathcal{C}} = \mathcal{O}_S$,

(2) $\pi_*(\Omega_{\mathcal{C}/S})$ is a vector bundle of rank g whose construction commutes with base change on S and $\mathrm{R}^1\pi_*(\Omega_{\mathcal{C}/S}) \cong \mathcal{O}_S$ while $\mathrm{R}^i\pi_*(\Omega_{\mathcal{C}/S}) = 0$ for $i \geq 2$, and

(3) for $k > 1$, the pushforward $\pi_*(\Omega_{\mathcal{C}/S}^{\otimes k})$ is a vector bundle of rank $(2k-1)(g-1)$ whose construction commutes with base change on S and $\mathrm{R}^i\pi_*(\Omega_{\mathcal{C}/S}^{\otimes k}) = 0$ for $i > 0$.

Proof. To see (1), observe that $\mathrm{H}^0(\mathcal{C}_s, \mathcal{O}_{\mathcal{C}_s}) = \kappa(s)$ for all $s \in S$ since $\mathcal{C}_s$ is proper and geometrically connected. It follows that $\phi_s^0\colon \pi_*\mathcal{O}_{\mathcal{C}} \otimes \kappa(s) \to \mathrm{H}^0(\mathcal{C}_s, \mathcal{O}_{\mathcal{C}_s})$ is surjective. Cohomology and Base Change (A.6.8(a)–(b)) with $i = 0$ implies that ϕ_s^0 is an isomorphism and that $\pi_*\mathcal{O}_{\mathcal{C}}$ is a line bundle. On a fiber over $s \in S$, the natural map $\mathcal{O}_S \to \pi_*\mathcal{O}_{\mathcal{C}}$ induces a surjective map $\kappa(s) \to \pi_*\mathcal{O}_{\mathcal{C}} \otimes \kappa(s)$ (as post-composing with $\phi_s^0\colon \pi_*\mathcal{O}_{\mathcal{C}} \otimes \kappa(s) \to \mathrm{H}^0(\mathcal{C}_s, \mathcal{O}_{\mathcal{C}_s}) = \kappa(s)$ is the identity). Thus $\mathcal{O}_S \to \pi_*\mathcal{O}_{\mathcal{C}}$ is a surjective morphism of line bundles, hence an isomorphism.

For (2), since $\Omega_{\mathcal{C}/S}$ is a relative dualizing sheaf (see [Liu02, §6.4]), Grothendieck–Serre Duality implies that $\mathrm{R}^1\pi_*\Omega_{\mathcal{C}/S} \cong \pi_*\mathcal{O}_{\mathcal{C}}$ and this is identified with $\mathcal{O}_S$ by (1). For $i \geq 2$, $\mathrm{H}^i(\mathcal{C}_s, \Omega_{\mathcal{C}/S} \otimes \kappa(s)) = 0$ (as $\dim \mathcal{C}_s = 1$), and A.6.8(a) implies that $\mathrm{R}^i\pi_*\Omega_{\mathcal{C}/S} = 0$. Applying A.6.8(b) with $i = 2$ yields that $\phi_s^1\colon \mathrm{R}^1\pi_*\Omega_{\mathcal{C}/S} \otimes \kappa(s) \to \mathrm{H}^1(\mathcal{C}_s, \Omega_{\mathcal{C}_s/\kappa(s)})$ is surjective for every $s \in S$ and applying A.6.8(a) with $i = 1$ shows that ϕ_s^1 is an isomorphism. Since $\mathrm{R}^1\pi_*\Omega_{\mathcal{C}/S}$ is a line bundle, applying A.6.8(b) with $i = 1$ shows that ϕ_s^0 is surjective, and applying A.6.8(a)–(b) with $i = 0$ implies that $\pi_*\Omega_{\mathcal{C}/S}$ is a vector bundle of rank $\mathrm{h}^0(\mathcal{C}_s, \Omega_{\mathcal{C}_s/\kappa(s)}) = g$ whose construction commutes with base change.

For (3), since $k > 1$, we have that $\deg(\Omega^{\otimes(1-k)}_{\mathcal{C}_s/\kappa(s)}) < 0$ for each $s \in S$, and Serre Duality (5.1.3) implies that $\mathrm{H}^1(\mathcal{C}_s, \Omega^{\otimes k}_{\mathcal{C}_s/\kappa(s)}) = \mathrm{H}^0(\mathcal{C}_s, \Omega^{\otimes(1-k)}_{\mathcal{C}_s/\kappa(s)}) = 0$. Observe that $\mathrm{H}^i(\mathcal{C}_s, \Omega^{\otimes k}_{\mathcal{C}_s/\kappa(s)}) = 0$ for $i \geq 2$ since $\dim \mathcal{C}_s = 1$. Cohomology and Base Change (A.6.8(a)) gives $\mathrm{R}^i\pi_*(\Omega^{\otimes k}_{\mathcal{C}/S}) = 0$ for $i > 0$. On the other hand, $\mathrm{h}^0(\mathcal{C}_s, \Omega^{\otimes k}_{\mathcal{C}_s/\kappa(s)}) = \deg(\Omega^{\otimes k}_{\mathcal{C}_s/\kappa(s)}) + 1 - g = (2k-1)(g-1)$ by Riemann–Roch (5.1.2). Applying Cohomology and Base (A.6.8(b)) with $i = 0$ yields that $\pi_*(\Omega^{\otimes k}_{\mathcal{C}/S})$ is a vector bundle of rank $(2k-1)(g-1)$. □

Similarly, we can apply Cohomology and Base Change to establish properties of families of coherent sheaves, which we will need for instance for the Algebraicity of $\mathcal{B}un(C)$ (3.1.21).

Proposition A.6.11. *Let $p \colon X \to S$ be a proper morphism of schemes and F be a finitely presented quasi-coherent sheaf on X flat over S. Suppose that $\dim X_s \leq d$ for all $s \in S$. The subset S' of points $s \in S$ such that $\mathrm{H}^j(X_s, F_s) = 0$ for all $j > 0$ is open. Denoting $X' = p^{-1}(S')$, $p' := p|_{X'} \colon X' \to S$, and $F' = F|_{X'}$, we have that $\mathrm{R}^j p'_* F' = 0$ for all $j > 0$ and that $p'_* F'$ is a vector bundle whose construction commutes with base change.*

Proof. For each $j = 1, \ldots, d$, A.6.8(a) implies that the locus of points $s \in S$ such that $\mathrm{H}^j(X_s, F_s) = 0$ is open and the comparison map $\phi^j_s \colon \mathrm{R}^j p_* F \otimes \kappa(s) \to \mathrm{H}^j(X_s, F_s)$ is an isomorphism. It follows that $\mathrm{R}^j p'_* F = 0$ which allows us to apply A.6.8(b) with $i = 1$ to conclude that $\phi^0_s \colon p'_* F' \otimes \kappa(s) \to \mathrm{H}^0(X_s, F_s)$ is surjective. Applying A.6.8(a)–(b) with $i = 0$ gives the final statement. □

A.6.3 Applications to line bundles

Given a proper flat morphism $f \colon X \to Y$, when is a line bundle L on X the pullback of a line bundle on Y? More generally, is there a largest subscheme $Z \subseteq Y$ where L_Z on $X_Z = X \times_Y Z$ is the pullback of a line bundle on Z? In this section, we provide three answers in increasing complexity. As the results depend on properties of the fibers X_y, we first discuss relationships between various conditions.

Lemma A.6.12. *Let $f \colon X \to Y$ be a proper flat morphism of noetherian schemes. Consider the following conditions:*

(1) the geometric fibers of $f \colon X \to Y$ are non-empty, connected, and reduced;

(2) $\mathrm{h}^0(X_y, \mathcal{O}_{X_y}) = 1$ for all $y \in Y$; and

(3) $\mathcal{O}_Y = f_* \mathcal{O}_X$ and this holds after arbitrary base change (i.e., $\mathcal{O}_T = f_{T,*}\mathcal{O}_{X_T}$ for a morphism $T \to Y$ of schemes).

Then (1) ⇒ (2) ⟺ (3).

Proof. If (1) holds, then $\mathrm{H}^0(X_y, \mathcal{O}_{X_y}) \otimes_{\kappa(y)} \overline{\kappa(y)} = \mathrm{H}^0(X \times_Y \overline{\kappa(y)}, \mathcal{O}_{X \times_Y \overline{\kappa(y)}})$ by Flat Base Change (A.2.12, and since a connected, reduced, and proper scheme over an algebraically closed field has only constant functions, we conclude that $\mathrm{h}^0(X_y, \mathcal{O}_{X_y}) = 1$. If (2) holds, then the comparison map $\phi^0_y \colon f_* \mathcal{O}_X \otimes \kappa(y) \to \mathrm{H}^0(X_y, \mathcal{O}_{X_y}) = \kappa(y)$ is necessarily surjective as there is a global section $1 \in \mathrm{H}^0(Y, f_* \mathcal{O}_X)$. Applying Theorem A.6.8 with $i = 0$ shows that $f_* \mathcal{O}_X$ is a line bundle. As $\mathcal{O}_Y \to f_* \mathcal{O}_X$ is a surjection of line bundles, it is an isomorphism. Since the same argument applies to the base change $X_T \to T$, this gives (3). The converse (3) ⇒ (2) by considering the map $T = \operatorname{Spec} \kappa(y) \to Y$. □

When the base is reduced, Grauert's Theorem provides a complete answer to when a line bundle is a pullback.

Proposition A.6.13 (Version 1)**.** *Let $f\colon X \to Y$ be a proper flat morphism of noetherian schemes such that $\mathrm{h}^0(X_y, \mathcal{O}_{X_y}) = 1$ for all $y \in Y$. Let L be a line bundle on X. If Y is reduced, then $L = f^*M$ for a line bundle M on Y if and only if L_y is trivial for all $y \in Y$. Moreover, if these conditions hold, then $M = f_*L$ and the adjunction morphism $f^*f_*L \to L$ is an isomorphism.*

Proof. The condition on geometric fibers implies that $\mathrm{h}^0(X_y, L_y) = 1$ and Grauert's Theorem (A.6.7) implies that f_*L is a line bundle and that $f_*L \otimes \kappa(y) \xrightarrow{\sim} \mathrm{H}^0(X_y, L_y)$ is an isomorphism. We claim that $f^*f_*L \to L$ is surjective. It suffices to show that $(f^*f_*L)|_{X_y} \to L|_{X_y}$ is surjective. Denoting $f_y\colon X_y \to \operatorname{Spec}\kappa(y)$, we have identifications $(f^*f_*L)|_{X_y} = f_y^*(f_*L \otimes \kappa(y)) = f_y^*(\mathcal{O}_{\operatorname{Spec}\kappa(y)}) = \mathcal{O}_{X_y}$ and the claim follows. Since $f^*f_*L \to L$ is a surjection of line bundles, it is an isomorphism. $\square$

Exercise A.6.14. Show that if Y is a connected and reduced noetherian scheme and E is a vector bundle on Y, then $\operatorname{Pic}(\mathbb{P}(E)) = \operatorname{Pic}(Y) \times \mathbb{Z}$. See also [Har77, Exc. III.12.5].

Proposition A.6.15 (Version 2)**.** *Let $f\colon X \to Y$ be a proper flat morphism of noetherian schemes with integral geometric fibers. For a line bundle L on X, the locus $\{y \in Y \mid L_y \text{ is trivial on } X_y\}$ is a closed subset of Y.*

Proof. The important observation here is that for a geometrically integral and proper scheme Z over field $\Bbbk$, a line bundle M is trivial if and only if $\mathrm{h}^0(Z, M) > 0$ and $\mathrm{h}^0(Z, M^\vee) > 0$. To see that the latter condition is sufficient, observe that we have nonzero homomorphisms $\mathcal{O}_Z \to M$ and $\mathcal{O}_Z \to M^\vee$, the latter of which dualizes to a nonzero map $M \to \mathcal{O}_Z$. Since Z is integral, the composition $\mathcal{O}_Z \to M \to \mathcal{O}_Z$ is also nonzero and thus an isomorphism as it corresponds to a nonzero constant in $\mathrm{H}^0(Z, \mathcal{O}_Z) = \Bbbk$. It follows that $M \to \mathcal{O}_Z$ is a surjective map of line bundles, hence an isomorphism. By the Semicontinuity Theorem (A.6.4) the condition that $\mathrm{h}^0(X_y, L_y) > 0$ and $\mathrm{h}^0(X_y, L_y^\vee) > 0$ is closed, and the statement follows. See also [Mum70a, Cor. 6, p. 54]. $\square$

Remark A.6.16. If the geometric fibers are only connected and reduced, the locus may fail to be closed. For example, giving a smooth family $f\colon X \to Y$ of curves over a smooth curve, and consider the blowup $\operatorname{Bl}_x X \to X$ at a closed point $x \in X$ with exceptional divisor E. Then $\operatorname{Bl}_x X \to Y$ is a proper flat morphism, and the fiber over $f(x) \in Y$ is connected and reduced but reducible. Setting $L = \mathcal{O}_{\operatorname{Bl}_x X}(E)$, the fiber L_y is trivial if and only if $y \neq f(x)$.

For moduli-theoretic applications, it is essential that we allow for the base Y to be non-reduced, and provide the locus $Z \subseteq Y$ with a functorial description. For many applications (e.g., to families of stable curves), it is also necessary to allow for reducible fibers X_y. Our final and strongest version incorporates both versions above and is proved using the algebraic formulation of Cohomology and Base Change (A.6.2). This result will be applied in the proof of Algebraicity of $\mathcal{M}_g$ (3.1.17) to exhibit a locally closed subscheme of the Hilbert scheme parameterizing smooth curves that are tricanonically embedded.

Proposition A.6.17 (Version 3)**.** *Let $f\colon X \to Y$ be a proper flat morphism of noetherian schemes such that $\mathrm{h}^0(X_y, \mathcal{O}_{X_y}) = 1$ for all $y \in Y$ (resp., the geometric fibers are integral). For a line bundle L on X, there is a unique locally closed (resp., closed) subscheme $Z \subseteq Y$ such that*

(1) L_Z on $X_Z = X \times_Y Z$ is the pullback of a line bundle on Z, and

(2) if $T \to Y$ is a morphism of schemes such that L_T on X_T is the pullback of a line bundle on T, then $T \to Y$ factors through Z.

In other words, the functor

$$\mathrm{Sch}/Y \to \mathrm{Sets},$$
$$(T \to Y) \mapsto \begin{cases} \{*\} & \text{if } L_T \text{ is the pullback of a line bundle on } T \\ \emptyset & \text{otherwise} \end{cases}$$

is representable by a locally closed (resp., closed) subscheme of Y.

Proof. We begin with the observation that L is the pullback of a line bundle if and only if f_*L is a line bundle and the adjunction map $f_*f^*L \to L$ is an isomorphism. Indeed, if $L = f^*M$ for a line bundle M on Y, then the projection formula and the equality $\mathcal{O}_Y = f_*\mathcal{O}_X$ (Lemma A.6.12) shows that

$$f_*L \cong f_*f^*M \cong f_*\mathcal{O}_X \otimes M \cong M$$

is a line bundle and that $f^*f_*L \to L$ is an isomorphism. As the same holds for the base change $X_T \to T$, we see that the question is Zariski-local on Y and T. We will show that every point $y \in Y$ has an open neighborhood where the proposition holds.

By the Semicontinuity Theorem (A.6.4), the locus $V = \{y \in Y \mid \mathrm{h}^0(X_y, L_y) \leq 1\}$ is open. Since L_y is not trivial for points $y \notin V$, we may replace Y with V and assume that $\mathrm{h}^0(X_y, L_y) \leq 1$ for all $y \in Y$. By Cohomology and Base Change (A.6.2) and after replacing Y with an open affine neighborhood of y, we may assume that there is a homomorphism $d\colon A^{r_0} \xrightarrow{d} A^{r_1}$ of finitely generated and free A-modules such that for every ring map $A \to B$, $\mathrm{H}^0(X_B, L_B) = \ker(d \otimes B)$. Using the dual $d^\vee$ of d, we define M as the cokernel in the sequence

$$A^{r_1} \xrightarrow{d^\vee} A^{r_0} \to M \to 0.$$

Tensoring over $A \to B$ yields a right exact sequence

$$B^{r_1} \xrightarrow{d^\vee \otimes B} B^{r_0} \to M \otimes_A B \to 0,$$

which after applying the contravariant left-exact functor $\mathrm{Hom}_B(-, B)$ becomes

$$0 \to \mathrm{Hom}_B(M \otimes_A B, B) \to B^{r_0} \xrightarrow{d \otimes_A B} B^{r_1}.$$

We conclude that

$$\mathrm{H}^0(X_B, L_B) = \mathrm{Hom}_B(M \otimes_A B, B) = \mathrm{Hom}_A(M, B). \tag{A.6.18}$$

Applying this to $A \to \kappa(y)$ for $y \in Y$, we obtain that $\mathrm{H}^0(X_y, L_y) = \mathrm{Hom}_A(M, \kappa(y)) = (M \otimes_A \kappa(y))^\vee$.

If $\mathrm{h}^0(X_y, L_y) = 0$, then L_y is not trivial and since $M \otimes_A \kappa(y) = 0$, there is an open neighborhood U of y such that $\widetilde{M}|_U = 0$. The proposition holds over U. If $\mathrm{h}^0(X_y, L_y) = 1$, then $M \otimes_A \kappa(y) = \kappa(y)$ and by Nakayama's lemma, after replacing Y with an open affine neighborhood of y, there is a surjection $A \to M$. Write $M = A/I$ for an ideal I and define the closed subscheme $Z = V(I) \subseteq Y$. Observe that $\mathrm{H}^0(Z, L_Z) = \mathrm{Hom}_A(A/I, A/I) = A/I$ so that $f_{Z,*}L_Z$ is the trivial line bundle. To see that the construction of $f_{Z,*}L_Z$ commutes with base change, let B

be an A/I-algebra and observe that $\mathrm{H}^0(X_B, L_B) = \mathrm{Hom}_A(A/I, B) = B$ and that $\mathrm{H}^0(X_Z, L_Z) \otimes_{A/I} B \to \mathrm{H}^0(X_B, L_B)$ is an isomorphism.

We claim that $T \to Y$ factors through Z if and only if $f_{T,*}L_T$ is a line bundle. The ($\Rightarrow$) implication is clear: $f_{Z,*}L_Z$ is a line bundle and its construction commutes with base change. The converse is Zariski-local on T and we may assume that $T = \operatorname{Spec} B$ is affine and $f_{T,*}L_T = \mathcal{O}_T$. Then (A.6.18) implies that $B = \mathrm{Hom}_A(A/I, B)$. Thus, $I \subseteq \ker(A \to B)$ or, in other words, $A \to B$ factors as $A \to A/I \to B$. Finally, considering the adjunction morphism $\lambda\colon f_Z^* f_{Z,*}L_Z \to L_Z$ on X_Z, we claim that for $y \in Z$, L_y is trivial if and only if $\lambda|_{X_y}$ is surjective. If $\lambda|_{X_y}$ is surjective, then using that $f_{Z,*}L_Z = \mathcal{O}_Z$, we have a surjection $\mathcal{O}_{X_y} \to L_y$ of line bundles, hence an isomorphism. For converse, since $f_{Z,*}L_Z$ commutes with base change, the comparison map $f_{Z,*}L_Z \otimes \kappa(y) = \mathrm{H}^0(X_y, L_y)$ is an isomorphism. Denoting $f_y\colon X_y \to \operatorname{Spec} \kappa(y)$, we have identifications $(f_Z^* f_{Z,*}L_Z)|_{X_y} = f_y^*(f_{Z,*}L_Z \otimes \kappa(y)) = f_y^* f_{y,*}L_y$ and $\lambda|_{X_y}$ corresponds to the adjunction map $f_y^* f_{y,*}L_y \to L_y$, which is an isomorphism (as L_y is trivial). Replacing Z with $Z \smallsetminus \operatorname{Supp}(\operatorname{coker}(\lambda))$ establishes the proposition. If, in addition, the fibers X_y are geometrically integral, then Proposition A.6.15 implies that Z is closed. See also [Mum70a, p. 90], [Vie95, Lem. 1.19], and [SP, Tags 0BEZ and 0BF0]. □

Remark A.6.19. For a proper flat morphism $X \to S$, the *relative Picard functor* is defined as

$$\mathrm{Pic}^{\mathrm{rel}}_{X/S}\colon \mathrm{Sch}/S \to \mathrm{Sets}, \quad T \mapsto \mathrm{Pic}(X_T)/\mathrm{Pic}(T);$$

see §6.4.7 for an exposition of Picard functors. If $f\colon X \to S$ has geometrically reduced (resp., integral) fibers, then the existence of a locally closed (resp., closed) subscheme $Z \subseteq Y$ characterized by Proposition A.6.17 is equivalent to the diagonal of $\mathrm{Pic}^{\mathrm{rel}}_{X/S}$ being representable by locally closed immersions (resp., closed immersions). In the case of geometrically integral fibers, this translates to the separatedness of $\mathrm{Pic}^{\mathrm{rel}}_{X/S} \to S$. In this language, the above result was established in [FGAV, Thm. 3.1].

A.7 Quasi-finite morphisms and Zariski's Main Theorem

A locally of finite type morphism $f\colon X \to Y$ of schemes is *locally quasi-finite at* $x \in X$ if x is isolated in the fiber $X_{f(x)} = X \times_Y \operatorname{Spec} \kappa(f(x))$. When $f\colon X \to Y$ is also quasi-compact, then this is equivalent to the finiteness of the set $f^{-1}(f(x))$, and we say that $f\colon X \to Y$ is *quasi-finite*.

A.7.1 Étale Localization of Quasi-Finite Morphisms

Theorem A.7.1 (Étale Localization of Quasi-Finite Morphisms)**.** *Let $f\colon X \to S$ be a separated and finite type morphism of schemes. Suppose that f is quasi-finite at every preimage of $s \in S$. There exists an étale neighborhood $(S', s') \to (S, s)$ with $\kappa(s') = \kappa(s)$ and a decomposition $X \times_S S' = Z \amalg W$ into open and closed subschemes such that $Z \to S'$ is finite and the fiber $W_{s'}$ is empty. Moreover, it can be arranged that Z factors as $Z_1 \amalg \cdots \amalg Z_n$ where each Z_i contains precisely one point z_i over s with $\kappa(z_i)/\kappa(s)$ purely inseparable.*

Proof. See [EGA, IV.8.12.3] or [SP, Tag 04HF]. □

The statement implies that any quasi-finite algebra A over a henselian local ring R is a product $A \cong B \times C$ with B finite over R and $C \otimes_R R/\mathfrak{m}_R = 0$, a property that, in fact, characterizes henselian local rings (Proposition B.5.9). It also provides the key technical input in factoring quasi-finite morphisms.

A.7.2 Factorizations of quasi-finite morphisms

Proposition A.7.2. *A quasi-finite and separated morphism $f\colon X \to Y$ of schemes factors as*

$$f\colon X \to \mathcal{S}\mathrm{pec}_Y\, f_*\mathcal{O}_X \to Y$$

where $X \hookrightarrow \mathcal{S}\mathrm{pec}_Y\, f_*\mathcal{O}_X$ is an open immersion and $\mathcal{S}\mathrm{pec}_Y\, f_*\mathcal{O}_X \to Y$ is affine.

Proof. As $f_*\mathcal{O}_X$ commutes with étale (even flat) base changes on Y, so does the factorization. Therefore, it suffices to show that every point $y \in Y$ has an étale neighborhood where the proposition holds. By Theorem A.7.1 we may assume that $X = X_1 \amalg X_2$ with X_1 finite over Y and $(X_2)_y = \emptyset$. After replacing Y with $\mathcal{S}\mathrm{pec}_Y\, f_*\mathcal{O}_X$, we may also assume that $f_*\mathcal{O}_X = \mathcal{O}_Y$. As $\mathcal{O}_X = \mathcal{A}_1 \times \mathcal{A}_2$ is the product of quasi-coherent $\mathcal{O}_X$-algebras, $\mathcal{O}_Y = f_*\mathcal{O}_X = f_*\mathcal{A}_1 \times f_*\mathcal{A}_2$ and thus Y decomposes as $Y_1 \amalg Y_2$ such that $y \in Y_1$ and $f(X_i) \subseteq Y_i$ for $i = 1, 2$. After replacing Y with Y_1, we see that $X \to Y$ is finite. Thus X is affine and $X = Y = \mathcal{S}\mathrm{pec}_Y\, f_*\mathcal{O}_X$. $\square$

In the above factorization, $f_*\mathcal{O}_Y$ may not be a finite type $\mathcal{O}_Y$-algebra; even if Y is a noetherian affine scheme, then $\Gamma(X, \mathcal{O}_X)$ may not be a noetherian ring (see [Ols16, Ex. 7.2.15]). However, we may modify the factorization to arrange that $X \to Y$ factors as an open immersion followed by a *finite* morphism.

Theorem A.7.3 (Zariski's Main Theorem)**.**

Proof. If $\mathcal{A} \subseteq f_*\mathcal{O}_X$ denotes the integral closure of $\mathcal{O}_Y \to f_*\mathcal{O}_{\mathcal{X}}$, f factors as the composition of

$$f\colon X \xrightarrow{j} \mathcal{S}\mathrm{pec}_Y\, \mathcal{A} \to Y.$$

We claim that j is an open immersion. It suffices to show that for every point $x \in X$, there is an open neighborhood $V \subseteq \mathcal{S}\mathrm{pec}_Y\, \mathcal{A}$ of $j(x)$ such that $j^{-1}(V) \to V$ is an isomorphism. Since normalization commutes with étale base change (Proposition A.7.4) and since being an open immersion is an fpqc local property (Proposition 2.1.26), we are free to replace Y by an étale neighborhood of $f(x)$. By Theorem A.7.1, we can assume that $X = F \amalg W$ with F finite over Y and $x \in F$. In this case, the normalization $\mathcal{S}\mathrm{pec}_Y\, \mathcal{A}$ of Y in X is $F \amalg \widetilde{W}$ where $\widetilde{W}$ is the normalization of Y in W. As $j^{-1}(F) \xrightarrow{\sim} F$, the claim follows. By construction, $\mathcal{S}\mathrm{pec}_Y\, \mathcal{A} \to Y$ is integral. We can write $\mathcal{A} = \operatorname{colim} \mathcal{A}_\lambda$, where each $\mathcal{A}_\lambda$ is a finite type $\mathcal{O}_Y$-algebra. Since open immersions descent under limits (Proposition B.3.7), $X \to \mathcal{S}\mathrm{pec}_Y\, \mathcal{A}_\lambda$ is an open immersion for $\lambda \gg 0$. Since $\mathcal{S}\mathrm{pec}_Y\, \mathcal{A}_\lambda \to Y$ is integral and of finite type, it is finite. See also [EGA, IV.8.12.6] or [SP, Tag 05K0]. $\square$

The following algebra result was used above and will also be used in the generalizations of Zariski's Main Theorem to algebraic spaces (Theorem 4.5.9) and stacks (Theorem 6.1.18).

Proposition A.7.4. *Let Y be a scheme, $\mathcal{B}$ be a quasi-coherent $\mathcal{O}_Y$-algebra and $\widetilde{\mathcal{B}}$ be the integral closure of $\mathcal{O}_Y$ in $\mathcal{B}$. If $f\colon X \to Y$ is a smooth morphism, then $f^*\widetilde{\mathcal{B}}$ is identified with the integral closure of $\mathcal{O}_X$ in $f^*\mathcal{B}$.*

Proof. See [SP, Tag 03GG] or [LMB00, Prop. 16.2]. $\square$

Zariski's Main Theorem has some useful corollaries.

Corollary A.7.5. *A quasi-finite and proper morphism (resp., proper monomorphism) of schemes is finite (resp., a closed immersion).*

Proof. If $f\colon X \to Y$ is a quasi-finite and proper, Zariski's Main Theorem (A.7.3) gives a factorization $f\colon X \hookrightarrow \widetilde{X} \to Y$ and the dense open immersion $X \hookrightarrow \widetilde{X}$ is also closed, thus an isomorphism. On the other hand, if $f\colon X \to Y$ is a proper monomorphism, then it is also quasi-finite, thus finite. The statement reduces to the algebraic fact that a finite epimorphism of rings is surjective (c.f.,[SP, Tag 04VT]). $\square$

Remark A.7.6. As universally closed morphisms are necessarily quasi-compact [SP, Tag 04XU], every universally closed and locally of finite type monomorphism is a closed immersion; see also [SP, Tag 04XV].

Appendix B

Further topics in scheme theory

Algebraic geometry seems to have acquired the reputation of being esoteric, exclusive, and very abstract, with adherents who are secretly plotting to take over all the rest of mathematics. In one respect this last point is accurate.

David Mumford

B.1 Algebraic groups

We provide a crash course in group schemes, algebraic groups, and their actions. For a more detailed exposition for algebraic groups, we recommend [Bor91], [Hum75], [Spr98], [Wat79], and [Mil17], while for group schemes over a general base, we recommend [SGA3_{I}],[SGA3_{II}], [SGA3_{III}], [DG70], and [Con14].

B.1.1 Group schemes and their actions

Definition B.1.1. A *group scheme over a scheme S* is a morphism $\pi\colon G \to S$ of schemes together with a multiplication morphism $m\colon G \times_S G \to G$, an inverse morphism $\iota\colon G \to G$, and an identity morphism $e\colon S \to G$ (with each morphism over S) such that the following diagrams commute:

$$\begin{tikzcd} G \times_S G \times_S G \arrow[r, "\mathrm{id} \times m"] \arrow[d, "m \times \mathrm{id}"'] & G \times_S G \arrow[d, "m"] \\ G \times_S G \arrow[r, "m"] & G \end{tikzcd} \qquad \begin{tikzcd} G \arrow[r, "(\mathrm{id},\iota)"] \arrow[d, "(\iota,\mathrm{id})"'] \arrow[dr, "e\circ\pi"] & G \times_S G \arrow[d, "m"] \\ G \times_S G \arrow[r, "m"] & G \end{tikzcd} \qquad \begin{tikzcd} G \arrow[r, "(e\circ\pi,\mathrm{id})"] \arrow[d, "(\mathrm{id},e\circ\pi)"'] \arrow[dr, "\mathrm{id}"] & G \times_S G \arrow[d, "m"] \\ G \times_S G \arrow[r, "m"] & G \end{tikzcd}$$

Associativity — Law of inverse — Law of identity

For group schemes H and G over S, a *morphism of group schemes* is a morphism $\phi\colon H \to G$ of schemes over S such that $m_G \circ (\phi \times \phi) = \phi \circ m_H$. A *(closed) subgroup* of G is a nonempty (closed) subscheme $H \subseteq G$ such that $H \times H \hookrightarrow G \times G \xrightarrow{m_G} G$ and $H \hookrightarrow G \xrightarrow{\iota} G$ factor through H. We say that a subgroup $H \subseteq G$ is normal if for every S-scheme T, the subgroup $H(T) \subseteq G(T)$ is normal.

Remark B.1.2. If G and S are affine, then by reversing the arrows above gives $\Gamma(G, \mathcal{O}_G)$ the structure of a *Hopf algebra* over $\Gamma(S, \mathcal{O}_S)$.

Exercise B.1.3. Show that a group scheme over S is equivalently defined as a scheme G over S together with a factorization

$$\begin{tikzcd} \mathrm{Sch}/S \arrow[r, dashed] \arrow[dr, "\mathrm{Mor}_S(-,G)"'] & \mathrm{Gps} \arrow[d] \\ & \mathrm{Sets} \end{tikzcd}$$

where Gps $\to$ Sets is the forgetful functor.

(*We are not requiring that there exists a factorization; the factorization is part of the data! Indeed, the same scheme can have multiple structures as a group scheme, e.g.,* $\mathbb{Z}/4$ *and* $\mathbb{Z}/2 \times \mathbb{Z}/2$ *over* $\mathbb{C}$.)

Example B.1.4 (Important examples)**.** Let $S = \operatorname{Spec} R$.

(1) The *multiplicative group scheme over* R is $\mathbb{G}_{m,R} = \operatorname{Spec} R[t]_t$ with comultiplication $m^*\colon R[t]_t \to R[t]_t \otimes_R R[t']_{t'}$ given by $t \mapsto tt'$. The product $\mathbb{G}_{m,R}^n$ is called the *split torus of rank* n.

(2) The *group scheme of* n*th roots of unity* is $\boldsymbol{\mu}_{n,R} = \operatorname{Spec} R[t]/(t^n - 1)$ with comultiplication given by $t \mapsto tt'$.

(3) The *additive group scheme over* R is $\mathbb{G}_{a,R} = \operatorname{Spec} R[t]$ with comultiplication $m^*\colon R[t] \to R[t] \otimes_R R[t']$ given by $t \mapsto t + t'$.

Let V be a free R-module of finite rank.

(3) The *general linear group on* V is

$$\mathrm{GL}(V) = \operatorname{Spec}(\operatorname{Sym}^*(\operatorname{End}(V))_{\det}),$$

where det denotes the determinant polynomial and where the comultiplication $m^*\colon \operatorname{Sym}^*(\operatorname{End}(V)) \to \operatorname{Sym}^*(\operatorname{End}(V)) \otimes_R \operatorname{Sym}^*(\operatorname{End}(V))$ is defined as following: for a basis $v_1, \dots, v_n$ of V, then for $i, j = 1, \dots, n$, the endomorphisms $x_{ij}\colon V \to V$ defined by $v_i \mapsto v_j$ and $v_k \mapsto 0$ if $k \neq i$ define a basis of $\operatorname{End}(V)$, and we define $m^*(x_{ij}) = x_{i1} \otimes x_{1j} + \cdots + x_{in} \otimes x_{nj}$.

(4) The *special linear group on* V is the closed subgroup $\mathrm{SL}(V) \subseteq \mathrm{GL}(V)$ defined by $\det = 1$.

(5) The *projective linear group* $\mathrm{PGL}(V)$ is the *affine* group scheme

$$\operatorname{Proj}(\operatorname{Sym}^*(\operatorname{End}(V)))_{\det}$$

with the comultiplication analogously to $\mathrm{GL}(V)$.

We write $\mathrm{GL}_{n,R} = \mathrm{GL}(R^n)$, $\mathrm{SL}_{n,R} = \mathrm{GL}(R^n)$, and $\mathrm{PGL}_{n,R} = \mathrm{PGL}(R^n)$. We often simply write $\mathbb{G}_m$, GL_n, SL_n, and PGL_n when the base ring is understood.

Exercise B.1.5.

(a) Provide functorial descriptions of each of example above.

(b) Show that every abstract group G can be given the structure of a group scheme $\amalg_{g \in G} S$ over a base scheme S. Provide both explicit and functorial descriptions. By abuse of notation, this group scheme is sometimes denoted as $G \to S$.

(c) Show that if n is invertible in $\Gamma(S, \mathcal{O}_S)$, then $\mu_{n,S}$ is isomorphic to the group scheme induced by the finite group $\mathbb{Z}/n\mathbb{Z}$.

Example B.1.6 (Diagonalizable group schemes)**.** Let R be a ring and A be a finitely generated abelian group. If we define $R[A]$ as the free R-module generated by elements of A, then $R[A]$ inherits an R-algebra structure with multiplication on generators induced from multiplication in A. The comultiplication $R[A] \to R[A] \otimes_R R[A]$ defined by $a \mapsto a \otimes a$ defines a group scheme $D_R(A) = \operatorname{Spec} R[A]$ over $\operatorname{Spec} R$. A group scheme G over $\operatorname{Spec} R$ is *diagonalizable* if $G \cong D_R(A)$ for some A.

The group scheme $D_R(\mathbb{Z}^r) = \mathbb{G}_{m,R}^r$ is the r-dimensional split torus while $D_R(\mathbb{Z}/n) = \boldsymbol{\mu}_{n,R} = \operatorname{Spec} R[t]/(t^n - 1)$ is the group of nth roots of unity; this holds for any ring R and integer n even if n is not invertible in R. The classification of finitely generated abelian groups implies that every diagonalizable group scheme is a product of $\mathbb{G}_m^r \times \boldsymbol{\mu}_{n_1} \times \cdots \times \boldsymbol{\mu}_{n_k}$.

A group scheme $G \to S$ is of *multiplicative type* if it becomes diagonalizable after étale cover of S.

Exercise B.1.7.

(1) Describe $D_R(A)$ as a functor $\mathrm{Sch}/R \to \mathrm{Gps}$.

(2) Show that a homomorphism $D_R(A) \to D_R(B)$ of group schemes over R is equivalent to a group homomorphism $B \to A$.

We recall the following general properties of group schemes.

Proposition B.1.8. *Let $G \to S$ be a locally of finite type group scheme.*

(1) If $S = \operatorname{Spec} \Bbbk$, then $\dim G = \dim_e G$, where $e \in G(\Bbbk)$ denotes the identity.

(2) The function
$$S \to \mathbb{Z}, \quad s \mapsto \dim G_s$$
is upper semicontinuous.

(3) $G \to S$ is trivial if and only if the fiber G_s is trivial for each $s \in S$.

(4) $G \to S$ is unramified (resp., separated, quasi-separated) if and only if the identity section $e\colon S \to G$ is an open immersion (resp., a closed immersion, quasi-compact).

Proof. For (1), we may assume that $\Bbbk$ is algebraically closed. In this case, an element $g \in G(\Bbbk)$ defines an isomorphism $g\colon G \to G$, so that $\dim_e G = \dim_g G$. For (2), for any locally of finite type morphism $\pi\colon G \to S$, the function $G \to \mathbb{Z}$, defined by $g \mapsto \dim G_{\pi(g)}$, is upper semicontinuous [EGA, IV.13.1.3]. As $G \to S$ is a group scheme, there is an identity section $S \to G$ and therefore the composition $S \to G \to \mathbb{Z}$, defined by $s \mapsto \dim_{e(s)} G_s = \dim G_s$, is upper semicontinuous. For (3), $G \to S$ is unramified since every fiber is. Therefore $\Omega_{G/S} = 0$ and the diagonal $G \to G \times_S G$ is an open immersion. It follows that the identity section $S \to G$ is a surjective open immersion, thus an isomorphism. For (4), $G \to S$ is unramified (resp., separated, quasi-separated) if and only if $\Delta_{G/S}\colon G \to G \times_S G$ is an open immersion (resp., a closed immersion, quasi-compact), and the cartesian diagram

$$\begin{tikzcd}
S \arrow[r, "e"] \arrow[d, "e"] \arrow[dr, phantom, "\square"] & G \arrow[r, "\pi"] \arrow[d, "\Delta_{G/S}"] \arrow[dr, phantom, "\square"] & S \arrow[d, "e"] \\
G \arrow[r, "\Delta_{G/S}"'] & G \times_S G \arrow[r, "(g,h)\mapsto g^{-1}h"'] & G
\end{tikzcd}$$

implies that this is equivalent to $e\colon S \to G$ being an open immersion (resp., a closed immersion, quasi-compact). $\square$

Actions.

Definition B.1.9 (Actions)**.** Let $G \to S$ be a group scheme with multiplication m and identity e. An *action* of G on a scheme $p\colon X \to S$ is a morphism $a\colon G \times_S X \to X$ over S such that the following diagrams commute:

$$\begin{tikzcd} G \times_S G \times_S X \arrow[r, "\mathrm{id} \times a"] \arrow[d, "m \times \mathrm{id}"'] & G \times_S X \arrow[d, "a"] \\ G \times_S X \arrow[r, "a"] & G \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "(e \circ p, \mathrm{id})"] \arrow[dr, "\mathrm{id}"'] & G \times_S X \arrow[d, "a"] \\ & X \end{tikzcd}$$

If X and Y are S-schemes with actions of G, a morphism $f\colon X \to Y$ of S-schemes is *G-equivariant* if $a_Y \circ (\mathrm{id} \times f) = f \circ a_X$, and is *$G$-invariant* if G-equivariant and Y has the trivial G-action.

For example, a group scheme $G \to S$ acts on itself via left multiplication $g \cdot g' = gg'$ or via right multiplication $g \cdot g' = g' g^{-1}$.

Exercise B.1.10. Show that giving a group action of $G \to S$ on $X \to S$ is the same as giving an action of the functor $\mathrm{Mor}_S(-, G)\colon \mathrm{Sch}/S \to \mathrm{Gps}$ on the functor $\mathrm{Mor}_S(-, X)\colon \mathrm{Sch}/S \to \mathrm{Sets}$.

(This requires first spelling out what it means for a functor to groups to act on a functor to sets.)

Definition B.1.11 (Stabilizers and Orbits)**.** Given an action σ of a group scheme $G \to S$ on $X \to S$, the *stabilizer of f* of an S-morphism $f\colon T \to X$ is the group scheme G_f (sometimes written as $\mathrm{Stab}^G(f)$ or $\mathrm{Stab}(f)$) over T defined as the fiber product

$$\begin{tikzcd} G_f \arrow[rr] \arrow[d] \arrow[drr, phantom, "\square"] & & T \arrow[d, "f"] \\ G \times_S T \arrow[r] & G \times_S X \arrow[r, "\sigma"] & X, \end{tikzcd}$$

while the *orbit of f* is defined set-theoretically as the image of $G \times_S T \to G \times_S X \xrightarrow{\sigma} X$.

The *stabilizer group scheme* $S_X \to X$ is defined as the stabilizer of the identity $\mathrm{id}\colon X \to X$ and is identified with the fiber product

$$\begin{tikzcd} S_X \arrow[r] \arrow[d] & X \arrow[d, "\Delta"] \\ G \times_S X \arrow[r, "(\sigma, p_2)"] & X \times_S X. \end{tikzcd}$$

When $S = \operatorname{Spec} \Bbbk$ for a field $\Bbbk$ and $x \in X(\Bbbk)$, then the stabilizer $G_x = \mathrm{Stab}^G(x)$ is preimage of x under the map $\sigma_x\colon G \to X$, given by $g \mapsto x$, and is identified with the fiber of the stabilizer group scheme $S_X \to X$. On the other hand, the orbit Gx (sometimes written as $O(x)$) is the image of σ_x. When G and X are of finite type, the image $Gx \subseteq X$ is locally closed (Proposition B.1.16(5)), and thus Gx has a natural scheme structure inherited from the scheme-theoretic image of σ_x; note that when G is smooth (e.g., $\mathrm{char}(\Bbbk) = 0$), the orbit Gx has the reduced scheme structure.

Quotients. Constructing quotients of subgroups or group actions is a subtle business in algebraic geometry. If $G \to S$ is an fppf group scheme acting freely on a scheme $X \to S$, the *quotient functor* is defined as the fppf sheafification of the functor

$$(X/G)^{\mathrm{pre}}\colon \mathrm{Sch}/S \to \mathrm{Sets}, \qquad T \mapsto X(T)/G(T).$$

While the quotient sheaf X/G is not always representable by a scheme, it is a theorem that X/G is an algebraic space (Theorem 6.3.1). It is also a theorem that if $G \to S$ is smooth, then X/G is identified with the *étale sheafification* of $(X/G)^{\mathrm{pre}}$ (Corollary 6.3.3).

In special situation however, the quotient is known to exist as a scheme. Here are three examples, each of which plays a prominent role in this text:

- If G is an algebraic group over $\Bbbk$ and $H \subseteq G$ is a subgroup, then G/H is a quasi-projective scheme (Proposition B.1.16(7)).
- If G is a finite group acting freely on an affine scheme X, then X/G is affine (Proposition 4.3.3). In fact, if G is a finite group acting (not necessarily freely) on an affine (resp. projective, quasi-projective) scheme X, there exists a *geometric quotient* (see Definition 4.2.2); see Exercise 4.2.14. The quotient is also denoted as X/G but it does *not* necessarily represent the sheafification of $(X/G)^{\mathrm{pre}}$.
- If G is a linearly reductive group over $\Bbbk$ acting on an affine scheme $X = \operatorname{Spec} A$, there exists a *good quotient* $X/\!\!/G = \operatorname{Spec} A^G$ which has desirable geometric properties (Definition 7.1.1 and Proposition 7.1.3). Given an action of G on a projective scheme X, Geometric Invariant Theory addresses how to identify open subschemes $U \subseteq X$ which admit projective good quotients $U/\!\!/G$; see Chapter 7.

Under quite general conditions, a geometric quotient X/G exists as an algebraic space, e.g., if G is an affine algebraic group and the action map $G \times X \to X \times X$ is proper (Corollary 4.4.12).

Inducing actions and the Borel construction. If $H \subseteq G$ is an inclusion of algebraic groups over $\Bbbk$ and X is a scheme with an H-action, then H acts freely on $G \times X$ via $h \cdot (g, x) = (gh^{-1}, hx)$, and we define

$$G \times^H X := (G \times X)/H$$

as the quotient *algebraic space*, which inherits an action of G via $g \cdot (g', x) = (gg', x)$. This is sometimes referred to as the *Borel construction*. In this case, the quotient stacks $[X/H]$ and $[G \times^H X/G]$ are isomorphic (Exercise 3.4.19).

Representations.

Definition B.1.12 (Representations)**.** Let $S = \operatorname{Spec} R$ be an affine scheme, and let $G \to S$ be a group scheme with multiplication m and identity e. A *representation* (or *comodule*) of G is an R-module V together with a homomorphism $\sigma \colon V \to \Gamma(G, \mathcal{O}_G) \otimes_R V$ of R-modules (referred to as a *coaction*) such that the following diagrams commute:

$$\begin{tikzcd} V \arrow[rr, "\sigma"] \arrow[d, "\sigma"] & & \Gamma(G,\mathcal{O}_G) \otimes_R V \arrow[d, "\mathrm{id} \otimes \sigma"] \\ \Gamma(G,\mathcal{O}_G) \otimes_R V \arrow[rr, "m^* \otimes \mathrm{id}"] & & \Gamma(G,\mathcal{O}_G) \otimes_R \Gamma(G,\mathcal{O}_G) \otimes_R V \end{tikzcd} \qquad \begin{tikzcd} V \arrow[r, "\sigma"] \arrow[dr, "\mathrm{id}"'] & \Gamma(G,\mathcal{O}_G) \otimes_R V \arrow[d, "e^* \otimes \mathrm{id}"] \\ & V \end{tikzcd}$$

Morphisms of representations and subrepresentations are defined in the natural way. If V is a G-representation, the invariant subspace is defined as $V^G = \{v \in V \mid \sigma(v) = 1 \otimes v\}$.

Example B.1.13.

(1) Given an R-module V, the *trivial representation* on V is defined using the coaction $\sigma(v) = 1 \otimes v$.

(2) The *regular representation* on $\Gamma(G, \mathcal{O}_G)$ is defined using the comultiplication $m^*\colon \Gamma(G, \mathcal{O}_G) \to \Gamma(G, \mathcal{O}_G) \otimes_R \Gamma(G, \mathcal{O}_G)$.

(3) The *standard representation* of $\mathrm{GL}_{n,R} = \operatorname{Spec} R[x_{ij}]_{\det}$ (or a subgroup scheme of $\mathrm{GL}_{n,R}$) on $V = R^n$ is given by the coaction $\sigma\colon V \to \Gamma(\mathrm{GL}_{n,R}) \otimes_R V$ defined by $\sigma(e_i) = \sum_{j=1}^n x_{ij} \otimes e_j$ where $(e_1, \ldots, e_n)$ is the standard basis of V.

A representation V of G induces an action of G on $\mathbb{A}(V) = \operatorname{Spec}(\operatorname{Sym}^* V)$, which we refer to as a *linear action*.

Exercise B.1.14. If G is a group scheme over a field $\Bbbk$, show that a G-representation of a finite dimensional vector space V is equivalent to a homomorphism $G \to \mathrm{GL}(V)$ of group schemes.

Proposition B.1.15. *Let $G = D_{\Bbbk}(A)$ be a diagonalizable group scheme over a ring $\Bbbk$. Every representation of G is a direct sum of one dimensional representations.*

Proof. Let V be a representation of G with coaction $\sigma\colon V \to \Bbbk[A] \otimes_{\Bbbk} V$. Each $a \in A$ defines a one dimensional representation $W_a = \Bbbk$ of G defined by the coaction $W_a \to \Bbbk[A] \otimes_{\Bbbk} W_a$ defined by $1 \mapsto a \otimes 1$. For $a \in A$, the subspace

$$V_a := \{v \in V \mid \sigma(v) = a \otimes v\}$$

is isomorphic to $W_a \otimes V_a$ as G-representations, where V_a is viewed as the trivial representation; if V_a is finite dimensional, then $V_a \cong W_a^{\dim V_a}$. Note that when $a = 0$, W_a is the trivial one dimensional representation and $V^G = V_0$. We leave the reader to check that $V \cong \bigoplus_{a \in A} V_a$ as G-representations. See also [Mil17, Thm 12.30] and [SGA3$_{\mathrm{I}}$, Thm. 5.3.3]. $\square$

B.1.2 Algebraic groups

An *algebraic group over a field* $\Bbbk$ is a group scheme G of finite type over $\Bbbk$.

Proposition B.1.16. *Let G be a group scheme locally of finite type over field $\Bbbk$ (e.g., an algebraic group).*

(1) G is separated.

(2) (Cartier's Theorem) If $\operatorname{char}(\Bbbk) = 0$, then G is smooth.

(3) If $\Bbbk$ is perfect, then G is smooth if and only if G is reduced if and only if G is geometrically reduced, and moreover $G_{\mathrm{red}} \subseteq G$ is a subgroup scheme.

(4) The connected component $G^0 \subseteq G$ containing the identity element is an open and closed irreducible subgroup scheme of finite type over $\Bbbk$. Moreover, the construction of G^0 commutes with field extensions of $\Bbbk$, and G/G^0 is an etale algebraic group over $\Bbbk$.

(5) If G acts on a finite type $\Bbbk$-scheme X and $x \in X$ is a closed point, the orbit Gu, defined set-theoretically as the image of $G \to X, g \mapsto g \cdot x$, is open in its closure $\overline{Gx}$. In particular, orbits of minimal dimension are clsoed. Moreover,

$$\dim G = \dim Gx + \dim G_x,$$

and the function $x \mapsto \dim G_x$ is upper semicontinuous while $x \mapsto \dim Gx$ is lower semicontinuous.

(6) *Every subgroup $H \subseteq G$ is closed.*

(7) *If G is of finite type and $H \subseteq G$ is a subgroup, then G/H is quasi-projective. In particular, every algebraic group is quasi-projective.*

(8) *(Barsotti–Chevalley's Structure Theorem) If G is smooth and connected, then there is a unique connected, affine, and normal subgroup $H \trianglelefteq G$, which is smooth if $\Bbbk$ is perfect, such that G/H is an abelian variety.*

In (8), an *abelian variety* by definition is a smooth proper algebraic group over a field. It is necessarily projective and has a commutative group law [Mum70a, pp. 39, 59]. An *elliptic curve* is an abelian variety of dimension 1.

