Stacks and Moduli

working draft

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January 31, 2023

Abstract

These notes provide the foundations of moduli theory in algebraic geometry with the goal of providing self-contained proofs of the following theorems:

Theorem A. *The moduli space* $\overline{\mathcal{M}}_g$ *of stable curves of genus* $g \geq 2$ *is a smooth, proper and irreducible Deligne–Mumford stack of dimension* $3g - 3$ *which admits a projective coarse moduli space.*

Theorem B. *The moduli space* $\mathcal{B}\mathrm{un}^{\mathrm{ss}}_{r,d}(C)$ *of semistable vector bundles of rank* r *and degree* d *over a smooth, connected, and projective curve* C *of genus* g *is a smooth, universally closed and irreducible algebraic stack of dimension* $r^2(g-1)$ *which admits a projective good moduli space.*

Along the way we develop the foundations of algebraic spaces and stacks, which provide a convenient language to discuss and establish geometric properties of moduli spaces. Introducing these foundations requires developing several themes at the same time including:

- using the functorial and groupoid perspective in algebraic geometry: we will introduce the new algebro-geometric structures of algebraic spaces and stacks,

- replacing the Zariski topology on a scheme with the étale topology: we will introduce Grothendieck topologies proving a generalization of topological spaces, and we will systematically use descent theory for étale morphisms, and

- relying on several advanced topics not typically seen in a first algebraic geometry course: properties of flat, étale and smooth morphisms of schemes, algebraic groups and their actions, deformation theory, and the birational geometry of surfaces.

Choosing a linear order in presenting the foundations is no easy task. We attempt to mitigate this challenge by relegating much of the background to appendices. We keep the main body of the notes always focused on developing moduli theory with the above two theorems in mind.

Contents

Chapter 0

Introduction and motivation

A *moduli space* is a space M (e.g. topological space, complex manifold or algebraic variety) where there is a natural one-to-one correspondence between points of M and isomorphism classes of certain types of algebro-geometric objects (e.g. smooth curves or vector bundles on a fixed curve). While every space M is the moduli space parameterizing points of M, it is much more interesting when alternative descriptions can be provided. For instance, projective space $\mathbb{P}^1$ can be described as the set of points in $\mathbb{P}^1$ (not so interesting) or as the set of lines in the plane passing through the origin (more interesting).

Moduli spaces arise as an attempt to answer one of the most fundamental problems in mathematics, namely the classification problem. In algebraic geometry, we may wish to classify all projective varieties, all vector bundles on a fixed variety or any number of other structures. The moduli space itself is the solution to the classification problem.

Depending on what objects are being parameterized, the moduli space could be discrete or continuous, or a combination of the two. For instance, the moduli space parameterizing line bundles on $\mathbb{P}^1$ is the discrete set $\mathbb{Z}$: every line bundle on $\mathbb{P}^1$ is isomorphic to $\mathcal{O}(n)$ for a unique integer $n \in \mathbb{Z}$. On the other hand, the moduli space parameterizing quadric plane curves $C \subset \mathbb{P}^2$ is the connected space $\mathbb{P}^5$: a plane curve defined by $a_0x^2 + a_1xy + a_2xz + a_3y^2 + a_4yz + a_5z^2$ is uniquely determined by the point $[a_0, \ldots, a_5] \in \mathbb{P}^5$, and as a plane curve varies continuously (i.e. by varying the coefficients a_i), the corresponding point in $\mathbb{P}^5$ does too.

The moduli space parameterizing smooth projective abstract curves has both a discrete and continuous component. While the genus of a smooth curve is a discrete invariant, smooth curves of a fixed genus vary continuously. For instance, varying the coefficients of a homogeneous degree d polynomial in x, y, z describes a continuous family of mostly non-isomorphic curves of genus $(d-1)(d-2)/2$. After fixing the genus $g \geq 2$, the moduli space M_g parameterizing genus g curves is a connected (even irreducible) variety of dimension $3g-3$, a deep fact providing the underlying motivation of these notes. Similarly, the moduli space of vector bundles on a fixed curve has a discrete component corresponding to the rank r and degree d of the vector bundle, and it turns out that after fixing these invariants, the moduli space is also irreducible.

An inspiring feature of moduli spaces and one reason they garner so much attention is that their properties inform us about the properties of the objects themselves that are being classified. For instance, knowing that M_g is unirational (i.e. there is a dominant rational map $\mathbb{P}^N \dashrightarrow M_g$) for a given genus g tells us that a general genus g curve can be written down explicitly in a similar way to how a general genus 3 curve can be expressed as the solution set to a plane quartic whose coefficients are general complex numbers.

Before we can get started discussing the geometry of moduli spaces such as M_g, we need to ask: why do they even exist? We develop the foundations of moduli theory with this single question in mind. Our goal is to establish the truly spectacular result that there is a projective variety whose points are in natural one-to-one correspondence with isomorphism classes of curves (or vector bundles on a fixed curve). In this chapter, we motivate our approach for constructing projective moduli spaces through the language of algebraic stacks.

0.1 Moduli sets

A *moduli set* is a set where elements correspond to isomorphism classes of certain types of algebraic, geometric or topological objects. To be more explicit, defining a moduli set entails specifying two things:

1. a class of certain types of objects, and
2. an equivalence relation on objects.

The word 'moduli' indicates that we are viewing an element of the set as an equivalence class of certain objects. In the same vein, we will discuss *moduli groupoids*, *moduli varieties/schemes* and *moduli stacks* in the forthcoming sections. Meanwhile, the word 'object' here is intentionally vague as the possibilities are quite broad: one may wish to discuss the moduli of really any type of mathematical structure, e.g. complex structures on a fixed space, flat connections, quiver representations, solutions to PDEs, or instantons. In these notes, we will entirely focus our study on moduli problems appearing in algebraic geometry although many of the ideas we present extend similarly to other branches of mathematics.

The two central examples in these notes are the moduli of curves and the moduli of vector bundles on a fixed curve—two of the most famous and studied moduli spaces in algebraic geometry. While there are simpler examples such as projective space and the Grassmanian that we will study first, the moduli spaces of curves and vector bundles are both complicated enough to reveal many general phenomena of moduli and simple enough that we can provide a self-contained exposition. Certainly, before you hope to study moduli of higher dimensional varieties or moduli of complexes on a surface, you better have mastered these examples.

0.1.1 Moduli of curves

Here's our first attempt at defining M_g:

Example 0.1.1 (Moduli set of smooth curves)**.** The *moduli set of smooth curves*, denoted as M_g, is defined as followed: the objects are smooth, connected, and projective curves of genus g over $\mathbb{C}$ and the equivalence relation is given by isomorphism.

There are alternative descriptions. We could take the objects to be complex structures on a fixed oriented compact surface Σ of genus g and the equivalence relation to be biholomorphism. Or we could take the objects to be pairs (X, ϕ)

where X is a hyberbolic surface and $\phi\colon \Sigma \to X$ is a diffeomorphism (the set of such pairs is the Teichmüller space) and the equivalence relation is isotopy (induced from the action of the mapping class group of Σ).

Each description hints at different additional structures that M_g should inherit.

There are many related examples parameterizing curves with additional structures as well as different choices for the equivalences relations.

Example 0.1.2 (Moduli set of plane curves)**.** The objects here are degree d plane curves $C \subset \mathbb{P}^2$ but there are several choices for how we could define two plane curves C and C' to be equivalent:

(1) C and C' are equal as subschemes;

(2) C and C' are projectively equivalent (i.e. there is an automorphism of $\mathbb{P}^2$ taking C to C'); or

(3) C and C' are abstractly isomorphic.

The three equivalence relations define three different moduli sets. The moduli set (1) is naturally bijective to the projectivization $\mathbb{P}(\operatorname{Sym}^d \mathbb{C}^3)$ of the space of degree d homogeneous polynomials in x, y, z while the moduli set (2) is naturally bijective to the quotient set $\mathbb{P}(\operatorname{Sym}^d \mathbb{C}^3)/\operatorname{Aut}(\mathbb{P}^2)$. The moduli set (3) is the subset of the moduli set of (possibly singular) abstract curves which admit planar embeddings.

Example 0.1.3 (Moduli set of curves with level n structure)**.** The objects are smooth, connected, and projective curves C of genus g over $\mathbb{C}$ together with a basis $(\alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g)$ of $H_1(C, \mathbb{Z}/n\mathbb{Z})$ such that the intersection pairing is symplectic. We say that $(C, \alpha_i, \beta_i) \sim (C', \alpha_i', \beta_i')$ if there is an isomorphism $C \to C'$ taking α_i and β_i to α_i' and β_i'.

A rational function f/g on a curve C defines a map $C \to \mathbb{P}^1$ given by $x \mapsto [f(x), g(x)]$. Visualizing a curve as a cover of $\mathbb{P}^1$ is extremely instructive providing a handle to its geometry. Likewise it is instructive to consider the moduli of such covers.

Example 0.1.4 (Moduli of branched covers)**.** We define the *Hurwitz moduli set* $\operatorname{Hur}_{d,g}$ where an object is a smooth, connected, and projective curve of genus g together with a finite morphisms $f\colon C \to \mathbb{P}^1$ of degree d, and we declare $(C \xrightarrow{f} \mathbb{P}^1) \sim (C' \xrightarrow{f'} \mathbb{P}^1)$ if there is an isomorphism $\alpha\colon C \to C'$ over $\mathbb{P}^1$ (i.e. $f' = f \circ \alpha$). By Riemann–Hurwitz, every such map $C \to \mathbb{P}^1$ has $2d + 2g - 2$ branch points. Conversely, given a general collection of $2d + 2g - 2$ points of $\mathbb{P}^1$, there exist a genus g curve C and a map $C \to \mathbb{P}^1$ branched over precisely these points. In fact there are only finitely many such covers $C \to \mathbb{P}^1$ as every cover is uniquely determined by the ramification type over the branched points and the finite number of permutations specifying how the unramified covering over the complement of the branched locus is obtained by gluing trivial coverings. In other words, the map $\operatorname{Hur}_{d,g} \to \operatorname{Sym}^{2d+2g-2} \mathbb{P}^1$, assigning a cover to its branched points, has dense image and finite fibers.

Likewise, for a fixed curve C, we could consider the moduli set $\operatorname{Hur}_{d,C}$ parameterizing degree d covers $C \to \mathbb{P}^1$ where the equivalence relation is equality. There is a map $\operatorname{Hur}_{d,g} \to M_g$ defined by $(C \to \mathbb{P}^1) \mapsto C$, and the fiber over a curve C is precisely $\operatorname{Hur}_{d,C}$. Equivalently, $\operatorname{Hur}_{d,C}$ can be described as parameterizing line bundles L on C together with linearly independent sections s_1, s_2 where $(L, s_1, s_2) \sim (L', s_1', s_2')$ if there exists an isomorphism $\alpha\colon L \to L'$ such that $s_i' = \alpha(s_i)$.

Application: number of moduli of M_g

Even before we attempt to give M_g the structure of a variety, we can use a parameter count to determine the *number of moduli of* M_g or in modern terminology the *dimension of the local deformation spaces* for $g \geq 2$. In other words, we can compute the number of moduli before we can even make sense of the dimension of M_g as a variety. Historically Riemann computed the number of moduli in the mid 19th century (in fact using several different methods) well before it was known that M_g is a variety. Following [Rie57], the main idea is to compute the number of moduli of $\mathrm{Hur}_{d,g}$ in two different ways using the diagram

$$\begin{tikzcd} & \mathrm{Hur}_{d,C} \arrow[r, hook] \arrow[dl] & \mathrm{Hur}_{d,g} \arrow[dl] \arrow[drr, "\text{finite fibers}"] & & \\ \{C\} \arrow[r, hook] & M_g & & & \mathrm{Sym}^{2d+2g-2}\mathbb{P}^1 \end{tikzcd} \tag{0.1.1}$$

We first compute the number of moduli of $\mathrm{Hur}_{d,C}$ and we might as well assume that d is sufficiently large (or explicitly $d > 2g$). For a fixed curve C, a degree d map $f : C \to \mathbb{P}^1$ is determined by an effective divisor $D := f^{-1}(0) = \sum_i p_i \in \mathrm{Sym}^d C$ and a section $t \in \mathrm{H}^0(C, \mathcal{O}(D))$ (so that $f(p) = [s(p), t(p)]$ where $s \in \Gamma(C, \mathcal{O}(D))$ defines D). Using that $\mathrm{H}^1(C, \mathcal{O}(D)) = \mathrm{H}^0(C, \mathcal{O}(K_C - D)) = 0$, Riemann–Roch implies that $\mathrm{h}^0(\mathcal{O}(D)) = d - g + 1$. Thus the number of moduli of $\mathrm{Hur}_{d,C}$ is the sum of the number of parameters determining D and the section t

$$\#\text{ of moduli of } \mathrm{Hur}_{d,C} = d + (d - g + 1) = 2d - g + 1.$$

Using (0.1.1), we compute that

$$\begin{aligned} \#\text{ of moduli of } M_g &= \#\text{ of moduli of } \mathrm{Hur}_{d,g} - \#\text{ of moduli of } \mathrm{Hur}_{d,C} \\ &= \#\text{ of moduli of } \mathrm{Sym}^{2d+2g-2}\mathbb{P}^1 - \#\text{ of moduli of } \mathrm{Hur}_{d,C} \\ &= (2d + 2g - 2) - (2d - g + 1) \\ &= 3g - 3. \end{aligned}$$

One goal of these notes is to put this calculation on a more solid footing. The interested reader may wish to consult [GH78, pg. 255-257] or [Mir95, pg. 211-215] for further discussion on the number of moduli of M_g, or [AJP16] for a historical background of Riemann's computations.

0.1.2 Moduli of vector bundles

The moduli of vector bundles on a fixed curve provides our second primary example of a moduli set:

Example 0.1.5 (Moduli set of vector bundles on a curve)**.** Let C be a fixed smooth, connected, and projective curve over $\mathbb{C}$, and fix integers $r \geq 0$ and d. The objects of interest are vector bundles E (i.e. locally free $\mathcal{O}_C$-modules of finite rank) of rank r and degree d, and the equivalence relation is isomorphism.

There are alternative descriptions. If V is a fixed C^∞-vector bundle V on C, we can take the objects to be connections on V and the equivalence relation to be gauge equivalence. Or we can take the objects to be representations $\pi_1(C) \to \mathrm{GL}_n(\mathbb{C})$ of the fundamental group $\pi_1(C)$ and declare two representations to be equivalent if they have the same dimension n and are conjugate under an element of $\mathrm{GL}_n(\mathbb{C})$. This last description uses the observation that a vector bundle induces a monodromy representation of $\pi_1(C)$ and conversely that a representation V of $\pi_1(C)$ induces a vector bundle $(\widetilde{C} \times V)/\pi_1(C)$ on C, where $\widetilde{C}$ denotes the universal cover of C.

Specializing to the rank one case is a model for the general case: the moduli set $\mathrm{Pic}^d(C)$ of line bundles on C of degree d is identified (non-canonically) with the abelian variety $\mathrm{H}^1(C, \mathcal{O}_C)/\mathrm{H}^1(C, \mathbb{Z})$ by means of the cohomology of the exponential exact sequence

$$\begin{array}{ccccccccc} \mathrm{H}^1(C,\mathbb{Z}) & \longrightarrow & \mathrm{H}^1(C,\mathcal{O}_C) & \longrightarrow & \mathrm{Pic}(C) & \longrightarrow & \mathrm{H}^2(C,\mathbb{Z}) & \longrightarrow & 0 \\ & & & & L & \mapsto & \deg(L) & & \end{array}$$

There is a group structure on $\mathrm{Pic}^0(C)$ corresponding to the tensor product of line bundles.

Example 0.1.6 (Moduli of vector bundles on $\mathbb{P}^1$)**.** Since all vector bundles on $\mathbb{A}^1$ are trivial, a vector bundle of rank n on $\mathbb{P}^1$ is described by an element of $\mathrm{GL}_n(k[x]_x)$ specifying how trivial vector bundles on $\{x \neq 0\}$ and $\{y \neq 0\}$ are glued. We can thus describe this moduli set by taking the objects to be elements of $\mathrm{GL}_n(k[x]_x)$ where two elements g and g' are declared equivalent if there exists $\alpha \in \mathrm{GL}_n(k[x])$ and $\beta \in \mathrm{GL}_n(k[1/x])$ (i.e. automorphisms of the trivial vector bundles on $\{x \neq 0\}$ and $\{y \neq 0\}$) such that $g' = \alpha g \beta$.

The Birkhoff–Grothendieck theorem asserts that every vector bundle E on $\mathbb{P}^1$ is isomorphic to $\mathcal{O}(a_1) \oplus \cdots \oplus \mathcal{O}(a_r)$ for unique integers $a_1 \leq \cdots \leq a_r$.[1] This implies that the moduli set of degree d vector bundles of rank r on $\mathbb{P}^1$ is bijective to the set of increasing tuples $(a_1, \ldots, a_r) \in \mathbb{Z}^r$ of integers with $\sum_i a_i = d$. One would be mistaken though to think that the moduli space of vector bundles on $\mathbb{P}^1$ with fixed rank and degree is discrete. For instance, if $d = 0$ and $r = 2$, the group of extensions

$$\mathrm{Ext}^1_{\mathcal{O}_{\mathbb{P}^1}}(\mathcal{O}_{\mathbb{P}^1}(1), \mathcal{O}_{\mathbb{P}^1}(-1)) = \mathrm{H}^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2)) = \mathrm{H}^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) = \mathbb{C}$$

is one-dimensional and the universal extension (see Example 0.4.26) is a vector bundle $\mathcal{E}$ on $\mathbb{P}^1 \times \mathbb{A}^1$ such that $\mathcal{E}|_{\mathbb{P}^1 \times \{t\}}$ is the non-trivial extension $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}$ for $t \neq 0$ and the trivial extension $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(1)$ for $t = 0$. This shows that $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}$ and $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(1)$ should be in the same connected component of the moduli space.

0.1.3 Why just sets?

It is indeed a bit silly to define these moduli spaces as sets. After all, any two complex projective varieties are bijective so we should be demanding a lot more structure than a variety whose points are in bijective correspondence with isomorphism classes. However, spelling out what properties we desire of the moduli

[1] Birkhoff proved this in 1909 using linear algebra by explicitly showing that an element $\mathrm{GL}_n(k[x]_x)$ can be multiplied on the left and right by elements of $\mathrm{GL}_n(k[x])$ and $\mathrm{GL}_n(k[1/x])$ to be a diagonal matrix $\mathrm{diag}(x^{a_1}, \ldots, x^{a_r})$ [Bir09] while Grothendieck proved this in 1957 via induction and cohomology by exhibiting a line subbundle $\mathcal{O}(a) \subset E$ such that the corresponding short exact sequence splits [Gro57a].

space is by no means easy. What we would really like is a quasi-projective variety M_g with a universal family $U_g \to M_g$ such that the fiber of a point $[C] \in M_g$ is precisely that curve. This is where the difficulty lies—automorphisms of curves obstruct the existence of such a family—and this is the main reason we want to expand our notion of a geometric space from schemes to algebraic stacks. Algebraic stacks provide a nice approach ensuring the existence of a universal family but it is by no means the only approach.

Historically, it was not clear what structure M_g should have. Riemann introduced the word 'Mannigfaltigkeiten' (or 'manifoldness') but did not specify what this means–complex manifolds were only introduced in the 1940s following Teichmüller, Chern and Weil. The first claim that M_g exists as an algebraic variety was perhaps due to Weil in [Wei58]: "As for M_g there is virtually no doubt that it can be provided with the structure of an algebraic variety." Grothendieck, aware that the functor of smooth families of curves was not representable, studied the functor of smooth families of curves with level structure $r \geq 3$ [Gro61]. While he could show representability, he struggled to show quasi-projectivity. It was only later that Mumford proved that M_g is a quasi-projective variety, an accomplishment for which he was awarded the Field Medal in 1974, by introducing and then applying Geometric Invariant Theory (GIT) to construct M_g as a quotient [GIT]. For further historical background, we recommend [JP13], [AJP16] and [Kol18].

In these notes, we take a similar approach to Mumford's original construction and integrate later influential results due to Deligne, Kollár, Mumford and others such as the seminal paper [DM69] which simultaneously introduced stable curves and stacks with the application of irreducibility of M_g in every characteristic. In this chapter, we motivate our approach by gradually building in additional structure: first as a groupoid (Section 0.3), then as a presheaf (i.e. contravariant functor) (Section 0.4), then as a stack (Section 0.6) and then ultimately as a projective variety (Section 0.8).

One of the challenges of learning moduli stacks is that it requires simultaneously extending the theory of schemes in several orthogonal directions including:

(1) the functorial approach: thinking of a scheme X not as topological space with a sheaf of rings but rather in terms of the functor Sch $\to$ Sets defined by $T \mapsto \mathrm{Mor}(T, X)$. For moduli problems, this means specifying not just objects but families of objects; and
(2) the groupoid approach: rather than specifying just the points we also specify their symmetries. For moduli problems, this means specifying not just the objects but their automorphism groups.

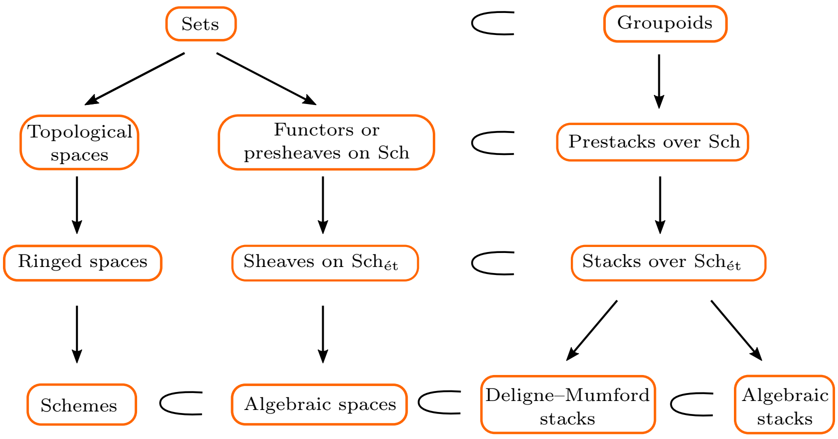


Figure 1: Schematic diagram featuring algebro-geometric enrichments of sets and groupoids where arrows indicate additional geometric conditions.

0.2 Toy example: moduli of triangles

Before we dive deeper into the moduli of curves or vector bundles, we will study the simple yet surprisingly fruitful example of the moduli of triangles which is easy both to visualize and construct. In fact, we present several variants of the moduli of triangles that highlight various concepts in moduli theory. The moduli spaces of labelled triangles and labelled triangles up to similarity have natural functorial descriptions and universal families while the moduli space of *unlabelled* triangles does not admit a universal family due to the presence of symmetries—in exploring this example, we are led to the concept of a moduli groupoid and ultimately to moduli stacks. Michael Artin is attributed to remarking that you can understand most concepts in moduli through the moduli space of triangles.

0.2.1 Labelled triangles

A *labelled triangle* is a triangle in $\mathbb{R}^2$ where the vertices are labelled with '1', '2' and '3', and the distances of the edges are denoted as a, b, and c. We require that triangles have non-zero area or equivalently that their vertices are not colinear.

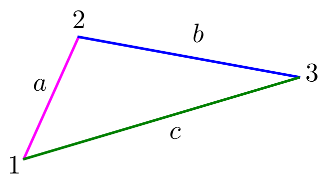


Figure 2: To keep track of the labelling, we color the edges as above.

We define the *moduli set of labelled triangles* M as the set of labelled triangles where two triangles are said to be equivalent if they are the same triangle in $\mathbb{R}^2$ with the same vertices and same labeling. By writing (x_1, y_1), (x_2, y_2) and (x_3, y_3)

as the coordinates of the labelled vertices, we obtain a bijection

$$M \cong \left\{(x_1, y_1, x_2, y_2, x_3, y_3) \mid \det \begin{pmatrix} x_2 - x_1 & x_3 - x_1 \\ y_2 - y_1 & y_3 - y_1 \end{pmatrix} \neq 0\right\} \subset \mathbb{R}^6 \tag{0.2.1}$$

with the open subset of $\mathbb{R}^6$ whose complement is the codimension 1 closed subset defined by the condition that the vectors $(x_2, y_2) - (x_1, y_1)$ and $(x_3, y_3) - (x_1, y_1)$ are linearly dependent.

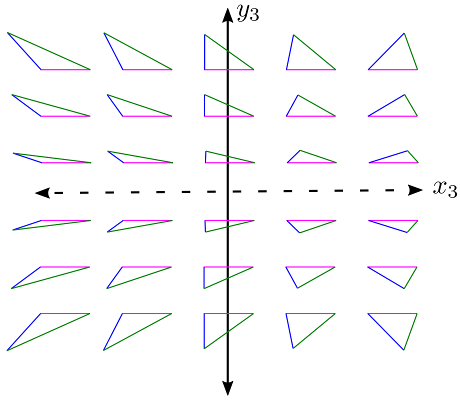


Figure 3: Picture of the slice of the moduli space M where $(x_1, y_1) = (0, 0)$ and $(x_2, y_2) = (1, 0)$. Triangles are described by their third vertex (x_3, y_3) with $y_3 \neq 0$. We've drawn representative triangles for a handful of points in the x_3y_3- plane.

0.2.2 Labelled triangles up to similarity

We define the *moduli set of labelled triangles up to similarity*, denoted by M^{lab}, by taking the same class of objects as in the previous example—labelled triangles—but changing the equivalence relation to label-preserving similarity.

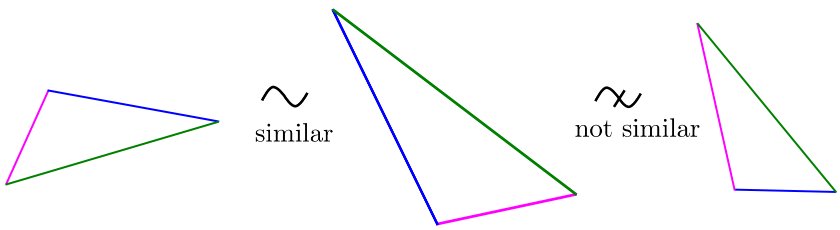


Figure 4: The two triangles on the left are similar, but the third is not.

Every labelled triangle is similar to a unique labelled triangle with perimeter $a + b + c = 2$. We have the description

$$M^{\text{lab}} = \left\{ (a, b, c) \;\middle|\; \begin{array}{c} a + b + c = 2 \\ 0 < a < b + c \\ 0 < b < a + c \\ 0 < c < a + b \end{array} \right\}. \tag{0.2.2}$$

By setting $c = 2 - a - b$, we may visualize M^{lab} as the analytic open subset of $\mathbb{R}^2$ defined by pairs (a, b) satisfying $0 < a, b < 1$ and $a + b > 1$.

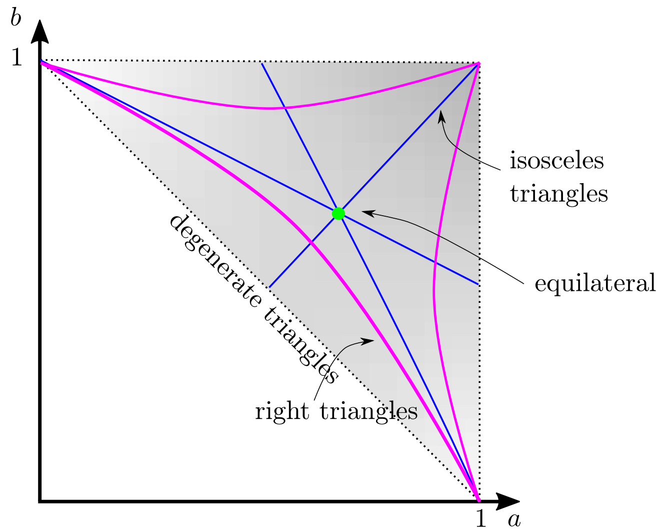


Figure 5: M^{lab} is the shaded area above. The pink lines represent the right triangles defined by $a^2 + b^2 = c^2$, $a^2 + c^2 = b^2$ and $b^2 + c^2 = a^2$, the blue lines represent isosceles triangles defined by $a = b$, $b = c$ and $a = c$, and the green point is the unique equilateral triangle defined by $a = b = c$.

0.2.3 Unlabelled triangles up to similarity

We now turn to the moduli of unlabelled triangles up to similarity, which reveals a new feature not seen in to the two above examples: symmetry!

We define the *moduli set of unlabelled triangles up to similarity*, denoted by M^{unl}, where the objects are unlabelled triangles in $\mathbb{R}^2$ and the equivalence relation is symmetry. We can describe a unlabelled triangle uniquely by the ordered tuple (a, b, c) of increasing side lengths as follows:

$$M^{\text{unl}} = \left\{ (a,b,c) \;\middle|\; \begin{array}{l} 0 < a \leq b \leq c < a + b \\ a + b + c = 2 \end{array} \right\}. \tag{0.2.3}$$

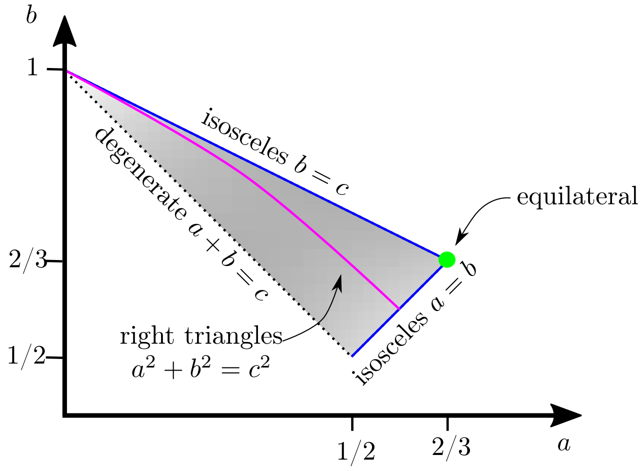


Figure 6: Picture of M^{unl} where $c = 2 - a - b$.

The isosceles triangles with $a = b$ or $b = c$ and the equilateral triangle with $a = b = c$ have symmetry groups of $\mathbb{Z}/2$ and S_3, respectively. This is unfortunately not encoded into our description M^{unl} above. However, we can identify M^{unl} as the quotient M^{lab}/S_3 of the moduli set of labelled triangles up to similarity modulo the natural action of S_3 on the labellings. Under this action, the stabilizers of isosceles and equilateral triangles are precisely their symmetry groups $\mathbb{Z}/2$ and S_3. The action of S_3 on the complement of the set of isosceles and equilateral triangles is free.

0.3 Moduli groupoids

We now change our perspective: rather than specifying *when* two objects are identified, we specify *how*! One of the most desirable properties of a moduli space is the existence of a universal family (see §0.4.6) and the presence of automorphisms obstructs its existence (see §0.4.7). Encoding automorphisms into our descriptions will allow us to get around this problem. A convenient mathematical structure to encode this information is a *groupoid*.

Definition 0.3.1. A *groupoid* is a category $\mathcal{C}$ where every morphism is an isomorphism.

0.3.1 Specifying a moduli groupoid

A *moduli groupoid* is described by

1. a class of certain algebraic, geometric or topological objects; and
2. a set of equivalences between two objects.

where (1) describes the objects and (2) the morphisms of a groupoid. In particular, the moduli groupoid encodes $\mathrm{Aut}(E)$ for every object E.

We say that two groupoids $\mathcal{C}_1$ and $\mathcal{C}_2$ are *equivalent* if there is an equivalence of categories (i.e. a fully faithful and essentially surjective functor) $\mathcal{C}_1 \to \mathcal{C}_2$. Moreover, we say that a groupoid $\mathcal{C}$ *is equivalent to a set* Σ if there is an equivalence of categories $\mathcal{C} \to \mathcal{C}_\Sigma$ (where $\mathcal{C}_\Sigma$ is defined in Example 0.3.2).

0.3.2 Examples

We will return to our two main examples—curves and vector bundles—in a moment but it will be useful first to consider a number of simpler examples.

Example 0.3.2. If Σ is a set, the category $\mathcal{C}_\Sigma$, whose objects are elements of Σ and whose morphisms consist of only the identity morphism, is a groupoid.

Example 0.3.3. If G is a group, the *classifying groupoid* BG of G, defined as the category with one object $\star$ such that $\mathrm{Mor}(\star, \star) = G$, is a groupoid.

Example 0.3.4. The category FB of finite sets where morphisms are bijections is a groupoid. Observe that the isomorphism classes of FB are in bijection with $\mathbb{N}$ but the groupoid FB retains the information of the permutation groups S_n.

Example 0.3.5 (Projective space)**.** Projective space can be defined as a moduli groupoid where the objects are lines $L \subset \mathbb{A}^{n+1}$ through the origin and whose morphisms consist of only the identity, or alternatively where the objects are non-zero linear maps $x = (x_0, \dots, x_n) \colon \mathbb{C} \to \mathbb{C}^{n+1}$ such that there is a unique morphism $x \to x'$ if $\mathrm{im}(x) = \mathrm{im}(x') \subset \mathbb{C}^{n+1}$ (i.e. there exists a $\lambda \in \mathbb{C}^*$ such that $x' = \lambda x$) and no morphisms otherwise.

0.3.3 Moduli groupoid of orbits

Example 0.3.6 (Moduli groupoid of orbits)**.** Given an action of a group G on a set X, we define the *moduli groupoid of orbits* $[X/G]$[2] by taking the objects to be all elements $x \in X$ and by declaring $\mathrm{Mor}(x, x') = \{g \in G \,|\, x' = gx\}$.

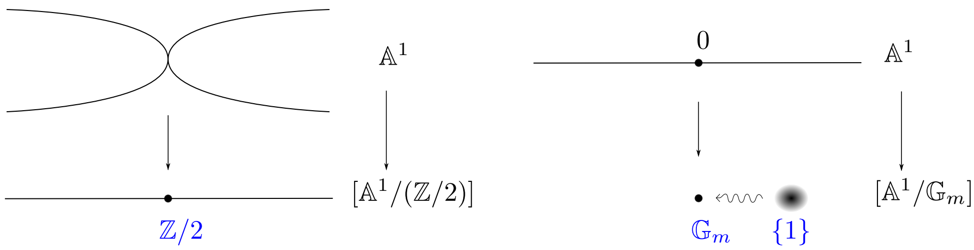


Figure 7: Pictures of the scaling actions of $\mathbb{Z}/2 = \{\pm 1\}$ and $\mathbb{G}_m$ on $\mathbb{A}^1$ over $\mathbb{C}$ with the automorphism groups listed in blue. Note that $[\mathbb{A}^1/\mathbb{G}_m]$ has two isomorphism classes of objects—0 and 1—corresponding to the two orbits—0 and $\mathbb{A}^1 \setminus 0$—such that $0 \in \overline{\{1\}}$ if the set $\mathbb{A}^1/\mathbb{G}_m$ is endowed with the quotient topology.

Exercise 0.3.7.

(a) Show that the moduli groupoid of orbits $[X/G]$ in Example 0.3.6 is equivalent to a set if and only if the action of G on X is free.

(b) Show that a groupoid $\mathcal{C}$ is equivalent to a set if and only if $\mathcal{C} \to \mathcal{C} \times \mathcal{C}$ is fully faithful.

Example 0.3.8. Consider the category $\mathcal{C}$ with two objects x_1 and x_2 such that $\mathrm{Mor}(x_i, x_j) = \{\pm 1\}$ for $i, j = 1, 2$ where composition of morphisms is given by multiplication. Then $\mathcal{C}$ is equivalent $B\mathbb{Z}/2$.

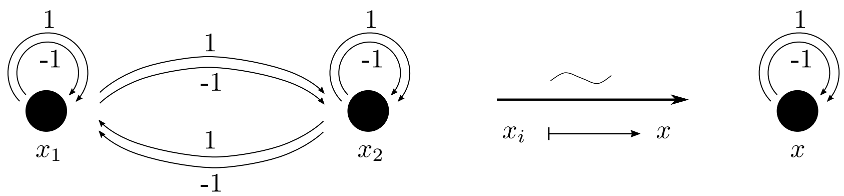


Figure 8: An equivalence of groupoids

Exercise 0.3.9. In Example 0.3.8, show that there is an equivalence of categories inducing a bijection on objects between $\mathcal{C}$ and either $[(\mathbb{Z}/2)/(\mathbb{Z}/4)]$ or $[(\mathbb{Z}/2)/(\mathbb{Z}/2 \times \mathbb{Z}/2)]$ where the action is given by the surjections $\mathbb{Z}/4 \to \mathbb{Z}/2$ or $\mathbb{Z}/2 \times \mathbb{Z}/2 \to \mathbb{Z}/2$.

[2] We use brackets to distinguish the groupoid quotient $[X/G]$ from the set quotient X/G. Later when G and X are enriched with more structure (e.g. an affine algebraic group acting on a variety), then $[X/G]$ will be correspondingly enriched (e.g. as an algebraic stack).

Example 0.3.10 (Projective space as a quotient)**.** The moduli groupoid of projective space (Example 0.3.5) can also be described as the moduli groupoid of orbits $[(\mathbb{A}^{n+1} \setminus 0)/\mathbb{G}_m]$.

We can also consider the quotient groupoid $[\mathbb{A}^{n+1}/\mathbb{G}_m]$, which is equivalent to the groupoid whose objects are (possibly zero) linear maps $x = (x_0, \ldots, x_n) \colon \mathbb{C} \to \mathbb{C}^{n+1}$ such that $\mathrm{Mor}(x, x') = \{t \in \mathbb{C}^* \mid x'_i = tx_i \text{ for all } i\}$. We can thus view $\mathbb{P}^n$ as a subgroupoid of $[\mathbb{A}^{n+1}/\mathbb{G}_m]$.

Exercise 0.3.11. If a group G acts on a set X and $x \in X$ is a point, there exists a fully faithful functor $BG_x \to [X/G]$. If the action is transitive, show that it is an equivalence.

A morphisms of groupoids $\mathcal{C}_1 \to \mathcal{C}_2$ is simply a functor, and we define the category $\mathrm{MOR}(\mathcal{C}_1, \mathcal{C}_2)$ whose objects are functors and whose morphisms are natural transformations.

Exercise 0.3.12. If $\mathcal{C}_1$ and $\mathcal{C}_2$ are groupoids, show that $\mathrm{MOR}(\mathcal{C}_1, \mathcal{C}_2)$ is a groupoid.

Exercise 0.3.13. If H and G are groups, show that there is an equivalence

$$\mathrm{MOR}(BH, BG) = \coprod_{\phi \in \mathrm{Conj}(H,G)} BC^G(\operatorname{im}\phi)$$

where $\mathrm{Conj}(H, G)$ denotes a set of representatives of homomorphisms $H \to G$ up to conjugation by G, and $C^G(\operatorname{im}\phi)$ denotes the centralizer of $\operatorname{im}\phi$ in G.

Exercise 0.3.14. Provide an example of group actions of H and G on sets X and Y and a map $[X/H] \to [Y/G]$ of groupoids that *does not* arise from a group homomorphism $\phi \colon H \to G$ and a ϕ-equivariant map $X \to Y$.

0.3.4 Moduli groupoids of curves and vector bundles

We return to the two main examples in these notes.

Example 0.3.15 (Moduli groupoid of smooth curves)**.** In this case, the objects are smooth, connected, and projective curves of genus g over $\mathbb{C}$ and for two curves C, C', the set of morphisms is defined as the set of isomorphisms

$$\mathrm{Mor}(C, C') = \{\text{isomorphisms } \alpha \colon C \xrightarrow{\sim} C'\}.$$

Example 0.3.16 (Moduli groupoid of vector bundles on a curve)**.** Let C be a fixed smooth, connected, and projective curve over $\mathbb{C}$, and fix integers $r \geq 0$ and d. The objects are vector bundles E of rank r and degree d, and the morphisms are isomorphisms of vector bundles.

0.3.5 Moduli groupoid of unlabelled triangles up to similarity

We now revisit Section 0.2.3 of the moduli set M^{unl} of unlabelled triangles up to similarity. We will show later that this moduli set does not admit a natural functorial descriptions nor universal family due to presence of symmetries (Example 0.4.39). Since these are such desirable properties, we will pursue a work around where we encode the symmetries into the definition.

We define the *moduli groupoid of unlabelled triangles up to similarity*, denoted by $\mathcal{M}^{\text{unl}}$ (note the calligraphic font), where the objects are unlabelled triangles in $\mathbb{R}^2$ and where for triangles $T_1, T_2 \subset \mathbb{R}^2$, the set $\text{Mor}(T_1, T_2)$ consists of the symmetries σ (corresponding to the permutations of the vertices) such that T_1 is similar to $\sigma(T_2)$. For example, an isosceles triangle (resp. equilateral triangle) has automorphism group $\mathbb{Z}/2$ (resp. S_3).

We can draw essentially the same picture as Figure 6 except we mark the automorphisms of triangles.

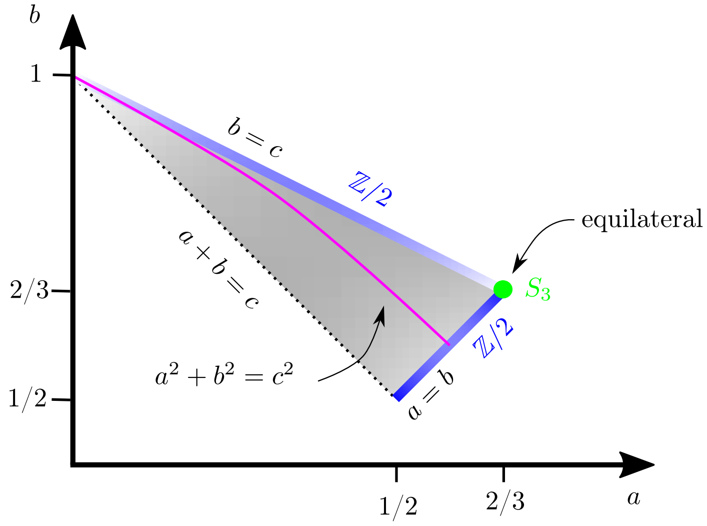


Figure 9: Picture of the moduli groupoid $\mathcal{M}^{\text{unl}}$ with non-trivial automorphism groups labelled.