Proof. Proposition B.1.8(4) implies (1) since any $\Bbbk$-point of a locally of finite type $\Bbbk$-scheme is closed. For (2), see [Car62, §15], [Oor66], [Mum66a, p.167], [Wat79, §11.4], [Mil17, Thm. 3.23 and Cor. 8.39], and [SP, Tag 047N]. For (3), see [Mil17, Prop. 1.26 and Cor. 1.39] and [SP, Tags 047P and 047R]. For (4), see [Wat79, §6.7], [Hum75, §7.3], [Spr98, Prop. 2.2.1], [Mil17, Prop. 1.34], and [SP, Tag 0B7R]. What may seem surprising here is that G^0 is automatically quasi-compact.[1] This follows from a simple argument: reduce to the case that $\Bbbk$ is algebraically closed and choose a nonempty open affine subscheme $U \subseteq G$. After shrinking, we may assume that U is closed under taking inverses. The quasi-compactness of G follows from the surjectivity of the multiplication map $U \times U \to G$ is surjective. If $g \in G(\Bbbk)$, then since U is dense, the intersection $U \cap gU$ contains an element h. If we write $h = gu$, then $g = hu^{-1}$.

For the first part of (5), see [Bor91, §I.1.8], [Hum75, §8.3], [Spr98, Lem. 2.3.3], and [Mil17, Prop. 1.68]. The identity $\dim G = \dim Gx + \dim G_x$ follows from the identification $Gx \cong G/G_x$, while the semicontinuity statements follow from Proposition B.1.8(2) applied to the stabilizer group scheme $S_X \to X$. Part (6) follows from (5) by considering the action of H on G. For (7), see [Cho57, p.128], [Bor91, Thm. 6.8], [Ray70, Cor VI.2.6], [Bri17, Thm. 5.2.2], [Hum75, §12], [Spr98, Thm. 5.5.5], and [Mil17, Thm. 8.4.4]. Chevalley announced a proof of (8) in 1953, but a proof did not appear until [Che60]. In the meantime, Barsotti provided an independent proof [Bar55a], [Bar55b]. Rosenlicht provided a more elementary argument in [Ros56]. See also [Con02], [Bri17, Thms. 1 and 2], and [Mil17, Thm. 8.27]. □

B.1.3 Affine algebraic groups.

We are particularly interested in *affine algebraic groups*, which are sometimes also called *linear algebraic groups*, as justified by (2) below.

Proposition B.1.17. *Let G be an affine algebraic group over a field $\Bbbk$.*

(1) *Every representation V of G is a union of its finite dimensional subrepresentations.*

(2) *There exists a finite dimensional representation V and a closed immersion $G \hookrightarrow \mathrm{GL}(V)$ of group schemes.*

Proof. For (1), let $\sigma\colon V \to \Gamma(G, \mathcal{O}_G) \otimes_{\Bbbk} V$ be the coaction. It suffices to show that every finite dimensional subspace $W \subseteq V$ is contained in a finite dimensional sub-G-representation $W' \subseteq V$. If $w_1, \ldots, w_n$ is a basis of W and $\sigma(w_i) = \sum_j f_{ij} \otimes v_{ij}$, then one checks that the subspace generated by v_{ij} is G-invariant and contains W. For (2),

[1] We use this in the text to show the boundedness of $\underline{\mathrm{Pic}}^0_X$; see Theorem 6.4.51.

we consider the regular representation $\Gamma(G, \mathcal{O}_G)$ of G and apply (1) to construct a finite dimensional subrepresentation V containing $\Bbbk$-algebra generators. One checks that this gives a closed immersion $G \hookrightarrow \mathrm{GL}(V)$. See [Bor91, §I.1.9-10], [Hum75, §8.6], [Spr98, Prop. 2.3.6 and Thm. 2.3.7], and [Mil17, Prop. 4.7, Cor. 4.10]. □

We will repeatedly use the following simple consequence of Proposition B.1.17(1).

Proposition B.1.18. *Let G be an affine algebraic group over a field $\Bbbk$. Let X be an affine scheme of finite type over $\Bbbk$ with an action of G.*

(1) There exists a G-equivariant closed immersion $X \hookrightarrow \mathbb{A}(V)$ where V is a finite dimensional G-representation.

(2) For every G-invariant closed subscheme $Z \subseteq X$, there exists a G-equivariant morphism $f\colon X \to \mathbb{A}(W)$, where W is a finite dimensional G-representation, such that $f^{-1}(0) = Z$.

Proof. Write $X = \operatorname{Spec} A$ and let $f_1, \ldots, f_n$ be $\Bbbk$-algebra generators. By B.1.17(1) there is a finite dimensional G-invariant subspace $V \subseteq A$ containing each f_i. The surjection $\operatorname{Sym}^* V \to A$ induces a G-equivariant embedding $X \hookrightarrow \mathbb{A}(V)$. For (2), let $Z = \operatorname{Spec} A/I$ and let $g_1, \ldots, g_m \in I$ be generators. Letting $W \subseteq I$ be a finite dimensional G-invariant subspace containing each g_i, the G-invariant morphism $f\colon X \to \mathbb{A}(W)$ has the desired property that $f^{-1}(0) = Z$. □

Tori. A subgroup $T \subseteq G$ of an affine algebraic group over a field $\Bbbk$ is called a *torus* (resp., *split torus*) of rank n if $T_{\bar{\Bbbk}} \cong \mathbb{G}^n_{m,\bar{\Bbbk}}$ (resp., $T \cong \mathbb{G}^n_{m,\Bbbk}$ over $\Bbbk$), and a *maximal torus* if it T is not contained in a larger subtorus of G. For example, the set of diagonal matrices in GL_n is a split maximal torus of rank n.

Proposition B.1.19. *Let G be an affine algebraic group over a field $\Bbbk$.*

(1) G contains a maximal torus T such that $T_{\Bbbk'} \subseteq G_{\Bbbk'}$ is a maximal torus for every field extension $\Bbbk \to \Bbbk'$.

(2) If $\Bbbk$ is algebraically closed, all maximal tori are conjugate.

Proof. See [Bor91, §III.8], [Hum75, §34.3-5], [Spr98, Thm. 13.3.6], and [Mil17, Thms. 17.82 and 17.105]. □

There is of course much more to the theory of affine algebraic groups. We quickly mention a few facts that we use.

Jordan decompositions. Recall that an element $g \in \mathrm{GL}_n(\Bbbk)$ is *semisimple* if it becomes diagonalizable after an extension of $\Bbbk$ and *unipotent* if $g - 1$ is nilpotent, i.e., $(1-g)^n = 0$ for some n. If G is an affine algebraic group over a perfect field $\Bbbk$, then for every element $g \in G(\Bbbk)$, there are unique elements $g_s, g_u \in G(\Bbbk)$, called the *semisimple* and *unipotent parts* of g, such that $g = g_s g_u = g_u g_s$ and such that the images of g_s and g_u under any representation $G \to \mathrm{GL}_n$ are semisimple and unipotent, respectively. See [Bor91, §4], [Hum75, §15.3], [Spr98, §2.4], and [Mil17, Thm. 9.17].

Example B.1.20 (Unipotent groups)**.** An affine algebraic group G over a field $\Bbbk$ is *unipotent* if there is a faithful representation V and a basis $V \cong \Bbbk^n$ such that the image of the induced map $G \hookrightarrow \mathrm{GL}(V) \cong \mathrm{GL}_n$ is contained in the subgroup $\mathbb{U}_n$ of upper triangle matrices with 1's along the diagonal. For example, $\mathbb{G}_a$ is unipotent. We have the following equivalences:

$$\begin{aligned} G \text{ is unipotent} &\iff \begin{array}{l} G \text{ has a filtration } 1 = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_n = G \\ \text{of normal subgroups with } G_i/G_{i-1} \cong \mathbb{G}_a \end{array} \\ &\iff V^G \neq 0 \text{ for every nonzero representation } V \\ &\iff \text{every element } g \in G \text{ is unipotent, i.e., } g = g_u. \end{aligned}$$

Every unipotent group is isomorphic to affine space as a scheme. See [Bor91, §4.8], [Hum75, §17.5], [Spr98, §2.4], and [Mil17, §14].

B.1.4 One-parameter subgroups, centralizers, and parabolics

Definition B.1.21 (One-parameter subgroups and characters)**.** If G is an algebraic group over a field $\Bbbk$, a *one-parameter subgroup* (also called a *cocharacter*) is a homomorphism $\lambda\colon \mathbb{G}_m \to G$ of algebraic groups (which is not required to be a subgroup). A *character* is a homomorphism $\chi\colon G \to \mathbb{G}_m$.

Despite the terminology, we do not require that a one-parameter subgroup is injective, e.g., $\mathbb{G}_m \to \mathbb{G}_m, t \mapsto t^2$ is a one-parameter subgroup. We let $\mathbb{X}_*(G)$ be the set of one-parameter subgroups and $\mathbb{X}^*(G)$ be the group of characters. Since any character $\mathbb{G}_m \to \mathbb{G}_m$ is given by $t \mapsto t^d$ for some $d \in \mathbb{Z}$, there is a pairing

$$\langle -,- \rangle\colon \mathbb{X}_*(G) \times \mathbb{X}^*(G) \to \mathbb{X}^*(\mathbb{G}_m) \cong \mathbb{Z}, \quad (\lambda, \chi) \mapsto \chi \circ \lambda.$$

Example B.1.22 (Tori)**.** If $T \cong \mathbb{G}_m^n$ is an n-dimensional torus, then any one-parameter subgroup $\lambda\colon \mathbb{G}_m \to T$ is given by $t \mapsto (t^{\lambda_1}, \cdots, t^{\lambda_n})$ for integers λ_i while a character of T is given by $(t_1, \ldots, t_n) \mapsto t_1^{\chi_1} \cdots t_n^{\chi_n}$ for integers χ_i. We thus have bijections $\mathbb{X}_*(T) \cong \mathbb{Z}^n$ and $\mathbb{X}^*(T) \cong \mathbb{Z}^n$ such that $\langle -,- \rangle\colon \mathbb{X}_*(T) \times \mathbb{X}^*(T) \to \mathbb{Z}$ is the standard inner product.

Example B.1.23 (GL_n)**.** Every one-parameter subgroup λ is contained in a maximal torus, and since maximal tori are conjugate (Proposition B.1.19), there exists $g \in G(\Bbbk)$ such that $g\lambda g^{-1}$ is contained in the maximal torus consisting of diagonal matrices.

Definition B.1.24 (Centralizers, parabolics, and unipotents)**.** Given a one-parameter subgroup $\lambda\colon \mathbb{G}_m \to G$ of an algebraic group, we define the subgroups:

$$\begin{array}{lll} C_\lambda = & \{g \in G \mid \lambda(t)g = g\lambda(t) \text{ for all } t\} & \text{(centralizer)} \\ P_\lambda = & \{g \in G \mid \lim_{t\to 0} \lambda(t)g\lambda(t)^{-1} \text{ exists}\} & \text{(parabolic)} \\ U_\lambda = & \{g \in G \mid \lim_{t\to 0} \lambda(t)g\lambda(t)^{-1} = 1\} & \text{(unipotent).} \end{array}$$

More precisely, there is a subgroup C_λ (resp., P_λ, U_λ) of G which represent the functor assigning a $\Bbbk$-algebra R to the subgroup of elements $g \in G(R)$ such that $\lambda_R = g^{-1}\lambda_R g$ (resp., $\lim_{t\to 0} \lambda_R(t)g\lambda(t)^{-1}$ exists, $\lim_{t\to 0} \lambda(t)g\lambda_R(t)^{-1} = 1$). Note that, by definition, the limit of $\lambda_R(t)g\lambda_R(t)^{-1}$ exists as $t \to 0$ if the natural map $\mathbb{G}_{m,R} \to G$, $t \mapsto \lambda_R(t)g\lambda_R(t)^{-1}$ extends to $\mathbb{A}^1_R \to G$, and the limit is the composition $\operatorname{Spec} R \xhookrightarrow{0} \mathbb{A}^1_R \to G$.

Under the conjugation action of λ on G, C_λ is precisely the fixed locus, while P_λ is the attractor locus G^+_λ as defined in §6.8.1. There is a homomorphism $P_\lambda \to C_\lambda$ defined by $g \mapsto \lim_{t\to 0} \lambda(t)g\lambda(t)^{-1}$ which is the identity on C_λ. This yields a split short exact sequence

$$1 \to U_\lambda \to P_\lambda \to C_\lambda \to 1.$$

Example B.1.25 (GL_n)**.** Let $\lambda\colon \mathbb{G}_m \to \mathrm{GL}_n$ be a one-parameter subgroup. After a change of basis, we can assume that $\lambda(t) = \mathrm{diag}(t^{\lambda_1}, \cdots, t^{\lambda_n})$ with $\lambda_1 \le \cdots \le \lambda_n$. Given $(g_{ij}) \in \mathrm{GL}_n$, $\lambda(t)(g_{ij})\lambda(t)^{-1} = (t^{\lambda_i - \lambda_j} g_{ij})$. If $n_1, \ldots, n_s$ are integers with $\sum_i n_i = n$ such that

$$\lambda_1 = \cdots = \lambda_{n_1} < \lambda_{n_1+1} = \cdots = \lambda_{n_1+n_2} < \cdots < \lambda_{n-n_s+1} = \cdots = \lambda_n,$$

then $C_\lambda = \mathrm{GL}_{n_1} \times \cdots \times \mathrm{GL}_{n_s}$ is the subgroup of block diagonal matrices while P_λ is the subgroup of block upper triangular matrices. For example, if $\lambda(t) = (t^{-1}, t^2, t^2, t^7)$, then

$$U_\lambda = \begin{pmatrix} 1 & * & * & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 1 \end{pmatrix}, P_\lambda = \begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & 0 & 0 & * \end{pmatrix}, \text{ and } C_\lambda = \begin{pmatrix} * & 0 & 0 & 0 \\ 0 & * & * & 0 \\ 0 & * & * & 0 \\ 0 & 0 & 0 & * \end{pmatrix}.$$

We record the following properties of parabolic subgroups. Reductive groups are defined and discussed in §B.1.6.

Proposition B.1.26. *Let G be a connected reductive algebraic group over an algebraically closed field $\Bbbk$, and let $\lambda\colon \mathbb{G}_m \to G$ be a one-parameter subgroup.*

(a) The centralizer C_λ is connected and reductive.

(b) The subgroup P_λ is connected and parabolic, i.e., G/P_λ is projective, and $N_G(P_\lambda) = P_\lambda$.

(c) The subgroup U_λ is the unipotent radical of P_λ, and it acts freely and transitively on the set of one-parameter subgroups of P_λ which are conjugate (under P_λ) to λ.

(d) If $\lambda, \lambda'\colon \mathbb{G}_m \to G$ are one-parameter subgroups, the intersection $P_\lambda \cap P_{\lambda'}$ contains a maximal torus of G.

Proof. For (a)–(c), see [Spr98, §13.4], [Con14, Thm. 4.1.7 and Cor. 5.2.8]. For (d), see [Bor91, Prop. 20.7]. □

Remark B.1.27 (Spherical buildings). The set of one-parameter subgroups of a reductive group can be given the structure of an "ungainly but remarkable metric space" (as described by Mumford in [GIT, p.55]): first introduced by J. Tits, the *spherical building* is the quotient of $\mathbb{X}_*(G)$ by the equivalence relation where $\lambda \sim \rho$ if there exists $g \in P_\lambda(\Bbbk)$ such that $\rho(t^m) = g^{-1}\lambda(t^m)g$ for integers n, m.

B.1.5 Line bundles with G-actions

Definition B.1.28. If G is an algebraic group over a field $\Bbbk$ acting on a $\Bbbk$-scheme U via $\sigma\colon G \times U \to U$, a *line bundle with a G-action* (also called a *G-linearization*) is a line bundle L on U together with an isomorphism $\alpha\colon \sigma^* L \xrightarrow{\sim} p_2^* L$ satisfying the cocycle condition $p_{23}^*\alpha \circ (\mathrm{id} \times \sigma)^*\alpha = (m \times \mathrm{id})^*\alpha$, i.e., the diagram

$$\begin{tikzcd}
 & (\sigma \circ (\mathrm{id} \times \sigma))^* L \arrow[r, "(\mathrm{id} \times \sigma)^*\alpha"] & (p_2 \circ (\mathrm{id} \times \sigma))^* L \arrow[dr, equal] & \\
(\sigma \circ (m \times \mathrm{id}))^* L \arrow[ur, equal] \arrow[dr, "(m \times \mathrm{id})^*\alpha"'] & & & (\sigma \circ p_{23})^* L \arrow[dl, "p_{23}^*\alpha"] \\
 & (p_2 \circ (m \times \mathrm{id}))^* L \arrow[r, equal] & (p_2 \circ p_{23})^* L &
\end{tikzcd}$$

commutes.

When U is projective, a G-action on a very ample line bundle L corresponds to a finite dimensional G-representation $V = \mathrm{H}^0(U, L)$ and a G-equivariant closed immersion $U \hookrightarrow \mathbb{P}(V)$. The cocycle condition in the diagram above is analogous to the cocycle condition in Fpqc Descent for Quasi-Coherent Sheaves (2.1.4). As line bundles on algebraic stacks are defined in terms of fpqc descent, a line bundle on $[U/G]$ is precisely a line bundle on U with a G-action.

Example B.1.29. Under the PGL_{n+1} and SL_{n+1} action on $\mathbb{P}^n$, the line bundle $\mathcal{O}(1)$ admits an action by SL_{n+1} but not by PGL_{n+1}. However, $\mathcal{O}(n+1)$ does admit an action by PGL_{n+1}.

Theorem B.1.30 (Sumihiro's Theorem on Linearizations)**.** *Let G be a smooth, connected, and affine algebraic group over an algebraically closed field $\Bbbk$. Let U be a normal scheme of finite type over $\Bbbk$ with an action of G.*

(1) If L is a line bundle on U, there exists an integer $n > 0$ such that $L^{\otimes n}$ admits a G-action.

(2) If U is quasi-projective, there exists a locally closed embedding $U \hookrightarrow \mathbb{P}(V)$ where V is a finite dimensional G-representation.

(3) Every point $u \in U$ has a G-invariant quasi-projective open neighborhood.

Moreover, (1) holds more generally if $G \to S$ is a smooth affine group scheme with connected fibers (e.g., $G = \mathrm{GL}_{n,S}$) and U is a normal noetherian scheme.

Proof. For (1), see [Sum74, Thm. 1], [Sum75, Lem. 1.2], and [KKLV89, Prop. 2.4]. Part (2) is a direct consequence of (1). For (3), see [Sum74, Lem. 8] and [Sum75, Thm. 3.8]. □

When G is a torus, there is a G-invariant *affine* cover.

Theorem B.1.31 (Sumihiro's Theorem on Torus Actions)**.** *Let U be a normal scheme of finite type over an algebraically closed field $\Bbbk$ with an action of a torus T. Then any point $u \in U$ has a T-invariant affine open neighborhood.*

Proof. See [Sum74, Cor. 2] and [Sum75, Cor. 3.11]. □

Remark B.1.32. Theorems B.1.30 and B.1.31 can fail if U is not normal, e.g., the plane nodal cubic curve has a $\mathbb{G}_m$-action and no $\mathbb{G}_m$-invariant neighborhood of the origin can be embedded $\mathbb{G}_m$-equivariantly into projective space. There is nevertheless a $\mathbb{G}_m$-equivariant *étale* affine neighborhood $\operatorname{Spec} \Bbbk[x, y]/(xy) \to U$ (where x and y have weights 1 and -1). In fact, every non-normal scheme (and even algebraic space) with a $\mathbb{G}_m$-action admits such an étale neighborhood (see Theorem 6.7.24).

B.1.6 Reductivity

Linearly reductive groups are used in the development of Geometric Invariant Theory (GIT) in Chapter 7. In characteristic p, there are three distinct properties—linear reductive, reductive, and geometrically reductive—of algebraic groups:

$$\begin{tikzcd} \text{linearly reductive} \arrow[r, Rightarrow] & \text{geometrically reductive} \arrow[l, Rightarrow, bend right, "\text{char}=0"'] \arrow[r, Rightarrow, bend left, "G \text{ smooth}"] & \text{reductive.} \arrow[l, Rightarrow] \end{tikzcd} \tag{B.1.33}$$

Linear reductive groups are very restrictive in characteristic p: it is a theorem of Nagata [Nag62] that a smooth algebraic group G in characteristic p is linearly reductive if and only if the connected component G^0 is a torus and the order of G/G^0 is prime to p. While it is not much more difficult to develop GIT for geometrically reductive groups (see Remark 6.5.12), it is easier for students to first learn the theory in the context of linear reductive groups.

Linear reductive groups. We denote by $\mathrm{Rep}(G)$ the category of representations of an algebraic group G. If V is a G-representation with coaction $\sigma\colon V \to \Gamma(G, \mathcal{O}_G) \otimes V$, then the *invariants* are $V^G := \{v \in V \mid \sigma(v) = 1 \otimes v\}$. A representation V of G is *irreducible* if every subrepresentation $W \subseteq V$ is either 0 or V.

Definition B.1.34. An affine algebraic group G over a field $\Bbbk$ is *linearly reductive* if the functor $\mathrm{Rep}(G) \to \mathrm{Vect}_{\Bbbk}$, taking a G-representation V to its G-invariants V^G, is exact.

Proposition B.1.35. *Let G be an affine algebraic group over a field $\Bbbk$. The following are equivalent:*

(1) G is linearly reductive;

(1′) The functor $\mathrm{Rep}^{\mathrm{fd}}(G) \to \mathrm{Vect}_{\Bbbk}$, $V \mapsto V^G$, on the category of finite dimensional representations is exact;

(2) Every G-representation (resp., finite dimensional G-representation) is a direct sum of irreducible representations.

(3) Given a G-representation (resp., finite dimensional G-representation) V and a G-invariant subspace $W \subseteq V$, there exists a G-invariant subspace $W' \subseteq V$ such that $V = W \oplus W'$.

(4) For every finite dimensional representation V and fixed $\Bbbk$-point $x \in \mathbb{P}(V)^G$, there exists a G-invariant linear function $f \in \Gamma(\mathbb{P}(V), \mathcal{O}(1))^G$ such that $f(x) \neq 0$.

Proof. Condition (4) translates to: for every surjection $V \to \Bbbk$ onto the trivial representation, there exists $f \in V^G$ mapping to a nonzero element. To see that this implies (1), let $\pi\colon V \to W$ be a surjection of G-representations and $w \in W^G$. By apply Proposition B.1.17 to $\pi^{-1}(\langle w \rangle)$, there is a nonzero finite dimensional G-representation $V' \subseteq V$ surjecting onto $\langle w \rangle$ and (4) implies that there is an element $v \in V'^G \subseteq V^G$ mapping to w. We conclude that (1) $\Leftrightarrow$ (1′) $\Leftrightarrow$ (4).

Denote the finite dimensional conditions in (2) and (3) as (2′) and (3′). The implications (2) $\Rightarrow$ (2′) $\Rightarrow$ (4) and (3) $\Rightarrow$ (3′) $\Rightarrow$ (1′) are easy. As (3) $\Rightarrow$ (2) is also clear, it suffices to show that (1) $\Rightarrow$ (3). To this end, applying the exact functor

$$\mathrm{Mor}_{\mathrm{Rep}(G)}(V/W, -) = \mathrm{Mor}_{\mathrm{Rep}(G)}(\Bbbk, (V/W)^\vee \otimes -) = ((V/W)^\vee \otimes -)^G$$

to the exact sequence $0 \to W \to V \to V/W \to 0$ implies that $V \to V/W$ has a G-invariant section. $\square$

Remark B.1.36. With the terminology introduced in §6.5, G is linearly reductive if and only if $BG \to \operatorname{Spec} \Bbbk$ is cohomologically affine or, equivalently, a good moduli space.

For a field extension $\Bbbk \to \Bbbk'$, G is linearly reductive if and only if $G_{\Bbbk'}$ is; this is easy to see directly but also follows from the more general statement of Lemma 6.5.17. Linear reductive groups are closed under quotients and extensions (Proposition 6.5.19).

Example B.1.37 (Diagonalizable groups)**.** Since every representation of a diagonalizable group scheme is a direct sum of one dimensional representations (Proposition B.1.15), every diagonalizable group scheme is linearly reductive.

Proposition B.1.38 (Maschke's Theorem)**.** *Let G be a finite abstract group viewed as a finite group scheme over a field $\Bbbk$. If the order of G is prime to* $\operatorname{char}(\Bbbk)$*, then G is linearly reductive.*

Proof. If V is a G-representation, averaging over translates gives a G-equivariant $\Bbbk$-linear map

$$R_V\colon V \to V^G, \qquad v \mapsto \frac{1}{|G|}\sum_{g\in G} g\cdot v, \tag{B.1.39}$$

which is the identity on V^G. These maps are functorial with respect to maps $f\colon V \to W$ of G-representations, i.e., $R_W \circ f = f \circ R_V$. It follows that a surjection $V \to W$ of G-representations induces a surjection $V^G \to W^G$ on invariants. □

Example B.1.40 ($\mathbb{Z}/p\mathbb{Z}$)**.** In characteristic p, $G = \mathbb{Z}/p\mathbb{Z}$ is not linearly reductive. To see this, let V be the two-dimensional representation of G where a generator acts via the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. The surjection $V \to \Bbbk$ onto the first component is a surjection of G-representations, but the induced map $V^G \to \Bbbk$ on invariants is the zero map. Geometrically, this corresponds to an action of G on $\mathbb{A}^2 = \mathbb{A}(V)$ such that the G-fixed point $(1,0)$ is contained in every invariant hyperplane. Note however that taking invariants of the pth power map $\operatorname{Sym}^p V \to \operatorname{Sym}^p \Bbbk = \Bbbk$ is surjective, or, in other words, there is a G-invariant *hypersurface* not containing $(1,0)$.

Example B.1.41 ($\mathbb{G}_a$)**.** Over any field $\Bbbk$, the additive group $\mathbb{G}_a$ is not linearly reductive $\mathbb{G}_a$. This time, let $V = \Bbbk^2$ be the two-dimensional representation given by $t \mapsto \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$. The projection $V \to \Bbbk$, defined by $(x,y) \mapsto x$, is a surjection of $\mathbb{G}_a$-representations with no complement. In this case, not only is there no $\mathbb{G}_a$-invariant hyperplane avoiding $(1,0) \in \mathbb{A}(V)$, there is no such $\mathbb{G}_a$-invariant hypersurface.

Remark B.1.42 (Reynolds operator). The map (B.1.39) is called a *Reynolds operator* for the action of G on V. If G is linearly reductive, the canonical projections $R_V\colon V \to V^G$ are Reynold operators, i.e., $\Bbbk$-linear maps which are the identity on V^G and compatible with maps of G-representations. For an action of G on a $\Bbbk$-scheme $\operatorname{Spec} A$ with dual action $A \to \Gamma(G, \mathcal{O}_G) \otimes A$, there is a projection $R_A\colon A \to A^G$. This is not a ring map, but since multiplication $A^G \otimes A \to A$ is a map of G-representations commuting with the Reynold operators, we have that

$$R_A(xy) = xR_A(y) \quad \text{for } x \in A^G,\ y \in A.$$

This is called the *Reynolds identity* and implies that $R_A\colon A \to A^G$ is an A^G-module homomorphism.

In Remark 6.5.10, the Reynolds operator was applied to show that A^G is finitely generated whenever A is. While we use the exactness of the invariant functor to prove the properties of affine GIT quotients in Corollary 6.5.8, the Reynolds operator can also be used; see [GIT, §1.2]. Moreover, an effective method to establish linearly reductivity is to construct a Reynolds operator. This was the proof technique in Maschke's Theorem (B.1.38) and it will be used again in Theorem B.1.43.

Reductive groups. A smooth affine algebraic group G over an algebraically closed field $\Bbbk$ is called *reductive* if every smooth, connected, unipotent, and normal subgroup of G is trivial.[2] Over $\mathbb{C}$, an G is reductive if and only if it is the complexification of any maximal compact subgroup [Hoc65, XVII.5]. Over an arbitrary field $\Bbbk$, G is called reductive if $G_{\overline{\Bbbk}}$ is. Reductive groups are a particularly nice class of algebraic groups appearing in many branches of mathematics. They admit an explicit classification in terms of their root data. See [Bor91, Hum75, Spr98, Mil17].

For a smooth affine algebraic group G, there are subgroups $R(G)$ and $R_u(G)$ of G, called the *radical* and *unipotent radical*, which are maximal among connected, normal, and solvable (resp., connected, normal, and unipotent) subgroups, which commute with separable field extensions. Over a perfect field $\Bbbk$, G is reductive if and only if $R_u(G)$ is trivial, and the quotient $G/R_u(G)$ is reductive. On the other hand, G is defined to be *semisimple* if $R(G)$ is trivial. For a reductive group G, the center $Z(G)$ is diagonalizable and contains $R(G)$ as its largest subtorus, and the quotient $G/R(G)$ is semisimple.

The classical algebraic groups GL_n, PGL_n, SL_n, and SP_{2n} are reductive in every characteristic. As we develop GIT for actions by linearly reductive groups, it is imperative to know that these groups are linearly reductive in characteristic 0.

Theorem B.1.43. *In characteristic* 0*, a reductive algebraic group is linearly reductive. The converse is true in every characteristic for smooth algebraic groups.*

Proof. In [Hil90], Hilbert established the linearly reductivity for SL_n and GL_n over $\mathbb{C}$ using a explicit differential operator well-known to 19th century invariant theorists: the Ω-*process*. We will sketch the argument for $G = \mathrm{GL}_n$ over $\mathbb{C}$. Write $\Gamma(\mathrm{GL}_n, \mathcal{O}_{\mathrm{GL}_n}) = \mathbb{C}[X_{ij}]_{\det}$. Let V be a finite dimensional GL_n-representation such that the scalar matrices act with weight k, and let $\sigma\colon \mathrm{Sym}^* V \to \mathbb{C}[X_{ij}]_{\det} \otimes \mathrm{Sym}^* V$ be the dual action on $\mathbb{A}(V) = \operatorname{Spec} \mathrm{Sym}^* V$. The differential operator

$$\Omega := \det\left(\frac{\partial}{\partial X_{ij}}\right)$$

acts linearly on $\mathbb{C}[X_{ij}]_{\det}$ and $\mathbb{C}[X_{ij}]_{\det} \otimes \mathrm{Sym}^* V$. One checks that the map

$$V \to V^{\mathrm{GL}_n}, \quad f \mapsto \frac{1}{\Omega^k\big(\det(X_{ij})^k\big)}\Omega^k\big(\det(X_{ij})^k \sigma(f)\big)$$

defines a Reynolds operator, which implies that GL_n is linearly reductive. The argument is algebraic and works over every field of characteristic 0. See [Stu08, §4.3], [Dol03, §2.1] and [DK15, §4.5.3].

Extending an integral procedure developed by Hurwitz, Schur, and Cartan, Weyl proved that every reductive algebraic group over $\mathbb{C}$ is linearly reductive [Wey26, Wey25]. The technique is now referred to as 'Weyl's unitarian trick'. A compact Lie group K has a left K-invariant finite measure μ, called the *left Haar measure*. For a finite dimensional K-representation V, averaging gives a $\Bbbk$-linear map

$$V \to V^K, \quad v \mapsto \frac{1}{\int_K d\mu(g)} \int_K (g \cdot v) d\mu(g)$$

constant on V^K and compatible with maps of K-representations. This is a Reynolds operator (Remark B.1.42) just as in Maschke's Theorem (B.1.38), and implies that

[2] Sometimes G is also assumed to be connected. For a *reductive group scheme* $G \to S$, there is no such ambiguity in the literature: G is smooth and affine over S with connected and reductive geometric fibers [SGA3$_{\mathrm{III}}$, Exp. XIX, Defn. 2.7].

$V \mapsto V^K$ is exact. For a reductive algebraic group G over $\mathbb{C}$, there is a real compact Lie subgroup $K \subseteq G(\mathbb{C})$ which is dense in the Zariski topology. For example, for GL_n, $K = U_n$ is the subgroup of unitary matrices (hence the name 'unitarian trick'). For a finite dimensional G-representation V, there is an identification $V^K = V^G$, and since the functor taking K-invariant is exact, so is the functor taking G-invariants. See also [Dol03, §3.2] and [Bum13, Thm. 14.3].[3]

There is also an algebraic argument using the *Casimir operator*. First, one reduces to the case that G is semisimple because every reductive group is an extension of a torus by a semisimple group. Given a Lie algebra representation $\rho\colon \mathfrak{g} \to \mathfrak{gl}(V)$ of G, there is a symmetric bilinear form on $\mathfrak{g}$ defined by $\langle x, y\rangle = \mathrm{Tr}(\rho(x) \circ \rho(y))$. Letting $\{e_i\}$ be a basis of $\mathfrak{g}$ and $\{e_i'\}$ be a dual basis with respect to $\langle -, -\rangle$, the *Casimir operator* is the $\mathfrak{g}$-endomorphism $c_V := \sum_{i=1}^n \rho(e_i) \circ \rho(e_i')$ on V. To show that G is linearly reductive, it suffices to find a complement of any codimension one irreducible subspace $W \subseteq V$. As G is semisimple, G acts trivial on V/W and therefore so does $\mathfrak{g}$. It follows that $\mathfrak{g}$ takes V into W and therefore so does c_V, i.e., $c_V(V) \subseteq W$. On the other hand, since W is irreducible, c_V acts on W by multiplication by a scalar (Schur's lemma). It follows that $\ker(c_V) \subseteq V$ is a complement of W. See also [Mil17, Thm. 22.42], [Muk03, §4.3], [Hum78, §6.2], and [DK15, §4.5.2].

For the converse, we need to show that the unipotent radical $R_u(G)$ of a linearly reductive group G is trivial. Since $G/R_u(G)$ is affine, Matsushima's Theorem (6.5.21) for *linearly reductive groups* implies that $R_u(G)$ is linearly reductive. However, a non-trivial unipotent group is not linearly reductive. Indeed, by the structure of unipotent groups, it suffices to show that $\mathbb{G}_a$ is not linearly reductive (Example B.1.41). See also [NM64]. □

Example B.1.44. The algebraic groups such as GL_n, PGL_n, SL_n, or SP_{2n} are not linearly reductive in characteristic p. For example, in characteristic 2, consider the action of SL_2 acts on the space $V = \mathrm{Sym}^2(\Bbbk^2) = \{Ax^2 + Bxy + Cy^2\}$ of degree 2 binary forms. The subspace W consisting of squares L^2 of linear forms is a GL_2-invariant subspace with no complement; the quotient $V \to V/W = \mathbb{C}$ is given by $(A, B, C) \mapsto B$. While there is no invariant *linear* function not vanishing at $(0, 1, 0)$, the discriminant $\Delta = B^2 \in \mathrm{Sym}^2 V^\vee$ is an invariant function nonzero at $(0, 1, 0)$ (verifying the geometric reductivity condition given below).

Theorem B.1.45 (Matsushima's Theorem)**.** *Let G be a reductive group over a field $\Bbbk$. Then a subgroup $H \subseteq G$ is reductive if and only if G/H is affine.*

Proof. See Proposition 6.5.21 for a proof when G is linearly reductive. The general case can be proven in a similar way relying on a generalization of Serre's Criterion for Affineness (4.5.16): an algebraic space U is affine if for every surjection $\mathcal{A} \to \mathcal{B}$ of $\mathcal{O}_U$-algebras, every global section of $\mathcal{B}$ has a positive power that lifts to a global section of $\mathcal{A}$. See also [Mat60], [BB63], [Ric77], [FS82], and [Alp14, Thm 9.4.1]. □

Geometrically reductive groups. An affine algebraic group G over a field $\Bbbk$ of characteristic $p \geq 0$ is called *geometrically reductive* if for every surjection $V \to W$ of G-representations and $w \in W^G$, there exists $n > 0$ such that w^{p^n} is in the image of $\mathrm{Sym}^{p^n} V \to \mathrm{Sym}^{p^n} W$. This condition translates to the geometric

[3] By the limit methods of §B.3, this analytic argument suffices to show the linear reductivity of a reductive group G over every characteristic 0 field: by limit methods (§B.3), there is a subfield $\Bbbk' \subseteq \Bbbk$ of finite transcendence degree over $\mathbb{Q}$ and a group scheme $G' \to \operatorname{Spec} \Bbbk'$ such that $G'_{\Bbbk} = G$. Choosing an embedding $\Bbbk' \hookrightarrow \mathbb{C}$ and using that reductivity and linear reductivity are insensitive to separable field extensions, we see that if $G'_{\mathbb{C}}$ is linearly reductive, so is G.

property analogous to Proposition B.1.35(4): for a fixed $\Bbbk$-point $x \in \mathbb{P}(V)^G$, there is an invariant homogenous polynomial $f \in \Gamma(\mathbb{P}(V), \mathcal{O}(p^n))^G$ for $n > 0$ such that $f(x) \neq 0$. In characteristic 0, linear reductivity is equivalent to geometric reductivity.

In an effort to extend GIT to actions by reductive groups such as SL_n and GL_n in positive characteristic, Mumford conjectured in [GIT, preface] a reductive group is geometrically reductive. This conjecture was resolved by Haboush.

Theorem B.1.46 (Haboush's Theorem)**.** *A reductive group G over a field $\Bbbk$ is geometrically reductive.*

Proof. See [Hab75]. See also [SS11]. $\square$

The converse is true when G is smooth. In fact, an affine algebraic group G is geometrically reductive if and only if G_{red} is reductive. A smooth algebraic group G in characteristic p is linearly reductive if and only if the connected component G^0 is a torus and the order of G/G^0 is prime to p [Nag62]. Every finite (possibly non-reduced) group scheme G is geometrically reductive, and is linearly reductive if and only if G^0 is diagonalizable and G/G^0 has order prime to p [HR15, Thm. 1.2]. A commutative algebraic group G is reductive if and only if it is diagonalizable. We also point out that reductivity of a smooth algebraic group G is characterized by the condition that the ring of invariants is finitely generated for every coaction on a finitely generated $\Bbbk$-algebra.

B.1.7 Principal G-bundles

A principal G-bundle is an algebraic version of a topological fiber bundle $P \to T$ where G acts freely and transitively on P with quotient $T = P/G$, e.g., $\mathbb{A}^2 \setminus 0 \to \mathbb{P}^1$ is a principal $\mathbb{G}_m$-bundle. Principal G-bundles and their properties are essential in the development of the theory of algebraic stacks. For instance, we define an object of a quotient stack $[U/G]$ over a scheme T as a principal G-bundle $P \to T$ together with a G-equivariant map $P \to U$. The reader may consult [Poo17, §5.12] and [Bal09] for additional background on principal G-bundles.

Definition B.1.47. Let $G \to S$ be an fppf affine group scheme. A *principal G-bundle over an S-scheme X* is a scheme P over X with an action of G via $\sigma\colon G \times_S P \to P$ such that $P \to X$ is a G-invariant fppf morphism (where X has the trivial action) and

$$(\sigma, p_2)\colon G \times_S P \to P \times_X P, \qquad (g, p) \mapsto (gp, p)$$

is an isomorphism.

Observe that a principal G-bundle over X is the same data as a principal $G \times_S X$-bundle over X. *Morphisms of principal G-bundles* are G-equivariant morphisms of schemes. A principal G-bundle $P \to X$ is *trivial* if there is an G-equivariant isomorphism $P \cong G \times_S X$, where G acts on $G \times_S X$ via multiplication on the first factor.

A principal G-bundle $P \to X$ can also be viewed as a *G-torsor*,(Definition 6.4.1), which is a general concept for a sheaf P of sets on a site with a free and transitive action of a sheaf G of groups. When $G \to S$ is an fppf affine group scheme, there is an equivalence of categories between principal G-bundles and G-torsors; see Example 6.4.5. In these notes, we will always distinguish between these two notions, but in conversation or the literature, they are often conflated.

Exercise B.1.48. Show that a morphism of principal G-bundles is necessarily an isomorphism.

Principal G-bundles can be trivialized fppf locally, and even étale locally if G is smooth.

Proposition B.1.49. *Let $G \to S$ be an fppf affine group scheme and $P \to X$ be a G-equivariant morphism of S-schemes where X has the trivial action. Then $P \to X$ is a principal G-bundle if and only if there exists an fppf morphism $X' \to X$ such that $P \times_X X'$ is isomorphic to the trivial principal G-bundle $G \times_S X'$ over X'. Moreover, if $G \to S$ is smooth, we can arrange that $X' \to X$ is surjective and étale.*

Proof. The ($\Rightarrow$) direction follows from the definition by taking $X' = P \to X$. For ($\Leftarrow$), after base changing $G \to S$ by $X \to S$, we may assume that G is defined over X. Let $G_{X'}$ and $P_{X'}$ be the base changes of G and P along $X' \to X$. The base change of the action map $(\sigma, p_2)\colon G \times_X P \to P \times_X P$ along $X' \to X$ is the action map $G_{X'} \times_{X'} P_{X'} \to P_{X'} \times_{X'} P_{X'}$ of $G_{X'}$ acting on $P_{X'}$ over X', which is trivial as $P_{X'}$ is trivial. Since the property of being an isomorphism is an Fpqc Local Property on the Target (2.1.26), we conclude that $(\sigma, p_2)\colon G \times_X P \to P \times_X P$ is an isomorphism. If G is smooth, then $X' = P \to S$ is a surjective smooth morphism such that $P_{X'}$ is trivial. Since there is a section of $X' \to S$ after a surjective étale morphism $S' \to S$ (Corollary A.3.5), $P_{S'}$ is also trivial. $\square$

Examples of principal G-bundles.

Exercise B.1.50. Let L/K be a finite Galois extension and $G = \mathrm{Gal}(L/K)$ be its Galois group viewed as a finite group scheme over $\operatorname{Spec} K$. Show that $\operatorname{Spec} L \to \operatorname{Spec} K$ is a principal G-bundle.

Exercise B.1.51 (Line bundles vs $\mathbb{G}_m$-bundles)**.** If X is a scheme, show that there is a covariant equivalence of categories

$$\begin{aligned}\{\text{line bundles on } X\} &\xrightarrow{\sim} \{\text{principal } \mathbb{G}_m\text{-bundles on } X\} \\ L &\mapsto \mathbb{A}(L^\vee) \smallsetminus X = \operatorname{Spec}(\operatorname{Sym}^* L^\vee) \smallsetminus \text{zero section}\end{aligned}$$

between the *groupoids* of line bundles on X (where the only morphisms allowed are isomorphisms) and principal $\mathbb{G}_m$-bundles on X.

Exercise B.1.52 (S_d-bundles)**.** If X is a scheme and $d \geq 1$, show that there there is an equivalence of *groupoids*

$$\begin{aligned}\{\text{finite, étale, and degree } d \text{ covers of } X\} &\xrightarrow{\sim} \{\text{principal } S_d\text{-bundles over } X\} \\ (Y \to X) &\mapsto (\underbrace{Y \times_X \cdots \times_X Y}_{d \text{ times}} \smallsetminus \Delta \to X) \\ (P/S_{d-1} \to X) &\leftarrow\!\shortmid (P \to X).\end{aligned}$$

For the rightward map, the symmetric group S_d acts on the d-fold fiber product $Y \times_X \cdots \times_X Y$ by permutation, and Δ denotes the big diagonal, i.e., the S_d-equivariant closed locus of d-tuples where at least two points coincide. Alternatively, $Y \times_X \cdots \times_X Y \smallsetminus \Delta$ can be identified with the scheme $\underline{\mathrm{Isom}}_X(X \times \{1, \ldots, d\}, Y)$ parameterizing isomorphisms between the trivial degree d cover and Y. For the leftward map, P/S_{d-1} denotes the quotient of the free action by the subgroup $S_{d-1} \subseteq S_d$ fixing the dth index.

Exercise B.1.53.

(a) Show that the standard projection $\mathbb{A}^{n+1} \smallsetminus 0 \to \mathbb{P}^n$ is a principal $\mathbb{G}_m$-bundle.

(b) For each line bundle $\mathcal{O}(d)$ on $\mathbb{P}^n$, explicitly determine the corresponding principal $\mathbb{G}_m$-bundle. In particular, which $\mathcal{O}(d)$ correspond to the principal $\mathbb{G}_m$-bundle of (a)?

Exercise B.1.54. Let $G \to S$ be an fppf affine group scheme.

(a) For principal G-bundles P and Q over an S-scheme X, show that the functor

$$\underline{\mathrm{Isom}}_X(P, Q) \colon \mathrm{Sch}/X \to \mathrm{Sets},$$

assigning a X-scheme T to the set of isomorphisms of the principal G-bundles $P \times_X T$ and $Q \times_X T$, is representable by a principal G-bundle over X.

(b) For a principal G-bundle $P \to X$, show that $\underline{\mathrm{Aut}}_X(P) := \underline{\mathrm{Isom}}_X(P, P)$ is isomorphic to $G \times^G P := (G \times P)/G$, where $h \cdot (g, p) = (h^{-1}gh, h \cdot p)$.

Exercise B.1.55 (Frame bundles)**.** Let T be a scheme and E be a vector bundle over X of rank n.

(a) The *frame bundle* Fr_E is the functor $\underline{\mathrm{Isom}}_{\mathcal{O}_X}(\mathcal{O}_X^{\oplus n}, E)$ on Sch/X, i.e

$$\begin{aligned} \mathrm{Fr}_E \colon \mathrm{Sch}/X &\to \mathrm{Sets} \\ (T \to X) &\mapsto \{\text{trivializations } \mathcal{O}_T^{\oplus n} \xrightarrow{\sim} E_T\}. \end{aligned}$$

Recalling from Exercise 0.3.20that the functor $\underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{O}_X^{\oplus n}, E)$ is representable by the scheme $H := \mathbb{A}(\mathscr{H}om_{\mathcal{O}_X}(\mathcal{O}_X^{\oplus n}, E)^\vee) = \mathbb{A}((E^\vee)^{\oplus n})$, show that Fr_E is representable by the open subscheme of H defined by $\det(u) \neq 0$, where $u \colon \mathcal{O}_H^{\oplus n} \to E_H$ is the universal homomorphism. Moreover, show that $\mathrm{Fr}_E \to X$ is a principal GL_n-bundle.

(b) The *projectivized frame bundle* $\mathbb{P}\mathrm{Fr}_E$ is the functor

$$\begin{aligned} \mathbb{P}\mathrm{Fr}_E \colon \mathrm{Sch}/X &\to \mathrm{Sets} \\ (T \to X) &\mapsto \left\{ (L, \alpha) \;\middle|\; \begin{array}{l} L \text{ is a line bundle on } T \text{ and} \\ \alpha \colon \mathcal{O}_T^{\oplus n} \xrightarrow{\sim} E_T \otimes L \text{ is an isomorphism} \end{array} \right\} \Big/ \sim, \end{aligned}$$

where $(L, \alpha) \sim (L', \alpha')$ if there is an isomorphism $\beta \colon L \to L'$ with $\alpha' = (\mathrm{id} \otimes \beta) \circ \alpha$. Show that $\mathbb{P}\mathrm{Fr}_E$ is representable by the open subscheme of $\mathbb{P} := \mathbb{P}(\mathscr{H}om_{\mathcal{O}_X}(\mathcal{O}_X^{\oplus n}, E)^\vee) = \mathbb{P}((E^\vee)^{\oplus n})$ defined by $\det(u) \neq 0$, where

$$u \colon \mathcal{O}_{\mathbb{P}}^{\oplus n} \to E_{\mathbb{P}} \otimes \mathcal{O}_{\mathbb{P}}(1)$$

is the homomorphism corresponding to the universal quotient $\mathscr{H}om_{\mathcal{O}_{\mathbb{P}}}(\mathcal{O}_{\mathbb{P}}^{\oplus n}, E_{\mathbb{P}})^\vee \twoheadrightarrow \mathcal{O}_{\mathbb{P}}(1)$. Moreover, show that $\mathbb{P}\mathrm{Fr}_E \to X$ is a principal PGL_n-bundle.

Exercise B.1.56 (Vector bundles vs GL-bundles)**.** For a scheme X, show that the assignment of a vector bundle E to the frame bundle Fr_E defines an equivalence of *groupoids*

$$\begin{aligned} \{\text{vector bundles over } X\} &\xrightarrow{\sim} \{\text{principal } \mathrm{GL}_n\text{-bundles over } X\} \\ E &\mapsto \mathrm{Fr}_W \end{aligned}$$

with the inverse defined by assigning $P \to X$ to the vector bundle whose total space is the quotient $P \times^{\mathrm{GL}_n} \mathbb{A}^n := (P \times \mathbb{A}^n)/\mathrm{GL}_n$ of the diagonal GL_n-action on $P \times \mathbb{A}^n$.

Exercise B.1.57 (SL-bundles)**.** Show that the groupoid of principal SL_n-bundles over a scheme X is equivalent to the groupoid of pairs (V, α) where V is a vector bundle on X of rank n and $\alpha\colon \mathcal{O}_X \xrightarrow{\sim} \det V$ is a trivialization. A morphism $(V', \alpha') \to (V, \alpha)$ of pairs is an isomorphism $\phi\colon V' \to V$ such that $\alpha' = \alpha \circ \det \phi$.

Exercise B.1.58 (Orthogonal group)**.** Let $\Bbbk$ be a field with $\mathrm{char}(\Bbbk) \neq 2$, and let V be an n dimensional vector space with a non-degenerate quadratic form q. Let $O(q) \subseteq \mathrm{GL}(V)$ be the subgroup of invertible matrices preserving the quadratic form. If $q = x_1^2 + \cdots + x_n^2$ is the diagonalized quadratic form, $O(q) = O_n$ is the set of orthogonal matrices A (i.e., $AA^\top = I$). Show that there is a bijection between principal $O(q)$-bundles over a $\Bbbk$-scheme X and vector bundles of rank n on X with a non-degenerate quadratic form.

Example B.1.59 (Special groups)**.** An affine algebraic group G over a field $\Bbbk$ is *special* if every principal G-bundle $P \to T$ is Zariski locally trivial. For example, GL_n (e.g., $\mathbb{G}_m = \mathrm{GL}_n$) is a special group as principal GL_n-bundles correspond to vector bundles (Exercise B.1.56). It is also true that $\mathbb{G}_a$ is special. One way to see this is to use that quasi-coherent cohomology can be computed either in the Zariski or small étale site (Proposition 4.1.43). Viewing the structure sheaf $\mathcal{O}_T$ as $\mathbb{G}_a$, this implies that $\mathrm{H}^1(T, \mathbb{G}_a) = \mathrm{H}^1(T_{\text{ét}}, \mathbb{G}_a)$ and it follows that there is a bijective correspondence between $\mathbb{G}_a$-torsors in the Zariski and small étale topology.