There is a functor

$$\mathcal{M}^{\text{unl}} \to M^{\text{unl}}$$

which is the identity on objects and collapses all morphisms to the identity. This could be called a *coarse moduli set* where by forgetting some information (i.e. the symmetry groups of isosceles and equilateral triangles), we can study the moduli problem as a more familiar object (i.e. a set rather than groupoid).

Exercise 0.3.17. Recall that the moduli set M^{lab} of labelled triangles up to similarity has the description as the set of tuples (a, b, c) such that $a + b + c = 2$, $0 < a < b + c$, $0 < b < a + c$, and $0 < c < a + b$ (see (0.2.1)). Show that there is a natural action of S_3 on the moduli set M^{lab} of labelled triangles up to similarity and that the functor obtained by forgetting the labelling

$$[M^{\text{lab}}/S_3] \to \mathcal{M}^{\text{unl}}$$

is an equivalence of categories.

Exercise 0.3.18. Define a moduli groupoid of *oriented triangles* and investigate its relation to the moduli sets and groupoids of triangles we've defined above.

0.4 Moduli functors

We now undertake the challenging task of motivating moduli functors, which will be our approach for endowing moduli sets with the enriched structure of a topological space or scheme. This will require a leap in abstraction that is not at all the most intuitive, especially if you are seeing this for the first time. The idea due to Grothendieck is to study a scheme X by studying all maps to it!

It may seem that this leap made life more difficult for us: rather than just specifying the points of a moduli space, we need to define all maps to the moduli space. In fact, it is easier than you may expect. Let's take M_g as an example. If S is a scheme and $f\colon S \to M_g$ is a map of sets, then for every point $s \in S$, the image $f(s) \in M_g$ corresponds to an isomorphism class of a curve C_s. But we don't want to consider arbitrary maps of sets. If M_g is enriched as a topological space (resp. scheme), then a continuous (resp. algebraic) map $f\colon S \to M_g$ should mean that the curves C_s are varying continuously (resp. algebraically). A nice way of packaging this is via *families of curves*, i.e. smooth and proper morphisms $\mathcal{C} \to S$ such that every fiber $\mathcal{C}_s$ is a curve.

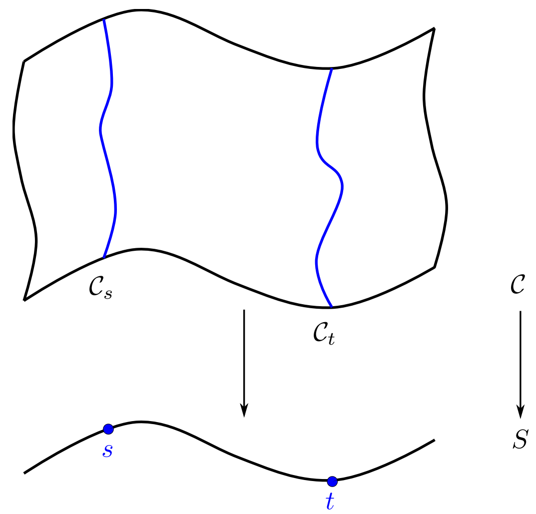


Figure 10: A family of curves over a curve S.

This suggests we define M_g as a functor Sch $\to$ Sets assigning a scheme S to the set of families of curves over S.

0.4.1 Yoneda's lemma

The fact that schemes are determined by maps into it follows from a completely formal argument that holds in every category. If X is an object of a category $\mathcal{C}$, the contravariant functor

$$h_X\colon \mathcal{C} \to \text{Sets}, \qquad S \mapsto \text{Mor}(S, X)$$

recovers the object X itself: this is the content of Yoneda's lemma:

Lemma 0.4.1 (Yoneda's lemma)**.** *Let $\mathcal{C}$ be a category and X be an object. For every contravariant functor $G\colon \mathcal{C} \to \mathrm{Sets}$, the map*

$$\mathrm{Mor}(h_X, G) \to G(X), \qquad \alpha \mapsto \alpha_X(\mathrm{id}_X)$$

is bijective and functorial with respect to both X and G.

Remark 0.4.2. The set $\mathrm{Mor}(h_X, G)$ consists of morphisms or natural transformations $h_X \to G$, and α_X denotes the map $h_X(X) = \mathrm{Mor}(X, X) \to G(X)$.

Caution 0.4.3. We will consistently abuse notation by conflating an element $g \in G(X)$ and the corresponding morphism $h_X \to G$, which we will often write simply as $X \to G$.

Exercise 0.4.4.

(a) Spell out precisely what 'functorial with respect to both X and G' means.

(b) Prove Yoneda's lemma.

Remark 0.4.5. It is instructive to imagine constructive proofs of Yoneda's lemma. Here we try to explicitly recover a *variety* X over $\mathbb{C}$ from its functor $h_X\colon \mathrm{Sch}/\mathbb{C} \to \mathrm{Sets}$. Clearly, we can recover the closed points of X by simply evaluating $h_X(\operatorname{Spec}\mathbb{C})$. To get all points, we need to allow points whose residue fields are extensions of $\mathbb{C}$. The underlying set of X is

$$\Sigma_X := \coprod_{\mathbb{C}\subset k} h_X(\operatorname{Spec} k)/\sim$$

where we say $x \in h_X(k)$ and $x' \in h_X(k')$ are equivalent if there is a further field extension $\mathbb{C} \subset k''$ containing both k and k' such that the images of x and x' in $h_X(k'')$ are equal under the natural maps $h_X(k) \to h_X(k'')$ and $h_X(k') \to h_X(k'')$. Later, we will follow the same approach when *defining* points of algebraic spaces and stacks (see Definition 3.3.16).

How can we recover the topological space? Here's a tautological way: we say a subset $A \subset \Sigma_X$ is open if there is an open immersion $U \hookrightarrow X$ with image A. Here's a better approach: we say a subset $A \subset \Sigma_X$ is open if for every map $f\colon S \to X$ of schemes, the subset $f^{-1}(A) \subset S$ is open.

What about recovering the sheaf of rings $\mathcal{O}_X$? For an open subset $U \subset \Sigma_X$, we define the functions on U as continuous maps $U \to \mathbb{A}^1$ such that for every morphism $f\colon S \to X$ of schemes, the composition (as a continuous map) $f^{-1}(U) \to U \to \mathbb{A}^1$ is an algebraic function (i.e. corresponds to an element $\Gamma(S, f^{-1}(U))$.

Exercise 0.4.6.

(a) Can the above argument be extended if X is non-reduced?

(b) Is it possible to explicitly recover a scheme X from its *covariant* functor $\mathrm{Sch} \to \mathrm{Sets}, S \mapsto \mathrm{Mor}(X, S)$?

0.4.2 Specifying a moduli functor

Defining a moduli functor requires specifying:

(1) families of objects;

(2) when two families of objects are isomorphic; and

(3) and how families pull back under morphisms.

In defining a moduli functor $F\colon \mathrm{Sch} \to \mathrm{Sets}$, then (1) and (2) specify $F(S)$ for a scheme S and (3) specifies the pull back $F(S) \to F(S')$ for maps $S' \to S$.

Example 0.4.7 (Moduli functor of smooth curves)**.** A *family of smooth curves* (of genus g) is a smooth, proper morphism $\mathcal{C} \to S$ of schemes such that for every $s \in S$, the fiber $\mathcal{C}_s$ is a connected curve (of genus g). The *moduli functor of smooth curves of genus g* is

$$F_{M_g}\colon \mathrm{Sch} \to \mathrm{Sets}, \quad S \mapsto \{\text{families of smooth curves } \mathcal{C} \to S \text{ of genus } g\} / \sim,$$

where two families $\mathcal{C} \to S$ and $\mathcal{C}' \to S$ are equivalent if there is a S-isomorphism $\mathcal{C} \to \mathcal{C}'$. If $S' \to S$ is a map of schemes and $\mathcal{C} \to S$ is a family of curves, the pull back is defined as the family $\mathcal{C} \times_S S' \to S'$.

Example 0.4.8 (Moduli functor of vector bundles on a curve)**.** Let C be a fixed smooth, connected, and projective curve over $\mathbb{C}$, and fix integers $r \geq 0$ and d. A *family of vector bundles* (of rank r and degree d) over a scheme S is a vector bundle $\mathcal{E}$ on $C \times S$ (such that for all $s \in S$, the restriction $\mathcal{E}_s := \mathcal{E}|_{C \times \operatorname{Spec} \kappa(s)}$ has rank r and degree d on $C_{\kappa(s)}$). The *moduli functor of vector bundles on C of rank r and degree d* is

$$\mathrm{Sch} \to \mathrm{Sets} \qquad S \mapsto \left\{ \begin{array}{l} \text{families of vector bundles } \mathcal{E} \text{ on } C \times S \\ \text{of rank } r \text{ and degree } d \end{array} \right\} / \sim,$$

where equivalence $\sim$ is given by isomorphism. If $S' \to S$ is a map of schemes and E is a vector bundle on $C \times S$, the pull back is defined as the vector bundle $(\mathrm{id} \times f)^* \mathcal{E}$ on $C \times S'$

Example 0.4.9 (Moduli functor of orbits)**.** Revisiting Example 0.3.6, consider an algebraic group G acting on a scheme X. For every scheme S, the abstract group $G(S)$ acts on the set $X(S)$ (in fact, giving such actions functorial in S uniquely specifies the group action). We can consider the functor

$$\mathrm{Sch} \to \mathrm{Sets} \qquad S \mapsto X(S)/G(S).$$

This is a naive candidate for a moduli functor of a quotient. Even with the action is free, this functor will not be representable. We will modify this example in §0.7.1.

To gain intuition of a moduli functor $F\colon \mathrm{Sch} \to \mathrm{Sets}$, it is always useful to plug in special test schemes. For instance, plugging in a field K should give the K-points of the moduli problem, plugging in $\mathbb{C}[\epsilon]/(\epsilon^2)$ should give pairs of $\mathbb{C}$-points together with tangent vectors, and plugging in a curve (e.g. a DVR) gives families of objects over the curve.

In some cases, even though you may know exactly what objects you want to parameterize, it is not always clear how to define families of objects. In fact, there may be several candidates for families corresponding to different scheme structures on the same topological space. This is the case for instance for the moduli of higher dimensional varieties.

0.4.3 Representable functors

Definition 0.4.10. We say that a functor $F\colon \mathrm{Sch} \to \mathrm{Sets}$ is *representable by a scheme* if there exists a scheme X and an isomorphism of functors $F \xrightarrow{\sim} h_X$.

We would like to know when a given a moduli functor F is representable by a scheme. Unfortunately, each of the functors considered in Examples 0.4.7 to 0.4.9 is *not* representable; see Section 0.4.7. We begin though by considering a few simpler moduli functors which are in fact representable.

Theorem 0.4.11 (Projective space as a functor)**.** *[Har77, Thm. II.7.1] There is a functorial bijection*

$$\mathrm{Mor}(S, \mathbb{P}^n_{\mathbb{Z}}) \cong \left\{ (L, s_0, \dots, s_n) \,\middle|\, \begin{array}{l} L \textit{ is a line bundle on } S \textit{ globally} \\ \textit{generated by } s_0, \dots, s_n \in \Gamma(S, L) \end{array} \right\} / \sim,$$

where $(L, (s_i)) \sim (L', (s'_i))$ *if there exists* $t \in \Gamma(S, \mathcal{O}_S)^*$ *such that* $s'_i = ts_i$ *for all* i*.*

In other words, the theorem states the functor defined on the right is representable by the scheme $\mathbb{P}^n_{\mathbb{Z}}$. The condition that the sections s_i are globally generated translates to the condition that for every $x \in S$, at least one section $s_i(x) \in L \otimes \kappa(t)$ is non-zero, or equivalently to the surjectivity of $(s_0, \dots, s_n)\colon \mathcal{O}_S^{n+1} \to L$.

This perspective of viewing projective space as parameterizing rank 1 quotients of the trivial bundle will be generalized can be generalized to the Grassmanian $\mathrm{Gr}(k, n)$ parameterizing k-dimensional *quotients* of n-dimensional space.[3] But what are families of k-dimensional quotients over a scheme S? As motivated by Example 0.4.18, they should be locally free quotients of $\mathcal{O}_S^n$. We thus define:

Definition 0.4.12. The *Grassmanian functor* is

$$\begin{aligned} \mathrm{Gr}(k, n)\colon\ & \mathrm{Sch} \to \mathrm{Sets} \\ & S \mapsto \left\{ [\mathcal{O}_S^n \twoheadrightarrow Q] \,\middle|\, Q \text{ is a vector bundle of rank } k \right\} / \sim \end{aligned}$$

where $[\mathcal{O}_S^n \overset{q}{\twoheadrightarrow} Q] \sim [\mathcal{O}_S^n \overset{q'}{\twoheadrightarrow} Q']$ if there exists an isomorphism $\Psi\colon Q \xrightarrow{\sim} Q'$ such that

$$\begin{tikzcd} \mathcal{O}_S^n \arrow[r, "q"] \arrow[dr, "q'"'] & Q \arrow[d, "\Psi"] \\ & Q' \end{tikzcd}$$

commutes (i.e. $q' = \Psi \circ q$), or equivalently if $\ker(q) = \ker(q')$.

Remark 0.4.13. Pullbacks are defined in the obvious manner. Observe that if $k = 1$, then $\mathrm{Gr}(1, n)$ is isomorphic to projective space $\mathbb{P}^{n-1}_{\mathbb{Z}}$, whose functorial description was given in Theorem 0.4.11.

We will later show that $\mathrm{Gr}(k, n)$ is representable by a scheme projective over $\mathbb{Z}$ (Theorem 1.1.3). The proof of this result is a good illustration of the utility of the functorial approach and a warmup for the representability of Hilb and Quot

[3] Alternatively, the points could be considered as k-dimensional *subspaces* but in these notes, we will follow Grothendieck's convention of quotients.

(Theorems 1.1.2 and 1.1.3). Since the Grassmanian parameterizes quotients V of a fixed vector space, this moduli problem does not have non-trivial symmetries, i.e. automorphisms. The same is true for the Hilb and Quot functors and thus we do not need the language of groupoids or stacks.

The projectivity of $\mathrm{Gr}(k,n)$ will be established by showing that the Plücker embedding $\mathrm{Gr}(k,n) \to \mathbb{P}(\bigwedge^k \mathcal{O}^n_{\operatorname{Spec}\mathbb{Z}})$ is a closed immersion and taking advantage of the representability of projective space (Theorem 0.4.11). Meanwhile, the representability and projectivity of Quot (and therefore also Hilb) we be established by using a suitable embedding into a Grassmanian.

Exercise 0.4.14.

(a) If S is a scheme and E is a vector bundle on S, show that the projectivization $\mathbb{P}_S(E) := \operatorname{Proj}_S \operatorname{Sym}^* E^\vee$ of E represents the functor

$$\begin{aligned}\mathbb{P}_S(E)\colon \mathrm{Sch}/S &\to \mathrm{Sets}\\ (T \xrightarrow{f} S) &\mapsto \{\text{quotients } f^*E \overset{q}{\twoheadrightarrow} L \text{ where } L \text{ is a line bundle on } T\}/\sim\end{aligned}$$

where $[f^*E \overset{q}{\twoheadrightarrow} L] \sim [f^*E \overset{q'}{\twoheadrightarrow} L']$ if $\ker(q) = \ker(q')$ (or equivalently there is an isomorphism $\alpha\colon L \to L'$ with $q' = \alpha \circ q$).

(b) Show that the same holds if E is a finite type quasi-coherent sheaf on S (e.g. a coherent sheaf if S is noetherian).

Note that there is an isomorphism $\mathbb{P}^n_{\mathbb{Z}} \cong \mathbb{P}(\mathcal{O}^{n+1}_{\operatorname{Spec}\mathbb{Z}})$ of functors.

Exercise 0.4.15. Provide functorial descriptions of:

(a) $\mathbb{A}^n \setminus 0$;

(b) the blowup $\mathrm{Bl}_p \mathbb{P}^n$ of $\mathbb{P}^n$ at a point;

(c) $\mathcal{S}\mathrm{pec}_S \mathcal{A}$ where $\mathcal{A}$ is a quasi-coherent sheaf of algebras on a scheme S; and

(d) $\operatorname{Proj} R$ where R is a positively graded ring.

Exercise 0.4.16.

(a) Let X be a scheme and E, F be $\mathcal{O}_X$-modules. Show that the functor

$$\underline{\mathrm{Hom}}_{\mathcal{O}_X}(E,F)\colon \mathrm{Sch} \to \mathrm{Sets}, \quad T \mapsto \mathrm{Hom}_{\mathcal{O}_{X\times T}}(E_T, F_T),$$

where E_T and F_T denote the pullbacks of E and F under $X \times T \to T$, is representable by $\operatorname{Spec} \operatorname{Sym} \operatorname{Ext}^1_{\mathcal{O}_X}(E,F)^\vee$.

(b) Let $E \to F$ be a homomorphism of coherent sheaves on a noetherian scheme X. Show that the subfunctor of X (or more precisely of $h_X = \mathrm{Mor}(-,X)$) defined by

$$\mathrm{Sch} \to \mathrm{Sets}, \quad T \mapsto \{\text{morphisms } T \to X \text{ such that } E_T \to F_T \text{ is zero}\}$$

is representable by a closed subscheme of X.

(c) Let $X \to S$ be a projective morphism of noetherian schemes and let E and F be coherent sheaves on X. Show that the functor

$$\underline{\mathrm{Hom}}_{\mathcal{O}_X/S}(E,F)\colon \mathrm{Sch} \to \mathrm{Sets}, \quad (T \to S) \mapsto \mathrm{Hom}_{\mathcal{O}_{X\times_S T}}(E_T, F_T),$$

where E_T and F_T are the pullbacks of E and F to $X \times_S T$, is representable by a scheme of finite type over S. Moreover, if $\mathcal{F}$ is flat over S, show that $\underline{\mathrm{Hom}}_{\mathcal{O}_X/S}(E,F)$ is representable by a linear scheme over S, i.e. the relative affine space $\mathbb{A}(V)$ of a vector bundle V on S.

Exercise 0.4.17. Let X be a scheme, and let E and G be $\mathcal{O}_X$-modules. The group $\mathrm{Ext}^1_{\mathcal{O}_X}(G, E)$ classifies extensions $0 \to E \to F \to G \to 0$ of $\mathcal{O}_X$-modules where two extensions are identified if there is an isomorphism of short exact sequences inducing the identity map on E and G [Har77, Exer. III.6.1].

Show that the affine scheme $\underline{\mathrm{Ext}}^1_{\mathcal{O}_X}(G, E) := \operatorname{Spec} \operatorname{Sym} \mathrm{Ext}^1_{\mathcal{O}_X}(G, E)^\vee$ represents the functor

$$\mathrm{Sch} \to \mathrm{Sets}, \quad T \mapsto \mathrm{Ext}^1_{\mathcal{O}_{X \times T}}(p_1^* G, p_1^* E)$$

where $p_1 \colon X \times T \to X$.

0.4.4 Schemes are sheaves

If $F \colon \mathrm{Sch} \to \mathrm{Sets}$ is representable by a scheme X (i.e. $F = \mathrm{Mor}(-, X)$), then F is necessarily a *sheaf in the big Zariski topology*, that is, for every scheme S, the presheaf on the Zariski topology of S defined by assigning to an open subset $U \subset S$ the set $F(U)$ is a sheaf on the Zariski topology of S. This is simply stating that morphisms into the fixed scheme X glue uniquely.

The failure to be a sheaf therefore provides an obstruction to the representability of a given moduli functor F: if F is not a sheaf in the big Zariski topology, then F can not be representable.

Example 0.4.18. Consider the functor

$$F \colon \mathrm{Sch} \to \mathrm{Sets}, \quad S \mapsto \{\text{quotients } q \colon \mathcal{O}_S^n \twoheadrightarrow \mathcal{O}_S^k\}/\sim$$

where quotients q and q' are identified if there exists an automorphism Ψ of $\mathcal{O}_S^k$ such that $q' = \Psi \circ q$ or equivalently if $\ker(q) = \ker(q')$.

If F were representable by a scheme, then since morphisms glue in the Zariski topology, sections of F should also glue. But it easy to see that this fails: specializing to $k = 1$ and $S = \mathbb{P}^1$ (with coordinates x and y), consider the cover $S_1 = \{y \neq 0\} = \operatorname{Spec} \mathbb{C}[\frac{x}{y}]$ and $S_2 = \{x \neq 0\} = \operatorname{Spec} \mathbb{C}[\frac{y}{x}]$. The quotients

$$[(1, \frac{x}{y}, 0, \cdots, 0) \colon \mathcal{O}_{S_1}^{\oplus n} \to \mathcal{O}_{S_1}] \in F(S_1) \quad \text{and} \quad [(\frac{y}{x}, 1, 0, \cdots, 0) \colon \mathcal{O}_{S_2}^{\oplus n} \to \mathcal{O}_{S_2}] \in F(S_2)$$

become equivalent in $F(S_1 \cap S_2)$ under the automorphism $\Psi = \frac{y}{x}$ of $\mathcal{O}_{S_1 \cap S_2}$ and *do not* glue to a section of $F(\mathbb{P}^1)$. Of course, the issue is that the structure sheaves on S_1 and S_2 glue to $\mathcal{O}_{\mathbb{P}^1}(1)$—not $\mathcal{O}_{\mathbb{P}^1}$—under Ψ.

Given the action of an affine algebraic group G on a scheme X, there is a functor $S \mapsto X(S)/G(S)$; see Example 0.4.9. Even in simple examples of free actions, this functor is not a sheaf.

Exercise 0.4.19. Consider $\mathbb{G}_m$ acting on $\mathbb{A}^{n+1} \setminus 0$ with the usual scaling action. Show that the functor $S \mapsto (\mathbb{A}^{n+1} \setminus 0)(S)/\mathbb{G}_m(S)$ is not a sheaf.

0.4.5 Working with functors

We can form a category $\mathrm{Fun}(\mathrm{Sch}, \mathrm{Sets})$ whose objects are contravariant functors $F \colon \mathrm{Sch} \to \mathrm{Sets}$ and whose morphisms are natural transformations. This category has fiber products: given morphisms $F \xrightarrow{\alpha} G$ and $G' \xrightarrow{\beta} G$, we define

$$\begin{aligned} F \times_G G' \colon \mathrm{Sch} &\to \mathrm{Sets} \\ S &\mapsto \{(a, b) \in F(S) \times G'(S) \mid \alpha_S(a) = \beta_S(b)\} \end{aligned}$$

Exercise 0.4.20. Show that $F \times_G G'$ satisfies the universal property for fiber products in $\mathrm{Fun}(\mathrm{Sch}, \mathrm{Sets})$.

Definition 0.4.21.

(1) We say that a morphism $F \to G$ of contravariant functors is *representable by schemes* if for every map $S \to G$ from a scheme S, the fiber product $F \times_G S$ is representable by a scheme.

(2) We say that a morphism $F \to G$ is an *open immersion* or that a subfunctor $F \subset G$ is *open* if for every morphism $S \to G$ from a scheme S, $F \times_G S$ is representable by an open subscheme of S.

(3) We say that a set of open subfunctors $\{F_i\}$ of F is a *Zariski open cover* if for every morphism $S \to F$ from a scheme S, $\{F_i \times_F S\}$ is a Zariski open cover of S (and in particular each F_i is an open subfunctor of F).

Each of these conditions can be checked on *affine* schemes

These definitions together with the following exercise provide us a recipe for checking that a given functor F is representable by a scheme: find a Zariski open cover $\{F_i\}$ where each F_i is representable.

Exercise 0.4.22.

(a) Let $F\colon \mathrm{Sch} \to \mathrm{Sets}$ be a functor which is a sheaf in the big Zariski topology and $\{F_i\}$ be a Zariski open cover of F. Show that if each F_i is representable by a scheme, then so is F.

(b) Show that a collection of open subfunctors $\{F_i\}$ of F is a Zariski open cover if and only if the map $\coprod_i F_i(\Bbbk) \to F(\Bbbk)$ is surjective for each algebraically closed field $\Bbbk$.

(c) Given morphisms of schemes $X \to Y$ and $Y' \to Y$, reprove the existence of the fiber product $X \times_Y Y'$ in the category of schemes by exhibiting a Zariski open cover $\{F_i\}$ of $X \times_Y Y'$ where each F_i is representable by an affine scheme.

Exercise 0.4.23. Show that a scheme can be equivalently defined as a contravariant functor $F\colon \mathrm{AffSch} \to \mathrm{Sets}$ on the category of affine schemes (or covariant functor on the category of rings) as follows. Let $\mathcal{C}$ be a full subcategory of the category $\mathrm{Fun}(\mathrm{AffSch}, \mathrm{Sets})$ of contravariant functors. Generalizing Definitions 0.4.10 and 0.4.21, we define a functor $F\colon \mathrm{AffSch} \to \mathrm{Sets}$ to be *representable by* $\mathcal{C}$ if there exist an object $X \in \mathcal{C}$ and a functorial equivalence $F(S) = \mathrm{Mor}(S, X)$ for every $S \in \mathrm{AffSch}$. We say that a map $F \to G$ of functors from AffSch to Sets is *representable by open immersions in* $\mathcal{C}$ if for every morphism $\operatorname{Spec} B \to G$, the fiber product $F \times_G \operatorname{Spec} B$ is representable by an object $X \in \mathcal{C}$ which is an open subscheme of $\operatorname{Spec} B$. Finally, we say that a collection $\{F_i\}$ of subfunctors of F is a *Zariski open* $\mathcal{C}$*-cover* if each $F_i \to F$ is representable by open immersions in $\mathcal{C}$ and for each algebraically closed field $\Bbbk$, the map $\coprod_i F_i(\Bbbk) \to F(\Bbbk)$ is surjective.

(a) Letting $\mathcal{C} = \mathrm{AffSch}$, show that a scheme with affine diagonal can be equivalently defined as a functor $F\colon \mathrm{AffSch} \to \mathrm{Sets}$ such that there exists a Zariski open AffSch-cover $\{F_i\}$ of F with each F_i representable by AffSch.

(b) Give a similar characterization of separated schemes.

(c) Letting $\mathcal{C}$ be the category of schemes with affine diagonal, show that a scheme can be equivalently defined as a functor $F\colon \mathrm{AffSch} \to \mathrm{Sets}$ such that there exists a Zariski open $\mathcal{C}$-cover $\{F_i\}$ with each F_i representable by $\mathcal{C}$.

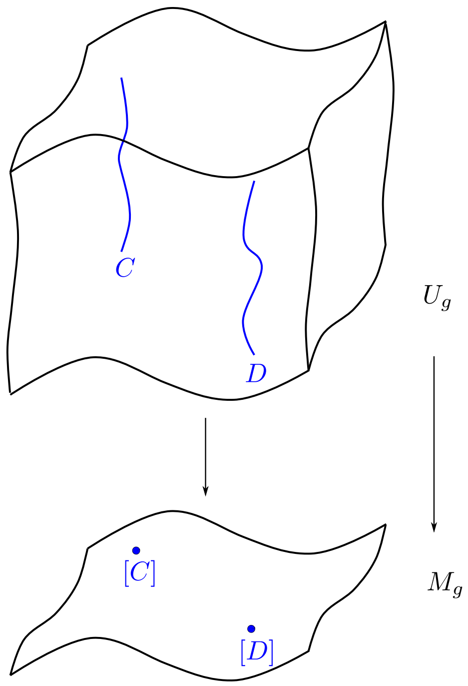


Figure 11: Visualization of a (non-existent) universal family over M_g.

Replacing Zariski opens with étale opens (see §0.5) leads to the definition of an algebraic space (Definition 3.1.2).

0.4.6 Universal families

Definition 0.4.24. Let $F\colon \mathrm{Sch} \to \mathrm{Sets}$ be a moduli functor representable by a scheme X via an isomorphism $\alpha\colon F \xrightarrow{\sim} h_X$ of functors. The *universal family* of F is the object $U \in F(X)$ corresponding under α to the identity morphism $\mathrm{id}_X \in h_X(X) = \mathrm{Mor}(X, X)$.

Suspend your skepticism for a moment and suppose that there actually exists a scheme M_g representing the moduli functor of smooth curves of genus g (Example 0.4.7). Then corresponding to the identity map $M_g \to M_g$ is a family of genus g curves $U_g \to M_g$ satisfying the following universal property: for every smooth family of curves $\mathcal{C} \to S$ over a scheme S, there is a unique map $S \to M_g$ and cartesian diagram

$$\begin{tikzcd} \mathcal{C} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & U_g \arrow[d] \\ S \arrow[r] & M_g. \end{tikzcd}$$

The map $S \to M_g$ sends a point $s \in S$ to the curve $[\mathcal{C}_s] \in M_g$.

Example 0.4.25. The universal family of the moduli functor of projective space (Theorem 0.4.11) is the line bundle $\mathcal{O}(1)$ on $\mathbb{P}^n$ together with the sections $x_0, \dots, x_n \in \Gamma(\mathbb{P}^n, \mathcal{O}(1))$.

Example 0.4.26 (Universal extensions)**.** If X is a scheme with vector bundles E and G, the universal family for the moduli functor $\underline{\mathrm{Ext}}^1_{\mathcal{O}_X}(G,E)$ of extensions of Exercise 0.4.17 is the extension $0 \to p_1^* G \to \mathcal{F} \to p_1^* E \to 0$ of vector bundle on $X \times \underline{\mathrm{Ext}}^1_{\mathcal{O}_X}(G,E)$. The restriction of this extension to $X \times \{t\}$ is the extension corresponding to $t \in \mathrm{Ext}^1_{\mathcal{O}_X}(G,E)$.

Example 0.4.27 (Classifying spaces in algebraic topology)**.** Let G be a topological group and $\mathrm{Top}^{\mathrm{para}}$ be the category of paracompact topological spaces where morphisms are defined up to homotopy. It is a theorem in algebraic topology that the functor

$$\mathrm{Top}^{\mathrm{para}} \to \mathrm{Sets}, \qquad S \mapsto \{\text{principal } G\text{-bundles } P \to S\}/\sim,$$

where $\sim$ denotes isomorphism, is represented by a topological space, which we denote by BG and call the *classifying space*. The universal family is usually denoted by $EG \to BG$.

For example, the classifying space $B\mathbb{C}^*$ is the infinite-dimensional manifold $\mathbb{CP}^\infty$; in algebraic geometry however the classifying stack $B\mathbb{G}_{m,\mathbb{C}}$ is an algebraic stack of dimension -1.

0.4.7 Non-representability of some moduli functors

For an algebraically closed field $\Bbbk$, suppose $F\colon \mathrm{Sch}/\Bbbk \to \mathrm{Sets}$ is a moduli functor parameterizing isomorphism classes of objects, and let's suppose that there is an object $E \in F(\Bbbk)$ with a non-trivial automorphism α. This can obstruct the representability of F as the automorphism α can sometimes be used to construct non-trivial families: namely, if $S = S_1 \cup S_2$ is an open cover of a scheme S, we can glue the trivial families $E \times S_1$ and $E \times S_2$ using α to obtain a family $\mathcal{E}$ over S which might be non-trivial. More precisely, we have:

Proposition 0.4.28. *Let $F\colon \mathrm{Sch}/\Bbbk \to \mathrm{Sets}$ be a moduli functor parameterizing isomorphism classes of objects. Suppose there is a family of objects $\mathcal{E} \in F(S)$ over a variety S. For a point $s \in S(\Bbbk)$, denote by $\mathcal{E}_s \in F(\Bbbk)$ the pull back of $\mathcal{E}$ along $s\colon \mathrm{Spec}\,\Bbbk \to S$. If*

(a) the fibers $\mathcal{E}_s$ are isomorphic for $s \in S(\Bbbk)$; and

(b) the family $\mathcal{E}$ is non-trivial, i.e. is not equal to the pull back of an object $E \in F(\Bbbk)$ along the structure map $S \to \mathrm{Spec}\,\Bbbk$,

then F is not representable.

Proof. Suppose by way of contradiction that F is represented by a scheme X. By condition (a), the restriction $E := \mathcal{E}_s$ is independent of $s \in S(\Bbbk)$ and defines a unique point $x \in X(\Bbbk)$. As S is reduced, the map $S \to X$ factors as $S \to \mathrm{Spec}\,\Bbbk \xrightarrow{x} X$. This implies that the family $\mathcal{E}$ is the pullback under the constant map $S \to \mathrm{Spec}\,\Bbbk \xrightarrow{x} X$, i.e. $\mathcal{E}$ is a trivial family, which contradicts condition (b). ∎

Example 0.4.29 (Moduli of vector bundles over a point)**.** Consider the moduli functor $F\colon \mathrm{Sch}/\Bbbk \to \mathrm{Sets}$ assigning a scheme S to the set of isomorphism classes of vector bundles over S. Note that $F(\mathrm{Spec}\,\Bbbk) = \coprod_{r\geq 0}\{\mathcal{O}^r_{\mathrm{Spec}\,\Bbbk}\}$. Since we know there exist non-trivial vector bundles (of any positive rank), we see that F cannot be representable by a scheme.

Exercise 0.4.30. Show that the moduli functor of vector bundles over a curve C is not representable.

Example 0.4.31 (Moduli of elliptic curves)**.** An *elliptic curve* over a field K is a pair (E, p) where E is a smooth, geometrically connected (i.e. $E_{\overline{K}}$ is connected), and projective curve E of genus 1 and $p \in E(K)$. A *family of elliptic curves* over a scheme S is a pair $(\mathcal{E} \to S, \sigma)$ where $\mathcal{E} \to S$ is smooth proper morphism with a section $\sigma : S \to \mathcal{E}$ such that for every $s \in S$, the fiber $(\mathcal{E}_s, \sigma(s))$ is an elliptic curve over the residue field $\kappa(s)$. The *moduli functor of elliptic curves* is

$$\begin{aligned} F_{M_{1,1}} \colon \mathrm{Sch} &\to \mathrm{Sets} \\ S &\mapsto \{\text{families } (\mathcal{E} \to S, \sigma) \text{ of elliptic curves }\}/\sim, \end{aligned}$$

where $(\mathcal{E} \to S, \sigma) \sim (\mathcal{E}' \to S, \sigma')$ if there is a S-isomorphism $\alpha \colon \mathcal{E} \to \mathcal{E}'$ compatible with the sections (i.e. $\sigma' = \alpha \circ \sigma$).

Exercise 0.4.32. Consider the family of elliptic curves defined over $\mathbb{A}^1 \setminus 0$ (with coordinate t) by

$$\begin{tikzcd} \mathcal{E} := V(y^2z - x^3 + tz^3) \arrow[r, hook] \arrow[d] & (\mathbb{A}^1 \setminus 0) \times \mathbb{P}^2 \\ \mathbb{A}^1 \setminus 0 & \end{tikzcd}$$

with section $\sigma \colon \mathbb{A}^1 \setminus 0 \to \mathcal{E}$ given by $t \mapsto [0, 1, 0]$. Show that $(\mathcal{E} \to \mathbb{A}^1 \setminus 0, \sigma)$ satisfies (a) and (b) in Proposition 0.4.28.

Example 0.4.33 (Moduli functor of smooth curves)**.** Let C be a curve with a non-trivial automorphism $\alpha \in \mathrm{Aut}(C)$ and let N be the nodal cubic curve, which we can think of as $\mathbb{P}^1$ with the points 0 and ∞ glued together. We can construct a family $\mathcal{C} \to N$ by taking the trivial family $\pi \colon C \times \mathbb{P}^1 \to \mathbb{P}^1$ and gluing the fiber $\pi^{-1}(0)$ with $\pi^{-1}(\infty)$ via the automorphism α.

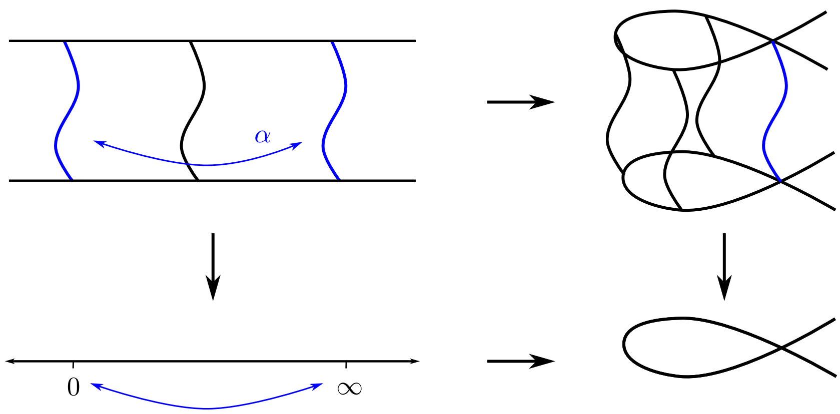


Figure 12: Family of curves over the nodal cubic obtaining by gluing the fibers over 0 and ∞ of the trivial family over $\mathbb{P}^1$ via α. (It would be more illustrative to draw a Mobius band as the family of curves over the nodal cubic.)

To show that the moduli functor of curves is not representable, it suffices to show that $\mathcal{C} \to N$ is non-trivial.

Exercise 0.4.34. Show that $\mathcal{C} \to N$ is a non-trivial family.

Remark 0.4.35. While Proposition 0.4.28 show that the existence of non-trivial automorphisms provides an obstruction to the representatibility of moduli functors, we saw in §0.4.4 that the failure of the functor to be a sheaf in the big Zariski topology provides another obstruction. In fact, these obstructions are intimately related as the existence of a non-trivial automorphism often implies that a given moduli functor is not a sheaf.

Consider the moduli functor F_{M_g} of smooth curves from Example 0.4.7. Let $\{S_i\}$ be a Zariski open covering of a scheme S. Suppose we have families of smooth curves $\mathcal{C}_i \to S_i$ and isomorphisms $\alpha_{ij} \colon \mathcal{C}_i|_{S_{ij}} \xrightarrow{\sim} \mathcal{C}_j|_{S_{ij}}$ on the intersection $S_{ij} := S_i \cap S_j$. The requirement that F_{M_g} be a sheaf (when restricted to the Zariski topology on S) implies that the families $\mathcal{C}_i \to S_i$ glue uniquely to a family of curves $\mathcal{C} \to S$. However, we have not required the isomorphisms α_i to be compatible on the triple intersection (i.e. $\alpha_{ij}|_{S_{ijk}} \circ \alpha_{jk}|_{S_{ijk}} = \alpha_{ik}|_{S_{ijk}}$) as is usual with gluing of schemes [Har77, Exercise II.2.12]. For this reason, F_{M_g} fails to be a sheaf.

Exercise 0.4.36. Show that the moduli functors of smooth curves and elliptic curves are not sheaves by explicitly exhibiting a scheme S, an open cover $\{S_i\}$ and families of curves over S_i that do not glue to a family over S.

0.4.8 Moduli functors of triangles

We will now attempt to define moduli functors of labelled and unlabelled triangles. Since we are primarily interested in constructing these moduli spaces as topological spaces, we will consider the category Top of topological spaces and consider representability as a topological space.

Example 0.4.37 (Labelled embedded triangles)**.** If S is a topological space, then we define a *family of labelled embedded triangles over* S as a tuple $(\mathcal{T}, \sigma_1, \sigma_2, \sigma_3)$ where $\mathcal{T} \subset S \times \mathbb{R}^2$ is a closed subset and $\sigma_i \colon S \to \mathcal{T}$ are continuous sections for $i = 1, 2, 3$ of the projection $\mathcal{T} \to S$ such that for every $s \in S$, the subset $\mathcal{T}_s \subset \mathbb{R}^2$ is a labelled triangle with vertices $\sigma_1(s)$, $\sigma_2(s)$, and $\sigma_3(s)$.

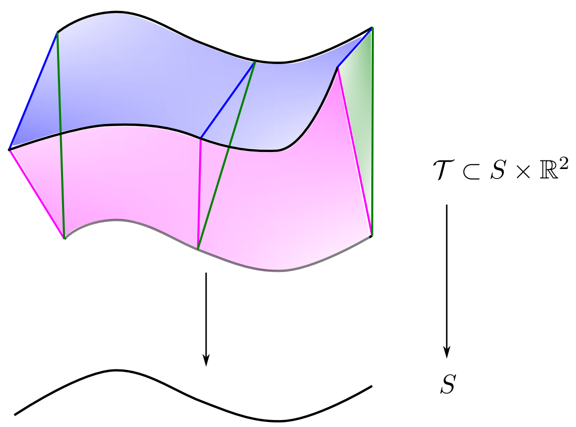


Figure 13: A family of labelled triangles over a curve.

Likewise, we define the *moduli functor of labelled triangles* as

$$F_M \colon \text{Top} \to \text{Sets}, \quad S \mapsto \{\text{families } (\mathcal{T}, \sigma_1, \sigma_2, \sigma_3) \text{ of labelled triangles}\}$$

We claim this functor is represented by the topological space of full rank 2×3 matrices

$$M := \left\{(x_1, y_1, x_2, y_2, x_3, y_3) \mid \det \begin{pmatrix} x_2 - x_1 & x_3 - x_1 \\ y_2 - y_1 & y_3 - y_1 \end{pmatrix} \neq 0\right\} \subset \mathbb{R}^6.$$

There is a bijection $F_M(\text{pt}) \to M$ given by taking the coordinates of the vertices. It is easy to see that this bijection can be promoted to an equivalence of functors $F_M \xrightarrow{\sim} h_M$, i.e. to a functorial bijection

$$F_M(S) \xrightarrow{\sim} \text{Mor}(S, M)$$

for each $S \in \text{Top}$, which assigns a family $(\mathcal{T}, \sigma_i)$ of labelled triangles to the map $S \to M$ where $s \mapsto (\sigma_1(s), \sigma_2(s), \sigma_3(s)) \in \mathcal{T}$.

Since F_M is representable by the topological space M, we have a universal family $\mathcal{T}_{\text{univ}} \subset M \times \mathbb{R}^2$ with $\sigma_1, \sigma_2, \sigma_3 \colon M \to \mathcal{T}_{\text{univ}}$. This universal family can be visualized over the locus $(x_1, y_1) = (0, 0)$ and $(x_2, y_2) = (1, 0)$ by taking Figure 3 and drawing the triangles *above* each point rather than *at* each point.

Example 0.4.38 (Labelled triangles up to similarity)**.** If S is a topological space, we define a *family of labelled triangles over* S as a tuple $(\mathcal{T}, \sigma_1, \sigma_2, \sigma_3)$ where $\mathcal{T} \to S$ is a fiber bundle with three sections $\sigma_i \colon S \to \mathcal{T}$ equipped with a continuous distance function $d \colon \mathcal{T} \times_S \mathcal{T} \to \mathbb{R}_{\geq 0}$ such that for every point $s \in S$, the restriction $d_s \colon \mathcal{T}_s \times \mathcal{T}_s \to \mathbb{R}_{\geq 0}$ is a metric on the fiber $\mathcal{T}_s$ with $\mathcal{T}_s$ isometric to a triangle with vertices $\sigma_i(s)$. We say two families $(\mathcal{T}, (\sigma_i))$ and $(\mathcal{T}', (\sigma_i'))$ of labelled triangles over $S \in \text{Top}$ are *similar* if there is a homeomorphism $f \colon \mathcal{T} \to \mathcal{T}'$ over S compatible with the sections (i.e. $f \circ \sigma_i = \sigma_i'$) such that for each $s \in S$, the induced map $\mathcal{T}_s \to \mathcal{T}_s'$ on fibers is a similarity of triangles, i.e. an isometry after rescaling.

We define the functor

$$F_{M^{\text{lab}}} \colon \text{Top} \to \text{Sets}, \quad S \mapsto \{\text{families } (\mathcal{T} \to S, \sigma_i) \text{ of labelled triangles}\}/\sim$$

where $\sim$ denotes similarity. Recall from (0.2.2) that the assignment of a triangle to its side lengths yields a bijection between $F_{M^{\text{lab}}}$ and

$$M^{\text{lab}} = \left\{ (a, b, c) \,\middle|\, \begin{array}{c} a + b + c = 2 \\ 0 < a < b + c \\ 0 < b < a + c \\ 0 < c < a + b \end{array} \right\};$$

As in the previous example, this extends to an isomorphism of functors $F_{M^{\text{lab}}} \to \text{Mor}(-, M^{\text{lab}})$. Therefore the topological space M^{lab} represents the functor $F_{M^{\text{lab}}}$, and there is a universal family $\mathcal{T}_{\text{univ}} \subset M^{\text{lab}} \times \mathbb{R}^2$ with $\sigma_i \colon M^{\text{lab}} \to \mathcal{T}_{\text{univ}}$.

Example 0.4.39 (Unlabelled triangles up to similarity)**.** If S is a topological space, a *family of triangles* is a fiber bundle $\mathcal{T} \to S$ equipped with a continuous distance function $d \colon \mathcal{T} \times_S \mathcal{T} \to \mathbb{R}_{\geq 0}$ that restricts to a metric on every fiber $\mathcal{T}_s$ with $\mathcal{T}_s$ isometric to a triangle. Two families $\mathcal{T} \to S$ and $\mathcal{T}' \to S$ are *similar*

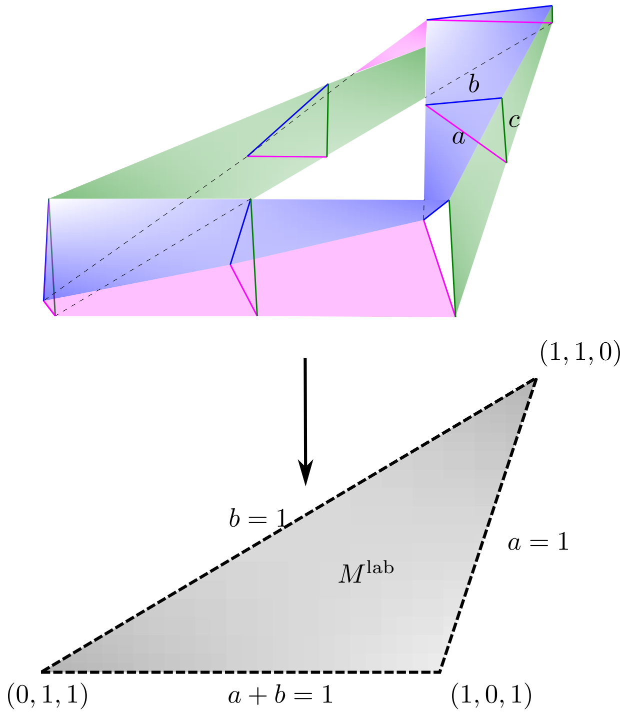


Figure 14: The universal family $U^{\mathrm{lab}} \to M^{\mathrm{lab}}$ of labelled triangles up to similarity.

if there is a homeomorphism $f\colon \mathcal{T} \to \mathcal{T}'$ over S compatible with the sections inducing similarities of triangles on fibers.

We define the functor

$$F\colon \text{Top} \to \text{Sets}, \quad S \mapsto \{\text{families } \mathcal{T} \subset S \times \mathbb{R}^2 \text{ of triangles}\}/\sim$$

where $\sim$ denotes similarity. This functor is *not* representable as there are non-trivial families of triangles $\mathcal{T}$ such that all fibers are similar triangles (Proposition 0.4.28). For instance, we construct a non-trivial family of triangles over S^1 by gluing two trivial families via a symmetry of an equilateral triangle.

Figure 15: A trivial (left) and non-trivial (right) family of equilateral triangles. Image taken from a video produced by Jonathan Wise: see http://math.colorado.edu/~jonathan.wise/visual/moduli/index.html.

0.5 Motivation: why the étale topology?

Why is the Zariski topology not sufficient for our purposes? The short answer is that there are not enough Zariski open subsets and that étale morphisms are an algebro-geometric replacement of analytic open subsets.

0.5.1 What is an étale morphism anyway?

I'm always baffled when a student is intimidated by étale morphisms, especially when she has already mastered the conceptually more difficult notions of say properness and flatness. One reason may be due to the fact that the definition is buried in [Har77, Exercises III.10.3-6] and its importance is not highlighted there.

The geometric picture of étaleness that you should have in your head is a covering space. The precise definition of an étale morphism is of course more algebraic, and there are in fact many equivalent formulations. This is possibly another point of intimidation for students as it is not at all obvious why the different notions are equivalent, and indeed some of the proofs are quite involved. Nevertheless, if you can take the equivalences on faith, it requires very little effort to not only internalize the concept, but to master its use.

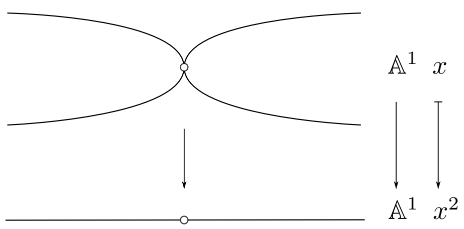


Figure 16: Picture of an étale double cover of $\mathbb{A}^1 \setminus 0$

For a morphism $f\colon X \to Y$ of schemes of finite type over $\mathbb{C}$, the following are equivalent characterizations of étaleness:

- f is smooth of relative dimension 0 (i.e. f is flat and all fibers are smooth of dimension 0);
- f is flat and unramified (i.e. for all $y \in Y(\mathbb{C})$, the scheme-theoretic fiber X_y is isomorphic to a disjoint union $\coprod_i \operatorname{Spec}\mathbb{C}$ of points);
- f is flat and $\Omega_{X/Y} = 0$;
- for all $x \in X(\mathbb{C})$, the induced map $\widehat{\mathcal{O}}_{Y,f(x)} \to \widehat{\mathcal{O}}_{X,x}$ on completions is an isomorphism; and
- (assuming in addition that X and Y are smooth) for all $x \in X(\mathbb{C})$, the induced map $T_{X,x} \to T_{Y,f(x)}$ on tangent spaces is an isomorphism.

We say that f is *étale at* $x \in X$ if there is an open neighborhood U of x such that $f|_U$ is étale.

Exercise 0.5.1. Show that $f\colon \mathbb{A}^1 \to \mathbb{A}^1, x \mapsto x^2$ is étale over $\mathbb{A}^1 \setminus 0$ but is not étale at the origin.

Try to show this for as many of the above definitions as you can.

Étale and smooth morphisms are discussed in much greater detail and generality in §A.3.

0.5.2 What can you see in the étale topology?

Working with the étale topology is like putting on a better pair of glasses allowing you to see what you couldn't before. Or perhaps more accurately, it is like getting magnifying lenses for your algebraic geometry glasses allowing you to visualize what you already could using your differential geometry glasses.

Example 0.5.2 (Irreducibility of the node)**.** Consider the plane nodal cubic C defined by $y^2 = x^2(x-1)$ in the plane. While there is an analytic open neighborhood of the node $p = (0,0)$ which is reducible, there is no such Zariski open neighborhood. However, taking a 'square root' of $x-1$ yields a reducible étale neighborhood. More specifically, define $C' = \operatorname{Spec} k[x,y,t]_t/(y^2-x^3+x^2, t^2-x+1)$ and consider

$$C' \to C, \qquad (x,y,t) \mapsto (x,y)$$

Since $y^2 - x^3 + x^2 = (y-xt)(y+xt)$, we see that C' is reducible.

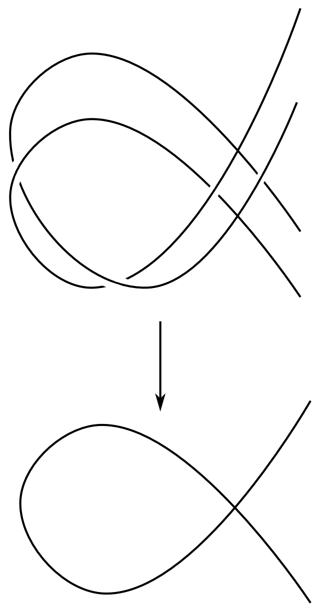


Figure 17: After an étale cover, the nodal cubic becomes reducible.

Example 0.5.3 (Étale cohomology)**.** Sheaf cohomology for the Zarisk-topology can be extended to the étale topology leading to the extremely robust theory of *étale cohomology.* As an example, consider a smooth projective curve C over $\mathbb{C}$ (or equivalently a Riemann surface of genus g), then the étale cohomology $\mathrm{H}^1(C_{\text{ét}}, \mathbb{Z}/n)$ of the finite constant sheaf is isomorphic to $(\mathbb{Z}/n)^{2g}$ just like the ordinary cohomology groups, while the sheaf cohomology $\mathrm{H}^1(C, \mathbb{Z}/n)$ in the Zariski-topology is 0.

Finally, we would be remiss without mentioning the spectacular application of étale cohomology to prove the Weil conjectures.

Example 0.5.4 (Étale fundamental group)**.** Have you ever thought that there is a similarity between the bijection in Galois theory between intermediate field extensions and subgroups of the Galois group, and the bijection in algebraic topology between covering spaces and subgroups of the fundamental group? Well, you're in good company—Grothendieck also considered this and developed a beautiful theory of the *étale fundamental group* which packages Galois groups and fundamental groups in the same framework.

We only point out here that this connection between étale morphisms and Galois theory is perhaps not so surprising given that a finite field extension L/K is étale (i.e. $\operatorname{Spec} L \to \operatorname{Spec} K$ is étale) if and only if L/K is separable. While we only defined étaleness above for $\mathbb{C}$-varieties, the general notion is not much more complicated; see Étale Equivalences A.3.2.

For the reader interested in reading more about étale cohomology or the étale fundamental group, we recommend [Mil80].

Example 0.5.5 (Quotients by free actions of finite groups)**.** If G is a finite group acting freely on a projective variety X, then there exists a quotient X/G as a projective variety. The essential reason for this is that every G-orbit (or in fact every finite set of points) is contained in an affine variety U, which is the complement of some hypersurface. Then the intersection $V = \bigcap_g gU$ of the G-translates is a G-invariant affine open containing Gx. One can then show that $V/G = \operatorname{Spec} \Gamma(V, \mathcal{O}_V)^G$ and that these local quotients glue to form X/G.

However, if X is not projective, the quotient does not necessarily exist as a scheme. As with most phenomenon for smooth proper varieties that are not-projective, a counterexample is provided by Hironaka's examples of smooth, proper 3-folds; [Har77, App. B, Ex. 3.4.1]. One can construct an example which has a free action by $G = \mathbb{Z}/2$ such that there is an orbit Gx not contained in any G-invariant affine open. This shows that X/G cannot exist as a scheme; indeed, if it did, then the image of x under the finite morphism $X \to X/G$ would be contained in some affine and its inverse would be an affine open containing Gx. See [Knu71, Ex. 1.3] or [Ols16, Ex. 5.3.2] for details.

Nevertheless, for every free action of a finite group G on a scheme X, there does exist a G-invariant étale morphism $U \to X$ from an affine scheme, and the quotients U/G can be glued in the *étale topology* to construct X/G as an *algebraic space*. The upshot is that we can always take quotients of free actions by finite groups, a very desirable feature given the ubiquity of group actions in algebraic geometry; this however comes at the cost of enlarging our category from schemes to algebraic spaces.

Example 0.5.6 (Artin approximation)**.** *Artin approximation* is a powerful and extremely deep result, due to Michael Artin, which implies that most properties which hold for the completion $\widehat{\mathcal{O}}_{X,x}$ of the local ring is also true in an étale neighborhood of x. More precisely, let $F\colon \operatorname{Sch}/X \to \operatorname{Sets}$ be a functor locally of finite presentation (i.e. satisfying the functorial property of Proposition A.1.3), $\widehat{a} \in F(\widehat{\mathcal{O}}_{X,x})$ and N a positive integer. Under the weak hypothesis of excellency on X (which holds if X is locally of finite type over $\mathbb{Z}$ or a field), Artin approximation states that there exists an étale neighborhood $(X', x') \to (X, x)$ with $\kappa(x') = \kappa(x)$ and an element $a' \in F(X')$ agreeing with a on the Nth order neighborhood of x.

For example, in Example 0.5.2, it's not hard to use properties of power series rings to establish that $\widehat{\mathcal{O}}_{C,p} \cong \mathbb{C}[\![x, y]\!]/(y^2 - x^2)$ (e.g. take a power series expansion of $\sqrt{x-1}$), which is reducible. If we consider the functor

$$F\colon \operatorname{Sch}/C \to \operatorname{Sets}, \qquad (C' \xrightarrow{\pi} C) \mapsto \{\text{decompositions } C' = C'_1 \cup C'_2\}$$

then applying Artin approximation yields an étale cover $C' \to C$ with C' reducible. Of course, we already knew this from an explicit construction in Example 0.5.2, but hopefully this example shows the potential power of Artin approximation.

0.5.3 Working with the étale topology: descent theory

Another reason why the étale topology is so useful is that many properties of schemes and their morphisms can be checked on étale covers. For instance, you already know that to check if a scheme X is noetherian, finite type over $\mathbb{C}$, reduced or smooth, it suffices to find a Zariski open cover $\{U_i\}$ such that the property holds for each U_i. *Descent theory* implies the same with respect to a collection $\{U_i \to U\}$ of étale morphisms such that $\coprod_i U_i \to U$ is surjective: X has the property if and only if each U_i does. Descent theory is developed in Chapter B and is used to prove just about everything concerning algebraic spaces and stacks.

0.6 Moduli stacks: moduli with automorphisms

The failure of the representability of the moduli functors of curves and vector bundles is a motivating factor for introducing moduli stacks, which encode the automorphisms groups as part of the data. We will synthesize the approaches from Section 0.3 on moduli groupoids and Section 0.4 on moduli functors.

0.6.1 Specifying a moduli stack

To define a moduli stack, we need to specify

1. families of objects;
2. how two families of objects are isomorphic; and
3. how families pull back under morphisms.

Notice the difference from specifying a moduli functor (Section 0.4.2) is that rather than specifying *when* two families are isomorphic, we specify *how*.

To specify a moduli stack in the algebro-geometric setting, we need to specify for each scheme T a *groupoid* Fam_T of families of objects over T. As a natural generalization of functors to sets, we could consider assignments

$$F\colon \mathrm{Sch} \to \mathrm{Groupoids}, \quad T \mapsto \mathrm{Fam}_T\,.$$

This presents the technical difficulty of considering functors between the category of schemes and the 'category' of groupoids. Morphisms of groupoids are functors but there are also morphisms of functors (i.e. natural transformations) which we call *2-morphisms*. This leads to a '2-category' of groupoids.

What is actually involved in defining such an assignment F? In addition to defining the groupoids Fam_T over each scheme T, we need pullback functors $f^*\colon \mathrm{Fam}_T \to \mathrm{Fam}_S$ for each morphism $f\colon S \to T$. But what should be the compatibility for a composition $S \xrightarrow{f} T \xrightarrow{g} U$ of schemes? Well, there should be an isomorphism of functors (i.e. a 2-morphism) $\mu_{f,g}\colon (f^* \circ g^*) \xrightarrow{\sim} (g \circ f)^*$. Should the isomorphisms $\mu_{f,g}$ satisfy a compatibility condition under triples $S \xrightarrow{f} T \xrightarrow{g} U \xrightarrow{h} V$? Yes, but we won't spell it out here (although we encourage the reader to work it out). Altogether this leads to the concept of a *pseudo-functor* (see [SP, Tag 003N]). We will take another approach however in specifying prestacks that avoids specifying such compatibility data.

0.6.2 Motivating the definition of a prestack

Instead of trying to define an assignment $T \mapsto \mathrm{Fam}_T$, we will build one massive category $\mathcal{X}$ encoding all of the groupoids Fam_T which will live over the category Sch of schemes. Loosely speaking, the objects of $\mathcal{X}$ will be a family a of objects over a scheme S, i.e. $a \in \mathrm{Fam}_S$. If $a \in \mathrm{Fam}_S$ and $b \in \mathrm{Fam}_T$, a morphism $a \to b$ in $\mathcal{X}$ will be a morphism $f\colon S \to T$ together with an isomorphism $a \xrightarrow{\sim} f^*b$.

A *prestack* over Sch is a category $\mathcal{X}$ together with functor $p\colon \mathcal{X} \to \mathrm{Sch}$, which we visualize as

$$\begin{tikzcd} \mathcal{X} \arrow[d, "p"] \\ \mathrm{Sch} \end{tikzcd} \qquad \begin{tikzcd} a \arrow[r, "\alpha"] \arrow[d, mapsto] & b \arrow[d, mapsto] \\ S \arrow[r, "f"] & T \end{tikzcd}$$

where the lower case letters a, b are objects in $\mathcal{X}$ and the upper case letters S, T are objects in Sch. We say that a is over S and $\alpha\colon a \to b$ is over $f\colon S \to T$. Moreover, we need to require certain natural axioms to hold for $\mathcal{X} \xrightarrow{p} \mathrm{Sch}$. This will be given in full later but vaguely we need to require the existence and uniqueness of pullbacks: given a map $S \to T$ and object $b \in \mathcal{X}$ over T, there should exist an arrow $a \xrightarrow{\alpha} b$ over f satisfying a suitable universal property. See Definition 2.3.1 for a precise definition.

Given a scheme S, the *fiber category* $\mathcal{X}(S)$ is the category of objects over S whose morphisms are over id_S. If $\mathcal{X}$ is built from the groupoids Fam_S as above, then the fiber category $\mathcal{X}(S) = \mathrm{Fam}_S$.

Example 0.6.1 (Viewing a moduli functor as a moduli prestack)**.** A moduli functor $F\colon \mathrm{Sch} \to \mathrm{Sets}$ can be encoded as a moduli prestack as follows: we define the category $\mathcal{X}_F$ of pairs (S, a) where S is a scheme and $a \in F(S)$. A map $(S', a) \to (S, a)$ is a map $f\colon S' \to S$ such that $a' = f^*a$, where f^* is convenient shorthand for $F(f)\colon F(S) \to F(S')$. Observe that the fiber categories $\mathcal{X}_F(S)$ are equivalent (even equal) to the set $F(S)$.

Example 0.6.2 (Moduli prestack of smooth curves)**.** We define the *moduli prestack of smooth curves* as the category $\mathscr{M}_g$ of families of smooth curves $\mathcal{C} \to S$ together with the functor $p\colon \mathscr{M}_g \to \mathrm{Sch}$ where $(\mathcal{C} \to S) \mapsto S$. A map $(\mathcal{C}' \to S') \to (\mathcal{C} \to S)$ is the data of maps $\alpha\colon \mathcal{C}' \to \mathcal{C}$ and $f\colon S' \to S$ such that the diagram

$$\begin{tikzcd} \mathcal{C}' \arrow[r, "\alpha"] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{C} \arrow[d] \\ S' \arrow[r, "f"] & S \end{tikzcd}$$

is cartesian.

Example 0.6.3 (Moduli prestack of vector bundles)**.** Let C be a fixed smooth, connected, and projective curve over $\mathbb{C}$, and fix integers $r \geq 0$ and d. We define the *moduli prestack of vector bundles on* C as the category $\mathcal{B}\mathrm{un}_{r,d}(C)$ of pairs (E, S) where S is a scheme and E is a vector bundle on $C_S = C \times_{\mathbb{C}} S$ together with the functor $p\colon \mathcal{B}\mathrm{un}_{r,d}(C) \to \mathrm{Sch}/\mathbb{C}, (E,S) \mapsto S$. A map $(E',S') \to (E,S)$ consists of a map of schemes $f\colon S' \to S$ together with a map $E \to (\mathrm{id} \times f)_* E'$ of $\mathcal{O}_{C_S}$-modules whose adjoint is an isomorphism (i.e. for every choice of pull back $(\mathrm{id} \times f)^* E$, the adjoint map $(\mathrm{id} \times f)^* E \to E'$ is an isomorphism). Note that a map $(E',S) \to (E,S)$ over the identity map id_S consists simply of an isomorphism $E' \to E$.

Remark 0.6.4. We have formulated morphisms using the adjoint because the pull back is only defined up to isomorphism while the pushforward is canonical. If we were to instead parameterize the total spaces of vector bundles (i.e. $\mathbb{A}(E)$ rather than E), then a morphism $(V',S') \to (V,S)$ would consist of morphisms $\alpha\colon V' \to V$ and $f\colon S' \to S$ such that $V' \to V \times_{C_S} C_{S'}$ is an isomorphism of vector bundles.

0.6.3 Motivating the definition of a stack

A stack is to a prestack as a sheaf is to a presheaf. The concept could not be more intuitive: we require that objects and morphisms glue uniquely.

Example 0.6.5 (Moduli stack of sheaves over a point)**.** Define the category $\mathcal{X}$ over Sch of pairs (E,S) where E is a sheaf of abelian groups on a scheme S, and the functor $p\colon \mathcal{X} \to \mathrm{Sch}$ given by $(E,S) \mapsto S$. A map $(E',S') \to (E,S)$ in $\mathcal{X}$ is a map of schemes $f\colon S' \to S$ together with a map $E \to f_* E'$ of $\mathcal{O}_{S'}$-modules whose adjoint is an isomorphism.

You already know that morphisms of sheaves glue [Har77, Exer. II.1.15]: let E and F be sheaves on schemes S and T, and let $f\colon S \to T$ be a map. If $\{S_i\}$ is a Zariski open cover of S, then giving a morphism $\alpha\colon (E,S) \to (F,T)$ is the same data as giving morphisms $\alpha_i\colon (E|_{S_i}, S_i) \to (F,T)$ such that $\alpha_i|_{S_{ij}} = \alpha_j|_{S_{ij}}$.

You also know how sheaves themselves glue [Har77, Exer. II.1.22]—it is more complicated than gluing morphisms since sheaves have automorphisms and given two sheaves, we prefer to say that they are isomorphic rather than equal. If $\{S_i\}$ is a Zariski open cover of a scheme S, then giving a sheaf E on S is equivalent to giving a sheaf E_i on S_i and isomorphisms $\phi_{ij}\colon E_i|_{S_{ij}} \to E_j|_{S_{ij}}$ such that $\phi_{ik} = \phi_{jk} \circ \phi_{ij}$ on the triple intersection S_{ijk}.

In an identical way, we could have considered the moduli stack of $\mathcal{O}$-modules, quasi-coherent sheaves or vector bundles.

The definition of a stack simply axiomitizes these two natural gluing concepts; it is postponed until Definition 2.4.1.

Exercise 0.6.6. Convince yourself that Examples 0.6.2 and 0.6.3 satisfy the same gluing axioms. (See also Propositions 2.4.11 and 2.4.13.)

0.6.4 Motivating the definition of an algebraic stack

There are functors $F\colon \mathrm{Sch} \to \mathrm{Sets}$ that are sheaves when restricted to the Zariski topology on a scheme T but that are not necessarily representable by schemes; see for instance Examples 3.9.1 and 3.9.2. In a similar way, there are prestacks $\mathcal{X}$ that are stacks but that are not sufficiently algebro-geometric. If we wish to bring our algebraic geometry toolkit (e.g. coherent sheaves, commutative algebra, cohomology, ...) to study stacks in a similar way that we study schemes, we must impose an algebraicity condition.

The condition we impose on a stack to be algebraic is very natural. Recall that a functor $F\colon \mathrm{Sch} \to \mathrm{Sets}$ is representable by a scheme if and only if there is a Zariski open cover $\{U_i \subset F\}$ such that U_i is an affine scheme. Similarly, we will say that a stack $\mathcal{X} \to \mathrm{Sch}$ is *algebraic* if

- there is a smooth cover $\{U_i \to \mathcal{X}\}$ where each U_i is an affine scheme.

To make this precise, we need to define what it means for $\{U_i \to \mathcal{X}\}$ to be a smooth cover. Just like in the definition of Zariski open cover (Definition 0.4.21(3)), we require that for every morphism $T \to \mathcal{X}$ from a scheme T, the fiber product (fiber products of prestacks will be formally introduced in §2.3.5) $U_i \times_{\mathcal{X}} T$ is representable (by an algebraic space) such that $\coprod_i U_i \times_{\mathcal{X}} T \to T$ is a smooth and surjective morphism. See Definition 3.1.6 for the precise definition of an algebraic stack.

Constructing a smooth cover of a given moduli stack is a geometric problem inherent to the moduli problem. It can often be solved by ridigifying the moduli problem by parameterizing additional information. This concept is best absorbed in examples.

Example 0.6.7 (Moduli stack of elliptic curves)**.** Over $\mathbb{C}$ (or any field $\Bbbk$ with $\mathrm{char}(\Bbbk) \neq 2, 3$), an elliptic curve (E, p) is embedded into $\mathbb{P}^2$ via $\mathcal{O}_E(3p)$ such that E is defined by a Weierstrass equation $y^2z = x(x-z)(x-\lambda z)$ for some $\lambda \neq 0, 1$ [Har77, Prop. IV.4.6]. Let $U = \mathbb{A}^1 \setminus \{0, 1\}$ with coordinate λ. The family $\mathcal{E} \subset U \times \mathbb{P}^2$ of elliptic curves defined by the Weierstrass equation gives a smooth (even étale) cover $U \to \mathcal{M}_{1,1}$.

Example 0.6.8 (Moduli stack of smooth curves)**.** For every smooth, connected, and projective curve C of genus $g \geq 2$, the third tensor power $\omega_C^{\otimes 3}$ is very ample and gives an embedding $C \hookrightarrow \mathbb{P}(\mathrm{H}^0(C, \omega_c^{\otimes 3})) \cong \mathbb{P}^{5g-6}$. There is a Hilbert scheme H parameterizing closed subschemes of $\mathbb{P}^{5g-6}$ with the same Hilbert polynomial as $C \subset \mathbb{P}^{5g-6}$, and there is a locally closed subscheme $H' \subset H$ parameterizing *smooth* subschemes such that $\omega_C^{\otimes 3} \cong \mathcal{O}_C(1)$. The universal subscheme over H' yields a smooth cover $H' \to \mathscr{M}_g$.

Example 0.6.9 (Moduli stack of vector bundles)**.** For every vector bundle E of rank r and degree d on a smooth, connected, and projective curve C, the twist $E(m)$ is globally generated for sufficiently large m. Taking $N = \mathrm{h}^0(C, E(m))$, we can view E as a quotient $\mathcal{O}_C(-m)^N \twoheadrightarrow E$. There is a Quot scheme Q_m parameterizing quotients $\mathcal{O}_C(-m)^N \xrightarrow{\pi} F$ with the same Hilbert polynomial as E and a locally closed subscheme $Q'_m \subset Q$ parameterizing quotients where E is a vector bundle and such that the induced map $\mathrm{H}^0(\pi \otimes \mathcal{O}_C(m))\colon \mathbb{C}^N \to \mathrm{H}^0(C, E(m))$ is an isomorphism. The universal quotient over Q'_m defines a smooth map $Q'_m \to \mathcal{B}\mathrm{un}_{r,d}(C)$ and the collection $\{Q'_m \to \mathcal{B}\mathrm{un}_{r,d}(C)\}$ over $m \gg 0$ defines a smooth cover.

0.6.5 Deligne–Mumford stacks and algebraic spaces

A *Deligne–Mumford stack* can be defined in two equivalent ways:

- a stack $\mathcal{X}$ such that there exists an étale (rather than smooth) cover $\{U_i \to \mathcal{X}\}$ by schemes; or
- an algebraic stack such that all automorphisms groups of field-valued points are étale, i.e. discrete (e.g. finite) and reduced.

The moduli stacks $\mathscr{M}_g$ and $\overline{\mathcal{M}}_g$ are Deligne–Mumford for $g \geq 2$, but $\mathcal{B}\mathrm{un}_{r,d}(C)$ is not. Similarly, an *algebraic space* can be defined in two equivalent ways:

- a sheaf (i.e. a contravariant functor $F\colon$ Sch $\to$ Sets that is a sheaf in the big étale topology) such that there exists an étale cover $\{U_i \to F\}$ by schemes; or
- an algebraic stack such that all automorphisms groups of field-valued points are trivial.

In other words, an algebraic space is an algebraic stack without any stackiness.

Table 1: Schemes, algebraic spaces, Deligne–Mumford stacks, and algebraic stacks are obtained by gluing affine schemes in certain topologies

Algebro-geometric space	Type of object	Obtained by gluing
Schemes	sheaf	affine schemes in the Zariski topology
Algebraic spaces	sheaf	affine schemes in the étale topology
Deligne–Mumford stacks	stack	affine schemes in the étale topology
Algebraic stacks	stack	affine schemes in the smooth topology

Example 0.6.10 (Quotients by finite groups)**.** Quotients by free actions of finite groups exist as algebraic spaces! See Corollary 3.1.12.

0.7 Moduli stacks and quotients

One of the most important examples of a stack is a quotient stack $[X/G]$ arising from an action of a smooth algebraic group G on a scheme X. The geometry of $[X/G]$ couldn't be simpler: it's the G-equivariant geometry of X.

Similar to how toric varieties provide concrete examples of schemes, quotient stacks provide both concrete examples useful to gain geometric intuition of general algebraic stacks and a fertile testing ground for conjectural results. On the other hand, it turns out that many algebraic stacks are quotient stacks (or at least locally quotient stacks) and therefore any (local) property that holds for quotient stacks also holds for many algebraic stacks.

0.7.1 Motivating the definition of the quotient stack

The quotient functor Sch $\to$ Sets defined by $S \mapsto X(S)/G(S)$ is not a sheaf even when the action is free (see Exercise 0.4.19). We therefore first need to consider a better notion for a family of orbits.

For simplicity, let's assume that G and X are defined over $\mathbb{C}$. For $x \in X(\mathbb{C})$, there is a G-equivariant map $\sigma_x\colon G \to X$ defined by $g \mapsto g \cdot x$. Note that two points

x, x' are in the same G-orbit (say $x = hx'$), if and only if there is a G-equivariant morphism $\varphi\colon G \to G$ (say by $g \mapsto gh$) such that $\sigma_x = \sigma_{x'} \circ \varphi$.

We can try the same thing for a T-point $T \xrightarrow{f} X$ by considering

$$\begin{tikzcd} G \times T \arrow[r, "f"] \arrow[d, "p_2"] & X, \\ T \end{tikzcd} \qquad (g, t) \longmapsto g \cdot f(t)$$

and noting that $f\colon G \times T \to X$ is a G-equivariant map. If we define a prestack consisting of such families, it fails to be a stack as objects don't glue: given a Zariski-cover $\{T_i\}$ of T, maps $T_i \xrightarrow{f_i} X$ and isomorphisms of the restrictions to T_{ij}, the trivial bundles $G \times T_i \to T_i$ will glue to a principal G-bundle $P \to T$ but it will *not* necessarily be trivial (i.e. $P \cong G \times T$). It is clear then how to correct this using the language of principal G-bundles (see Section C.2):

Definition 0.7.1 (Quotient stack)**.** We define $[X/G]$ as the category over Sch whose objects over a scheme S are diagrams

$$\begin{tikzcd} P \arrow[r, "f"] \arrow[d] & X \\ S \end{tikzcd}$$

where $P \to S$ is a principal G-bundle and $f\colon P \to X$ is a G-equivariant morphism. A morphism $(P' \to S', P' \xrightarrow{f'} X) \to (P \to S, P \xrightarrow{f} X)$ consists a maps $g\colon S' \to S$ and $\varphi\colon P' \to P$ of schemes such that the diagram

$$\begin{tikzcd} P' \arrow[r, "\varphi"] \arrow[d] \arrow[rr, bend left, "f'"] \arrow[dr, phantom, "\square"] & P \arrow[r, "f"] \arrow[d] & X \\ S' \arrow[r, "g"] & S \end{tikzcd}$$

commutes with the left square cartesian.

There is an object of $[X/G]$ over X given by the diagram

$$\begin{tikzcd} G \times X \arrow[r, "\sigma"] \arrow[d, "p_2"] & X \\ X, \end{tikzcd}$$

where σ denotes the action map. This corresponds to a map $X \to [X/G]$ via a 2-categorical version of Yoneda's lemma.

The map $X \to [X/G]$ is a principal G-bundle even if the action of G on X is not free. We state that again: **the map $X \to [X/G]$ is a principal G-bundle even if the action of G on X is not free**. Pause for a moment to appreciate how remarkable that is!

In particular, the map $X \to [X/G]$ is smooth and it follows that $[X/G]$ is algebraic. At the expense of enlarging our category from schemes to algebraic stacks, we are able to (tautologically) construct the quotient $[X/G]$ as a 'geometric space' with desirable geometric properties.

Example 0.7.2. Specializing to the case that $X = \operatorname{Spec}\mathbb{C}$ is a point, we define the *classifying stack of* G as the category $BG := [\operatorname{Spec}\mathbb{C}/G]$ of principal G-bundles $P \to S$. The projection $\operatorname{Spec}\mathbb{C} \to BG$ is not only a principal G-bundle; it is the *universal principal* G*-bundle*. Given any other principal G-bundle $P \to S$, there is a unique map $S \to BG$ and a cartesian diagram

$$\begin{tikzcd} P \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \operatorname{Spec}\mathbb{C} \arrow[d] \\ S \arrow[r] & BG. \end{tikzcd}$$

Exercise 0.7.3. What is the universal family over the quotient stack $[X/G]$?

0.7.2 Moduli as quotient stacks

Moduli stacks can often be described as quotient stacks, and these descriptions can be leveraged to establish properties of the moduli stack.

Example 0.7.4 (Moduli stack of smooth curves)**.** In Example 0.6.8, the embedding of a smooth curve C via $C \overset{|\omega_C^{\otimes 3}|}{\hookrightarrow} \mathbb{P}^{5g-6}$ depends on a choice of basis $\mathrm{H}^0(C, \omega_C^{\otimes 3}) \cong \mathbb{C}^{5g-5}$ and therefore is only unique up to a projective automorphism, i.e. an element of $\mathrm{PGL}_{5g-5} = \mathrm{Aut}(\mathbb{P}^{5g-6})$. The action of the algebraic group PGL_{5g-5} on the scheme H', parameterizing *smooth* subschemes such that $\omega_C \cong \mathcal{O}_C(3)$, yields an identification $\mathscr{M}_g \cong [H'/\mathrm{PGL}_{5g-6}]$. See Theorem 3.1.15.

Example 0.7.5 (Moduli stack of vector bundles)**.** In Example 0.6.9, the presentation of a vector bundle E as a quotient $\mathcal{O}_C(-m)^N \twoheadrightarrow E$ depends on a choice of basis $\mathrm{H}^0(C, E(m)) \cong \mathbb{C}^N$. The algebraic group PGL_{N-1} acts on the scheme Q'_m, parameterizing vector bundle quotients of $\mathcal{O}_C(-m)^N$ such that $\mathbb{C}^N \xrightarrow{\sim} \mathrm{H}^0(C, E(m))$, yields an identification $\mathcal{B}\mathrm{un}_{r,d}(C) \cong \bigcup_{m \gg 0}[Q'_m/\mathrm{PGL}_{N-1}]$. See Theorem 3.1.19.

0.7.3 Geometry of $[X/G]$

While the definition of the quotient stack $[X/G]$ may appear abstract, its geometry is very familiar. The table below provides a dictionary between the geometry of a quotient stack $[X/G]$ and the G-equivariant geometry of X. The stack-theoretic concepts on the left-hand side will be introduced later. For simplicity we work over $\mathbb{C}$.

Table 2: Dictionary

Geometry of $[X/G]$	G-equivariant geometry of X
$\mathbb{C}$-point $\overline{x} \in [X/G]$	orbit Gx of $\mathbb{C}$-point $x \in X$ (with $\overline{x}$ the image of x under $X \to [X/G]$)
automorphism group $\mathrm{Aut}(\overline{x})$	stabilizer G_x
function $f \in \Gamma([X/G], \mathcal{O}_{[X/G]})$	G-equivariant function $f \in \Gamma(X, \mathcal{O}_X)^G$
map $[X/G] \to Y$ to a scheme Y	G-equivariant map $X \to Y$
line bundle	G-equivariant line bundle (or G-linearization)
quasi-coherent sheaf	G-equivariant quasi-coherent sheaf
tangent space $T_{[X/G],\overline{x}}$	normal space $T_{X,x}/T_{Gx,x}$ to the orbit
coarse moduli space $[X/G] \to Y$	geometric quotient $X \to Y$
good moduli space $[X/G] \to Y$	good GIT quotient $X \to Y$

0.8 Constructing moduli spaces as projective varieties

One of the primary reasons for introducing algebraic stacks to begin with is to ensure that a given moduli problem $\mathcal{M}$ is in fact represented by a bona fide algebro-geometric space equipped with a universal family. Many geometric questions can be answered (and arguably should be answered) by studying the moduli stack $\mathcal{M}$ itself. However, even in the presence of automorphisms, there still may exist a scheme—even a projective variety— that closely approximates the moduli problem. If we are willing to sacrifice some desirable properties (e.g. a universal family), we can sometimes construct a more familiar algebro-geometric space—namely a projective variety—where we have the much larger toolkit of projective geometry (e.g., Hodge theory, birational geometry, intersection theory, ...) at our disposal.

In this section, we present a general strategy for a constructing a moduli space specifically as a *projective variety.*

0.8.1 Boundedness

The first potential problem is that our moduli problem may simply have too many objects so that there is no hope of representing it by a *finite type* or *quasi-compact* scheme. We say that a moduli functor or stack $\mathcal{M}$ over $\mathbb{C}$ is *bounded* if there exists a scheme X of finite type over $\mathbb{C}$ and a family of objects $\mathcal{E}$ over X such that every object E of $\mathcal{M}$ is isomorphic to a fiber $E \cong \mathcal{E}_x$ for some (not necessarily unique) $x \in X(\mathbb{C})$.

Example 0.8.1. Let Vect be the algebraic stack over $\mathbb{C}$ where objects over a scheme S consist of vector bundles. Since we have not specified the rank, $\mathrm{Vect}_{\mathbb{C}}$ is not bounded. In fact, if we let $\mathrm{Vect}_r \subset \mathrm{Vect}$ be the substack parameterizing vector bundles of rank r, then $\mathrm{Vect} = \coprod_{r \geq 0} \mathrm{Vect}_r$ While Vect is locally of finite type over $\mathbb{C}$, it is not of finite type (or equivalently quasi-compact).

Exercise 0.8.2. Show that Vect_r is isomorphic to the classifying stack $B\,\mathrm{GL}_r$ (Example 0.7.2).

Example 0.8.3. Let $\mathcal{V}$ be the stack of *all* vector bundles over a smooth, connected, and projective curve C. The stack $\mathcal{V}$ is clearly not bounded since we haven't specified the rank and degree. But even the substack $\mathcal{B}\mathrm{un}_{r,d}(C)$ of vector bundles with prescribed rank and degree is not bounded! For example, on $\mathbb{P}^1$, there are vector bundles $\mathcal{O}(-d) \oplus \mathcal{O}(d)$ of rank 2 and degree 0 for every $d \in \mathbb{Z}$, and not all of them can arise as the fibers of a single vector bundle on a finite type $\mathbb{C}$-scheme.

Exercise 0.8.4. Prove that $\mathcal{B}\mathrm{un}_{r,d}(C)$ is not bounded for every curve C.

Although $\mathcal{B}\mathrm{un}_{r,d}(C)$ is not bounded, we will study the substack $\mathcal{B}\mathrm{un}_{r,d}(C)^{\mathrm{ss}}$ of *semistable* vector bundles which is bounded. Semistable vector bundles admit a number of remarkable properties with boundedness being one of the most important.

0.8.2 Compactness

Projective varieties are compact so if we are going to have any hope to construct a projective moduli space, the moduli stack better be compact as well. However, many moduli stacks such as $\mathscr{M}_g$ are not compact as they don't have *enough* objects. This is in contrast to the issue of non-boundedness where there may be *too many* objects.