If $1 \to K \to G \to Q \to 1$ is an exact sequence of affine algebraic groups and both K and Q are special, it is not hard to see that G is also special. It follows from the characterization of unipotent groups as extensions of $\mathbb{G}_a$ that every unipotent group is special.

Inducing G-bundles and reduction of structure group.

Definition B.1.60 (Induced G-bundles)**.** Let $H \to G$ be a homomorphism of fppf affine groups schemes over a scheme S. If $P \to X$ is a principal H-bundle, the *principal G-bundle induced by P via $H \to G$* is

$$G \times^H P \to X,$$

where $G \times^H P$ is the quotient $(G \times P)/H$ of the action $h \cdot (g, p) = (gh^{-1}, hp)$, and where G acts on $G \times^H P$ via $g' \cdot (g, p) = (g'g, p)$.

Exercise B.1.61. Verify that $G \times^H P \to X$ is a principal G-bundle.

Definition B.1.62 (Reduction of structure group)**.** Let $H \to G$ be a homomorphism of fppf affine groups schemes over a scheme S. If $Q \to X$ is a principal G-bundle, a *reduction of structure group of Q by $H \to G$* is a principal H-bundle $P \to X$ and an isomorphism $Q \xrightarrow{\sim} G \times^H P$ of principal G-bundles.

Lemma B.1.63. *Let $H \to G$ be a monomorphism of fppf affine groups schemes over a scheme S, and let $Q \to X$ be a principal G-bundle. A reduction of structure group of Q by $H \to G$ is equivalent to giving a section of Q/H over X.*

Proof. A section $s\colon X \to Q/H$ induces a principal H-bundle $P \to X$ via pullback

$$\begin{tikzcd} P \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & Q \arrow[d] \\ X \arrow[r, hook, "s"] & Q/H, \end{tikzcd}$$

and the induced map $G \times^H P \to Q$, defined by $(g, p) \mapsto g \cdot p$, is an isomorphism of principal G-bundles. Conversely, by precomposing an isomorphism $G \times^H P \xrightarrow{\sim} Q$ with the H-equivariant inclusion $P \to G \times^H P$, given by $p \mapsto (\mathrm{id}, p)$, defines an H-equivariant map $P \to Q$ which descends under the H-action to a section $X = P/H \to Q/H$. □

Isomorphism classes of principal G-bundles over a scheme X is classified by the étale cohomology group $\mathrm{H}^1(X_{\text{ét}}, G)$; see Exercise 6.4.30.

Exercise B.1.64. Let $1 \to K \to G \to Q \to 1$ be an exact sequence of abstract abelian groups, which induces a short exact sequence

$$\mathrm{H}^1(X_{\text{ét}}, K) \to \mathrm{H}^1(X_{\text{ét}}, G) \to \mathrm{H}^1(X_{\text{ét}}, Q)$$

of étale cohomology groups over a scheme X. Show that a principal G-bundle $P \to X$ admits a reduction of structure group to K if and only if the class of $[P] \in P \in \mathrm{H}^1(X_{\text{ét}}, G)$ maps to $0 \in \mathrm{H}^1(X_{\text{ét}}, Q)$.

Exercise B.1.65. Let $G \to S$ be an fppf affine group scheme, and let P_1 and P_2 be principal G-bundles over an S-scheme X. Show that a reduction of structure of $P_1 \times P_2 \to X \times X$ by the diagonal $G \to G \times G$ corresponds to an isomorphism $P_1 \to P_2$ of principal G-bundles.

Brauer–Severi schemes and Azumaya algebras.

Exercise B.1.66 (Brauer–Severi schemes)**.** A morphism $P \to X$ of schemes is a *Brauer–Severi scheme of relative dimension* r if there exists an étale cover $X' \to X$ and an isomorphism $P \times_X X' \cong \mathbb{P}^r_{X'}$. An example of a non-trivial Brauer-Severi scheme is $\operatorname{Proj} \mathbb{R}[x, y, z]/(x^2 + y^2 + z^2) \to \operatorname{Spec} \mathbb{R}$. Show that

$$\begin{aligned} \{\text{Brauer–Severi schemes of rel. dim. } r \text{ over } X\} &\to \{\text{principal } \mathrm{PGL}_r\text{-bundles over } X\} \\ X &\mapsto \underline{\mathrm{Isom}}_X(\mathbb{P}^r_X, X) \\ (P \times \mathbb{P}^r)/\mathrm{PGL}_r &\leftarrow\!\shortmid P \end{aligned}$$

defines an equivalence of groupoids.

Exercise B.1.67. Let $P \to X$ be a proper, flat, and finitely presented morphism of schemes. Assume that for every geometric point $\operatorname{Spec} \Bbbk \to X$, the geometric fiber $P \times_X \Bbbk$ is isomorphic to $\mathbb{P}^1_{\Bbbk}$. Show that $P \to X$ is a Brauer–Severi scheme of relative dimension 1.

Approach 1 (local-to-global): Show that for every point $x \in X$, there is a finite and separable field extension $\kappa(s) \to K$ such that $P \times_X K \cong \mathbb{P}^1_K$. Then show that there an étale neighborhood $(X', x') \to (X, x)$ such that $K \cong \kappa(x')$ over $\kappa(x)$. Assuming now that $P \times_s \kappa(s) \cong \mathbb{P}^1_{\kappa(s)}$, use deformation theory (Proposition C.2.4) to show that there are compatible isomorphisms $P \times_X \mathcal{O}_{X,x}/\mathfrak{m}_x^n \cong \mathbb{P}^1_{\mathcal{O}_{X,x}/\mathfrak{m}_x^n}$ for $n > 0$. Use Grothendieck's Existence Theorem (C.5.3) to show that $P \times_X \widehat{\mathcal{O}}_{X,x} \cong \mathbb{P}^1_{\widehat{\mathcal{O}}_{X,x}}$. Finally, apply Artin Approximation (B.5.18) to show that there is an étale neighborhood $(X', x') \to (X, x)$ such that $P \times_X X' \cong \mathbb{P}^1_{X'}$.

Approach 2 (direct): Assuming that there is a section $\sigma \colon X \to P$ of $\pi \colon P \to X$, show that every point $x \in X$ has an open neighborhood $U \subseteq X$ such that $P \times_X U \cong \mathbb{P}^1_U$. Letting L be the line bundle on P corresponding to the Cartier divisor σ, use

Cohomology and Base Change (A.6.8) to show that $\mathcal{E} := \pi_* \mathcal{L}$ is a rank 2 vector bundle on X, that $\pi^* \mathcal{E} \twoheadrightarrow \mathcal{L}$ is surjective, and that $P \cong \mathbb{P}(\mathcal{E})$ over X. Conclude by choosing an open neighborhood of $x \in X$ where $\mathcal{E}$ is trivial. Returning to the general case, show that there is an effective divisor D associated to $\Omega^\vee_{P/X}$ such that $D \to X$ is étale. Reduce to the case where $P \to X$ has a section by base changing by $D \to X$. See also [Har77, Prop. 25.3 and Exc. 25.2].

Exercise B.1.68 (Azumaya algebras)**.** An *Azumaya algebra of rank* r^2 over a scheme X is a (possibly non-commutative) associative $\mathcal{O}_X$-algebra A, which is coherent as an $\mathcal{O}_X$-module, such that there is an étale covering $X' \to X$ with $A \otimes_{\mathcal{O}_X} \mathcal{O}_{X'}$ isomorphic to the matrix algebra $M_r(\mathcal{O}_X)$; see [Mil80, §IV.2]. An Azumaya algebra over a field $\Bbbk$ is a central simple algebra (i.e., a finite dimensional associative $\Bbbk$-algebra which is simple and whose center is $\Bbbk$); the quaternions defines a central simple algebra over $\mathbb{R}$. Show that the assignment

$$A \mapsto \underline{\mathrm{Isom}}_X(M_r(\mathcal{O}_X), A)$$

defines a bijection between Azumaya algebras of rank r^2 over X and PGL_n-torsors over X.

Remark B.1.69. Exercises B.1.66 and B.1.68 provide bijections

$$\begin{aligned} \{\text{Azumaya algebras of rank } r^2\} &\simeq \{\text{principal } \mathrm{PGL}_n\text{-torsors}\} \\ &\simeq \{\text{Brauer–Severi schemes of relative dimension } r\} \end{aligned}$$

on sets of isomorphism classes of objects over a scheme X. The composition of these bijections can be interpreted as the map taking an Azumaya algebra $\mathcal{A}$ over X to the Brauer–Severi scheme defined as the closed subscheme $X \subseteq \mathrm{Gr}_X(r, \mathcal{A})$ classifying rank r right ideals. The *Brauer group* $\mathrm{Br}(X)$ is defined in terms of Azumaya algebras and closely related to $\mathbb{G}_m$-gerbes; see Remark 6.4.37 and Exercise 6.4.38.

In this appendix, we cover several important topics needed in the development of moduli theory that are not always covered in a first course in algebraic geometry.

B.2 Birational geometry and positivity

We summarize basic facts of birational geometry needed in our development of moduli theory. In Stable Reduction (5.5.1) for $\overline{M}_g$, the birational geometry of surfaces in §B.2.1 is used crucially . To prove Kollár's Criterion for Ampleness (5.9.2), which we apply to prove that $\overline{M}_g$ is projective, we appeal to the Nakai–Moishezon Criterion for Ampleness (B.2.28) for ampleness and properties of nef vector bundles in §B.2.4.

B.2.1 Birational geometry of surfaces

For an integral noetherian scheme X, a *resolution of singularities* is a proper birational morphism $X' \to X$ from an integral regular scheme X'.

Theorem B.2.1 (Existence of Resolutions)**.** *Every two dimensional integral noetherian scheme X has a resolution of singularities.*

Proof. This was shown by Zariski in characteristic 0 [Zar39], by Abhyankar in characteristic p [Abh56], and by Lipman in mixed characteristic [Lip78]. See also [Kol07, §2] and [SP, Tag 0BGP]. □

Theorem B.2.2 (Existence of Minimal Resolutions)**.** *Let X be a two dimensional integral noetherian scheme. There exists a resolution of singularities $\pi\colon \widetilde{X} \to X$ such that every other resolution of singularities $Y \to X$ factors as $Y \to \widetilde{X} \to X$. Moreover, $K_{\widetilde{X}} \cdot E \geq 0$ for every π-exceptional curve E.*

Proof. See [Kol07, Thm. 2.16]. □

Theorem B.2.3 (Existence of Embedded Resolutions)**.** *Let X be a regular scheme of dimension 2 and $Y \subseteq X$ be a subscheme of pure dimension one. Assume that for every irreducible component $Z \subseteq Y$, the normalization $\widetilde{Z} \to Z$ is finite. Then there is a finite sequence of blowups*

$$X_n \to \cdots \to X_1 \to X$$

at reduced closed points such that the preimage $Y_n \subseteq X_n$ of Y is an effective Cartier divisor supported on a normal crossings divisor, i.e., $(Y_n)_{\text{red}}$ is nodal.

Proof. See [Har77, Thm V.3.9], [Kol07, Thm. 1.47], and [SP, Tag 0BIC]. □

Theorem B.2.4 (Factorization of Birational Maps)**.** *Let X and Y be regular, integral, and noetherian schemes of dimension two. Every proper birational morphism $f\colon X \to Y$ is the composition of blowups at reduced closed points.*

Proof. See [Har77, Thm V.5.5], [Kol07, Thm 2.13], and [SP, Tag 0C5R]. □

Theorem B.2.5 (Hodge Index Theorem for Exceptional Curves)**.** *Let $f\colon X \to Y$ be a projective and generically finite morphism of noetherian schemes of dimension 2, where X is regular and Y is quasi-projective over a field or DVR. Let $E_1, \ldots, E_n$ be the exceptional curves. Then the intersection form matrix $(E_i \cdot E_j)$ is negative-definite. In particular, $E_i^2 < 0$ for each i.*

Proof. See [Kol07, Thm 2.12]. □

Theorem B.2.6 (Castelnuovo's Contraction Theorem)**.** *Let X be a regular scheme of dimension 2 which is projective over either a field $\Bbbk$ or a DVR R with residue field $\Bbbk$, and let $E = \mathbb{P}^1_{\Bbbk} \subseteq X$ be a smooth rational curve with $E^2 < 0$. Then there is a projective morphism $X \to Y$ to a projective surface and a point $y \in Y$ such that $f^{-1}(y) = E$ and $X \setminus E \to Y \setminus \{y\}$ is an isomorphism. If $E^2 = -1$, then Y is smooth.*

Proof. See [Har77, Thm. V.5.7, Exc. V.5.2] and [Kol07, Thm. 2.14, Rmk. 2.15]. □

One can show that the process of repeatedly contracting smooth rational -1 curves in a smooth projective surface terminates (see [Har77, Thm 5.8]). Thus by applying Castelnuovo's Contractibility Criterion a finite number of times, one obtains:

Corollary B.2.7 (Existence of Minimal Models)**.** *A smooth surface X admits a projective birational morphism $X \to X_{\min}$ to a smooth surface such that every projective birational morphism $X_{\min} \to Y$ to a smooth surface is an isomorphism. In particular, $X_{\min}$ has no smooth rational -1 curves.* □

B.2.2 Positivity

We discuss positivity properties of line bundles, some of which are extended to algebraic spaces in §5.9.2. An excellent reference for this material is [Laz04a, Laz04b].

Ample line bundles. A line bundle L on a scheme X is *ample* if X is quasi-compact and for every $x \in X$, there exists a section $s \in \Gamma(X, L)$ such that $X_s = \{s \neq 0\}$ is affine and contains x.

Proposition B.2.8 (Characterizations of Ampleness)**.** *For a line bundle L on a noetherian scheme X, the following are equivalent:*

(1) L is ample,

(2) the natural map $X \to \operatorname{Proj} \bigoplus_{d \geq 0} \Gamma(X, L^{\otimes d})$ is well-defined and an open immersion, and

(3) for every coherent sheaf F, the tensor product $F \otimes L^{\otimes m}$ is base point free for $m \gg 0$.

If in addition X is proper over a noetherian ring R, then the above are also equivalent to:

(4) for some $m > 0$, $L^{\otimes m}$ is very ample, i.e., defines a closed embedding $|L^{\otimes m}| \colon X \hookrightarrow \mathbb{P}^N_R$ into projective space, and

(5) for every coherent sheaf F on X, $\mathrm{H}^i(X, F \otimes L^{\otimes m}) = 0$ for $m \gg 0$ and $i > 0$.

Proof. See [Har77, §II.7 and III.5.3], [EGA, II.4.5 and III.2.6], or [SP, Tags 01PR and 0B5U]. □

Proposition B.2.9 (Properities of Ampleness)**.** *Let X be a proper scheme over a field $\Bbbk$ and L be a line bundle on X.*

(1) If $f \colon X' \to X$ is a finite surjective morphism, L is ample if and only if f^*L is.

(2) For a field extension $\Bbbk \to \Bbbk'$, L is ample on X if and only if $L_{\Bbbk'}$ is ample on $X_{\Bbbk'}$.

Proof. Both follow from the cohomological characterization of ampleness. For (1), see [Har77, Exer III.5.7], [EGA, III.2.6.2], and [SP, Tag 0B5V]. Part (2) also follows directly from Fpqc Descent for Ampleness (B.2.12). □

Part (1) implies that a line bundle L on X is ample if and only if its restriction $L|_{(X_i)_{\mathrm{red}}}$ to the reduced subscheme of each irreducible component X_i is ample.

Proposition B.2.10 (Openness of Ampleness)**.** *Let $f \colon X \to S$ be a proper and finitely presented morphism of schemes, and L be a line bundle on X. If for some $s \in S$, the restriction L_s of L to the fiber X_s is ample (resp., very ample and $\mathrm{H}^i(X_s, L_s) = 0$ for $i > 0$), then there exists an open neighborhood $U \subseteq S$ of s such that the restriction L_U on X_U is relatively ample (resp., relatively very ample) over U. In particular, for all $u \in U$, L_u is ample (resp., very ample) on X_u.*

Proof. We present a proof under the condition that $X \to S$ is flat (which suffices for many applications). If L_s is ample on X_s, then for $n \gg 0$, $L_s^{\otimes n}$ is very ample and $\mathrm{H}^i(X_s, L_s^{\otimes n}) = 0$ for $i > 0$. It therefore suffices to handle the very ample case. By Cohomology and Base Change (A.6.8), after replacing S with an open neighborhood of s, f_*L is a vector bundle and the comparison map $f_*L \otimes \kappa(t) \to \mathrm{H}^i(X_t, L_t)$ is an isomorphism for $t \in S$. By further replacing S with an affine open neighborhood, we can arrange that $\mathrm{H}^0(X, L)$ is freely generated by sections $t_0, \dots, t_n$ that restrict to a basis in $\mathrm{H}^0(X_s, L_s)$. The vanishing locus $V := V(t_0, \dots, t_n) \subseteq X$ is closed and disjoint from X_s. By replacing S with an affine open neighborhood of s contained in $S \smallsetminus f(V)$, we may assume that the sections t_i generate L and that they define a morphism $g\colon X \to \mathbb{P}^n_S$ over S that restricts to a closed immersion $g_s\colon X_s \hookrightarrow \mathbb{P}^n_{\kappa(s)}$. By upper semicontinuity of fiber dimension, there is a closed locus $Z \subseteq \mathbb{P}^n_S$ consisting of points z such that $\dim g^{-1}(z) > 0$. Since Z is disjoint from $\mathbb{P}^n_{\kappa(s)}$, we may shrink S further so that $g\colon X \to \mathbb{P}^n_S$ is quasi-finite, and hence finite (as g is proper). The cokernel of $\mathcal{O}_{\mathbb{P}^n_S} \to g_*\mathcal{O}_X$ is coherent and its support is a closed subscheme of $\mathbb{P}^n$ disjoint from $\mathbb{P}^n_{\kappa(s)}$. By shrinking S further, we may arrange that $g\colon X \to \mathbb{P}^n_S$ is a closed immersion, and hence $L = g^*\mathcal{O}_{\mathbb{P}^n_S}(1)$ is very ample. See also [Laz04a, Thm. 1.2.17, Thm. 1.7.8], [EGA, III_1.4.7.1, IV_3.9.6.4], [KM98, Prop. 1.41] and [SP, Tag 0D3D]. □

Example B.2.11 (suggested by Brian Nugent)**.** It is not true that very ampleness is an open condition. If C is a non-hyperelliptic curve of genus 3, then K_C is very ample and we can write $K_C = \mathcal{O}_C(p_1 + \cdots + p_4)$. Considering the constant family $C \times C \to C$ with the constant sections p_1, p_2, and p_3 and the diagonal section Δ, then the fiber of $\mathcal{O}_{C\times C}(p_1 + p_2 + p_3 + \Delta)$ is very ample over $p_4 \in C$ but the fiber over a point $s \in C$ near to p_4 is not very ample.

We will also need the fact that ampleness is an fpqc local property on the target.

Proposition B.2.12 (Fpqc Descent of Ampleness)**.** *Let $f\colon X \to S$ be a morphism of schemes and L be a line bundle on X. If $S' \to S$ is an fpqc morphism of schemes, then L is relatively ample over S if and only if the pullback of L to $X \times_S S'$ is relatively ample over S'.*

Proof. We know that relative ampleness is stable under base change. If the pullback L' to $X' := X \times_S S'$ is relatively ample, then $X' \to S'$ is quasi-compact and quasi-separated. Since these are Fpqc Local Properties on the Target (2.1.26), $f\colon X \to S$ is also quasi-compact and quasi-separated. We may therefore form the quasi-coherent graded $\mathcal{O}_S$-algebra $\mathcal{A} = \bigoplus_{d\geq 0} f_*L^{\otimes d}$ and the corresponding morphism $X \to \mathcal{P}\mathrm{roj}_S\, \mathcal{A}$. We appeal to the fact that L is relatively ample if and only if $X \to \mathcal{P}\mathrm{roj}_S\, \mathcal{A}$ is well-defined and an open immersion. Since $X \to \mathcal{P}\mathrm{roj}_S\, \mathcal{A}$ base changes to $X' \to \mathcal{P}\mathrm{roj}_{S'}\, \mathcal{A}'$ with $\mathcal{A}' = \bigoplus_{d\geq 0} f'_*L'^{\otimes d}$, the statement follows from an open immersion being an Fpqc Local Properties on the Target (2.1.26). See also [EGA, IV_2.2.7.2] and [SP, Tag 0D2P]. □

Nef line bundles. A line bundle L on a proper scheme X over a field $\Bbbk$ is *nef* (or *numerically effective*) if

$$\int_C c_1(L) \geq 0$$

for every integral closed curve $C \subseteq X$. Here $\int_C c_1(L)$ denote the same quantity as $c_1(L)\cdot C$, $L \cdot C$, or $\deg L|_C$. We say that L is *strictly nef* if $c_1(L)\cdot C > 0$ for every integral closed curve.

Theorem B.2.13 (Kleiman's Theorem)**.** *If L is a line bundle on a proper scheme X over a field $\Bbbk$, then L is nef if and only if for every integral subscheme $Z \subseteq X$ of dimension k,*

$$\int_Z c_1(L)^k \geq 0.$$

Proof. See [Laz04a, Thm. 1.4.9], [Kol96, Thm. 2.17], or the original source [Kle66]. □

It is often convenient to write line bundles in additive notation, so that $mL + H$ corresponds to $L^{\otimes m} \otimes H$.

Corollary B.2.14 (Characterization of Nefness)**.** *Let X be a projective scheme over a field $\Bbbk$ and H be an ample line bundle. A line bundle L on X is nef if and only if $mL + H$ is ample for $m \gg 0$.*

Proof. See [Laz04a, Cor. 1.4.10]. □

Proposition B.2.15 (Properties of Nefness)**.** *Let X be a proper scheme over a field $\Bbbk$ and L be a line bundle on X.*

(1) If $f\colon X' \to X$ is a surjective proper morphism, then L is nef if and only if f^*L is.

(2) For a field extension $\Bbbk \to \Bbbk'$, L is nef on X if and only if $L_{\Bbbk'}$ is nef on $X_{\Bbbk'}$.

Proof. For (1), if $C' \subseteq X'$ is an integral curve and d is the degree of the induced map $C' \to f(C')$, then

$$f^*L \cdot C' = d(L \cdot f(C')) \tag{B.2.16}$$

by the projection formula. This gives the ($\Rightarrow$) implication. Conversely, if $C \subseteq X$ is an integral curve, we may choose an integral curve $C' \subseteq X'$ with $C = f(C')$, and thus (B.2.16) also implies the ($\Leftarrow$) implication. For (2), by Chow's Lemma and (1), we may assume that X is projective. In this case, the Characterization of Nefness (B.2.14) reduces us to the corresponding statement for ampleness (Proposition B.2.9(2)). □

Proposition B.2.17 (Nefness is Stable under Generization)**.** *Let X be a proper flat scheme over a DVR R and L be a line bundle on X. If the restriction L_0 of L to the central fiber X_0 is nef, then so is the restriction L_η to the generic fiber X_η.*

Proof. By Chow's Lemma and Proposition B.2.15(1), we may assume that X is projective with an ample line bundle H. By the Characterization of Nefness (B.2.14), $mL_0 + H_0$ is ample for $m \gg 0$. By Openness of Ampleness (B.2.10), $mL + H$ is ample, and thus so is $mL_\eta + H_\eta$. By applying again the Characterization of Nefness, we conclude that L_η is nef. □

Remark B.2.18. For a surjective proper morphism $X \to S$ of varieties and a line bundle L on X whose fiber L_s over $s \in S$ is nef, there exists a countable union $B \subseteq S$ of proper subschemes not containing s such that L_t is nef for every $t \in S \setminus B$ [Laz04a, Prop 1.4.14]. It is not true that nefness is open in general; see [Lan13, Ex. 5.3] and [Les14, Thm. 1.2].

Remark B.2.19 (Ample and nef cones). The ample and nef line bundles generate cones $\mathrm{Amp}(X), \mathrm{Nef}(X) \subseteq N^1(X)_{\mathbb{R}}$, called the *ample cone* and *nef cone*. For a projective variety, Kleiman's Theorem (B.2.13) implies that the nef cone is the closure of the ample cone and the ample cone is the interior of the nef cone; see [Laz04a, Thm. 1.4.23].

Effective, base point free, and semiample line bundles. We have the following notions for a line bundle L on a proper scheme X over a field $\Bbbk$:

- L is *effective* if $\Gamma(X, L) \neq 0$,
- L is *base point free* (or *globally generated*) if for every $x \in X$, there exists $s \in \Gamma(X, L)$ with $s(x) \neq 0$, or equivalent the complete linear series $|L|$ defines a morphism $X \to \mathbb{P}(\mathrm{H}^0(X, L))$, and

- L is *semiample* if for some $m > 0$, $L^{\otimes m}$ is base point free.

A semiample line bundle L is necessarily nef; indeed if for some $m > 0$, $L^{\otimes m}$ defines a morphism $f\colon X \to \mathbb{P}^N$ with $f^*\mathcal{O}(1) \cong c_1(L^{\otimes m})$, then the projection formula implies that $C \cdot L^{\otimes m} = f(C) \cdot c_1(\mathcal{O}(1)) \geq 0$. We thus have the implications

$$\text{base point free} \Rightarrow \text{semiample} \Rightarrow \text{nef}.$$

Big line bundles. A line bundle L on a proper scheme X over a field $\Bbbk$ is *big* if there exists a constant $C > 0$ such that $\mathrm{h}^0(X, L^{\otimes m}) > C \cdot m^{\dim(X)}$ for $m \gg 0$.

Proposition B.2.20 (Kodaira's Lemma)**.** *Let X be a projective scheme over a field $\Bbbk$ and L be a big line bundle on X. If E is an effective line bundle, then $mL - E$ is effective for m sufficiently divisible.*

Proof. See [Laz04a, Prop. 2.2.6]. □

Proposition B.2.21 (Characterizations of Bigness)**.** *For a projective scheme X over a field $\Bbbk$ and a line bundle L on X, the following are equivalent:*

(1) L is big,

(2) for every ample divisor A on X, there exists a positive integer $m > 0$ and an effective divisor N on X such that $mL = A + N$ (linear equivalence), and

(3) there exists an ample divisor A on X, a positive integer $m > 0$, and an effective divisor N on X such that $mL \equiv A + N$ (numerical equivalence).

If in addition X is normal, then the above are also equivalent to:

(4) for some $m > 0$, $|L^{\otimes m}|$ defines a rational map $X \dashrightarrow \mathbb{P}(\mathrm{H}^0(X, L^{\otimes m}))$ which is birational onto its image.

Proof. See [Laz04a, Cor. 2.2.7]. □

As a consequence, we see that up to scaling (i.e., taking positive tensor powers), a big line bundle is the same as the sum of an ample and effective line bundle. This implies that the sum of a big and effective line bundle is also big. To summarize,

$$\text{big} \xLeftrightarrow{\text{up to scaling}} \text{ample} + \text{effective}$$

$$\text{big} + \text{effective} \Rightarrow \text{big}.$$

Theorem B.2.22 (Asymptotic Riemann–Roch)**.** *Let X be a proper scheme over a field $\Bbbk$ of dimension n, and let L be a line bundle on X. Then the Euler characteristic*

$$\chi(X, L^{\otimes m}) = \frac{(c_1(L)^n)}{n!} m^n + O(m^{n-1})$$

is a polynomial of degree $\leq n$ in m. If in addition L is nef, then $\mathrm{h}^i(X, L^{\otimes m}) = O(m^{n-1})$ and

$$\mathrm{h}^0(X, L^{\otimes m}) = \frac{(c_1(L)^n)}{n!} m^n + O(m^{n-1}).$$

Proof. See [Laz04a, Cor. 1.4.41] (projective case), [Kol96, Thm. VI.2.14-15], and [SP, Tag 0BJ8]. □

Accepting that $\chi(X, L^{\otimes m})$ is a polynomial, one can define the intersection number $c_1(L)^n$ as the normalized leading coefficient. It can also be defined in several other ways, c.f., [Kol96, Thm. VI.2]. Asymptotic Riemann–Roch provides a useful characterization of bigness for nef line bundles, implying for instance that ample line bundles are big.

Corollary B.2.23 (Characterization of Bigness II)**.** *Let X be a proper scheme over a field $\Bbbk$ of dimension n. A nef line bundle L on X is big if and only if $c_1(L)^n > 0$.* □

Proposition B.2.24 (Properties of Bigness)**.** *Let X be an integral proper scheme over a field $\Bbbk$ and L be a line bundle on X.*

(1) Let $f\colon X' \to X$ be a generically quasi-finite and proper morphism of schemes. Then L is big if and only if f^*L is big.

(2) For a field extension $\Bbbk \to \Bbbk'$, L is big on X if and only if $L_{\Bbbk'}$ is big on $X_{\Bbbk'}$.

Proof. For (1), the projection formula implies that

$$\mathrm{H}^0(X', f^*L^{\otimes m}) = \mathrm{H}^0(X, f_*f^*L^{\otimes m}) = \mathrm{H}^0(X, L^{\otimes m} \otimes f_*\mathcal{O}_{X'}).$$

As $\mathcal{O}_X \hookrightarrow f_*\mathcal{O}_{X'}$ is injective, we have an inclusion $\mathrm{H}^0(X, L^{\otimes m}) \hookrightarrow \mathrm{H}^0(X, L^{\otimes m} \otimes f_*\mathcal{O}_{X'})$. Since $\dim X' = \dim X$, the bigness of L implies the bigness of f^*L. Note that if $X' \to X$ is birational and X is normal, then $\mathcal{O}_X = f_*\mathcal{O}_{X'}$ and $\mathrm{H}^0(X', f^*L^{\otimes m}) = \mathrm{H}^0(X, L^{\otimes m})$, which gives the converse. In general, the projection formula for intersection numbers implies that $c_1(f^*L)^{\dim X'} = \deg(f) c_1(L)^{\dim X}$. Since L is nef if and only if f^*L is nef (Proposition B.2.15), the characterization of Bigness II (B.2.24) shows that L is big if and only if f^*L is.

Part (2) follows from the identification $\mathrm{H}^0(X', f^*L^{\otimes m}) = \mathrm{H}^0(X, L^{\otimes m}) \otimes_{\Bbbk} \Bbbk'$. □

Remark B.2.25 (Big and pseudo-effective cones). Big and effective divisors generate the *big cone* $\mathrm{Big}(X)$ and *effective cone* $\mathrm{Eff}(X)$ in $N^1(X)_{\mathbb{R}}$, and the closure $\overline{\mathrm{Eff}}(X)$ is called the *pseudo-effective cone*. The big cone $\mathrm{Big}(X)$ is contained in the interior of $\overline{\mathrm{Eff}}(X)$, and $\overline{\mathrm{Eff}}(X) = \overline{\mathrm{Big}}(X)$ [Laz04a, Thm. 2.2.6].

B.2.3 Ampleness criteria

We review techniques to verify ampleness of a line bundle on a proper scheme. The first strategy to keep in mind is: semiample and strictly nef $\Rightarrow$ ample.

Lemma B.2.26. *On a proper scheme over a field $\Bbbk$, a line bundle L is ample if and only if it is strictly nef and semiample.*

Proof. For the non-trivial direction, for some $m > 0$, $L^{\otimes m}$ defines a morphism $f\colon X \to \mathbb{P}^N$ which does not contract any curves. It follows that $f\colon X \to \mathbb{P}^N$ is a proper and quasi-finite, thus finite. Therefore, $L^{\otimes m} = f^*\mathcal{O}(1)$ is ample. □

Remark B.2.27. The semiampleness condition can be very challenging to verify in practice. There are powerful base point free theorems in birational geometry that can sometimes be applied to reduce semiampleness to bigness and nefness. For instance, Kawamata's base point freeness theorem states that if (X, Δ) is a proper klt pair with Δ effective and D is a nef Cartier divisor such that $aD - K_X - \Delta$ is nef and big for some $a > 0$, then D is semiample [KM98, Thm. 3.3]. One can contrast this result with the Abundance Conjecture that states that if (X, Δ) is a proper log canonical pair with Δ effective, then the nefness of $K_X + \Delta$ implies semiampleness [KM98, Conj. 3.12]. Alternatively, it is a classical result of Zariski and Wilson that if X is a normal projective variety and D is a nef and big divisor, then D is semiample if and only if its graded section ring $\bigoplus_n \Gamma(X, \mathcal{O}_X(nD))$ is finitely generated; see [Laz04a, Thm. 2.3.15]. While [BCHM10] can sometimes be applied to verify the finite generation, this result already presumes the projectivity of X. Nevertheless, it is sometimes useful. In positive characteristic, Keel's theorem [Kee99] provides another technique: on a projective variety X over a field $\Bbbk$ of charaacteristic p, a nef line bundle L is semiample if and only if the restriction of L to the *exceptional locus* E is semiample, where the exceptional locus E is defined as the union of irreducible subvarieties $Z \subseteq X$ satisfying $L^{\dim Z} \cdot Z = 0$.

Numerical criteria for ampleness. The Nakai–Moishezon Criterion for Ampleness[4] for ampleness provides a convenient method to establish projectivity. We will extend this criterion to algebraic spaces in Theorem 5.9.10, as it enters into the proof of Kollár's Criterion (5.9.2), which in turn is used for the projectivity of $\overline{M}_g$.

Theorem B.2.28 (Nakai–Moishezon Criterion for Ampleness)**.** *Let X be a proper scheme over an algebraically closed field $\Bbbk$, and let L be a line bundle on X. The following are equivalent:*

(1) L is ample;

(2) for every integral closed subscheme $Z \subseteq X$, $c_1(L)^{\dim Z} \cdot Z > 0$;

(3) L is nef and for every integral closed subscheme $Z \subseteq X$, $L|_Z$ is big; and

(4) L is strictly nef and for every integral closed subscheme $Z \subseteq X$, $L|_Z^{\otimes m}$ is effective for some $m > 0$.

Proof. A nef line bundle L is big if and only if the top intersection is positive (Corollary B.2.23). This gives the equivalence between (2) and (3). We therefore have: (1) $\Rightarrow$ (2) $\iff$ (3) $\Rightarrow$ (4). For (4) $\Rightarrow$ (1), since L is strictly nef, it suffices by Lemma B.2.26 to verify that L is semiample. Write $L = \mathcal{O}_X(D)$ for a divisor D. Since D is big on X, some positive multiple mD is effective. After replacing D by mD, let $s \in \mathrm{H}^0(X, \mathcal{O}_X(D))$ be a nonzero section. Then $\mathcal{O}_X(D)$ has no base points on $X \smallsetminus D$. We will show that for $m \gg 0$, $\mathcal{O}_X(mD)$ also has no base points on D. By induction on $\dim X$, we can assume that $\mathcal{O}_X(D)|_D$ is ample. Consider the exact sequence

$$0 \to \mathcal{O}_X((m-1)D) \to \mathcal{O}_X(mD) \to \mathcal{O}_X(mD)|_D \to 0.$$

For $m \gg 0$, $\mathcal{O}_X(mD)|_D$ is base point free and $\mathrm{H}^1(X, \mathcal{O}_X(mD)|_D) = 0$. It follows that $\mathrm{H}^1(X, \mathcal{O}_X((m-1)D)) \twoheadrightarrow \mathrm{H}^1(X, \mathcal{O}_X(mD))$ is surjective, but since each vector space is finite dimensional, we see that these surjections eventually become isomorphisms for $m \gg 0$. Thus, for $m \gg 0$, $\mathrm{H}^0(X, \mathcal{O}_X(mD)) \to \mathrm{H}^0(D, \mathcal{O}_X(mD)|_D)$ is surjective, and $\mathcal{O}_X(mD)$ has no base points on D. See also [Har77, Thm. V.1.10] (surface case), [Laz04a, Thm. 1.2.23] (projective case), [Kol96, Thm VI.2.18], [KM98, Thm. 1.37], and [Kle66, §III.1]. □

While we will not apply the following criteria in this text, they are often useful in other contexts.

Theorem B.2.29 (Kleiman's Criterion)**.** *If X is a projective scheme or a $\mathbb{Q}$-factorial (e.g., smooth) proper scheme over an algebraically closed field $\Bbbk$, a line bundle L on X is ample if and only if for all $C \in \overline{\mathrm{Eff}}(X)$, $c_1(L) \cdot C > 0$.*

[4] This is also known as the Nakai Criterion or the Nakai–Moishezon–Kleiman Criterion. See [Laz04a, §1.2.B] for a historical account and further references.

Proof. See [Kle66, §III.1], [Kol96, Thm. VI.2.19], [KM98, Thm. 1.18], and [Laz04a, Thm. 1.4.23]. □

Note that it is not enough to check that $c_1(L) \cdot C > 0$ for only integral curves $C \subseteq X$; one must check it on curve classes in the closure $\overline{\mathrm{Eff}(X)}$ of the effective cone of curves. See [Har70, p.50-56] for a counterexample due to Mumford.

Theorem B.2.30 (Sesahdri's criterion)**.** *If X is a proper scheme over an algebraically closed field $\Bbbk$, a line bundle L on X is ample if and only if there exists an $\epsilon > 0$ such that for every point $x \in X$ and every integral curve $C \subseteq X$, $c_1(L) \cdot C > \epsilon \operatorname{mult}_x(C)$, where $\operatorname{mult}_x(C)$ denotes the multiplicity of C at x.*

Proof. See [Laz04a, Thm. 1.4.13] and [Kol96, Thm. 2.18]. □

B.2.4 Nef vector bundles

In Kollár's Criterion for Ampleness (5.9.2), nefness of vector bundles plays an essential role.

Definition B.2.31. A vector bundle E on a scheme X is called *nef* (or *semipositive*) if $\mathcal{O}_{\mathbb{P}(E)}(1)$ is nef on $\mathbb{P}(E)$.

There is a related notion of an *ample vector bundle*, which we will not need, defined by requiring $\mathcal{O}_{\mathbb{P}(E)}(1)$ to be ample on $\mathbb{P}(E)$; see [Har66a] and [Laz04b, §6].

Proposition B.2.32 (Characterization of Nefness for Bundles)**.** *Let E be a vector bundle on a proper scheme X over an algebraically closed field $\Bbbk$. Then the following are equivalent:*

(1) E is nef,

(2) for every map $f \colon C \to X$ from a smooth proper curve, every quotient line bundle of $f^*E \twoheadrightarrow L$ has nonnegative degree, and

(3) for every map $f \colon C \to X$ from a smooth proper curve, every quotient vector bundle $f^*E \twoheadrightarrow W$ has nonnegative degree.

Proof. See [Bar71, p.437], [Laz04b, Prop. 6.1.18], and [Kol90, Def.-Prop. 3.3] □

Proposition B.2.33 (Properties of Nefness for Bundles)**.** *Let X be a proper scheme over a field $\Bbbk$ and E be a vector bundle on X.*

(1) If $f \colon X' \to X$ is a surjective proper morphism, then E is nef if and only if f^*E is.

(2) For a field extension $\Bbbk \to \Bbbk'$, E is nef on X if and only if $E_{\Bbbk'}$ is nef on $X_{\Bbbk'}$.

(3) Quotients, extensions, and tensor products of nef vector bundles are nef. If E is nef, so is $\bigwedge^k E$, $\operatorname{Sym}^k E$, and $\operatorname{Sym}^k(E^\vee)^\vee$ for $k \geq 0$.

Proof. Parts (1) and (2) follow from the analogous properties of nef line bundles (Proposition B.2.15). Part (3) requires some work. By (2), we can assume that $\Bbbk$ is algebraically closed. From the Characterization of Nefness for Bundles (B.2.32), it suffices to assume that X is a smooth curve of genus g. This characterization makes it clear that quotients of nef bundles are nef, and it is not hard to show that an extension of nef bundles is nef.

Before we prove the remaining parts, we claim that if E is a nef vector bundle on X and L is a line bundle with $\deg L \geq 2g-1$ (resp., $\deg L \geq 2g$), then $\mathrm{H}^1(X, E \otimes L) = 0$ (resp., $E \otimes L$ is globally generated). By Serre–Duality (5.1.3), $\mathrm{H}^1(X, E \otimes L) =$

$\mathrm{Hom}_{\mathcal{O}_X}(E \otimes L, \omega_X)$. If $E \otimes L \to \omega_X$ is a nonzero map, the image $I \subseteq \omega_X$ is a line bundle with $\deg I \leq \deg \omega_X = 2g - 2$. If $\deg L \geq 2g - 1$, the induced quotient $E \twoheadrightarrow I \otimes L^\vee$ would have negative degree, contradicting the nefness of E, hence $\mathrm{H}^1(X, E \otimes L) = 0$. If $\deg L \geq 2g$, then for every $p \in X(base)$, $\mathrm{H}^1(X, E \otimes L(-p)) = 0$ and $\mathrm{H}^0(X, E \otimes L) \to \mathrm{H}^0(X, E \otimes L \otimes \kappa(p))$ is surjective.

Assume that E and F be nef bundles on X of rank e and f. Let L be a line bundle of degree $d \geq 2g$. Since $E \otimes L$ is globally generated, we may choose e global sections of $E \otimes L$ that restrict to a basis of $E \otimes L \otimes \kappa(\xi)$ over the generic point $\xi \in X$. This gives a generically surjective map $(L^\vee)^{\oplus e} \to E$. Similarly, there is a generically surjective map $(L^\vee)^{\oplus f} \to F$. Taking the tensor product of these maps gives a generically surjective map

$$(L^\vee)^{\oplus ef} \to E \otimes F.$$

If Q is a quotient line bundle of $E \otimes F$, then the image of $Q' \subseteq Q$ of $(L^\vee)^{\oplus ef}$ satisfies $\deg Q \geq \deg Q' \geq -d$. This shows that every quotient line bundle Q of the tensor product $E \otimes F$ of *any two* nef bundles E and F of rank e and f satisfies $\deg Q \geq -d$.

Assume that $\mathrm{char}(\Bbbk) = p$. Suppose that there is a line bundle quotient $E \otimes F \twoheadrightarrow Q$ with $\deg Q < 0$. Denote $\mathrm{Fr}^N : X \to X$ be the Nth power of the absolute Frobenius, the quotient $(\mathrm{Fr}^N)^* E \otimes (\mathrm{Fr}^N)^* F \twoheadrightarrow (\mathrm{Fr}^N)^* Q$ is a quotient line bundle of degree $Np \deg Q$. By Proposition B.2.33(1), $(\mathrm{Fr}^N)^* E$ and $(\mathrm{Fr}^N)^* F$ are nef. Taking N such that $Np \deg Q < -d$ gives a quotient line bundle whose degree is less than $-d$, contradicting the fact above.

Assume that $\mathrm{char}(\Bbbk) = 0$. Since X is of finite type over $\Bbbk$, its defining equations involve finitely many coefficients of $\Bbbk$. Thus there exists a finitely generated $\mathbb{Z}$-subalgebra $A \subseteq \Bbbk$, a scheme $\widetilde{X}$ of finite type over A, and a cartesian diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \widetilde{X} \arrow[d] \\ \operatorname{Spec} \Bbbk \arrow[r] & \operatorname{Spec} A. \end{tikzcd}$$

By Descent of Properties of Morphisms under Limits (B.3.7), we may further arrange that $\widetilde{X} \to \operatorname{Spec} A$ is a family of smooth curves. Finally, by restricting along a map $\operatorname{Spec} R \to \operatorname{Spec} A$, we may assume that A is a DVR whose closed and generic points have characteristic p and 0, respectively. We may therefore reduce to the positive characteristic case by using that Nefness for Bundles is Stable under Generization (B.2.34). The nefness of $\bigwedge^k E$, $\mathrm{Sym}^k E$, and $\mathrm{Sym}^k(E^\vee)^\vee$ follow from the same argument, and in fact shows more generally that for nef vector bundle $E_1, \dots, E_m$ of ranks $r_1, \dots, r_m$ and for a representation $\rho\colon \mathrm{GL}_{r_1} \times \cdots \times \mathrm{GL}_{r_m} \to \mathrm{GL}_N$ which is *semipositive*, i.e., extends to a map $\mathrm{Mat}_{r_1,r_1} \times \cdots \times \mathrm{Mat}_{r_m,r_m} \to \mathrm{Mat}_{N,N}$, then the induced vector bundle $\rho(E_1, \dots, E_m)$ is also nef. See also [Kol90, Prop. 3.5], [Bar71, Thm. 3.3], and [Har66a, Thm. 5.2]. □

Proposition B.2.34 (Nefness for Bundles is Stable under Generization)**.** *Let X be a proper and flat scheme over a DVR R and E be a vector bundle on X. If the restriction E_0 of E to the central fiber X_0 is nef, then so is the restriction E_η to the generic fiber X_η.*

Proof. This follows from Proposition B.2.17. □

B.3 Limits of schemes

In moduli theory, we often need to deal with non-noetherian schemes for the simple reason that moduli functors and stacks are defined over the category of all schemes. Trying to work instead with the category of locally noetherian schemes has the limitation that it is not closed under fiber products, while the category of schemes finite type over a field or $\mathbb{Z}$ doesn't contain local rings of schemes or their completions. In any case, it is usually straightforward to reduce properties of schemes and their morphisms to the noetherian case using the *limit methods* introduced in this section.

B.3.1 Existence of limits and noetherian approximation

The limit of an inverse system of schemes with affine transition maps exists.

Proposition B.3.1 (Existence of Limits)**.** *If $(S_\lambda, f_{\lambda\mu})_{\lambda \in \Lambda}$ is an inverse system of schemes with affine transition maps, then the limit $S = \lim_{\lambda \in \Lambda} S_\lambda$ exists in the category of schemes such that each morphism $f_\lambda \colon S \to S_\lambda$ is affine.*

Proof. If each $S_\lambda = \operatorname{Spec} A_\lambda$ is affine, one takes $S = \operatorname{Spec}(\operatorname{colim}_\lambda A_\lambda)$. In general, choose an element $0 \in \Lambda$ and set $S = \mathcal{S}\mathrm{pec}_{S_0}(\operatorname{colim}_{\lambda \geq 0} f_{\lambda 0,*}\mathcal{O}_{S_\lambda})$. Details can be found in [EGA, IV.8.2] and [SP, Tag 01YX]. $\square$

Every affine scheme $\operatorname{Spec} A$ is the limit of affine schemes $\operatorname{Spec} A_\lambda$ of finite type over $\mathbb{Z}$. This follows from the fact that the ring A is the union of its finitely generated $\mathbb{Z}$-subalgebras. More generally, we have:

Proposition B.3.2 (Relative Noetherian Approximation)**.** *Let $X \to S$ be a morphism of schemes with X quasi-compact and quasi-separated and with S quasi-separated. Then $X = \lim_{\lambda \in \Lambda} X_\lambda$ is a limit of an inverse system $(X_\lambda, f_{\lambda\mu})$ of schemes of finite presentation over S with affine transition maps over S.*

Proof. See [SP, Tag 09MV]. When $S = \operatorname{Spec} \mathbb{Z}$, this is often referred to as *Absolute Noetherian Approximation* and was first established in [TT90, Thm. C.9]. $\square$

Proposition B.3.3 (Descent of Morphisms under Limits)**.** *Let $S = \lim_{\lambda \in \Lambda} S_\lambda$ be a limit of an inverse system of quasi-compact and quasi-separated schemes with affine transition maps.*

(1) For a finitely presented morphism $X \to S$ of schemes, there exists an index $0 \in \Lambda$ and a finitely presented morphism $X_0 \to S_0$ of schemes such that $X \cong X_0 \times_{S_0} S$. Moreover, if we define $X_\lambda := X_0 \times_{S_0} S_\lambda$ for $\lambda > 0$, then $X = \lim_{\lambda \geq 0} X_\lambda$ is the limit of the inverse system $(X_\lambda, f_{\lambda\mu})$ where the (affine) transition map $f_{\lambda\mu} \colon X_\lambda \to X_\mu$ is the base change of $S_\lambda \to S_\mu$ for $\lambda \geq \mu$.

(2) Let X_0 and Y_0 be finitely presented schemes over S_0 for some index $0 \in \Lambda$. For $\lambda > 0$, set $X_\lambda = X_0 \times_{S_0} S_\lambda$ and $Y_\lambda = Y_0 \times_{S_0} S_\lambda$, and let $X = \lim_\lambda X_\lambda$ and $Y = \lim_\lambda Y_\lambda$ be the limits (Proposition B.3.1). Then the natural map

$$\operatorname{colim}_{\lambda \geq 0} \operatorname{Mor}_{S_\lambda}(X_\lambda, Y_\lambda) \to \operatorname{Mor}_S(X, Y)$$

is bijective.

In other words, the category of schemes finitely presented over S is the colimit of the categories of schemes finitely presented over S_λ.

Proof. See [EGA, IV.8.8] and [SP, Tag 01ZM]. $\square$

Quasi-coherent sheaves also descend under limits.

Proposition B.3.4 (Descent of Quasi-Coherent Sheaves under Limits)**.** *Let $(S_\lambda, f_{\lambda\mu})$ be an inverse system of quasi-compact and quasi-separated schemes with affine transition maps and limit $S = \lim_{\lambda \in \Lambda} S_\lambda$. Denote the projection maps by $f_\lambda \colon S \to S_\lambda$.*

(1) If F is a quasi-coherent $\mathcal{O}_S$-module of finite presentation (resp., vector bundle, line bundle), then there exists an index $\lambda \in \Lambda$ and an $\mathcal{O}_{S_\lambda}$ module F_λ of finite presentation (resp., vector bundle, line bundle) such that $F \cong f_\lambda^* F_\lambda$.

(2) For an index $0 \in \Lambda$, let F_0 and G_0 be $\mathcal{O}_{S_0}$-modules of finite presentation, and let F and G be the pullbacks to S via f_0 and F_λ and G_λ be the pullbacks to S_λ via $f_{0\lambda}$. The natural map

$$\operatorname{colim}_{\lambda \geq 0} \operatorname{Hom}_{\mathcal{O}_{S_\lambda}}(F_\lambda, G_\lambda) \to \operatorname{Hom}_{\mathcal{O}_S}(F, G)$$

is bijective.

(3) For an index $0 \in \Lambda$, let $f_0 \colon X_0 \to Y_0$ be a finitely presented morphism of schemes over S_0 and let F_0 be a quasi-coherent sheaf on X_0 of finite presentation. If the pullback of F_0 under $X_0 \times_{S_0} S \to X_0$ is flat over $Y_0 \times_{S_0} S$, then the pullback of F_0 under $X_0 \times_{S_0} S_\lambda \to X_0$ is flat over $Y_0 \times_{S_0} S_\lambda$ for $\lambda \gg 0$.

In other words, the category of finitely presented modules over S is the colimit of the categories of finitely presented modules over S_λ. Note that applying (2) with $F_0 = \mathcal{O}_{S_0}$ implies $\Gamma(S, F) = \operatorname{colim}_{\lambda \geq 0} \Gamma(S_\lambda, F_\lambda)$.

Proof. See [EGA, IV.8.5.2] and [SP, Tags 01ZR, 0B8W, and 05LY]. □

B.3.2 Descent of properties under limits

Proposition B.3.5 (Descent of Properties of Schemes under Limits)**.** *Let $S = \lim_\lambda S_\lambda$ be a limit of an inverse system of quasi-compact and quasi-separated schemes with affine transition maps. If S is affine (resp., quasi-affine, separated), then so is S_λ for $\lambda \gg 0$.*

Proof. See [SP, Tags 01Z6, 086Q, and 01Z5] and [TT90, Props C.6-7]. □

Definition B.3.6. We say that a property $\mathcal{P}$ of morphisms of schemes *descends under limits* if for every limit $S = \lim_{\lambda \in \Lambda} S_\lambda$ of an inverse system of quasi-compact and quasi-separated schemes with affine transition maps, the following holds: for every index $0 \in \Lambda$, and for every morphism $g_0 \colon X_0 \to Y_0$ of quasi-compact and quasi-separated schemes with base changes $g_\lambda \colon X_\lambda \to Y_\lambda$ over S_λ and $g \colon X \to Y$ over S, if g has $\mathcal{P}$, then g_λ has $\mathcal{P}$ for $\lambda \gg 0$.

Proposition B.3.7 (Descent of Properties of Morphisms under Limits)**.** *The following properties of morphisms of schemes descend under limits: isomorphism, closed immersion, open immersion, affine, quasi-affine, finite, quasi-finite, proper, projective, quasi-projective, separated, monomorphism, surjective, flat, locally of finite presentation, unramified, étale, smooth, syntomic, and the property that every fiber is connected and has pure dimension d for a fixed integer d.*

Proof. See [EGA, IV 8.10.5] and [SP, Tags 081C and 05M5]. □

B.3.3 Spreading out and other applications

Limits methods of schemes allows us to 'spread out' objects defined over colimits of rings (e.g., a local ring $R_{\mathfrak{p}}$) to an object defined over a finite type ring (e.g., a localization R_f).

Proposition B.3.8 (Spreading out)**.** *Let R be a ring and $\mathfrak{p} \subseteq R$ be a prime ideal. If $X \to \operatorname{Spec} R_{\mathfrak{p}}$ is a finitely presented morphism, there exists an element $f \notin \mathfrak{p}$ and a finitely presented morphism $X' \to \operatorname{Spec} R_f$ such that $X \cong X' \times_{R_f} R_{\mathfrak{p}}$.*

Proof. This is a direct consequence of Descent of Morphisms under Limits (B.3.3) applied to $R_{\mathfrak{p}} = \operatorname{colim}_{f \notin \mathfrak{p}} R_f$. □

For example, if Y is an integral scheme with function field $K(Y)$, every finite presented scheme defined over $K(Y)$ can be extended to a scheme defined over an nonempty open subscheme. Similarly, a finitely presented scheme over the henselization $R^{\mathrm{h}}_{\mathfrak{p}}$ (resp., strict henselization $R^{\mathrm{sh}}_{\mathfrak{p}}$) as defined in §B.5.3 can be spread out to a finitely presented scheme X' over an étale neighborhood (resp., residually trivial étale neighborhood) $\operatorname{Spec} R' \to \operatorname{Spec} R$ of $\mathfrak{p}$.

For another typical application of noetherian approximation, we illustrate how properties of an arbitrary family of curves can be reduced to a family over a noetherian base.

Proposition B.3.9. *Let S be a quasi-compact and quasi-separated scheme (e.g., an affine scheme), and let $\mathcal{C} \to S$ be a proper, flat, and finitely presented morphism of schemes such that every geometric fiber has dimension at most 1. Then there exists a cartesian diagram*

$$\begin{tikzcd} \mathcal{C} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{C}' \arrow[d] \\ S \arrow[r] & S' \end{tikzcd}$$

where S' is a scheme of finite type over $\mathbb{Z}$ and $\mathcal{C}' \to S'$ is a proper flat morphism of schemes such that every geometric fiber has dimension at most 1. Moreover, if $\mathcal{C} \to S$ is smooth, then $\mathcal{C}' \to S'$ can also be arranged to be smooth.

Proof. Write $S = \lim_{\lambda \in \Lambda} S_\lambda$ as a limit of an inverse system of schemes of finite type over $\mathbb{Z}$ (Proposition B.3.1). Since $\mathcal{C} \to S$ is finitely presented, there exists an index $0 \in \Lambda$ and a finitely presented morphism $\mathcal{C}_0 \to S_0$ such that $\mathcal{C} \cong \mathcal{C}_0 \times_{S_0} S$ (Proposition B.3.3). For each $\lambda > 0$, we can define $\mathcal{C}_\lambda = \mathcal{C}_0 \times_{S_0} S_\lambda$ and we have a cartesian diagram

$$\begin{tikzcd} \mathcal{C} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{C}_\lambda \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{C}_0 \arrow[d] \\ S \arrow[r] & S_\lambda \arrow[r] & S_0. \end{tikzcd}$$

Since $\mathcal{C} \to S$ is flat and proper with fiber of dimension at most 1 (resp., smooth), then there exists $\lambda_0 \in \Lambda$ such that the same is true for $\mathcal{C}_\lambda \to S_\lambda$ for all $\lambda \geq \lambda_0$ (Proposition B.3.7). We now take $S' = S_\lambda$ and $\mathcal{C}' = \mathcal{C}_\lambda$ for some $\lambda \geq \lambda_0$. □

The upshot is that if we can establish properties of the morphism $\mathcal{C}' \to S'$ of *noetherian* schemes and the properties are stable under base change, then they hold for $\mathcal{C} \to S$. In Lemma 5.2.24, we show the same property for nodal families of curves.

B.4 Pushouts of schemes

Pushouts are the dual notion of fiber products. Unlike fiber products, pushouts may not always exist. However, Ferrand identified a general situation where they do exist: one map is a closed immersion and the other is affine. Ferrand pushouts of Artin rings are especially important in deformation theory such as in the homogeneity conditions appearing in Rim–Schlessinger's Criteria (C.4.6) and Artin's Axioms for Algebraicity (C.7.4). We also use pushouts to construct the Gluing Morphisms (5.6.16) between moduli spaces of stable curves.

B.4.1 Existence of pushouts

Theorem B.4.1 (Ferrand Pushout)**.** *Consider a diagram*

$$\begin{tikzcd} X_0 \arrow[r, hook, "i"] \arrow[d, "f_0"'] & X \arrow[d, dashed, "f"] \\ Y_0 \arrow[r, hook, dashed, "j"] & Y \end{tikzcd} \tag{B.4.2}$$

of schemes where $i\colon X_0 \hookrightarrow X$ is a closed immersion and $f_0\colon X_0 \to Y_0$ is affine. If

($\star$) *for every point $y_0 \in Y_0$, there exists an affine open subscheme $U \subseteq X$ with $f_0^{-1}(y) \subseteq U \cap X_0$,*

then there exists a scheme Y, a closed immersion $j\colon Y_0 \hookrightarrow Y$, and an affine morphism $f\colon X \to Y$ of schemes such that (B.4.2) is a pushout. Moreover,

(a) the square (B.4.2) is cartesian, $X \to Y$ restricts to an isomorphism $X \smallsetminus X_0 \to Y \smallsetminus Y_0$, and $|Y|$ is identified with the pushout $|X| \coprod_{|X_0|} |Y_0|$ as a topological space,

(b) the induced map

$$\mathcal{O}_Y \to j_* \mathcal{O}_{Y_0} \times_{(j \circ f_0)_* \mathcal{O}_{X_0}} f_* \mathcal{O}_X$$

is an isomorphism of sheaves, and

(c) if f_0 is finite, then so is f. In this case, if X_0, X, and Y_0 are locally of finite type over a noetherian scheme, then so is Y.

Proof. See [Fer03, Thm. 5.4 and 7.1], [Art70, Thm. 6.1], and [SP, Tag 0ECH]. □

We call $Y = X \coprod_{X_0} Y_0$ the *Ferrand pushout.* Note that if there is a cartesian diagram of schemes as in (B.4.2) with $f\colon X \to Y$ affine, then condition (($\star$)) is satisfied.

Remark B.4.3 (Existence of pushouts in general). Condition ($\star$) does not always hold: for example, consider $f_0\colon X_0 \to \operatorname{Spec} \Bbbk$ and $i\colon X_0 \hookrightarrow X$, where X is a smooth proper (but not projective) 3-fold X over an algebraically closed field $\Bbbk$ such that there is a set X_0 two $\Bbbk$-points not contained in an affine. However, the pushout always exists as an algebraic space and is a pushout in the category of algebraic spaces.

Example B.4.4 (Affine case)**.** In the affine case where $X = \operatorname{Spec} A$, $X_0 = \operatorname{Spec} A_0$, $Y_0 = \operatorname{Spec} B_0$, then $\operatorname{Spec}(A \times_{A_0} B_0)$ is the pushout $X \amalg_{X_0} Y_0$.

Example B.4.5 (Gluing and pinching)**.** If $X_0 \hookrightarrow X$ and $X_0 \hookrightarrow Y_0$ are closed immersions, the pushout $X \amalg_{X_0} Y_0$ can be viewed as the gluing of X and Y_0 along X_0. For example, the nodal curve $\operatorname{Spec} \Bbbk[x,y]/xy$ is the union of $\mathbb{A}^1$ and $\mathbb{A}^1$ along their origins. If $X_0 = Z \amalg Z$ is the union of two isomorphic disjoint subschemes of X and $X_0 \to Z$ is the projection, then the pushout $X \amalg_{Z \amalg Z} Z$ can be viewed as the pinching of the two copies of Z in X. For example, the nodal cubic curve is the pinching of 0 and ∞ in $\mathbb{P}^1$.

Exercise B.4.6. If X is the scheme-theoretic union of two closed subschemes Z_1 and Z_2, show that $X = Z_1 \amalg_{Z_1 \cap Z_2} Z_2$.

Example B.4.7 (Non-noetherianness)**.** When $f_0 \colon X_0 \to Y_0$ is affine but not finite, the pushout $X \amalg_{X_0} Y_0$ is often not noetherian. For example, if $X_0 = V(x) \subseteq X = \mathbb{A}^2_{\Bbbk}$ and $f_0 \colon X_0 \to \operatorname{Spec} \Bbbk$, the pushout is the non-noetherian affine scheme defined by

$$\Bbbk[x, y] \times_{\Bbbk[x]} \Bbbk = \Bbbk[x, xy, xy^2, xy^3, \ldots] \subseteq \Bbbk[x, y].$$

On the other hand, we shouldn't expect a finite type pushout: the y-axis in $\mathbb{A}^2_{\Bbbk}$ cannot be contracted.

B.4.2 Properties of pushouts

Proposition B.4.8 (Properties of Pushouts)**.** *Let $X_0 \hookrightarrow X$ be a closed immersion and $X_0 \to Y_0$ be an affine morphism of schemes.*

(1) If $Y \cong X \coprod_{X_0} Y_0$ is a Ferrand pushout of schemes, then the natural functor

$$\operatorname{QCoh}(Y) \to \operatorname{QCoh}(Y_0) \times_{\operatorname{QCoh}(X_0)} \operatorname{QCoh}(X),$$

restricts to an equivalence on the full subcategories of flat $\mathcal{O}$-modules (resp., finite type and flat $\mathcal{O}$-modules, finitely presented and flat $\mathcal{O}$-modules).

Consider a commutative cube of schemes

$$\begin{tikzcd}
& X_0' \arrow[rr, hook] \arrow[dl] \arrow[dd] & & X' \arrow[dl] \arrow[dd] \\
X_0 \arrow[rr, hook, crossing over] \arrow[dd] & & X & \\
& Y_0' \arrow[rr, hook] \arrow[dl] & & Y' \arrow[dl] \\
Y_0 \arrow[rr, hook] & & Y \arrow[from=uu, crossing over] &
\end{tikzcd}$$

of schemes where $X_0 \hookrightarrow X$ is a closed immersion and $X_0 \to Y_0$ is affine.

(2) Assume that $Y' \to Y$ is flat such that X_0', Y_0', and X' are the base changes. If $Y \cong X \coprod_{X_0} Y_0$, then $Y' \cong X' \coprod_{X_0'} Y_0'$. If $Y' \to Y$ is fppf, the converse is true.

(3) If the top and left faces are cartesian, and the front and back faces are Ferrand pushouts, then all faces are cartesian. Moreover, if $Y_0' \to Y_0$ and $X' \to X$ are étale (resp., smooth), so is $Y' \to Y$.

(4) Suppose that Y is defined over a scheme S. Let $S' \to S$ be morphisms of schemes, and let Y', X_0', Y_0', and X' be the base changes. If $X_0 \to S$ is flat and $Y \cong X \coprod_{X_0} Y_0$, then $Y' \cong X' \coprod_{X_0'} Y_0'$.

Proof. Parts (1)–(3) follow from [Fer03, Thm. 2.2]; see also [SP, Tag 0D2K] and [AHHLR24, §4]. Part (4) is elementary: one reduces to the affine case $Y = \operatorname{Spec} B_0 \times_{A_0} A$ and $S = \operatorname{Spec} R$, and since A_0 is flat over R, the exact sequence $0 \to B \to A \times B_0 \to A_0 \to 0$ of R-modules remains exact after tensoring with an R-algebra. $\square$

B.5 Completions, henselizations, and Artin Approximation

After reviewing properties of complete, henselian, and strictly henselian local rings, we discuss Artin Approximation (Theorem B.5.18) which can vaguely formulated as: every algebraic object defined over the completion $\widehat{\mathcal{O}}_{S,s}$ of the local ring of a finite type scheme S at a point s can be approximated to an object defined over a residually-trivially étale neighborhood $(S', s') \to (S, s)$. While difficult to prove (and we do not prove it here!), the statement of Artin Approximation is at least very easy to digest and teaches us how to think about étale maps. It is a fundamental tool in local-to-global arguments in moduli theory, but its use can usually be avoided (by more direct but less conceptual methods).

B.5.1 Complete rings

Definition B.5.1. A ring R is *complete with respect to an ideal I* if the natural map

$$R \to \varprojlim_n R/I^n \tag{B.5.2}$$

is an isomorphism. The *completion with respect to an ideal I* of R is defined as

$$\widehat{R} = \varprojlim_n R/I^n.$$

More generally, we say that an R-module M is *complete* if $M \to \varprojlim_n M/I^n M$ is an isomorphism, and we define the *completion* of M as $\widehat{M} = \varprojlim_n M/I^n M$.

The most important case is when $I = \mathfrak{m} \subseteq R$ is a maximal ideal.

Caution B.5.3. When R is non-noetherian, the completion of a local ring may not even be complete; see [SP, Tag 05JC]. In the literature, 'complete' sometimes refers to only the surjectivity of (B.5.2), while 'separated' refers to the injectivity, i.e., $\bigcap_n I^n = 0$.

The Artin–Rees Lemma plays an important role in establishing basic properties of complete noetherian local rings.

Lemma B.5.4 (Artin–Rees Lemma)**.** *Let R be a noetherian ring, $I \subseteq R$ be an ideal, M be a finitely generated A-module, and $M = M_0 \supseteq M_1 \supseteq M_2 \supseteq \cdots$ be a stable I-filtration (i.e., $IM_n = M_{n+1}$ for $n \gg 0$). If $M' \subseteq M$ is a submodule, then $M' = M' \cap M_0 \supseteq M' \cap M_1 \supseteq \cdots$ is a stable I-filtration. In particular, there exists an integer k such that*

$$M' \cap (I^n M) = I^{n-k}(M' \cap (I^k M))$$

for all $n \geq k$.

Proof. See [AM69, Prop. 10.9 and Cor. 10.10] and [Eis95, Lem. 5.1]. ☐

Proposition B.5.5 (Properties of Noetherian Complete Local Rings)**.** *Let $(R, \mathfrak{m})$ be a noetherian local ring.*

(1) $\widehat{R}$ is a complete noetherian local ring with maximal ideal $\widehat{\mathfrak{m}} = \mathfrak{m}\widehat{R} = \mathfrak{m} \otimes_R \widehat{R}$;

(2) $\widehat{\mathfrak{m}^n} = \widehat{\mathfrak{m}}^n$ and $\mathfrak{m}^n/\mathfrak{m}^{n+1} = \widehat{\mathfrak{m}}^n/\widehat{\mathfrak{m}}^{n+1}$;

(3) $R \to \widehat{R}$ is faithfully flat; and

(4) For a finitely generated R-module M, $\widehat{M} = M \otimes_R \widehat{R}$.

Proof. See [AM69, Prop. 10.13–16]. □

The following provides variants of Nakayama's lemma that hold for complete local rings without finite generated hypotheses.

Lemma B.5.6 (Complete Nakayma's Lemma)**.**

(1) If $(A, \mathfrak{m})$ is a complete noetherian local rings and M is a (possibly not finitely generated) A-module such that $\bigcap_k \mathfrak{m}^k M = 0$ and $m_1, \ldots, m_n \in M$ generate $M/\mathfrak{m}M$, then $m_1, \ldots, m_n$ also generate M.

(2) If $(A, \mathfrak{m}_A)$ is a local ring and $M \to N$ is a homomorphism of A-modules such that $M/\mathfrak{m}_A M \to N/\mathfrak{m}_A N$ is surjective, then $\widehat{M} \to \widehat{N}$ is surjective.

(3) Let $(A, \mathfrak{m}_A) \to (B, \mathfrak{m}_B)$ be a local homomorphism of complete noetherian local rings such that $A/\mathfrak{m}_A \cong B/\mathfrak{m}_B$. If $\mathfrak{m}_A/\mathfrak{m}_A^2 \to \mathfrak{m}_B/\mathfrak{m}_B^2$ is surjective, so is $A \to B$. If in addition $A = B$, then $A \to B$ is an isomorphism.

Proof. For (1) and (2), see [Eis95, Exc. 7.2] and [SP, Tag 0315]. To see (3), (2) implies that the inclusion $\mathfrak{m}_A B \hookrightarrow \mathfrak{m}_A$ is surjective. Thus, $\mathfrak{m}_A B = \mathfrak{m}_A$ and B is complete as an A-module with respect to $\mathfrak{m}_A$. As $A/\mathfrak{m}_A \to B/\mathfrak{m}_B$ is surjective, applying (2) again shows that $A \to B$ is surjective. The final statement follows from the fact that a surjective endomorphism of a noetherian ring is an isomorphism. □

Theorem B.5.7 (Cohen Structure Theorem)**.** *If $(R, \mathfrak{m})$ is a complete noetherian local ring containing a field, then $R \cong (R/\mathfrak{m})[\![y_1, \ldots, y_r]\!]/J$. If in addition R is regular, then $R \cong (R/\mathfrak{m})[\![y_1, \ldots, y_r]\!]$.*

Proof. See [Eis95, Thm. 7.7] and [SP, Tags 032A and 0C0S]. □

B.5.2 Henselian and strictly henselian local rings

Let $(R, \mathfrak{m})$ be a local ring with residue field κ. We will denote the image of $a \in R$ (resp., $f \in R[x]$) as $\overline{a} \in \kappa$ (resp., $\overline{f} \in \kappa[x]$). If $f \in R[t]$, we denote its derivative by $f' \in R[t]$. Note that $\overline{f'} = \overline{f}'$.

Definition B.5.8. Let $(R, \mathfrak{m})$ be a local ring with residue field κ.

(1) We say that R is *henselian* if for every monic polynomial $f \in R[t]$, every root $\alpha_0 \in \kappa$ of $\overline{f}$ with $\overline{f'}(\alpha_0) \neq 0$ lifts to a root $\alpha \in R$ of f.

(2) We say that R is *strictly henselian* if R is henselian and κ is separably closed.

Hensel's lemma states that every complete DVR R, e.g., $\mathbb{Z}_p$, is henselian.

Proposition B.5.9 (Henselian Equivalences)**.** *The following are equivalent for a local ring $(R, \mathfrak{m})$ with residue field κ:*

(1) R is henselian;

(2) for every polynomial $f \in R[t]$, every factorization $\overline{f} = g_0 h_0$ with $\gcd(g_0, h_0) = 1$ lifts to a factorization $f = gh$ with $\overline{g} = g_0$ and $\overline{h} = h_0$;

(3) every finite R-algebra is a finite product of local rings finite over R;

(4) every quasi-finite R-algebra A is isomorphic to a product $A \cong B \times C$ where B is a finite over R and $C \otimes_R \kappa = 0$;

(5) every étale ring homomorphism $\phi \colon R \to A$ and a prime $\mathfrak{p} \subseteq A$ with $\phi^{-1}(\mathfrak{p}) = \mathfrak{m}$ and $\kappa = \kappa(\mathfrak{p})$ has a unique section $s \colon A \to R$ with $s^{-1}(\mathfrak{m}) = \mathfrak{p}$.

Moreover, R is strictly henselian if and only if for every étale ring homomorphism $\phi\colon R \to A$ and prime $\mathfrak{p} \subseteq A$ with $\phi^{-1}(\mathfrak{p}) = \mathfrak{m}$, there is a unique section $s\colon A \to R$ with $s^{-1}(\mathfrak{m}) = \mathfrak{p}$.

Proof. See [EGA, IV.18.5.11], [Mil80, Thm. I.4.2], and [SP, Tag 04GG]. □

Proposition B.5.10. *Let $(R, \mathfrak{m})$ be a henselian (resp., strictly henselian) local ring with residue field κ.*

(1) Every finite R-algebra is a product of finite henselian local (resp., strictly henselian) R-algebras.

(2) Every complete local ring is henselian.

(3) The functor $A \mapsto A \otimes_R \kappa$ gives an equivalence of categories between finite étale A-algebras and finite étale κ-algebras.

Proof. See [EGA, IV.18.5.10-15], [Mil80, 4.3-4.5], and [SP, Tag 04GE]. □

Remark B.5.11. Although it is not used in this text, there is a general notion of henselian pairs that is sometimes useful. A pair (X, X_0) consisting of a scheme X and a closed subscheme $X_0 \subseteq X$ is *henselian* if every finite morphism $f\colon U \to X$ induces a bijection $\mathrm{ClOpen}(U) \to \mathrm{ClOpen}(f^{-1}(X_0))$ between open and closed subschemes of U and those of $f^{-1}(X_0)$. If $(R, \mathfrak{m})$ is a henselian local ring, then $(\operatorname{Spec} R, \operatorname{Spec}(R/\mathfrak{m}))$ is a henselian pair by Proposition B.5.9(3). See [EGA, IV.18.5.5] or [SP, Tag 09XD] for a further discussion and equivalences.

B.5.3 Henselizations and strict henselizations

Definition B.5.12. Let $(R, \mathfrak{m})$ be a local ring with residue field κ. The *henselization of R* is a local homomorphism $R \to R^{\mathrm{h}}$ into a henselian local ring R^{h} such that every other local homomorphism $R \to A$ into a henselian local ring factors uniquely through $R \to R^{\mathrm{h}}$.

Given a separable closure $\kappa \to \kappa^s$, the *strict henselization of R with respect to $\kappa \to \kappa^s$* is a local homomorphism $R \to R^{\mathrm{sh}}$ into a strictly henselian local ring $(R^{\mathrm{sh}}, \mathfrak{m}^{\mathrm{sh}})$ inducing $\kappa \to \kappa^s$ on residue fields such that every other local homomorphism $R \to A$ into a strictly henselian local ring $(A, \mathfrak{m}_A)$ factors through $R \to R^{\mathrm{sh}}$ and the factorization is uniquely determined by the inclusion $R^{\mathrm{sh}}/\mathfrak{m}^{\mathrm{sh}} \to A/\mathfrak{m}_A$ of residue fields.

Proposition B.5.13. *Let $(R, \mathfrak{m}_R)$ be a local ring with residue field. The henselization $R \to R^{\mathrm{h}}$ (resp., strict henselization $R \to R^{\mathrm{sh}}$) exist and can be constructed as* $\operatorname{colim} A$, *where the colimit is taken over all étale local R-algebras A with $R/\mathfrak{m}_R \cong A/\mathfrak{m}_A$ (resp., over diagrams $R \to A \to (R/\mathfrak{m}_R)^s$ where $(R/\mathfrak{m}_R)^s$ is a fixed separable closure of $R/\mathfrak{m}_R$ and A is an étale local R-algebra). Moreover,*

(1) the residue fields of R^{h} and R^{sh} are $R/\mathfrak{m}_R$ and $(R/\mathfrak{m}_R)^s$, respectively,

(2) the maps $R \to R^{\mathrm{h}}$ and $R \to R^{\mathrm{sh}}$ are faithfully flat local ring homomorphisms, and

(3) if R is noetherian, then so is R^{h} and R^{sh}.

Proof. See [EGA, IV.18.5-8], [Mil80, I.4], and [SP, Tags 0BSK and 07QL]. □

For a scheme X and a point $x \in X$ with a choice of separable closure $\kappa(x) \to \kappa^s$, the henselization $\mathcal{O}^{\mathrm{h}}_{X,x}$ and strict henselization $\mathcal{O}^{\mathrm{sh}}_{X,x}$ are the colimits of $\Gamma(U, \mathcal{O}_U)$

taken over diagrams

$$\begin{tikzcd} \operatorname{Spec} \kappa(x) \arrow[r] \arrow[dr, "x"'] & U \arrow[d, "\text{ét}"] \\ & X \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \operatorname{Spec} \kappa(x)^s \arrow[r] \arrow[d] & U \arrow[d, "\text{ét}"] \\ \operatorname{Spec} \kappa(x) \arrow[r, "x"] & X, \end{tikzcd}$$

where $U \to X$ is étale. We can view $\mathcal{O}^{\mathrm{h}}_{X,x}$ and $\mathcal{O}^{\mathrm{sh}}_{X,x}$ as local rings in the étale topology.

B.5.4 Néron–Popescu Desingularization

Artin Approximation (B.5.18) is closely related to Néron–Popescu Desingularization (B.5.15), another equally deep and powerful theorem. We do not attempt to prove Néron–Popescu Desingularization, but we do show how it implies Artin Approximation.

Definition B.5.14. A ring homomorphism $A \to B$ of noetherian rings is called *geometrically regular* if $A \to B$ is flat and for every prime ideal $\mathfrak{p} \subseteq A$ and every finite field extension $\kappa(\mathfrak{p}) \to \kappa'$ (where $\kappa(\mathfrak{p}) = A_{\mathfrak{p}}/\mathfrak{p}$), the fiber $B \otimes_A \kappa'$ is regular.

If $A \to B$ is of finite type, then $A \to B$ is geometrically regular if and only if $\operatorname{Spec} B \to \operatorname{Spec} A$ is smooth (Theorem A.3.1). A $\Bbbk$-algebra B is geometrically regular if and only if $A \otimes_{\Bbbk} \Bbbk'$ is regular for every field extension (equivalently, for every finite purely inseparable extension) $\Bbbk \to \Bbbk'$; see [EGA, IV_2.6.7.8], [Mat89, Thm. 23.7], or [SP, Tag 0381]. Note that a field extension is separable if and only if it is geometrically regular.

Theorem B.5.15 (Néron–Popescu Desingularization)**.** *A homomorphism $A \to B$ of noetherian rings is geometrically regular if and only if there is a directed system B_λ of smooth A-algebras over a directed set Λ such that $B = \operatorname{colim} B_{\lambda \in \Lambda}$.*

Proof. This result was proved by Néron in [Nér64] in the case of DVRs and in general by Popescu in [Pop85], [Pop86], and [Pop90]. We recommend [Swa98] and [SP, Tag 07GC] for expositions. □

Definition B.5.16. A noetherian local ring A is a *G-ring* if the homomorphism $A \to \widehat{A}$ is geometrically regular.

One of the defining properties of an excellent scheme is that the local rings are G-rings. Fortunately, most local rings that we care about in algebraic geometry are G-rings.

Theorem B.5.17. *The localization of a finitely generated algebra over a field or $\mathbb{Z}$ is a G-ring.*

Proof. While substantially easier than Néron–Popescu Desingularization, this result also requires some effort. See [EGA, IV.7.4.4] or [SP, Tag 07PX]. □

B.5.5 Artin Approximation

Recall from Definition A.1.4 that a contravariant functor $F\colon \mathrm{Sch}/S \to \mathrm{Sets}$ is *limit preserving* (or *locally of finite presentation*) if $\operatorname{colim} F(A_\lambda) \xrightarrow{\sim} F(\operatorname{colim} A_\lambda)$ for all inverse systems $\{\operatorname{Spec} A_\lambda\}$ over S. When F is representable by X, this is equivalent to $X \to S$ being of locally of finite presentation.

Theorem B.5.18 (Artin Approximation)**.** *Let S be a scheme and $s \in S$ be a point such that $\mathcal{O}_{S,s}$ is a G-ring (Definition B.5.16), e.g., a scheme of finite type over a field or $\mathbb{Z}$. Let*

$$F\colon \mathrm{Sch}/S \to \mathrm{Sets}$$

be a limit preserving contravariant functor and $\widehat{\xi} \in F(\operatorname{Spec} \widehat{\mathcal{O}}_{S,s})$. For every integer $N \geq 0$, there exists an étale morphism $(S', s') \to (S, s)$ with $\kappa(s) = \kappa(s')$ and an object $\xi' \in F(S')$ such that the restrictions of $\widehat{\xi}$ and ξ' to $\operatorname{Spec}(\mathcal{O}_{S,s}/\mathfrak{m}_s^{N+1})$ are equal.

The restriction ξ' to $\operatorname{Spec}(\mathcal{O}_{S,s}/\mathfrak{m}_s^{N+1})$ is well-defined because of the identification $\mathcal{O}_{S,s}/\mathfrak{m}_s^{N+1} \cong \mathcal{O}_{S',s'}/\mathfrak{m}_{s'}^{N+1}$. It is not possible in general to find $\xi' \in F(S')$ that precisely restricts to $\widehat{\xi}$, or even such that the restrictions of ξ' and $\widehat{\xi}$ to $\operatorname{Spec} \mathcal{O}_{S,s}/\mathfrak{m}_s^{n+1}$ agree for all $n \geq 0$. For instance, consider $F = \operatorname{Mor}(-, \mathbb{A}^1)$ and a non-algebraic power series $\widehat{\xi} \in \widehat{\mathcal{O}}_{\mathbb{A}^1,0}$.

Proof. The theorem was originally proven in [Art69a, Cor. 2.2] in the case that S is of finite type over a field or an excellent dedekind domain. We also recommend [BLR90, §3.6] for an accessible account of the case of excellent and henselian DVRs. We prove only how Artin Approximation follows from Néron–Popescu Desingularization (B.5.15). By Néron–Popescu, we may write $\widehat{\mathcal{O}}_{S,s} = \operatorname{colim}_{\lambda \in \Lambda} B_\lambda$ as a directed colimit of smooth $\mathcal{O}_{S,s}$-algebras. Since F is limit preserving, there exists $\lambda \in \Lambda$, a factorization $\mathcal{O}_{S,s} \to B_\lambda \to \widehat{\mathcal{O}}_{S,s}$, and an element $\xi_\lambda \in F(\operatorname{Spec} B_\lambda)$ whose restriction to $F(\operatorname{Spec} \widehat{\mathcal{O}}_{S,s})$ is $\widehat{\xi}$. Letting $B = B_\lambda$ and $\xi = \xi_\lambda$, we have a commutative diagram

$$\begin{tikzcd}
\operatorname{Spec} \widehat{\mathcal{O}}_{S,s} \arrow[r, "g"] \arrow[rr, bend left, "\widehat{\xi}"] \arrow[dr] & \operatorname{Spec} B \arrow[r, "\xi"] \arrow[d] & F \\
& \operatorname{Spec} \mathcal{O}_{S,s} &
\end{tikzcd}$$

where $\operatorname{Spec} B \to \operatorname{Spec} \mathcal{O}_{S,s}$ is smooth. We claim that we can find a commutative diagram

$$\begin{tikzcd}
S' \arrow[r, hook] \arrow[dr] & \operatorname{Spec} B \arrow[d] \\
& \operatorname{Spec} \mathcal{O}_{S,s}
\end{tikzcd} \tag{B.5.19}$$

where $S' \hookrightarrow \operatorname{Spec} B$ is a closed immersion, $(S', s') \to (\operatorname{Spec} \mathcal{O}_{S,s}, s)$ is étale with $\kappa(s) = \kappa(s')$ such that $\operatorname{Spec} \mathcal{O}_{S,s}/\mathfrak{m}_s^{N+1} \to S' \to \operatorname{Spec} B$ agrees with the restriction of $g\colon \operatorname{Spec} \widehat{\mathcal{O}}_{S,s} \to \operatorname{Spec} B$.[5] To see this, since $\Omega_{B/\mathcal{O}_{S,s}}$ is a locally free B-module, after replacing $\operatorname{Spec} B$ with an affine open neighborhood of $g(s)$, we may assume that $\Omega_{B/\mathcal{O}_{S,s}}$ is free with basis $db_1, \ldots, db_r$. This induces a homomorphism $\mathcal{O}_{S,s}[x_1, \ldots, x_r] \to B$ defined by $x_i \mapsto b_i$ and provides a factorization

$$\operatorname{Spec} B \to \mathbb{A}^r_{\mathcal{O}_{S,s}} \to \operatorname{Spec} \mathcal{O}_{S,s}$$

[5]This is where the approximation occurs. It is not possible to find a an étale map $S' \to \operatorname{Spec} B \to \operatorname{Spec} \mathcal{O}_{S,s}$ such that $\operatorname{Spec} \widehat{\mathcal{O}}_{S,s} \to S' \to \operatorname{Spec} B$ is equal to g.

such that $\operatorname{Spec} B \to \mathbb{A}^r_{\mathcal{O}_{S,s}}$ is étale. Choosing a lift of the composition

$$\begin{tikzcd}
 & & & \mathcal{O}_{S,s} \arrow[d] \\
\mathcal{O}_{S,s}[x_1,\ldots,x_r] \arrow[r] \arrow[rrru, dashed] & B \arrow[r] & \widehat{\mathcal{O}}_{S,s} \arrow[r] & \mathcal{O}_{S,s}/\mathfrak{m}_s^{N+1}
\end{tikzcd}$$

defines a section $s\colon \operatorname{Spec} \mathcal{O}_{S,s} \to \mathbb{A}^r_{\mathcal{O}_{S,s}}$ and we define S' as the fibered product

$$\begin{tikzcd}
S' \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\square"] & \operatorname{Spec} \mathcal{O}_{S,s} \arrow[d, hook, "s"] \\
\operatorname{Spec} B \arrow[r] & \mathbb{A}^r_{\mathcal{O}_{S,s}}.
\end{tikzcd}$$

This gives the desired diagram (B.5.19), and the composition $\xi'\colon S' \to \operatorname{Spec} B \xrightarrow{\xi} F$ is an element which agrees with $\widehat{\xi}$ up to order N. Finally, we use limit methods to 'spread out' the étale map $(S', s') \to (\operatorname{Spec} \mathcal{O}_{S,s}, s)$ and element $\xi' \in F(S')$ to an étale map $(S'', s'') \to (S, s)$ and an element $\xi'' \in F(S'')$. Assuming $S = \operatorname{Spec} A$ is affine and writing $\mathcal{O}_{S,s} = \operatorname{colim}_{g \notin \mathfrak{m}_s} A_g$, we may use Propositions B.3.3, B.3.5 and B.3.7 (or a direct argument) to construct $g \notin \mathfrak{m}_s$ and an étale affine morphism $S'' \to \operatorname{Spec} A_g$ such that $S'' \times_{A_g} A_{\mathfrak{m}_s} \cong S'$. As F is limit preserving and $\Gamma(S', \mathcal{O}_{S'}) = \operatorname{colim}_{h \notin \mathfrak{m}_s} \Gamma(S''_h, \mathcal{O}_{S''_h})$, after replacing g with hg for some $h \notin \mathfrak{m}_s$, we can find an element $\xi'' \in F(S'')$ restricting to ξ' and, in particular, agreeing with $\widehat{\xi}$ up to order N. $\square$

Exercise B.5.20 (Alternative formulations)**.** Let $(A, \mathfrak{m})$ be a henselian local G-ring.

(1) Let $f_1, \ldots, f_m \in A[x_1, \ldots, x_n]$ and $\widehat{a} = (\widehat{a}_1, \ldots, \widehat{a}_n) \in \widehat{A}^{\oplus n}$ be a solution. Show that for every $N \geq 0$, there is a solution $a = (a_1, \ldots, a_n) \in A^{\oplus n}$ such that $a \cong \widehat{a} \mod \mathfrak{m}^{N+1}$.

Hint: Apply Artin Approximation to the functor representing $\operatorname{Spec} A[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$.

(2) Show that (1) implies Artin Approximation.

Hint: Use that F is limit preserving to find a finitely generated A-subalgebra $B \subseteq \widehat{\mathcal{O}}_{S,s}$ and an element $\xi \in F(B)$ restricting to $\widehat{\xi}$.

B.5.6 A first application of Artin Approximation

Two pointed schemes with isomorphic completions have isomorphic étale neighborhoods.

Corollary B.5.21. *Let X and Y be schemes of finite type over a scheme S and let $s \in S$ be a point such that $\mathcal{O}_{S,s}$ is a G-ring. If $x \in X$ and $y \in Y$ are points over s such that $\widehat{\mathcal{O}}_{X,x}$ and $\widehat{\mathcal{O}}_{Y,y}$ are isomorphic as $\mathcal{O}_S$-algebras, then there exists étale morphisms*

$$\begin{tikzcd}
 & (U, u) \arrow[dl] \arrow[dr] & \\
(X, x) & & (Y, y)
\end{tikzcd}$$

with $\kappa(x) = \kappa(u) = \kappa(y)$.

Proof. The functor

$$F\colon \mathrm{Sch}/X \to \mathrm{Sets}, \quad (T \to X) \mapsto \mathrm{Mor}_S(T, Y)$$

is limit preserving as it can be identified with the representable functor $\mathrm{Mor}_X(-, Y \times_S X)$. The isomorphism $\widehat{\mathcal{O}}_{X,x} \cong \widehat{\mathcal{O}}_{Y,y}$ gives an element of $F(\operatorname{Spec} \widehat{\mathcal{O}}_{X,x})$. Applying Artin Approximation with $N = 1$ yields an étale map $(U, u) \to (X, x)$ with $\kappa(x) = \kappa(u)$ and a map $(U, u) \to (Y, y)$ such that $\mathcal{O}_{Y,y}/\mathfrak{m}_y^2 \to \mathcal{O}_{U,u}/\mathfrak{m}_u^2$ is an isomorphism. Since $\widehat{\mathcal{O}}_{U,u}$ is abstractly isomorphic to $\widehat{\mathcal{O}}_{Y,y}$, Complete Nakayama's Lemma (B.5.6(3)) implies that $\widehat{\mathcal{O}}_{Y,y} \to \widehat{\mathcal{O}}_{U,u}$ is an isomorphism. This further implies that $U \to Y$ is étale at u and the statement follows after replacing U with an open neighborhood of u. See also [SP, Tag 0CAV]. □

If $\phi\colon \widehat{\mathcal{O}}_{Y,y} \xrightarrow{\sim} \widehat{\mathcal{O}}_{X,x}$ is a specified isomorphism, it is not always possible to find étale neighborhoods $(U, u) \to (X, x)$ and $(U, u) \to (Y, y)$ such that the composition $\widehat{\mathcal{O}}_{Y,y} \cong \widehat{\mathcal{O}}_{U,u} \cong \widehat{\mathcal{O}}_{X,x}$ agrees with ϕ.

Appendix C

Deformation theory

Deformation theory is the study of the local geometry of a moduli space $\mathcal{M}$ near an object $E_0 \in \mathcal{M}(\Bbbk)$. We focus primarily on the following three deformation problems:

(A) *Embedded deformations of a closed subscheme Z_0 in a fixed projective scheme X over $\Bbbk$.* Here the moduli problem is the Hilbert functor $\mathrm{Hilb}^P(X)$ and the object is $E_0 = [Z_0 \subseteq X] \in \mathrm{Hilb}^P(X)(\Bbbk)$.

(B) *Deformations of a scheme E_0 over $\Bbbk$.* The motivating example for us is when E_0 is a smooth curve, in which case the moduli problem is $\mathcal{M}_g$ and $[E_0] \in \mathcal{M}_g(\Bbbk)$, or more generally when E_0 is a proper curve, in which case the moduli problem is the stack $\mathcal{M}_g^{\mathrm{all}}$ of all curves.

(C) *Deformations of a coherent sheaf E_0 on a fixed projective scheme X over $\Bbbk$.* The moduli problem is $\underline{\mathrm{Coh}}(X)$ and $[E_0] \in \underline{\mathrm{Coh}}(X)(\Bbbk)$.

Deformation theory provides a local-to-global perspective of moduli. By zooming in around $E_0 \in \mathcal{M}(\Bbbk)$, we studying successively first-order neighborhoods of $\mathcal{M}$ at E_0, higher-order deformations of E_0, formal neighborhoods of E_0, and finally étale or smooth neighborhoods of E_0. Before getting started, we give a quick overview of the seven sections of this appendix.

(1) A *first-order deformation* of E_0 is an object $E \in \mathcal{M}(\Bbbk[\epsilon])$ over the dual numbers $\Bbbk[\epsilon] := \Bbbk[\epsilon]/(\epsilon^2)$ together with an isomorphism $\alpha\colon E_0 \to E|_{\operatorname{Spec} \Bbbk}$, or in other words a commutative diagram

$$\begin{tikzcd}
\operatorname{Spec} \Bbbk \arrow[r, hook] \arrow[dr, "{[E_0]}"'] & \operatorname{Spec} \Bbbk[\epsilon] \arrow[d, dashed, "{[E]}"] \\
& \mathcal{M},
\end{tikzcd}$$

allowing us to view E as a tangent vector of $\mathcal{M}$ at E_0. We classify first-order deformations of Problems (A)–(C) in §C.1.

(2) Given a surjection $A' \twoheadrightarrow A$ of artinian local $\Bbbk$-algebras with residue field $\Bbbk$ and an object $E \in \mathcal{M}(A)$ with an isomorphism $E_0 \to E|_{\operatorname{Spec} \Bbbk}$, a *deformation of E over A'* is an object $E' \in \mathcal{M}(A')$ with an isomorphism $\alpha\colon E \to E'|_{\operatorname{Spec} A}$. Pictorially, this corresponds to a commutative diagram

$$\begin{tikzcd}
\operatorname{Spec} \Bbbk \arrow[r, hook] \arrow[drr, "{[E_0]}"'] & \operatorname{Spec} A \arrow[r, hook] \arrow[dr, "{[E]}"] & \operatorname{Spec} A' \arrow[d, dashed, "{[E']}"] \\
& & \mathcal{M}.
\end{tikzcd}$$

In §C.2, for Problems (A)–(C), we determine the obstruction for the existence of a deformation E' of E over A', and we classify all such deformations in the case that there is no obstruction.

These first two sections are essential for our study of $\overline{\mathcal{M}}_g$ and $\mathcal{B}un(C)$, e.g., for establishing smoothness and computing their dimensions; see Theorem 5.4.14. Since the study of deformation theory is inextricably connected to stacks and moduli, we have included §C.3 - C.7 for completeness.

(3) In §C.3, we offer a survey of the cotangent complex, and in particular how it governs infinitesimal deformation theory, recovering some of the results of §C.1-C.2.

(4) Given a complete noetherian local $\Bbbk$-algebra $(R, \mathfrak{m})$, a *formal deformation of* E_0 *over* R is a compatible collection of deformations $E_n \in \mathcal{M}(R/\mathfrak{m}^{n+1})$ of E_0, and a formal deformation $\{E_n\}$ is *versal* if every deformation over a thickening of artin rings factors through one of the E_n (see Definition C.4.2 for a precise definition). Rim–Schlessinger's Criteria (Theorem C.4.6) provides criteria for the existence of a versal deformation $\{E_n\}$ of E_0, and in §C.4 we verify the criteria for the Problems (A)–(C).

(5) A formal deformation $\{E_n\}$ over $(R, \mathfrak{m})$ is *effective* if there exists an object $\widehat{E} \in \mathcal{M}(R)$ extending the $\{E_n\}$, or in other words there exists a commutative diagram

$$\begin{tikzcd}
\operatorname{Spec} R/\mathfrak{m} \arrow[r, hook] \arrow[drrrr, "{[E_0]}"'] & \operatorname{Spec} R/\mathfrak{m}^2 \arrow[r, hook] \arrow[drrr, "{[E_1]}"'] & \operatorname{Spec} R/\mathfrak{m}^3 \arrow[r, hook] \arrow[drr, "{[E_2]}"] & \cdots \arrow[r, hook] & \operatorname{Spec} R \arrow[d, dashed, "{[\widehat{E}]}"] \\
 & & & & \mathcal{M}.
\end{tikzcd}$$

In §C.5, we prove Grothendieck's Existence Theorem (C.5.3) and show how it implies that formal deformations are effective for Problems (A)–(C).

(6) Given an effective versal formal deformation $\widehat{E}$ over R, Artin Algebraization (C.6.8) ensures the existence of a *finite type* $\Bbbk$-scheme U with a point $u \in U(\Bbbk)$ and an object $E \in \mathcal{M}(U)$ such that $R \cong \widehat{\mathcal{O}}_{U,u}$ and $\widehat{E}|_{\operatorname{Spec} R/\mathfrak{m}^{n+1}} \cong E|_{\operatorname{Spec} R/\mathfrak{m}^{n+1}}$ for all n.

(7) Artin's Axioms for Algebraicity (C.7.1 and C.7.4) provide criteria to verify the algebraicity of a moduli problem $\mathcal{M}$. Namely, it provides conditions to ensure that the morphism $[E]\colon U \to \mathcal{M}$ constructed above is a smooth morphism in an open neighborhood of E_0.

For additional algebraic treatments of deformation theory, we recommend [Art76], [Kol96, §I.2], [Vis97], [FGI+05, §6], [Ser06], [Nit09], [Har10], and [SP, Tag 0ELW]. We also recommend [Kod86] for deformations of manifolds, and [Ill71, Ill72] for an exhaustive treatment of the cotangent complex.

C.1 First order deformations

For a field $\Bbbk$, denote the *dual numbers* by $\Bbbk[\epsilon] := \Bbbk[\epsilon]/(\epsilon^2)$.

C.1.1 First order embedded deformations

Definition C.1.1. Let X be a projective scheme over a $\Bbbk$ and $Z_0 \subseteq X$ be a closed subscheme. A *first-order deformation of* $Z_0 \subseteq X$ is a closed subscheme

$Z \subseteq X_{\Bbbk[\epsilon]} := X \times_{\Bbbk} \Bbbk[\epsilon]$ flat over $\Bbbk[\epsilon]$ such that $Z_0 = Z \times_{\Bbbk[\epsilon]} \Bbbk$. Pictorially, a first-order deformation is a filling of the diagram

$$\begin{tikzcd}
 & X \arrow[rr, hook] \arrow[dd] & & X_{\Bbbk[\epsilon]} \arrow[dd] \\
Z_0 \arrow[ur, hook, "\mathrm{cl}"] \arrow[rr, dashed] \arrow[dr] & & Z \arrow[ur, hook, dashed, "\mathrm{cl}"] \arrow[dr, dashed, "\mathrm{flat}"] & \\
 & \operatorname{Spec} \Bbbk \arrow[rr, hook] & & \operatorname{Spec} \Bbbk[\epsilon].
\end{tikzcd}$$

with a scheme Z flat over $\Bbbk[\epsilon]$ and dotted arrows making the diagram cartesian.

Note that since both Z_0 and the central fiber $Z \times_{\Bbbk[\epsilon]} \Bbbk$ of Z are embedded in X, it makes sense in the definition to require that they are equal. We say that $Z \subseteq X_{\Bbbk[\epsilon]}$ is *trivial* if $Z = Z_0 \times_{\Bbbk} \Bbbk[\epsilon]$.

Remark C.1.2. The closed subscheme $Z_0 \subseteq X$ defines a $\Bbbk$-point $[Z_0 \subset X] \in \operatorname{Hilb}^P(X)$ of the Hilbert scheme, where P is the Hilbert polynomial of Z_0 with respect to a fixed ample line bundle on X. A first-order deformation corresponds to a commutative diagram

$$\begin{tikzcd}
\operatorname{Spec} \Bbbk \arrow[r, "{[Z_0 \subseteq X]}"] \arrow[d, hook] & \operatorname{Hilb}^P(X) \\
\operatorname{Spec} \Bbbk[\epsilon], \arrow[ur, "{[Z \subseteq X_{\Bbbk[\epsilon]}]}"'] &
\end{tikzcd}$$

or in other words a tangent vector $[Z \subseteq X_{\Bbbk[\epsilon]}] \in T_{\operatorname{Hilb}^P(X),[Z_0 \subseteq X]}$.

Proposition C.1.3. *Let X be a scheme over a $\Bbbk$ and $Z_0 \subseteq X$ be a closed subscheme defined by a sheaf of ideals $I_0 \subseteq \mathcal{O}_X$. There is a bijection*

$$\{\text{first-order deformations } Z \subseteq X_{\Bbbk[\epsilon]}\} \cong \mathrm{H}^0(Z_0, N_{Z_0/X})$$

where $N_{Z_0/X} = \mathscr{H}om_{\mathcal{O}_{Z_0}}(I_0/I_0^2, \mathcal{O}_{Z_0})$ is the normal sheaf. Under this correspondence, the trivial deformation corresponds to $0 \in \mathrm{H}^0(Z_0, N_{Z_0/X})$.

Remark C.1.4. In light of Remark C.1.2, this proposition gives a bijection $T_{\operatorname{Hilb}^P(X),[Z_0 \subseteq X]} \cong \mathrm{H}^0(Z_0, N_{Z_0/X})$.