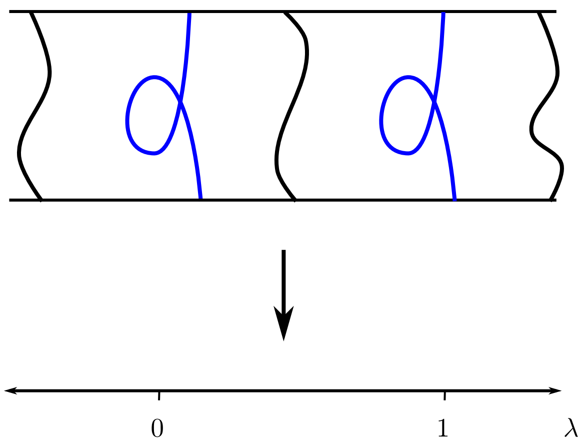


Figure 18: The family of elliptic curves $y^2z = x(x-z)(x-\lambda z)$ degenerates to the nodal cubic over $\lambda = 0, 1$.

The scheme-theoretic notion for compactness is *properness*—universally closed, separated and of finite type. There is a conceptual criterion to test properness called the *valuative criterion* which loosely speaking requires one-dimensional limits to exist. The usefulness of the valuative criterion is arguably best witnessed through studying moduli problems.

More precisely, a moduli stack $\mathcal{M}$ of finite type over $\mathbb{C}$ is *proper* (resp. *universally closed*, *separated*) if for every DVR R with fraction field K and for every diagram

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & \mathcal{M} \\ \operatorname{Spec} R, \arrow[ur, dashed] & \end{tikzcd} \tag{0.8.1}$$

after possibly allowing for an extension of R, there exists a unique extension (resp. there exists an extension, resp. there exists at most one extension) of the above diagram.[4] Since $\mathcal{M}$ is a moduli stack, a map $\operatorname{Spec} K \to \mathcal{M}$ corresponds to an object $E^\times$ over $\operatorname{Spec} K$ and a dotted arrow corresponds to a family of objects E over $\operatorname{Spec} R$ and an isomorphism $E|_{\operatorname{Spec} K} \cong E^\times$. In other words, properness of $\mathcal{M}$ means that every object E^* over the punctured disk $\operatorname{Spec} K$ extends uniquely (after possibly allowing for an extension of R) to a family E of objects over the entire disk $\operatorname{Spec} R$.

Example 0.8.5. The moduli stack $\mathscr{M}_g$ of smooth curves is not proper as exhibited in Figure 18. The pioneering insight of Deligne and Mumford is that there is a *moduli-theoretic compactification*! Namely, there is an algebraic stack $\overline{\mathscr{M}_g}$ parameterizing *Deligne–Mumford stable curves*, i.e. proper curves C with at worst nodal singularities such that every smooth rational subcurve $\mathbb{P}^1 \subset C$ intersects the rest of the curve along at least three points. The stack $\overline{\mathcal{M}}_g$ is a proper algebraic stack (due to the stable reduction theorem for curves) and contains $\mathscr{M}_g$ as an open substack.

Example 0.8.6. Let $\mathcal{B}\mathrm{un}_{r,d}(C)^{\mathrm{ss}}$ be the moduli stack parameterizing semistable vector bundles over a curve of prescribed rank and degree. We will later show that $\mathcal{B}\mathrm{un}_{r,d}(C)^{\mathrm{ss}}$ is an algebraic stack of finite type over $\mathbb{C}$. Langton's semistable reduction theorem states that $\mathcal{B}\mathrm{un}_{r,d}(C)^{\mathrm{ss}}$ is universally closed, i.e. satisfies the existence part of the above valuative criterion.

However $\mathcal{B}\mathrm{un}_{r,d}(C)^{\mathrm{ss}}$ is not separated as there may exist several non-isomorphic extensions of a vector bundle on C_K to C_R. Indeed, let E be vector bundle and consider the trivial family E_K on C_K. This extends to trivial family E_R over C_R but the data of an extension

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r, "{[E_K]}"] \arrow[d] & \mathcal{B}\mathrm{un}_{r,d}(C)^{\mathrm{ss}} \\ \operatorname{Spec} R, \arrow[ur, dashed, "{[E_R]}"'] & \end{tikzcd}$$

also consists of an isomorphism $E_R|_{C_K} = E_K \xrightarrow{\sim} E_K$ or equivalently a K-point of $\operatorname{Aut}(E)$. There are many such isomorphisms and some don't extend to R-points. The automorphism group of a vector bundle is a positive dimensional affine algebraic group containing a copy of $\mathbb{G}_m$ corresponding to scaling. For instance, if $\pi \in K$ is a uniformizing parameter, the automorphism $1/\pi \in \mathbb{G}_m(K)$ does not extend to $\mathbb{G}_m(R)$ so (E_R, id) and $(E_R, 1/\pi)$ give non-isomorphic extensions of E_K. In a similar way, every moduli stack which has an object with a positive dimensional affine automorphism group is not separated.

[4]The valuative criterion can be equivalently formulated by replacing the local curve $\operatorname{Spec} R$ with a smooth curve C and $\operatorname{Spec} K$ with a puncture curve $C \setminus p$.

0.8.3 Enlarging a moduli stack

It is often useful to consider enlargements $\mathcal{X} \subset \mathcal{M}$ of a given moduli stack $\mathcal{X}$ by parameterizing a larger collection of objects. For instance, rather than just considering smooth or Deligne–Mumford stable curve, you could consider all curves, or rather than considering semistable vector bundles, you could consider all vector bundles or even all coherent sheaves.

Let's call an object of $\mathcal{M}$ *semistable* if is isomorphic to an object of $\mathcal{X}$; in this way, we can view $\mathcal{X} = \mathcal{M}^{\mathrm{ss}} \subset \mathcal{M}$ as the substack of semistable objects. Often it is easier to show properties (e.g. algebraicity) for $\mathcal{M}$ and then infer the corresponding property for $\mathcal{M}^{\mathrm{ss}}$.

0.8.4 The six steps toward projective moduli

In the setting of a moduli stack $\mathcal{M}^{\mathrm{ss}}$ of semistable objects and an enlargement $\mathcal{M}^{\mathrm{ss}} \subset \mathcal{M}$, we outline the steps to construct a projective moduli scheme M^{ss} approximating $\mathcal{M}^{\mathrm{ss}}$.[5]

Step 1 (Algebraicity): $\mathcal{M}$ is an algebraic stack locally of finite type over $\mathbb{C}$.

This requires first defining $\mathcal{M}$ by specifying both (1) families of objects over an arbitrary $\mathbb{C}$-scheme S, (2) how two families are isomorphic, and (3) how families pull back; see Section 0.6.1. One must then check that $\mathcal{M}$ is a stack.

To check that $\mathcal{M}$ is an algebraic stack locally of finite type over $\mathbb{C}$ entails finding a smooth cover of $\{U_i \to \mathcal{M}\}$ by affine schemes (see Section 0.6.4) where each U_i is of finite type over $\mathbb{C}$.

An alternative approach is to verify 'Artin's criteria' for algebraicity which essentially amounts to verifying local properties of the moduli problem and in particular requires an understanding of the deformation and obstruction theory.

Step 2 (Openness of semistability): semistability is an open condition, i.e. $\mathcal{M}^{\mathrm{ss}} \subset \mathcal{M}$ is an open substack.

If E is an object of $\mathcal{M}$ over T, one must show that the locus of points $t \in T$ such that the restriction E_t is semistable is an open subset of T. Indeed, just like in the definition of an open subfunctor, a substack $\mathcal{M}^{\mathrm{ss}} \subset \mathcal{M}$ is open if and only if for all maps $T \to \mathcal{M}$, the fiber product $\mathcal{M}_{\mathrm{ss}} \times_{\mathcal{M}} T$ is an open subscheme of T. This ensures in particular that $\mathcal{M}^{\mathrm{ss}}$ is also an algebraic stack locally of finite type over $\mathbb{C}$.

Step 3 (Boundedness of semistability): semistability is bounded, i.e. $\mathcal{M}^{\mathrm{ss}}$ is of finite type over $\mathbb{C}$.

One must verify the existence of a scheme T of finite type over $\mathbb{C}$ and a family $\mathcal{E}$ of objects over T such that every semistable object $E \in \mathcal{M}^{\mathrm{ss}}(\mathbb{C})$ appears as a fiber of $\mathcal{E}$; see Section 0.8.1. In other words, one must exhibit a morphism $U \to \mathcal{M}$ from a scheme U of finite type whose image contains $\mathcal{M}^{\mathrm{ss}}$. It is worth noting that since we already know $\mathcal{M}$ is locally of finite type, the finite typeness of $\mathcal{M}$ is equivalent to quasi-compactness; *boundedness* is casual term often used to refer to this property.

[5]The calligraphic font $\mathcal{M}^{\mathrm{ss}}$ denotes an algebraic stack while the Roman font M^{ss} denotes an algebraic space. This notation will be continued throughout the notes.

Step 4 (Existence of coarse/good moduli space): there exists either a coarse or good moduli space $\mathcal{M}^{\text{ss}} \to M^{\text{ss}}$ where M^{ss} is a separated algebraic space.

The algebraic space M^{ss} can be viewed as the best possible approximation of $\mathcal{M}^{\text{ss}}$ which is an algebraic space. If automorphisms are finite and $\mathcal{M}^{\text{ss}}$ is a proper Deligne–Mumford stack, the Keel–Mori theorem ensures that there exists a *coarse moduli space* $\pi\colon \mathcal{M}^{\text{ss}} \to M^{\text{ss}}$ with M^{ss} proper; this means that (1) π is universal for maps to algebraic spaces and (2) π induces a bijection between the isomorphism classes of $\mathbb{C}$-points of $\mathcal{M}^{\text{ss}}$ and the $\mathbb{C}$-points of M^{ss}.

In the case of infinite automorphisms, we often cannot expect the existence of a coarse moduli space (as defined above) and we therefore relax the notion to a *good moduli space* $\pi\colon \mathcal{M}^{\text{ss}} \to M^{\text{ss}}$ which may identify non-isomorphic objects. In fact, it identifies precisely the $\mathbb{C}$-points whose closures in $\mathcal{M}^{\text{ss}}$ intersect in an analogous way to the orbit closure equivalence relation in GIT. A good moduli space is also universal for maps to algebraic spaces even if this property is not obvious from the definitions. We will use an analogue of the Keel–Mori theorem which ensures the existence of a proper good moduli space as long as $\mathcal{M}^{\text{ss}}$ can be verified to be both 'S-complete' and 'Θ-complete'.

Step 5 (Semistable reduction): $\mathcal{M}^{\text{ss}}$ is universally closed, i.e. satisfies the existence part of the valuative criterion for properness.

This requires checking that every family of objects $E^{\times}$ over a punctured DVR or smooth curve $C^{\times} = C \setminus p$ has at least one extension to a family of objects over C after possibly taking an extension of C; see Section 0.8.2. For moduli problems with finite automorphisms, the uniqueness of the extension can usually be verified, which implies the properness of $\mathcal{M}$. For moduli problems with infinite affine automorphism groups, the extension is never unique. While $\mathcal{M}$ is therefore not separated, you can often still verify a condition called 'S-completeness', which enjoys properties analogous to separatedness. This property is often referred to as *stable or semistable reduction*.

As a consequence, we conclude that M^{ss} is a proper algebraic space.

Step 6 (Projectivity): a tautological line bundle on $\mathcal{M}^{\text{ss}}$ descends to an ample line bundle on M^{ss}.

This is often the most challenging step in this process. It requires a solid understanding of the geometry of the moduli problem and often relies on techniques in higher dimensional geometry.

0.8.5 An alternative approach using Geometric Invariant Theory

The approach outlined above is by no means the only way to construct moduli spaces. One alternative approach is Mumford's Geometric Invariant Theory, which has been wildly successful in both constructing and studying moduli spaces. The main idea is to rigidify the moduli stack $\mathcal{M}^{\text{ss}}$ (e.g. $\overline{\mathcal{M}}_g$) by parameterizing additional data (e.g. a stable curve C *and* an embedding $C \overset{|\omega_C^{\otimes 3}|}{\hookrightarrow} \mathbb{P}^N$) in such way that it represented by a projective scheme X and such that the different choices of additional data correspond to different orbits for the action of an algebraic group G acting on X. This provides an identification of the moduli stack $\mathcal{M}^{\mathrm{ss}}$ as an open substack of the quotient stack $[X/G]$. Given a *choice* of equivariant embedding $X \hookrightarrow \mathbb{P}^n$, GIT constructs the quotient as the projective variety

$$X/\!\!/G := \operatorname{Proj} \bigoplus_{d \geq 0} \Gamma(X, \mathcal{O}(d))^G$$

The rational map $X \dashrightarrow X/\!\!/G$ is defined on an open subscheme X^{ss}, which we call the *GIT semistable locus*. To make this procedure work (and this is the hard part!), one must show that an element $x \in X$ is GIT semistable if and only if the corresponding object of $[X/G]$ is semistable (i.e. is in $\mathcal{M}^{\mathrm{ss}}$).

One of the striking features of GIT is that it handles all six steps at once and in particular constructs the moduli space as a projective variety. Moreover, if we do not know a priori how to compactify a moduli problem, GIT can sometimes tell you how.

Example 0.8.7 (Deligne–Mumford stable curves)**.** Using the quotient presentation $\overline{\mathcal{M}}_g = [H'/\operatorname{PGL}_{5g-6}]$ of Example 0.7.4, the closure $\overline{H}'$ of H' in the Hilbert scheme inherits an action of $\operatorname{PGL}_{5g-6}$ and one must show than an element in H' is GIT semistable if and only if the corresponding curve is Deligne–Mumford stable.

Example 0.8.8 (Semistable vector bundles)**.** Using the quotient presentation $\mathcal{B}\mathrm{un}_{r,d}(C)^{\mathrm{ss}} = [Q'_m/\operatorname{PGL}_{N-1}]$ of Example 0.7.5, the closure $\overline{Q}'_m$ has a PGL_{N-1}-action and one must show that an element in $\overline{Q}'_m$ is GIT semistable if and only if the corresponding quotient is semistable.

0.8.6 Trichotomy of moduli spaces

Table 3: The trichotomy of moduli

	No Auts	Finite Auts	Infinite Auts
Type of space	Algebraic variety / space	Deligne–Mumford stack	algebraic stack
Defining property	Zariski/étale locally an affine scheme	étale locally an affine scheme	smooth-locally an affine scheme
Examples	$\mathbb{P}^n$, $\operatorname{Gr}(k,n)$, Hilb, Quot	$\mathscr{M}_g$	$\mathcal{B}\mathrm{un}_{r,d}(C)$
Quotient stacks $[X/G]$	action is free	finite stabilizers	any action
Existence of moduli varieties / spaces	already an algebraic variety/space	coarse moduli space	good moduli space

Notes

For a more detailed exposition of the moduli stack of triangles, we recommend Behrend's notes [Beh14].

Chapter 1

Hilbert and Quot schemes

We prove that the Grassmanian, Hilbert and Quot functors are representable by projective schemes. These results serve as the backbone of many results in moduli theory and more widely algebraic geometry. In particular, they are essential for establishing properties about the moduli stacks $\overline{\mathcal{M}}_g$ of stable curves and $\mathcal{V}^{\mathrm{ss}}_{r,d}$ of vector bundles over a curve. While the reader could safely treat these results as black boxes (and we encourage some readers to do this), it is also worthwhile to dive into the details. We follow Mumford's simplification [Mum66] of the Grothendieck's original construction of Hilbert of Quot schemes [FGA_{IV}]. Specifically, we exploit the theory of Castelnuovo–Mumford regularity (Section 1.3) and flattening stratifications (Theorem A.2.14), which are interesting results on their own with wide-ranging applications outside moduli theory.

1.1 The Grassmanian, Hilbert and Quot functors

1.1.1 The main results

The representability theorems below are formulated for a *strongly projective* morphism $X \to S$ of noetherian schemes, i.e. there exists a closed immersion $X \hookrightarrow \mathbb{P}_S(E)$ over S where E is a vector bundle on S. This is a stronger condition than the *projectivity* of $X \to S$ which only requires that E is a coherent sheaf [EGA, §II.5], [SP, Tag 01W8]. On the other hand, the definition of projectivity in [Har77, II.4] requires that X embeds into projective space $\mathbb{P}^n_S$ over S.

Theorem 1.1.1. *Let S be a noetherian scheme and V be a vector bundle of rank n. For an integer $0 < k < n$, the functor*

$$\mathrm{Gr}_S(k, V) \colon \mathrm{Sch}/S \to \mathrm{Sets}$$
$$(T \xrightarrow{f} S) \mapsto \{ \text{ vector bundle quotients } V_T = f^*V \to Q \text{ of rank } k \ \}$$

is represented by a scheme strongly projective over S.

If $S = \operatorname{Spec} \mathbb{Z}$ and $V = \mathcal{O}_S^n$, then $\mathrm{Gr}_S(k, V)$ is equal to the functor $\mathrm{Gr}(k, n)$ defined in Definition 0.4.12. In addition, when $k = 1$, the Grassmanian $\mathrm{Gr}_S(1, V)$ is identified with the projectivization $\mathbb{P}_S(V)$ of V as discussed in Exercise 0.4.14. For arbitrary S, we sometimes denote $\mathrm{Gr}_S(k, n) := \mathrm{Gr}_S(k, \mathcal{O}_S^n)$ and we sometimes drop the subscript S when we are working over a fixed base such as $S = \operatorname{Spec} \Bbbk$ or $S = \operatorname{Spec} \mathbb{Z}$.

In the formulation of the following two theorems, we will use the convention that if $X \to S$ and $T \to S$ a morphisms of schemes, then $X_T := X \times_S T$. Similarly, if F is a sheaf on X, then F_T denotes the pullback of F under $X_T \to X$. If $s \in S$ is a point, then $X_s := X \times_S \operatorname{Spec} \kappa(s)$ and $F_s := F|_{X_s} = F_{\operatorname{Spec} \kappa(s)}$. If $X \to S$ is a projective morphism, $\mathcal{O}_X(1)$ is relatively ample and $s \in S$ is a point, the *Hilbert polynomial of F_s* is

$$P_{F_s}(z) = \chi(X_s, F_s(z)),$$

where $F_s(z) = F_s \otimes \mathcal{O}_{X_s}(z)$. It is a fact that this defines a polynomial $P_{F_s} \in \mathbb{Q}[z]$ (c.f. [Har77, Exer III.5.2]]); for $z \gg 0$, we have $P_{F_s}(z) = \mathrm{h}^0(X_s, F_s(z))$.

Theorem 1.1.2. *Let $X \to S$ be a strongly projective morphism of noetherian schemes and $\mathcal{O}_X(1)$ be a relatively ample line bundle on X. For every polynomial $P \in \mathbb{Q}[z]$, the functor*

$$\begin{aligned}
&\operatorname{Hilb}^P(X/S) \colon \mathrm{Sch}/S \to \mathrm{Sets} \\
&\qquad (T \to S) \mapsto \left\{ \begin{array}{l} \textit{subschemes } Z \subset X_T \textit{ flat and finitely presented} \\ \textit{over } T \textit{ such that } Z_t \subset X_t \textit{ has Hilbert} \\ \textit{polynomial } P \textit{ for all } t \in T \end{array} \right\}
\end{aligned}$$

is represented by a scheme strongly projective over S.

Theorem 1.1.3. *Let $\pi \colon X \to S$ be a strongly projective morphism of noetherian schemes, $\mathcal{O}_X(1)$ be a relatively ample line bundle on X, and F be a coherent sheaf on X which is the quotient of $\pi^*(W)(q)$ for a vector bundle W on S and an integer q. For every polynomial $P \in \mathbb{Q}[z]$, the functor*

$$\begin{aligned}
&\operatorname{Quot}^P(F/X/S) \colon \mathrm{Sch}/S \to \mathrm{Sets} \\
&\qquad (T \to S) \mapsto \left\{ \begin{array}{l} \textit{quasi-coherent and finitely presented} \\ \textit{quotients } F_T \to Q \textit{ on } X_T \textit{ such that } Q \textit{ is} \\ \textit{flat over } T \textit{ and } Q|_{X_t} \textit{ on } X_t \textit{ has} \\ \textit{Hilbert polynomial } P \textit{ for all } t \in T \end{array} \right\}
\end{aligned}$$

is represented by a scheme strongly projective over S.

The Grassmanian and the Hilbert scheme are special cases of the Quot scheme: $\operatorname{Gr}_S(k, V) \cong \operatorname{Quot}^P(V/S/S)$ where $P(z) = k$ is the constant polynomial and $\operatorname{Hilb}^P(X/S) = \operatorname{Quot}^P(\mathcal{O}_X/X/S)$.

Remark 1.1.4.

(1) In the definition of the Grassmanian and Quot functor above, two quotients $V_T \xrightarrow{q} Q$ and $V_T \xrightarrow{q'} Q'$ are identified if $\ker(q) = \ker(q')$ as subsheaves of V_T, or equivalently there exists an isomorphism $Q \xrightarrow{\alpha} Q'$ such that the composition $V_T \xrightarrow{q} Q \xrightarrow{\alpha} Q'$ is equal to $V_T \xrightarrow{q'} Q'$. In the Hilbert functor, two subschemes of X_T are identified if they are equal as subschemes (or equivalently their ideal sheaves are equal as subsheaves of $\mathcal{O}_{X_T}$).

(2) The definitions $\operatorname{Hilb}^P(X/S)$ and $\operatorname{Quot}^P(F/X/S)$ depend on the relatively ample line bundle $\mathcal{O}_X(1)$ but we have suppressed this from the notation.

(3) When T is noetherian, the conditions that Z be finitely presented and Q be of finite presentation in the definitions of $\operatorname{Hilb}^P(X/S)$ and $\operatorname{Quot}^P(F/X/S)$ are superfluous.

(4) If we do not fix P, then $\operatorname{Hilb}(X/S)$ and $\operatorname{Quot}(F/X/S)$ are representable by schemes *locally* of finite type, and there are decompositions

$$\operatorname{Hilb}(X/S) = \coprod_P \operatorname{Hilb}^P(X/S) \quad \text{and} \quad \operatorname{Quot}(F/X/S) = \coprod_P \operatorname{Quot}^P(F/X/S);$$

these functorial decompositions follows from the flatness of the quotient Q and the local constancy of the Hilbert polynomial (Proposition A.2.4).

(5) Suppose that S satisfies the *resolution property*, i.e. every coherent sheaf is the quotient of a vector bundle. This is satisfied if S has an ample line bundle or if S is regular. Then a projective morphism $X \to S$ is necessarily strongly projective. Moreover, if F is a coherent sheaf on X, then $\pi^*\pi_*(F(q)) \to F(q)$ is surjective for $q \gg 0$ and choosing a surjection $W \twoheadrightarrow \pi_*(F(q))$ from a vector bundle W on S, we have a surjection $\pi^*(W(-q)) \mapsto F$. Theorem 1.1.3 therefore implies that $\operatorname{Quot}^P(F/X/S)$ is strongly projective over S if $X \to S$ is projective and F is coherent.

Caution 1.1.5. We will abuse notation by using $\operatorname{Hilb}^P(X/S)$, $\operatorname{Quot}^P(F/X/S)$ and $\operatorname{Gr}_S(k, V)$ to denote both the functor and the scheme that represents it.

1.1.2 Strategy of proof

In §1.2, we show that $\operatorname{Gr}_S(k, V)$ is representable by a projective scheme by using the functorial Plücker embedding $\operatorname{Gr}_S(k, V) \to \mathbb{P}(\bigwedge^k V)$ which over an S-scheme T sends a quotient $V_T \to Q$ to the line bundle quotient $\bigwedge^k V_T \to \bigwedge^k Q$.

In §1.3, we introduce Castelnuovo–Mumford regularity and exploit Mumford's result on Boundedness of Regularity (Theorem 1.3.8) to show that under the hypotheses of Theorem 1.1.3, then for $d \gg 0$, the morphism of functors

$$\begin{aligned} \operatorname{Quot}^P(F/X/S) &\to \operatorname{Gr}_S(P(d), \pi_* F(d)) \\ [F_T \twoheadrightarrow Q] &\mapsto [\pi_{T,*}(F_T(d)) \to \pi_{T,*}(Q(d))], \end{aligned} \tag{1.1.1}$$

defined over an S-scheme T, is well-defined. Note that for a field-valued point $s\colon \operatorname{Spec} \Bbbk \to S$ a quotient $[F_s \mapsto Q]$ is mapped to $[\mathrm{H}^0(X_s, F_s(d)) \to \mathrm{H}^0(X_s, Q(d))]$.

In fact, we show that the above functor is representable by locally closed immersions (Proposition 1.4.1). This is established by reducing to the special case where $X = \mathbb{P}_S(V)$ and $F = \pi^* W$ where V and W are vector bundles on S; this is where Boundedness of Regularity (Theorem 1.3.8) is applied.

Since $\operatorname{Gr}_S(P(d), \pi_* F(d))$ is representable by a projective scheme over S (Theorem 1.1.1), this already establishes the representability and quasi-projectivity of $\operatorname{Quot}^P(F/X/S)$. Finally, we establish that $\operatorname{Quot}^P(F/X/S)$ is proper over S (Proposition 1.4.2) by checking the valuative criterion which implies that $\operatorname{Quot}^P(F/X/S)$ is projective over S.

1.2 Representability and projectivity of the Grassmanian

The Grassmanian provides a warmup to the functorial approach of constructing projective moduli spaces in these notes and is also used in the proof of the representability of Hilb and Quot. Given its importance, we present a slow-paced expository account of the representability and projectivity of the Grassmanian. We focus first on the Grassmanian $\mathrm{Gr}(k,n)$ over $\mathbb{Z}$ parameterizing k-dimension quotients of a trivial vector bundle of rank n; see Definition 0.4.12. The proof of the projectivity and representability of the relative Grassmanian $\mathrm{Gr}_S(k,V)$ is shown in §1.2.3.

1.2.1 Representability by a scheme

In this subsection, we show that $\mathrm{Gr}(k,n)$ is representable by a scheme (Proposition 1.2.3). Our strategy will be to find a Zariski open cover of $\mathrm{Gr}(k,n)$ by representable subfunctors; see Definition 0.4.21. Given a subset $I \subset \{1,\ldots,n\}$ of size k, let $\mathrm{Gr}_I \subset \mathrm{Gr}(k,n)$ be the subfunctor where for a scheme S, $\mathrm{Gr}(k,n)_I(S)$ is the subset of $\mathrm{Gr}(k,n)(S)$ consisting of surjections $\mathcal{O}_S^n \overset{q}{\twoheadrightarrow} Q$ such that the composition

$$\mathcal{O}_S^I \xrightarrow{e_I} \mathcal{O}_S^n \overset{q}{\twoheadrightarrow} Q$$

is an isomorphism, where e_I is the canonical inclusion.

Lemma 1.2.1. *For each $I \subset \{1,\ldots,n\}$ of size k, the functor Gr_I is representable by affine space $\mathbb{A}_{\mathbb{Z}}^{k\times(n-k)}$*

Proof. We may assume that $I = \{1,\ldots,k\}$. We define a map of functors $\phi\colon \mathbb{A}^{k\times(n-k)} \to \mathrm{Gr}_I$ where over a scheme S, a $k\times(n-k)$ matrix

$$f = \big(f_{i,j}\big)_{1\le i\le k\,,\,1\le j\le n-k}$$

of global functions on S is mapped to the quotient

$$\left(\begin{array}{cccc|ccc} 1 & & & & f_{1,1} & \cdots & f_{1,n-k} \\ & 1 & & & f_{2,1} & \cdots & f_{2,n-k} \\ & & \ddots & & \vdots & & \\ & & & 1 & f_{k,1} & \cdots & f_{k,n-k} \end{array}\right) : \mathcal{O}_S^n \to \mathcal{O}_S^k. \tag{1.2.1}$$

The injectivity of $\phi(S)\colon \mathbb{A}^{k\times(n-k)}(S) \to \mathrm{Gr}_I(S)$ follows from the fact that any two quotients written in the form of (1.2.1) which are equivalent in Gr_I are necessarily defined by the same equations. To see surjectivity, let $[\mathcal{O}_S^n \xrightarrow{q} Q] \in \mathrm{Gr}_I(S)$ where by definition $\mathcal{O}_S^I \xrightarrow{e_I} \mathcal{O}_S^n \overset{q}{\twoheadrightarrow} Q$ is an isomorphism. The tautological commutative diagram

$$\begin{tikzcd} \mathcal{O}_S^n \arrow[r, "q"] \arrow[dr, "(q\circ e_I)^{-1}\circ q"'] & Q \arrow[d, "(q\circ e_I)^{-1}"] \\ & \mathcal{O}_S^I \end{tikzcd}$$

shows that $[\mathcal{O}_S^n \xrightarrow{q} Q] = [\mathcal{O}_S^n \overset{(q\circ e_I)^{-1}\circ q}{\twoheadrightarrow} \mathcal{O}_S^I] \in \mathrm{Gr}(k,n)(S)$. Since the composition $\mathcal{O}_S^I \xrightarrow{e_I} \mathcal{O}_S^n \overset{(q\circ e_I)^{-1}}{\twoheadrightarrow} \mathcal{O}_S^I$ is the identity, the $k\times n$ matrix corresponding to $(q\circ e_I)^{-1}\circ q$ is necessarily of the same form as (1.2.1) for functions $f_{i,j} \in \Gamma(S, \mathcal{O}_S)$. Therefore $\phi(S)(\{f_{i,j}\}) = [\mathcal{O}_S^n \overset{q}{\twoheadrightarrow} Q] \in \mathrm{Gr}(k,n)(S)$. □

Lemma 1.2.2. *$\{\mathrm{Gr}_I\}$ is a Zariski open cover of $\mathrm{Gr}(k,n)$ where I ranges over all subsets of size k.*

Proof. For a fixed subset I, we first show that $\mathrm{Gr}_I \subset \mathrm{Gr}(k,n)$ is an open subfunctor. To this end, we consider a scheme S and a morphism $S \to \mathrm{Gr}(k,n)$ corresponding to a quotient $q\colon \mathcal{O}_S^n \to Q$. Let C denote the cokernel of the composition $q\circ e_I\colon \mathcal{O}_S^I \to Q$. Notice that if $C = 0$, then $q\circ e_I$ is an isomorphism. The fiber product

$$\begin{tikzcd} F_I \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & S \arrow[d, "{[\mathcal{O}_S^n \overset{q}{\twoheadrightarrow} Q]}"] \\ \mathrm{Gr}_I \arrow[r] & \mathrm{Gr}(k,n) \end{tikzcd}$$

of functors is representable by the open subscheme $U = S \setminus \mathrm{Supp}(C)$ (the reader is encouraged to verify this claim). Note that if S is not noetherian, then $\mathrm{Supp}(C) \subset S$ is still closed as C is finitely presented as a quasi-coherent sheaf.

To check the surjectivity of $\coprod_I F_I \to S$, let $s \in S$ be a point. Since $\kappa(s)^n \overset{q\otimes\kappa(s)}{\twoheadrightarrow} Q\otimes\kappa(s)$ is a surjection of vector spaces, there is a non-zero $k\times k$ minor, given by a subset I, of the $k\times n$ matrix $q\otimes\kappa(s)$. This implies that $[\kappa(s)^n \overset{q\otimes\kappa(s)}{\twoheadrightarrow} Q\otimes\kappa(s)] \in F_I(\kappa(s))$. □

Lemmas 1.2.1 and 1.2.2 together imply:

Proposition 1.2.3. *The functor $\mathrm{Gr}(k,n)$ is representable by a scheme.* □

Exercise 1.2.4. Show that $\mathrm{Gr}(k,n)$ is an integral scheme of finite type over $\mathbb{Z}$.

Exercise 1.2.5. Use the valuative criterion of properness to show that $\mathrm{Gr}(k,n) \to \operatorname{Spec}\mathbb{Z}$ is proper.

1.2.2 Projectivity of the Grassmanian

We show that the Grassmanian scheme $\mathrm{Gr}(k,n)$ is projective (Proposition 1.2.6) by explicitly providing a projective embedding. The *Plücker embedding* is the map of functors

$$P\colon\ \mathrm{Gr}(k,n) \to \mathbb{P}(\textstyle\bigwedge^k \mathcal{O}^n_{\operatorname{Spec}\mathbb{Z}})$$
$$[\mathcal{O}_S^n \overset{q}{\twoheadrightarrow} Q] \mapsto [\textstyle\bigwedge^k \mathcal{O}_S^n \to \bigwedge^k Q]$$

defined above over a scheme S. As both sides are representable by schemes, the morphism P corresponds to a morphism of schemes via Yoneda's lemma.

Proposition 1.2.6. *The morphism $P\colon \mathrm{Gr}(k,n) \to \mathbb{P}(\bigwedge^k \mathcal{O}^n_{\operatorname{Spec}\mathbb{Z}})$ of schemes is a closed immersion. In particular, $\mathrm{Gr}(k,n)$ is a strongly projective scheme over $\mathbb{Z}$.*

Proof. A subset $I \subset \{1, \ldots, n\}$ corresponds to a coordinate x_I on $\mathbb{P}(\bigwedge^k \mathcal{O}^n_{\operatorname{Spec}\mathbb{Z}})$, and we set $\mathbb{P}(\bigwedge^k \mathcal{O}^n_{\operatorname{Spec}\mathbb{Z}})_I$ to be the open locus where $x_I \neq 0$. Note that $\mathbb{P}(\bigwedge^k \mathcal{O}^n_{\operatorname{Spec}\mathbb{Z}})_I \subset \mathbb{P}(\bigwedge^k \mathcal{O}^n_{\operatorname{Spec}\mathbb{Z}})$ is the subfunctor parameterizing line bundle quotients $\bigwedge^k \mathcal{O}^n_S \to L$ such that the composition $\mathcal{O}_S \xrightarrow{e_I} \bigwedge^k \mathcal{O}^n_S \to L$ (where the first map is the inclusion of the Ith term) is an isomorphism, or in other words $\mathbb{P}(\bigwedge^k \mathcal{O}^n_{\operatorname{Spec}\mathbb{Z}})_I \cong \operatorname{Gr}(1, \binom{n}{k})_{\{I\}}$ viewing $\{I\}$ as the corresponding subset of $\{1, \ldots, \binom{n}{k}\}$ of size 1. Using these functorial descriptions, one can check that there is a cartesian diagram of functors

$$\begin{tikzcd}
\operatorname{Gr}(k,n)_I \arrow[r, "P_I"] \arrow[d] \arrow[dr, phantom, "\square"] & \mathbb{P}(\bigwedge^k \mathcal{O}^n_{\operatorname{Spec}\mathbb{Z}})_I \arrow[d] \\
\operatorname{Gr}(k,n) \arrow[r, "P"] & \mathbb{P}(\bigwedge^k \mathcal{O}^n_{\operatorname{Spec}\mathbb{Z}}).
\end{tikzcd}$$

Since $\{\mathbb{P}(\bigwedge^k \mathcal{O}^n_{\operatorname{Spec}\mathbb{Z}})_I\}$ is a Zariski open cover, it suffices to show that each $P_I \colon \operatorname{Gr}(k,n)_I \to \mathbb{P}(\bigwedge^k \mathcal{O}^n_{\operatorname{Spec}\mathbb{Z}})_I$ is a closed immersion.

For simplicity, assume that $I = \{1, \ldots, k\}$. Under the isomorphisms $\operatorname{Gr}(k,n)_I \cong \mathbb{A}^{k \times (n-k)}_{\mathbb{Z}}$ of Lemma 1.2.1 and $\mathbb{P}(\bigwedge^k \mathcal{O}^n_{\operatorname{Spec}\mathbb{Z}})_I \cong \mathbb{A}^{\binom{n}{k}-1}_{\mathbb{Z}}$, the morphism P_I corresponds to the map

$$\mathbb{A}^{k \times (n-k)}_{\mathbb{Z}} \to \mathbb{A}^{\binom{n}{k}-1}_{\mathbb{Z}}$$

assigning a $k \times (n-k)$ matrix $A = \{x_{i,j}\}$ to the element of $\mathbb{A}^{\binom{n}{k}-1}_{\mathbb{Z}}$ whose Jth coordinate, where $J \subset \{1, \ldots, n\}$ is a subset of length k distinct from I, is the $\{1, \ldots, k\} \times J$ minor of the $k \times n$ block matrix

$$\left(\begin{array}{cccc|ccc}
1 & & & & x_{1,1} & \cdots & x_{1,n-k} \\
& 1 & & & x_{2,1} & \cdots & x_{2,n-k} \\
& & \ddots & & \vdots & & \\
& & & 1 & x_{k,1} & \cdots & x_{k,n-k}
\end{array}\right).$$

The coordinate $x_{i,j}$ on $\mathbb{A}^{k \times (n-k)}_{\mathbb{Z}}$ is the pull back of the coordinate corresponding to the subset $\{1, \cdots, \widehat{i}, \cdots, k, k+j\}$ (see Figure 1.1). This shows that the corresponding ring map is surjective thereby establishing that P_I is a closed immersion. $\square$

$$x_{i,j} = \det \left(\begin{array}{ccccc|ccccc}
1 & & & & & x_{1,1} & \cdots & x_{1,j} & \cdots & x_{1,n-k} \\
& \ddots & & & & & & \vdots & & \vdots \\
& & 1 & & & x_{i,1} & \cdots & x_{i,j} & \cdots & x_{i,n-k} \\
& & & \ddots & & & & \vdots & & \vdots \\
& & & & 1 & x_{k,1} & \cdots & x_{k,j} & \cdots & x_{k,n-k}
\end{array}\right)$$

Figure 1.1: The minor obtained by removing the ith column and all columns $k+1, \ldots, n$ other than $k+j$ is precisely $x_{i,j}$.

Exercise 1.2.7. For a field $\Bbbk$, let $\mathrm{Gr}(k,n)_{\Bbbk}$ be the $\Bbbk$-scheme $\mathrm{Gr}(k,n) \times_{\mathbb{Z}} \Bbbk$, and $p \in \mathrm{Gr}(k,n)_{\Bbbk}$ be the point corresponding to a quotient $Q = \Bbbk^n/K$. Show that there is a natural bijection of the tangent space

$$T_p \mathrm{Gr}(k,n)_{\Bbbk} \xrightarrow{\sim} \mathrm{Hom}_{\Bbbk}(K,Q).$$

with the vector space of $\Bbbk$-linear maps $K \to Q$.

Exercise 1.2.8. Provide an alternative proof of the projectivity of $\mathrm{Gr}(k,n)$ as follows.

(a) Show that the functor $P\colon \mathrm{Gr}(k,n) \to \mathbb{P}(\bigwedge^k \mathcal{O}^n_{\mathrm{Spec}\,\mathbb{Z}})$ is injective on points and tangent spaces.

(b) Use a criterion for being a closed immersion (c.f. [Har77, Prop. II.7.3]) to show that $P\colon \mathrm{Gr}(k,n) \to \mathbb{P}(\bigwedge^k \mathcal{O}^n_{\mathrm{Spec}\,\mathbb{Z}})$ is a closed immersion.

(Alternatively, you could show that $P\colon \mathrm{Gr}(k,n) \to \mathbb{P}(\bigwedge^k \mathcal{O}^n_{\mathrm{Spec}\,\mathbb{Z}})$ is a proper monomorphism and conclude that $\mathrm{Gr}(k,n)$ is projective over $\mathbb{Z}$.)

1.2.3 Relative version

We now prove the relative version of the representability and strong projectivity of the Grassmanian.

Proof of Theorem 1.1.1. If V is a vector bundle over S of rank n, there is the relative Plücker embedding

$$\begin{aligned} P\colon \mathrm{Gr}_S(k,V) &\to \mathbb{P}_S(\textstyle\bigwedge^k V) \\ [V_T \overset{q}{\twoheadrightarrow} Q] &\mapsto \big[\textstyle\bigwedge^k V_T \to \bigwedge^k Q\big] \end{aligned} \tag{1.2.2}$$

defined above over a S-scheme T. This is a morphism of functors over S. Since $\mathbb{P}_S(\bigwedge^k V)$ is projective over S, it suffices to show that this morphism is representable by closed immersions. This property can be checked Zariski-locally: if $U \subset S$ is an open subscheme where V is trivial, then the base change of $\mathrm{Gr}_S(k,V) \to \mathbb{P}_S(\bigwedge^k V)$ over U is the Plücker embedding $\mathrm{Gr}_U(k, \mathcal{O}_U^n) \to \mathbb{P}_S(\bigwedge^k \mathcal{O}_U^N)$ which is a closed immersion (Proposition 1.2.6). $\square$

Since the Grassmanian functor is representable, there is a *universal quotient* $\mathcal{O}_{\mathrm{Gr}_S(k,V)} \otimes_S V \to Q_{\mathrm{univ}}$; here $\mathcal{O}_{\mathrm{Gr}_S(k,V)} \otimes_S V$ denotes the pullback of V under the structure morphism $\mathrm{Gr}_S(k,V) \to S$. Under the Plücker embedding (1.2.2), the pullback of $\mathcal{O}(1)$ is identified with $\det(Q_{\mathrm{univ}})$, which we sometimes call the *Plücker line bundle*. Thus, we obtain:

Corollary 1.2.9. *The determinant* $\det(Q_{\mathrm{univ}})$ *of the universal quotient is a very ample line bundle on* $\mathrm{Gr}_S(k,V)$. $\square$

Remark 1.2.10. For projective space $\mathbb{P}^n = \mathrm{Gr}(1,n)$, the universal quotient yields an exact sequence $0 \to \Omega_{\mathbb{P}^n}(1) \to \mathcal{O}^{n+1}_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^n}(1) \to 0$, which is the dual of the Euler sequence [Har77, Ex. 8.20.1] twisted by $\mathcal{O}_{\mathbb{P}^n}(1)$.

1.3 Castelnuovo–Mumford regularity

The Cartan–Serre–Grothendieck theorem states that if F is a coherent sheaf on a projective variety $(X, \mathcal{O}_X(1))$, then for $d \gg 0$

(1) $F(d)$ is globally generated;

(2) $\mathrm{H}^i(X, F(d)) = 0$ for $i > 0$; and

(3) the multiplication map

$$\mathrm{H}^0(X, F(d)) \otimes \mathrm{H}^0(X, \mathcal{O}(p)) \to \mathrm{H}^0(X, F(d+p))$$

is surjective for all $p \geq 0$.