Proof. We first handle the case when $X = \operatorname{Spec} B$ and $Z_0 = \operatorname{Spec} B/I_0$, and show that the set of first-order deformations is bijective to

$$\mathrm{H}^0(Z_0, N_{Z_0/X}) \cong \operatorname{Hom}_{B/I_0}(I_0/I_0^2, B/I_0) \cong \operatorname{Hom}_B(I_0, B/I_0).$$

Given a first-order deformation $Z = \operatorname{Spec} B[\epsilon]/I$, the flatness of Z over $\Bbbk[\epsilon]$ implies that tensoring the exact sequence $0 \to I \to B[\epsilon] \to B[\epsilon]/I \to 0$ of $\Bbbk[\epsilon]$-modules with $\Bbbk = \Bbbk[\epsilon]/(\epsilon)$ yields an exact sequence $0 \to I_0 \to B \to B/I_0 \to 0$. We define $\alpha \colon I_0 \to B/I_0$ as follows: for $x_0 \in I_0$, choose a preimage $x = a + b\epsilon \in I$ and set $\alpha(x_0) := \bar{b} \in B/I_0$. Conversely, given a B-module homomorphism $\alpha \colon I_0 \to B/I_0$, we define

$$I = \{a + b\epsilon \mid a \in I_0, b \in B \text{ such that } \bar{b} = \alpha(a) \in B/I_0\} \subseteq B[\epsilon].$$

Then $(B[\epsilon]/I) \otimes_{\Bbbk[\epsilon]} \Bbbk = B/I_0$. To see that $B[\epsilon]/I$ is flat over $\Bbbk[\epsilon]$, it suffices by the flatness criterion for the dual numbers (see Remark A.2.7) to check that the map

$B/I_0 \xrightarrow{\epsilon} B[\epsilon]/I$ is injective: given $b \in B$ with $b\epsilon \in I$, then $b \in I_0$ by the definition of I. Thus $Z = \operatorname{Spec} B[\epsilon]/I$ defines a first-order deformation of Z_0. For a general $\Bbbk$-scheme X, after choosing an affine cover $\{U_i\}$, one checks that the bijections between deformations of U_i and $\mathrm{H}^0(U_i \cap Z_0, N_{U_i \cap Z_0/U_i})$ glue to the desired bijection. See also [Art76, Thm. 6.1], [Kol96, Thm. 2.8], [Ser06, Prop. 3.2.1], [Har10, Prop. 2.3] for details. ☐

C.1.2 First-order deformations of schemes

Definition C.1.5. Let X_0 be a scheme over a field $\Bbbk$. A *first-order deformation of* X_0 is a scheme X flat over $\Bbbk[\epsilon]$ together with an isomorphism $\alpha\colon X_0 \to X \times_{\Bbbk[\epsilon]} \Bbbk$, or in other words a cartesian diagram

$$\begin{tikzcd} X_0 \arrow[r, hook, dashed] \arrow[d] \arrow[dr, phantom, "\square"] & X \arrow[d, dashed, "\text{flat}"] \\ \operatorname{Spec} \Bbbk \arrow[r, hook] & \operatorname{Spec} \Bbbk[\epsilon]. \end{tikzcd} \tag{C.1.6}$$

A *morphism of first-order deformations* (X, α) *and* (X', α') is a morphism $\beta\colon X \to X'$ of schemes over $\Bbbk[\epsilon]$ such that $(\beta \times_{\Bbbk[\epsilon]} \Bbbk) \circ \alpha = \alpha'$, or in other words considering X and X' in cartesian diagrams (C.1.6), we require the restriction of β to central fiber X_0 to be the identity.

We say that X is *trivial* if X is isomorphic to $X_0 \times_{\Bbbk} \Bbbk[\epsilon]$ as first-order deformations and *locally trivial* if there exists a Zariski-cover $X = \bigcup_i U_i$ such that U_i is a trivial first-order deformation of $U_i \times_{\Bbbk[\epsilon]} \Bbbk \subseteq X_0$.

Every morphism of deformations is necessarily an isomorphism. This is a consequence of the following algebraic fact.

Lemma C.1.7. *Let A be a ring, $\mathfrak{m} \subseteq A$ be a nilpotent ideal (e.g., $(A, \mathfrak{m})$ is an artinian local ring), and $M \to N$ be a homomorphism of A-modules. Assume that N is flat over A. If $M/\mathfrak{m}M \to N/\mathfrak{m}N$ is an isomorphism, so is $M \to N$.*

Proof. If $C := \operatorname{coker}(M \to N)$, then $C/\mathfrak{m}C = \operatorname{coker}(M/\mathfrak{m}M \to N/\mathfrak{m}N) = 0$. As $\mathfrak{m}^n = 0$ for some n, we obtain that $C = \mathfrak{m}C = \mathfrak{m}^2 C = \cdots = \mathfrak{m}^n C = 0$. If $K := \ker(M \to N)$, then the flatness of N implies that $K/\mathfrak{m}K = \ker(M/\mathfrak{m}M \to N/\mathfrak{m}N) = 0$. Thus $K = \mathfrak{m}K = \cdots = \mathfrak{m}^n K = 0$, and we see that $M \to N$ is an isomorphism. ☐

Proposition C.1.8. *Every first-order deformation of a smooth affine scheme X_0 over $\Bbbk$ is trivial. In other words, X_0 is rigid.*

Proof. Let X be a first-order deformation of X_0. Since $X_0 \to \operatorname{Spec} \Bbbk$ is smooth, we may apply the Infinitesimal Lifting Criterion for Smoothness (A.3.1) to construct a lift $X \to X_0$ making the diagram

$$\begin{tikzcd} X_0 \arrow[r, "\mathrm{id}"] \arrow[d, hook] & X_0 \arrow[d, "\text{smooth}"] \\ X \arrow[r] \arrow[ur, dashed] & \operatorname{Spec} \Bbbk \end{tikzcd}$$

commute. This induces a morphism $X \to X_0 \times_{\Bbbk} \Bbbk[\epsilon]$ over $\Bbbk[\epsilon]$ which restricts to the identity on X_0, and thus is an isomorphism by Lemma C.1.7. See also [Har77, Exc. II.8.7]. ☐

If X_0 is not smooth or affine, then first-order deformations are not necessarily trivial. For example, if $X_0 = \operatorname{Spec} \Bbbk[x, y]/(xy)$ is the nodal affine plane curve, then $X = \operatorname{Spec} \Bbbk[x, y, \epsilon]/(xy - \epsilon) \to \operatorname{Spec} \Bbbk[\epsilon]$ is a non-trivial first-order deformation. On the other hand, considering an elliptic curve $E_0 = V(y^2z - x(x - z)(x - \lambda z)) \subseteq \mathbb{P}^2$ for $\lambda \neq 0, 1$,

$$E_\alpha = V(y^2z - x(x - z)(x - (\lambda + \alpha\epsilon)z)) \subseteq \mathbb{P}^2_{\Bbbk[\epsilon]} \tag{C.1.9}$$

defines a first-order deformation of E_0 for every $\alpha \in \Bbbk$, and the assignment $\alpha \mapsto E_\alpha$ defines a bijection between $\Bbbk$ and the set of isomorphism classes of first-order deformations; this follows from Proposition C.1.11 since $\mathrm{H}^1(E_0, T_{E_0}) = \mathrm{H}^1(E_0, \mathcal{O}_{E_0}) = \Bbbk$. In fact, the same formula (C.1.9) defines a global deformation of E_0 over $\mathbb{A}^1$ with two singular fibers.

To get a handle on locally trivial deformations, we need to understand automorphisms of the trivial deformation.

Lemma C.1.10. *Let $X_0 = \operatorname{Spec} A$ be an affine scheme over $\Bbbk$ and $X = \operatorname{Spec} A[\epsilon]$ be the trivial first-order deformation. There are identifications*

$$\{\textit{automorphisms } X \to X \textit{ of first-order defs}\} \cong \operatorname{Der}_\Bbbk(A, A) \cong \operatorname{Hom}_A(\Omega_{A/\Bbbk}, A).$$

Proof. The second equivalence is given by the universal property of the module of differentials. An automorphism of X (as a first-order deformation) corresponds to a $\Bbbk[\epsilon]$-algebra isomorphism $\phi\colon A \oplus A\epsilon \to A \oplus A\epsilon$ which is the identity modulo ϵ. Therefore, ϕ is determined by the images $\phi(a) = a + d(a)\epsilon$ of elements $a \in A$ where $d\colon A \to A$ is $\Bbbk$-linear map. The map ϕ is a ring homomorphism if and only if $aa' + d(aa')\epsilon = (a + d(a)\epsilon)(a' + d(a')\epsilon) = aa' + (ad(a') + a'd(a))\epsilon$ for elements $a, a' \in A$, which this translates into the condition that $d\colon A \to A$ is a $\Bbbk$-derivation. □

For a scheme X_0 over $\Bbbk$, let $\operatorname{Def}(X_0)$ and $\operatorname{Def}^{\mathrm{lt}}(X_0)$ denote the sets of isomorphism classes of first-order and locally trivial first-order deformations.

Proposition C.1.11. *For a scheme X_0 of finite type over $\Bbbk$ with affine diagonal, there is a bijection*

$$\operatorname{Def}^{\mathrm{lt}}(X_0) \cong \mathrm{H}^1(X_0, T_{X_0}),$$

where $T_{X_0} = \mathscr{H}om_{\mathcal{O}_{X_0}}(\Omega_{X_0/\Bbbk}, \mathcal{O}_{X_0})$. The trivial deformation corresponds to $0 \in \mathrm{H}^1(X_0, T_{X_0})$. If in addition X_0 is smooth over $\Bbbk$, there is a bijection

$$\operatorname{Def}(X_0) \cong \mathrm{H}^1(X_0, T_{X_0}).$$

Proof. Let $X \to \operatorname{Spec} \Bbbk[\epsilon]$ be a locally trivial first-order deformation of X_0. Choose an affine cover $\{U_i\}$ of X_0 and isomorphisms $\phi_i\colon U_i \times_\Bbbk \Bbbk[\epsilon] \xrightarrow{\sim} X \cap U_i$. Letting $U_{ij} = U_i \cap U_j$, we have automorphisms $\phi_j^{-1}|_{U_{ij} \times_\Bbbk \Bbbk[\epsilon]} \circ \phi_i|_{U_{ij} \times_\Bbbk \Bbbk[\epsilon]}$ of the trivial deformation $U_{ij} \times_\Bbbk \Bbbk[\epsilon]$, which corresponds by Lemma C.1.10 to elements $\phi_{ij} \in T_{X_0}(U_{ij})$. Since $\phi_{ij} \circ \phi_{jk} = \phi_{ik}$ on $U_{ijk} := U_i \cap U_j \cap U_k$, we have that $\phi_{ij} + \phi_{jk} = \phi_{ik} \in T_{X_0}(U_{ijk})$. Recall that $\mathrm{H}^1(X_0, T_{X_0})$ can be computed using the Čech complex

$$0 \longrightarrow \bigoplus_i T_{X_0}(U_i) \xrightarrow{d_0} \bigoplus_{i,j} T_{X_0}(U_{ij}) \xrightarrow{\quad d_1 \quad} \bigoplus_{i,j,k} T_{X_0}(U_{ijk})$$

$$(s_{ij}) \longmapsto (s_{ij}|_{U_{ijk}} - s_{ik}|_{U_{ijk}} + s_{jk}|_{U_{ijk}})_{ijk}$$

As $\{\phi_{ij}\} \in \bigoplus_{i,j} T_{X_0}(U_{ij})$ is in the kernel of d_1, it defines an element of $\mathrm{H}^1(X_0, T_{X_0}) = \ker(d_1)/\operatorname{im}(d_0)$. Conversely, given an element of $\mathrm{H}^1(X_0, T_{X_0})$ and a choice of representative $\{\phi_{ij}\} \in \ker(d_1)$, then viewing each ϕ_{ij} as an automorphism of the trivial deformation of U_{ij}, we may glue together the trivial deformations $U_i \times_{\Bbbk} \Bbbk[\epsilon]$ along $U_{ij} \times_{\Bbbk} \Bbbk[\epsilon]$ via ϕ_{ij} to construct a global first-order deformation X of X_0. The final statement follows as all first-order deformations are locally trivial when X_0 is smooth (Proposition C.1.8). See also [Har77, Exc. III.4.10 and Ex. III.9.13.2]. $\square$

Example C.1.12. If C is a smooth projective curve of genus $g \geq 2$, then

$$T_{\mathcal{M}_g,[C]} = \mathrm{H}^1(C, T_C) \overset{\mathrm{SD}}{=} \mathrm{H}^0(C, \Omega_{C/\Bbbk}^{\otimes 2}),$$

which by Riemann–Roch is a $3g-3$ dimensional vector space.

Exercise C.1.13 (easy)**.** Use the Euler exact sequence to show that $\mathrm{H}^1(\mathbb{P}^n, T_{\mathbb{P}^n}) = 0$ and conclude that every first-order deformation of $\mathbb{P}^n$ is trivial, i.e., $\mathbb{P}^n$ is rigid.

Exercise C.1.14 (moderate)**.** Let C_0 be a smooth and proper curve over $\Bbbk$ with marked points $p_1, \ldots, p_n \in C_0(\Bbbk)$. Let $\mathrm{Def}(C_0, p_i)$ denote the set of first-order deformations of (C_0, p_i), i.e., flat morphisms $C \to \operatorname{Spec} \Bbbk[\epsilon]$ with n sections σ_i extending (C_0, p_i). Show that

$$\mathrm{Def}(C_0, p_i) \cong \mathrm{H}^1\big(C_0, T_{C_0}(-\textstyle\sum_i p_i)\big).$$

Remark C.1.15. More generally, if X_0 is generically smooth and a local complete intersection over $\Bbbk$, then

$$\mathrm{Def}(X_0) = \mathrm{Ext}^1_{\mathcal{O}_{X_0}}(\Omega_{X_0}, \mathcal{O}_{X_0}).$$

Observe that when X_0 is smooth, this recovers Proposition C.1.11. This also holds if $X_0 = \operatorname{Spec} R$ is the spectrum a complete noetherian local ring with an isolated singularity such that R is a complete intersection ring. It is easy to see how a first-order deformation X of X_0 induces an extension class: the closed immersion $X_0 \hookrightarrow X$ is defined by an ideal $J \subset \mathcal{O}_X$ with $J^2 = 0$ such that $J \cong \mathcal{O}_{X_0}$ as an $\mathcal{O}_{X_0}$-module, and the usual right exact sequence

$$\mathcal{O}_{X_0} \to \Omega_X|_{X_0} \to \Omega_{X_0} \to 0$$

is also left exact because the kernel of the left map is on one hand torsion free (as a subsheaf of $\mathcal{O}_{X_0}$) and on the other hand torsion (as generic smoothness implies that the map is generically injective).

Exercise C.1.16 (good practice)**.**

(1) Show that there is a bijection $\mathrm{Def}(\Bbbk[x,y]/(xy)) \cong \Bbbk$, where an element $t \in \Bbbk$ corresponds to the first-order deformation $\operatorname{Spec} \Bbbk[x,y,\epsilon]/(xy - t\epsilon)$.

(2) Classify first-order deformations of the A_k-singularity $\Bbbk[x,y]/(y^2 - x^{k+1})$.

C.1.3 First order deformations of vector bundles and coherent sheaves

Definition C.1.17. Let X be a scheme over $\Bbbk$ and E_0 be a coherent sheaf. A *first-order deformation of* E_0 is a pair (E, α) where E is a coherent sheaf on $X \times_{\Bbbk} \Bbbk[\epsilon]$ flat over $\Bbbk[\epsilon]$ and $\alpha \colon E_0 \xrightarrow{\sim} E|_X$ is an isomorphism. Pictorially, we have

$$\begin{tikzcd} E_0 \arrow[d, dash] & E \arrow[d, dash, "\text{flat}/\Bbbk[\epsilon]"] \\ X \arrow[r, hook] & X_{\Bbbk[\epsilon]}. \end{tikzcd}$$

A *morphism* $(E, \alpha) \to (E', \alpha')$ *of first-order deformations* is a morphism $\beta \colon E \to E'$ (equivalently an isomorphism by Lemma C.1.7) of coherent sheaves on X' such that $\alpha' = \beta|_X \circ \alpha$. We say that (E, α) is *trivial* if it isomorphic as first-order deformations to $(p^* E_0, \mathrm{id})$ where $p \colon X_{\Bbbk[\epsilon]} \to X$.

Proposition C.1.18. *Let X be a scheme over $\Bbbk$ and E_0 be a coherent sheaf. There is a bijection*

$$\{\textit{first-order deformations } (E, \alpha) \textit{ of } E_0\}/\!\sim \;\cong \mathrm{Ext}^1_{\mathcal{O}_X}(E_0, E_0).$$

Under this correspondence, the trivial deformation corresponds to $0 \in \mathrm{Ext}^1_{\mathcal{O}_X}(E_0, E_0)$. If in addition E_0 is a vector bundle (resp., line bundle), then each first-order deformation is a vector bundle (resp., line bundle), and the set of isomorphism classes of first-order deformations of E_0 is bijective to $\mathrm{H}^1(X, \mathscr{E}nd_{\mathcal{O}_X}(E_0))$ (resp., $\mathrm{H}^1(X, \mathcal{O}_X)$).

Proof. If (E, α) is a first-order deformation, then since E is flat over $\Bbbk[\epsilon]$, we may tensor the exact sequence $0 \to \Bbbk \xrightarrow{\epsilon} \Bbbk[\epsilon] \to \Bbbk \to 0$ of $\Bbbk[\epsilon]$-modules with E to obtain an exact sequence $0 \to E_0 \xrightarrow{\epsilon} E \to E_0 \to 0$ (after identifying $E \otimes_{\Bbbk[\epsilon]} \Bbbk$ with E_0 via α). Since $\mathrm{Ext}^1_{\mathcal{O}_X}(E_0, E_0)$ parameterizes isomorphism classes of extensions [Har77, Exc. III.6.1], we have constructed an element of $\mathrm{Ext}^1_{\mathcal{O}_X}(E_0, E_0)$. Conversely, given an exact sequence $0 \to E_0 \xrightarrow{\alpha} E \to E_0 \to 0$, then E is a coherent sheaf on $X_{\Bbbk[\epsilon]}$ and is flat over $\Bbbk[\epsilon]$ by the flatness criterion over the dual numbers (see Remark A.2.7). The restriction $E|_X$ is isomorphic to E_0 via α. See also [Har10, Thm. 2.7]. □

Remark C.1.19. The classifications of Proposition C.1.3, C.1.11, and C.1.18 give a vector space structure to the set of isomorphism classes of first-order deformations. This vector space structure can also be witnessed as a consequence of Rim–Schlessinger's homogeneity condition; see Lemma C.4.9. For each case, the operations of scalar multiplication and addition afford geometric descriptions.

C.2 Higher-order deformations and obstructions

We need some new obstructions!

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Let $\mathcal{M}$ be a moduli problem, $E \in \mathcal{M}(A)$ be an object defined over a ring A, and let $A' \twoheadrightarrow A$ be a surjection of rings with square-zero kernel. This section addresses the following two questions:

(1) Does E deform to an object $E' \in \mathcal{M}(A')$?

(2) If so, can we classify all such deformations?

Pictorially, we have:

$$\begin{tikzcd} E \arrow[d, dash] & E' \arrow[d, dash] \\ \operatorname{Spec} A \arrow[r, hook] & \operatorname{Spec} A'. \end{tikzcd}$$

Note that since $J = \ker(A' \to A)$ is square-zero, $J = J/J^2$ is naturally a module over $A = A'/J$. Question (1) asks whether there is an 'obstruction' to the existence of a deformation E' while (2) seeks to classify all higher-order deformations given that there is no obstruction.

An interesting case is when A and A' are local artinian algebras with residue field $\Bbbk$ and the kernel $J = \ker(A' \to A)$ satisfies $\mathfrak{m}_{A'} J = 0$ (which implies that $J^2 = 0$). In this case, $J = J/\mathfrak{m}_{A'} J$ is naturally a vector space over $\Bbbk = A'/\mathfrak{m}_{A'}$. Setting $E_0 := E|_{\Bbbk} \in \mathcal{M}(\Bbbk)$, we can view E as a deformation of E_0 over A, and we are attempting to classify the higher-order deformations over A'. If there are no obstructions to deforming, then the Infinitesimal Lifting Criterion for Smoothness (3.7.1) implies that $\mathcal{M}$ is smooth at $[E_0]$.

The previous section studied the specific case when $A = \Bbbk$ and $A' = \Bbbk[\epsilon]$ in which case deformations of an object $E_0 \in \mathcal{M}(\Bbbk)$ over A' correspond to first-order deformations. In this case, the obstruction vanishes as there is always the trivial deformation (i.e., the pullback of E_0 along $\operatorname{Spec} \Bbbk[\epsilon] \to \operatorname{Spec} \Bbbk$). Other examples of $A' \to A$ to keep in mind are $\Bbbk[x]/x^{n+1} \twoheadrightarrow \Bbbk[x]/x^n$ and $\mathbb{Z}/p^{n+1} \twoheadrightarrow \mathbb{Z}/p^n$ where we inductively attempt to deform E_0 over the nilpotent thickenings $\operatorname{Spec} \Bbbk[x]/x^{n+1} \hookrightarrow \mathbb{A}^1$ and $\operatorname{Spec} \mathbb{Z}/p^{n+1} \hookrightarrow \operatorname{Spec} \mathbb{Z}$.

C.2.1 Higher-order embedded deformations

Definition C.2.1. Let $A' \twoheadrightarrow A$ be a surjection of rings. Let X' be a scheme over A' and set $X := X' \times_{A'} A$. Let $Z \subseteq X$ be a closed subscheme flat over A. A *deformation of $Z \subseteq X$ over A'* is a closed subscheme $Z' \subseteq X'$ flat over A' such that $Z' \times_{A'} A = Z$ as closed subschemes of X. Pictorially, a deformation is a filling of the cartesian diagram

$$\begin{tikzcd}
& X \arrow[rr, hook] \arrow[dd] & & X' \arrow[dd] \\
Z \arrow[ur, hook, "\mathrm{cl}"] \arrow[rr, dashed, hook] \arrow[dr, "\mathrm{flat}"'] & & Z' \arrow[ur, dashed, hook, "\mathrm{cl}"] \arrow[dr, dashed, "\mathrm{flat}"] & \\
& \operatorname{Spec} A \arrow[rr, hook] & & \operatorname{Spec} A'.
\end{tikzcd}$$

The formulation of the next proposition uses the following notion: a *torsor* of an abstract group G is set with a free and transitive of G.

Proposition C.2.2. *Let X be a scheme over $\Bbbk$ with affine diagonal (e.g., separated). Let $A' \twoheadrightarrow A$ be a surjection of artinian $\Bbbk$-algebras with residue field $\Bbbk$ such that $\mathfrak{m}_{A'} J = 0$ where $J = \ker(A' \to A)$. Let $Z \subseteq X_A$ be a closed subscheme flat over A, and let $Z_0 = Z \times_A \Bbbk$. Then*

(1) If there exists a deformation $Z' \subseteq X_{A'}$ of $Z \subseteq X_A$ over A', then the set of such deformations is a torsor under $\operatorname{Ext}^0_{\mathcal{O}_X}(I_{Z_0}, \mathcal{O}_{Z_0} \otimes_{\Bbbk} J)$.

(2) There exists an element $\mathrm{ob}_Z \in \operatorname{Ext}^1_{\mathcal{O}_X}(I_{Z_0}, \mathcal{O}_{Z_0} \otimes_{\Bbbk} J)$ (depending on Z and $A' \twoheadrightarrow A$) such that there exists a deformation of $Z \subseteq X$ over A' if and only if $\mathrm{ob}_Z = 0$.

More generally, if $A' \twoheadrightarrow A$ is a surjection of $\Bbbk$-algebras with square-zero kernel J, then deformations and obstructions of $Z \subset X_A$ are classified by $\operatorname{Ext}^i_{\mathcal{O}_{X_A}}(I_Z, \mathcal{O}_Z \otimes_A J)$ for $i = 0, 1$.

Note that if $Z_0 \subseteq X$ is a local complete intersection, then I_{Z_0}/I_{Z_0} is a vector bundle and $\operatorname{Ext}^i_{\mathcal{O}_X}(I_{Z_0}, \mathcal{O}_{Z_0} \otimes_{\Bbbk} J) = \mathrm{H}^i(Z_0, \mathcal{N}_{Z_0/X} \otimes_{\Bbbk} J)$.

Proof. We first handle the affine case. Write $X = \operatorname{Spec} B_0$, $X = \operatorname{Spec} B$, $X' = \operatorname{Spec} B'$, $Z = \operatorname{Spec} B/I$, and $Z_0 = \operatorname{Spec} B_0/I_0$. If there exists a deformation $Z' =$

Spec B'/I', then there is an exact diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & I \otimes_A J \arrow[r] \arrow[d] & I' \arrow[r] \arrow[d] & I \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & B \otimes_A J \arrow[r] \arrow[d] & B' \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & (B/I) \otimes_A J \arrow[r] \arrow[d] & B'/I' \arrow[r] \arrow[d] & B/I \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

The exactness of the bottom row (resp., middle row) is equivalent to the flatness of B'/I' (resp., B') over A' by the Local Criterion of Flatness (A.2.6), while the exactness of the left column follows from the flatness of B/I over A. Conversely, an exact diagram defines a deformation $Z' = \operatorname{Spec} B'/I'$.

We will define an action $\operatorname{Hom}_B(I, (B/I) \otimes_A J)$ on the set of deformations. Given $\phi \in \operatorname{Hom}_B(I, (B/I) \otimes_A J)$ and a deformation $Z' = \operatorname{Spec} B'/I'$, define $I'' \subseteq B'$ as the set of elements $x'' \in B'$ with the following property: the image $\overline{x}'' \in B$ of x'' lies in I and if $x' \in I'$ is a lifting of $\overline{x}'' \in I$, the image of $x'' - x' \in B \otimes_A J$ in $(B/I) \otimes_A J$ is equal to $\phi(\overline{x}'')$ (noting that this condition is independent of the choice of lifting x'). One checks that $\operatorname{Spec} B'/I''$ is another deformation. On the other hand, given two deformations defined by ideals I' and I'', we define $\phi \colon I \to (B/I) \otimes_A J$ by $\phi(x) = \overline{x'' - x'}$, where $x' \in I'$ and $x'' \in I''$ are lifts of x (which forces $x'' - x' \in B \otimes_A J$). One checks that this is a B-module homomorphism providing an inverse to the above construction. Since J is a $\Bbbk$-vector space, there is an identification $\operatorname{Hom}_B(I, (B/I) \otimes_A J) = \operatorname{Hom}_{B_0}(I_0, B_0/I_0 \otimes_{\Bbbk} J)$, and this construction globalizes to X to establish (1).

We will prove (2) under the hypothesis that there is an open cover $\{U_i\}$ of X and deformations $Z'_i \subseteq X_{A'} \cap U_i$ of $Z \cap U_i \hookrightarrow X_A \cap U_i$ over A'. This is satisfies if $Z_0 \hookrightarrow X$ is a local complete intersection; see [Kol96, Lem. 2.12]. The restrictions $Z'_i \cap U_{ij}$ and $Z'_j \cap U_{ij}$ are related by an element $\phi_{ij} \in \mathrm{H}^0(U_{ij}, N_{Z_0/X} \otimes_A J)$ which in turn defines a Čech 1-cocycle $(\phi_{ij}) \in \mathrm{H}^1(X, N_{Z_0/X} \otimes_A J)$. We leave the reader to check that the vanishing of (ϕ_{ij}) characterizes whether there is a deformation of $Z \subseteq X_A$ over A'. See also [FGAIII, §5], [Art69b, Lem. 6.7], [Kol96, Prop. 2.5], [Vis97, Thm. 2.5], and [Har10, Thm. 6.2]. □

C.2.2 Higher-order deformations of schemes

Definition C.2.3. Let $A' \twoheadrightarrow A$ be a surjection of rings and $X \to \operatorname{Spec} A$ be a flat morphism of schemes. A *deformation of* $X \to \operatorname{Spec} A$ *over* A' is a flat morphism $X' \to \operatorname{Spec} A'$ together with an isomorphism $\alpha \colon X \xrightarrow{\sim} X' \times_{A'} A$ over A, or in other words a cartesian diagram

$$\begin{tikzcd}
X \arrow[r, hook, dashed] \arrow[d, "\text{flat}"'] \arrow[dr, phantom, "\square"] & X' \arrow[d, dashed, "\text{flat}"] \\
\operatorname{Spec} A \arrow[r, hook] & \operatorname{Spec} A'.
\end{tikzcd}$$

A *morphism of deformations over A'* is a morphism of schemes over A' restricting to the identity on X. By Lemma C.1.7, every morphism of deformations is an isomorphism.

Proposition C.2.4 (Higher-order Deformations of Complete Intersections)**.** *Let X_0 be a scheme of finite type over a field $\Bbbk$ such that X_0 is generically smooth and a local complete intersection. Let $A' \twoheadrightarrow A$ be a surjection of artinian local rings with residue field $\Bbbk$ such that $\mathfrak{m}_{A'} J = 0$ where $J := \ker(A' \to A)$. If $X \to \operatorname{Spec} A$ is a deformation of X_0, then:*

(1) The group of automorphisms of a deformation $X \to \operatorname{Spec} A$ over A' is bijective to $\operatorname{Ext}^0_{\mathcal{O}_{X_0}}(\Omega_{X_0}, J)$.

(2) If there exists a deformation of $X \to \operatorname{Spec} A$ over A', then the set of isomorphism classes of all such deformations is a torsor under $\operatorname{Ext}^1_{\mathcal{O}_{X_0}}(\Omega_{X_0}, J)$.

(3) There is an element $\mathrm{ob}_X \in \operatorname{Ext}^2_{\mathcal{O}_{X_0}}(\Omega_{X_0}, J)$ with the property that there exists a deformation of $X \to \operatorname{Spec} A$ over A' if and only if $\mathrm{ob}_X = 0$.

In particular, if X_0 is smooth, then automorphisms, deformations and obstructions are classified by $\mathrm{H}^i(X_0, T_{X_0} \otimes_\Bbbk J)$ for $i = 0, 1, 2$.

Proof. We will prove the smooth case. For the general case, an explicit argument is given in [Vis97, Thm. 4.4]; alternatively, since X_0 is generically smooth and a local complete intersection, the cotangent complex $_{X_0}$ is quasi-isomorphic to Ω_{X_0} (Theorem C.3.1(3)) and thus the result follows from the fact that the cotangent complex controls automorphisms, deformations, and obstructions (Theorem C.3.6).

When $X_0 = \operatorname{Spec} B_0$ is an affine scheme, the same argument of Lemma C.1.10 shows that group of automorphisms of X' is identified with $\operatorname{Hom}_B(\Omega_{B/\Bbbk}, J)$. Since $U \mapsto \operatorname{Hom}_{\mathcal{O}_U}(\Omega_{X_0/\Bbbk}|_U, J)$ and $U \mapsto \operatorname{Aut}(X' \cap U / X \cap U)$ are sheaves, Part (1) follows.

Part (2) follows from a similar argument to Proposition C.1.11. Indeed, fix a deformation $X' \to \operatorname{Spec} A'$. If $\{U_i\}$ is an affine cover of X_0, by the Infinitesimal Lifting Criterion for Smoothness (A.3.1), there are trivializations $\phi_i \colon U_i \times_\Bbbk A' \xrightarrow{\sim} X' \cap U_i$. Then $\phi_{ij} = \phi_j^{-1} \circ \phi_i$ defines an automorphism of the trivial deformation corresponding by (1) to an element $\phi_{ij} \in \mathrm{H}^0(U_{ij}, T_{X_0} \otimes J)$. An element of $\mathrm{H}^1(X_0, T_{X_0} \otimes_\Bbbk J)$ defines a Čech 1-cocycle (ψ_{ij}) with respect to the covering $\{U_i\}$, and $\phi_{ij} + \Psi_{ij}$ defines isomorphisms of the restrictions of the trivial deformations over U_i, which glue to a global deformation $X'' \to \operatorname{Spec} A'$. Conversely, if $X'' \to \operatorname{Spec} A'$ is another deformation, there are isomorphisms $\phi_i \colon X' \cap U_i \to X'' \cap U_i$ for each i, and $\phi_{ij} = \phi_j^{-1}|_{X'|_{U_{ij}}} \circ \phi_i|_{X'|_{U_{ij}}} \in \mathrm{H}^0(U_{ij}, T_{X_0} \otimes J)$ defines a Čech 1-cycle (ϕ_{ij}) in $\mathrm{H}^1(X_0, T_{X_0} \otimes J)$.

For (3), we again let $\{U_i\}$ be an affine cover. Each deformation $X \cap U_i$ of $X_0 \cap U_i$ is trivial, and induces automorphisms $\phi_{ij} \colon U_{ij} \times_\Bbbk A \xrightarrow{\sim} U_{ij} \times_\Bbbk A$. By the Infinitesimal Lifting Criterion for Smoothness (A.3.1), we may choose extensions $\phi'_{ij} \colon U_{ij} \times_\Bbbk A' \xrightarrow{\sim} U_{ij} \times_\Bbbk A'$ of ϕ_{ij}. This defines gluing data for a deformation X' of X if the cocycle condition $\phi'_{jk} \circ \phi'_{ij} = \phi'_{ik}$ on the triple intersections $U_{ijk} \times_\Bbbk A'$ is satisfied. The automorphism $\Psi'_{ijk} = (\phi'_{ik})^{-1} \circ \phi'_{jk} \circ \phi'_{ij}$ restricts to the identity on $U_{ijk} \times_\Bbbk A$ and thus defines an element of $\mathrm{H}^0(U_{ijk}, T_{X_0} \otimes J)$. Consider the Čech complex for $F = T_{X_0} \otimes J$ with respect to $\{U_i\}$:

$$\begin{tikzcd} \bigoplus_{i,j} F(U_{ij}) \arrow[r, "d_1"] & \bigoplus_{i,j,k} F(U_{ijk}) \arrow[r, "d_2"] & \bigoplus_{i,j,k,l} F(U_{ijkl}) \end{tikzcd}$$

$$(s_{ij}) \longmapsto (s_{ij} - s_{ik} + s_{jk})_{ijk}$$

$$(s_{ijk}) \longmapsto (s_{ijk} - s_{ijl} + s_{ikl} - s_{jkl})_{ijkl}.$$

One checks that $d_2(\Psi'_{ijk}) = 0$ and thus (Ψ'_{ijk}) is a Čech 2-cocycle defining an element of $\mathrm{H}^2(X, T_{X/A} \otimes J)$. If this element is zero, i.e., there exists (s_{ij}) mapping to (Ψ'_{ijk}), then modifying the automorphisms ϕ'_{ij} by s_{ij} defines isomorphisms ϕ''_{ij} satisfying the cocycle condition. See also [Ser06, Prop. 1.2.12] and [Har10, Cor. 10.3]. $\square$

Exercise C.2.5 (Interpretation of deformations and obstruction using gerbes)**.** Let $X \to \operatorname{Spec} A$ be a smooth morphism of schemes. Consider the category $\mathcal{G}$ over Sch/X whose objects over $S \to X$ are cartesian diagrams

$$\begin{tikzcd} S \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & S' \arrow[d] \\ \operatorname{Spec} A \arrow[r, hook] & \operatorname{Spec} A' \end{tikzcd}$$

where $S \to \operatorname{Spec} A$ is the composition $S \to X \to \operatorname{Spec} A$. A morphism $(S \to X, S \hookrightarrow S' \to \operatorname{Spec} A') \to (T \to X, T \hookrightarrow T' \to \operatorname{Spec} A')$ is the data of a morphism $\phi\colon S' \to T'$ over A' such that ϕ restricts to a morphism $S \to T$ over X.

(a) Show that $\mathcal{G}$ is a gerbe banded by the sheaf of groups $T_{X/A} \otimes_A J$ on X. (*Hint: See Definition 6.4.10 for the definition of a banded gerbe.*)

(b) Give an alternate proof of Proposition C.2.4. (*Hint:* For part C.2.4(3), use Exercise 6.4.31.)

Exercise C.2.6 (Deformations of principal G-bundles)**.** Let G be a smooth affine algebraic group over a field $\Bbbk$ with Lie algebra $\mathfrak{g}$. Let $X \hookrightarrow X'$ be a closed immersion of finite type $\Bbbk$-schemes defined by a square-zero sheaf of ideals J and assume that X has affine diagonal. Show that

(1) The group of automorphisms of a deformation $P' \to X'$ of $P \to X$ is bijective to $\mathrm{H}^0(X, \mathfrak{g} \otimes_\Bbbk J)$.

(2) If there exists a deformation over X', then the set of isomorphism classes of all such deformations is a torsor under $\mathrm{H}^1(X, \mathfrak{g} \otimes_\Bbbk J)$.

(3) There is an element $\mathrm{ob}_X \in \mathrm{H}^2(X, \mathfrak{g} \otimes_\Bbbk J)$ with the property that there exists a deformation over X' if and only if $\mathrm{ob}_X = 0$.

Example C.2.7 (Abelian varieties)**.** If X_0 is an abelian variety over $\mathbb{C}$ of dimension n, then it turns out that deforming X_0 as an abstract scheme is equivalent to deforming it as an abelian variety, and thus obstructions to deforming X_0 as an abelian variety also live in $\mathrm{H}^2(X_0, T_{X_0})$. Using that $\Omega_{X_0} = \mathcal{O}_{X_0}^n$ is trivial and the Hodge symmetries,

$$\mathrm{H}^2(X_0, T_{X_0}) = \mathrm{H}^2(X_0, \mathcal{O}_{X_0})^{\oplus n} = \mathrm{H}^0(X_0, \bigwedge^2 \mathcal{O}_{X_0}^n)^{\oplus n}$$

is nonzero. Nevertheless, Grothendieck and Mumford showed that the obstruction $\mathrm{ob}_X \in \mathrm{H}^2(X, T_{X/A} \otimes_A J)$ vanishes for every deformation problem! This shows that abelian varieties are unobstructed, and their moduli space is smooth. See [Oor71].

C.2.3 Higher-order deformations of morphisms

For the deformation theory of *pointed* stable curves, we will need the following enhancement of Higher-order Deformations of Complete Intersections (C.2.4) to the following deformation problem where we are assuming that a deformation Z' over A' already exists:

$$\begin{tikzcd}
Z \arrow[r, hook] \arrow[d, hook] \arrow[dr, phantom, "\square"] & Z' \arrow[d, hook, dashed] \\
X \arrow[r, hook, dashed] \arrow[d] \arrow[dr, phantom, "\square"] & X' \arrow[d] \\
\operatorname{Spec} A \arrow[r, hook] & \operatorname{Spec} A'.
\end{tikzcd}$$

Proposition C.2.8 (Higher-order Deformations of Morphisms with Fixed Source)**.** *Let X_0 be a scheme of finite type over a field $\Bbbk$ such that X_0 is generically smooth and a local complete intersection, and let $Z_0 \subset X_0$ be a closed subscheme with ideal sheaf I_{Z_0}. Let $A' \twoheadrightarrow A$ be a surjection of artinian local rings with residue field $\Bbbk$ such that $\mathfrak{m}_{A'} J = 0$ where $J := \ker(A' \to A)$. If $X \to \operatorname{Spec} A$ is a deformation of X_0 and $Z' \to \operatorname{Spec} A'$ is a deformation of Z_0, then:*

(1) The group of automorphisms of a deformation $Z' \subseteq X'$ of $Z \subseteq X$ over A' is bijective to $\operatorname{Ext}^0_{\mathcal{O}_{X_0}}(\Omega_{X_0}, I_{Z_0} \otimes_{\Bbbk} J)$.

(2) If there exists a deformation $Z' \subseteq X'$ over A', then the set of isomorphism classes of all such deformations is a torsor under $\operatorname{Ext}^1_{\mathcal{O}_{X_0}}(\Omega_{X_0}, I_{Z_0} \otimes_{\Bbbk} J)$.

(3) There is an element $\mathrm{ob}_{Z/X} \in \operatorname{Ext}^2_{\mathcal{O}_{X_0}}(\Omega_{X_0}, I_{Z_0} \otimes_{\Bbbk} J)$ with the property that there exists a deformation $Z' \subseteq X'$ over A' if and only if $\mathrm{ob}_X = 0$.

Proof. See [Vis97, Thm. 5.4]. □

Exercise C.2.9 (Higher-order deformations of morphisms)**.** Assume that X and Y are proper A, and that $f_*\mathcal{O}_X = \mathcal{O}_Y$ and $\mathrm{R}^1 f_*\mathcal{O}_X = 0$. Show that the functor taking a deformation $f' \colon X' \to Y'$ of $f \colon X \to Y$ over A' to the deformation X' over X over A' induces an isomorphism of categories.

Hint: Given a deformation X' over X, define Y' as the ringed space $(Y, f_*\mathcal{O}_{X'})$. Use the conditions of f and the flatness of X' over A' to show that Y' is a scheme flat over A'. See also [Ran89, Thm. 3.3], [Vak06, §5.3], and [SP, Tag 0E3X].

For further background on deformations of morphisms, see [Ser06, §3.4].

C.2.4 Higher-order deformations of vector bundles

Definition C.2.10. Let $A' \twoheadrightarrow A$ be a surjection of rings. Let X' be a scheme over A' and set $X := X' \times_{A'} A$. Given a coherent sheaf E on X flat over A, a *deformation of E over $A' \to A$* is a pair (E', α) where E' is a coherent sheaf on X' flat over A' and $\alpha \colon E \to E'|_X$ is an isomorphism. Pictorially, we have

$$\begin{tikzcd}
E \arrow[d, dash, "\text{flat}/A"] & E' \arrow[d, dash, "\text{flat}/A'"] \\
X \arrow[r, hook] & X'.
\end{tikzcd}$$

A *morphism* $(E, \alpha) \to (E', \alpha')$ *of deformations* is a morphism $\beta \colon E \to E'$ of coherent sheaves on $X_{A'}$ such that $\alpha' = \beta|_X \circ \alpha$. By Lemma C.1.7, every morphism of deformations is an isomorphism.

Proposition C.2.11. *Let X be a scheme over a field $\Bbbk$. Let $A' \twoheadrightarrow A$ be a surjection of artinian local rings with residue field $\Bbbk$ such that $\mathfrak{m}_{A'}J = 0$ where $J := \ker(A' \to A)$. Let E be a coherent sheaf on X_A and set $E_0 = E|_X$.*

(1) The group of automorphisms of a deformation E' of E over A' is bijective to $\operatorname{Ext}^0_{\mathcal{O}_X}(E_0, E_0 \otimes_\Bbbk J)$.

(2) If there exists a deformation of E over A', then the set of isomorphism classes of all such deformations is a torsor under $\operatorname{Ext}^1_{\mathcal{O}_X}(E_0, E_0 \otimes_\Bbbk J)$..

(3) There is an element $\mathrm{ob}_E \in \operatorname{Ext}^2_{\mathcal{O}_X}(E_0, E_0 \otimes_\Bbbk J)$ with the property that there exists a deformation of E over A' if and only if $\mathrm{ob}_E = 0$.

More generally, if $A' \twoheadrightarrow A$ is a surjection of $\Bbbk$-algebras with square-zero kernel J, automorphisms, deformations, and obstructions are classified by $\operatorname{Ext}^i_{\mathcal{O}_{X_A}}(E, E \otimes_\Bbbk J)$ for $i = 0, 1, 2$.

Remark C.2.12. If E is a vector bundle (resp., line bundle), automorphisms, deformations, and obstructions are classified by $\mathrm{H}^i(X, \mathscr{E}nd_{\mathcal{O}_X}(E_0) \otimes_\Bbbk J)$ (resp., $\mathrm{H}^i(X, J)$) for $i = 0, 1, 2$. In the case of line bundles, the obstruction can be realized by the exact sequence $\operatorname{Pic}(X') \to \operatorname{Pic}(X) \to \mathrm{H}^2(X, J)$ induced from taking cohomology of the short exact sequence $0 \to J \to \mathbb{G}_{m,X'} \to \mathbb{G}_{m,X} \to 1$.

Proof. For (1), since E' is flat over A', tensoring the exact sequence $0 \to \mathcal{O}_{X'} \otimes_\Bbbk J \to \mathcal{O}_{X'} \to \mathcal{O}_X \to 0$ with E yields a short exact sequence

$$0 \to E \otimes_\Bbbk J \to E' \to E \to 0.$$

If $\alpha\colon E' \xrightarrow{\sim} E'$ is an automorphism with $\alpha|_E = \mathrm{id}$, then $\alpha - \mathrm{id}$ defines a map $E' \to E \otimes_\Bbbk J$, which factors to give a map $\phi\colon E_0 \to E_0 \otimes_\Bbbk J$. Conversely, if ϕ is a homomorphism, then the sum of the identity and $E \to E_0 \xrightarrow{\phi} E_0 \otimes_\Bbbk J$ defines an automorphism.

For the rest of the proof, we will assume that E is a vector bundle. For (2), let E' be a deformation. Let $\{U_i\}$ be an affine covering of X with trivializations $\phi_i\colon E'|_{U_i} \xrightarrow{\sim} \mathcal{O}^{\oplus r_i}_{U_i \times_\Bbbk A'}$. Then $\phi_{ij} = \phi_j^{-1} \circ \phi_i$ is an automorphism of $E'|_{U_{ij}}$, which by (1) corresponds to an element of $\mathrm{H}^0(U_{ij}, \mathscr{E}nd_{\mathcal{O}_X}(E))$. A element of $\mathrm{H}^1(X, \mathscr{E}nd_{\mathcal{O}_X}(E))$ defines a Čech 1-cocycle (ψ_{ij}) with $\psi_{ij} \in \mathrm{H}^0(U_{ij}, \mathscr{E}nd_{\mathcal{O}_X}(E))$, and $\phi_{ij} + \psi_{ij}$ defines isomorphisms of the trivial vector bundle over $U_{ij} \times_\Bbbk A'$ which glue to a deformation E' over $X_{A'}$. Conversely, if E' and E'' are two deformations, there exists a covering $\{U_i\}$ and isomorphisms $\alpha_i\colon E'|_{U_i} \to E''|_{U_i}$. The automorphisms $\phi_{ij} = \phi_j^{-1} \circ \phi_i$ defines elements in $\mathrm{H}^0(U_{ij}, \mathscr{E}nd_{\mathcal{O}_X}(E))$, and (ϕ_{ij}) defines a Čech 1-cycle in $\mathrm{H}^1(X, \mathscr{E}nd_{\mathcal{O}_X}(E))$.

For (3), let $\{U_i\}$ be an affine covering with trivializations $\phi_i\colon \mathcal{O}^{\oplus r_i}_{U_i \times \operatorname{Spec} A} \xrightarrow{\sim} E|_{U_i}$ yielding automorphisms $\phi_{ij} = \phi_j^{-1} \circ \phi_i$ of the trivial vector bundles on $U_{ij} \times_\Bbbk A$. Choose automorphisms ϕ'_{ij} of the trivial vector bundles on $U_{ij} \times_\Bbbk A$ extending ϕ_{ij}. The automorphisms $\Psi'_{ijk} = (\phi'_{ik})^{-1} \circ \phi'_{jk} \circ \phi'_{ij}$ correspond to elements of $\mathrm{H}^0(U_{ijk}, T_{X_0} \otimes J)$ and define a Čech 2-cocycle $\mathrm{ob}_E := (\Psi'_{ijk}) \in \mathrm{H}^2(X, \mathscr{E}nd_{\mathcal{O}_X}(E))$. See also [Har10, Thm. 7.1], [HL10, §2.A.6], and [SP, Tag 08VW]. □

Exercise C.2.13. Give an alternative proof of Proposition C.2.11 using the technique outlined in Exercise C.2.5. bundle case

C.3 Cotangent complex

Computers are useless. They can only give you answers.

Pablo Picasso

In this chapter, we summarize properties of the cotangent complex of a morphism of schemes as introduced in [Ill71], globalizing work of André [And67] and Quillen [Qui68, Qui70] on the cotangent complex of a ring homomorphism. A major advantage of the cotangent complex is that it allows us to describe the deformations and obstruction of singular schemes (Theorem C.3.6).

C.3.1 Properties of the cotangent complex

Theorem C.3.1. *For every morphism $f\colon X \to Y$ of schemes (resp., finite type morphism of noetherian schemes), there exists a complex*

$$\mathrm{L}_{X/Y}\colon \ \cdots \to L^{-1}_{X/Y} \to L^{0}_{X/Y} \to 0$$

of flat $\mathcal{O}_X$-modules with quasi-coherent (resp., coherent) cohomology, whose image in $D^-_{\mathrm{QCoh}}(\mathcal{O}_X)$ (resp., $D^-_{\mathrm{Coh}}(\mathcal{O}_X)$) is also denoted by $\mathrm{L}_{X/Y}$. It satisfies the following properties.

(1) $\mathrm{H}^0(X, \mathrm{L}_{X/Y}) \cong \Omega_{X/Y}$.

(2) The morphism f is smooth if and only if f is locally of finite presentation and $\mathrm{L}_{X/Y}$ is a perfect complex supported in degree 0. In this case, there is a quasi-isomorphism $\mathrm{L}_{X/Y} \xrightarrow{\sim} \Omega_{X/Y}$ with $\Omega_{X/Y}$ in degree 0.

(3) If f is flat and finitely presented, then f is syntomic (Definition A.3.17) if and only if $\mathrm{L}_{X/Y}$ is a perfect complex supported in degrees $[-1, 0]$. Explicitly, if f factors as a complete intersection $X \hookrightarrow \widetilde{X}$ defined by a sheaf of ideals I and a smooth morphism $\widetilde{X} \to Y$, then $\mathrm{L}_{X/Y}$ is quasi-isomorphic to $0 \to I/I^2 \to \Omega_{\widetilde{X}/Y}|_X \to 0$ with $\Omega_{X/Y}$ in degree 0. If in addition f is generically smooth, then $L_{X/Y} \simeq \Omega_{X/Y}$.

(4) If

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d] \arrow[dr, phantom, "\square"] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

is a cartesian diagram with either f or g flat (or more generally f and g are tor-independent), then there is a quasi-isomorphism $g'^*\mathrm{L}_{X/Y} \to \mathrm{L}_{X'/Y'}$. (Note that without any flatness condition $g'^*\Omega_{X/Y} \simeq \Omega_{X'/Y'}$.)

(5) If $X \xrightarrow{f} Y \to Z$ is a composition of morphisms of schemes, then there is an exact triangle in $D^-_{\mathrm{QCoh}}(\mathcal{O}_X)$

$$f^*\mathrm{L}_{Y/Z} \to \mathrm{L}_{X/Z} \to \mathrm{L}_{X/Y} \to f^*\mathrm{L}_{Y/Z}[1].$$

This induces a long exact sequence on cohomology

$$\begin{tikzcd}
& \cdots \arrow[r] & \mathrm{H}^{-2}(\mathrm{L}_{X/Y}) \arrow[dll, out=-10, in=170] \\
\mathrm{H}^{-1}(f^*\mathrm{L}_{X/Z}) \arrow[r] & \mathrm{H}^{-1}(\mathrm{L}_{X/Z}) \arrow[r] & \mathrm{H}^{-1}(\mathrm{L}_{X/Y}) \arrow[dll, out=-10, in=170] \\
f^*\Omega_{Y/Z} \arrow[r] & \Omega_{X/Z} \arrow[r] & \Omega_{X/Y} \arrow[r] & 0,
\end{tikzcd}$$

extending the usual right exact sequence on differentials [Har77, II.8.12]. (Note that if f is smooth, then $H^{-1}(\mathrm{L}_{X/Y}) = 0$ and $f^*\Omega_{Y/Z} \to \Omega_{X/Z}$ is injective.)