Castelnuovo–Mumford regularity provides a quantitative measure of the size of d necessary so that the twist $F(d)$ has the three above desired cohomological properties and in particular that the Hilbert polynomial $\chi(X, F(d))$ of F evaluated at d agrees with $\mathrm{h}^0(X, F(d))$.

1.3.1 Definition and basic properties

Definition 1.3.1. Let F be a coherent sheaf on projective space $\mathbb{P}^n$ over a field $\Bbbk$. For an integer m, we say that F is *m-regular* if

$$\mathrm{H}^i(\mathbb{P}^n, F(m-i)) = 0$$

for all $i \geq 1$.

The *regularity of* F is the smallest integer m such that $F(m)$ is m-regular.

While the requirement that the ith cohomology of the $(m-i)$th twist vanishes may appear mysterious at first, this definition is very convenient for induction arguments on the dimension n as indicated for instance by the following result.

Lemma 1.3.2. *Let F be an m-regular coherent sheaf on $\mathbb{P}^n$ over a field $\Bbbk$. If $H \subset \mathbb{P}^n$ is a hyperplane avoiding the associated points of F, then $F|_H$ is also m-regular.*

Proof. The hypotheses imply that over an affine open subscheme $U \subset \mathbb{P}^n$, the defining equation of H is a nonzerodivisor for the module $\Gamma(U, F)$. Thus $F(-1) \xrightarrow{H} F$ is injective and for an integer $i > 0$ we have a short exact sequence

$$0 \to F(m-i-1) \to F(m-i) \to F|_H(m-i) \to 0$$

inducing a long exact sequence on cohomology

$$\cdots \to \mathrm{H}^i(\mathbb{P}^n, F(m-i)) \to \mathrm{H}^i(H, F|_H(m-i)) \to \mathrm{H}^{i+1}(\mathbb{P}^n, F(m-i-1)) \to \cdots$$

If F is m-regular, then $\mathrm{H}^i(\mathbb{P}^n, F(m-i)) = \mathrm{H}^{i+1}(\mathbb{P}^n, F(m-i-1)) = 0$. It follows that $\mathrm{H}^i(H, F|_H(m-i)) = 0$ for all $i > 0$, and thus $F|_H$ is also m-regular. $\square$

Remark 1.3.3. It follows from the definition of regularity that if F is m-regular, then $F(d)$ is $(m-d)$-regular. We will show in Lemma 1.3.6 that if F is m-regular, it also d-regular for all $d \geq m$.

Exercise 1.3.4.

(a) Show that $\mathcal{O}(d)$ is $(-d)$-regular on $\mathbb{P}^n$.

(b) Show that the structure sheaf of a hypersurface $H \subset \mathbb{P}^n$ of degree d is $(d-1)$-regular.

(c) Show that the structure sheaf of a smooth curve $C \subset \mathbb{P}^n$ of genus g is $(2g-1)$-regular.

Exercise 1.3.5. Let F be a coherent sheaf on $\mathbb{P}^n$ resolved by a long exact sequence of coherent sheaves. Show that if each F_i is $(m+i)$-regular, then F is m-regular.

$$\cdots \to F_2 \to F_1 \to F_0 \to F \to 0$$

Another advantage of regularity is the following lemma due to Castelnuovo.

Lemma 1.3.6. *Let F be an m-regular coherent sheaf on $\mathbb{P}^n$.*

(1) For $d \geq m$, F is d-regular.

(2) The multiplication map

$$\mathrm{H}^0(\mathbb{P}^n, F(d)) \otimes \mathrm{H}^0(\mathbb{P}^n, \mathcal{O}(k)) \to \mathrm{H}^0(\mathbb{P}^n, F(d+k))$$

is surjective if $d \geq m$ and $k \geq 0$.

(3) For $d \geq m$, $F(d)$ is globally generated and $\mathrm{H}^i(\mathbb{P}^n, F(d)) = 0$ for $i \geq 1$.

Proof. If $\Bbbk \to \Bbbk'$ is a field extension, then flat base change implies that $\mathrm{H}^i(\mathbb{P}^n_{\Bbbk}, F) \otimes_{\Bbbk} \Bbbk' = \mathrm{H}^i(\mathbb{P}^n_{\Bbbk'}, F \otimes_{\Bbbk} \Bbbk')$. As $\Bbbk \to \Bbbk'$ is faithfully flat, the assertions (1)–(3) can be checked after base change. We can thus assume that $\Bbbk$ is algebraically closed and in particular infinite.

For (1) and (2), we will argue by induction on n with the base case of $n = 0$ being clear. If $n > 0$, since $\Bbbk$ is infinite, we may choose a hyperplane $H \subset \mathbb{P}^n$ avoiding the associated points of F. Since the restriction $F|_H$ is m-regular (Lemma 1.3.2) on $H \cong \mathbb{P}^{n-1}$, the inductive hypothesis implies that (1) and (2) hold for $F|_H$.

We prove (1) by using induction also on d. The base case $d = m$ holds by hypothesis. For $d > m$, the short exact sequence $0 \to F(d-i-1) \to F(d-i) \to F|_H(d-i) \to 0$ induces a long exact sequence on cohomology

$$\cdots \to \mathrm{H}^i(\mathbb{P}^n, F(d-i-1)) \to \mathrm{H}^i(\mathbb{P}^n, F(d-i)) \to \mathrm{H}^i(H, F|_H(d-i)) \to \cdots$$

For $i > 0$, the first term vanishes by the induction hypothesis on d (F is $(d-1)$-regular so $\mathrm{H}^i(\mathbb{P}^n, F(d-1-i)) = 0$) and the third term vanishes by the inductive hypothesis on n ($F|_H$ is m-regular by Lemma 1.3.2 and thus d-regular by the inductive hypothesis on n so $\mathrm{H}^i(H, F|_H(d-i)) = 0$). Thus, the second term vanishes and we have established (1).

To show (2), we use induction on k in addition to n. We denote the multiplication map by

$$\mu_{d,k} \colon \mathrm{H}^0(\mathbb{P}^n, F(d)) \otimes \mathrm{H}^0(\mathbb{P}^n, \mathcal{O}(k)) \to \mathrm{H}^0(\mathbb{P}^n, F(d+k)).$$

While the base case $k = 0$ is clear, the inductive argument will require us to directly establish the case $k = 1$. To this end, we consider the commutative diagram

$$\begin{tikzcd}[column sep=small]
 & \mathrm{H}^0(\mathbb{P}^n, F(d)) \otimes \mathrm{H}^0(\mathbb{P}^n, \mathcal{O}(1)) \arrow[r, "\nu_d \otimes \mathrm{res}"] \arrow[d, "\mu_{d,1}"] & \mathrm{H}^0(H, F|_H(d)) \otimes \mathrm{H}^0(H, \mathcal{O}_H(1)) \arrow[d] \\
\mathrm{H}^0(\mathbb{P}^n, F(d)) \arrow[ur, "\mathrm{id} \otimes H"] \arrow[r, "\alpha"] & \mathrm{H}^0(\mathbb{P}^n, F(d+1)) \arrow[r, "\nu_{d+1}"] & \mathrm{H}^0(H, F|_H(d+1)).
\end{tikzcd} \tag{1.3.1}$$

As the map α is given by multiplication by $H \in \mathrm{H}^0(\mathbb{P}^n, \mathcal{O}(1))$, α factors through the the map $\mathrm{id} \otimes H$ defined by $v \mapsto v \otimes H$. It follows that $\mathrm{im}(\alpha) \subset \mathrm{im}(\mu_{d,1})$. Since $\mathrm{H}^1(\mathbb{P}^n, F(d)) = 0$ by (2), the restriction map $\nu_d \colon \mathrm{H}^0(\mathbb{P}^n, F(d)) \to \mathrm{H}^0(H, F|_H(d))$ is surjective. Likewise, since $\mathrm{H}^1(\mathbb{P}^n, \mathcal{O}) = 0$, $\mathrm{res} \colon \mathrm{H}^0(\mathbb{P}^n, \mathcal{O}(1)) \to \mathrm{H}^0(H, \mathcal{O}_H(1))$ is surjective. We conclude that the top horizontal arrow is surjective. The inductive hypothesis applied to $H = \mathbb{P}^{n-1}$ implies that the right vertical arrow is surjective. Therefore, the composition $\nu_{d+1} \circ \mu_{d,1}$ is surjective and it follows that $\mathrm{im}(\mu_{d,1})$ surjects onto $\mathrm{H}^0(H, F|_H(d+1))$. By exactness of the bottom row, we have that

$$\mathrm{H}^0(\mathbb{P}^n, F(d+1)) = \mathrm{im}(\mu_{d,1}) + \ker(\beta) = \mathrm{im}(\mu_{d,1}) + \mathrm{im}(\alpha) = \mathrm{im}(\mu_{d,1}),$$

which shows that $\mu_{d,1}$ is surjective.

If $k > 1$, we consider the commutative square

$$\begin{tikzcd}
\mathrm{H}^0(\mathbb{P}^n, F(d)) \otimes \mathrm{H}^0(\mathbb{P}^n, \mathcal{O}(k-1)) \otimes \mathrm{H}^0(\mathbb{P}^n, \mathcal{O}(1)) \arrow[r] \arrow[d, "\mu_{d,k-1} \otimes \mathrm{id}"] & \mathrm{H}^0(\mathbb{P}^n, F(d)) \otimes \mathrm{H}^0(\mathbb{P}^n, \mathcal{O}(k)) \arrow[d, "\mu_{d,k}"] \\
\mathrm{H}^0(\mathbb{P}^n, F(d+k-1)) \otimes \mathrm{H}^0(\mathbb{P}^n, \mathcal{O}(1)) \arrow[r, "\mu_{d+k-1,1}"] & \mathrm{H}^0(\mathbb{P}^n, F(d+k)).
\end{tikzcd}$$

The left vertical map and bottom horizontal arrow are surjective by the inductive hypothesis applied to $k-1$ and $k = 1$, respectively. It follows that $\mu_{d,k}$ is surjective.

To show (3), we know that for $k \gg 0$, $F(d+k)$ is globally generated, i.e. $\gamma_{F(d+k)} \colon \mathrm{H}^0(\mathbb{P}^n, F(d+k)) \otimes \mathcal{O}_{\mathbb{P}^n} \to F(d+k)$ is surjective. Consider the commutative square

$$\begin{tikzcd}
\mathrm{H}^0(\mathbb{P}^n, F(d)) \otimes \mathrm{H}^0(\mathbb{P}^n, \mathcal{O}(k)) \otimes \mathcal{O}_{\mathbb{P}^n} \arrow[r, "\mu_{d,k} \otimes \mathrm{id}"] \arrow[d, "\gamma_{F(d)} \otimes \mathrm{id}"] & \mathrm{H}^0(\mathbb{P}^n, F(d+k)) \otimes \mathcal{O}_{\mathbb{P}^n} \arrow[d, "\gamma_{F(d+k)}"] \\
F(d) \otimes \big(\mathrm{H}^0(\mathbb{P}^n, \mathcal{O}(k)) \otimes \mathcal{O}_{\mathbb{P}^n}\big) \arrow[r, "\mathrm{id} \otimes \gamma_{\mathcal{O}(k)}"] & F(d) \otimes \mathcal{O}(k).
\end{tikzcd}$$

Since top horizontal arrow is surjective by (1), the composition from the top left to the bottom right is surjective. Given the nature of the bottom horizontal map, we see that $\gamma_{F(d)}$ must be surjective (indeed, if $V = \mathrm{im}(\gamma_{F(d)}) \subset F(d)$, then $\mathrm{im}(\mathrm{id} \otimes \gamma_{\mathcal{O}(k)} \circ \gamma_{F(d)} \otimes \mathrm{id}) = V \otimes \mathcal{O}(k))$. Finally, to see the vanishing of the higher cohomology of $F(d)$ observe that for each $i > 0$, the sheaf F is $(d+i)$-regular by (2) and thus $\mathrm{H}^i(\mathbb{P}^n, F(d)) = 0$. $\square$

One easy consequence of (1) is that if F is m-regular, then the restriction map

$$\nu_d \colon : \mathrm{H}^0(\mathbb{P}^n, F(d)) \to \mathrm{H}^0(H, F|_H(d))$$

is surjective for all $d \geq m$. Indeed, (1) implies that F is also d-regular and the surjectivity follows from the vanishing of $\mathrm{H}^1(\mathbb{P}^n, F(d-1))$. The following lemma—which will be used in the proof of Theorem 1.3.8—shows that we can still arrange for the surjectivity of ν_d under weaker hypotheses.

Lemma 1.3.7. *Let F be a coherent sheaf on $\mathbb{P}^n$ and H be a hyperplane avoiding the associated points of F. If $F|_H$ is m-regular and ν_d is surjective for some $d \geq m$, then ν_p is surjective for all $p \geq d$.*

Proof. By staring at the square in diagram (1.3.1), we see that the top arrow $\nu_d \otimes \mathrm{res}$ is surjective (as both ν_d and res are surjective) and the vertical right multiplication morphism is surjective (by applying Lemma 1.3.6(2) to the m-regular sheaf $F|_H$). The statement follows. □

1.3.2 Regularity bounds

We now turn to the following bound on the regularity of subsheaves of the trivial vector bundle established by Mumford in [Mum66, p.101].

Theorem 1.3.8 (Boundedness of Regularity)**.** *For every pair of non-negative integers r and n, and for every polynomial $P \in \mathbb{Q}[z]$, there exists an integer m_0 with the following property: for every field $\Bbbk$, every subsheaf $F \subset \mathcal{O}^r_{\mathbb{P}^n_\Bbbk}$ with Hilbert polynomial P is m_0-regular.*

Proof. As in the proof of Lemma 1.3.6, we can assume that $\Bbbk$ is infinite. We will argue by induction on n. The base case of $n = 0$ holds as every sheaf F on $\mathbb{P}^0$ is m-regular for every integer m.

For $n \geq 1$ and a subsheaf $F \subset \mathcal{O}^r_{\mathbb{P}^n}$ with Hilbert polynomial P, we can choose a hyperplane $H \subset \mathbb{P}^n$ avoiding all associated points of $\mathcal{O}^r_{\mathbb{P}^n}/F$. This ensures that $\mathrm{Tor}^1_{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{O}_H, \mathcal{O}^r_{\mathbb{P}^n}/F) = 0$ and that the short exact sequence $0 \to F \to \mathcal{O}^r_{\mathbb{P}^n} \to \mathcal{O}^r_{\mathbb{P}^n}/F \to 0$ restricts to a short exact sequence

$$0 \to F|_H \to \mathcal{O}^r_H \to \mathcal{O}^r_H/F \to 0. \tag{1.3.2}$$

As $H \cong \mathbb{P}^{n-1}$, this will allow us to apply the inductive hypothesis to $F|_H \subset \mathcal{O}^r_H$.

On the other hand, since $F \subset \mathcal{O}^r_{\mathbb{P}^n}$ is torsion-free, we have a short exact sequence

$$0 \to F(-1) \xrightarrow{H} F \to F|_H \to 0, \tag{1.3.3}$$

and the Hilbert polynomial of $F|_H$ is $\chi(F|_H(d)) = \chi(F(d)) - \chi(F(d-1)) = P(d) - P(d-1)$. In particular, the Hilbert polynomial of $F|_H$ only depends on P and the inductive hypothesis applied to $F|_H \subset \mathcal{O}^r_H$ gives an integer m_1 such that $F|_H$ is m_1-regular.

For $m \geq m_1 - 1$, since $\mathrm{H}^i(H, F|_H(m)) = 0$ for all $i \geq 1$, we have a long exact sequence

$$\begin{multline} 0 \to \mathrm{H}^0(\mathbb{P}^n, F(m-1)) \to \mathrm{H}^0(\mathbb{P}^n, F(m)) \to \mathrm{H}^0(H, F|_H(m)) \to \\ \mathrm{H}^1(\mathbb{P}^n, F(m-1)) \to \mathrm{H}^1(\mathbb{P}^n, F(m)) \to 0. \end{multline} \tag{1.3.4}$$

For $i \geq 2$, we also have isomorphisms $\mathrm{H}^i(\mathbb{P}^n, F(m-1)) \to \mathrm{H}^i(\mathbb{P}^n, F(m))$, and since $\mathrm{H}^i(\mathbb{P}^n, F(d))$ vanishes for $d \gg 0$, we can conclude that $\mathrm{H}^i(\mathbb{P}^n, F(m-1)) = 0$.

To handle H^1, we use the inequalities $\mathrm{h}^1(\mathbb{P}^n, F(m_1)) \geq \mathrm{h}^1(\mathbb{P}^n, F(m_1+1)) \geq \cdots$, which eventually stabilize to 0. We claim that in fact that the inequalities $\mathrm{h}^1(\mathbb{P}^n, F(m_1)) > \mathrm{h}^1(\mathbb{P}^n, F(m_1+1)) > \cdots$ are strict until they become 0. To see this, we observe that there is an equality $\mathrm{h}^1(\mathbb{P}^n, F(m-1)) = \mathrm{h}^1(\mathbb{P}^n, F(m))$ for $m \geq m_1$ if and only if $\nu_m \colon \mathrm{H}^0(\mathbb{P}^n, F(m)) \to \mathrm{H}^0(H, F|_H(m))$ is surjective. If $\mathrm{h}^1(\mathbb{P}^n, F(m-1)) = \mathrm{h}^1(\mathbb{P}^n, F(m))$ for some $m \geq m_1$, then ν_m is surjective. Since $F|_H$ is m_1-regular, we may apply Lemma 1.3.7 to conclude that $\nu_{m'}$ is surjective for all $m' \geq m$, which in turn implies that $\mathrm{h}^1(\mathbb{P}^n, F(m'))$ is constant for $m' \geq m$, and therefore zero. This establishes the claim. Setting $m_2 = m_1 + 1 + \mathrm{h}^1(\mathbb{P}^n, F(m_1))$, we see that $\mathrm{h}^1(\mathbb{P}^n, F(m_2 - 1)) = 0$ and that F is m_2-regular.

We now show that m_2 is bounded above by a constant m_0 independent of F. Since $F \subset \mathcal{O}_{\mathbb{P}^n}^r$, we have that $\mathrm{h}^0(\mathbb{P}^n, F(d)) \leq r\mathrm{h}^0(\mathbb{P}^n, \mathcal{O}(d)) = r\binom{n+d}{n}$ for any $d \geq 0$. Using the vanishing of $\mathrm{h}^i(\mathbb{P}^n, F(m_1))$ for $i \geq 2$, we have

$$\begin{aligned} \mathrm{h}^1(\mathbb{P}^n, F(m_1)) &= \mathrm{h}^0(\mathbb{P}^n, F(m_1)) - \chi(F(m_1)) \\ &\leq r\binom{n+m_1}{n} + P(m_1). \end{aligned}$$

Thus, defining $m_0 := m_1 + 1 + r\binom{n+m_1}{n} + P(m_1)$, we have that $m_2 \leq m_0$. □

Remark 1.3.9. The above proof establishes in fact a stronger statement. In order to formulate the result, we recall that every numerical polynomial $P \in \mathbb{Q}[z]$ (i.e. $P(d) \in \mathbb{Z}$ for integers $d \gg 0$) of degree n can be uniquely written as

$$P(d) = \sum_{i=0}^{n} a_i \binom{d}{i}$$

for $a_i \in \mathbb{Z}$; this follows from a straightforward inductive argument (c.f. [Har77, Prop. I.7.3]). For non-negative integers r and n, there exists a polynomial $\Lambda_{r,n} \in \mathbb{Z}[x_0, \ldots, x_n]$ with the following property: for every field $\Bbbk$, every subsheaf $F \subset \mathcal{O}_{\mathbb{P}^n_{\Bbbk}}^r$ with Hilbert polynomial $P(d) = \sum_{i=0} a_i \binom{d}{i}$ is m_0-regular for $m_0 = \Lambda_{r,n}(a_0, \ldots, a_n)$.

Remark 1.3.10 (Optimal bounds)**.** Although Mumford's result on Boundedness of Regularity (Theorem 1.3.8) provides an explicit bound and is sufficient for many applications including the construction of the Quot scheme as well as for other applications, there is a more optimal bound established by Gotzmann: for a projective scheme $X \subset \mathbb{P}^N$ over a field $\Bbbk$ with Hilbert polynomial P, there are unique integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \geq 1$ such that P can be expressed as

$$P(d) = \binom{d + \lambda_1 - 1}{\lambda_1 - 1} + \binom{d + \lambda_2 - 2}{\lambda_2 - 1} + \cdots + \binom{d + \lambda_r - r}{\lambda_r - 1},$$

and the ideal sheaf $\mathcal{I}_X$ of X is r-regular. See [Got78], [Gre89], [Gre98, §3] and [BH93, §4.3].

Exercise 1.3.11. Let $C \subset \mathbb{P}^n$ be a curve of degree d and genus g. Show that Gotzmann's bound implies that the ideal sheaf I_C of C is $(\binom{d}{2} + 1 - g)$-regular. Can you compare this to the bound given by the proof of Theorem 1.3.8, i.e. can you compute $\Lambda_{1,n}(1-g, d)$ for an explicit polynomial satisfying Theorem 1.3.8?

Remark 1.3.12. It was shown in [GLP83] that the ideal sheaf I_C of an integral, non-degenerate curve $C \subset \mathbb{P}^N$ of degree d is $(d-N+2)$-regular. It is conjectured more generally that the ideal sheaf of a smooth, non-degenerate projective variety $X \subset \mathbb{P}^N$ of dimension n and degree d is $(d-(N-n))+1)$-regular; see [GLP83] and [EG84].

Corollary 1.3.13. *Let $\pi \colon X \to S$ be a strongly projective morphism of noetherian schemes and $\mathcal{O}_X(1)$ be a relatively ample line bundle on X. Let F be quotient sheaf of $\pi^*(W)(q)$ for some vector bundle W on S and integer q. Let $P \in \mathbb{Q}[z]$ be a polynomial. There exists an integer m_0 satisfying the following property for every $d \geq m_0$: for every morphism $f \colon T \to S$ inducing a cartesian square*

$$\begin{tikzcd} X_T \arrow[r, "f_T"] \arrow[d, "\pi_T"'] & X \arrow[d, "\pi"] \\ T \arrow[r, "f"] & S \end{tikzcd}$$

and every finitely presented quotient $Q = F_T/K$ flat over S such that every fiber Q_t on X_t has Hilbert polynomial P, then

1. *(1) $\pi_{T,*}Q(d)$ is a vector bundles of rank $P(d)$*
2. *(2) the comparison maps $f^*\pi_*Q(d) \to \pi_{T,*}f_T^*Q(d)$, $f^*\pi_*F(d) \to \pi_{T,*}f_T^*F(d)$ and $f^*\pi_*K(d) \to \pi_{T,*}f_T^*K(d)$ are isomorphisms;*
3. *(3) $R^1\pi_{T,*}K(d) = 0$ for $i > 0$; and*
4. *(4) the adjunction maps $\pi_T^*\pi_{T,*}Q(d) \to Q(d)$, $\pi_T^*\pi_{T,*}F_T(d) \to F_T(d)$ and $\pi_T^*\pi_{T,*}K(d) \to K(d)$ are surjective.*

Proof. For (2), since $\pi \colon X \to S$ is strongly projective, there is a closed immersion $i \colon X \hookrightarrow \mathbb{P}_S(V)$ where V is a vector bundle on S. Since the statement is local on S (and S is quasi-compact), we may assume that S is affine and that V is the trivial vector bundle of rank $n+1$. We are given a surjection $\pi^*(W)(q) \twoheadrightarrow F$, and if Q is a quotient of i_*F with Hilbert polynomial P, then $Q(-q)$ is a quotient of π^*W with Hilbert polynomial P' where $P'(z) = P(z+d)$. We can therefore replace (F, X, P) with $(\pi^*(W), \mathbb{P}_S(V), P')$. In particular, for every field-valued point $s \colon \operatorname{Spec} \Bbbk \to S$, $\mathbb{P}_S(V)_s \cong \mathbb{P}^n_\Bbbk$ and $F_s \cong \mathcal{O}^k_{\mathbb{P}^n_\Bbbk}$ where $\operatorname{rk}(V) = n+1$ and $\operatorname{rk}(W) = k$.

By Boundedness of Regularity (Theorem 1.3.8), there exists an integer m_0 depending on n, r and P such that for every every field-valued point $s \colon \operatorname{Spec} \Bbbk \to S$, the kernel K_s is m_0-regular. As K_s is also (m_0+2)-regular (Lemma 1.3.6) and $F_s \cong \mathcal{O}^k_{\mathbb{P}^n_\Bbbk}$ is (m_0+1)-regular (in fact, it is 0-regular), it follows that Q_s is m_0-regular (Exercise 1.3.4). By Lemma 1.3.6, for $d \geq m_0+2$, $K_s(d)$, $F_s(d)$ and $Q_s(d)$ are each globally generated with vanishing higher cohomology. Since K, F and Q are flat over S, statements (1)–(3) follow from applying Cohomology and Base Change in the form of Corollary A.7.7. For (4), to verify the surjectivity of the adjunction map $\pi_T^*\pi_{T,*}K(d) \to K(d)$ (and likewise for F_T and Q), it suffices to check that the restriction

$$(\pi_T^*\pi_{T,*}K(d))|_{X_t} \to K_t(d) \tag{1.3.5}$$

is surjective one each fiber X_t over $t \in T$. Using (2), we have identifications

$$(\pi_T^*\pi_{T,*}K(d))|_{X_t} \cong \pi_t^*(\pi_{T,*}K(d) \otimes \kappa(t)) \cong \pi_t^*\pi_{t,*}K_t(d),$$

where $\pi_t \colon X_t \to \operatorname{Spec} \kappa(t)$ and thus (1.3.5) corresponds to the adjunction map $\pi_t^* \pi_{t,*} K_t(d) \to K_t(d)$, which we know is surjective as $K_t(d)$ is globally generated. $\square$

1.4 Representability and projectivity of Hilb and Quot

In this section, we prove the representability and projectivity of Quot (Theorem 1.1.3) and as a consequence we obtain the same for the Hilbert scheme (Theorem 1.1.2).

As before, $\pi \colon X \to S$ is a strongly projective morphism of noetherian schemes, $\mathcal{O}_X(1)$ is a relatively ample line bundle on X, F is a quotient sheaf of $\pi^*(W)(q)$ for some vector bundle W on S and integer q, and $P \in \mathbb{Q}[z]$ is a polynomial. Our strategy is to use the morphism of functors

$$\begin{aligned} \operatorname{Quot}^P(F/X/S) &\to \operatorname{Gr}_S(P(d), \pi_* F(d)) \\ [F_T \twoheadrightarrow Q] &\mapsto [\pi_{T,*} F_T(d) \to \pi_{T,*} Q(d)], \end{aligned}$$

defined above over an S-scheme T. For $d \gg 0$, Corollary 1.3.13 implies that the above morphism is well-defined: indeed part (1) shows that $\pi_{T,*} Q(d)$ is a vector bundle of rank $P(d)$, part (2) shows the pullback of the coherent sheaf $\pi_* F(d)$ under $T \to S$ is identified with $\pi_{T,*} F_T(d)$, and part (3) shows that $R^1 \pi_{T,*} K(d) = 0$ which implies the surjectivity of $\pi_{T,*} F_T(d) \to \pi_{T,*} Q(d)$.

1.4.1 $\operatorname{Quot}^P(F/X/S) \to \operatorname{Gr}_S(P(d), \pi_* F(d))$ is a locally closed immersion

Proposition 1.4.1. *Let $\pi \colon X \to S$ be a strongly projective morphism of noetherian schemes, $\mathcal{O}_X(1)$ be a relatively ample line bundle on X, and F be a coherent sheaf on X which is the quotient of $\pi^*(W)(q)$ for a vector bundle W on S and an integer q. For $d \gg 0$, the morphism $\operatorname{Quot}^P(F/X/S) \to \operatorname{Gr}_S(P(d), \pi_* F(d))$ is representable by locally closed immersions, i.e. for every morphism $T \to \operatorname{Gr}_S(P(d), \pi_* F(d))$ from a scheme, the fiber product*

$$T \times_{\operatorname{Gr}_S(P(d), \pi_* F(d))} \operatorname{Quot}^P(F/X/S)$$

is representable by a locally closed subscheme of T.

Proof. We first reduce to the special case that $X = \mathbb{P}_S(V)$ and $F = \pi^* W$ for trivial vector bundles V and W. Let $i \colon X \hookrightarrow \mathbb{P}_S(V)$ be a closed immersion where V is a vector bundle on S. The morphism of functors $\operatorname{Quot}^P(F/X/S) \to \operatorname{Gr}_S(P(d), \pi_* F(d))$ is defined over S and its base change to an open subscheme $U \subset S$ is identified with the morphism $\operatorname{Quot}^P(F_U/X_U/U) \to \operatorname{Gr}_S(P(d), \pi_{U,*} F_U(d))$. Since the property of being a locally closed immersion is Zariski-local on the target, the statement is Zariski-local on S. We may therefore assume that S is affine and that V is the trivial vector bundle of rank $n+1$.

First, observe that since there is an isomorphism of functors

$$\operatorname{Quot}^P(F/X/S) \to \operatorname{Quot}^P(i_* F/\mathbb{P}_S(V)/S),$$

we may replace (F, X) with $(i_*F, \mathbb{P}_S(V))$. Next using the surjection $\pi^*(W)(q) \twoheadrightarrow F$, we obtain a morphism of functors

$$\begin{aligned} \operatorname{Quot}^P(F/\mathbb{P}_S(V)/S) &\to \operatorname{Quot}^{P'}(\pi^*W/\mathbb{P}_S(V)/S) \\ [F_T \to Q] &\mapsto [(\pi^*W)_T \to F(-q)_T \to Q(-q)], \end{aligned}$$

defined over an S-scheme T, where $P'(z) = P(z-q)$. We claim that this morphism is representable by closed immersions. This claims boils down to the statement that for an S-scheme T and quotient $\pi^*W(q)_T \twoheadrightarrow Q$, there is a closed subscheme $Z \subset T$ such that a morphism $U \to T$ factors through Z if and only if the restriction $\pi^*W(q)_U \twoheadrightarrow G_U$ factors through F_U. Defining $K = \ker(\pi^*W(q)_T \to F_T)$ and considering the diagram

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r] \arrow[dr] & \pi^*W(q)_T \arrow[r] \arrow[d, two heads] & F_T \arrow[r] & 0 \\ & & G, & & \end{tikzcd}$$

we see that the claim is satisfied by taking $Z \subset T$ to be vanishing scheme of the morphism $K \to G$ (see Exercise 0.4.16(b)).

Finally, using that $\pi_*(\pi^*W(d)) = W \otimes \operatorname{Sym}^d V$, we have a commutative diagram

$$\begin{tikzcd} \operatorname{Quot}^P(F/\mathbb{P}_S(V)/S) \arrow[r, hook] \arrow[d] & \operatorname{Quot}^{P'}(\pi^*W/\mathbb{P}_S(V)/S) \arrow[d, hook] \\ \operatorname{Gr}_S(P(d), \pi_*F(d)) \arrow[r] & \operatorname{Gr}_S(P'(d), W \otimes \operatorname{Sym}^d V). \end{tikzcd}$$

By the above claim, the top horizontal map is a closed immersion. As $\operatorname{Gr}_S(P(d), \pi_*F(d))$ and $\operatorname{Gr}_S(P'(d), W \otimes \operatorname{Sym}^d V)$ are projective (Theorem 1.1.1), the bottom horizontal map is projective and in particular separated. If the proposition holds for $\operatorname{Quot}^{P'}(\pi^*W/\mathbb{P}_S(V)/S)$ and the right vertical map is a locally closed immersion, then the left vertical map is also a closed immersion by the cancellation property.

We now handle the special case. We first claim that $\operatorname{Quot}^P(F/X/S) \to \operatorname{Gr}_S(P(d), W \otimes \operatorname{Sym}^d V)$ is a monomorphism, i.e.

$$\operatorname{Quot}^P(F/X/S)(T) \to \operatorname{Gr}_S(P(d), \pi_*F(d))(T)$$

is injective for each scheme T. To see this, observe that if $F_T = Q/K$ is a quotient with Hilbert polynomial P, then Corollary 1.3.13 implies that there is a map of short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & \pi_T^*\pi_{T,*}K(d) \arrow[r] \arrow[d] & \pi_T^*\pi_{T,*}F_T(d) \arrow[r] \arrow[d] & \pi_T^*\pi_{T,*}Q(d) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & K(d) \arrow[r] & F_T(d) \arrow[r] & Q(d) \arrow[r] & 0 \end{tikzcd}$$

where the vertical maps are surjections. Thus $F_T(d) \to Q(d)$ can be recovered from $\pi_{T,*}F_T(d) \to \pi_{T,*}Q(d)$ by taking the cokernel of the composition $\pi_T^*\pi_{T,*}K(d) \to \pi_T^*\pi_{T,*}F_T(d) \to F_T(d)$.

Let $T \to \mathrm{Gr}_S(P(d), W \otimes \mathrm{Sym}^d V)$ be a morphism determined by a vector bundle quotient $\gamma\colon \pi_{T,*}F_T(d) = W_T \otimes \mathrm{Sym}^d V_T \to G$ of rank $P(d)$. Define Q as the quotient sheaf of F_T with the property that $F_T(d) \twoheadrightarrow Q(d)$ is identified with the cokernel of $\ker(\pi_T^*\gamma) \to \pi_T^*\pi_{T,*}F_T(d) \to F_T(d)$. The fiber product

$$\begin{tikzcd}
Z \arrow[r] \arrow[d] & T \arrow[d] \\
\mathrm{Quot}^P(F/X/S) \arrow[r] & \mathrm{Gr}_S(P(d), W \otimes \mathrm{Sym}^d V)
\end{tikzcd}$$

is identified with the subfunctor of T (or more precisely the subfunctor of $\mathrm{Mor}_S(-, T)$) consisting of morphisms $T' \to T$ such that $Q_{T'}$ is flat over T' with Hilbert polynomial P (in other words, a map $T' \to T$ factors through Z if and only if $Q_{T'}$ is flat over T' with Hilbert polynomial P). By Existence of Flattening Stratifications (Theorem A.2.14), Z is representable by a locally closed subscheme of T. □

1.4.2 Valuative Criterion for Quot

In order to establish that Quot is projective, it will be sufficient to know that it is proper.

Proposition 1.4.2. *For every projective morphism $X \to S$ of noetherian schemes, relatively ample line bundle $\mathcal{O}_X(1)$, coherent sheaf F on X and polynomial $P \in \mathbb{Q}[x]$, the functor $\mathrm{Quot}^P(F/X/S)$ satisfies the valuative criterion for properness, i.e. for every DVR R over S with fraction field K, every flat coherent quotient $F_K \to Q^\times$ on X_K with Hilbert polynomial P extends uniquely to a flat coherent quotient $F_R \to Q$ on X_R with Hilbert polynomial P.*

Remark 1.4.3. In other words, the proposition implies that for every commutative diagram

$$\begin{tikzcd}
\operatorname{Spec} K \arrow[r] \arrow[d] & \mathrm{Quot}^P(F/X/S) \arrow[d] \\
\operatorname{Spec} R \arrow[r] \arrow[ur, dashed] & S,
\end{tikzcd}$$

of solid arrows, there is a unique dotted arrow filling in the diagram. See §3.8 for a further discussion of the valuative criterion for functors and stacks.

Proof. If we write $j\colon X_K \hookrightarrow X_R$ as the open immersion, we define Q as the image of the composition $F_R \to j_*F_K \to j_*Q^\times$ (where the first map is given by the adjunction $F_R \to j_*j^*F_R = j_*F_K$). Since Q is a subsheaf of $j_*Q^\times$, it is torsion free over R and thus flat (as R is a DVR). (Locally, if $S = \operatorname{Spec} B$ is affine and $U = \operatorname{Spec} A \subset X$ is an affine open, then we can write $F|_U = \widetilde{M}$ for a finitely generated A-module M and we have a quotient $M \otimes_B K \to N^\times$ of $A \otimes_B K$-modules where $Q^\times|_{U_K} = \widetilde{N^\times}$. Then $Q = \widetilde{N}$ where N is the $A \otimes_B R$-module defined by $N := \mathrm{im}(M \otimes_B R \to M \otimes_B K \to N^\times)$. Since the R-module N is a subsheaf of the K-module $N^\times$, we see that N is torsion free and thus flat.) Finally, since Q if flat over R and $\operatorname{Spec} R$ is connected, its Hilbert polynomial is constant. □

Remark 1.4.4. For $\mathrm{Hilb}^P(X/S)$, the argument translates into the following: the unique extension of a closed subscheme $Z^\times \subset X_K$ is the scheme-theoretic image $Z = \mathrm{im}(Z^\times \to X_K \hookrightarrow X_R)$. The scheme Z is flat over R as all associated points live over the generic point of $\operatorname{Spec} R$.

1.4.3 Projectivity

The proof of the main theorem of this section (Theorem 1.1.3) follows from the following proposition.

Proposition 1.4.5. *Let $\pi\colon X \to S$ be a strongly projective morphism of noetherian schemes, $\mathcal{O}_X(1)$ be a relatively ample line bundle on X, and F be a coherent sheaf on X which is the quotient of $\pi^*(W)(q)$ for a vector bundle W on S and an integer q. For $d \gg 0$, the morphism $\mathrm{Quot}^P(F/X/S) \to \mathrm{Gr}_S(P(d), \pi_*F(d))$ is a closed immersions.*

Proof. For $d \gg 0$, the morphism

$$\mathrm{Quot}^P(F/X/S) \to \mathrm{Gr}_S(P(d), \pi_*F(d))$$

is a locally closed immersion of schemes defines over S (Proposition 1.4.1). Since $\mathrm{Quot}^P(F/X/S)$ is proper over S (Proposition 1.4.2), this map is a closed immersion. Since $\mathrm{Gr}_S(P(d), \pi_*F(d))$ is strongly projective over S (Theorem 1.1.1), so is $\mathrm{Quot}^P(F/X/S)$. □

Consider the diagram

$$\begin{tikzcd}
 & X \times_S \mathrm{Quot}^P(F/X/S) \arrow[dl, "p_1"'] \arrow[d, "p_2"] & \\
X \arrow[dr, "\pi"] & \mathrm{Quot}^P(F/X/S) \arrow[r, hook] \arrow[d] & \mathrm{Gr}_S(P(d), \pi_*F(d)) \arrow[dl, "g"] \\
 & S &
\end{tikzcd}$$

As $\mathrm{Quot}^P(F/X/S)$ represents the Quot functor, there is a *universal quotient* $p_1^*F \to \mathcal{Q}_{\mathrm{univ}}$ on $X \times_S \mathrm{Quot}^P(F/X/S)$. For $d \gg 0$, we also have the universal quotient $g^*\pi_*F(d) \to Q_{\mathrm{univ}}$ on $\mathrm{Gr}_S(P(d), \pi_*F(d))$ and a composition of closed immersions

$$\begin{array}{ccccc}
\mathrm{Quot}^P(F/X/S) & \hookrightarrow & \mathrm{Gr}_S(P(d), \pi_*F(d)) & \hookrightarrow & \mathbb{P}_S(\bigwedge^{P(d)}(\pi_*F(d))) \\
[F_T \twoheadrightarrow Q] & \longmapsto & [\pi_{T,*}F_T(d) \to \pi_{T,*}Q(d)] & \longmapsto & \big[\bigwedge^{P(d)}(\pi_{T,*}F_T(d)) \to \bigwedge^{P(d)}(\pi_{T,*}Q(d))\big]
\end{array}$$

The pullback of $\mathcal{O}(1)$ on $\mathbb{P}_S(\bigwedge^{P(d)}(\pi_*F(d)))$ pulls back to the Plücker line bundle $\det(Q_{\mathrm{univ}})$ (Corollary 1.2.9) which in turn pulls back to $\det\big(p_{2,*}(\mathcal{Q}_{\mathrm{univ}}(d))\big)$ on $\mathrm{Quot}^P(F/X/S)$. We obtain:

Corollary 1.4.6. *For $d \gg 0$, the line bundle $\det\big(p_{2,*}(\mathcal{Q}_{\mathrm{univ}}(d))\big)$ is very ample on $\mathrm{Quot}^P(F/X/S)$.* □

Exercise 1.4.7.

(a) Show that if S is a noetherian scheme and V is a *coherent* sheaf on S, then functor $\mathrm{Gr}_S(k, V)$ defined analogously to Theorem 1.1.1 is represented by a scheme projective (but not necessarily strongly projective) over S.

(b) Show that if $X \to S$ is a projective morphism of noetherian scheme and F is a coherent sheaf on X *flat* over S, then $\mathrm{Quot}^P(F/X/S) \to \mathrm{Gr}_S(P(d), \pi_* F(d))$ is well-defined for $d \gg 0$ and $\mathrm{Quot}^P(F/X/S)$ is projective over S.