Proof. See [Ill71, II.1.2.3] and [SP, Tag 08T2] for the definition and construction of the cotangent complex of a morphism of schemes (and more generally for morphisms of ringed topoi). For (1)–(5), see [Ill71, II.1.2.4.2, II.3.1.2, II.3.2.6, II.2.2.3 and II.2.1.2] and [SP, Tags 08UV, 0D0N, 0FK3, 08QQ, and 08T4] (noting that [SP, Tag 08RB] relates the *naive cotangent complex* $N\mathrm{L}_{X/Y}$ to $\mathrm{L}_{X/Y}$). For the final statement of (3), we observe that the right exact sequence

$$I/I^2 \xrightarrow{d} \Omega_{\widetilde{X}/Y}|_X \to \Omega_{X/Y} \to 0, \tag{C.3.2}$$

is also left exact: as $X \hookrightarrow \widetilde{X}$ is a complete intersection, I/I^2 is a vector bundle and thus $\ker(d)$ is torsion free, but, on the other hand, since $X \to Y$ is generically smooth, d is generically injective so that $\ker(d)$ is torsion. It follows $\ker(d) = 0$ and that $L_{X/Y} = [I/I^2 \xrightarrow{d} \Omega_{\widetilde{X}/Y}|_X]$ is quasi-isomorphic to $\Omega_{X/Y}$. □

C.3.2 Truncations of the cotangent complex

The definition of the cotangent complex relies on simplicial techniques and we will not attempt an exposition here. We will however give an explicit description of its truncation, which often suffices for applications. First, if $X \to Y$ factors as a closed immersion $X \hookrightarrow P$ defined by a sheaf of ideals I and a smooth morphism $P \to Y$, then the truncation $\tau_{\geq -1}(\mathrm{L}_{X/Y})$ of $\mathrm{L}_{X/Y}$ in degrees $[-1, 0]$ is quasi-isomorphic to $0 \to I/I^2 \to \Omega_{X/Y} \to 0$ (with $\Omega_{X/Y}$ in degree 0). If $X \to Y$ is syntomic (e.g., smooth), then $X \hookrightarrow \widetilde{Y}$ is a regular immersion, I/I^2 is a vector bundle, and $\mathrm{L}_{X/Y} \cong \tau_{\geq -1}(\mathrm{L}_{X/Y})$ (Theorem C.3.1(3)).

For a morphism $X = \operatorname{Spec} A \to \operatorname{Spec} B = Y$ of affine schemes, Lichtenbaum–Schlessinger [LS67] offer an explicit description of the truncation $\tau_{\geq -2}(\mathrm{L}_{A/B})$ of $L_{X/Y} = L_{B/A}$. Choose a polynomial ring $P = B[x_i]$ (with possibly infinitely many generators) and a surjection $P \twoheadrightarrow A$ as B-algebras with kernel I. Choose a free P-module F and a surjection $p \colon F \twoheadrightarrow I$ of P-modules with kernel $K = \ker(p)$. Let $K' \subseteq K$ be the submodule generated by $p(x)y - p(y)x$ for $x, y \in F$. Then $\tau_{\geq -2}(\mathrm{L}_{B/A})$ is quasi-isomorphic to the complex of A-modules

$$K/K' \to F \otimes_P A \to \Omega_{P/B} \otimes_P A, \tag{C.3.3}$$

with the last term in degree 0. See also [SP, Tag 09CG].

For $i = 0, 1, 2$, one defines the T^i *functors* on the category of A-modules by

$$T^i(A/B, -) := \mathrm{H}^i(\operatorname{Hom}_A(\mathrm{L}_{A/B}, -)) = \mathrm{H}^i(\operatorname{Hom}_A(\tau_{\geq 2}\mathrm{L}_{A/B}, -)),$$

which describe deformations of schemes. See also [LS67, §2.3] and [Har10, §1.3].

C.3.3 Extensions of algebras and schemes

Definition C.3.4. An *extension* of a morphism $X \to S$ of schemes by a quasi-coherent $\mathcal{O}_X$-module J is a short exact sequence

$$0 \to J \to \mathcal{O}_{X'} \to \mathcal{O}_X \to 0$$

where $X \hookrightarrow X'$ is a closed immersion of schemes defined by the sheaf of ideals $J \subseteq \mathcal{O}_{X'}$ with $J^2 = 0$. (Note that the condition $J^2 = 0$ implies that the $J \subset \mathcal{O}_{X'}$ is naturally a $\mathcal{O}_X$-module.) The *trivial extension* is $X[J] := (X, \mathcal{O}_X \oplus J)$ where the ring structure is defined by $J^2 = 0$.

A *morphism of extensions* is a morphism of short exact sequences which is the identity on J and $\mathcal{O}_X$. We let $\underline{\mathrm{Exal}}_S(X, J)$ be the category of extensions of $X \to S$ by J, and $\mathrm{Exal}_S(X, J)$ be the set of isomorphism classes. If $X = \operatorname{Spec} A$ and $S = \operatorname{Spec} R$ is affine, we write $\mathrm{Exal}_R(A, J)$.

Geometrically, an extension is a commutative diagram of schemes

$$\begin{tikzcd}
X \arrow[r, hook] \arrow[d] & X' \arrow[dl] \\
\operatorname{Spec} R &
\end{tikzcd}$$

such that $J \cong \ker(\mathcal{O}_{X'} \to \mathcal{O}_X)$ and $J^2 = 0$. The set of extensions $\mathrm{Exal}_S(X, J)$ is functorial with respect to $\mathcal{O}_X$-module maps $J \to J'$ and morphisms $X' \to X$ of S-schemes, and inherits the structure of a module over $\Gamma(X, \mathcal{O}_X)$. In fact, the groupoid $\underline{\mathrm{Exal}}_S(X, J)$ is a *Picard category*, and the prestack over Sch/S whose fiber category over $f\colon T \to S$ is $\underline{\mathrm{Exal}}_T(X_T, f^*J)$ is a *Picard stack*; see [Ill71, III.1.1.5] and [SGA4, XVIII.1.4]. Given an exact sequence $0 \to J' \to J \to J'' \to 0$ of $\mathcal{O}_X$-modules, there is an exact sequence

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Der}_{\mathcal{O}_S}(\mathcal{O}_X, J') \arrow[r] & \mathrm{Der}_{\mathcal{O}_S}(\mathcal{O}_X, J) \arrow[r] & \mathrm{Der}_{\mathcal{O}_S}(\mathcal{O}_X, J'') \arrow[dll, out=-10, in=170] \\
& \mathrm{Exal}_S(X, J) \arrow[r] & \mathrm{Exal}_S(X, J) \arrow[r] & \mathrm{Exal}_S(X, J'').
\end{tikzcd}$$

Given a morphism $f\colon X \to Y$ of S-schemes, there is an exact sequence

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Der}_{\mathcal{O}_Y}(\mathcal{O}_X, J) \arrow[r] & \mathrm{Der}_{\mathcal{O}_S}(\mathcal{O}_X, J) \arrow[r] & \mathrm{Der}_{\mathcal{O}_S}(\mathcal{O}_Y, f_*J) \arrow[dll, out=-10, in=170] \\
& \mathrm{Exal}_Y(X, J) \arrow[r] & \mathrm{Exal}_S(X, J) \arrow[r] & \mathrm{Exal}_S(Y, f_*J).
\end{tikzcd}$$

See [EGA, 0.20.2.3] and [Ill71, III.1.2.4.3, III.1.2.5.4]. The top row of the second diagram above is realized by applying $\mathrm{Hom}_{\mathcal{O}_X}(-, J)$ to the right exact sequence $f^*\Omega_{Y/S} \to \Omega_{X/S} \to \Omega_{X/Y} \to 0$ and using the identifications $\mathrm{Hom}_{\mathcal{O}_X}(\Omega_{X/Y}, J) = \mathrm{Der}_{\mathcal{O}_Y}(\mathcal{O}_X, J)$, $\mathrm{Hom}_{\mathcal{O}_X}(\Omega_{X/S}, J) = \mathrm{Der}_{\mathcal{O}_S}(\mathcal{O}_X, J)$, and $\mathrm{Hom}_{\mathcal{O}_X}(f^*\Omega_{Y/S}, J) = \mathrm{Hom}_{\mathcal{O}_Y}(\Omega_{Y/S}, f_*J) = \mathrm{Der}_{\mathcal{O}_S}(\mathcal{O}_Y, f_*J)$.

C.3.4 The cotangent complex and deformation theory

Theorem C.3.5. *If $X \to Y$ is a morphism of schemes and J is a quasi-coherent $\mathcal{O}_Y$-module, there is a natural isomorphism*

$$\mathrm{Exal}_Y(X, J) \cong \mathrm{Ext}^1_{\mathcal{O}_X}(\mathrm{L}_{X/Y}, J).$$

Proof. See [Ill71, III.1.2.3]. □

By applying $\mathrm{Hom}_{\mathcal{O}_X}(\mathrm{L}_{X/Y}, -)$ to the exact sequence $0 \to J' \to J \to J''$ and $\mathrm{Hom}_{\mathcal{O}_X}(-, J)$ to the exact triangle $f^*\mathrm{L}_{Y/Z} \to \mathrm{L}_{X/Z} \to \mathrm{L}_{X/Y}$ allows us to extend the two above six-term exact sequences to long exact sequences. When $X = \operatorname{Spec} A \to \operatorname{Spec} B = Y$ is a morphism of affine schemes, using the T^i functors of §C.3.2, the above equivalence translates to $\mathrm{Exal}_B(A, J) = T^1(A/B, J)$, which can also be explicitly using the truncated cotangent complex (C.3.3); see [LS67, 4.2.2] and [Har10, Thm. 5.1]. See [LS67, 2.3.5-6] and [Har10, Thms. 3.4-5] for how the T^i functors extend the above six-term sequences nine-term sequences.

Theorem C.3.6. *Consider the following deformation problem*

$$\begin{tikzcd} X \arrow[r, hook, dashed] \arrow[d, "f"] & X' \arrow[d, dashed, "f'"] \\ Y \arrow[r, hook, "i"] & Y' \end{tikzcd}$$

where $f\colon X \to Y$ is a morphism of schemes and $i\colon Y \hookrightarrow Y'$ is a closed immersion of schemes defined by an ideal sheaf $J \subseteq \mathcal{O}_{Y'}$ with $J^2 = 0$. A deformation *is a morphism $f'\colon X' \to Y'$ making the above diagram cartesian, and a morphism of deformations is a morphism over Y' restricting to the identity on X.*

(1) The group of automorphisms of a deformation $f'\colon X' \to Y'$ is isomorphic to $\mathrm{Ext}^0_{\mathcal{O}_X}(\mathrm{L}_{X/Y}, f^*J)$.

(2) If there exists a deformation, then the set of deformations is a torsor under $\mathrm{Ext}^1_{\mathcal{O}_X}(\mathrm{L}_{X/Y}, f^*J)$.

(3) There exists an element $\mathrm{ob}_X \in \mathrm{Ext}^2_{\mathcal{O}_X}(\mathrm{L}_{X/Y}, f^*J)$ with the property that there exists a deformation if and only if $\mathrm{ob}_X = 0$.

Proof. See [Ill71, III.2.1.7] and [SP, Tag 08UZ]. See also [LS67, 4.2.5] and [Har10, Thm. 10.1] for descriptions in the affine case using the truncated cotangent complex. □

As a reality check, if $f\colon X \to \operatorname{Spec} A$ is syntomic (e.g., smooth) and $A' \twoheadrightarrow A$ is a surjection of rings with square-zero kernel J, then the identification

$$\mathrm{Ext}^i_{\mathcal{O}_X}(\mathrm{L}_{X/A}, f^*J) = \mathrm{Ext}^i_{\mathcal{O}_X}(\Omega_{X/A}, J)$$

recovers Proposition C.2.4.

Remark C.3.7. There are analogous results for other deformation problems. For instance, for the deformation problem

$$\begin{tikzcd} X \arrow[r, hook] \arrow[dr, "f"] \arrow[ddr] & X' \arrow[dr, dashed] \arrow[ddr] & \\ & Y \arrow[r, hook] \arrow[d] & Y' \arrow[d] \\ & Z \arrow[r, hook] & Z', \end{tikzcd}$$

where the horizontal morphisms are closed immersions defined by square-zero ideal sheaves J_X, J_Y and J_Z, then automorphisms, deformations, and obstructions are classified by $\mathrm{Ext}^i_{\mathcal{O}_X}(Lf^*\mathrm{L}_{Y/Z}, J_X)$ for $i = -1, 0, 1$ [Ill71, III.2.2.4].

C.4 Versal formal deformations and Rim–Schlessinger's Criteria

C.4.1 Functors and prestacks over artin rings

We work over a fixed field $\Bbbk$ for simplicity, but the definitions below and Rim–Schlessinger's Criteria can be formulated more generally (see Remark C.4.5). Let $\mathrm{Art}_{\Bbbk}$ denote the category of artinian local $\Bbbk$-algebras with residue field $\Bbbk$.

Definition C.4.1 (Prorepresentability)**.** We say that a covariant functor $F\colon \mathrm{Art}_{\Bbbk} \to \mathrm{Sets}$ is *prorepresentable* if there exists a complete noetherian local $\Bbbk$-algebra R and an isomorphism $F \xrightarrow{\sim} h_R$ where $h_R := \mathrm{Hom}_{\Bbbk-\mathrm{alg}}(R, -)$.

If $F\colon \mathrm{Sch}/\Bbbk \to \mathrm{Sets}$ is a contravariant functor and $x_0 \in F(\Bbbk)$, then we can consider the induced functor of artin rings

$$F_{x_0}\colon \mathrm{Art}_{\Bbbk} \to \mathrm{Sets}, \quad A \mapsto \{x \in F(A) \mid x|_{\Bbbk} = x_0 \in F(\Bbbk)\}$$

where $x|_{\Bbbk}$ denotes the image of x under $F(A) \to F(A/\mathfrak{m}_A)$. If F is representable by a scheme X and $x \in X$ is the $\Bbbk$-point corresponding to x_0, then F_{x_0} is prorepresentable by $\widehat{\mathcal{O}}_{X,x}$. It is possible that F_{x_0} be prorepresentable, but F not be representable.

Many functors of artin rings are not prorepresentable. For example, if C_0 is a smooth, connected, and projective curve with a non-trivial automorphism group, then the covariant functor $F_{C_0}\colon \mathrm{Art}_{\Bbbk} \to \mathrm{Sets}$ where $F_{C_0}(A)$ consists of isomorphism classes of smooth proper families of curves $\mathcal{C} \to \operatorname{Spec} A$ such that $\mathcal{C} \times_A A/\mathfrak{m}_A$ is isomorphic to C_0, is not prorepresentable. Nevertheless, many moduli functors admit *versal* deformations. As it is important to keep track of automorphisms, we will formulate the definition for prestacks over $\mathrm{Art}_{\Bbbk}^{\mathrm{op}}$.

Definition C.4.2 (Versality)**.** Let $\mathcal{X}$ be a prestack over $\mathrm{Art}_{\Bbbk}^{\mathrm{op}}$ such that the groupoid $\mathcal{X}(\Bbbk)$ is equivalent to a set $\{x_0\}$.

(1) A *formal deformation* $(R, \{x_n\})$ *of* x_0 is the data of a complete noetherian local $\Bbbk$-algebra $(R, \mathfrak{m}_R)$ together with objects $x_n \in \mathcal{X}(R/\mathfrak{m}_R^{n+1})$ and morphisms $x_{n-1} \to x_n$ over $\operatorname{Spec} R/\mathfrak{m}_R^n \to \operatorname{Spec} R/\mathfrak{m}_R^{n+1}$.

(2) A formal deformation $(R, \{x_n\})$ is *versal* if for every surjection $A \twoheadrightarrow A_0$ in $\mathrm{Art}_{\Bbbk}$ with $\mathfrak{m}_A^{n+1} = 0$, object $\eta \in \mathcal{X}(A)$, and $\Bbbk$-algebra homomorphism $\phi_0\colon R/\mathfrak{m}_R^{n+1} \to A_0$ with an isomorphism $\alpha_0\colon x_n|_{A_0} \xrightarrow{\sim} \eta|_{A_0}$ in $\mathcal{X}(A_0)$, there exists a $\Bbbk$-algebra homomorphism $\phi\colon R/\mathfrak{m}_R^{n+1} \to A$ extending ϕ_0 and an isomorphism $\alpha\colon x_n|_A \xrightarrow{\sim} \eta$ in $\mathcal{X}(A)$ extending α_0.

(3) A versal formal deformation $(R, \{x_n\})$ is *miniversal* (or *a prorepresentable hull*) if the induced map $h_R(\Bbbk[\epsilon]) \to \mathcal{X}(\Bbbk[\epsilon])/\!\sim$ on isomorphism classes is bijective.

In other words, a formal deformation is an element of $\varprojlim \mathcal{X}(R/\mathfrak{m}^n)$. When $\mathcal{X} = F$ is a covariant functor $\mathrm{Art}_{\Bbbk} \to \mathrm{Sets}$, a formal deformation is a compatible sequence of elements $x_n \in F(R/\mathfrak{m}_R^{n+1})$. If F is prorepresentable by R and $x_n \in F(R/\mathfrak{m}_R^{n+1})$ is the corresponding element, then $\{x_n\}$ is a miniversal formal deformation, in which case there is a unique lift in (C.4.3). Just as a deformation $x_n \in \mathcal{X}(R/\mathfrak{m}_R^{n+1})$ can be viewed via Yoneda's 2-Lemma as a morphism $\operatorname{Spec} R/\mathfrak{m}_R^{n+1} \to \mathcal{X}$, a formal deformation can be viewed as a morphism $\{x_n\}\colon h_R \to \mathcal{X}$ of prestacks. From this perspective, $\{x_n\}$ is versal if there exists a lift for every commutative diagram

$$\begin{tikzcd} \operatorname{Spec} A_0 \arrow[r] \arrow[d, hook] & h_R \arrow[d, "\{x_n\}"] \\ \operatorname{Spec} A \arrow[r, "\eta"] \arrow[ur, dashed] & \mathcal{X}, \end{tikzcd} \tag{C.4.3}$$

where $A \twoheadrightarrow A_0$ is a surjection in $\mathrm{Art}_{\Bbbk}$. This should be compared with the Infinitesimal Lifting Criterion for Smoothness (A.3.1 and 3.7.1). Note that since every surjection $A \to A_0$ factors as a composition of surjections with one dimensional kernels, versality can be checked on surjections with $\ker(A \to A_0) \cong \Bbbk$. A versal deformation is miniversal if it induces an isomorphism on tangent spaces $h_R(\Bbbk[\epsilon]) \to \mathcal{X}(\Bbbk[\epsilon])/\sim$.

Remark C.4.4 (Global prestacks to local deformation prestacks)*.* If $\mathcal{X}$ is a prestack over $\mathrm{Sch}/\Bbbk$ and $x_0 \in \mathcal{X}(\Bbbk)$, we define the *local deformation prestack* $\mathcal{X}_{x_0}$ *at* x_0 as the prestack, where an object is a morphism $\alpha\colon x_0 \to x$ in $\mathcal{X}$ to an object x over a ring $A \in \mathrm{Art}_{\Bbbk}^{\mathrm{op}}$; in other words, an object is a pair (x, α) where $x \in \mathcal{X}(A)$ and $\alpha\colon x_0 \to x|_{\Bbbk}$ is an isomorphism. A morphism $(\alpha\colon x_0 \to x) \to (\alpha'\colon x_0 \to x')$ is a morphism $\beta\colon x \to x'$ such that $\alpha' = \alpha \circ \beta$. Note that the fiber category $\mathcal{X}_{x_0}(\Bbbk)$ is equivalent to the set $\{\mathrm{id}\colon x_0 \to x_0\}$.

If $\mathcal{X}$ is an algebraic stack with a smooth presentation $U \to \mathcal{X}$ from a $\Bbbk$-scheme U and $u \in U(\Bbbk)$ is a lift of x_0, setting $x_n \in \mathcal{X}(\mathcal{O}_{U,u}/\mathfrak{m}_u^{n+1})$ to be the composition $\operatorname{Spec} \mathcal{O}_{U,u}/\mathfrak{m}_u^{n+1} \hookrightarrow U \to \mathcal{X}$ defines a versal formal deformation $\{x_n\}$. Rim–Schlessinger's Criteria (Theorem C.4.6) provides criteria for the existence of a versal formal deformation and Artin's Axioms for Algebraicity (Theorem C.7.1) provides further criteria for a versal formal deformation to be induced by a smooth presentation $U \to \mathcal{X}$ as above.

Remark C.4.5*.* To work over a more general base, consider a complete noetherian local ring Λ with residual field $\Bbbk$, and define Art_{Λ} to be the category of artinian local Λ-algebras $(A, \mathfrak{m})$ with an identification $\Bbbk \xrightarrow{\sim} A/\mathfrak{m}$. In practice, Λ is often taken to be a ring of Witt vectors, e.g., $\Lambda = \mathbb{Z}_p$. This generality is important for many applications, e.g., for lifting objects from characteristic p to characteristic 0; see §C.5.3. Even more generally, one can consider the setup where $A \to \Bbbk$ is a finite (but not necessarily surjective); see [SP, Tag 06GB].

C.4.2 Rim–Schlessinger's Criteria

Rim–Schlessinger's Criteria provides necessary and sufficient conditions for a prestack $\mathcal{X}$ over $\mathrm{Art}_{\Bbbk}^{\mathrm{op}}$ (or a covariant functor $F\colon \mathrm{Art}_{\Bbbk} \to \mathrm{Sets}$ as in Schlessinger's original formulation) to admit a versal formal deformation.

Theorem C.4.6 (Rim–Schlessinger's Criteria)**.** *Let $\mathcal{X}$ be a prestack over $\mathrm{Art}_{\Bbbk}^{\mathrm{op}}$ such that the groupoid $\mathcal{X}(\Bbbk)$ is equivalent to the set $\{x_0\}$. For morphisms $B_0 \to A_0$ and $A \to A_0$ in $\mathrm{Art}_{\Bbbk}$, consider the natural functor*

$$\mathcal{X}(B_0 \times_{A_0} A) \to \mathcal{X}(B_0) \times_{\mathcal{X}(A_0)} \mathcal{X}(A) \tag{C.4.7}$$

Then $\mathcal{X}$ admits a miniversal formal deformation if and only if

(RS$_1$) *the functor (C.4.7) is essentially surjective whenever $A \twoheadrightarrow A_0$ is surjection with kernel $\Bbbk$;*

(RS$_2$) *the map* (C.4.7) *is essentially surjective when* $A_0 = \Bbbk$ *and* $A = \Bbbk[\epsilon]$*, and given two commutative diagrams*

$$\begin{tikzcd} x_0 \arrow[r] \arrow[d] & y_0 \arrow[d, "\alpha"'] \arrow[dr, "\alpha'"] & \\ x \arrow[r, "\gamma"] \arrow[rr, bend right, "\gamma'"'] & y \arrow[r, dashed, "\beta"] & y' \end{tikzcd} \qquad \textit{over} \qquad \begin{tikzcd} \operatorname{Spec}\Bbbk \arrow[r, hook] \arrow[d] & \operatorname{Spec} B_0 \arrow[d] \\ \operatorname{Spec}\Bbbk[\epsilon] \arrow[r, hook] & \operatorname{Spec}(\Bbbk[\epsilon] \times_{\Bbbk} B_0), \end{tikzcd}$$

there exists an isomorphism $\beta\colon y \to y'$ *in* $\mathcal{X}(\Bbbk[\epsilon] \times_{\Bbbk} B_0)$ *such that* $\alpha' = \beta \circ \alpha$ *(but we do not require that* $\gamma' = \gamma \circ \beta$*).*

(RS$_3$) $\dim_{\Bbbk} T_{\mathcal{X}} < \infty$ *where* $T_{\mathcal{X}} := \mathcal{X}(\Bbbk[\epsilon])/\sim$ *(which is a vector space by Lemma C.4.9).*

Moreover, $\mathcal{X}$ *is prorepresentable if and only if* $\mathcal{X}$ *is equivalent to a functor and*

(RS$_4$) *the map* (C.4.7) *is an equivalence whenever* $A \twoheadrightarrow A_0$ *is a surjection with kernel* $\Bbbk$*.*

Conditions (RS$_2$)–(RS$_3$) (sometimes referred to as *semi-homogeneity*) may be difficult to parse, but in practice, it is in fact often just as easy to verify the stronger condition (RS$_4$) (called *homogeneity*), and in fact the even stronger condition (RS$_4^*$) (called *strong homogeneity*); see §C.4.3. When $A \to A_0$ is surjective, $\operatorname{Spec} B_0 \times_{A_0} A$ is the pushout of $\operatorname{Spec} A_0 \hookrightarrow \operatorname{Spec} A$ and $\operatorname{Spec} B_0 \to \operatorname{Spec} B$ in the category of schemes (see §B.4). Therefore, homogeneity conditions translate into gluing conditions of objects over the pushout.

Remark C.4.8 (Schlessinger's Criteria). When $\mathcal{X}$ is a covariant functor $F\colon \mathrm{Art}_{\Bbbk} \to \mathrm{Sets}$ with $F(\Bbbk) = \{x_0\}$, then (RS$_1$)–(RS$_4$) translate into Schlessinger's conditions as introduced in [Sch68]:

(H$_1$) the map (C.4.7) is surjective whenever $A \twoheadrightarrow A_0$ is a surjection with kernel $\Bbbk$;

(H$_2$) the map (C.4.7) is bijective when $A_0 = \Bbbk$ and $A = \Bbbk[\epsilon]$;

(H$_3$) $\dim_{\Bbbk} F(\Bbbk[\epsilon]) < \infty$; and

(H$_4$) the map (C.4.7) is bijective whenever $A \twoheadrightarrow A_0$ is a surjection with kernel $\Bbbk$.

The functor F admits a miniversal formal deformation if (H$_1$)–(H$_3$) hold and is prorepresentable if (H$_3$)–(H$_4$) hold.

If a prestack $\mathcal{X}$ over $\mathrm{Art}_{\Bbbk}$ satisfies (RS$_1$)–(RS$_3$), then the functor $F_{\mathcal{X}}\colon \mathrm{Art}_{\Bbbk} \to \mathrm{Sets}$ parameterizing isomorphism classes of objects satisfies (H$_1$)–(H$_3$) but the converse does not always hold. Moreover, the essential surjectivity of $\mathcal{X}(B_0 \times_{A_0} A) \to \mathcal{X}(B_0) \times_{\mathcal{X}(A_0)} \mathcal{X}(A)$ implies the surjectivity of $F_{\mathcal{X}}(B_0 \times_{A_0} A) \to F_{\mathcal{X}}(B_0) \times_{F_{\mathcal{X}}(A_0)} F_{\mathcal{X}}(A)$ and the fully faithfulness for $\mathcal{X}$ implies the injectivity of $F_{\mathcal{X}}$ as long as $\mathrm{Aut}_{\mathcal{X}(B_0)}(y_0) \to \mathrm{Aut}_{\mathcal{X}(A_0)}(y_0|_{A_0})$ is surjective for an object $y_0 \in \mathcal{X}(B_0)$. This latter condition holds in the case when $F_{\mathcal{X}}(A_0)$ is a set, e.g., when $A_0 = \Bbbk$. If $\mathcal{X}$ is the local deformation prestack arising from an object $x_0 \in \widetilde{\mathcal{X}}(\Bbbk)$ of an algebraic stack $\widetilde{\mathcal{X}}$ over $\mathrm{Sch}/\Bbbk$ as in Remark C.4.4, then the surjectivity condition on automorphisms translates by the Infinitesimal Lifting Criterion for Smoothness (3.7.1) to the smoothness of the inertia stack $I_{\mathcal{X}} \to \mathcal{X}$ at $e(x_0)$, where $e\colon \mathcal{X} \to I_{\mathcal{X}}$ is the identity section.

While the existence of a miniversal formal deformation of $F_{\mathcal{X}}$ suffices for many applications, for moduli problems with automorphisms it is more natural to ask for the existence of a miniversal formal deformation of $\mathcal{X}$ and this generality is needed for some applications, e.g., Artin's Algebraization (Theorem C.6.8) and Artin's Axioms for Algebraicity (Theorem C.7.4).

Before proceeding to the proof, we first indicate some properties of the conditions (RS$_1$)–(RS$_4$).

Lemma C.4.9. *Let $\mathcal{X}$ be a prestack over $\mathrm{Art}_{\Bbbk}^{\mathrm{op}}$ such that the groupoid $\mathcal{X}(\Bbbk)$ is equivalent to the set $\{x_0\}$, and let $F_{\mathcal{X}} \colon \mathrm{Art}_{\Bbbk} \to \mathrm{Sets}$ be the covariant functor assigning $A \in \mathrm{Art}_{\Bbbk}$ to the set of isomorphism classes $\mathcal{X}(A)/\sim$. Assume that Condition (RS$_2$) holds for $\mathcal{X}$.*

(1) The tangent space $T_{\mathcal{X}} = F_{\mathcal{X}}(\Bbbk[\epsilon])$ has a natural structure of a $\Bbbk$-vector space. More generally, for every finite dimensional $\Bbbk$-vector space V, denoting $\Bbbk[V]$ as the $\Bbbk$-algebra $\Bbbk \oplus V$ defined by $V^2 = 0$, the set $F_{\mathcal{X}}(\Bbbk[V])$ has a natural structure of a $\Bbbk$-vector space and there is a functorial bijection $F_{\mathcal{X}}(\Bbbk[V]) = T_{\mathcal{X}} \otimes_{\Bbbk} V$.

(2) Consider a surjection $A \twoheadrightarrow A_0$ in $\mathrm{Art}_{\Bbbk}$ with square-zero kernel I and an element $x_0 \in \mathcal{X}(A_0)$, and let $\mathrm{Lift}_x(A)$ be the set of morphisms $\alpha \colon x_0 \to y$ over $\operatorname{Spec} A_0 \to \operatorname{Spec} A$ where $(\alpha \colon x_0 \to x) \sim (\alpha' \colon x_0 \xrightarrow{\alpha'} x')$ if there is an isomorphism $\beta \colon x \to x'$ such that $\alpha' = \beta \circ \alpha$. There is an action of $T_{\mathcal{X}} \otimes I$ on $\mathrm{Lift}_{x_0}(A)$ which is functorial in $\mathcal{X}$. Assume that $\mathrm{Lift}_{x_0}(A)$ is non-empty. Then the action is transitive if Condition (RS$_1$) holds for $\mathcal{X}$, and is free and transitive (i.e., $\mathrm{Lift}_{x_0}(A)$ is a torsor under $T_{\mathcal{X}} \otimes I$) if Condition (RS$_4$) holds for $\mathcal{X}$.

Proof. We first note if V is a finite dimensional vector space, then $\Bbbk[V] = \Bbbk[\epsilon] \times_{\Bbbk} \cdots \times_{\Bbbk} \Bbbk[\epsilon]$ and by applying (RS$_2$) inductively, we see that the conclusion of (RS$_2$) also holds for $A_0 = \Bbbk$ and $A = \Bbbk[V]$. For $B_0 \in \mathrm{Art}_{\Bbbk}$, the first part of (RS$_2$) implies that $F_{\mathcal{X}}(B_0 \times_{\Bbbk} \Bbbk[V]) \xrightarrow{\sim} F_{\mathcal{X}}(B_0) \times F_{\mathcal{X}}(\Bbbk[V])$ is a bijection. In particular, $F_{\mathcal{X}}(\Bbbk[V] \times_{\Bbbk} \Bbbk[W]) \xrightarrow{\sim} F_{\mathcal{X}}(\Bbbk[V]) \times F_{\mathcal{X}}(\Bbbk[W])$ is bijective for every pair of finite dimensional vector spaces, or in other words the functor $V \mapsto F_{\mathcal{X}}(\Bbbk[V])$ commutes with finite products. The vector space structure of $T_{\mathcal{X}} = F_{\mathcal{X}}(\Bbbk[\epsilon])$ follows from the bijectivity of

$$F_{\mathcal{X}}(\Bbbk[\epsilon] \times_{\Bbbk} \Bbbk[\epsilon']) \xrightarrow{\sim} F_{\mathcal{X}}(\Bbbk[\epsilon]) \times F_{\mathcal{X}}(\Bbbk[\epsilon']). \tag{C.4.10}$$

Indeed, if $\tau_1, \tau_2 \in F_{\mathcal{X}}(\Bbbk[\epsilon])$, then we may use (C.4.10) to view $(\tau_1, \tau_2) \in F_{\mathcal{X}}(\Bbbk[\epsilon] \times_{\Bbbk} \Bbbk[\epsilon'])$, and we define $\tau_1 + \tau_2$ as the image of (τ_1, τ_2) under $F(\Bbbk[\epsilon] \times_{\Bbbk} \Bbbk[\epsilon']) \to F(\Bbbk[\epsilon])$ induced by the ring map $\Bbbk[\epsilon] \times_{\Bbbk} \Bbbk[\epsilon'] \to \Bbbk[\epsilon]$ taking $(\epsilon, 0)$ and $(0, \epsilon')$ to ϵ. Scalar multiplication of $c \in \Bbbk$ on $\tau \in F_{\mathcal{X}}(\Bbbk[\epsilon])$ is defined by taking the image of τ under $F_{\mathcal{X}}(\Bbbk[\epsilon]) \to F_{\mathcal{X}}(\Bbbk[\epsilon])$ induced by the map $\Bbbk[\epsilon] \to \Bbbk[\epsilon]$ taking ϵ to $c\epsilon$.

The same argument shows that $V \mapsto F_{\mathcal{X}}(\Bbbk[V])$ defines a $\Bbbk$-linear functor $\mathrm{Vect}_{\Bbbk}^{\mathrm{fd}} \to \mathrm{Vect}_{\Bbbk}$. The natural map

$$F_{\mathcal{X}}(\Bbbk[\epsilon]) \times \operatorname{Hom}_{\Bbbk}(\Bbbk[\epsilon], \Bbbk[V]) \to F_{\mathcal{X}}(\Bbbk[V]), \qquad (\tau, \phi) \mapsto \phi^* \tau := F_{\mathcal{X}}(\phi)(\tau)$$

For (2), observe that the natural map

$$A \times_{A_0} A \to \Bbbk[I] \times_{\Bbbk} A, \qquad (a_1, a_2) \mapsto (\overline{a}_1 + a_2 - a_1, a_1)$$

is an isomorphism. We therefore have a diagram

$$\mathcal{X}(\Bbbk[I]) \times \mathcal{X}(A) \twoheadleftarrow \mathcal{X}(\Bbbk[I] \times_{\Bbbk} A) \cong \mathcal{X}(A \times_{A_0} A) \xrightarrow{p_1^*} \mathcal{X}(A)$$

where the left functor is essentially surjective by the first part of (RS$_2$). To define the action, let $\tau \in T_{\mathcal{X}} \otimes I = F_{\mathcal{X}}(\Bbbk[I])$ and $(\alpha \colon x_0 \to x) \in \mathrm{Lift}_{x_0}(A)$. Choose a representative $\widetilde{\tau} \in \mathcal{X}(\Bbbk[I])$ of τ. We define $\tau \cdot (\alpha \colon x_0 \to x) \in \mathrm{Lift}_{x_0}(A)$ as $p_1^* y$, where

$y \in \mathcal{X}(\Bbbk[I] \times_{\Bbbk} A)$ is a preimage of $(\widetilde{\tau}, x)$. To see that this is well-defined, suppose that $y' \in \mathcal{X}(\Bbbk[I] \times_{\Bbbk} A)$ is another preimage. This yields a diagram

$$\begin{tikzcd} x_0 \arrow[r] \arrow[d] & x \arrow[d] \arrow[dr] & \\ \tau \arrow[r] \arrow[rr, bend right] & y \arrow[r, dashed, "\beta"] & y' \end{tikzcd} \quad \text{over} \quad \begin{tikzcd} \operatorname{Spec} \Bbbk \arrow[r, hook] \arrow[d] & \operatorname{Spec} A \arrow[d] \\ \operatorname{Spec} \Bbbk[I] \arrow[r, hook] & \operatorname{Spec}(\Bbbk[I] \times_{\Bbbk} A). \end{tikzcd}$$

By the second part of (RS$_2$), there exists $\beta \colon y \to y'$ filling in the diagram, and thus $p_1^* y$ and $p_1^* y'$ in $\mathcal{X}(A)$ define the same element in $\mathrm{Lift}_{x_0}(A)$. Finally, if (RS$_1$) holds (resp., (RS$_4$) holds), then $\mathcal{X}(A \times_{A_0} A) \to \mathcal{X}(A) \times_{\mathcal{X}(A_0)} \mathcal{X}(A)$ is essentially surjective (resp., an equivalence), and we see that the action is transitive (resp., free and transitive). □

Proof Theorem C.4.6. We establish the sufficiency of the criteria, leaving the necessity to the reader. The tangent space $T_{\mathcal{X}} := \mathcal{X}(\Bbbk[\epsilon])/\!\sim$ has the structure of a vector space by Lemma C.4.9(1) and is finite dimensional by (RS$_2$). Let $N = \dim_{\Bbbk} T_{\mathcal{X}}$ and $x_1, \ldots, x_N \in T_{\mathcal{X}}$ be a basis. Define $S = \Bbbk[\![x_1, \ldots, x_N]\!]$ with $\mathfrak{m}_S = (x_1, \ldots, x_n)$. We will construct inductively a decreasing sequence of ideals $J_0 \supseteq J_1 \supseteq \cdots$ and objects $\eta_n \in \mathcal{X}(S/J_n)$ together with maps $\eta_n \to \eta_{n+1}$ over $\operatorname{Spec} S/J_n \hookrightarrow \operatorname{Spec} S/J_{n+1}$. We set $J_0 = \mathfrak{m}_S$ and $\eta_0 = x_0 \in \mathcal{X}(\Bbbk)$. We also set $J_1 = \mathfrak{m}_S^2$ so that $S/J_1 \cong \Bbbk[T_{\mathcal{X}}]$. Using the bijection $F_{\mathcal{X}}(\Bbbk[T_{\mathcal{X}}]) \cong T_{\mathcal{X}} \otimes_{\Bbbk} T_{\mathcal{X}}$ of Lemma C.4.9(1), the element $\sum_i x_i \otimes x_i$ defines an isomorphism class of an object $\eta_1 \in \mathcal{X}(S/J_1)$ such that the induced map $\operatorname{Spec} S/J_1 \to \mathcal{X}$ induces a bijection on tangent spaces. By construction,there is a map $\eta_0 \to \eta_1$ over $\operatorname{Spec} \Bbbk \hookrightarrow \operatorname{Spec} S/J_1$.

Suppose we have constructed J_n and $\eta_{n-1} \to \eta_n$. We claim that the set of ideals

$$\Sigma = \left\{ J \subseteq S \;\middle|\; \begin{array}{l} \mathfrak{m}_S J_n \subseteq J \subseteq J_n \text{ and there exists } \eta_n \to \eta \\ \text{over } \operatorname{Spec} S/J_n \hookrightarrow \operatorname{Spec} S/J \end{array} \right\} \tag{C.4.11}$$

has a minimal element. Indeed, it is non-empty since $J_n \in \Sigma$ so it suffices to check that $J \cap K \in \Sigma$ for $J, K \in \Sigma$. To achieve this, note that $J_n/\mathfrak{m}_S J_n$ is a $\Bbbk$-vector space with subspaces $J/\mathfrak{m}_S J_n$ and $K/\mathfrak{m}_S J_n$. We may therefore choose an ideal $J \subseteq J' \subseteq J_n$ with $J \cap K = J' \cap K$ and $J' + K = J_n$. We have a diagram

$$\begin{tikzcd} \eta_{J\cap K} & \eta_J|_{S/J'} \arrow[l, dashed] \\ \eta_K \arrow[u, dashed] & \eta_n \arrow[l] \arrow[u] \end{tikzcd} \quad \text{over} \quad \begin{tikzcd} S/(J\cap K) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & S/J' \arrow[d] \\ S/K \arrow[r] & S/J_n, \end{tikzcd}$$

where $\eta_J \in \mathcal{X}(S/J)$ and $\eta_K \in \mathcal{X}(S/K)$ are the objects corresponding to J and K. Condition (RS$_1$) implies that the diagram can be filled in, which shows that $J \cap K \in \Sigma$.

Define $J = \bigcap_n J_n$, $R = S/J$, and $I_n = J_n/J$. We claim that for every n, there exists N_n with $J_{N_n} \subseteq \mathfrak{m}_S^{n+1} + J$, or in other words the topology on R defined by (I_n) is the $\mathfrak{m}_R$-adic topology on R. For each n, since $S/\mathfrak{m}_S^n$ is artinian, there exists N_n with $J_{N_n} + \mathfrak{m}_S^n = J_{N_{n+1}} + \mathfrak{m}_S^n = \cdots$. We set $E = \varprojlim (J_{N_n} + \mathfrak{m}_S^n)/\mathfrak{m}_S^n \subseteq \varprojlim S/\mathfrak{m}_S^n = S$. We claim that $E \subseteq J$: for $f \in E$ and any m, $f \in J_m + \mathfrak{m}_S^n$ for $n \gg 0$, and the claim follows from Krull's intersection theorem. Since E surjects onto $(J_{N_n} + \mathfrak{m}_S^{n+1})/\mathfrak{m}_S^{n+1}$, the natural map $J \to (J_{N_n} + \mathfrak{m}_S^{n+1})/\mathfrak{m}_S^{n+1}$ is also surjective, and the claim follows. Since $I_{N_n} \subseteq \mathfrak{m}_R^{n+1}$, we can define $\xi_n := \eta_{N_n}|_{R/\mathfrak{m}_R^{n+1}}$.

It remains to show that the formal deformation $\xi := \{\xi_n\}$ over R is versal. Suppose $B \twoheadrightarrow A$ is a surjection in $\mathrm{Art}_{\Bbbk}$ with kernel $\Bbbk$ and that we have a diagram

$$\begin{tikzcd} x \arrow[r] \arrow[d] & \xi \\ y \arrow[ur, dashed] & \end{tikzcd} \qquad \text{over} \qquad \begin{tikzcd} \operatorname{Spec} A \arrow[r, "g"] \arrow[d, hook] & h_R \\ \operatorname{Spec} B. \arrow[ur, dashed, "\widetilde{g}"'] & \end{tikzcd}$$

We must construct a morphism $y \to \xi$ extending $x \to \xi$. We claim that it suffices to construct a morphism $\widetilde{g}\colon \operatorname{Spec} B \to h_R$ (i.e., a ring map $R \to B$) extending g. Since $h_R(\Bbbk[\epsilon]) \to T_{\mathcal{X}}$ is bijective, Lemma C.4.9(2) implies that there are actions of $T_{\mathcal{X}}$ on the sets $\mathrm{Lift}_x(B)$ and $\mathrm{Lift}_g(B)$ of isomorphism classes of lifts of x and g to objects in $\mathcal{X}(B)$ and $h_R(B)$. Moreover, since (RS$_1$) holds for $\mathcal{X}$, the action on $\mathrm{Lift}_x(B)$ is transitive. Thus, we can find $\tau \in T_{\mathcal{X}}$ such that $y = \tau \cdot (\widetilde{g}^*\xi) = (\tau \cdot \widetilde{g})^*\xi$. This gives an arrow $y \to \xi$ over $\tau \cdot \widetilde{g}\colon \operatorname{Spec} B \to h_R$.

To construct $\widetilde{g}$, choose n such that $R \to A$ factors as $R \to R/I_n = S/J_n \to A$. It suffices to show that $\operatorname{Spec} A \to \operatorname{Spec} S/J_n$ extends to a map $\operatorname{Spec} B \to \operatorname{Spec} S/J_{n+1}$ and for this, it suffices to show the existence of a dotted arrow making the diagram

$$\begin{tikzcd} \operatorname{Spec} A \arrow[r] \arrow[d, hook] & \operatorname{Spec} S/J_n \arrow[d, hook] \arrow[dr] & \\ \operatorname{Spec} B \arrow[r] & \operatorname{Spec} B \times_A (S/J_n) \arrow[r, dashed] & \operatorname{Spec} S/J_{n+1} \end{tikzcd}$$

commutative. Note that since $\ker(B \to A) = \Bbbk$, $\ker(B \times_A (S/J_n) \to S/J_n) = \Bbbk$. As S is a power series ring, we may choose an extension $S \to B$ of $S \to S/J_n \to A$. This induces a map $S \to B \times_A (S/J_n)$. If this map is not surjective, then every element of J_n must map to 0 in $B \times_A (S/J_n)$, which implies that $B \times_A (S/J_n) \to S/J_n$ has a section giving the desired lift. Otherwise, $B \times_A (S/J_n) = S/K$ where $K = \ker(S \to B \times_A (S/J_n))$. The ideal K lies in the set of ideals defined in (C.4.11): the inclusion $K \subset J_n$ is clear, the inclusion $\mathfrak{m}_S J_n \subseteq K$ is implied by the equality $\ker(B \to A) = \Bbbk$, and the existence of $\eta_n \to \eta$ over $\operatorname{Spec} S/J_n \hookrightarrow \operatorname{Spec} S/K$ follows from applying (RS$_1$) to the above square. By minimality of J_{n+1}, we have a containment $J_{n+1} \subseteq K$ and thus a ring map $S/J_{n+1} \to S/K = B \times_A (S/J_n)$ inducing the desired dotted arrow.

Finally, suppose that $\mathcal{X}$ is equivalent to a functor F and (RS$_4$) holds. Given a surjection $B \to A$ with kernel $\Bbbk$ and $x \in F(A)$, we need to show the existence of a *unique* lift in every diagram

$$\begin{tikzcd} \operatorname{Spec} A \arrow[r, "g"] \arrow[d, hook] & h_R \arrow[d] \\ \operatorname{Spec} B \arrow[r] \arrow[ur, dashed] & F. \end{tikzcd}$$

By Lemma C.4.9(2), the map $\mathrm{Lift}_g(B) \to \mathrm{Lift}_x(B)$ is bijective as both are torsors under $T_{\mathcal{X}}$. This implies the existence of a unique lift. See also [Sch68, Thm. 2.11], [SGA7-I, Thm. VI.1.11], [Har10, Thm. 16.2], [Ser06, Thm. 2.3.2], and [SP, Tag 06IX]. $\square$

C.4.3 Verifying Rim–Schlessinger's Criteria

We apply Rim–Schlessinger's Criteria (C.4.6) to construct miniversal formal deformations for our three main moduli problems by verifying (RS$_1$)-(RS$_3$). In fact, we will verify the following *strong homogeneity* condition:

(RS$_4^*$) $\mathcal{X}(B_0 \times_{A_0} A) \to \mathcal{X}(B_0) \times_{\mathcal{X}(A_0)} \mathcal{X}(A)$ is an equivalence for every map $B_0 \to A_0$ and surjection $A \twoheadrightarrow A_0$ of (not necessarily artinian) rings with square-zero kernel.

It is often just as easier to verify (RS$_4^*$) as the weaker conditions (RS$_1$)-(RS$_2$). Strong homogeneity will also appear as one of the axioms in our second version of Artin's Axioms for Algebraicity (C.7.4), as it will be useful to verify openness of versality. It turns out that every algebraic stack satisfies (RS$_4^*$) (see [SP, Tag 07WN]), or, in other words, the Ferrand pushout $\operatorname{Spec}(B_0 \times_{A_0} A)$ is a pushout in the category of algebraic stacks.

Verifying (RS$_4^*$) relies on properties of modules over fiber products of rings.

Lemma C.4.12. *Let $A \to A_0$ be a surjection of rings with square-zero kernel, and $B_0 \to A_0$ be a maps of rings. Let M, M_0, N_0 be flat modules over A, A_0, B_0, $M \to M_0$ be an A-module map, and $N_0 \to M_0$ be a B_0-module map. Assume that $M \otimes_A A_0 \to M_0$ and $N_0 \otimes_{B_0} A_0 \to M_0$ are isomorphisms. Set $B := B_0 \times_{A_0} A$ and $N = N_0 \times_{M_0} M$. Then*

(1) the maps $N \otimes_B B_0 \to N_0$ and $N \otimes_B A \to M$ are isomorphisms,

(2) N is flat over B, and

(3) The modules N_0 and M are finitely presented if and only if N is.

(4) Let B_0' be a B_0-algebra, A' be an A-algebra, and $B_0' \otimes_{B_0} A_0 \to A' \otimes_A A_0$ be an isomorphism. Set $B' = B_0' \times_{A_0'} A$. Then $B \to B'$ is flat and of finite presentation if and only if $B_0 \to B_0'$ and $A \to A'$ are.

Proof. We verify (2), leaving the remaining claims to the reader. Let $J = \ker(A \to A_0)$. Since M is flat over A, the Local Criterion of Flatness (A.2.6) implies that $0 \to J \otimes_{A_0} M_0 \to M \to M_0 \to 0$ is exact, and thus

$$0 \to J \otimes_{A_0} M_0 \to N \to N_0 \to 0 \tag{C.4.13}$$

is also exact. To show the flatness of N, by the Local Criterion of Flatness (A.2.6), it suffices to verify that (a) $N \otimes_B B_0$ is flat over B_0 and (b) $J \otimes_{B_0} N_0 \to N$ is injective. The module in (a) is identified with N_0 by (2), which is flat over B_0 by hypothesis. Since $J \otimes_{A_0} M_0 = J \otimes_{A_0} (N_0 \otimes_{B_0} A_0) = J \otimes_{B_0} N_0$, (b) follows from (C.4.13). See [Sch68, Lem. 3.4], [Har10, Prop. 16.4], and [SP, Tags 08KG, 0D2G, and 0D2K]. $\square$

We say that a prestack $\mathcal{X}$ over $\mathrm{Sch}/\Bbbk$ *admits miniversal formal deformations* (resp., *is locally prorepresentable*, *satisfies* (RS$_3$)) if for every $x_0 \in \mathcal{X}(\Bbbk)$, the local deformation prestack $\mathcal{X}_{x_0}$ (as defined in Remark C.4.4) admits a miniversal formal deformation (resp., is prorepresentable, satisfies (RS$_3$)).

Proposition C.4.14. *Let X be a proper scheme over a field $\Bbbk$.*

(1) The Hilbert functor $\mathrm{Hilb}(X)\colon \mathrm{Sch}/\Bbbk \to \mathrm{Sets}$, whose objects over S are closed subschemes $Z \subseteq X_S$ flat and finitely presented over S, satisfies (RS$_3$) and (RS$_4^*$), and is therefore locally prorepresentable.

(2) The stack $\mathcal{F}\mathrm{am}$ over $\mathrm{Sch}/\Bbbk$, whose objects over S are proper, flat, and finitely presented morphisms $\mathcal{Y} \to S$ of algebraic spaces,[1] admits miniversal formal deformations. In particular, the stack $\mathcal{M}_g^{\mathrm{all}}$ of all curves and the stack $\mathcal{M}_g$ of smooth curves admit miniversal formal deformations.

[1]We need to allow $\mathcal{Y}$ to be an algebraic space if we want $\mathcal{F}\mathrm{am}$ to be a stack; see Example 2.5.12 and Exercise 4.5.15. On the other hand, Rim–Schlessinger's Criteria (C.4) equally applies to the prestack parameterizing proper, flat, and finitely presented morphisms $\mathcal{Y} \to S$ of *schemes*.

(3) The stack $\underline{\mathrm{Coh}}(X)$ over $\mathrm{Sch}/\Bbbk$, whose objects over S are finitely presented quasi-coherent $\mathcal{O}_{X_S}$-modules flat over S, satisfies (RS$_3$) *and* (RS$_4^*$)*, and therefore admits miniversal formal deformations. In particular, if C is a smooth, connected, and projective curve, $\mathcal{B}un(C)$ admits miniversal formal deformations.*

Proof. For (1), Proposition C.1.3 identifies the tangent space of $\mathrm{Hilb}(X)$ at $Z_0 \subseteq X$ with $\mathrm{H}^1(Z, N_{Z_0/X})$. Since X is proper, $\mathrm{H}^1(Z, N_{Z_0/X})$ is finite-dimensional and (RS$_3$) holds. To check (RS$_4^*$), $A \to A_0$ be a surjection of rings with square-zero kernel, and $B_0 \to A_0$ be a maps of rings, and suppose that $W_0 \subseteq X_{B_0}$ and $Z \subseteq X_A$ are closed subschemes flat over the base such that $Z_0 := W_0 \times_{B_0} A_0 = Z \times_A A_0 \subseteq X_{A_0}$. Then $\mathcal{O}_{W_0} \times_{\mathcal{O}_{Z_0}} \mathcal{O}_Z$ is a quotient of $\mathcal{O}_{X_B}$, and defines a closed subscheme $W \subseteq X_B$. For each affine open $\operatorname{Spec} B' \subseteq X_B$ with restrictions $\operatorname{Spec} B'_0 \subseteq X_{B_0}$, $\operatorname{Spec} A' \subseteq X_A$, and $\operatorname{Spec} A'_0 \subseteq X_{A_0}$, we have that $B' = B'_0 \times_{A'_0} A'$. By Lemma C.4.12(4), $B \to B'$ is flat and finitely presented, and thus W is flat and finitely presented over B.

For (2), the tangent space of $\mathcal{F}\mathrm{am}$ at Y_0 is identified with $\mathrm{Ext}^1_{\mathcal{O}_{Y_0}}(_{Y_0}, \mathcal{O}_{Y_0})$ by Theorem C.3.6, and is therefore finite dimensional. (When Y_0 is smooth, the tangent space is $\mathrm{H}^1(Y_0, T_{Y_0})$.) If $[\mathcal{Z}_0 \to \operatorname{Spec} B_0] \in \mathcal{F}\mathrm{am}(B_0)$ and $[\mathcal{Y} \to \operatorname{Spec} A] \in \mathcal{F}\mathrm{am}(A)$ with an isomorphism $(\mathcal{Z}_0)_{A_0} \xrightarrow{\sim} \mathcal{Y}_{A_0}$, then the Ferrand pushout $\mathcal{Z} := \mathcal{Z}_0 \amalg_{\mathcal{Y}_{A_0}} \mathcal{Y}$ exists by Theorem B.4.1. Applying Lemma C.4.12(4) to an affine cover of $\mathcal{Z}$ shows that $\mathcal{Z} \to \operatorname{Spec} B$ is flat and finitely presented. Moreover, since $\mathcal{Z}$ is a pushout, compatible isomorphisms of $\mathcal{Z}_0$ and $\mathcal{Y}$ extend uniquely to an isomoprhism of $\mathcal{Z}$.

For (3), the tangent space of $\underline{\mathrm{Coh}}(X)$ at a coherent sheaf is identified with $\mathrm{Ext}^1_{\mathcal{O}_X}(E, E)$ by Proposition C.1.18, which is finite dimensional. The base change X_B is the pushout of $X_{A_0} \to X_{B_0}$ and $X_{A_0} \hookrightarrow X_A$, and for each affine open $\operatorname{Spec} B' \subseteq X_B$ with restrictions $\operatorname{Spec} B'_0 \subseteq X_{B_0}$, $\operatorname{Spec} A' \subseteq X_A$, and $\operatorname{Spec} A'_0 \subseteq X_{A_0}$, $B' = B'_0 \times_{A'_0} A'$. For $G_0 \in \underline{\mathrm{Coh}}(X)(B_0)$ and $F \in \underline{\mathrm{Coh}}(X)(A)$ restricting to $F_0 \in \underline{\mathrm{Coh}}(X)(A)$, we define $G := G_0 \times_{F_0} F$ on X_B. This is finitely presented by Lemma C.4.12(3) (applied to $G|_{\operatorname{Spec} B'}$ over $B' = B'_0 \times_{A'_0} A'$) and flat over B by Lemma C.4.12(2) (applied to $G|_{\operatorname{Spec} B'}$ over $B = B_0 \times_{A_0} A$). This gives essential surjectivity for (RS$_4^*$), and the fully faithfullness is clear. □

Exercise C.4.15. If X_0 is a smooth proper scheme over $\Bbbk$ with no infinitesimal automorphisms, i.e., $\mathrm{H}^0(X_0, T_{X_0}) = 0$, show that the functor of deformations of X_0 is prorepresentable.

Exercise C.4.16. Let $X = \operatorname{Spec} A$ be an affine scheme with isolated singularities over a field $\Bbbk$, and let $F_X \colon \mathrm{Art}_\Bbbk \to \mathrm{Sets}$ be the functor, where $F_X(R)$ is the set of isomorphism classes of pairs (B, β) where B is a flat R-algebra and $\beta \colon A \to B \otimes_R R/\mathfrak{m}_R$ is an isomorphism. Show that F_X admits a miniversal formal deformations.

C.5 Effective formal deformations and Grothendieck's Existence Theorem

Grothendieck's Existence Theorem is a powerful technique for showing that formal deformations are effective. It is sometimes referred to as Formal GAGA, as it is analogous to Serre's GAGA Theorem [Ser56] that for a proper scheme X over $\mathbb{C}$ with analytification X^{an}, the natural functor $\mathrm{Coh}(X) \to \mathrm{Coh}(X^{\mathrm{an}})$, taking a coherent sheaf F to its analytification F^{an}, is an equivalence. The proofs follow very similar strategies.

C.5.1 Effective formal deformations

Definition C.5.1. Let $\mathcal{X}$ be a prestack (or functor) over $\mathrm{Sch}/\Bbbk$. Let $x_0 \in \mathcal{X}(\Bbbk)$ and consider a formal deformation $\{x_n\}$ over a complete noetherian local $\Bbbk$-algebra with residue field $\Bbbk$ of x_0 (or more precisely a formal deformation of the deformation stack $\mathcal{X}_{x_0}$ at x_0 as defined in Remark C.4.4). We say that $\{x_n\}$ is *effective* if there exists an object $\widehat{x} \in \mathcal{X}(R)$ and compatible isomorphisms $x_n \xrightarrow{\sim} \widehat{x}|_{\operatorname{Spec} R/\mathfrak{m}^{n+1}}$.

A formal deformation $(R, \{x_n\})$ is effective if it is in the essential image of the natural functor $\mathcal{X}(R) \to \varprojlim \mathcal{X}(R/\mathfrak{m}^n)$, or in other words if there exists a dotted arrow making the diagram

$$\begin{tikzcd}
\operatorname{Spec} R/\mathfrak{m} \arrow[r, hook] \arrow[drrrr, "x_0"'] & \operatorname{Spec} R/\mathfrak{m}^2 \arrow[r, hook] \arrow[drrr, "x_1"] & \operatorname{Spec} R/\mathfrak{m}^3 \arrow[r, hook] \arrow[drr, "x_2"] & \cdots \arrow[r, hook] & \operatorname{Spec} R \arrow[d, dashed, "\widehat{x}"] \\
 & & & & \mathcal{X}
\end{tikzcd}$$

commutative.

Example C.5.2. If $F\colon \mathrm{Sch}/\Bbbk \to \mathrm{Sets}$ is a contravariant functor representable by a scheme X over $\Bbbk$, then every formal deformation $(R, \{x_n\})$ is effective. Indeed, x_n corresponds to a morphism $\operatorname{Spec} R/\mathfrak{m}^{n+1} \to X$ with image $x \in X(\Bbbk)$ and thus to a $\Bbbk$-algebra homomorphism $\phi_n\colon \widehat{\mathcal{O}}_{X,x} \to R/\mathfrak{m}^{n+1}$. By taking the inverse image of ϕ_n, we have a local homomorphism $\widehat{\mathcal{O}}_{X,x} \to R$ which in turn defines a morphism $\widehat{x}\colon \operatorname{Spec} R \to X$ extending $\{x_n\}$.

More generally, if $\mathcal{X}$ is an algebraic stack over $\Bbbk$, every formal deformation is effective. Indeed, there exists a smooth presentation $U \to \mathcal{X}$ and a lift $u \in U(\Bbbk)$ of $x_0 \in \mathcal{X}(\Bbbk)$. By applying the Infinitesimal Lifting Criterion for Smoothness (3.7.1), we may inductively construct lifts

$$\begin{tikzcd}
\operatorname{Spec} R/\mathfrak{m}^n \arrow[r] \arrow[d, hook] & U \arrow[d] \\
\operatorname{Spec} R/\mathfrak{m}^{n+1} \arrow[r, "x_n"] \arrow[ur, dashed] & \mathcal{X}.
\end{tikzcd}$$

Since U is a scheme, the maps $\operatorname{Spec} R/\mathfrak{m}^n \to U$ extend to a map $\operatorname{Spec} R \to U$, and the composition $\operatorname{Spec} R \to U \to \mathcal{X}$ effectivizes the formal deformation $\widehat{x} = \{x_n\}$. Since the diagonal of $\mathcal{X}$ is representable, it follows that a compatible automorphisms α_n of x_n extend to a unique automorphism of $\widehat{X}$, i.e., the functor

$$\mathcal{X}(R) \to \varprojlim \mathcal{X}(R/\mathfrak{m}^{n+1})$$

is an equivalence of categories.

C.5.2 Grothendieck's Existence Theorem

The following is frequently applied when $(R, \mathfrak{m})$ is a complete *local* ring.

Theorem C.5.3 (Grothendieck's Existence Theorem)**.** *Let X be a scheme proper over a noetherian ring A which is complete with respect to an ideal $\mathfrak{m} \subseteq A$. Let $X_n := X \times_R R/\mathfrak{m}^{n+1}$. The functor*

$$\operatorname{Coh}(X) \to \varprojlim \operatorname{Coh}(X_n), \qquad E \mapsto \{E/\mathfrak{m}^{n+1}E\} \tag{C.5.4}$$

is an equivalence of categories.

The essential surjectivity of (C.5.4) translates to an extension of the diagram

$$\begin{tikzcd}
E_0 \arrow[d, dash] & E_1 \arrow[d, dash] & E_2 \arrow[d, dash] & & E \arrow[d, dash] \\
X_0 \arrow[r, hook] \arrow[d] & X_1 \arrow[r, hook] \arrow[d] & X_2 \arrow[r, hook] \arrow[d] & \cdots \arrow[r, hook] & X \arrow[d] \\
\operatorname{Spec} R/\mathfrak{m} \arrow[r, hook] & \operatorname{Spec} R/\mathfrak{m}^2 \arrow[r, hook] & \operatorname{Spec} R/\mathfrak{m}^3 \arrow[r, hook] & \cdots \arrow[r, hook] & \operatorname{Spec} R.
\end{tikzcd}$$

Using the language of formal schemes and setting $\widehat{X} = X \times_{\operatorname{Spec} R} \operatorname{Spf} R$ to be the $\mathfrak{m}$-adic completion of X, then Grothendieck's Existence Theorem asserts that the functor $\operatorname{Coh}(X) \to \operatorname{Coh}(\widehat{X})$, defined by $E \mapsto \widehat{E}$, is an equivalence.

Remark C.5.5 (Limits of categories). An object of $\varprojlim \operatorname{Coh}(X_n)$ is a sequence $\{E_n\}$ of coherent $\mathcal{O}_{X_n}$-modules E_n together with isomorphisms $E_{n+1}/\mathfrak{m}^{n+1}E_{n+1} \xrightarrow{\sim} E_n$. A morphism $\phi = \{\phi_n\}\colon \{E_n\} \to \{F_n\}$ is the data of compatible $\mathcal{O}_{X_n}$-module homomorphisms $\phi_n\colon E_n \to F_n$. If $X \cong \operatorname{Spec} R$, then $\varprojlim \operatorname{Coh}(X_n)$ is equivalent to the category of finite A-modules.

The category $\varprojlim \operatorname{Coh}(X_n)$ is abelian and (C.5.4) is an exact functor. While the cokernel of a map $\{\phi_n\}\colon \{E_n\} \to \{F_n\}$ in $\varprojlim \operatorname{Coh}(X_n)$ is simply $\{\operatorname{coker} \phi_n\}$, kernels are more subtle. The Artin–Rees Lemma (B.5.4) implies that for each m, the image of $\ker(\alpha_l) \to E_m$ stabilizes for $l \gg m$ to a subsheaf $E'_m \subseteq E_m$, and that for each n, the quotients $E'_m/\mathfrak{m}^{n+1}E'_m$ stabilizes for $m \gg n$ to a coherent sheaf K_n on X_n. The kernel of $\{\phi_n\}$ is $\{K_n\}$ (which is usually different from $\{\ker \phi_n\}$). See [SP, Tag 0EHN].

Proof. Fully faithfulness of (C.5.4) translates into the bijection $\operatorname{Ext}^0_{\mathcal{O}_X}(E,F) \xrightarrow{\sim} \varprojlim_n \operatorname{Ext}^0_{\mathcal{O}_{X_n}}(E|_{X_n}, F|_{X_n})$, which is a consequence of Formal Functions (see Exercise A.5.5). It remains to show essential surjectivity.

Projective case: The key claim is that for every $\{E_n\} \in \operatorname{Coh}(X_n)$, there exists integers m and r together with compatible surjections $\mathcal{O}_{X_n}(-m)^{\oplus r} \twoheadrightarrow E_n$. Consider the finite type $\mathcal{O}_{X_0}$-algebra $\mathcal{A} := \bigoplus_{i\geq 0} \mathfrak{m}^i\mathcal{O}_{X_0}/\mathfrak{m}^{i+1}\mathcal{O}_{X_0}$ and the finitely generated $\mathcal{O}_{\mathcal{A}}$-module $\mathcal{G} := \bigoplus_{i\geq 0} \mathfrak{m}^i E_i$ (noting that when $\{E_n\} = \{E/\mathfrak{m}^{n+1}\}$ for a coherent sheaf E on X, then $\mathcal{G}$ is the associated graded $\bigoplus_{i\geq 0} \mathfrak{m}^i E/\mathfrak{m}^{i+1}E$). Viewing $\mathcal{G}$ as a coherent sheaf on $\operatorname{Spec}_{X_0} \mathcal{A}$ and applying Serre's Vanishing Theorem [Har77, Thm. II.5.2] to the projective morphism $\operatorname{Spec}_{X_0} \mathcal{A} \to \operatorname{Spec} \bigoplus_{i\geq 0} \mathfrak{m}^i/\mathfrak{m}^{i+1}$ gives an integer m_0 such that for all $m \geq m_0$ $\mathrm{H}^1(X_0, \mathcal{G}(m)) = 0$. This yields that $\mathrm{H}^1(X_0, \mathfrak{m}^{n+1}E_n(m)) = 0$ for all n, which in turn implies that

$$\mathrm{H}^0(X_0, E_{n+1}(m)) \twoheadrightarrow \mathrm{H}^0(X_0, E_n(m)) \tag{C.5.6}$$

is surjective. After possibly increasing m_0, we can assure that $E_0(m)$ is globally generated by sections $s_{0,1}, \ldots, s_{0,r}$. By the surjectivity of (C.5.6), we can find compatible lifts $s_{n,i} \in \mathrm{H}^0(X_0, E_n(m))$ of the sections $s_{0,i}$. This gives compatible maps $\mathcal{O}_{X_n}^{\oplus r} \to E_n(m)$, each which is surjective by Nakayama's Lemma.

With the claim established, let $\{K_n\} \in \varprojlim \operatorname{Coh}(X_n)$ be the kernel $\{\phi_n\colon \mathcal{O}_{X_n}(-m)^{\oplus r} \twoheadrightarrow E_n\}$ as in Remark C.5.5. Applying the claim again to $\{K_n\}$ induces compatible right exact sequences

$$\mathcal{O}_{X_n}(-m')^{\oplus r'} \xrightarrow{\alpha_n} \mathcal{O}_{X_n}(-m)^{\oplus r} \to E_n \to 0.$$

By fully faithfulness, the morphism $\{\alpha_n\}$ is induced by a morphism $\widetilde{\alpha}\colon \mathcal{O}_X(-m')^{\oplus r'} \to \mathcal{O}_X(-m)^{\oplus r}$, and it follows that $E := \operatorname{coker}(\widetilde{\alpha})$ is a coherent sheaf on X such that $\{E/\mathfrak{m}^{n+1}E\} \cong \{E_n\}$ in $\varprojlim \operatorname{Coh}(X_n)$.

Proper case: By Chow's Lemma [Har77, Exc. II.4.10], there exists a projective morphism $g\colon X' \to X$ which is an isomorphism over a dense open subset $U \subseteq X$ such that X' is projective over the $\operatorname{Spec} R$. Given $\{E_n\} \in \operatorname{Coh}(X_n)$, consider the pullback $\{g^*E_n\} \in \operatorname{Coh}(X'_n)$, where $X'_n := X' \times_R R/\mathfrak{m}^{n+1}$. Since X' is projective over R, there exists a coherent sheaf E' on X' an an isomorphism

$$\{\beta_n\}\colon \{g^*E_n\} \xrightarrow{\sim} \{E'/\mathfrak{m}^{n+1}E'\}$$

in $\varprojlim \operatorname{Coh}(X'_n)$. By Finiteness of Cohomology (A.5.3), g_*E' is coherent. Conceptually, the argument is now very straightforward: adjunction $E_n \to g_*g^*E_n$ induces an exact sequence

$$0 \to \{K_n\} \to \{E_n\} \xrightarrow{\{\alpha_n\}} \big\{(g_*E')/\mathfrak{m}^{n+1}(g_*E')\big\} \xrightarrow{\{q_n\}} \{Q_n\} \to 0 \qquad \text{(C.5.7)}$$

in $\varprojlim \operatorname{Coh}(X_n)$ such that the $\ker(\alpha) = \{K_n\}$ and $\operatorname{coker}(\alpha) = \{Q_n\}$ are supported on $X \smallsetminus U$, i.e., each K_n and Q_n are supported on $X \smallsetminus U$. By noetherian induction, we can assume that there are coherent sheaves K and Q on X and isomorphisms $\{K_n\} \cong \{K/\mathfrak{m}^{n+1}K\}$ and $\{Q_n\} \cong \{Q/\mathfrak{m}^{n+1}Q\}$. By full faithfulness and exactness of (C.5.4), the map $\{q_n\}$ is induced by a surjection $g_*E' \to Q$, and we define $F := \ker(g_*E' \to Q)$. Since $\operatorname{Ext}^i_{\mathcal{O}_X}(K,F) \xrightarrow{\sim} \varprojlim_n \operatorname{Ext}^i_{\mathcal{O}_{X_n}}(K_n, F/\mathfrak{m}^{n+1}F)$, there is an extension $0 \to K \to E \to F \to 0$ giving a coherent sheaf E on X such that $\{E_n\} \cong (E/\mathfrak{m}^{n+1}E)$.

The existence of $\{\alpha_n\}$ in (C.5.7), however, takes some work. The formalism of formal schemes can be useful here as one can consider the map of formal schemes $\widehat{g}\colon \widehat{X}' \to \widehat{X}$ over $\operatorname{Spf} R$, the coherent sheaf $\widehat{E} = \varprojlim E_n$ on $\widehat{X}$, and the adjunction morphism $\widehat{E} \to \widehat{g}_*\widehat{g}^*\widehat{E}$ in $\operatorname{Coh}(\widehat{X})$, and apply a version of formal functions [EGA, III$_1$.4.1.5]—sometimes called the 'comparison theorem'—giving identifications $\mathrm{R}^i\widehat{g}_*(\widehat{g}^*\widehat{E}) \cong \varprojlim \mathrm{R}^n g_*(g^*E_n) \cong (\mathrm{R}^n g_*E')\widehat{\ }$.

We argue more directly. We first show that there are unique maps α_n filling in the diagram

$$\begin{tikzcd} E_n \arrow[r, "c_n"] \arrow[d, dashed, "\alpha_n"] & g_*g^*E_n \arrow[d, "g_*\beta_n"', "\wr"] \\ (g_*E')/\mathfrak{m}^{n+1}(g_*E') \arrow[r, "d_n"] & g_*(E'/\mathfrak{m}^{n+1}E'), \end{tikzcd}$$

where c_n and d_n are the natural maps. By the uniqueness, the existence of $\{\alpha_n\}$ is local on X, so we may assume that $X = \operatorname{Spec} B$. Since $B \to \widehat{B} := \varprojlim B/\mathfrak{m}^{n+1}B$ is flat and all the coherent sheaves in the diagram are annihilated by a power of $\mathfrak{m}$, Flat Base Change (A.2.12) further reduces us to the case that B is complete with respect to $\mathfrak{m}B$. In this case $E := \varprojlim E_n$ corresponds to a coherent sheaf on X mapping to $\{E_n\}$ in $\varprojlim \operatorname{Coh}(X_n)$. Applying Formal Functions (A.5.4) to $X' \to \operatorname{Spec} B$ and E' yields that $g_*E' \cong \varprojlim g_*(E'/\mathfrak{m}^{n+1}E')$, which shows that $\varprojlim d_n$ is an isomorphism. Therefore, the composition $\alpha := (\varprojlim d_n)^{-1} \circ \varprojlim(g_*\beta_n \circ c_n)$ defines a map $E \to g_*E'$, which induces the desired maps α_n. We now show that the kernel and cokernel of (C.5.7) are supported on $X \smallsetminus U$. Since $g^*E \cong \varprojlim g^*E_n$, the isomorphism $\{\beta_n\}\colon (g^*E_n) \to (E'/\mathfrak{m}^{n+1}E')$ comes from an isomorphism $g^*E \xrightarrow{\sim} E'$ such that $\{\alpha_n\}\colon \{E_n\} \to \big((g_*E')/\mathfrak{m}^{n+1}(g_*E')\big)$ comes from $E \to g_*g^*E \xrightarrow{\sim} g_*E'$. Since g is an isomorphism over U, the adjunction map $E \to g_* g^* E$ is an isomorphism over U, and it follows that $\ker\{\alpha_n\}$ and $\operatorname{coker}\{\alpha_n\}$ are supported on $X \smallsetminus U$. See also [EGA, III$_1$.5.1.4], [Ill05, Thm. 8.4.2], and [SP, Tag 088E]. $\square$

Corollary C.5.8. *Let $(R, \mathfrak{m})$ be a complete noetherian local ring and $X_n \to \operatorname{Spec} R/\mathfrak{m}^{n+1}$ be a sequence of proper morphisms such that $X_n \times_{R/\mathfrak{m}^{n+1}} R/\mathfrak{m}^n \cong X_{n-1}$. If L_n is a compatible sequence of line bundles on X_n such that L_0 is ample, then there exists a projective morphism $X \to \operatorname{Spec} R$ and an ample line bundle L on X and compatible isomorphisms $X_n \cong X \times_R R/\mathfrak{m}^{n+1}$ and $L_n \xrightarrow{\sim} L|_{X_n}$.*

In other words, there is an extension in the cartesian diagram

$$\begin{tikzcd}
X_0 \arrow[r, hook] \arrow[d] & X_1 \arrow[r, hook] \arrow[d] & X_2 \arrow[r, hook] \arrow[d] & \cdots \arrow[r, hook, dashed] & X \arrow[d, dashed] \\
\operatorname{Spec} R/\mathfrak{m} \arrow[r, hook] & \operatorname{Spec} R/\mathfrak{m}^2 \arrow[r, hook] & \operatorname{Spec} R/\mathfrak{m}^3 \arrow[r, hook] & \cdots \arrow[r, hook] & \operatorname{Spec} R
\end{tikzcd}$$

with X projective over R. We say that the formal deformation $\{X_n \to \operatorname{Spec} R/\mathfrak{m}^{n+1}\}$ of X_0 is *effective*.[2]

Proof. Let $\Bbbk = R/\mathfrak{m}$. Consider the finitely generated graded $\Bbbk$-algebra $B = \bigoplus \mathfrak{m}^n/\mathfrak{m}^{n+1}$ and the quasi-coherent graded $\mathcal{O}_{X_0}$-algebra $\mathcal{A} = B \otimes_{\Bbbk} \mathcal{O}_{X_0}$. By applying Serre's Vanishing Theorem to $\operatorname{Spec}_{X_0} \mathcal{A}$ and the ample line bundle $L_0 \otimes_{\mathcal{O}_{X_0}} \mathcal{O}_{X_0'}$, there exists d_0 such that $\mathrm{H}^1(X_0, \mathcal{A} \otimes L_0^{\otimes d}) = 0$ for $d \geq d_0$. By possibly enlarging d_0, we can assume that $L_0^{d_0}$ is very ample. Let $s_{0,0}, \ldots, s_{0,N}$ of $\mathrm{H}^0(X_0, L_0^{\otimes d_0})$ be sections defining a closed immersion $X_0 \hookrightarrow \mathbb{P}^N$. There is an exact sequence

$$0 \to \mathfrak{m}^n \mathcal{O}_{X_{n+1}}/\mathfrak{m}^{n+1}\mathcal{O}_{X_{n+1}} \to \mathcal{O}_{X_{n+1}} \to \mathcal{O}_{X_n} \to 0.$$

Tensoring by $L_{n+1}^{\otimes d}$ yields a short exact sequence

$$0 \to (\mathfrak{m}^n \mathcal{O}_{X_{n+1}}/\mathfrak{m}^{n+1}\mathcal{O}_{X_{n+1}}) \otimes L_0^{\otimes d} \to L_{n+1}^{\otimes d} \to L_n^{\otimes d} \to 0,$$

where we have used that $(\mathfrak{m}^n \mathcal{O}_{X_{n+1}}/\mathfrak{m}^{n+1}\mathcal{O}_{X_{n+1}})$ is supported on X_0 along with the identifications $L_{n+1} \otimes \mathcal{O}_{X_m} \cong L_m$ for $m \leq n$. The vanishing of $\mathrm{H}^1(X_0, \mathcal{A} \otimes L_0^{\otimes d})$ implies that we may lift the sections $s_{0,0}, \ldots, s_{0,N}$ inductively to compatible sections $s_{n,0}, \ldots, s_{n,N}$ of $\mathrm{H}^0(X_n, L_n^{\otimes d})$. By Nakayama's Lemma, the induced morphisms $X_n \hookrightarrow \mathbb{P}^N_{R/\mathfrak{m}^{n+1}}$ are closed immersions giving a commutative diagram

$$\begin{tikzcd}
 & \mathbb{P}^N \arrow[rr, hook] \arrow[dd] & & \mathbb{P}^n_{R/\mathfrak{m}^2_R} \arrow[rr, hook] \arrow[dd] & & \cdots \arrow[rr, hook] & & \mathbb{P}^N_R \arrow[dd] \\
X_0 \arrow[ur, hook, "\mathrm{cl}"] \arrow[rr, hook] \arrow[dr] & & X_1 \arrow[ur, hook, "\mathrm{cl}"] \arrow[rr, hook] \arrow[dr] & & \cdots \arrow[rr, hook, dashed] & & X \arrow[ur, hook, dashed, "\mathrm{cl}"] \arrow[dr, dashed] & \\
 & \operatorname{Spec} \Bbbk \arrow[rr, hook] & & \operatorname{Spec} R/\mathfrak{m}^2 \arrow[rr, hook] & & \cdots \arrow[rr, hook] & & \operatorname{Spec} R
\end{tikzcd}$$

Grothendieck's Existence Theorem (C.5.3) gives an equivalence $\operatorname{Coh}(\mathbb{P}^N_R) \to \varprojlim \operatorname{Coh}(\mathbb{P}^N_{R/\mathfrak{m}^{n+1}})$. Essential surjectivity gives a coherent sheaf E on $\mathbb{P}^N_R$ extending $\{\mathcal{O}_{X_n}\}$ and full faithfulness gives a surjection $\mathcal{O}_{\mathbb{P}^N_R} \to E$ extending $\mathcal{O}_{\mathbb{P}^N_{R/\mathfrak{m}^{n+1}}} \to \mathcal{O}_{X_n}$. We take $X \subseteq \mathbb{P}^N_R$ to be the closed subscheme defined by $\ker(\mathcal{O}_{\mathbb{P}^N_R} \to E)$. See also [EGA, III.5.4.5], [Ill05, Thm. 8.4.10], and [SP, Tag 089A]. $\square$

[2]This is also sometimes referred to as *algebraizable*, but we reserve this term a deformation over a *finite type $\Bbbk$-scheme*, e.g., the output of Artin's Algebraization (Theorem C.6.8).

Remark C.5.9. Assume in addition that each X_n is flat over $R/\mathfrak{m}^{n+1}$. If we are only given an ample line bundle L_0 on X_0 (but not the line bundles L_n), then the obstruction to deforming L_{n-1} to L_n is an element $\mathrm{ob}_{L_{n-1}} \in \mathrm{H}^2(X, \mathcal{O}_X \otimes_{\Bbbk} \mathfrak{m}^n)$ (Proposition C.2.11). If these cohomology groups vanish (e.g., if X is of dimension 1), then there exists compatible extensions L_n, and thus the formal deformation $\{X_n \to \operatorname{Spec} R/\mathfrak{m}^{n+1}\}$ are effective.

Without the existence of deformations L_n of L_0, it is not necessarily true that formal deformations are effective. For instance, there is a projective K3 surface (X_0, L_0) and a first-order deformation $X_1 \to \operatorname{Spec} \Bbbk[\epsilon]$ which is not projective (so L_0 does not deform to X_1), and a formal deformation which is not effective; see [Har10, Ex. 21.2.1], [Sta06, Claim 3.5], and [SP, Tag 0D1Q]. Similarly, formal deformations of abelian varieties may not be effective. Note that for the moduli of abstract K3 surfaces or abelian varieties, Rim–Schlessinger's Criteria (C.4.6) applies to construct versal formal deformations, but the lack of effectivity implies that the corresponding stacks are not algebraic (Example C.5.2).

We apply Grothendieck's Existence Theorem to the Hilbert functor $\operatorname{Hilb}(X)$, the stack $\mathcal{M}_g^{\mathrm{all}}$ of all curves, and the stack $\underline{\mathrm{Coh}}(X)$ of coherent sheaves, each defined over $\mathrm{Sch}/\Bbbk$ as Proposition C.4.14.

Proposition C.5.10. *Every formal deformation is effective for the* $\operatorname{Hilb}(X)$*,* $\mathcal{M}_g^{\mathrm{all}}$ *and* $\underline{\mathrm{Coh}}(X)$*. In particular, there exist effective miniversal formal deformations.*

Proof. For $\operatorname{Hilb}(X)$, let $\{Z_n \subseteq X_{R/\mathfrak{m}^{n+1}}\}$ be a formal deformation. Grothendieck's Existence Theorem (C.5.3) implies the existence of a coherent sheaf E on X_R extending the structure sheaves $\{\mathcal{O}_{Z_n}\}$, and moreover that there is a surjection $\mathcal{O}_{X_R} \to E$ extending $\{\mathcal{O}_{X_n} \to \mathcal{O}_{Z_n}\}$. The subscheme $Z \subseteq X_R$ defined by $\ker(\mathcal{O}_{X_R} \to E)$ effectivizes the formal deformation.

For $\mathcal{M}_g^{\mathrm{all}}$, let $\{C_n \to \operatorname{Spec} R/\mathfrak{m}^{n+1}\}$ be a formal deformation. As C_0 is a proper curve over a field, it is projective. Let L_0 be an ample line bundle on C_0. The obstruction to deforming a line bundle L_n on C_n to a line bundle L_{n+1} on C_{n+1} is an element $\mathrm{ob}_{L_{n-1}} \in \mathrm{H}^2(X, \mathcal{O}_X \otimes_{\Bbbk} \mathfrak{m}^n)$ (Proposition C.2.11). Since $\dim C = 1$, this cohomology group is zero and thus there are a compatible family of line bundle $\{L_n\}$. We may therefore apply Corollary C.5.8.

For $\underline{\mathrm{Coh}}(X)$, the effectivity of a formal deformation follows directly from Grothendieck's Existence Theorem (C.5.3). The last statement follows from the existence of miniversal formal deformations (Proposition C.4.14). ☐

Exercise C.5.11. Let X and Y be proper schemes over the spectrum $S = \operatorname{Spec} R$ of a complete noetherian local ring R. Denote by X_n and Y_n the restrictions of X and Y to $S_n = \operatorname{Spec} R/\mathfrak{m}_R^{n+1}$. Show that a compatible sequence of morphisms $f_n \colon X_n \to Y_n$ over S_n extends to a unique morphism $f \colon X \to Y$.

C.5.3 Lifting to characteristic 0

One striking application of deformation theory is to "lift" schemes X_0 over a field $\Bbbk$ of $\mathrm{char}(\Bbbk) = p$ to characteristic 0. We say that X_0 is *liftable to characteristic* 0 if there exists a complete noetherian local ring $(R, \mathfrak{m})$ of characteristic 0 such that $R/\mathfrak{m} = \Bbbk$ and a smooth scheme $X \to \operatorname{Spec} R$ such that $X_0 \cong X \times_R \Bbbk$.[3] One can hope to then use characteristic 0 techniques (e.g., Hodge theory) on X and deduce properties of X_0. The strategy to lift X_0 is to inductively deform X_0 to schemes

[3]There are some variants to this definition, e.g., when R is already given as a complete DVR with residue field $\Bbbk$.

X_n over $R/\mathfrak{m}^{n+1}$ and then apply Grothendieck's Existence Theorem to effective the formal deformation.

Smooth curves are liftable as obstructions to deforming both the curve and the ample line bundle both vanish. Serre produced an example of a non-liftable projective threefold (see [Har10, Thm. 22.4]), which Mumford extended to a non-liftable projective surface (see [Ill05, Cor. 8.6.7]). On the other hand, Mumford showed that principally polarized abelian varieties are liftable [Mum69] while Deligne showed that K3 surfaces are liftable [Del81]. These examples are quite interesting as, in both cases, formal deformations are not necessarily effective (see Remark C.5.9).

C.6 Artin Algebraization

Artin Algebraization states that every effective versal formal deformation "algebraizes", i.e., extends to an an object over a *finite type* $\Bbbk$-scheme. In this section, we show how Artin Algebraization follows from Artin Approximation following the ideas of Conrad and de Jong [CdJ02].

C.6.1 Limit preserving prestacks

Extending the definition of a limit preserving functor §B.5.5, we say that a prestack $\mathcal{X}$ over $\mathrm{Sch}/\Bbbk$ is *limit preserving* (or *locally of finite presentation*) if for every system B_λ of $\Bbbk$-algebras, the natural functor

$$\operatorname{colim} \mathcal{X}(B_\lambda) \to \mathcal{X}(\operatorname{colim} B_\lambda)$$

is an equivalence of categories. When $\mathcal{X}$ is an algebraic stack over $\Bbbk$, then this equivalent to the morphism $\mathcal{X} \to \operatorname{Spec} \Bbbk$ being locally of finite presentation; see Exercise 3.3.33.

Exercise C.6.1. Use the Limit Methods of §B.3 to show that $\mathrm{Hilb}(X)$, $\mathcal{M}_g^{\mathrm{all}}$ and $\underline{\mathrm{Coh}}(X)$ are each limit preserving over $\mathrm{Sch}/\Bbbk$.

C.6.2 Conrad–de Jong Approximation

In Artin Approximation (B.5.18), the initial data is an object over a complete noetherian local $\Bbbk$-algebra which is assumed to be the completion of a finitely generated $\Bbbk$-algebra at a maximal ideal. We will now see that a similar approximation result still holds if this latter hypothesis is dropped. The idea is to approximate *both* the complete local ring and the object.

Recall also that if $(A, \mathfrak{m})$ is a local ring and M is an A-module, then the *associated graded module of* M is defined as $\mathrm{Gr}_{\mathfrak{m}}(M) = \bigoplus_{n \geq 0} \mathfrak{m}^n M/\mathfrak{m}^{n+1}M$; it is a graded module over the graded ring $\mathrm{Gr}_{\mathfrak{m}}(A)$.

Theorem C.6.2 (Conrad–de Jong Approximation)**.** *Let $\mathcal{X}$ be a limit preserving prestack over $\mathrm{Sch}/\Bbbk$, $(R, \mathfrak{m}_R)$ be a complete noetherian local $\Bbbk$-algebra with residue field $\Bbbk$, and $\xi \in \mathcal{X}(R)$. Then for every integer $N \geq 0$, there exist*

1. *an affine scheme $\operatorname{Spec} A$ of finite type over $\Bbbk$ and a $\Bbbk$-point $u \in \operatorname{Spec} A$;*
2. *an object $\eta \in \mathcal{X}(A)$;*
3. *an isomorphism $\phi_N \colon R/\mathfrak{m}_R^{N+1} \cong A/\mathfrak{m}_u^{N+1}$;*
4. *an isomorphism of $\xi|_{R/\mathfrak{m}_R^{N+1}}$ and $\eta|_{A/\mathfrak{m}_u^{N+1}}$ via ϕ_N; and*
5. *an isomorphism $\mathrm{Gr}_{\mathfrak{m}_R}(R) \cong \mathrm{Gr}_{\mathfrak{m}_u}(A)$ of graded $\Bbbk$-algebras.*

The proof of this theorem will proceed by simultaneously approximating equations and relations defining R and the object ξ. The statements (1)–(4) will be easily obtained as a consequence of Artin Approximation. It is a nice insight of Conrad and de Jong that condition (5) can also be ensured by Artin Approximation, and moreover that this condition suffices to imply the isomorphism of complete local $\Bbbk$-algebras in Artin Algebraization. Unsurprisingly, (5) takes the most work to establish.

We will need some preparatory results controlling the constant appearing in the Artin–Rees Lemma (B.5.4).

Definition C.6.3 (Artin–Rees Condition)**.** Let $(A, \mathfrak{m})$ be a noetherian local ring, $\varphi\colon M \to N$ be a morphism of finite A-modules, and $c \geq 0$ be an integer. We say that $(\mathrm{AR})_c$ *holds for* φ if

$$\varphi(M) \cap \mathfrak{m}^n N \subseteq \varphi(\mathfrak{m}^{n-c} M), \quad \forall n \geq c.$$

The Artin–Rees Lemma (B.5.4) implies that $(\mathrm{AR})_c$ holds for φ if $c \gg 0$.

Lemma C.6.4. *Let $(A, \mathfrak{m})$ be a noetherian local ring. Let*

$$L \xrightarrow{\alpha} M \xrightarrow{\beta} N \qquad \textit{and} \qquad L' \xrightarrow{\alpha'} M \xrightarrow{\beta'} N$$

be two complexes of finite A-modules. Let c be a positive integer. Assume that

(a) the first sequence is exact,

(b) the complexes are isomorphic modulo $\mathfrak{m}^{c+1}$, and

(c) $(\mathrm{AR})_c$ holds for α and β.

Then there exists an isomorphism $\mathrm{Gr}_{\mathfrak{m}}(\operatorname{coker}\beta) \to \mathrm{Gr}_{\mathfrak{m}}(\operatorname{coker}\beta')$ of graded $\mathrm{Gr}_{\mathfrak{m}}(A)$-modules.

Proof. The proof, while technical, is rather straightforward. First, by taking free presentations of L and L', we can assume that $L = L'$. One shows that $(\mathrm{AR})_c$ holds for β' and that the second sequence is exact. Then one establishes the equality

$$\mathfrak{m}^{n+1}N + \beta(M) \cap \mathfrak{m}^n N = \mathfrak{m}^{n+1}N + \beta'(M) \cap \mathfrak{m}^n N$$

by using that $(\mathrm{AR})_c$ holds for β to show the containment "$\subseteq$", and then using that $(\mathrm{AR})_c$ holds for β' to get the other containment. The statement then follows from the description $\mathrm{Gr}_{\mathfrak{m}}(\operatorname{coker}\beta)_n = \mathfrak{m}^n N/(\mathfrak{m}^{n+1}N + \beta(M) \cap \mathfrak{m}^n N)$ and the similar description of $\mathrm{Gr}_{\mathfrak{m}}(\operatorname{coker}\beta')_n$. For details, see [CdJ02, §3] and [SP, Tag 07VF]. $\square$

Proof of Conrad–de Jong Approximation (Theorem C.6.2). Since $\mathcal{X}$ is limit preserving and R is the colimit of its finitely generated $\Bbbk$-subalgebras, there is an affine scheme $V = \operatorname{Spec} B$ of finite type over $\Bbbk$ and an object γ of $\mathcal{X}$ over V together with a 2-commutative diagram

$$\begin{tikzcd} \operatorname{Spec} R \arrow[r] \arrow[rr, bend left, "\xi"] & V \arrow[r, "\gamma"] & \mathcal{X}. \end{tikzcd}$$

Let $v \in V$ be the image of the maximal ideal $\mathfrak{m} \subseteq R$. After adding generators to the ring B if necessary, we can assume that the composition $\widehat{\mathcal{O}}_{V,v} \to R \to R/\mathfrak{m}^2$ is surjective. This implies that $\widehat{\mathcal{O}}_{V,v} \to R$ is surjective by Complete Nakayama's Lemma (B.5.6(3)). The goal now is to simultaneously approximate over V the equations and relations defining the closed immersion $\operatorname{Spec} R \hookrightarrow \operatorname{Spec} \widehat{\mathcal{O}}_{V,v}$ and the object ξ. To accomplish this goal, we choose a resolution

$$\widehat{\mathcal{O}}_{V,v}^{\oplus r} \xrightarrow{\widehat{\alpha}} \widehat{\mathcal{O}}_{V,v}^{\oplus s} \xrightarrow{\widehat{\beta}} \widehat{\mathcal{O}}_{V,v} \to R \to 0 \tag{C.6.5}$$

as $\widehat{\mathcal{O}}_{V,v}$-modules and consider the functor

$$\begin{aligned} F\colon (\mathrm{Sch}/V) &\to \mathrm{Sets} \\ (T \to V) &\mapsto \big\{\text{complexes } \mathcal{O}_T^{\oplus r} \xrightarrow{\alpha} \mathcal{O}_T^{\oplus s} \xrightarrow{\beta} \mathcal{O}_T\big\}. \end{aligned}$$

It is not hard to check that this functor is limit preserving. The resolution in (C.6.5) yields an element of $F(\widehat{\mathcal{O}}_{V,v})$. Applying Artin Approximation (B.5.18) gives an étale morphism $(V' = \operatorname{Spec} B', v') \to (V, v)$ and an element

$$(B'^{\oplus r} \xrightarrow{\alpha'} B'^{\oplus s} \xrightarrow{\beta'} B') \in F(V') \tag{C.6.6}$$

such that α', β' are equal to $\widehat{\alpha}, \widehat{\beta}$ modulo $\mathfrak{m}^{N+1}$.

Let $U = \operatorname{Spec} A \hookrightarrow \operatorname{Spec} B' = V'$ be the closed subscheme defined by $\operatorname{im} \beta'$ and let $u = v' \in U$. Consider the composition

$$\eta\colon U \hookrightarrow V' \to V \xrightarrow{\gamma} \mathcal{X}$$

As $R = \operatorname{coker} \widehat{\beta}$ and $A = \operatorname{coker} \beta'$, we have an isomorphism $R/\mathfrak{m}^{N+1} \cong A/\mathfrak{m}_u^{N+1}$ together with an isomorphism of $\xi|_{R/\mathfrak{m}^{N+1}}$ and $\eta|_{A/\mathfrak{m}_u^{N+1}}$. This gives statements (1)–(4).

To establish (5), we need to show that there are isomorphisms $\mathfrak{m}^n/\mathfrak{m}^{n+1} \cong \mathfrak{m}_u^n/\mathfrak{m}_u^{n+1}$. For $n \leq N$, this is guaranteed by the isomorphism $R/\mathfrak{m}^{N+1} \cong A/\mathfrak{m}_u^{N+1}$. On the other hand, for $n \gg 0$, this can be seen to be a consequence of the Artin–Rees Lemma (B.5.4). To handle the middle range of n, we need to control the constant appearing in the Artin–Rees Lemma. First, note that before we applied Artin Approximation, we could have increased N to ensure that $(\mathrm{AR})_N$ holds for $\widehat{\alpha}$ and $\widehat{\beta}$. We are thus free to assume this. Now statement (5) follows directly if we apply Lemma C.6.4 to the exact complex $\widehat{\mathcal{O}}_{V,v}^{\oplus r} \xrightarrow{\widehat{\alpha}} \widehat{\mathcal{O}}_{V,v}^{\oplus s} \xrightarrow{\widehat{\beta}} \widehat{\mathcal{O}}_{V,v}$ of (C.6.5) and the complex $\widehat{\mathcal{O}}_{V,v}^{\oplus r} \xrightarrow{\widehat{\alpha}'} \widehat{\mathcal{O}}_{V,v}^{\oplus s} \xrightarrow{\widehat{\beta}'} \widehat{\mathcal{O}}_{V,v}$ obtained by restricting (C.6.6) to $F(\widehat{\mathcal{O}}_{V,v})$. See also [CdJ02] and [SP, Tag 07XB]. $\square$

Exercise C.6.7. Show that Conrad–de Jong Approximation implies Artin Approximation.

C.6.3 Artin Algebraization

Artin Algebraization has a stronger conclusion than Artin Approximation or Conrad–de Jong Approximation in that no approximation is necessary. It guarantees the existence of an object η over a pointed affine scheme $(\operatorname{Spec} A, u)$ of finite type over $\Bbbk$, which agrees with the given effective formal deformation ξ to all orders. To ensure this, we need to impose that ξ is *versal at* u, i.e., that the restrictions $\xi_n = \xi|_{A/\mathfrak{m}_u^{n+1}}$ define a versal formal deformation $\{\xi_n\}$ over A (Definition C.4.2).

Theorem C.6.8 (Artin Algebraization)**.** *Let $\mathcal{X}$ be a limit preserving prestack over* $\mathrm{Sch}/\Bbbk$*. Let $(R, \mathfrak{m})$ be a complete noetherian local $\Bbbk$-algebra and $\xi \in \mathcal{X}(R)$ be an effective versal formal deformation. There exist*

(1) *an affine scheme* $\operatorname{Spec} A$ *of finite type over* $\Bbbk$ *and a* $\Bbbk$*-point* $u \in \operatorname{Spec} A$*;*

(2) *an object* $\eta \in \mathcal{X}(A)$*;*

(3) *an isomorphism* $\alpha\colon R \xrightarrow{\sim} \widehat{A}_{\mathfrak{m}_u}$ *of* $\Bbbk$*-algebras; and*

(4) *a compatible family of isomorphisms* $\xi|_{R/\mathfrak{m}^{n+1}} \cong \eta|_{A/\mathfrak{m}_u^{n+1}}$ *(under the identification* $R/\mathfrak{m}^{n+1} \cong A/\mathfrak{m}_u^{n+1}$*) for* $n \geq 0$*.*

Note that we are not asserting that $\xi \cong \eta|_{\widehat{A}_{\mathfrak{m}_u}}$, although this does hold in some settings, e.g., if $\mathcal{X}$ is an algebraic stack. See Example C.7.17 for an example where the effective versal formal deformation is not unique, in which case there are two algebraizations which are not étale-locally isomorphic.

Remark C.6.9. If R is known to be the completion of a finitely generated $\Bbbk$-algebra, this theorem can be viewed as an easy consequence of Artin Approximation. Indeed, one applies Artin Approximation with $N = 1$ and then uses versality to obtain compatible maps $R \to A/\mathfrak{m}_u^{n+1}$ and therefore a map $R \to \widehat{A}_{\mathfrak{m}_u}$ which is an isomorphism modulo $\mathfrak{m}^2$. As R and $\widehat{A}_{\mathfrak{m}_u}$ are abstractly isomorphic, the homomorphism $R \to \widehat{A}_{\mathfrak{m}_u}$ is an isomorphism by Complete Nakayama's Lemma (B.5.6(3)) and the statement follows. The argument in the general case is analogous, except we use Conrad–de Jong Approximation instead of Artin Approximation.

Proof. Applying Conrad–de Jong Approximation (C.6.2) with $N = 1$, we obtain an affine scheme $\operatorname{Spec} A$ of finite type over $\Bbbk$ with a $\Bbbk$-point $u \in \operatorname{Spec} A$, an object $\eta \in \mathcal{X}(A)$, an isomorphism $\phi_2\colon \operatorname{Spec} A/\mathfrak{m}_u^2 \to \operatorname{Spec} R/\mathfrak{m}^2$, an isomorphism $\alpha_2\colon \xi|_{R/\mathfrak{m}^2} \to \eta|_{A/\mathfrak{m}_u^2}$, and an isomorphism $\operatorname{Gr}_{\mathfrak{m}}(R) \cong \operatorname{Gr}_{\mathfrak{m}_u}(A)$ of graded $\Bbbk$-algebras. We claim that ϕ_2 and α_2 can be extended inductively to a compatible family of morphisms $\phi_n\colon \operatorname{Spec} A/\mathfrak{m}_u^{n+1} \to \operatorname{Spec} R$ and isomorphisms $\alpha_n\colon \xi|_{A/\mathfrak{m}_u^{n+1}} \to \eta|_{A/\mathfrak{m}_u^{n+1}}$. Indeed, given ϕ_n and α_n, versality of ξ implies that there is a lift ϕ_{n+1} filling in the commutative diagram

$$\begin{tikzcd}
\operatorname{Spec} A/\mathfrak{m}_u^n \arrow[r, "\phi_n"] \arrow[d] & \operatorname{Spec} R \arrow[d, "\xi"] \\
\operatorname{Spec} A/\mathfrak{m}_u^{n+1} \arrow[r, "\eta|_{A/\mathfrak{m}_u^{n+1}}"'] \arrow[ur, dashed, "\phi_{n+1}"] & \mathcal{X}.
\end{tikzcd}$$

By taking the limit, we have a homomorphism $\widehat{\phi}\colon R \to \widehat{A}_{\mathfrak{m}_u}$ which is surjective by Complete Nakayama's Lemma (B.5.6(3)). On the other hand, for each n the $\Bbbk$-vector spaces $\mathfrak{m}^N/\mathfrak{m}^{N+1}$ and $\mathfrak{m}_u^N/\mathfrak{m}_u^{N+1}$ have the same dimension. This implies that $\widehat{\phi}$ is an isomorphism. See also [Art69b, Thm. 1.6] and [CdJ02, §4]. □

C.7 Artin's Axioms for Algebraicity

As a general fact, our knowledge of nonprojective existence theorems is exceedingly poor, and I hope this will change eventually.