1.4.4 Generalizations

If $\pi \colon X \to S$ is a strongly quasi-projective morphism of noetherian schemes (i.e. there is a locally closed immersion $X \hookrightarrow \mathbb{P}_S(V)$ where V is a vector bundle on S), $\mathcal{O}_X(1)$ is a relatively ample line bundle, F is a coherent sheaf on X which is a quotient of $\pi^*(W)(q)$ for a vector bundle W on S and integer q, and $P \in \mathbb{Q}[z]$ is a polynomial, we can modify the functors of Hilb and Quot as follows:

$$\mathrm{Hilb}^P(X/S) \colon \mathrm{Sch}/S \to \mathrm{Sets}$$
$$(T \to S) \mapsto \left\{ \begin{array}{l} \text{subschemes } Z \subset X_T \text{ flat, proper and finitely} \\ \text{presented over } T \text{ such that } Z_t \subset X_t \\ \text{has Hilbert polynomial } P \text{ for all } t \in T \end{array} \right\}$$

$$\mathrm{Quot}^P(F/X/S) \colon \mathrm{Sch}/S \to \mathrm{Sets}$$
$$(T \to S) \mapsto \left\{ \begin{array}{l} \text{quasi-coherent quotients } F_T \to Q \text{ on } X_T \\ \text{of finite presentation with proper support} \\ \text{over } T \text{ such that } Q|_{X_t} \text{ on } X_t \\ \text{has Hilbert polynomial } P \text{ for all } t \in T \end{array} \right\}$$

Then $\mathrm{Hilb}^P(X/S)$ and $\mathrm{Quot}^P(F/X/S)$ are represented by strongly quasi-projective schemes over S; see [FGA_{IV}, §4], [AK80] or [FGI+05, §5.6]

If $X \to S$ is merely a separated morphism of noetherian schemes, then one can define functors $\mathrm{Hilb}(X/S)$ and $\mathrm{Quot}(F/X/S)$ as above dropping the condition on the Hilbert polynomial P. These functors are representable by algebraic spaces separated and locally of finite type over S; see [Art69b, Thm. 6.1][1] and [SP, Tag 09TQ]. Examples of Hironaka produce smooth proper (but not projective) 3-folds X over $\mathbb{C}$ such that $\mathrm{Hilb}^P(X/S)$ is not a scheme.

There are further variants and generalizations:

- Vistoli's Hilbert stack parameterizing finite and unramified morphisms to a separated scheme X (or stack) [Vis91].
- Alexeev and Knutson's moduli of branch varieties parameterizing finite morphisms from a geometrically reduced proper scheme to a separated scheme X [AK10].
- If $X \to S$ is not separated, then Hall and Rydh show that there is an algebraic stack locally of finite type over S parameterizing quasi-finite morphism $Z \to X$ from a proper scheme [HR14].

[1] As pointed out in [Art74, Appendix], the representability is not true without the separated hypothesis on $X \to S$.

Exercise 1.4.8 (Schemes of morphisms)**.** For projective morphisms $X \to S$ and $Y \to S$ of noetherian schemes, consider the functor

$$\begin{aligned}\underline{\mathrm{Mor}}_S(X,Y)\colon \mathrm{Sch}/S &\to \mathrm{Sets}\\ (T \to S) &\mapsto \mathrm{Mor}_T(X_T, Y_T)\end{aligned}$$

assigning an S-scheme T to the set of T-morphisms $X_T \to Y_T$. By using a suitable Hilbert scheme $\mathrm{Hilb}^P(X \times_S Y/X)$ parameterizing graphs $X \subset X \times_S Y$ of morphisms $X \to Y$, show that $\underline{\mathrm{Mor}}_S(X,Y)$ is representable by a projective scheme over S. Can we weaken the hypothesis on X and Y?

1.5 An invitation to the geometry of Hilbert schemes

In this section, we work over an algebraically closed field $\Bbbk$.

The Hilbert polynomial $P(z) = \sum_{i=0}^d a_i z^i$ of a projective scheme $X \subset \mathbb{P}^n$ encodes invariants of X. For instance, $\dim X$ is the degree d of P and $\deg X$ is normalized leading coefficient $d!a_d$. Applying Riemann–Roch in the case of a smooth curve $C \subset \mathbb{P}^n$ gives $P(z) = \deg(C)z + (1-g)$ and for a surface $S \subset \mathbb{P}^n$ gives $P(z) = \frac{1}{2}(zH \cdot (zH - K)) + (1 - p_a)$ where H is a hyperplane divisor, K is the canonical divisor and $p_a = 1 - \chi(\mathcal{O}_S)$ is the arithmetic genus. In arbitrary dimension, Hirzebruch–Riemann–Roch implies that $P(z) = \int_X \mathrm{ch}(\mathcal{O}_X(z))\mathrm{td}(X)$, where $\mathrm{ch}(\mathcal{O}_X(z))$ is the Chern character and $\mathrm{td}(X)$ the Todd class.

1.5.1 Local properties

Exercise 1.5.1. Let X be a projective scheme over a field $\Bbbk$ and F be a coherent sheaf on X.

(a) Let $p \in \mathrm{Quot}^P(F/X/\Bbbk)$ be the point corresponding to a quotient $Q = F/K$. Show that $T_p \mathrm{Quot}^P(F/X/\Bbbk) \cong \mathrm{Hom}_{\mathcal{O}_X}(K, Q)$. This generalizes the exercise computing the tangent space of the Grassmanian (Exercise 1.2.7).

(b) Conclude that if $p \in \mathrm{Hilb}^P(X/\Bbbk)$ is a point corresponding to a closed subscheme $Z \subset X$ defined by a sheaf of ideals I, then $T_p \mathrm{Hilb}^P(X/\Bbbk) \cong \mathrm{H}^0(Z, N_{Z/X})$ where $N_{Z/X}$ is the normal sheaf $\mathrm{Hom}_{\mathcal{O}_Z}(I_Z/I_Z^2, \mathcal{O}_Z)$.

1.5.2 Hilbert scheme of hypersurfaces and linear subspaces

A hypersurface $H \subset \mathbb{P}^n$ of degree d has Hilbert polynomial

$$P(z) = \chi(\mathcal{O}_{\mathbb{P}^n}(z)) - \chi(\mathcal{O}_{\mathbb{P}^n}(z-d)) = \binom{n+z}{n} - \binom{n+z-d}{n}$$

(coming from the exact sequence $0 \to \mathcal{O}_{\mathbb{P}^n}(-d) \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_H \to 0$). We claim that $\mathrm{Hilb}^P(\mathbb{P}^n) \cong \mathbb{P}(\Gamma(\mathbb{P}^n, \mathcal{O}(d)))$. We encourage the reader to show this and in particular establish that every subscheme $Z \subset \mathbb{P}^n$ with Hilbert polynomial P is a hypersurface.

Similarly, a linear subspace $L \subset \mathbb{P}^n$ of dimension k has Hilbert polynomial $P(z) = \binom{z+k}{k}$ and $\mathrm{Hilb}^P(\mathbb{P}^n) = \mathrm{Gr}(k+1, n+1)$.

1.5.3 Hilbert scheme of points on a curve

If C is a smooth projective curve, then the Hilbert scheme of n points $\mathrm{Hilb}^n(C)$ (viewing n as the constant polynomial) is a smooth irreducible projective variety isomorphic to the symmetric product

$$\mathrm{Sym}^n C := \underbrace{C \times \cdots \times C}_{n} / S_n,$$

where S_n acts by permuting the factors. The quotient exists as a projective variety since $C \times \cdots \times C$ is projective; see Exercise 4.2.8.

1.5.4 Hilbert scheme of points on a surface

If S is a smooth irreducible projective surface, then the Hilbert scheme of n points $\mathrm{Hilb}^n(S)$ is a smooth irreducible projective variety [Fog68]. See also [Nak99a] and [Mac07, §4]. There is a birational morphism

$$\mathrm{Hilb}^n(S) \to \mathrm{Sym}^n(S) := \underbrace{S \times \cdots \times S}_{n} / S_n,$$

of projective varieties. The symmetric product $\mathrm{Sym}^n(S)$ is not smooth for $n > 1$ and this provides a resolution of singularities. For an unordered collection of (possibly non-distinct) points $(p_1, \ldots, p_n) \in \mathrm{Sym}^n(S)$, the fiber consists of all possible scheme structures on $\{p_1, \ldots, p_n\}$ of length n.

When $n = 1$, $\mathrm{Hilb}^1(S) = S$. For $n = 2$ and the points p_1 and p_2 are equal, there is a $\mathbb{P}^1$ of scheme structures given by $k[x,y]/(x^2, xy, y^2, ay - bx)$ (with coordinates such that $p_1 = p_2 = 0$) parameterized by their "tangent direction" $[a : b] \in \mathbb{P}^1$. In this case, $\mathrm{Hilb}^2(S) \to \mathrm{Sym}^2(S)$ is the blow-up of the diagonal $S \hookrightarrow \mathrm{Sym}^2(S)$ given by $p \mapsto (p, p)$. In fact, for $n > 2$, the map $\mathrm{Hilb}^n(S) \to \mathrm{Sym}^n(S)$ is a blow-up along some ideal sheaf [Hai98] but the description of the ideal sheaf is more complicated.

When X is of arbitrary dimension, $\mathrm{Hilb}^n(X)$ is smooth at (reduced) closed subschemes $Z \subset X$ consisting of n distinct smooth points of X. If X is reduced, there is an open subscheme of $\mathrm{Hilb}^n(X)$ dimension $n \dim(X)$ parameterizing n distinct smooth points. Another result of Fogarty is that $\mathrm{Hilb}^n(X)$ is connected as long as X is connected [Fog68]. Moreover, for every projective scheme X, there is an irreducible component $\mathrm{Hilb}^n(X)$, called the "good component," that can be identified with the blow-up of $\mathrm{Sym}^n(S)$ along some ideal sheaf [ES14].

1.5.5 Twisted cubics

The Hilbert scheme $\mathrm{Hilb}^{3z+1}(\mathbb{P}^3)$ consists of the union of two smooth rational irreducible components H and H' of dimensions 12 and 15 intersecting transversely along a smooth rational subvariety of dimension 11 [PS85].

The locus H is the closure of the locus H_0 consisting of *twisted cubics*, i.e. rational smooth curves in $\mathbb{P}^3$ of degree 3. Each twisted cubic can be represented by a map $\mathbb{P}^1 \to \mathbb{P}^3$ given by the line bundle $\mathcal{O}_{\mathbb{P}^1}(3)$ and a choice of basis of $\Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(3))$, and this representation is unique up to automorphisms of $\mathbb{P}^1$. All such curves are projectively equivalent, i.e. differ by an automorphism of $\mathbb{P}^3$, so we see that H_0 is identified with the homogeneous space $\mathrm{Aut}(\mathbb{P}^3)/\mathrm{Aut}(\mathbb{P}^1) = \mathrm{SL}_4 / \mathrm{SL}_2$, which is smooth and irreducible of dimension 12. The locus H_0 is not proper as it includes families such as $\mathbb{P}^1 \hookrightarrow \mathbb{P}^3$ given by $[x, y] \mapsto [x^3, x^2y, xy^2, ty^3]$ parameterized by

$t \in \mathbb{A}^1$ whose limit is a singular curve C_0 supported on a nodal cubic in $V(w) = \mathbb{P}^2$ (where w is the 4th coordinate) but with an embedded point at the node; see [Har77, Ex. 9.8.4].

The locus H' is the closure of the locus H'_0 consisting of subschemes $C \sqcup \{p\}$ where C is a smooth cubic curve contained in a hyperplane H and $p \in \mathbb{P}^3 \setminus C$. To count the dimension, observe that the choice of hyperplane $H \in \mathbb{P}(\mathrm{H}^0(\mathbb{P}^3, \mathcal{O}(1)))$ is given by 3 parameters, the choice of plane cubic $C \in \mathbb{P}(\mathrm{H}^0(H, \mathcal{O}_H(3)))$ is given by 9 parameters and the point $p \in \mathbb{P}^3 \setminus C$ is given by 3 parameters. The locus H'_0 is smooth and irreducible of dimension 15. Again, the locus H'_0 is not proper and its closure contains the limits of for instance degenerating the point p to lie on the curve whose limit can be curves like C_0.

The intersection $H \cap H'$ consists of plane, singular cubic curves with an embedded point at the singular point. This locus contains curves such as C_0 above but it also contains even more degenerate curves such as a triple line with an embedded point. Every curve $C \in H \cap H'$ is in fact projectively equivalent to the curve defined by $V(xz, yz, z^2, q(x, y, w))$ where $q(x, y, w)$ is a homogeneous cubic polynomial with a singular point at $(0, 0, 1)$. This depends on 11 parameters.

1.5.6 Non-emptiness

The Hilbert scheme $\mathrm{Hilb}^P(\mathbb{P}^n)$ is non-empty if and only if the Hilbert polynomial P can be written as as

$$P(z) = \binom{z + \lambda_1 - 1}{\lambda_1 - 1} + \binom{z + \lambda_2 - 2}{\lambda_2 - 1} + \cdots + \binom{z + \lambda_r - r}{\lambda_r - 1}, \qquad (1.5.1)$$

integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \geq 1$. This is a result of Hartshorne [Har66b, Cor. 5.7] The necessity of this condition was already mentioned in Remark 1.3.10 in the context of Gotzmann's bounds on regularity.

1.5.7 Connectedness

Hartshorne's Connectedness Theorem asserts that the Hilbert scheme $\mathrm{Hilb}^P(\mathbb{P}^n)$ is connected for every Hilbert polynomial P [Har66b]. More generally, for every connected noetherian scheme S, $\mathrm{Hilb}^P(\mathbb{P}^n_S/S)$ is connected.

The strategy of the argument is to show that every closed subscheme $Z \subset \mathbb{P}^n$ degenerates to a subscheme $V(I)$ defined by a monomial ideal. This reduces the question to the combinatorial question of connecting any two monomial ideals by a family over $\mathbb{A}^1$. This turns out to be a purely deformation and combinatorial question or as Hartshorne writes: "It also appears that the Hilbert scheme is never actually needed in the proof."

See also [Mac07, §3].

1.5.8 Murphy's Law

> Murphy's Law for Hilbert Schemes: There is no geometric possibility so horrible that it cannot be found generically on some component of the Hilbert scheme. [HM98, p.18]

The first pathology was exhibited by Mumford: there is an irreducible component of $\mathrm{Hilb}^{14z-23}(\mathbb{P}^3)$ which is generically non-reduced [Mum62]. Ellia—

Hirschowitz—Mezzetti show that the number of irreducible components in $\mathrm{Hilb}^{az+b}(\mathbb{P}^3)$ is not bounded by a polynomial in a, b [EHM92].

Murphy's Law was made precise by Vakil [Vak06]: for every scheme X finite type over $\mathbb{Z}$ and point $x \in X$, there exists a point $q = [Z \subset \mathbb{P}^n] \in \mathrm{Hilb}^P(\mathbb{P}^n)$ of some Hilbert scheme and an isomorphism

$$\widehat{\mathcal{O}}_{X,p}[\![x_1, \ldots, x_s]\!] \cong \widehat{\mathcal{O}}_{\mathrm{Hilb}^P(\mathbb{P}^n),q}[\![y_1, \ldots, y_t]\!]$$

for integers s, t. In other words, if we introduce the equivalence relation on pointed schemes (Z, z) generated by $(Z, z) \sim (Z', z')$ if there exists a smooth pointed morphism $(Z', z') \to (Z, z)$, then (X, p) is equivalent to to $(\mathrm{Hilb}^P(\mathbb{P}^n), q)$.

In fact, it can be arranged that the Hilbert scheme parameterizes smooth curves in $\mathbb{P}^n$ for some n, or that it parameterizes smooth surfaces in $\mathbb{P}^5$ (resp. surfaces in $\mathbb{P}^4$). It turns out that various other moduli spaces also satisfy Murphy's Law: Kontsevich's moduli space of maps, moduli of canonically polarized smooth surfaces, moduli of curves with linear systems and the moduli space of stable sheaves.

1.5.9 Smoothness

Despite Murphy's Law, many Hilbert schemes are in fact smooth. We've seen before that the Hilbert scheme of points on a smooth surface is smooth. Moreover, it is not hard to see that the Hilbert scheme $\mathrm{Hilb}^P(\mathbb{P}^n/\Bbbk)$ of projective space over a field is smooth at every complete intersection $Z \subset \mathbb{P}^n$ (despite the obstruction space $\mathrm{H}^1(Z, N_{Z/X})$ being potentially non-zero).

A theorem of Skjelnes–Smith [SS20] states that the Hilbert scheme $\mathrm{Hilb}^P(\mathbb{P}^n)$ is smooth if and only if $P(z)$ can be written as (1.5.1) for a partition $\lambda = (\lambda_1, \ldots, \lambda_r)$ of integers satisfying $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \geq 1$ such that one of the seven condition holds:

(1) $n = 2$;

(2) $\lambda_r \geq 2$;

(3) $\lambda = (1)$ or $\lambda = (n^{r-2}, \lambda_{r-1}, 1) = (\underbrace{n, \ldots, n}_{r-2}, \lambda_{r-1}, 1)$ where $r \geq 2$ and $n \geq \lambda_{r-1} \geq 1$;

(4) $\lambda = (n^{r-s-3}, \lambda_{r-s-2}^{s+2}, 1)$ where $r - s \geq s \geq 0$ and $n - 1 \geq \lambda_{r-s-2} \geq 3$;

(5) $\lambda = (n^{r-s-5}, 2^{s+4}, 1)$ where $r - 5 \geq s \geq 0$;

(6) $\lambda = (n^{r-3}, 1^s)$ where $r \geq 3$;

(7) $\lambda = (n + 1)$ or $r = 0$.

Notes

Grothendieck established both the representability and projectivity of $\mathrm{Quot}^P(F/\mathbb{P}^n_A/\operatorname{Spec} A)$ where F is coherent sheaf on $\mathbb{P}^n_A$ and A is a noetherian ring [FGA_{IV}, Thm. 3.2]. Our exposition follows Grothendieck's strategy deviating in only our use of Mumford–Castelnuovo regularity to establish boundedness. Grothendieck's original approach established the boundedness of $\mathrm{Quot}^P(F/\mathbb{P}^n_A/\operatorname{Spec} A)$ by reducing it to the case when $F = \mathcal{O}_X$ and relying on Chow's result on the boundedness of reduced, pure-dimensional subscheme $Y \subset X$

of fixed degree. We have followed Mumford's argument for Boundedness of Regularity (Theorem 1.3.8) in [Mum66] which Mumford applies to construct the Hilbert scheme of curves on a surface (but applies equally to $\mathrm{Quot}^P(F/\mathbb{P}^n_A/\operatorname{Spec} A)$). Our formulation of Theorem 1.1.3 using the strong projectivity of $X \to S$ follows [AK80, Thm. 2.6]. This chapter follows closely the excellent expositions of [Mum66, §14-15], [FGI$^+$05, §6], [Kol96, §1], [Laz04a, §1.8], and [AK80, §2].

Chapter 2

Sites, sheaves, and stacks

2.1 Grothendieck topologies and sites

We would like to form a topology on a scheme where étale morphisms replace Zariski open subsets. This doesn't quite make sense using the conventional notion of a topological space so we instead adapt our definitions. Grothendieck topologies and stacks were introduced in [SGA4]. Our exposition closely follows [Art62], [FGI$^+$05, Part 1], [Ols16, §2], and [SP, Tag 00UZ].

Definition 2.1.1 (Sites)**.** A *Grothendieck topology* on a category $\mathcal{S}$ consists of the following data: for each object $X \in \mathcal{S}$, there is a set $\mathrm{Cov}(X)$ consisting of *coverings* of X, i.e. collections of morphisms $\{X_i \to X\}_{i \in I}$ in $\mathcal{S}$. We require that:

(1) (identity) If $X' \to X$ is an isomorphism, then $(X' \to X) \in \mathrm{Cov}(X)$.

(2) (restriction) If $\{X_i \to X\}_{i \in I} \in \mathrm{Cov}(X)$ and $Y \to X$ is a morphism, then the fiber products $X_i \times_X Y$ exist in $\mathcal{S}$ and the collection $\{X_i \times_X Y \to Y\}_{i \in I} \in \mathrm{Cov}(Y)$.

(3) (composition) If $\{X_i \to X\}_{i \in I} \in \mathrm{Cov}(X)$ and $\{X_{ij} \to X_i\}_{j \in J_i} \in \mathrm{Cov}(X_i)$ for each $i \in I$, then $\{X_{ij} \to X_i \to X\}_{i \in I, j \in J_i} \in \mathrm{Cov}(X)$.

A *site* is a category $\mathcal{S}$ with a Grothendieck topology.

Example 2.1.2 (Topological spaces)**.** If X is a topological space, let $\mathrm{Op}(X)$ denote the category of open sets $U \subset X$. There is a unique morphism $U \to V$ if and only if $U \subset V$. We say that a covering of U (i.e. an element of $\mathrm{Cov}(U)$) is a collection of open immersions $\{U_i \to U\}_{i \in I}$ such that $U = \bigcup_{i \in I} U_i$. This defines a Grothendieck topology on $\mathrm{Op}(X)$.

In particular, if X is a scheme, the Zariski-topology on X defines a site X_{Zar}, called the *small Zariski site on X*

The most important sites for us will be the small and big étale sites.

Example 2.1.3 (Small étale site)**.** If X is a scheme, the *small étale site on X* is the category $X_{\text{ét}}$ of étale morphisms $U \to X$ such that a morphism $(U \to X) \to (V \to X)$ is simply an X-morphism $U \to V$ (which is necessarily étale). In other words, $X_{\text{ét}}$ is the full subcategory of Sch/X consisting of schemes étale over X. A covering of an object $(U \to X) \in X_{\text{ét}}$ is a collection of étale morphisms $\{U_i \to U\}$ such that $\coprod_i U_i \to U$ is surjective.

Later we will introduce the small étale site $\mathcal{X}_{\text{ét}}$ of an algebraic space or Deligne–Mumford stack (Definition 4.1.1) and use it to define sheaves on $\mathcal{X}$.

Big sites

Example 2.1.4 (Big topological site)**.** Let Top be the category of topological spaces. A covering of $U \in \text{Top}$ is a collection of open subspaces $\{U_i \hookrightarrow U\}_{i\in I}$ such that $U = \bigcup_{i\in I} U_i$.

The big étale site is the most frequently used site in these notes. It is used to define the most central notions in this book: an algebraic space is a sheaf on $\text{Sch}_{\text{ét}}$ that is étale locally a scheme (Definition 3.1.2) while an algebraic stack is a stack over $\text{Sch}_{\text{ét}}$ that is smooth-locally a scheme (Definition 3.1.6).

Example 2.1.5 (Big étale site)**.** The *big étale site* is the category Sch where a covering of a scheme U is a collection of étale morphisms $\{U_i \to U\}$ in Sch such that $\coprod_i U_i \to U$ is surjective. We denote this site as $\text{Sch}_{\text{ét}}$.

The following sites will be less important for us than the étale sites.

Example 2.1.6 (Big Zariski site)**.** Replacing étale morphisms in Example 2.1.5 with open immersions defines the *big Zariski site* Sch_{Zar}.

Example 2.1.7 (Big fppf site)**.** An fppf morphism of schemes is by definition a surjective and flat morphism locally of finite presentation; see also Definition A.2.18. The *big fppf site* Sch_{fppf} is the category Sch of schemes where a covering $\{U_i \to U\}$ is a collection of morphisms such that $\coprod_i U_i \to U$ is fppf, i.e. each $U_i \to U$ is flat and locally of finite presentation, and $\coprod_i U_i \to U$ is surjective.

Example 2.1.8 (Lisse-étale site)**.** On a scheme X, the *lisse-étale site* $X_{\text{lis-ét}}$ is the category of schemes smooth over X where morphisms in $X_{\text{lis-ét}}$ are (not necessarily smooth) morphisms of schemes over X. A covering $\{U_i \to U\}$ of an X-scheme U is a collection of X-morphisms such that $\coprod_i U_i \to U$ is surjective and étale.

We will later introduce the lisse-étale site of an algebraic stack (Definition 6.1.1)

Instead of defining the above sites on the category Sch of schemes, one can define the sites $\text{AffSch}_{\text{Zar}}$, $\text{AffSch}_{\text{ét}}$, $\text{AffSch}_{\text{fppf}}$ and $\text{AffSch}_{\text{lis-ét}}$ on the category of affine schemes with the same coverings. These variants sometimes appear in the literature.

Example 2.1.9 (Localized categories and sites)**.** If $\mathcal{S}$ is a category and $S \in \mathcal{S}$, define the category $\mathcal{S}/S$ whose objects are maps $T \to S$ in $\mathcal{S}$. A morphism $(T' \to S) \to (T \to S)$ is a map $T' \to T$ over S. If $\mathcal{S}$ is a site, $\mathcal{S}/S$ is also a site where a covering of $T \to S$ in $\mathcal{S}/S$ is a covering $\{T_i \to T\}$ in $\mathcal{S}$.

Applying this construction to a scheme S yields the big Zariski, étale, fppf and fpqc sites $(\text{Sch}/S)_{\text{Zar}}$, $(\text{Sch}/S)_{\text{ét}}$, $(\text{Sch}/S)_{\text{fppf}}$ and $(\text{Sch}/S)_{\text{fpqc}}$.

Example 2.1.10 (Grothendieck topolgoies on the category of affine schemes)**.** A variant of the big sites introduced above on the category Sch of all schemes are the sites $\text{AffSch}_{\text{Zar}}$, $\text{AffSch}_{\text{ét}}$, $\text{AffSch}_{\text{fppf}}$ and $\text{AffSch}_{\text{lis-ét}}$

2.2 Presheaves and sheaves

Recall that if X is a topological space, a presheaf of sets on X is simply a contravariant functor $F\colon \mathrm{Op}(X) \to \mathrm{Sets}$ on the category $\mathrm{Op}(X)$ of open sets. The sheaf axiom translates succinctly into the condition that for each covering $U = \bigcup_i U_i$, the sequence

$$F(U) \to \prod_i F(U_i) \rightrightarrows \prod_{i,j} F(U_i \cap U_j)$$

is exact (i.e. is an equalizer diagram), where the two maps $F(U_i) \rightrightarrows F(U_i \cap U_j)$ are induced by the two inclusions $U_i \cap U_j \subset U_i$ and $U_i \cap U_j \subset U_j$. Also note that the intersections $U_i \cap U_j$ can also be viewed as fiber products $U_i \times_X U_j$.

2.2.1 Definitions

Definition 2.2.1 (Presheaves)**.** A *presheaf* on a category $\mathcal{S}$ is a contravariant functor $\mathcal{S} \to \mathrm{Sets}$.

Remark 2.2.2. If $F\colon \mathcal{S} \to \mathrm{Sets}$ is a presheaf and $S \xrightarrow{f} T$ is a map in $\mathcal{S}$, then the pullback $F(f)(b)$ of an element $b \in F(T)$ is sometimes denoted as f^*b or $b|_S$.

Definition 2.2.3 (Sheaves)**.** A *sheaf* on a site $\mathcal{S}$ is a presheaf $F\colon \mathcal{S} \to \mathrm{Sets}$ such that for every object S and covering $\{S_i \to S\} \in \mathrm{Cov}(S)$, the sequence

$$F(S) \to \prod_i F(S_i) \rightrightarrows \prod_{i,j} F(S_i \times_S S_j) \tag{2.2.1}$$

is exact, where the two maps $F(S_i) \rightrightarrows F(S_i \times_S S_j)$ are induced by the two maps $S_i \times_S S_j \to S_i$ and $S_i \times_S S_j \to S_i$.

Remark 2.2.4. The exactness of (2.2.1) means that it is an equalizer diagram: $F(S)$ is precisely the subset of $\prod_i F(S_i)$ consisting of elements whose images under the two maps $F(S_i) \rightrightarrows F(S_i \times_S S_j)$ are equal.

Exercise 2.2.5. Let F be a presheaf on Sch.

(a) Show that F is a sheaf on $\mathrm{Sch}_{\text{ét}}$ (resp. $\mathrm{Sch}_{\text{fppf}}$, $\mathrm{Sch}_{\text{fpqc}}$) if and only if for every surjective étale (resp. fppf, fpqc) morphism $S' \to S$ of schemes, the sequence $F(S) \to F(S') \rightrightarrows F(S' \times_S S')$ is exact. *Hint: Given a covering* $\{S_i \to S\}$*, consider the map* $\coprod_i S_i \to S$*.*

(b) Show that F is a sheaf on $\mathrm{Sch}_{\text{ét}}$ if and only if

  (i) F is a sheaf in the big Zariski topology $\mathrm{Sch}_{\mathrm{Zar}}$; and

  (ii) or every étale surjective morphism $S' \to S$ of *affine* schemes, the sequence $F(S) \to F(S') \rightrightarrows F(S' \times_S S')$ is exact.

Proposition 2.2.6. *If* $X \to S$ *is a morphism of schemes, then* $\mathrm{Mor}_S(-, X)\colon \mathrm{Sch}/S \to \mathrm{Sets}$ *is a sheaf on* $(\mathrm{Sch}/S)_{\text{fpqc}}$ *and therefore also a sheaf on* $(\mathrm{Sch}/S)_{\text{ét}}$*.*

Proof. By Exercise 2.2.5, it suffices to show that if $T' \to T$ is an fpqc morphism over schemes over S, then the sequence

$$\mathrm{Mor}_S(T, X) \to \mathrm{Mor}_S(T', X) \rightrightarrows \mathrm{Mor}_S(T' \times_T T', X)$$

is exact. This follows from fpqc descent for morphisms of schemes (Proposition B.2.1). □

2.2.2 Morphisms and fiber products

A *morphism* of presheaves or sheaves is by definition a natural transformation. By Yoneda's lemma (Lemma 0.4.1), if X is a scheme and F is a presheaf on Sch, a morphism $\alpha\colon X \to F$ (which we interpret as a morphism of presheaves $\mathrm{Mor}(-, X) \to F$) corresponds to an element in $F(X)$, which by abuse of notation we also denote by α.

Exercise 2.2.7. Recall from Proposition 2.2.6 that a scheme can be viewed as a sheaf in the big fpqc topology.

(a) Show that a surjective étale (resp. fppf, fpqc) morphism of schemes is an epimorphism sheaves on $\mathrm{Sch}_{\text{ét}}$ (resp. $\mathrm{Sch}_{\text{fppf}}$, $\mathrm{Sch}_{\text{fpqc}}$).

(b) Show that a surjective smooth morphism of schemes is an epimorphism sheaves on $\mathrm{Sch}_{\text{ét}}$.

Given morphisms $F \xrightarrow{\alpha} G$ and $G' \xrightarrow{\beta} G$ of presheaves on a category $\mathcal{S}$, define the presheaf $F \times_G G'$ by $(F \times_G G')(S) = F(S) \times_{G(S)} G'(S)$, i.e.

$$
\begin{aligned}
F \times_G G' \colon \mathcal{S} &\to \mathrm{Sets} \\
S &\mapsto \{(a, b) \in F(S) \times G'(S) \mid \alpha_S(a) = \beta_S(b)\}\,.
\end{aligned}
\tag{2.2.2}
$$

Exercise 2.2.8.

(a) Show that (2.2.2) is a fiber product $F \times_G G'$ in $\mathrm{Pre}(\mathcal{S})$. (This is a generalization of Exercise 0.4.20 but the same proof should work.)

(b) Show that if F, G and G' are sheaves on a site $\mathcal{S}$, then so is $F \times_G G'$. In particular, (2.2.2) is also a fiber product $F \times_G G'$ in $\mathrm{Sh}(\mathcal{S})$.

2.2.3 Sheafification

Theorem 2.2.9 (Sheafification)**.** *Let $\mathcal{S}$ be a site. The forgetful functor* $\mathrm{Sh}(\mathcal{S}) \to \mathrm{Pre}(\mathcal{S})$ *admits a left adjoint $F \mapsto F^{\mathrm{sh}}$, called the sheafification.*

Proof. A presheaf F on $\mathcal{S}$ is called *separated* if for every covering $\{S_i \to S\}$ of an object S, the map $F(S) \to \prod_i F(S_i)$ is injective (i.e. if sections glue, they glue uniquely). Let $\mathrm{Pre}(\mathcal{S})$ and $\mathrm{Sh}(\mathcal{S})$ be the categories of presheaves and shaves, and let $\mathrm{Pre}^{\mathrm{sep}}(\mathcal{S}) \subset \mathrm{Pre}(\mathcal{S})$ be the full subcategory of separated presheaves. We will construct left adjoints

$$
\begin{tikzcd}
\mathrm{Sh}(\mathcal{S}) \arrow[r, hook] & \mathrm{Pre}^{\mathrm{sep}}(\mathcal{S}) \arrow[l, bend right, "\mathrm{sh}_2"'] \arrow[r, hook] & \mathrm{Pre}(\mathcal{S}). \arrow[l, bend right, "\mathrm{sh}_1"']
\end{tikzcd}
$$

For $F \in \mathrm{Pre}(\mathcal{S})$, we define $\mathrm{sh}_1(F)$ by $S \mapsto F(S)/\sim$ where $a \sim b$ if there exists a covering $\{S_i \to S\}$ such that $a|_{S_i} = b|_{S_i}$ for all i.

For $F \in \mathrm{Pre}^{\mathrm{sep}}(\mathcal{S})$, we define $\mathrm{sh}_2(F)$ by

$$
S \mapsto \left\{ (\{S_i \to S\}, \{a_i\}) \,\middle|\, \begin{array}{l} \text{where } \{S_i \to S\} \in \mathrm{Cov}(S) \text{ and } a_i \in F(S_i) \\ \text{such that } a_i|_{S_{ij}} = a_j|_{S_{ij}} \text{ for all } i, j \end{array} \right\} / \sim
$$

where $(\{S_i \to S\}, \{a_i\}) \sim (\{S'_j \to S\}, \{a'_j\})$ if $a_i|_{S_i \times_S S'_j} = a'_j|_{S_i \times_S S'_j}$ for all i, j. The details are left to the reader. $\square$

Remark 2.2.10 (Topos)**.** A *topos* is a category equivalent to the category of sheaves on a site. Two different sites may have equivalent categories of sheaves, and the topos can be viewed as a more fundamental invariant. While topoi are undoubtedly important in moduli theory, they will not play a role in these notes.

2.2.4 Effective descent for sheaves

Proposition 2.2.11 (Effective Descent)**.** *Let $\mathcal{P}$ be one of the following properties of morphisms of schemes: open immersion, closed immersion, locally closed immersion, affine, quasi-affine or separated and locally quasi-finite. Let $X \to Y$ be a surjective smooth (resp. fppf) morphism of schemes. Let F be a sheaf on $(\mathrm{Sch}/Y)_{\text{ét}}$ (resp, $(\mathrm{Sch}/Y)_{\mathrm{fppf}}$). Consider the fiber product*

$$\begin{tikzcd} F_X \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X \arrow[d] \\ F \arrow[r] & Y \end{tikzcd}$$

of sheaves. If F_X is a scheme and $F_X \to X$ has $\mathcal{P}$, then F is a scheme and $F \to Y$ has $\mathcal{P}$.

Proof. As F_X is the pullback of F, there is a canonical isomorphism $\alpha\colon p_1^* F_X \to F_2^* Q_X$ on $X \times_Y X$ satisfying the cocycle condition $p_{23}^*\alpha \circ p_{12}^*\alpha = p_{13}^*\alpha$. By Proposition B.3.1, there exists a morphism of schemes $W \to Y$ satisfying $\mathcal{P}$ that pulls back to $F_X \to X$. There is an induced injective morphism $F \to W$ of sheaves over Y that pulls back to an isomorphism under $X \to Y$. Since $X \to Y$ is smooth and surjective (resp. fppf), it is an epimorphism of sheaves (Exercise 2.2.7) and it follows that $F \to W$ is an epimorphism, thus isomorphism of sheaves. □

2.3 Prestacks

In Section 0.6.1, we motivated the concept of a prestack on a category $\mathcal{S}$ as a generalization of a presheaf $\mathcal{S} \to \mathrm{Sets}$. By trying to keep track of automorphisms, we were naively led to consider a 'functor' $F\colon \mathcal{S} \to \mathrm{Groupoids}$ but decided instead to package this data into one large category $\mathcal{X}$ over $\mathcal{S}$ parameterizing pairs (a, S) where $S \in \mathcal{S}$ and $a \in F(S)$.

2.3.1 Definition of a prestack

Let $\mathcal{S}$ be a category and $p\colon \mathcal{X} \to \mathcal{S}$ be a functor of categories. We visualize this data as

$$\begin{tikzcd} \mathcal{X} \arrow[d, "p"] \\ \mathcal{S} \end{tikzcd} \qquad \begin{tikzcd} a \arrow[r, "\alpha"] \arrow[d, mapsto] & b \arrow[d, mapsto] \\ S \arrow[r, "f"] & T \end{tikzcd}$$

where the lower case letters a, b are objects of $\mathcal{X}$ and the upper case letters S, T are objects of $\mathcal{S}$. We say that a is over S and $\alpha\colon a \to b$ is over $f\colon S \to T$.

Definition 2.3.1 (Prestacks)**.** A functor $p\colon \mathcal{X} \to \mathcal{S}$ is a *prestack over a category* $\mathcal{S}$ if

(1) (pullbacks exist) for every diagram

$$\begin{tikzcd} a \arrow[r, dashed] \arrow[d, dashed, mapsto] & b \arrow[d, mapsto] \\ S \arrow[r] & T \end{tikzcd}$$

of solid arrows, there exist a morphism $a \to b$ over $S \to T$; and

(2) (universal property for pullbacks) for every diagram

$$\begin{tikzcd} a \arrow[r, dashed] \arrow[d, mapsto] \arrow[rr, bend left] & b \arrow[r] \arrow[d, mapsto] & c \arrow[d, mapsto] \\ R \arrow[r] & S \arrow[r] & T \end{tikzcd}$$

of solid arrows, there exists a unique arrow $a \to b$ over $R \to S$ filling in the diagram.

Caution 2.3.2. When defining and discussing prestacks, we often simply write $\mathcal{X}$ instead of $\mathcal{X} \to \mathcal{S}$. In most examples it is clear what the functor $\mathcal{X} \to \mathcal{S}$ is. When necessary, we denote the projection by $p_{\mathcal{X}} \colon \mathcal{X} \to \mathcal{S}$.

Moreover, when defining a prestack $\mathcal{X}$, we often only define the objects and morphisms in $\mathcal{X}$, and we leave the definition of the composition law to the reader.

Remark 2.3.3. Axiom (2) above implies that the pullback in Axiom (1) is unique up to unique isomorphism. We often write f^*b or simply $b|_S$ to indicate a *choice* of a pullback.

Definition 2.3.4 (Fiber categories)**.** If $\mathcal{X}$ is a prestack over $\mathcal{S}$, the *fiber category* $\mathcal{X}(S)$ over $S \in \mathcal{S}$ is the category of objects in $\mathcal{X}$ over S with morphisms over id_S.

Exercise 2.3.5. Show that the fiber category $\mathcal{X}(S)$ is a groupoid.

Caution 2.3.6. Our terminology is not standard. Prestacks are usually referred to as *categories fibered in groupoids*. In the literature (c.f. [FGI+05, Part 1], [Ols16]) a prestack is sometimes defined as a category fibered in groupoids together with Axiom 2.4.1(1) of a stack.

It is also standard to call a morphism $b \to c$ in $\mathcal{X}$ *cartesian* if it satisfies the universal property in Axiom 2.4.1(2) and $p \colon \mathcal{X} \to S$ a *fibered category* if for every diagram as in Axiom 2.4.1(1), there exists a cartesian morphism $a \to b$ over $S \to T$. With this terminology, a prestack (as we've defined it) is a fibered category where every arrow is cartesian, or equivalently where every fiber category $\mathcal{X}(S)$ is a groupoid.

2.3.2 Examples

Example 2.3.7 (Presheaves are prestacks)**.** If $F \colon \mathcal{S} \to \mathrm{Sets}$ is a presheaf, we can construct a prestack $\mathcal{X}_F$ as the category of pairs (a, S) where $S \in \mathcal{S}$ and $a \in F(S)$. A map $(a', S') \to (a, S)$ is a map $f \colon S' \to S$ such that $a' = f^*a$, where f^* is convenient shorthand for $F(f) \colon F(S) \to F(S')$. Observe that the fiber categories $\mathcal{X}_F(S)$ are equivalent (even equal) to the set $F(S)$. We will often abuse notation by conflating F and $\mathcal{X}_F$.

Example 2.3.8 (Schemes are prestacks)**.** For a scheme X, applying the previous example to the functor $\mathrm{Mor}(-, X)\colon \mathrm{Sch} \to \mathrm{Sets}$ yields a prestack $\mathcal{X}_X$. This allows us to view a scheme X as a prestack and we will often abuse notation by referring to $\mathcal{X}_X$ as X.

Example 2.3.9 (Prestack of smooth curves)**.** We define the prestack $\mathscr{M}$ over Sch as the category of families of smooth curves $\mathcal{C} \to S$, i.e. smooth and proper morphisms $\mathcal{C} \to S$ (of finite presentation) of schemes such that every geometric fiber is a connected curve. A map $(\mathcal{C}' \to S') \to (\mathcal{C} \to S)$ is the data of maps $\alpha\colon \mathcal{C}' \to \mathcal{C}$ and $f\colon S' \to S$ such that the diagram

$$\begin{tikzcd} \mathcal{C}' \arrow[r, "\alpha"] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{C} \arrow[d] \\ S' \arrow[r, "f"] & S \end{tikzcd}$$

is cartesian.

The prestack $\mathscr{M}_g$ is the defined as the full subcategory of $\mathscr{M}$ consisting of families of smooth curves $\mathcal{C} \to S$ where every geometric fiber has genus g. Note that the fiber category $\mathscr{M}_g(\Bbbk)$ over a field $\Bbbk$ is the groupoid of smooth connected projective complex curves C of genus g such that $\mathrm{Mor}_{\mathscr{M}_g(\Bbbk)}(C, C') = \mathrm{Isom}_{\mathrm{Sch}/\Bbbk}(C, C')$.

Exercise 2.3.10. Verify that $\mathscr{M}$ and $\mathscr{M}_g$ are prestacks.

Example 2.3.11 (Prestack of coherent sheaves and vector bundles)**.** Let C be a fixed smooth connected projective curve over an algebraically closed field $\Bbbk$. We define the prestack $\underline{\mathrm{Coh}}(C)$ over $\mathrm{Sch}/\Bbbk$ where objects are pairs (E, S) where S is a scheme over $\Bbbk$ and E is a coherent sheaf on $C_S = C \times_{\Bbbk} S$ flat over S. A morphism $(E', S') \to (E, S)$ consists of a map of schemes $f\colon S' \to S$ together with a map $E \to (\mathrm{id} \times f)_* E'$ of $\mathcal{O}_{C_S}$-modules whose adjoint is an isomorphism (i.e. for every *choice* of pullback $(\mathrm{id} \times f)^* E$, the adjoint map $(\mathrm{id} \times f)^* E \to E'$ is an isomorphism).