Grothendieck, letter to Murre, 1962 [Mum10, p. 663]

Artin's Axioms for Algebraicity provide criteria, often verifiable in practice, ensuring that a given stack is algebraic. This foundational result was proved by Artin in the very same paper [Art74b] where he introduced algebraic stacks. We provide two versions below: Theorems C.7.1 and C.7.4. The first version is a fairly easy consequence of Artin Algebraization (C.6.8).

Theorem C.7.1. *(Artin's Axioms for Algebraicity—first version) Let $\mathcal{X}$ be a stack over $(\mathrm{Sch}/\Bbbk)_{\text{ét}}$. Then $\mathcal{X}$ is an algebraic stack locally of finite type over $\Bbbk$ if and only if the following conditions hold:*

(1) (Limit preserving) The stack $\mathcal{X}$ is limit preserving over $\mathrm{Sch}/\Bbbk$, i.e., for every system B_λ of $\Bbbk$-algebras, the functor

$$\operatorname{colim} \mathcal{X}(B_\lambda) \to \mathcal{X}(\operatorname{colim} B_\lambda)$$

is an equivalence of categories.

(2) (Representability of the diagonal) The diagonal $\mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is representable.

(3) (Existence of versal formal deformations) Every $x_0 \in \mathcal{X}(\Bbbk)$ has a versal formal deformation $\{x_n\}$ over a complete noetherian local $\Bbbk$-algebra $(R, \mathfrak{m})$ with residue field $\Bbbk$.

(4) (Effectivity) For every complete noetherian local $\Bbbk$-algebra $(R, \mathfrak{m})$ with residue field $\Bbbk$, the natural functor

$$\mathcal{X}(\operatorname{Spec} R) \to \varprojlim \mathcal{X}(\operatorname{Spec} R/\mathfrak{m}^n)$$

is an equivalence of categories.

(5) (Openness of versality) For every morphism $U \to \mathcal{X}$ from a finite type $\Bbbk$-scheme which is versal at $u \in U(\Bbbk)$ (i.e., the formal deformation $\{\operatorname{Spec} \widehat{\mathcal{O}}_{U,u}/\mathfrak{m}_u^{n+1} \to \mathcal{X}\}$ is versal), there exists an open neighborhood V of u such that $U \to \mathcal{X}$ is versal at every $\Bbbk$-point of V.

Proof. We first note that for a representable and locally of finite type morphism $U \to \mathcal{X}$ from a finite type $\Bbbk$-scheme U, the Infinitesimal Lifting Criterion for Smoothness (3.7.1) implies that $U \to \mathcal{X}$ is smooth if and only if it is versal at all $\Bbbk$-points $u \in U$. For $(\Rightarrow)$, (1) holds by Exercise 3.3.33, (2) holds by Theorem 3.2.1, and (4) holds by Example C.5.2. If $U \to \mathcal{X}$ is a morphism from a finite type $\Bbbk$-scheme, then it is necessarily representable and locally of finite type. Part (3) holds by choosing a smooth presentation $U \to \mathcal{X}$ and a preimage $u \in U(\Bbbk)$ of x_0 and taking the formal deformation $\{\operatorname{Spec} \mathcal{O}_{U,u}/\mathfrak{m}_u^{n+1} \to \mathcal{X}\}$. Part (5) holds by openness of smoothness.

For the converse, we first note that representability of the diagonal, i.e., condition (2), implies that every morphism $U \to \mathcal{X}$ from a scheme U is representable, and the limit preserving property (1) implies that $U \to \mathcal{X}$ is locally of finite type. For every object $x_0 \in \mathcal{X}(\Bbbk)$, we will construct a smooth morphism $U \to \mathcal{X}$ from a scheme and a preimage $u \in U(\Bbbk)$ of x_0. Conditions (3)–(4) guarantee that there exists an effective versal formal deformation $\widehat{x}\colon \operatorname{Spec} R \to \mathcal{X}$ of x_0 where $(R, \mathfrak{m})$ is a complete noetherian local $\Bbbk$-algebra with residue field $\Bbbk$. By Artin Algebraization (C.6.8), there exists a finite type $\Bbbk$-scheme U, a point $u \in U(\Bbbk)$, a morphism $p\colon U \to \mathcal{X}$, an isomorphism $R \cong \widehat{\mathcal{O}}_{U,u}$, and compatible isomorphisms $p|_{R/\mathfrak{m}^{n+1}} \xrightarrow{\sim} \widehat{x}|_{R/\mathfrak{m}^{n+1}}$. By (5), we can replace U with an open neighborhood of u so that $U \to \mathcal{X}$ is versal (or in other words smooth) at every $\Bbbk$-point of U. See also [Art74b, §5], [LMB00, Cor. 10.11], and [SP, Tag 07Y4]. □

Remark C.7.2. In practice, condition (1)–(4) are often easy to verify directly with (3) a consequence of Rim–Schlessinger's Criteria (C.4.6) and (4) a consequence of Grothendieck's Existence Theorem (C.5.3). Also note that (2) can sometimes be established by applying the theorem to the diagonal $\mathcal{X} \to \mathcal{X} \times \mathcal{X}$, i.e., to the Isom sheaves $\underline{\mathrm{Isom}}_T(x, y)$ of objects $x, y \in \mathcal{X}(T)$ over a scheme T. In some specialized cases, (5) can be checked directly, but it is frequently verified as a consequence of a well-behaved deformation and obstruction theory, as we explain in the next section.

C.7.1 Artin's Axioms via an obstruction theory

We state a refinement of Artin's Axioms for Algebraicity that is often easier to verify in practice. Namely, we show that openness of versality (C.7.1(5)) holds if the stack has a well-behaved deformation theory and if there exists a well-behaved obstruction theory.

To formulate the statements, we will need a bit of notation. Let $\xi \in \mathcal{X}(A)$ be an object over a finitely generated $\Bbbk$-algebra A. Let M be a finite A-module and denote by $A[M]$ the ring $A \oplus M$ defined by $M^2 = 0$. Let $\mathrm{Def}_\xi(M)$ the set of isomorphism classes of diagrams

$$\begin{tikzcd} \operatorname{Spec} A \arrow[r, "\xi"] \arrow[d, hook] & \mathcal{X} \\ \operatorname{Spec} A[M], \arrow[ur, dashed, "\eta"] & \end{tikzcd}$$

where an isomorphism of two extensions $\eta, \eta' \colon \operatorname{Spec} A[M] \to \mathcal{X}$ is by definition an isomorphism $\eta \xrightarrow{\sim} \eta'$ in $\mathcal{X}(A[M])$ restricting to the identity on ξ. Let $\mathrm{Aut}_\xi(M)$ be the group of automorphisms of the trivial deformation $\xi' \colon \operatorname{Spec} A[M] \to \operatorname{Spec} A \to \mathcal{X}$. Note that when $\xi \in \mathcal{X}(\Bbbk)$, then $\mathrm{Def}_\xi(\Bbbk)$ is precisely the tangent space of $\mathcal{X}$ at ξ, while $\mathrm{Aut}_\xi(\Bbbk)$ is the group of infinitesimal automorphism of ξ, i.e. the kernel of $\mathrm{Aut}_{\mathcal{X}(\Bbbk[\epsilon])}(\xi') \to \mathrm{Aut}_{\mathcal{X}(\Bbbk)}(\xi)$.

Lemma C.7.3. *Suppose that $\mathcal{X}$ is a prestack over* $\mathrm{Sch}/\Bbbk$ *satisfying the strong homogeneity condition* (RS_4^*). *Let $\xi \in \mathcal{X}(A)$ be an object over a finitely generated $\Bbbk$-algebra A.*

(1) For every A-module M, $\mathrm{Def}_\xi(M)$ *and* $\mathrm{Aut}_\xi(M)$ *are naturally A-modules, and the functors*

$$\mathrm{Aut}_\xi(-) \colon \mathrm{Mod}(A) \to \mathrm{Mod}(A)$$
$$\mathrm{Def}_\xi(-) \colon \mathrm{Mod}(A) \to \mathrm{Mod}(A)$$

are A-linear.

(2) Consider a surjection $B \twoheadrightarrow A$ of $\Bbbk$-algebras with square-zero kernel I, and let $\mathrm{Lift}_\xi(B)$ *be the set of morphisms $\xi \to \eta$ over* $\operatorname{Spec} A \to \operatorname{Spec} B$ *where $(\alpha \colon \xi \to \eta) \sim (\alpha' \colon \xi \to \eta')$ if there is an isomorphism $\beta \colon \eta \to \eta'$ such that $\alpha' = \beta \circ \alpha$. There is an action of* $\mathrm{Def}_\xi(I)$ *on* $\mathrm{Lift}_\xi(B)$ *which is functorial in B and I. Assuming* $\mathrm{Lift}_\xi(B)$ *is non-empty, this action is free and transitive.*

Proof. This can be established by arguing as in Lemma C.4.9. For instance, scalar multiplication by $x \in A$ is defined by pulling back along the morphism $\operatorname{Spec} A[M] \to \operatorname{Spec} A[M]$ induced by the A-algebra homomorphism $A[M] \to A[M], a+m \mapsto a+xm$. Condition (RS_4^*) implies that the functor $\mathcal{X}(A[M \oplus M]) \to \mathcal{X}(A[M]) \times_{\mathcal{X}(A)} \mathcal{X}(A[M])$ is an equivalence. Addition $M \oplus M \to M$ induces an A-algebra homomorphism $A[M \oplus M] \to A[M]$ and thus a functor

$$\mathcal{X}(A[M]) \times_{\mathcal{X}(A)} \mathcal{X}(A[M]) \cong \mathcal{X}(A[M \oplus M]) \to \mathcal{X}(A[M])$$

which defines addition on $\mathrm{Def}_\xi(M)$ and $\mathrm{Aut}_\xi(M)$. □

Unlike the automorphism $\mathrm{Def}_\xi(-)$ and deformation $\mathrm{Def}_\xi(-)$ functors which are intrinsic to the moduli problem, an obstruction functor is an additional piece of data.

Theorem C.7.4 (Artin's Axioms for Algebraicity—second version)**.** *A stack $\mathcal{X}$ over $(\mathrm{Sch}/\Bbbk)_{\text{ét}}$ is an algebraic stack locally of finite type over $\Bbbk$ if the following conditions hold.*

(AA_1) *(Limit preserving) The stack $\mathcal{X}$ is limit preserving.*

(AA_2) *(Representability of the diagonal) The diagonal $\mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is representable.*

(AA_3) *(Finiteness of tangent spaces) For every object $\xi\colon \operatorname{Spec} \Bbbk \to \mathcal{X}$, $\operatorname{Def}_\xi(\Bbbk)$ is a finite dimensional $\Bbbk$-vector space.*

(AA_4) *(Strong homogeneity) For every $\Bbbk$-algebra homomorphism $B_0 \to A_0$ and surjection $A \twoheadrightarrow A_0$ of $\Bbbk$-algebras with square-zero kernel, the functor*

$$\mathcal{X}(B_0 \times_{A_0} A) \to \mathcal{X}(B_0) \times_{\mathcal{X}(A_0)} \mathcal{X}(A)$$

is an equivalence, i.e., (RS_4^*) holds.

(AA_5) *(Effectivity) For every complete noetherian local $\Bbbk$-algebra $(R, \mathfrak{m})$ with residue field $\Bbbk$, the natural functor*

$$\mathcal{X}(\operatorname{Spec} R) \to \varprojlim \mathcal{X}(\operatorname{Spec} R/\mathfrak{m}^n)$$

is an equivalence of categories.

(AA_6) *(Coherent deformation theory) For every object $\xi \in \mathcal{X}(A)$ over a $\Bbbk$-algebra A, the functor $\operatorname{Def}_\xi(-)$ commutes with products.*

(AA_7) *(Existence of a coherent obstruction theory) For every object $\xi \in \mathcal{X}(A)$ over a $\Bbbk$-algebra A, there exists the following data*

(a) there is an A-linear functor

$$\operatorname{Ob}_\xi(-)\colon \operatorname{Mod}(A) \to \operatorname{Mod}(A),$$

and for every surjection $B \to A$ with square-zero kernel I, there is an element $\operatorname{ob}_\xi(B) \in \operatorname{Ob}_\xi(I)$ such that there is an extension

$$\begin{tikzcd} \operatorname{Spec} A \arrow[r, "\xi"] \arrow[d, hook] & \mathcal{X} \\ \operatorname{Spec} B \arrow[ur, dashed] & \end{tikzcd}$$

if and only if $\operatorname{ob}_\xi(B) = 0$, and

(b) for every composition $B \to B' \to A$ of $\Bbbk$-algebras such that $B \twoheadrightarrow A$ and $B' \twoheadrightarrow A$ are surjective with square-zero kernels I and I', the image of $\operatorname{ob}_\xi(B)$ under $\operatorname{Ob}_\xi(I) \to \operatorname{Ob}_\xi(I')$ is $\operatorname{ob}_\xi(B')$.

(c) For every object $\xi \in \mathcal{X}(A)$ over a $\Bbbk$-algebra A, the functor $\operatorname{Ob}_\xi(-)$ commutes with products.

Moreover, (AA_2) can be replaced with

$(\mathrm{AA}_{2'})$ *For every object $\xi\colon \operatorname{Spec} \Bbbk \to \mathcal{X}$, $\operatorname{Aut}_\xi(\Bbbk)$ is a finite dimensional $\Bbbk$-vector space, and for every object $\xi \in \mathcal{X}(A)$ over a $\Bbbk$-algebra A, the functor $\operatorname{Aut}_\xi(-)$ commutes with products.*

Proof. We verify the conditions of Theorem C.7.1. By (AA_3)–(AA_4), we may apply Rim–Schlessinger's Criteria (C.4.6) to deduce the existence of versal formal deformations, i.e., C.7.1(3) holds. It remains to check openness of versality, i.e., C.7.1(5). Let $\xi_0 \colon U_0 \to \mathcal{X}$ be a morphism from an affine scheme $U_0 = \operatorname{Spec} B_0$ of finite type over $\Bbbk$ such that ξ_0 is versal at a point $u_0 \in U_0(\Bbbk)$. By (AA$_1$)–(AA$_2$), the morphism $\xi_0 \colon U_0 \to \mathcal{X}$ is representable and locally of finite type. Let $\Sigma = \{u \in U_0(\Bbbk) \mid \xi_0 \colon U_0 \to \mathcal{X} \text{ is not versal at } u\}$. If openness of versality does not hold, then $u_0 \in \overline{\Sigma}$ and there exists a countably infinite subset $\Sigma' = \{u_1, u_2, \ldots\} \subseteq \Sigma$ of distinct points with $u_0 \in \overline{\Sigma'}$.

Step 1. We claim that there exists a commutative diagram

$$\begin{tikzcd}
U_0 \arrow[r, hook] \arrow[d, "\xi_0"'] & U_1 \arrow[r, hook] \arrow[dl, "\xi_1"'] & U_2 \arrow[r, hook] \arrow[dll, "\xi_2"] & \cdots \\
\mathcal{X}
\end{tikzcd}$$

where each closed immersion $U_{n-1} \hookrightarrow U_n$ is defined by a short exact sequence

$$0 \to \kappa(u_n) \to \mathcal{O}_{U_n} \to \mathcal{O}_{U_{n-1}} \to 0,$$

and there exists open neighborhoods $W_n \subseteq U_n \smallsetminus \{u_0, \ldots, u_{n-1}\}$ of u_n such that each restriction $\xi_n|_{W_n}$ is not the trivial deformation of $\xi_0|_{W_n \cap U_0}$. By construction, each $U_n = \operatorname{Spec} B_n$ is an affine scheme and each closed immersion $U_n \hookrightarrow U_m$ for $m \geq n$ is defined by a square-zero ideal. Suppose that we've already constructed $\xi_0, \ldots, \xi_{n-1}$. Since $\xi_0 \colon U_0 \to \mathcal{X}$ and $\xi_{n-1} \colon U_{n-1} \to \mathcal{X}$ are isomorphic in an open neighborhood of u_n, the morphism $\xi_{n-1} \colon U_{n-1} \to \mathcal{X}$ is also not versal at u_n. Therefore, there exists a surjection $A \to A_0$ in $\mathrm{Art}_{\Bbbk}$ with $\ker(A \to A_0) = \Bbbk$ and a commutative diagram

$$\begin{tikzcd}
\operatorname{Spec} \Bbbk \arrow[r, hook] \arrow[rr, bend left, "u_n"] & \operatorname{Spec} A_0 \arrow[r] \arrow[d, hook] & U_{n-1} \arrow[d, "\xi_{n-1}"] \\
& \operatorname{Spec} A \arrow[r] \arrow[ur, dashed, "\nexists"] & \mathcal{X},
\end{tikzcd} \tag{C.7.5}$$

such that u_n is the image of $\operatorname{Spec} A_0 \to U_{n-1}$, which does not admit a lift $\operatorname{Spec} A \to U_{n-1}$. Using strong homogeneity (AA$_4$), there exists an extension of the commutative diagram

$$\begin{tikzcd}
\operatorname{Spec} A_0 \arrow[r] \arrow[d, hook] & U_{n-1} = \operatorname{Spec} B_{n-1} \arrow[d, hook] \arrow[ddr, bend left, "\xi_{n-1}"] & \\
\operatorname{Spec} A \arrow[r] \arrow[drr, bend right] & U_n = \operatorname{Spec}(A \times_{A_0} B_{n-1}) \arrow[dr, dashed, "\xi_n"] & \\
& & \mathcal{X}
\end{tikzcd}$$

yielding an object ξ_n over $U_n = \operatorname{Spec} B_n$ with $B_n := A \times_{A_0} B_{n-1}$. If ξ_n were the trivial deformation of ξ_0 in an open neighborhood of u_n, then $\operatorname{Spec} A \to \mathcal{X}$ would be the trivial deformation of $\operatorname{Spec} A_0$ contradicting the obstruction to a lift of (C.7.5). Finally note that $\ker(B_n \to B_{n-1}) = \Bbbk$ since $\ker(A \to A_0) = \Bbbk$. This establishes the claim.

Step 2. Letting $\widehat{B} = \varprojlim B_n$ and $\widehat{U} = \operatorname{Spec} \widehat{B}$, we claim that there exists an object $\widehat{\xi} \in \mathcal{X}(\widehat{U})$ extending each $\xi_n \in \mathcal{X}(U_n)$. Let $M_n = \ker(B_n \to B_0)$ (noting that

$M_0 = 0$). Since $M_n^2 = 0$, we can view M_n as a B_0-module. The $\Bbbk$-algebra

$$\widetilde{B} := \left\{ (b_0, b_1, \dots) \in \prod_{n \geq 0} B_n \;\middle|\; \text{the image of each } b_n \text{ under } B_n \to B_0 \text{ is } b_0 \right\}$$

has the following properties.

- The surjective $\Bbbk$-algebra homomorphism $\widetilde{B} \to B_0$ defined by $(b_i) \mapsto b_0$ has kernel $M := \prod_{n \geq 0} M_n$.
- The map $\widetilde{B} \to B_0[M]$ defined by $(b_0, b_1, b_2, \dots) \mapsto (b_0, b_1 - b_0, b_2 - b_1, b_3 - b_2, \dots)$ is a surjective $\Bbbk$-algebra homomorphism with square-zero kernel.
- The composition $\widehat{B} \to \widetilde{B} \to B_0[M]$ induces a short exact sequence

$$\begin{tikzcd}
0 \arrow[r] & \ker(\widehat{B} \to B_0) \arrow[r] \arrow[d, equal] & \ker(\widetilde{B} \to B_0) \arrow[r] \arrow[d, equal] & \ker(B_0[M] \to B_0) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \varprojlim_{n \geq 0} M_n \arrow[r] & \prod_{n \geq 0} M_n \arrow[r] & \prod_{n \geq 0} M_n \arrow[r] & 0 \\
& & (b_0, b_1, \dots) \arrow[r, mapsto] & (b_1 - b_0, b_2 - b_1, \dots). &
\end{tikzcd}$$

- There is an identification $\widehat{B} = \widetilde{B} \times_{B_0[M]} B_0$.

Since the lift $\xi_n \in \mathcal{X}(B_n)$ of ξ_0 exists for each n, $\mathrm{ob}_{\xi_0}(B_n) = 0 \in \mathrm{Ob}_{\xi_0}(M_n)$. By (AA$_7$)(b), the element $\mathrm{ob}_{\xi_0}(\widetilde{B})$ maps to $\mathrm{ob}_{\xi_0}(B_n)$ under $\mathrm{Ob}_{\xi_0}(M) \to \mathrm{Ob}_{\xi_0}(M_n)$. By , the map $\mathrm{Ob}_{\xi_0}(M) \hookrightarrow \prod_n \mathrm{Ob}_{\xi_0}(M_n)$ is injective[4] and thus $\mathrm{ob}_{\xi_0}(\widetilde{B}) = 0 \in \mathrm{Ob}_{\xi_0}(M)$ which shows that there exists a lift $\widetilde{\xi} \in \mathcal{X}(\widetilde{B})$ of ξ_0.

The restrictions $\widetilde{\xi}|_{B_n}$ are not necessarily isomorphic to ξ_n. However, we may use the free and transitive action $\mathrm{Def}_{\xi_0}(M_n) = \mathrm{Lift}_{\xi_0}(B_0[M_n])$ on the non-empty set of liftings $\mathrm{Lift}_{\xi_0}(\widetilde{B_n})$ to find elements $t_n \in \mathrm{Def}_{\xi_0}(M_n)$ such that $\xi_n = t_n \cdot \widetilde{\xi}|_{B_n}$ (Lemma C.7.3). Since $\mathrm{Def}_{\xi_0}(M) \xrightarrow{\sim} \prod_n \mathrm{Def}_{\xi_0}(M_n)$ by (AA$_6$), there exists $\widetilde{t} \in \mathrm{Def}_{\xi_0}(M)$ mapping to (t_n). After replacing $\widetilde{\xi}$ with $\widetilde{t} \cdot \widetilde{\xi}$, we can arrange that $\widetilde{\xi}|_{B_n}$ and ξ_n are isomorphic for each n.

We now show that each restriction $\widetilde{\xi}|_{B_0[M_n]} \in \mathrm{Def}_{\xi_0}(M_n)$ under the composition $\widetilde{B} \to B_0[M] \to B_0[M_n]$ is the trivial deformation. Indeed, the map $M = \ker(\widetilde{B} \to B_0) \to \ker(B_0[M_n] \to B_0) = M_n$ induces a map $\mathrm{Def}_{\xi_0}(M) \to \mathrm{Def}_{\xi_0}(M_n)$ on deformation modules, which under the identification $\mathrm{Def}_{\xi_0}(M) \xrightarrow{\sim} \prod_n \mathrm{Def}_{\xi_0}(M_n)$ of (AA$_6$), sends an element $(\eta_0, \eta_1, \dots)$ to $(\eta_{n+1}|_{B_n} - \eta_n)$. The ring map $\widetilde{B} \to B_0[M_n]$ also induces a map $\mathrm{Lift}_{\xi_0}(\widetilde{B}) \to \mathrm{Lift}_{\xi_0}(B_0[M_n])$ which is equivariant with respect to $\mathrm{Def}_{\xi_0}(M) \to \mathrm{Def}_{\xi_0}(M_n)$. It follows that the image of $\widetilde{\xi}$ in $\mathrm{Lift}_{\xi_0}(B_0[M_n]) = \mathrm{Def}_{\xi_0}(M_n)$ is $\xi_{n+1}|_{B_n} - \xi_n = 0$.

The existence of $\widehat{\xi} \in \mathcal{X}(\widehat{B})$ extending $(\xi_n) \in \varprojlim \mathcal{X}(B_n)$ now follows from applying

[4] The hypotheses of (AA$_7$)(c) can be weakened to only require the injectivity of $\mathrm{Ob}_{\xi_0}(M) \hookrightarrow \prod_n \mathrm{Ob}_{\xi_0}(M_n)$, although in practice one usually verifies bijectivity.

the identity $\widehat{B} = \widetilde{B} \times_{B_0[M]} B_0$ and strong homogeneity (AA$_4$) to the diagram

$$\begin{tikzcd}
\operatorname{Spec} B_0[M] \arrow[r] \arrow[d, hook] & \operatorname{Spec} B_0 \arrow[d, hook] \arrow[ddr, bend left, "\xi_0"] & \\
\operatorname{Spec} \widetilde{B} \arrow[r] \arrow[drr, bend right, "\widetilde{\xi}"'] & \operatorname{Spec} \widehat{B} \arrow[dr, dashed, "\widehat{\xi}"] & \\
& & \mathcal{X}.
\end{tikzcd}$$

Step 3. We now use the versality of $\xi_0 \colon U_0 \to \mathcal{X}$ at u_0 to arrive at a contradiction. Since $\mathcal{X}$ is limit preserving (AA$_1$), there exists a finitely generated $\Bbbk$-subalgebra $B' \subseteq \widehat{B}$ and an object $\xi' \in \mathcal{X}(B')$ together with an isomorphism $\widehat{\xi} \xrightarrow{\sim} \xi'|_{\widehat{B}}$. After possibly enlarging B', we may assume that the composition $B' \hookrightarrow \widehat{B} \to B_0$ is surjective. Since $\ker(\widehat{B} \to B_0)$ is square-zero, so is $\ker(B' \to B_0)$. This defines a closed immersion $U_0 \hookrightarrow U' := \operatorname{Spec} B'$, and we can consider the commutative diagram

$$\begin{tikzcd}
U_0 \arrow[dr, "i"] \arrow[drr, hook, bend left] \arrow[ddr, bend right, "\mathrm{id}"'] & & \\
& U_0 \times_{\mathcal{X}} U' \arrow[r] \arrow[d] & U' = \operatorname{Spec} B' \arrow[d, "\xi'"] \\
& U_0 \arrow[r, "\xi_0"] & \mathcal{X}.
\end{tikzcd}$$

where the fiber product $U_0 \times_{\mathcal{X}} U'$ is an algebraic space locally of finite type over $\Bbbk$. Since $\xi_0 \colon U_0 \to \mathcal{X}$ is versal at u_0, it follows from (the artinian version of) the Infinitesimal Lifting Criterion for Smoothness (3.7.1) that $U_0 \times_{\mathcal{X}} U' \to U'$ is smooth at $i(u_0)$. After replacing U_0 and U' with affine open neighborhoods and $\{u_1, u_2, \ldots\}$ with an infinite subsequence contained in these open subsets, we can arrange that $U_0 \times_{\mathcal{X}} U' \to U'$ is smooth. The (non-artinian version) of the Infinitesimal Lifting Criterion for Smoothness (A.3.1) for schemes implies that the section of $U_0 \times_{\mathcal{X}} U' \to U'$ over U_0 extends to a global section $U' \to U_0 \times_{\mathcal{X}} U'$. This implies that ξ' is the trivial deformation of ξ_0, which in turn implies that each ξ_n is a trivial deformation of ξ_0, a contradiction.

For the addendum, we show that (AA$_{2'}$) implies (AA$_2$), i.e., the representability of the diagonal. By using the Limit Methods of §B.3, it suffices to consider maps $(a, b) \colon T \to \mathcal{X} \times_{\Bbbk} \mathcal{X}$ from a finite type $\Bbbk$-scheme. The deformation functor $\operatorname{Def}_{\xi}(-)$ for the base change $\underline{\operatorname{Isom}}_T(a, b) \to T$ corresponds to the automorphism functor $\operatorname{Aut}_{\xi}(-)$ for $\mathcal{X}$, and we take obstruction functor $\operatorname{Ob}_{\xi}(-)$ for $\underline{\operatorname{Isom}}_T(a, b)$ to be the deformation functor $\operatorname{Def}_{\xi}(-)$ for $\mathcal{X}$. Our exposition follows [SP, Tag 0CYF] and [Hal17, Thm. A]. See also [Art69b,], [Art74b, Thm. 5.3] and [HR19a, Main Thm.] for alternative versions of Artin's Criteria, and [Mur95, Thm. 1] and [Mur64, Thm. 1] for criteria for functors to abelian groups to be representable. □

Remark C.7.6. The converse of the theorem also holds. For the necessity of the conditions, we only need to check (AA$_3$), (AA$_4$), (AA$_6$), and (AA$_7$). Condition (AA$_3$) (finiteness of the tangent spaces) holds as $\mathcal{X}$ is of finite type over $\Bbbk$, and (AA$_4$) (strong homogeneity) holds by [SP, Tag 07WN]. Condition (AA$_7$) (existence of an obstruction theory) follows from the existence of a cotangent complex $\mathrm{L}_{\mathcal{X}/\Bbbk}$ for $\mathcal{X}$

satisfying properties analogous to Theorem C.3.1; see [Ols06]. If $\xi\colon \operatorname{Spec} A \to \mathcal{X}$ is a morphism from a finitely generated $\Bbbk$-algebra A and I is an A-module, then we set $\operatorname{Ob}_\xi(I) := \operatorname{Ext}^1_A(\xi^* L_{\mathcal{X}/\Bbbk}, I)$. Finally, it follows from cohomology and base change (see [Hal14a]) that $\operatorname{Def}_\xi(-)$ and $\operatorname{Ob}_\xi(-)$ commute with products.

C.7.2 Verifying Artin's Axioms

Theorem C.7.7. *Let X be a proper scheme over a field $\Bbbk$.*

(1) The Hilbert functor $\operatorname{Hilb}(X)\colon \mathrm{Sch}/\Bbbk \to \mathrm{Sets}$, whose objects over S are closed subschemes $Z \subseteq X_S$ flat and finitely presented over S, is an algebraic space locally of finite type over $\Bbbk$.

(2) The prestack $\mathcal{M}_g^{\mathrm{all}}$ over $\mathrm{Sch}/\Bbbk$, whose objects over S are proper, flat, and finitely presented morphisms $\mathcal{Y} \to S$ of algebraic spaces with one dimensional fibers, is an algebraic stack locally of finite type over $\Bbbk$.

(3) The prestack $\underline{\mathrm{Coh}}(X)$ over $\mathrm{Sch}/\Bbbk$, whose objects over S are finitely presented quasi-coherent $\mathcal{O}_{X_S}$-modules flat over S, is an algebraic stack locally of finite type over $\Bbbk$.

When X is projective, (1) and (3) are established by more explicit methods in Theorem 1.1.2 and Exercise 3.1.23, while (2) was established in Theorem 5.4.6.

Proof. Fpqc Descent for Quasi-Coherent Sheaves (2.1.4) implies that $\operatorname{Hilb}(X)$ is a sheaf and that $\underline{\mathrm{Coh}}(X)$ is a stack, and Exercise 4.5.15 implies that $\mathcal{M}_g^{\mathrm{all}}$ is a stack. We check the conditions of Theorem C.7.4. Condition (AA_1) (limit preserving) was verified in Exercise C.6.1. Condition (AA_3) and the first part of ($\mathrm{AA}_{2'}$), i.e., the finite dimensionality of $\operatorname{Aut}_\xi(\Bbbk)$ and $\operatorname{Def}_\xi(\Bbbk)$, follow from the identifications with 0 and $\operatorname{Ext}^0_{\mathcal{O}_X}(I_{Z_0}, \mathcal{O}_{Z_0})$ for $\xi = [Z_0 \subseteq X] \in \operatorname{Hilb}(X)(\Bbbk)$ (Proposition C.2.2), with $\operatorname{Ext}^i_{\mathcal{O}_C}(_C, \mathcal{O}_C)$ for $i = 0, 1$ for $\xi = [C] \in \mathcal{M}_g^{\mathrm{all}}(\Bbbk)$ (Theorem C.3.6), and with $\operatorname{Ext}^i_{\mathcal{O}_X}(E, E)$ for $\xi = [E] \in \underline{\mathrm{Coh}}(X)(\Bbbk)$ for $i = 0, 1$ (Proposition C.2.11). Condition (AA_4) (the strong homogeneity condition of (RS_4^*)) was verified in Proposition C.4.14. Condition (AA_5) (effectivity) was checked in Proposition C.5.10 as a consequence of Grothendieck's Existence Theorem. For (AA_7), we define obstruction theories as follows: for a $\Bbbk$-algebra A and an A-module M, we set

- $\operatorname{Ob}_\xi(M) := \operatorname{Ext}^1_{\mathcal{O}_{X_A}}(I_Z, \mathcal{O}_Z \otimes_A M)$ for $\xi = [Z \subseteq X_A] \in \operatorname{Hilb}^P(X)(A)$,
- $\operatorname{Ob}_\xi(M) := \operatorname{Ext}^2_{\mathcal{O}_{\mathcal{C}}}(L_{\mathcal{C}/A}, M) = 0$ for $\xi = [\mathcal{C} \to \operatorname{Spec} A] \in \mathcal{M}_g^{\mathrm{all}}(A)$, and
- $\operatorname{Ob}_\xi(M) := \operatorname{Ext}^2_{\mathcal{O}_{X_A}}(E, E \otimes_A M)$ for $\xi = [E] \in \underline{\mathrm{Coh}}(X)(A)$.

Condition (AA_7)(a)–(b) follow from Proposition C.2.2, Theorem C.3.6, and Proposition C.2.11, which also provides cohomological identifications with $\operatorname{Aut}_\xi(M)$ and $\operatorname{Def}_\xi(M)$. Condition ($\mathrm{AA}_6$), ($\mathrm{AA}_7$)(c), and the second part of ($\mathrm{AA}_{2'}$) ($\operatorname{Aut}_\xi(-)$, $\operatorname{Def}_\xi(-)$, and $\operatorname{Ob}_\xi(-)$ commutes with products) follows from Lemma C.7.8. See also [Art69b, Cor. 6.2], [Lie06, Thm. 2.11], and [SP, Tags 09TU, 0D5A, and 08KA]. □

Lemma C.7.8. *Let $X \to \operatorname{Spec} A$ be a flat proper morphism of schemes. Let E and F be coherent sheaves on X with F flat over A. The functors*

$$\begin{aligned} \mathrm{H}^i(X, F \otimes_A -)&\colon \operatorname{Mod}(A) \to \operatorname{Mod}(A) \\ \operatorname{Ext}^i_{\mathcal{O}_X}(L_{X/A}, -)&\colon \operatorname{Mod}(A) \to \operatorname{Mod}(A) \\ \operatorname{Ext}^i_{\mathcal{O}_X}(E, F \otimes_A -)&\colon \operatorname{Mod}(A) \to \operatorname{Mod}(A) \end{aligned}$$

commute with products.

Proof. Since F is flat over A, there is a perfect complex $K^\bullet$ of A-modules such that $\mathrm{H}^i(X, F \otimes_A -) \cong \mathrm{H}^i(K^\bullet \otimes_A -)$ (Theorem A.6.2). Write $K^d = A^{\oplus r_d}$. For every set of A-modules $\{M_\alpha\}$ we have an identification of complexes

$$\begin{tikzcd}
0 \arrow[r] & \prod_\alpha M_\alpha^{\oplus r_0} \arrow[r] \arrow[d, equal] & \prod_\alpha M_\alpha^{\oplus r_1} \arrow[r] \arrow[d, equal] & \cdots \arrow[r] & \prod_\alpha M_\alpha^{\oplus r_n} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & (\prod_\alpha M_\alpha)^{\oplus r_0} \arrow[r] & (\prod_\alpha M_\alpha)^{\oplus r_1} \arrow[r] & \cdots \arrow[r] & (\prod_\alpha M_\alpha)^{\oplus r_n} \arrow[r] & 0.
\end{tikzcd}$$

The top row is the product of the complexes $K^\bullet \otimes_A M_\alpha$ and its cohomology is identified with $\prod_\alpha \mathrm{H}^i(X, F \otimes_A M_\alpha)$, while the bottom row is $K^\bullet \otimes_A (\prod_\alpha M_\alpha)$ with cohomology groups $\mathrm{H}^i(X, F \otimes_A (\prod_\alpha M_\alpha))$. For the remaining statements, one needs to apply more sophisticated versions of cohomology and base change; see [EGA, III.7.7.5], [SP, Tag 08JR], and [Hal14b, Thm. E]. □

Exercise C.7.9 (Hom stacks, hard)**.** Let $\mathcal{X}$ and $\mathcal{Y}$ be proper Deligne–Mumford stacks over a field $\Bbbk$. Show that the stack $\underline{\mathrm{Mor}}(\mathcal{X}, \mathcal{Y})$ over $(\mathrm{Sch}/\Bbbk)_{\text{ét}}$, whose fiber category over a $\Bbbk$-scheme S is $\mathrm{Mor}(\mathcal{X}_S, \mathcal{Y}_S)$, is an algebraic stack locally of finite type.

Exercise C.7.10 (Weil restriction, hard)**.** Let $\mathcal{S}' \to S$ is be a proper flat morphism of Deligne–Mumford stacks locally of finite type over $\Bbbk$. Let $\mathcal{X}' \to \mathcal{S}'$ be a locally of finite type morphism. Show that the stack $\mathrm{Re}_{\mathcal{S}'/S}(X')$ over $(\mathrm{Sch}/S)_{\text{ét}}$ whose fiber category over an S-scheme T is $\mathcal{X}'(T \times_S \mathcal{S}')$ is representable by an algebraic stack locally of finite type over S. This is a generalization of Exercise 0.3.23.

C.7.3 Counterexamples

We prove examples of non-algebraic sheaves and stacks failing Artin's Axioms.

Example C.7.11 (Automorphism of modules)**.** For a module M over a ring A, consider

$$\underline{\mathrm{Aut}}_A(M)\colon \mathrm{Sch}/A \to \mathrm{Sets}, \quad S \mapsto \mathrm{Aut}_S(M \otimes_A S).$$

If M is flat and finitely presented, then $\mathrm{Aut}_A(M)$ is representable by a group scheme (and in fact a finite type affine group scheme). This fails if M is not flat or not finitely presented. For example, if $A = \Bbbk$ and $V = \bigoplus_{n \in \mathbb{N}} \Bbbk \cdot \langle e_i \rangle$, then $\mathrm{Aut}_\Bbbk(V)$ is not limit preserving: letting $B = \Bbbk[x_1, x_2, \ldots] = \bigcup_n B_n$ with $B_n = \Bbbk[x_1, \ldots, x_n]$, the B-automorphism of $V \otimes_\Bbbk B$, defined by $e_i \mapsto e_i + x_{i+1}e_{i+1}$, is not induced by a B_n-automorphism of $V \otimes_\Bbbk B_n$ for any n, i.e., $\mathrm{colim}_n \mathrm{Aut}_{B_n}(V \otimes_\Bbbk B_n) \to \mathrm{Aut}_B(V \otimes_\Bbbk B)$ is not surjective.

On the other hand, let $A = \Bbbk[x]$ and $M = A/(x) = \Bbbk$. Suppose that $G = \mathrm{Aut}_{\Bbbk[x]}(\Bbbk)$ is representable by an algebraic space. Sections of $G \to \operatorname{Spec} \Bbbk[x]$ correspond to $\mathrm{Aut}_{\Bbbk[x]}(\Bbbk) = \Bbbk^\times$. Two distinct sections must restrict to distinct sections over $\operatorname{Spec} \Bbbk[x]_x$, but this contradicts that G restricts to the trivial group scheme over $\operatorname{Spec} \Bbbk[x]_x$. The sheaf G fails the strong homogeneity condition (RS$_4^*$) of axiom (AA$_4$). In fact, it fails Rim–Schlessinger's Condition (RS$_2$) (or equivalently Schlessinger's Condition (H$_2$)): $G(\Bbbk[\epsilon] \times_\Bbbk \Bbbk[\epsilon)) \to G(\Bbbk[\epsilon]) \times_{G(\Bbbk)} G(\Bbbk[\epsilon])$ is not bijective, where $\Bbbk[\epsilon]$ has the $\Bbbk[x]$-algebra structure via $x \mapsto \epsilon$. Indeed, $G(\Bbbk[\epsilon]) = \mathrm{Aut}_{\Bbbk[\epsilon]}(\Bbbk) = \Bbbk^\times = G(\Bbbk)$, but $G(\Bbbk[\epsilon] \times_\Bbbk \Bbbk[\epsilon]) = \big(\Bbbk[\epsilon] \times_\Bbbk \Bbbk[\epsilon]/(\epsilon, \epsilon)\big)^\times \cong \Bbbk[\epsilon]^\times \cong \Bbbk^\times \times \Bbbk$.

Example C.7.12 (Stacks of quasi-coherent sheaves / non-flat coherent sheaves)**.** If X is a proper scheme over a field $\Bbbk$, the prestack $\underline{\mathrm{QCoh}}(X)$, whose objects over S are quasi-coherent sheaves F flat over S, is not algebraic. The previous example shows that $\underline{\mathrm{QCoh}}(X)$ is not limit preserving (because the requisite functor is not even fully faithful) and that the diagonal is not representable, i.e., both (AA$_1$) and (AA$_2$) fail. Similarly, the stack of finitely presented (but not necessarily flat) quasi-coherent sheaves is not algebraic nor limit preserving. By the previous example, (AA$_2$) and the fully faithfulness of (AA$_4$) both fail.

Example C.7.13 (Automorphisms of schemes)**.** If X is a scheme over $\Bbbk$, consider

$$\underline{\mathrm{Aut}}(X)\colon \mathrm{Sch}/\Bbbk \to \mathrm{Sets}, \quad S \mapsto \mathrm{Aut}_S(X_S).$$

If X is proper, this is representable by an algebraic space. Without properness, this may fail. For example, $\underline{\mathrm{Aut}}(\mathbb{A}^1)$ is not representable and fails (AA$_3$): if $\xi = \{\mathrm{id}\} \in \underline{\mathrm{Aut}}(\mathbb{A}^1)(\Bbbk)$, then by Proposition C.2.4, there is an identification of the tangent space $\mathrm{Def}_\xi(\Bbbk)$ with $\mathrm{H}^0(\mathbb{A}^1, T_{\mathbb{A}^1}) = \Bbbk[x]$.

Example C.7.14 (Stack of all algebraic spaces)**.** Let $\mathcal{X}$ be the prestack over $\mathrm{Sch}/\Bbbk$, whose objects over a $\Bbbk$-scheme S is a morphism $X \to S$ of algebraic spaces, and where a morphism $(X \to S) \to (X' \to S')$ is a cartesian diagram. This is a stack over $(\mathrm{Sch}/\Bbbk)_{\text{ét}}$, but it is not limit preserving and the diagonal $\mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is not representable, i.e., both (AA$_1$) and (AA$_2$) fail. It also fails (AA$_{2'}$) as $\mathrm{Aut}_\xi(\Bbbk)$ may be infinite dimensional, as we saw in the previous example.

Example C.7.15 (Stack of K3 surfaces)**.** The moduli stack $\mathcal{K}_3$ over $(\mathrm{Sch}/\Bbbk)_{\text{ét}}$, whose objects over a $\Bbbk$-scheme S are smooth and proper morphisms $\mathcal{X} \to S$ of algebraic spaces such that every fiber is a K3 surface, is not algebraic. It fails the effectivity axiom (AA$_5$); see Remark C.5.9.

We now describe Artin's counterexamples from [Art69c]. In each case, the setup is an inductive system of affine schemes

$$X_1 \to X_2 \to X_3 \to \cdots \qquad \text{with } X_q = \operatorname{Spec} A_q$$

and we consider the functor

$$\operatorname{colim} X_q\colon \mathrm{AffSch}/\Bbbk \to \mathrm{Sets}, \quad \operatorname{Spec} R \mapsto \operatorname{colim} \mathrm{Mor}_\Bbbk(\operatorname{Spec} R, X_i),$$

where an element of $F(R)$ is an equivalence class of a pair $(q, \phi\colon A_q \to R)$ of an positive integer and an $\Bbbk$-algebra homomorphism, where $(q, \phi) \sim (q', \phi')$ if there exists $Q \geq q, q'$ such that $A_Q \to A_q \xrightarrow{\phi} R$ and $A_Q \to A_{q'} \xrightarrow{\phi'} R$ agree.

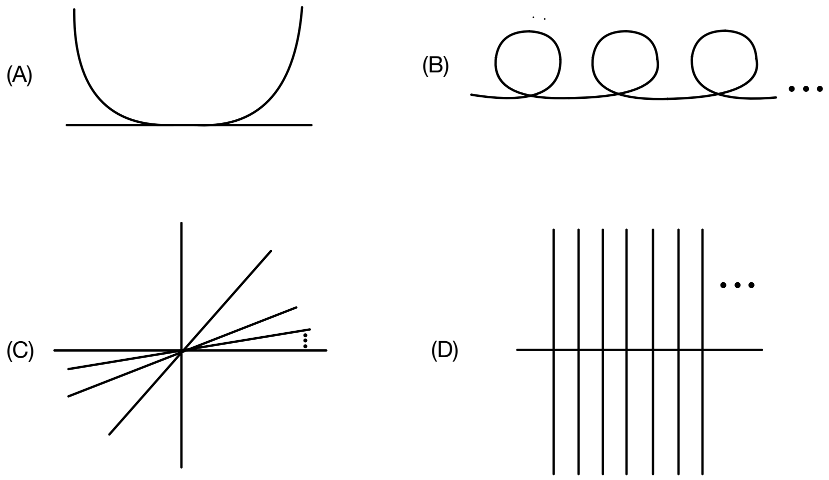

Figure 3.7.16: Counterexamples to Artin's Axioms

Example C.7.17 (Infinitely tangential curves)**.** For Figure 3.7.16(A), let $X_q = \operatorname{Spec} A_q$, where $A_q = \Bbbk[x,y]/(y(y-x^q))$ and $A_{q+1} \to A_q$ is defined by $(x,y) \mapsto (x, xy)$. In $F(\Bbbk[t]/t^{n+1})$, consider the elements $\alpha_n : A_q \to \Bbbk[t]/t^{n+1}$ (for any q), defined by $(x,y) \mapsto (t,0)$, and $\beta_n : A_q \to \Bbbk[t]/t^{n+1}$ (for any q), defined by $(x,y) \mapsto (t,t^q)$. For each n, $\alpha_n \sim \beta_n$ as we can take $q \geq n+1$. Then $\{\alpha_n\}$ is a versal formal deformation that effectivizes to two distinct elements $\widehat{\alpha}, \widehat{\beta} \in F(\Bbbk[\![t]\!])$ defined on A_q (for any q) by $\widehat{\alpha}(x,y) \mapsto (t,0)$ and $\widehat{\beta}(x,y) \mapsto (t,t^q)$. In this way, we see that the functor $F(\Bbbk[\![t]\!]) \to \varprojlim F(\Bbbk[t]/t^q)$ is not injective, so that the effectivity axiom (AA$_5$) fails. In this case, C.7.1(5) (openness of versality) also fails. Likewise, the diagonal on F is not representable: the formal deformation of $\underline{\mathrm{Isom}}_{\Bbbk[\![t]\!]}(\widehat{\alpha}, \widehat{\beta})$ given by $\{\alpha_n \xrightarrow{\sim} \beta_n\}$ is not effective.

Example C.7.18 (Infinitely many nodes)**.** For Figure 3.7.16(B), let X_q be the affine scheme over $\mathbb{C}$ with q nodes obtained from $\mathbb{A}^1$ by nodally-identifying q pairs of points. Then $\mathbb{A}^1 \to \operatorname{colim} X_q$ is formally versal at any point that doesn't map to a node, but this is not formally versal in any open neighborhood of $\mathbb{A}^1$ as one must remove infinitely many points. Thus C.7.1(5) (openness of versality) fails.

Similarly, for Figure 3.7.16(C), if $X_q = \operatorname{Spec} \Bbbk[x,y]/(y \prod_{i=1}^{1}(x-q))$, then the inclusion $\mathbb{A}^1 \to \operatorname{colim} X_q$ is formally versal at any non-nodal point, but is not formally versal in an open neighborhood.

Example C.7.19 (Infinitely many lines)**.** For Figure 3.7.16(D), let $X_q = \operatorname{Spec} \Bbbk[x,y]/\prod_{i=1}^{q}(iy-x)$ be the union of q lines in the plane, and $X_q \hookrightarrow X_{q+1}$ be the induced closed immersions. Then $\operatorname{colim} X_q$ satisfyies Schlessinger's Axioms (H$_1$)–(H$_4$) (or equivalent the Rim–Schlessinger Axioms (RS$_1$)–(RS$_4$)). Thus Schlessinger's Theorem applies to produce a formal versal deformation $\{x_n\}$, where $x_n \in (\operatorname{colim} X_q)(\Bbbk[x,y]/(x,y)^{n+1})$ is defined by the closed immersion $\operatorname{Spec} \Bbbk[x,y]/(x,y)^{n+1} \hookrightarrow X_{n+1}$. The effectivity axiom (AA$_5$) fails as there is no element of $(\operatorname{colim} X_q)(\Bbbk[\![x,y]\!])$ extending $\{x_n\}$.

Example C.7.20 (Hom stacks)**.** The Hom stack $\underline{\mathrm{Mor}}(\mathcal{X}, \mathcal{Y})$ over $(\mathrm{Sch}/\Bbbk)_{\text{ét}}$ is algebraic if $\mathcal{X}$ and $\mathcal{Y}$ are proper over $\Bbbk$ (Exercise C.7.9). This holds more generally over an arbitrary base if $\mathcal{X} \to S$ is proper and flat, and if $\mathcal{Y} \to S$ is only assumed to be locally of finite presentation, quasi-separated, and with *affine stabilizer groups*;

see [HR19b, Thm. 1.2] and [BHL17, Cor. 1.6]. In these settings, a version of Tannaka Duality (6.6.1) holds, i.e., $\mathrm{Mor}(\mathcal{X}, \mathcal{Y}) \xrightarrow{\sim} \mathrm{Mor}^{\otimes}(\mathrm{Coh}(\mathcal{Y}), \mathrm{Coh}(\mathcal{X}))$, and this reduces the effectivity axiom (AA_5) to a version of Grothendieck's Existence Theorem.

It is essential however that $\mathcal{Y}$ have affine stabilizers. If $\mathcal{Y}$ is the classifying stack of an abelian variety, then Tannaka Duality may not hold and the Hom stack $\underline{\mathrm{Mor}}(\mathcal{X}, \mathcal{Y})$ may fail to be algebraic; see [SP, Tag 0AF8] and [HR19b, §10].

Exercise C.7.21 (Sheafification of the functor of smooth curves)**.** Let F be the sheafification in $(\mathrm{Sch}/\Bbbk)_{\text{ét}}$ of the functor assigning S to the set $\mathcal{M}_g(S)/\sim$ of isomorphism classes of families of smooth curves over S. Which of Artin Axioms fails for F?

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