The substack $\mathcal{B}\mathrm{un}(C) \subset \underline{\mathrm{Coh}}(C)$ is the full subcategory consisting of pairs (E, S) such that E is a vector bundle (i.e. locally free sheaf of finite rank). For each integers $r \geq 0$ and d, the full subcategories $\underline{\mathrm{Coh}}_{r,d}(C) \subset \underline{\mathrm{Coh}}(C)$ (resp. $\mathcal{B}\mathrm{un}_{r,d}(C) \subset \mathcal{B}\mathrm{un}(C)$) are defined to contain only coherent sheaves (resp. vector bundles) of rank r and degree d.

Exercise 2.3.12. Verify that $\underline{\mathrm{Coh}}(C)$, $\mathcal{B}\mathrm{un}(C)$, $\underline{\mathrm{Coh}}_{r,d}(C)$ and $\mathcal{B}\mathrm{un}_{r,d}(C)$ are prestacks.

Let $G \to S$ be an fppf affine group scheme. A *principal G-bundle over an S-scheme T* is a scheme P with an action of G via $\sigma\colon G \times_S P \to P$ such that $P \to T$ is a G-invariant fppf morphism and

$$(\sigma, p_2)\colon G \times_S P \to P \times_T P, \qquad (g, p) \mapsto (gp, p)$$

is an isomorphism.

Example 2.3.13 (Classifying stack)**.** Let $G \to S$ be a smooth affine group scheme. A *principal G-bundle over an S-scheme T* is a scheme P with an action of G via $\sigma\colon G \times_S P \to P$ such that $P \to T$ is a G-invariant fppf morphism and

$(\sigma, p_2): G \times_S P \to P \times_T P, \qquad (g,p) \mapsto (gp, p)$ is an isomorphism. Equivalently, there is an étale cover $T' \to T$ such that $P \times_T T'$ is G-equivariant isomorphic to the trivial principal G-bundle $G \times T$ (Proposition C.2.4). See §C.2 for a further discussion with examples.

We define the *classifying stack* $\mathbf{B}G$ *of* G as the prestack over Sch/S where objects are principal G-bundles $P \to T$ and a morphism $(P' \to T') \to (P \to T)$ is the data of a G-equivariant morphism $P' \to P$ such that

$$\begin{tikzcd} P' \arrow[r] \arrow[d] & P \arrow[d] \\ T' \arrow[r] & T \end{tikzcd}$$

is cartesian. We show in Example 2.4.8 that $\mathbf{B}G$ is a stack over $(\mathrm{Sch}/S)_{\text{ét}}$. See §0.7.1 for motivation of the above definition.

Definition 2.3.14 (Quotient prestacks and stacks)**.** Let $G \to S$ be a smooth affine group scheme acting on a scheme U over S. We define the *quotient prestack* $[U/G]^{\mathrm{pre}}$ as the category over Sch/S where the fiber category over an S-scheme T is the quotient groupoid $[U(T)/G(T)]$ whose objects are elements $u \in U(T)$. A morphism from $u' \in U(T')$ to $u \in U(T)$ is the data of a map $f: T' \to T$ and an element $g \in G(T')$ such that $u' = \gamma \cdot (u \circ f)$.

We define the *quotient stack* $[U/G]$ as the prestack (which is shown to be a stack in Example 2.4.9) over Sch/S whose objects over an S-scheme T are diagrams

$$\begin{tikzcd} P \arrow[r] \arrow[d] & U \\ T & \end{tikzcd}$$

where $P \to T$ is a principal G-bundle and $P \to U$ is a G-equivariant morphism of schemes. A morphism $(P' \to T', P' \to U) \to (P \to T, P \to U)$ consists of a morphism $T' \to T$ and a G-equivariant morphism $P' \to P$ of schemes such that the diagram

$$\begin{tikzcd} P' \arrow[r] \arrow[d] \arrow[rr, bend left] \arrow[dr, phantom, "\square"] & P \arrow[r] \arrow[d] & U \\ T' \arrow[r] & T & \end{tikzcd}$$

is commutative and left square is cartesian.

Classifying stacks and quotient stacks for non-smooth or non-affine group schemes are discussed in Definitions 6.2.7 and 6.2.8.

Exercise 2.3.15. Verify that $[U/G]^{\mathrm{pre}}$ and $[U/G]$ are prestacks over Sch/S.

2.3.3 Morphisms of prestacks

Definition 2.3.16.

(1) A *morphism of prestacks* $f\colon \mathcal{X} \to \mathcal{Y}$ is a functor $f\colon \mathcal{X} \to \mathcal{Y}$ such that the diagram

$$\begin{tikzcd} \mathcal{X} \arrow[rr, "f"] \arrow[dr, "p_{\mathcal{X}}"'] & & \mathcal{Y} \arrow[dl, "p_{\mathcal{Y}}"] \\ & \mathcal{S} & \end{tikzcd}$$

strictly commutes, i.e. for every object $a \in \mathrm{Ob}(\mathcal{X})$, there is an *equality* $p_{\mathcal{X}}(a) = p_{\mathcal{Y}}(f(a))$ of objects in $\mathcal{S}$.

(2) If $f, g\colon \mathcal{X} \to \mathcal{Y}$ are morphisms of prestacks, a *2-morphism* (or *2-isomorphism*) $\alpha\colon f \to g$ is a natural transformation $\alpha\colon f \to g$ such that for every object $a \in \mathcal{X}$, the morphism $\alpha_a\colon f(a) \to g(a)$ in $\mathcal{Y}$ (which is an isomorphism) is over the identity in $\mathcal{S}$. We often describe the 2-morphism α schematically as

$$\begin{tikzcd} \mathcal{X} \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathcal{Y}. \end{tikzcd}$$

(3) We define the category $\mathrm{MOR}(\mathcal{X}, \mathcal{Y})$ whose objects are morphisms of prestacks and whose morphisms are 2-morphisms.

(4) A *2-commutative diagram* (which we often call simply a *commutative diagram*) is a diagram

$$\begin{tikzcd} \mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}' \arrow[r, "f'"] \arrow[d, "g'"] \arrow[dr, phantom, "\swarrow\alpha"] & \mathcal{Y}' \arrow[d, "g"] \\ \mathcal{X} \arrow[r, "f"] & \mathcal{Y} \end{tikzcd}$$

together with a 2-isomorphism $\alpha\colon g \circ f' \xrightarrow{\sim} f \circ g'$.

(5) A morphism $f\colon \mathcal{X} \to \mathcal{Y}$ of prestacks is a *monomorphism* (resp. *epimorphism*, *isomorphism*) if f is fully faithful (resp. essentially surjective, equivalence of categories).

Exercise 2.3.17. Show that every 2-morphism is an isomorphism of functors, or in other words that $\mathrm{MOR}(\mathcal{X}, \mathcal{Y})$ is a groupoid.

Exercise 2.3.18. Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a morphism of prestacks over a category $\mathcal{S}$.

(a) Show that f is a monomorphism if and only if $f_S\colon \mathcal{X}(S) \to \mathcal{Y}(S)$ is fully faithful for every $S \in \mathcal{S}$.

(b) Show that f is an isomorphism if and only if there exists a morphism $g\colon \mathcal{Y} \to \mathcal{X}$ and 2-isomorphisms $g \circ f \xrightarrow{\sim} \mathrm{id}_{\mathcal{X}}$ and $f \circ g \xrightarrow{\sim} \mathrm{id}_{\mathcal{Y}}$.

A prestack $\mathcal{X}$ is *equivalent to a presheaf* if there is a presheaf F and an isomorphism between $\mathcal{X}$ and the stack $\mathcal{X}_F$ corresponding to F (see Example 2.3.7).

Exercise 2.3.19. Show that G acts freely on U (i.e. the action map $(\sigma, p_2)\colon G \times_S U \to U \times_S U$ is a monomorphism) if and only if $[U/G]^{\mathrm{pre}}$ (resp. $[U/G]$) is equivalent to a presheaf. We often denote these presheaves by $(U/G)^{\mathrm{pre}}$ and U/G.

2.3.4 The 2-Yoneda lemma

Recall that Yoneda's lemma (Lemma 0.4.1) implies that for a presheaf $F\colon \mathcal{S} \to \mathrm{Sets}$ on a category $\mathcal{S}$ and an object $S \in \mathcal{S}$, there is a bijection $\mathrm{Mor}(S, F) \xrightarrow{\sim} F(S)$, where we view S as a presheaf via $\mathrm{Mor}(-, S)$. We will need an analogue of Yoneda's lemma for prestacks. First we recall that an object $S \in \mathcal{S}$ can also be viewed as a prestack over $\mathcal{S}$, which we also denote by S, whose objects over $T \in \mathcal{S}$ are morphisms $T \to S$ and a morphism $(T \to S) \to (T' \to S)$ is an S-morphism $T \to T'$.

Lemma 2.3.20 (The 2-Yoneda Lemma)**.** *Let $\mathcal{X}$ be a prestack over a category $\mathcal{S}$ and $S \in \mathcal{S}$. The functor*

$$\mathrm{MOR}(S, \mathcal{X}) \to \mathcal{X}(S), \qquad f \mapsto f_S(\mathrm{id}_S)$$

is an equivalence of categories.

Proof. We will construct a quasi-inverse $\Psi\colon \mathcal{X}(S) \to \mathrm{MOR}(S, \mathcal{X})$ as follows.

On objects: For $a \in \mathcal{X}(S)$, we define $\Psi(a)\colon S \to \mathcal{X}$ as the morphism of prestacks sending an object $(T \xrightarrow{f} S)$ (of the prestack corresponding to S) over T to a *choice* of pullback $f^*a \in \mathcal{X}(T)$ and a morphism $(T' \xrightarrow{f'} S) \to (T \xrightarrow{f} S)$ given by an S-morphism $g\colon T' \to T$ to the morphism $f'^*a \to f^*a$ uniquely filling in the diagram

$$\begin{tikzcd}
f'^*a \arrow[r, dashed] \arrow[d] \arrow[rr, bend left] & f^*a \arrow[r] \arrow[d, mapsto] & a \arrow[d, mapsto] \\
T' \arrow[r, "g"] & T \arrow[r, "f"] & S,
\end{tikzcd}$$

using Axiom (2) of a prestack.

On morphisms: If $\alpha\colon a' \to a$ is a morphism in $\mathcal{X}(S)$, then $\Psi(\alpha)\colon \Psi(a') \to \Psi(a)$ is defined as the morphism of functors which maps a morphism $T \xrightarrow{f} S$ (i.e. an object in S over T) to the unique morphism $f^*a' \to f^*a$ filling in the diagram

$$\begin{tikzcd}
f^*a' \arrow[r, dashed] \arrow[d] & f^*a \arrow[d] \\
a' \arrow[r, "\alpha"] & a
\end{tikzcd} \qquad \text{over} \qquad \begin{tikzcd}
T \arrow[d, "f"] \\
S
\end{tikzcd}$$

using again Axiom (2) of a prestack.

We leave the verification that Ψ is a quasi-inverse to the reader. □

We will use the 2-Yoneda lemma, often without mention, throughout these notes in passing between morphisms $S \to \mathcal{X}$ and objects of $\mathcal{X}$ over S.

Example 2.3.21 (Quotient stack presentations)**.** Consider the prestack $[U/G]$ in Definition 2.3.14 arising from a group action $\sigma\colon G \times_S U \to U$. The object of $[U/G]$ over U given by the diagram

$$\begin{tikzcd}
G \times_S U \arrow[r, "\sigma"] \arrow[d, "p_2"] & U \\
U
\end{tikzcd}$$

corresponds via the 2-Yoneda lemma (Lemma 2.3.20) to a morphism $U \to [U/G]$.

Exercise 2.3.22.

(a) Show that there is a morphism $p\colon U \to [U/G]^{\text{pre}}$ and a 2-commutative diagram

$$\begin{tikzcd} G \times_S U \arrow[r, "\sigma"] \arrow[d, "p_2"'] \arrow[dr, phantom, "\Swarrow\alpha"] & U \arrow[d, "p"] \\ U \arrow[r, "p"] & {[U/G]^{\text{pre}}} \end{tikzcd}$$

(b) Show that $U \to [U/G]^{\text{pre}}$ is a categorical quotient among prestacks, i.e. for every 2-commutative diagram

$$\begin{tikzcd} G \times_S U \arrow[r, "\sigma"] \arrow[d, "p_2"'] \arrow[dr, phantom, "\Swarrow\alpha"] & U \arrow[d, "p"] \arrow[ddr, bend left, "\varphi"] & \\ U \arrow[r, "p"] \arrow[drr, bend right, "\varphi"'] & {[U/G]^{\text{pre}}} \arrow[dr, phantom, "\Swarrow\tau"] & \\ & & \mathcal{Z} \end{tikzcd}$$

of prestacks, there exists a morphism $\chi\colon [U/G]^{\text{pre}} \to \mathcal{Z}$ and a 2-isomorphism $\beta\colon \varphi \xrightarrow{\sim} \chi \circ p$ which is compatible with α and τ (i.e. the two natural transformations $\varphi \circ \sigma \xrightarrow{\beta\circ\sigma} \chi \circ p \circ \sigma \xrightarrow{\chi\circ\alpha} \chi \circ p \circ p_2$ and $\varphi \circ \sigma \xrightarrow{\tau} \varphi \circ p_2 \xrightarrow{\beta\circ p_2} \chi \circ p \circ p_2$ agree.

2.3.5 Fiber products

We discuss fiber products for prestacks and in particular prove their existence. Recall that for morphisms $X \to Y$ and $Y' \to Y$ of presheaves on a category $\mathcal{S}$, the fiber product can be constructed as the presheaf mapping an object $S \in \mathcal{S}$ to the fiber product $X(S) \times_{Y(S)} Y'(S)$ of sets. Essentially the same construction works for morphisms $\mathcal{X} \to \mathcal{Y}$ and $\mathcal{Y}' \to \mathcal{Y}$ of prestacks but since we are dealing with groupoids rather than sets, the fiber category over an object $S \in \mathcal{S}$ should be the fiber product $\mathcal{X}(S) \times_{\mathcal{Y}(S)} \mathcal{Y}'(S)$ of groupoids.

The reader may first want to work on Item (a) and Exercise 2.3.28 on fiber products of groupoids as they not only provide a warmup to fiber products of prestacks but motivate its construction.

Construction 2.3.23. Let $f\colon \mathcal{X} \to \mathcal{Y}$ and $g\colon \mathcal{Y}' \to \mathcal{Y}$ be morphisms of prestacks over a category $\mathcal{S}$. Define the prestack $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}'$ over $\mathcal{S}$ as the category of triples (x, y', γ) where $x \in \mathcal{X}$ and $y' \in \mathcal{Y}'$ are objects over the *same* object $S := p_{\mathcal{X}}(x) = p_{\mathcal{Y}'}(y') \in \mathcal{S}$, and $\gamma\colon f(x) \xrightarrow{\sim} g(y')$ is an isomorphism in $\mathcal{Y}(S)$. A morphism $(x_1, y_1', \gamma_1) \to (x_2, y_2', \gamma_2)$ consists of a triple (f, χ, γ') where $f\colon p_{\mathcal{X}}(x_1) = p_{\mathcal{Y}'}(y_1') \to p_{\mathcal{Y}'}(y_2') = p_{\mathcal{X}}(x_2)$ is a morphism in $\mathcal{S}$, and $\chi\colon x_1 \xrightarrow{\sim} x_2$ and $\gamma'\colon y_1' \xrightarrow{\sim} y_2'$ are morphisms in $\mathcal{X}$ and $\mathcal{Y}'$ over f such that

$$\begin{tikzcd} f(x_1) \arrow[r, "f(\chi)"] \arrow[d, "\gamma_1"] & f(x_2) \arrow[d, "\gamma_2"] \\ g(y_1') \arrow[r, "g(\gamma')"] & g(y_2') \end{tikzcd}$$

commutes.

Let $p_1 : \mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}' \to \mathcal{X}$ and $p_2 : \mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}' \to \mathcal{X}$ denote the projections $(x, y', \gamma) \mapsto x$ and $(x, y', \gamma) \mapsto y'$. There is a 2-isomorphism $\alpha \colon f \circ p_1 \xrightarrow{\sim} g \circ p_2$ defined on an object $(x, y', \gamma) \in \mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}'$ by $\alpha_{(x,y',\gamma)} \colon f(x) \xrightarrow{\gamma} g(y')$. This yields a 2-commutative diagram

$$\begin{tikzcd}
\mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}' \arrow[r, "p_2"] \arrow[d, "p_1"] \arrow[dr, phantom, "\alpha \Nearrow"] & \mathcal{Y}' \arrow[d, "g"] \\
\mathcal{X} \arrow[r, "f"] & \mathcal{Y}
\end{tikzcd} \tag{2.3.1}$$

Theorem 2.3.24. *The prestack $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}'$ together with the morphisms p_1 and p_2 and the 2-isomorphism α as in (2.3.1) satisfy the following universal property: for every 2-commutative diagram*

$$\begin{tikzcd}
 & & \mathcal{Y}' \arrow[dr, "g"] & \\
\mathcal{T} \arrow[urr, bend left, "q_2"] \arrow[drr, bend right, "q_1"'] \arrow[r, phantom, "\tau \Uparrow"] & \mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}' \arrow[ur, "p_2"] \arrow[dr, "p_1"'] \arrow[rr, phantom, "\alpha \Uparrow"] & & \mathcal{Y} \\
 & & \mathcal{X} \arrow[ur, "f"'] &
\end{tikzcd}$$

with 2-isomorphism $\tau \colon f \circ q_1 \xrightarrow{\sim} g \circ q_2$, there exist a morphism $h \colon \mathcal{T} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}'$ and 2-isomorphisms $\beta \colon q_1 \to p_1 \circ h$ and $\gamma \colon q_2 \to p_2 \circ h$ yielding a 2-commutative diagram

$$\begin{tikzcd}
 & & \mathcal{Y}' \arrow[dr, "g"] & \\
\mathcal{T} \arrow[urr, bend left, "q_2"] \arrow[drr, bend right, "q_1"'] \arrow[r, "h"] \arrow[ur, phantom, "\Downarrow\gamma"] \arrow[dr, phantom, "\Uparrow\beta"] & \mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}' \arrow[ur, "p_2"] \arrow[dr, "p_1"'] \arrow[rr, phantom, "\alpha \Uparrow"] & & \mathcal{Y} \\
 & & \mathcal{X} \arrow[ur, "f"'] &
\end{tikzcd}$$

such that

$$\begin{tikzcd}
f \circ q_1 \arrow[r, "f(\beta)"] \arrow[d, "\tau"] & f \circ p_1 \circ h \arrow[d, "\alpha \circ h"] \\
g \circ q_2 \arrow[r, "g(\gamma)"] & g \circ p_2 \circ h
\end{tikzcd}$$

commutes. The data (h, β, γ) is unique up to unique isomorphism.

Proof. We define $h \colon \mathcal{T} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}'$ on objects by $t \mapsto \big(q_1(t), q_2(t), f(q_1(t)) \xrightarrow{\tau_t} g(q_2(t))\big)$ and on morphisms as $(t \xrightarrow{\Psi} t') \mapsto (p_{\mathcal{T}}(\Psi), q_1(\Psi), q_2(\Psi))$. There are *equalities* of functors $q_1 = p_1 \circ h$ and $q_2 = p_2 \circ h$ so we define β and γ as the identity natural transformation. The remaining details are left to the reader. $\square$

Definition 2.3.25. We say that a 2-commutative diagram

$$\begin{tikzcd} \mathcal{X}' \arrow[r] \arrow[d] \arrow[dr, phantom, "\swarrow_{\alpha}"] & \mathcal{Y}' \arrow[d] \\ \mathcal{X} \arrow[r] & \mathcal{Y} \end{tikzcd}$$

is *cartesian* if it satisfies the universal property of Theorem 2.3.24. We often write a cartesian diagram of stacks as

$$\begin{tikzcd} \mathcal{X}' \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{Y}' \arrow[d] \\ \mathcal{X} \arrow[r] & \mathcal{Y} \end{tikzcd}$$

where the existence of the 2-isomorphism α is implicit.

2.3.6 Examples of fiber products

In this section, the reader is asked to verify several convenient fiber product diagrams. To get started, it is instructive to first compute fiber products of groupoids (rather than prestacks).

Exercise 2.3.26.

(a) If $\mathcal{C} \xrightarrow{f} \mathcal{D}$ and $\mathcal{D}' \xrightarrow{g} \mathcal{D}$ are morphisms of groupoids, define the groupoid $\mathcal{C} \times_{\mathcal{D}} \mathcal{D}'$ whose objects are triples (c, d', δ) where $c \in \mathcal{C}$ and $d' \in \mathcal{D}'$ are objects, and $\delta\colon f(c) \xrightarrow{\sim} g(d')$ is an isomorphism in $\mathcal{D}$. A morphism $(c_1, d_1', \delta_1) \to (c_2, d_2', \delta_2)$ is the data of morphisms $\gamma\colon c_1 \xrightarrow{\sim} c_2$ and $\delta'\colon d_1' \xrightarrow{\sim} d_2'$ such that

$$\begin{tikzcd} f(c_1) \arrow[r, "f(\gamma)"] \arrow[d, "\delta_1"] & f(c_2) \arrow[d, "\delta_2"] \\ g(d_1') \arrow[r, "g(\delta')"] & g(d_2') \end{tikzcd}$$

commutes. Formulate a university property for fiber products of groupoids and show that $\mathcal{C} \times_{\mathcal{D}} \mathcal{D}'$ satisfies it.

(b) If $f\colon \mathcal{X} \to \mathcal{Y}$ and $g\colon \mathcal{Y}' \to \mathcal{Y}$ are morphisms of prestacks over a category $\mathcal{S}$, show that for every $S \in \mathcal{S}$, the fiber category $(\mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}')(S)$ is a fiber product $\mathcal{X}(S) \times_{\mathcal{Y}(S)} \mathcal{Y}'(S)$ of groupoids.

The following foreshadows analogous cartesian diagrams associated to quotient stacks.

Exercise 2.3.27.

(a) Let G be a group acting on a set U via $\sigma\colon G \times U \to U$. Let $[U/G]$ denote the quotient groupoid (Exercise 0.3.7) with projection $p\colon U \to [U/G]$. Show that there are cartesian diagrams

$$\begin{tikzcd} G \times U \arrow[r, "\sigma"] \arrow[d, "p_2"] \arrow[dr, phantom, "\square"] & U \arrow[d, "p"] \\ U \arrow[r, "p"] & {[U/G]} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} G \times U \arrow[r, "(\sigma, p_2)"] \arrow[d] \arrow[dr, phantom, "\square"] & U \times U \arrow[d, "p \times p"] \\ {[U/G]} \arrow[r, "\Delta"] & {[U/G] \times [U/G].} \end{tikzcd}$$

(b) Recall from Example 0.3.3 that the classifying groupoid BG of a group G is the category with one object $*$ with $\mathrm{Mor}(*,*) = G$. If $x \in U$, show that there is a morphism $BG_x \to [U/G]$ of groupoids and a cartesian diagram

$$\begin{tikzcd} Gx \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & U \arrow[d, "p"] \\ BG_x \arrow[r] & {[U/G]}. \end{tikzcd}$$

(See Proposition 3.5.16 and Remark 3.5.19 for the analogous diagrams for algebraic stacks.)

(c) Let $\phi\colon H \to G$ be a homomorphism of groups. Show that there is an induced morphism $BH \to BG$ of groupoids and that $BH \times_{BG} \mathrm{pt} \cong [G/H]$.

(d) If $K \lhd G$ is a normal subgroup with quotient $Q = G/K$, show that there is a cartesian diagram

$$\begin{tikzcd} Q \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & BK \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathrm{pt} \arrow[d] \\ \mathrm{pt} \arrow[r] & BG \arrow[r] & BQ \end{tikzcd}$$

.

The following exercise is essential for working with quotient stacks and in particular is used to verify the algebraicity of quotient stacks (Theorem 3.1.9).

Exercise 2.3.28.

(a) Let $G \to S$ be a smooth affine group scheme acting on a scheme U over S via $\sigma\colon G \times_S U \to U$. Let $[U/G]$ be the quotient stack (Definition 2.3.14). Show that there are cartesian diagrams

$$\begin{tikzcd} G \times_S U \arrow[r, "\sigma"] \arrow[d, "p_2"] \arrow[dr, phantom, "\square"] & U \arrow[d, "p"] \\ U \arrow[r, "p"] & {[U/G]} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} G \times_S U \arrow[r, "{(\sigma, p_2)}"] \arrow[d] \arrow[dr, phantom, "\square"] & U \times_S U \arrow[d, "p \times p"] \\ {[U/G]} \arrow[r, "\Delta"] & {[U/G] \times_S [U/G]}. \end{tikzcd}$$

(b) Show that if $P \to T$ is a principal G-bundle and $P \to U$ is a G-equivariant map, there is a morphism $T \to [U/G]$, unique up to unique isomorphism, and a cartesian diagram

$$\begin{tikzcd} P \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & U \arrow[d] \\ T \arrow[r] & {[U/G]}. \end{tikzcd}$$

(We will later see that $[U/G]$ is an algebraic stack and that $U \to [U/G]$ is principal G-bundle (Theorem 3.1.9). Therefore the principal G-bundle $U \to [U/G]$ with the identity map $U \to U$ is the universal family over $[U/G]$, corresponding to the identity map $[U/G] \to [U/G]$.

As with schemes, the following diagram is utilized extensively. As we will see in §3.2.2, the diagonal is used to define stabilizers and the inertia stack.

Exercise 2.3.29 (Magic Square)**.** Let $\mathcal{X}$ be a prestack. Show that for every morphism $a\colon S \to \mathcal{X}$ and $b\colon T \to \mathcal{X}$, there is a cartesian diagram

$$\begin{tikzcd} S \times_{\mathcal{X}} T \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & S \times T \arrow[d, "a \times b"] \\ \mathcal{X} \arrow[r, "\Delta"] & \mathcal{X} \times \mathcal{X}. \end{tikzcd}$$

Properties of the diagonal are used to define separation conditions on algebraic stacks (see §3.3.3) and translate into properties of Isom presheaves.

Exercise 2.3.30 (Isom presheaves)**.** Let $\mathcal{X}$ be a prestack over a category $\mathcal{S}$.

(a) For $S \in \mathcal{S}$, recall from Example 2.1.9 that $\mathcal{S}/S$ denotes the category whose objects are morphisms $T \to S$ in $\mathcal{S}$ and whose morphisms are S-morphisms. Show that for objects a and b of $\mathcal{X}$ over S that the functor

$$\begin{aligned} \underline{\mathrm{Isom}}_{\mathcal{X}(S)}(a,b)\colon \mathcal{S}/S &\to \mathrm{Sets} \\ (T \xrightarrow{f} S) &\mapsto \mathrm{Mor}_{\mathcal{X}(T)}(f^*a, f^*b), \end{aligned}$$

where f^*a and f^*b are choices of a pullback, defines a presheaf on $\mathcal{S}/S$.

(b) Show that there is a cartesian diagram

$$\begin{tikzcd} \underline{\mathrm{Isom}}_{\mathcal{X}(S)}(a,b) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & S \arrow[d, "(a,b)"] \\ \mathcal{X} \arrow[r, "\Delta"] & \mathcal{X} \times \mathcal{X}. \end{tikzcd}$$

(c) Show that the presheaf $\underline{\mathrm{Aut}}_{\mathcal{X}(T)}(a) = \underline{\mathrm{Isom}}_{\mathcal{X}(T)}(a,a)$ is naturally a presheaf in groups.

(d) Show that $\mathcal{X}$ is equivalent to a sheaf if and only if the diagonal $\mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is a monomorphism.

Exercise 2.3.31. If $n \geq 2$, show that $[\mathbb{A}^n/\mathbb{G}_m^n] \cong \underbrace{[\mathbb{A}^1/\mathbb{G}_m] \times \cdots \times [\mathbb{A}^1/\mathbb{G}_m]}_{n \text{ times}}$.

Exercise 2.3.32.

(a) Show that if $H \to G$ is a morphism of smooth affine group schemes over a scheme S, there is an induced morphism of prestacks $\mathbf{B}H \to \mathbf{B}G$ over Sch/S.

(b) Show that $\mathbf{B}H \times_{\mathbf{B}G} S \cong [G/H]$.

(c) If $1 \to K \to G \to Q \to 1$ is an exact sequence of smooth affine algebraic groups over a field $\Bbbk$, show that there is a cartesian diagram

$$\begin{tikzcd} Q \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathbf{B}K \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \operatorname{Spec} \Bbbk \arrow[d] \\ \operatorname{Spec} \Bbbk \arrow[r] & \mathbf{B}G \arrow[r] & \mathbf{B}Q. \end{tikzcd}$$

2.4 Stacks

A stack over a site $\mathcal{S}$ is a prestack $\mathcal{X}$ such that objects and morphisms glue uniquely in the Grothendieck topology of $\mathcal{S}$ (see Definition 2.4.1). Verifying a given prestack is a stack reduces to a *descent* condition on objects and morphisms with respect to the covers of $\mathcal{S}$. The theory of descent is discussed in Section B.1 and is essential for verifying the stack axioms.

2.4.1 Definition of a stack

Definition 2.4.1 (Stacks)**.** A prestack $\mathcal{X}$ over a site $\mathcal{S}$ is a *stack* if the following conditions hold for all coverings $\{S_i \to S\}$ of an object $S \in \mathcal{S}$:

(1) (morphisms glue) For objects a and b in $\mathcal{X}$ over S and morphisms $\phi_i \colon a|_{S_i} \to b$ such that $\phi_i|_{S_{ij}} = \phi_j|_{S_{ij}}$ as displayed in the diagram

$$\begin{tikzcd}
 & a|_{S_i} \arrow[dr] \arrow[drr, bend left, "\phi_i"] & & \\
a|_{S_{ij}} \arrow[ur] \arrow[dr] & & a \arrow[r, dashed] & b \\
 & a|_{S_j} \arrow[ur] \arrow[urr, bend right, "\phi_j"'] & &
\end{tikzcd}
\quad \text{over} \quad
\begin{tikzcd}
 & S_i \arrow[dr] & \\
S_{ij} \arrow[ur] \arrow[dr] & & S, \\
 & S_j \arrow[ur] &
\end{tikzcd}$$

there exists a unique morphism $\phi \colon a \to b$ with $\phi|_{S_i} = \phi_i$.

(2) (objects glue) For objects a_i over S_i and isomorphisms $\alpha_{ij} \colon a_i|_{S_{ij}} \to a_j|_{S_{ij}}$, as displayed in the diagram

$$\begin{tikzcd}
 & & a_i \arrow[dr, dashed] & \\
a_i|_{S_{ij}} \arrow[r, "\alpha_{ij}"] & a_j|_{S_{ij}} \arrow[ur] \arrow[dr] & & a \\
 & & a_j \arrow[ur, dashed] &
\end{tikzcd}
\quad \text{over} \quad
\begin{tikzcd}
 & S_i \arrow[dr] & \\
S_{ij} \arrow[ur] \arrow[dr] & & S \\
 & S_j \arrow[ur] &
\end{tikzcd}$$

satisfying the cocycle condition $\alpha_{jk}|_{S_{ijk}} \circ \alpha_{ij}|_{S_{ijk}} = \alpha_{ik}|_{S_{ijk}}$ on S_{ijk}, then there exists an object a over S and isomorphisms $\phi_i \colon a|_{S_i} \to a_i$ such that $\alpha_{ij} \circ \phi_i|_{S_{ij}} = \phi_j|_{S_{ij}}$ on S_{ij}.

Remark 2.4.2. There is an alternative description of the stack axioms analogous to the sheaf axiom of a presheaf $F \colon \mathcal{S} \to \mathrm{Sets}$, i.e. that $F(S) \to \prod_i F(S_i) \rightrightarrows \prod_{i,j} F(S_i \times_S S_j)$ is exact for coverings $\{S_i \to S\}$. Namely, by adding an additional layer corresponding to triple intersections, the stack axiom translates to the 'exactness' of

$$\mathcal{X}(S) \longrightarrow \prod_i \mathcal{X}(S_i) \rightrightarrows \prod_{i,j} \mathcal{X}(S_i \times_S S_j) \substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow} \prod_{i,j,k} \mathcal{X}(S_i \times_S S_j \times_S S_k).$$

Exercise 2.4.3. Show that Axiom (1) is equivalent to the condition that for all objects a and b of $\mathcal{X}$ over $S \in \mathcal{S}$, the Isom presheaf $\underline{\mathrm{Isom}}_{\mathcal{X}(S)}(a, b)$ (see Exercise 2.3.30) is a sheaf on $\mathcal{S}/S$.

A *morphism* of stacks is a morphism of prestacks.

Exercise 2.4.4 (Fiber product of stacks)**.** Show that if $\mathcal{X} \to \mathcal{Y}$ and $\mathcal{Y}' \to \mathcal{Y}$ are morphisms of stacks over a site $\mathcal{S}$, then $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}'$ is also a stack over $\mathcal{S}$.

2.4.2 First examples of stacks

Example 2.4.5 (Sheaves and schemes are stacks)**.** Recall that if F is a presheaf on a site $\mathcal{S}$, we can construct a prestack $\mathcal{X}_F$ over $\mathcal{S}$ as the category of pairs (a, S) where $S \in \mathcal{S}$ and $a \in F(S)$ (see Example 2.3.7). If F is a sheaf, then $\mathcal{X}_F$ is a stack. We often abuse notation by writing F also as the stack $\mathcal{X}_F$.

Since schemes are sheaves on $\mathrm{Sch}_{\text{ét}}$ (Proposition 2.2.6), a scheme X defines a stack over $\mathrm{Sch}_{\text{ét}}$ (where objects over a scheme S are morphisms $S \to X$), which we also denote as X.

Example 2.4.6 (Stack of sheaves)**.** Let $\underline{\mathrm{Sheaves}}$ be the prestack over Sch whose objects are pairs (T, F) where T is a scheme and F is a sheaf on the Zariski topology of T. A morphism $(T, F) \to (T', F')$ is the data of a morphism $f\colon T \to T'$ of schemes and a morphism $f_*F \to F'$ of sheaves on T' such that the adjoint $F \to f^{-1}F'$ is an isomorphism. Because sheaves and their morphisms glue in the Zariski topology ([Har77, Exer. II.1.15 and 22]), $\mathcal{X}$ is a stack over the big Zariski site $\mathrm{Sch}_{\mathrm{Zar}}$. The full subcategories $\underline{\mathrm{QCoh}}$ and $\underline{\mathrm{Bun}}$ of $\underline{\mathrm{Sheaves}}$ parameterizing quasi-coherent sheaves and vector bundles are also stacks over $\mathrm{Sch}_{\mathrm{Zar}}$.

Exercise 2.4.7.

(1) Formulate and prove a more general statement for sheaves over an arbitrary site.

(2) Use fppf descent to show that the prestack $\underline{\mathrm{QCoh}}$ (resp. $\underline{\mathrm{Bun}}$), parameterizing pairs (T, F) where T is a scheme and F is a quasi-coherent sheaf on T (resp. vector bundle on T), is a stack over $\mathrm{Sch}_{\mathrm{fppf}}$.

Example 2.4.8 (Classifying stacks)**.** Let $G \to S$ be a smooth affine group scheme. The classifying prestack $\mathbf{B}G$ is the category over Sch/S classifying principal G-bundles $P \to T$ (see Example 2.3.13). We claim that $\mathbf{B}G$ is a stack over $(\mathrm{Sch}/S)_{\text{ét}}$. Axiom (1) holds as morphisms to schemes glue uniquely in the étale topology (Proposition B.2.1). For Axiom (2), if $\{T_i \to T\}$ is an étale covering, $(P_i \to T_i, P_i \to U)$ are objects over T_i, and $\alpha_{ij}\colon P_i \times T_i T_{ij} \to P_j \times_{T_j} T_{ij}$ satisfying the cocycle condition $\alpha_{jk} \circ \alpha_{ij} = \alpha_{ik}$ on T_{ijk}, then the existence of a principal G-bundle $P \to T$ follows from Effective Descent for Principal G-bundles (Proposition C.2.5) and the existence of $P \to U$ follows from étale descent for morphisms of schemes (B.2.1).

Example 2.4.9 (Quotient stacks)**.** Let $G \to S$ be a smooth affine group scheme acting on a scheme U over S. Let $[U/G]$ be the prestack defined in Definition 2.3.14; an object over an S-scheme T is a diagram

$$\begin{tikzcd} P \arrow[r] \arrow[d] & U \\ T & \end{tikzcd}$$

where $P \to T$ is a principal G-bundle and $P \to U$ is a G-equiariant morphism of schemes. The prestack $[U/G]$ is a stack over $(\mathrm{Sch}/S)_{\text{ét}}$, which we call the *quotient* *stack*. Axiom (1) holds by étale descent for morphisms of schemes (B.2.1). Axiom (2) holds because in the étale topology principal G-bundles glue uniquely (as seen in the previous example) and morphisms of schemes do also (B.2.1).

Example 2.4.10 (Stack of schemes over $\mathrm{Sch}_{\mathrm{Zar}}$)**.** Define $\underline{\mathrm{Schemes}}$ as the prestack over Sch consists of morphisms $T \to S$ of schemes where a morphism $(T \to S) \to (T' \to S')$ consists of morphisms $T \to T'$ and $S \to S'$ of schemes such that the two compositions $T \to S'$ agree. The projection map takes $T \to S$ to S. Since schemes glue in the Zariski topology [Har77, Exer. II.2.12], $\underline{\mathrm{Schemes}}$ is a stack over $\mathrm{Sch}_{\mathrm{Zar}}$. However, $\underline{\mathrm{Schemes}}$ is *not* a stack over $\mathrm{Sch}_{\text{ét}}$. Schemes can be glued to algebraic spaces in the étale topology and there is a stack of algebraic spaces over $\mathrm{Sch}_{\text{ét}}$; see Exercise 4.4.14.

2.4.3 Moduli stack of curves

Let $\mathscr{M}_g$ denote the prestack of families of smooth curves $\mathcal{C} \to S$ of genus g; see Example 2.3.9.

Proposition 2.4.11 (Moduli stack of smooth curves)**.** *If* $g \geq 2$*, then* $\mathscr{M}_g$ *is a stack over* $\mathrm{Sch}_{\text{ét}}$*.*

Proof. Axiom (1) translates to: for families of smooth curves $\mathcal{C} \to S$ and $\mathcal{D} \to S$ of genus g and commutative diagrams

$$\begin{tikzcd}
\mathcal{C}_{S_{ij}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{C}_{S_i} \arrow[r] \arrow[d] \arrow[rr, bend left, "f_i"] \arrow[dr, phantom, "\square"] & \mathcal{C} \arrow[r, dashed, "f"] \arrow[d] & \mathcal{D} \arrow[dl] \\
S_{ij} \arrow[r] & S_i \arrow[r] & S &
\end{tikzcd}$$

of solid arrows for all i, j (i.e. morphisms $f_i \colon \mathcal{C}_{S_i} \to \mathcal{D}$ such that $f_i|_{\mathcal{C}_{S_{ij}}} = f_j|_{\mathcal{C}_{S_{ij}}}$), there exists a unique morphism filling in the diagram (i.e. $f_i = f|_{\mathcal{C}_{S_i}}$). The existence and uniqueness of f follows from étale descent for morphisms (Proposition B.2.1). The fact that f is an isomorphism also follows from étale descent (Proposition B.4.1).

Axiom (2) is more difficult: we must show that given diagrams

$$\begin{tikzcd}
\mathcal{C}_i|_{S_{ij}} \arrow[r, "\alpha_{ij}"] \arrow[dr] \arrow[rr, bend left] & \mathcal{C}_j|_{S_{ij}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{C}_j \arrow[r, dashed] \arrow[d, "\pi_j"] \arrow[dr, phantom, "\square"] & \mathcal{C} \arrow[d, dashed] \\
& S_{ij} \arrow[r] & S_j \arrow[r] & S
\end{tikzcd}$$

for all i, j where $\pi_i \colon \mathcal{C}_i \to S_i$ are families of smooth curves of genus g and $\alpha_{ij} \colon \mathcal{C}_i|_{S_{ij}} \to \mathcal{C}_j|_{S_{ij}}$ are isomorphisms satisfying the cocycle condition $\alpha_{jk} \circ \alpha_{ij} = \alpha_{ik}$, there is family of smooth curves $\mathcal{C} \to S$ and isomorphisms $\phi_i \colon \mathcal{C}|_{S_i} \to \mathcal{C}_i$ such that $\alpha_{ij} \circ \phi_i|_{\mathcal{C}_{S_{ij}}} = \phi_j|_{\mathcal{C}_{S_{ij}}}$.

We will use the following property of families of smooth curves (see Proposition 5.1.9): for a family of smooth curves $\pi \colon \mathcal{C} \to S$, $\omega_{\mathcal{C}/S}^{\otimes 3}$ is relatively very ample on S (as $g > 2$) and $F := \pi_* \omega_{\mathcal{C}/S}^{\otimes 3}$ is a vector bundle of rank $5(g-1)$. In particular, $\omega_{\mathcal{C}/S}^{\otimes 3}$ yields a closed immersion $\mathcal{C} \hookrightarrow \mathbb{P}(F)$ over S.

Therefore, if we set $E_i = (\pi_i)_*(\omega_{\mathcal{C}_i/S_i})$, there is a closed immersion $\mathcal{C}_i \hookrightarrow \mathbb{P}(E_i)$ over S_i. The isomorphisms α_{ij} induce isomorphisms $\beta_{ij} \colon E_i|_{S_{ij}} \to E_j|_{S_{ij}}$ satisfying the cocycle condition $\beta_{jk} \circ \beta_{ij} = \beta_{ik}$ on S_{ijk}. Descent for quasi-coherent sheaves (Proposition B.1.3) implies there is a quasi-coherent sheaf E on S and isomorphisms $\Psi_i \colon E|_{S_i} \to E_i$ such that $\beta_{ij} \circ \Psi_i|_{S_{ij}} = \Psi_j|_{S_{ij}}$. It follows again from descent that E is in fact a vector bundle (Proposition B.4.3). Pictorially, we have

$$\begin{tikzcd}
& \mathbb{P}(E_{ij}) \arrow[r] & \mathbb{P}(E_i) \arrow[r] & \mathbb{P}(E) \\
\mathcal{C}_i|_{S_{ij}} \arrow[ur, hook] \arrow[r] \arrow[d] & \mathcal{C}_i \arrow[ur, hook] \arrow[r, dashed] \arrow[d] & \mathcal{C} \arrow[ur, hook, dashed] \arrow[d, dashed] & \\
S_{ij} \arrow[r] & S_i \arrow[r] & S. &
\end{tikzcd}$$

Since the preimages of $\mathcal{C}_i \subset \mathbb{P}(E_i)$ and $\mathcal{C}_j \subset \mathbb{P}(E_j)$ in $\mathbb{P}(E_{ij})$ are equal, it follows from descent for closed subschemes (Proposition B.3.1) that there exists $\mathcal{C} \to S$ and isomorphisms ϕ_i such that $\alpha_{ij} \circ \phi_i|_{\mathcal{C}_{S_{ij}}} \to \phi_j|_{\mathcal{C}_{S_{ij}}}$. Since smoothness and properness are étale-local property on the target (Proposition B.4.1), $\mathcal{C} \to S$ is smooth and proper. The geometric fibers of $\mathcal{C} \to S$ are connected genus g curves since the geometric fibers of $\mathcal{C}_i \to S_i$ are. $\square$

Exercise 2.4.12.

(a) Show that the prestack $\mathcal{M}_0$ is a stack on $\mathrm{Sch}_{\text{ét}}$ isomorphic to $\mathbf{B}\,\mathrm{PGL}_2$ over $\operatorname{Spec}\mathbb{Z}$.

(b) Show that the moduli stack $\mathcal{M}_{1,1}$, whose objects are families of elliptic curves (see Example 0.4.31) is a stack on $\mathrm{Sch}_{\text{ét}}$.

(c) Is the prestack $\mathcal{M}_1$, whose objects over a scheme S are smooth families $\mathcal{C} \to S$ of genus 1 curves, a stack over $\mathrm{Sch}_{\text{ét}}$?

2.4.4 Moduli stack of coherent sheaves and vector bundles

Let C be a smooth connected projective curve over an algebraically closed field $\Bbbk$, and fix integers $r \geq 0$ and d. Recall from Example 2.3.11 that $\underline{\mathrm{Coh}}_{r,d}(C)$ (resp. $\mathcal{B}\mathrm{un}_{r,d}(C)$) denotes the prestack over $\mathrm{Sch}/\Bbbk$ consisting of pairs (E, S) where S is a $\Bbbk$-scheme and E is a coherent sheaf on C_S flat over S (resp. vector bundle on C_S) of rank r and degree d.

Proposition 2.4.13. *For all integers r and d with $r \geq 0$, $\underline{\mathrm{Coh}}_{r,d}(C)$ and $\mathcal{B}\mathrm{un}_{r,d}(C)$ are stacks over $(\mathrm{Sch}/\Bbbk)_{\text{ét}}$.*

Proof. Axioms (1) is precisely descent for morphisms of quasi-coherent sheaves (Proposition B.1.3(2)) while Axiom (2) is descent for quasi-coherent sheaves (Proposition B.1.3(1)) coupled with the fact that the property of a quasi-coherent sheaf being a coherent sheaf (resp. vector bundle) is étale-local (Proposition B.4.3). $\square$

2.4.5 Stackification

Given a presheaf F on a site $\mathcal{S}$, there is a sheafification $F \to F^{\mathrm{sh}}$ which is a left adjoint to the inclusion, i.e. $\mathrm{Mor}(F^{\mathrm{sh}}, G) \to \mathrm{Mor}(F, G)$ is bijective for every sheaf

G on $\mathcal{S}$ (Theorem 2.2.9). Similarly, there is a stackification $\mathcal{X} \to \mathcal{X}^{\mathrm{st}}$ of a prestack $\mathcal{X}$ over $\mathcal{S}$.

Theorem 2.4.14 (Stackification)**.** *If $\mathcal{X}$ is a prestack over a site $\mathcal{S}$, there exists a stack $\mathcal{X}^{\mathrm{st}}$, which we call the* stackification*, and a morphism $\mathcal{X} \to \mathcal{X}^{\mathrm{st}}$ of prestacks such that for every stack $\mathcal{Y}$ over $\mathcal{S}$, the induced functor*

$$\mathrm{MOR}(\mathcal{X}^{\mathrm{st}}, \mathcal{Y}) \to \mathrm{MOR}(\mathcal{X}, \mathcal{Y}) \tag{2.4.1}$$

is an equivalence of categories.

Proof. As in the construction of the sheafification (see the proof of Theorem 2.2.9), we construct the stackification in stages. Most details are left to the reader.

First, given a prestack $\mathcal{X}$, we can construct a prestack $\mathcal{X}^{\mathrm{st}_1}$ satisfying Axiom (1) and a morphism $\mathcal{X} \to \mathcal{X}^{\mathrm{st}_1}$ of prestacks such that

$$\mathrm{MOR}(\mathcal{X}^{\mathrm{st}_1}, \mathcal{Y}) \to \mathrm{MOR}(\mathcal{X}, \mathcal{Y})$$

is an equivalence for all prestacks $\mathcal{Y}$ satisfying Axiom (1). Specifically, the objects of $\mathcal{X}^{\mathrm{st}_1}$ are the same as $\mathcal{X}$, and for objects $a, b \in \mathcal{X}$ over $S, T \in \mathcal{S}$, the set of morphisms $a \to b$ in $\mathcal{X}^{\mathrm{st}_1}$ over a given morphism $f \colon S \to T$ is the global sections $\Gamma(S, \underline{\mathrm{Isom}}_{\mathcal{X}(S)}(a, f^*b)^{\mathrm{sh}})$ of the sheafification of the Isom presheaf (Exercise 2.3.30).

Second, given a prestack $\mathcal{X}$ satisfying Axiom (1), we construct a stack $\mathcal{X}$ and a morphism $\mathcal{X} \to \mathcal{X}^{\mathrm{st}}$ of prestacks such that (2.4.1) is an equivalence for all stacks $\mathcal{Y}$. An object of $\mathcal{X}^{\mathrm{st}}$ over $S \in \mathcal{S}$ is given by a triple consisting of a covering $\{S_i \to S\}$, objects a_i of $\mathcal{X}$ over S_i, and isomorphisms $\alpha_{ij} \colon a_i|_{S_{ij}} \to a_j|_{S_{ij}}$ satisfying the cocycle condition $\alpha_{jk}|_{S_{ijk}} \circ \alpha_{ij}|_{S_{ijk}} = \alpha_{ik}|_{S_{ijk}}$ on S_{ijk}. Morphisms

$$\big(\{S_i \to S\}, \{a_i\}, \{\alpha_{ij}\}\big) \to \big(\{T_\mu \to T\}, \{b_\mu\}, \{\beta_{\mu\nu}\}\big)$$

in $\mathcal{X}^{\mathrm{st}}$ over $S \to T$ are defined as follows: first consider the induced cover $\{S_i \times_S T_\mu \to S\}_{i,\mu}$ and choose pullbacks $a_i|_{S_i \times_S T_\mu}$ and $b_\mu|_{S_i \times_S T_\mu}$. A morphism is then the data of maps $\Psi_{i\mu} \colon a_i|_{S_i \times_S T_\mu} \to b_\mu|_{S_i \times_S T_\mu}$ for all i, μ which are compatible with α_{ij} and $\beta_{\mu\nu}$ (i.e. $\Psi_{j\nu} \circ \alpha_{ij} = \beta_{\mu\nu} \circ \Psi_{i\mu}$ on $S_{ij} \times_T T_{\mu\nu}$). $\square$

Exercise 2.4.15. Show that stackification commutes with fiber products: if $\mathcal{X} \to \mathcal{Y}$ and $\mathcal{Z} \to \mathcal{Y}$ are morphisms of prestacks, then $(\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z})^{\mathrm{st}} \cong \mathcal{X}^{\mathrm{st}} \times_{\mathcal{Y}^{\mathrm{st}}} \mathcal{Z}^{\mathrm{st}}$.

Exercise 2.4.16. Let $G \to S$ be a smooth affine group scheme acting on a scheme U over S. Recall from Definition 2.3.14 that the quotient prestack $[U/G]^{\mathrm{pre}}$ and quotient stack $[U/G]$ denote the prestacks over Sch/S classifying trivial principal G-bundles (resp. principal G-bundles) $P \to T$ and G-equivariant maps $P \to U$.

(a) Show that $[U/G]^{\mathrm{pre}}$ satisfies Axiom (1) of a stack over $(\mathrm{Sch}/S)_{\text{ét}}$.

(b) Show that the $[U/G]$ is isomorphic to the stackification of $[U/G]^{\mathrm{pre}}$ over $(\mathrm{Sch}/S)_{\text{ét}}$, and that $[U/G]^{\mathrm{pre}} \to [U/G]$ is fully faithful.

Exercise 2.4.17. Extending Exercise 2.3.22, show that $U \to [U/G]$ is a categorical quotient among stacks.

Chapter 3

Algebraic spaces and stacks

3.1 Definitions of algebraic spaces and stacks

What are algebraic spaces, Deligne–Mumford stacks and algebraic stacks? After giving their definitions, we will verify the algebraicity of quotient stacks $[U/G]$, the moduli stack of curves $\mathscr{M}_g$ and the moduli stack of vector bundles $\mathcal{B}\mathrm{un}_{r,d}(C)$.

3.1.1 Algebraic spaces

Definition 3.1.1 (Morphisms representable by schemes)**.** A morphism $\mathcal{X} \to \mathcal{Y}$ of prestacks (or presheaves) over Sch is *representable by schemes* if for every morphism $T \to \mathcal{Y}$ from a scheme, the fiber product $\mathcal{X} \times_{\mathcal{Y}} T$ is a scheme.

If $\mathcal{P}$ is a property of morphisms of schemes (e.g. surjective or étale), a morphism $\mathcal{X} \to \mathcal{Y}$ of prestacks representable by schemes *has property* $\mathcal{P}$ if for every morphism $T \to \mathcal{Y}$ from a scheme, the morphism $\mathcal{X} \times_{\mathcal{Y}} T \to T$ of schemes has property $\mathcal{P}$.

Definition 3.1.2. An *algebraic space* is a sheaf X on $\mathrm{Sch}_{\text{ét}}$ such that there exist a scheme U and a surjective étale morphism $U \to X$ representable by schemes.

The map $U \to X$ is called an *étale presentation.* Morphisms of algebraic spaces are by definition morphisms of sheaves. Every scheme is an algebraic space.

3.1.2 Deligne–Mumford stacks

Definition 3.1.3 (Representable morphisms)**.** A morphism $\mathcal{X} \to \mathcal{Y}$ of prestacks (or presheaves) over Sch is *representable* if for every morphism $T \to \mathcal{Y}$ from a scheme T, the fiber product $\mathcal{X} \times_{\mathcal{Y}} T$ is an algebraic space.

If $\mathcal{P}$ is a property of morphisms of schemes which is étale-local on the source (e.g., surjective, étale, or smooth), we say that a representable morphism $\mathcal{X} \to \mathcal{Y}$ of prestacks *has property* $\mathcal{P}$ if for every morphism $T \to \mathcal{Y}$ from a scheme and étale presentation $U \to \mathcal{X} \times_{\mathcal{Y}} T$ by a scheme, the composition $U \to \mathcal{X} \times_{\mathcal{Y}} T \to T$ has property $\mathcal{P}$.

Definition 3.1.4. A *Deligne–Mumford stack* is a stack $\mathcal{X}$ over $\mathrm{Sch}_{\text{ét}}$ such that there exist a scheme U and a surjective, étale, and representable morphism $U \to \mathcal{X}$.

The morphism $U \to \mathcal{X}$ is called an *étale presentation.* Morphisms of Deligne–Mumford stacks are by definition morphisms of stacks. Every algebraic space is a Deligne–Mumford stack via Example 2.3.7.

Remark 3.1.5. If the diagonal of $\mathcal{X}$ is separated and quasi-compact, then it is in fact representable by schemes and every presentation $U \to \mathcal{X}$ is representable by schemes; see Corollary 4.4.8.

3.1.3 Algebraic stacks

Definition 3.1.6. An *algebraic stack* is a stack $\mathcal{X}$ over $\mathrm{Sch}_{\text{ét}}$ such that there exist a scheme U and a surjective, smooth, and representable morphism $U \to \mathcal{X}$.

The morphism $U \to \mathcal{X}$ is called a *smooth presentation.* Morphisms of algebraic stacks are by definition morphisms of prestacks. Every scheme, algebraic space, or Deligne–Mumford stack is also an algebraic stack.

Caution 3.1.7. The definitions above are not standard as most authors also add a representability condition on the diagonal. They are nevertheless equivalent to the standard definitions: we show in Theorem 3.2.1 that the diagonal of an algebraic space is representable by schemes and that the diagonal of an algebraic stack is representable.

Exercise 3.1.8 (Fiber products)**.** Show that fiber products exist for algebraic spaces, Deligne–Mumford stacks and algebraic stacks

3.1.4 Algebraicity of quotient stacks

If $G \to S$ is a smooth affine group scheme acting on an algebraic space U over a base scheme S, the quotient stack $[U/G]$ is algebraic and $U \to [U/G]$ is a principal G-bundle (Theorem 3.1.9).

Since we want to allow for the case that U is not a scheme, we need to generalize a few definitions. An *action* of a smooth affine group scheme $G \to S$ on an algebraic space $U \to S$ is a morphism $\sigma\colon G \times_S U \to U$ satisfying the same axioms as in Definition C.1.7, and we define as in Definition 2.3.14 the quotient stack $[U/G]$ as the stackification of the prestack $[U/G]^{\mathrm{pre}}$, whose fiber category over an S-scheme T is the quotient groupoid $[U(T)/G(T)]$. Objects of $[U/G]$ over an S-scheme T are principal G-bundles $P \to T$ and G-equivariant morphisms $T \to U$. Since morphisms to algebraic spaces glue uniquely in the étale topology (by definition), the argument of Example 2.4.9 extends to show that $[U/G]$ is a stack. Using Definition 3.1.3, the morphism $U \to [U/G]$ is a *principal G-bundle* if for every morphism $T \to \mathcal{X}$ from a scheme T, the algebraic space $U \times_{\mathcal{X}} T$ with the induced G-action is a principal G-bundle over T.

Theorem 3.1.9 (Algebraicity of Quotient Stacks)**.** *If $G \to S$ is a smooth affine group scheme acting on an algebraic space $U \to S$, the quotient stack $[U/G]$ is an algebraic stack over S such that $U \to [U/G]$ is a principal G-bundle and in particular surjective, smooth and affine.*

Proof. If $T \to [U/G]$ is a morphism from an S-scheme corresponding to a principal

G-bundle $P \to T$ and a G-equivariant map $P \to U$, there is a cartesian diagram

$$\begin{tikzcd} P \arrow[r] \arrow[d] & U \arrow[d] \\ T \arrow[r] & {[U/G]} \end{tikzcd}$$

(see Exercise 2.3.28). This shows that $U \to [U/G]$ is a principal G-bundle. If $U' \to U$ is an étale presentation by a scheme, then $U' \to U \to [U/G]$ provides a smooth presentation. $\square$

Corollary 3.1.10. *If $G \to S$ is a smooth affine group scheme, then the classifying stack $\mathbf{B}G = [S/G]$ is algebraic.* $\square$

Example 3.1.11. In this example, we use the alternative geometric descriptions of principal G-bundles from §C.2.2 to give alternative descriptions of classifying stacks. The classifying stack $\mathbf{B}\mathbb{G}_m$ (resp. $\mathbf{B}\,\mathrm{GL}_n$) is the stacks over Sch whose objects are pairs (S, V) consisting of a scheme S and a line bundle V (resp. vector bundle of rank n) V on S. Objects of the classifying stack $\mathbf{B}\,\mathrm{PGL}_n$ over a scheme S can be described equivalently as either principal PGL_n-bundles, Brauer–Severi schemes, or Azumaya algebras over S.

Over a field $\Bbbk$ of $\mathrm{char}(\Bbbk) \neq 2$, recall that for a non-degenerate quadratic form q on an n-dimensional vector space V, the orthogonal group $O(q)$ is the subgroup of $\mathrm{GL}(V)$ containing matrices preserving q. For two different non-degenerate forms q and q', then $\mathbf{B}O(q) \cong \mathbf{B}O(q')$ as both classifying rank n vector bundles equipped with non-degenerate quadratic forms even though $O(q)$ and $O(q')$ may be non-isomorphic.

Corollary 3.1.12. *If G is a finite group acting freely on an algebraic space U, then the quotient sheaf U/G is an algebraic space.*

Proof. Since the action is free, the quotient stack $[U/G]$ is equivalent to a sheaf, which we denote by U/G (see Exercise 2.3.19). Theorem 3.1.9 implies that U/G is an algebraic stack and that $U \to U/G$ is a principal G-bundle so in particular finite, étale, surjective and representable by schemes. Taking $U' \to U$ to be an étale presentation by a scheme, the composition $U' \to U \to U/G$ yields an étale presentation of U/G. $\square$

Remark 3.1.13. This resolves the troubling issue from Example 0.5.5 where we saw that the quotient of a finite group acting freely on a scheme need not exist as a scheme. In addition, it shows that the category of algebraic spaces itself is closed under taking quotients by free actions of finite groups so that we don't need to enlarge our category even more.

Exercise 3.1.14. Let $G \to S$ be a smooth affine group scheme acting on S-schemes X and Y.

(a) Show that a G-equivariant morphism $X \to Y$ induces a morphism $[X/G] \to [Y/G]$ of algebraic stacks.

(b) Show that $[X/G] \to [Y/G]$ is induced by a G-equivariant morphism if and only if $[X/G] \to [Y/G]$ is a morphism over $\mathbf{B}G$.

3.1.5 Algebraicity of $\mathscr{M}_g$

The main reason that $\mathscr{M}_g$ is an algebraic stack is quite simple: every smooth connected projective curve C is tri-canonically embedded $C \hookrightarrow \mathbb{P}^{5g-6}$ by the very ample line bundle $\omega_C^{\otimes 3}$ and the locally closed subscheme $H' \subset \mathrm{Hilb}_P(\mathbb{P}^{5g-6})$ parameterizing smooth families of tri-canonically embedded curves provides a smooth presentation $H' \to \mathscr{M}_g$.

Theorem 3.1.15 (Algebraicity of the stack of smooth curves)**.** *If $g \geq 2$, then $\mathscr{M}_g$ is an algebraic stack over* $\operatorname{Spec} \mathbb{Z}$*.*

Proof. As in the proof that $\mathscr{M}_g$ is a stack (Proposition 2.4.11), we will use Properties of Families of Smooth Curves (5.1.9): for a family of smooth curves $p\colon \mathcal{D} \to S$, $\omega_{\mathcal{D}/S}^{\otimes 3}$ is relatively very ample on S and $p_*(\omega_{\mathcal{D}/S}^{\otimes 3})$ is a vector bundle of rank $5(g-1)$. It follows that $\omega_{\mathcal{D}/S}^{\otimes 3}$ defines a closed immersion $\mathcal{D} \hookrightarrow \mathbb{P}(p_*(\omega_{\mathcal{D}/S}^{\otimes 3}))$ over S. By Riemann–Roch, the Hilbert polynomial of a fiber $\mathcal{D}_s \hookrightarrow \mathbb{P}^{5g-6}_{\kappa(s)}$ is given by

$$P(n) := \chi(\mathcal{O}_{\mathcal{D}_s}(n)) = \deg(\omega_{\mathcal{D}_s}^{\otimes 3n}) + 1 - g = (6n-1)(g-1).$$

Let

$$H := \mathrm{Hilb}^P(\mathbb{P}^{5g-6}_{\mathbb{Z}}/\mathbb{Z})$$

be the (projective) Hilbert scheme parameterizing closed subschemes of $\mathbb{P}^{5g-6}$ with Hilbert polynomial P (Theorem 1.1.2). Let $\mathcal{C} \hookrightarrow \mathbb{P}^{5g-6} \times H$ be the universal closed subscheme and let $\pi\colon \mathcal{C} \to H$ be the projection. We claim that there is a unique locally closed subscheme $H' \subset H$ consisting of points $h \in H$ satisfying

(a) $\mathcal{C}_h \to \operatorname{Spec} \kappa(h)$ is smooth and geometrically connected; and

(b) $\mathcal{C}_h \hookrightarrow \mathbb{P}^{5g-6}_{\kappa(h)}$ is embedded by the complete linear series $\omega_{\mathcal{C}_h/\kappa(h)}^{\otimes 3}$.

(c) denote $\mathcal{C}' = \mathcal{C} \times_H H'$, the line bundles $\omega_{\mathcal{C}'/H'}^{\otimes 3}$ and $\mathcal{O}_{\mathcal{C}_{H'}}(1)$ differ by a pullback of a line bundle from H'.

Moreover, if $T \to H$ is a morphism schemes such that (a)–(c) hold for the family $\mathcal{C}_T \to T$, then $T \to H$ factors through H'.

Since the condition that a fiber of a proper morphism (of finite presentation) is smooth is an open condition on the target (Corollary A.3.10), the condition on H that $\mathcal{C}_h$ is smooth is open. Consider the Stein factorization [Har77, Cor. 11.5] $\mathcal{C} \to \widetilde{H} = \operatorname{Spec}_H \pi_* \mathcal{O}_{\mathcal{C}} \to H$ where $\mathcal{C} \to \widetilde{H}$ has geometrically connected fibers and $\widetilde{H} \to H$ is finite. Since the kernel and cokernel of $\mathcal{O}_H \to \pi_*\mathcal{O}_{\mathcal{C}}$ have closed support (as they are coherent), $\widetilde{H} \to H$ is an isomorphism over an open subscheme of H, which is precisely where the fibers of $\mathcal{C} \to H$ are geometrically connected. In summary, the set of $h \in H$ satisfying (a) is an open subscheme of H, which we will denote by H_1.

The relative canonical sheaf $\omega_{\mathcal{C}_1/H_1}$ of the family $\mathcal{C}_1 := \mathcal{C}_{H_1} \to H_1$ is a line bundle. By Proposition A.7.16, there exists a locally closed subscheme $H_2 \hookrightarrow H_1$ such that a morphism $T \to H_1$ factor through H_2 if and only if $\omega_{\mathcal{C}_1/H_1}|_{\mathcal{C}_T}$ and $\mathcal{O}_{\mathcal{C}}(1)|_{\mathcal{C}_T}$ differ by the pullback of a line bundle on T. In particular, (c) holds and for every $h \in H_2$, there is an isomorphism $\omega_{\mathcal{C}_h/\kappa(h)}^{\otimes 3} \cong \mathcal{O}_{\mathcal{C}_h}(1)$. To arrange (b), consider the restriction of the universal curve $\pi_2\colon \mathcal{C}_2 := \mathcal{C}_{H_2} \to H_2$. There is a canonical map $\alpha\colon \mathrm{H}^0(\mathbb{P}^{5g-6}_{\mathbb{Z}}, \mathcal{O}(1)) \otimes \mathcal{O}_{H_2} \to \pi_{2,*}\mathcal{O}_{\mathcal{C}_2}(1)$ of vector bundles of rank $5g-5$ on H_2 whose fiber over a point $h \in H_2$ is the map $\alpha_h\colon \mathrm{H}^0(\mathbb{P}^{5g-6}_{\kappa(h)}, \mathcal{O}(1)) \to \mathrm{H}^0(\mathcal{C}_h, \mathcal{O}_{\mathcal{C}_h}(1)) \cong \mathrm{H}^0(\mathcal{C}_h, \omega_{\mathcal{C}_h/\kappa(h)}^{\otimes 3})$. The closed locus defined by the support of coker(α) is precisely the locus where α_h is not an isomorphism (as the vector bundles have the same rank). The subscheme $H' = H_2 \setminus \operatorname{Supp}(\operatorname{coker}(\alpha))$ satisfies (a)–(c) along with the universal property.

The group scheme $\mathrm{PGL}_{5g-5} = \underline{\mathrm{Aut}}(\mathbb{P}^{5g-6}_{\mathbb{Z}})$ over $\mathbb{Z}$ acts naturally on H: if $g \in \mathrm{Aut}(\mathbb{P}^{5g-6}_S)$ and $[\mathcal{D} \subset \mathbb{P}^{5g-6}_S] \in H(S)$, then $g \cdot [\mathcal{D} \subset \mathbb{P}^{5g-6}_S] = [g(\mathcal{D}) \subset \mathbb{P}^{5g-6}_S]$. The closed subscheme $H' \subset H$ is PGL_{5g-5}-invariant and we claim that $\mathscr{M}_g \cong [H'/\mathrm{PGL}_{5g-5}]$. This establishes the theorem since $[H'/\mathrm{PGL}_{5g-5}]$ is algebraic (Theorem 3.1.9).

Consider the morphism $H' \to \mathscr{M}_g$ defined by the restriction $\mathcal{C}' \to H'$ of the universal family of the Hilbert scheme. This is the morphism which forgets the embedding, i.e. assigns a closed subscheme $\mathcal{D} \subset \mathbb{P}^{5g-6}_S$ to the family $\mathcal{D} \to S$. This morphism is PGL_{5g-5}-invariant and descends to a morphism $[H'/\mathrm{PGL}_{5g-5}]^{\mathrm{pre}} \to \mathscr{M}_g$ of prestacks. We claim that this map is fully faithful. To see this, observe that for a family $p\colon \mathcal{D} \to S$ in H' defined by a closed subscheme $\mathcal{D} \subset \mathbb{P}^{5g-6}_S$, by (c) there is an isomorphism $\mathcal{O}_{\mathcal{D}}(1) \cong \omega_{\mathcal{D}/S}^{\otimes 3} \otimes p^* M$ for a some line bundle M on S, and by (b), the canonical map

$$\mathrm{H}^0(\mathbb{P}^{5g-6}_{\mathbb{Z}}, \mathcal{O}(1)) \otimes \mathcal{O}_S \to p_*\mathcal{O}_{\mathcal{D}}(1) \cong p_*(\omega_{\mathcal{D}/S}^{\otimes 3} \otimes p^* M) \cong p_*\omega_{\mathcal{D}/S}^{\otimes 3} \otimes M$$

is an isomorphism. Every automorphism of $\mathcal{D} \to S$ induces an automorphism of $\omega_{\mathcal{D}/S}^{\otimes 3}$ and thus an automorphism of $p_*\omega_{\mathcal{D}/S}^{\otimes 3} \otimes M$, which in turn induces an automorphism of $\mathbb{P}^{5g-6}_S$ preserving $\mathcal{D}$. Since $\mathscr{M}_g$ is a stack (Theorem 3.1.9), the universal property of stackification yields a morphism $[H'/\mathrm{PGL}_{5g-5}] \to \mathscr{M}_g$; this map is fully faithful since $[H'/\mathrm{PGL}_{5g-5}]^{\mathrm{pre}} \to [H'/\mathrm{PGL}_{5g-5}]$ is fully faithful (Exercise 2.4.16). It remains to check that $[H'/\mathrm{PGL}_{5g-5}] \to \mathscr{M}_g$ is essentially surjective. For this, it suffices to check that if $p\colon \mathcal{D} \to S$ is a family of smooth curves, then there exists an étale cover $\{S_i \to S\}$ such that each $\mathcal{D}_{S_i}$ is in the image of $H'(S_i) \to \mathscr{M}_g(S_i)$. Since $\omega_{\mathcal{D}/S}^{\otimes 3}$ defines a closed immersion $\mathcal{D} \hookrightarrow \mathbb{P}(p_*\omega_{\mathcal{D}/S}^{\otimes 3})$ over S and $p_*\omega_{\mathcal{D}/S}^{\otimes 3}$ is locally free of rank $5g-5$, we may simply take $\{S_i\}$ to be a Zariski open cover (and thus étale cover) where the restriction of $p_*\omega_{\mathcal{D}/S}^{\otimes 3}$ is free. □

Remark 3.1.16. The entire stack $\mathscr{M}$ of smooth curves (as defined in Example 2.3.9) is also algebraic since $\mathscr{M} = \coprod_g \mathscr{M}_g$.

Exercise 3.1.17. Let $\mathcal{M}_{1,1}$ be the stack over Sch where an object over a scheme S is a *family of elliptic curves over* S, i.e. a pair $(\mathcal{E} \to S, \sigma)$ where $\mathcal{E} \to S$ is smooth proper morphism with a section $\sigma\colon S \to \mathcal{E}$ such that for every $s \in S$, the fiber $(\mathcal{E}_s, \sigma(s))$ is an elliptic curve over the residue field $\kappa(s)$.

(a) Show that $\mathcal{M}_{1,1}$ is an algebraic stack over $\mathbb{Z}$.

(b) Use the Weierstrass form $y^2 = x^3 + ax + b$ (see [Sil09, §3.1]) to show that if we invert the primes 2 and 3, there is an isomorphism

$$\mathcal{M}_{1,1} \times_{\mathbb{Z}} \mathbb{Z}[1/6] \cong [(\mathbb{A}^2 \smallsetminus V(\Delta))/\mathbb{G}_m],$$

where the action is given by $t \cdot (a, b) = (t^4 a, t^6 b)$ and Δ is the discriminant $4a^3 + 27b^2$.

(c) Define a *stable elliptic curve* over a field $\Bbbk$ as a pair (E, p) consisting of a projective curve E of arithmetic genus 1 with at worst nodal singularities and a rational point $p \in E(\Bbbk)$. Over a scheme S, a *family of stable elliptic curves over* S is a proper flat $\mathcal{E} \to S$ and a section $\sigma\colon S \to \mathcal{E}$ such that every fiber is a stable elliptic curves. Denoting $\overline{\mathcal{M}}_{1,1}$ as the stack over Sch classifying stable elliptic curves, show that

$$\mathcal{M}_{1,1} \times_{\mathbb{Z}} \mathbb{Z}[1/6] \cong [(\mathbb{A}^2 \smallsetminus 0)/\mathbb{G}_m]$$

with the same action as above.

Exercise 3.1.18. An *n-pointed family of genus* 0 *curves* is proper, flat, and finitely presented morphism $X \to S$ of schemes with n sections $\sigma_1, \dots, \sigma_n\colon S \to X$ such that for every geometric point $s\colon \operatorname{Spec} \Bbbk \to S$, the geometric fiber $X \times_S \Bbbk$ is a smooth genus 0 curve and $\sigma_1(s), \dots, \sigma_n(s) \in X(\Bbbk)$ are distinct.

(a) Show that the prestack $\mathcal{M}_{0,n}$ parameterizing n-pointed families of genus 0 curves is a stack over $\mathrm{Sch}_{\text{ét}}$.

(b) Show that $\mathcal{M}_{0,0} \cong \mathbf{B}\,\mathrm{PGL}_2$.

Hint: Use Exercise C.2.14 to show that $X \to S$ is a Brauer–Severi scheme (i.e. there exists an étale cover $S' \to S$ such that $X \times_S S' \cong \mathbb{P}^1_{S'}$), and use the correspondence between Brauer–Severi schemes and principal PGL_2*-torsors (Exercise C.2.13).*

(c) Show that $\mathcal{M}_{0,1} \cong \mathbf{B}U_2$ where $U_2 \subset \mathrm{PGL}_2$ is the two-dimensional subgroup of upper triangular matrices.

(d) Show that $\mathcal{M}_{0,2} \cong \mathbf{B}\mathbb{G}_m$.

(e) Show that $\mathcal{M}_{0,3} \cong \operatorname{Spec} \mathbb{Z}$.

(f) Show that for $n > 3$, $\mathcal{M}_{0,n} \cong (\mathbb{P}^1 \setminus \{0, 1, \infty\})^{n-3} \setminus \Delta$ where Δ is the closed subscheme where at least two of the $n-3$ points are equal.

3.1.6 Algebraicity of $\mathcal{B}\mathrm{un}_{r,d}(C)$

Theorem 3.1.19 (Algebraicity of the stack of vector bundles)**.** *Let C be a smooth, connected, and projective curve over a field $\Bbbk$, and let r and d be integers with $r \geq 0$. The stacks $\underline{\mathrm{Coh}}_{r,d}(C)$ and $\mathcal{B}\mathrm{un}_{r,d}(C)$ are algebraic stack over* $\operatorname{Spec} \Bbbk$.

Proof. For every vector bundle E on C of rank r and degree d, by Serre vanishing $E(m)$ is globally generated and $\mathrm{H}^1(C, E(m)) = 0$ for $m \gg 0$. In particular,

$$\Gamma(C, E(m)) \otimes \mathcal{O}_C \twoheadrightarrow E(m) \tag{3.1.1}$$

is surjective which by construction induces an isomorphism on global sections. By Riemann–Roch, the Hilbert polynomial of E is

$$P(n) = \chi(E(n)) = \deg(E(n)) + \mathrm{rk}(E(n))(1-g) = d + rn + r(1-g).$$

For each integer m, consider the substack $\mathcal{C}_m \subset \underline{\mathrm{Coh}}_{r,d}(C)$ parameterizing coherent sheaves E on C such that (3.1.1) is surjective and induces an isomorphism of global sections, and $\mathrm{h}^0(C, E(m)) = P(m)$ (or equivalently $\mathrm{H}^1(C, E(m)) = 0$). This is an open substack. Indeed, given scheme S and a coherent sheaf E on C_S flat over S, consider the diagram

$$\begin{tikzcd} & C \times S \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ C & & S. \end{tikzcd}$$

A simple application of Cohomology and Base Change (see Proposition A.7.9) implies that the locus of points $s \in S$ such that $\mathrm{H}^1(C, E_s(m)) = 0$ is open and that over this locus we have that $\mathrm{R}^1 p_{2,*}E(m) = 0$ and that $p_{2,*}E(m)$ is a vector bundle whose construction commutes with base change. The morphism $p_2^* p_{2,*}E(m) \to E(m)$ thus also commutes with base change and fails to be surjective on the closed subset given by the support of its cokernel.

For each m, consider the Quot scheme

$$Q_m := \mathrm{Quot}^P(\mathcal{O}_C(-m)^{P(m)}/C/\Bbbk)$$

parameterizing quotients $\mathcal{O}_C(-m)^{P(m)} \twoheadrightarrow F$ with Hilbert polynomial P (Theorem 1.1.3). Consider the subset Q'_m consisting of quotients $q\colon \mathcal{O}_C(-m)^{P(m)} \to F$ such that $\mathrm{H}^0(q(m))\colon \mathrm{H}^0(C, \mathcal{O}_C)^{P(m)} \to \mathrm{H}^0(C, F(m))$ is an isomorphism. Note that since we have specified the Hilbert polynomial, this implies that $\mathrm{H}^1(C, F(m)) = 0$. The subset Q'_m is an open subscheme; its complement is given by the support of the cokernel of $p_{2,*}\mathcal{O}_{C\times Q_m}^{P(m)} \to p_{2,*}(\mathcal{E}_m(m))$ where $\mathcal{O}_{C\times Q_m}(-m)^{P(m)} \to \mathcal{E}_m$ is the universal quotient on $C \times Q_m$.

The Quot scheme Q_m inherits a natural action from $\mathrm{GL}_{P(m)}$ such that Q'_m is invariant. The morphism $Q'_m \to \mathcal{C}_m$, defined by $[\mathcal{O}_C(-m)^{P(m)} \twoheadrightarrow F] \mapsto F$, factors to a yield a morphism $\Psi^{\mathrm{pre}}\colon [Q'_m/\,\mathrm{GL}_{P(m)}]^{\mathrm{pre}} \to \mathcal{C}_m$ of prestacks. The map Ψ^{pre} is fully faithful since every automorphism of a vector bundle F on $C \times S$ induces an automorphism of $p_{2,*}F(m) = \mathcal{O}_S^{P(m)}$, i.e. an element of $\mathrm{GL}_{P(m)}(S)$, and this element acts on $\mathcal{O}_C(-m)^{P(m)}$ preserving the quotient F.

Since $\underline{\mathrm{Coh}}_{r,d}(C)$ is a stack (Proposition 2.4.13), there is an induced morphism $\Psi\colon [Q'_m/\,\mathrm{GL}_{P(m)}] \to \mathcal{C}_m$ of stacks which is also fully faithful (Exercise 2.4.16) and by construction essentially surjective. We conclude that $\mathcal{C}_m = [Q'_m/\,\mathrm{GL}_{P(m}]$ and that

$$\underline{\mathrm{Coh}}_{r,d}(C) = \bigcup_m \,[Q'_m/\,\mathrm{GL}_{P(m)}\,].$$

The algebraicity of $\underline{\mathrm{Coh}}_{r,d}(C)$ follows from the algebraicity of quotient stacks (Theorem 3.1.9) and the algebraicity of $\mathcal{B}\mathrm{un}_{r,d}(C)$ follows as the property of being a vector bundle is an open condition. □

Remark 3.1.20. Note that the entire stack of coherent sheaves and vector bundles are also algebraic since

$$\underline{\mathrm{Coh}}(C) = \coprod_{r,d} \underline{\mathrm{Coh}}_{r,d}(C) \quad \text{and} \quad \mathcal{B}\mathrm{un}(C) = \coprod r, d\, \mathcal{B}\mathrm{un}_{r,d}(C)$$

Note also that while $\mathcal{B}\mathrm{un}_{r,d}(C)$ itself is not quasi-compact (Definition 3.3.20), the proof establishes that every quasi-compact open substack of $\mathcal{B}\mathrm{un}_{r,d}(C)$ is a quotient stack.

Exercise 3.1.21. Modify the above argument to show that $\underline{\mathrm{Coh}}(X)$ is an algebraic stack if X is a projective scheme over $\Bbbk$.

3.1.7 Desideratum

We will develop the foundations of algebraic spaces and stacks in the forthcoming chapters but it is worth first highlighting some of the most important results.

3.1.7.1 The importance of the diagonal

When overhearing others discussing algebraic stacks, you may have wondered what's all the fuss about the diagonal? Well, I'll tell you—the diagonal encodes the stackiness!

First and foremost, the diagonal $\mathcal{X} \to \mathcal{X} \times \mathcal{X}$ of an algebraic stack is representable and the diagonal $X \to X \times X$ of an algebraic space is representable by schemes (Theorem 3.2.1).

The stabilizer G_x of a field-valued point $x\colon \operatorname{Spec} K \to \mathcal{X}$ is defined as the sheaf $\underline{\mathrm{Aut}}_{\mathcal{X}(K)}(x) = \underline{\mathrm{Aut}}_{\mathcal{X}(K)}(x,x)$ and is identified with the fiber product $\mathcal{X} \times_{\mathcal{X}\times\mathcal{X}} \operatorname{Spec} K$ by Exercise 2.3.30. By Representability of the Diagonal, the stabilizer G_x is representable by a group algebraic space over K. We later show that G_x is in fact a group scheme of finite type over K (Corollary 4.4.28). See §3.2.2 for a further discussion of stabilizers.

For schemes (resp. separated schemes), the diagonal is an immersion (resp. closed immersion). For algebraic stacks, the diagonal is not necessarily even a monomorphism as the fiber over $(x,x)\colon \operatorname{Spec} K \to \mathcal{X} \times \mathcal{X}$, or in other words the stabilizer G_x, may be non-trivial. Properties of the diagonal in fact characterize algebraic spaces and Deligne–Mumford stacks: an algebraic stack is an algebraic space (resp. Deligne–Mumford stack) if and only if $\mathcal{X} \to \mathcal{X} \to \mathcal{X}$ is a monomorphism (resp. unramified)—see Theorems 3.6.4 and 3.6.5. An equivalent characterization is given by properties of the stabilizer groups as in the table below:

Table 3.1: Characterization of algebraic spaces and Deligne–Mumford stacks

Type of space	Property of the diagonal	Property of stabilizers
algebraic space	monomorphism	trivial
Deligne–Mumford stack	unramified	reduced finite groups[1]
algebraic stack	arbitrary	arbitrary

As a consequence of these characterizations, we will generalize Corollary 3.1.12: the quotient of a free action of a smooth algebraic group on an algebraic space exists as an algebraic space (Corollary 3.6.7). We will also be able to establish that $\mathscr{M}_g$ is Deligne–Mumford (Corollary 3.6.8) rather than just algebraic (Theorem 3.1.15).

We now summarize additional important properties of algebraic spaces, Deligne–Mumford stacks and algebraic stacks. The reader may also wish to consult Table 3 for a brief recap of the trichotomy of moduli spaces.

3.1.7.2 Properties of algebraic spaces

- If $R \rightrightarrows U$ is an étale equivalence relation of schemes, the quotient sheaf U/R is an algebraic space (Theorem 3.4.11).
- If X is a quasi-separated algebraic space, there exists a dense open subspace $U \subset X$ which is a scheme (Theorem 4.4.1).
- If $X \to Y$ is a separated and quasi-finite morphism of noetherian algebraic spaces, then there exists a factorization $X \hookrightarrow \widetilde{X} \to Y$ where $X \hookrightarrow \widetilde{X}$ is

[1] If the diagonal is not quasi-compact, the stabilizers will only be discrete and reduced.

an open immersion and $\widetilde{X} \to Y$ is finite (Zariski's Main Theorem). In particular, $X \to Y$ is quasi-affine.

3.1.7.3 Properties of Deligne–Mumford stacks

- If $R \rightrightarrows U$ is an étale groupoid of schemes, the quotient stack $[U/R]$ is a Deligne–Mumford stack (Theorem 3.4.11).
- If $\mathcal{X}$ is a Deligne–Mumford stack and $x \in \mathcal{X}(k)$ is a field-valued point, there exists an étale neighborhood $[\operatorname{Spec}(A)/G] \to \mathcal{X}$ of x where G is a finite group, which can be arranged to be the stabilizer of x (Local Structure of Deligne–Mumford Stacks, Theorem 4.2.11).
- If $\mathcal{X}$ is a separated Deligne–Mumford stack, there exists a coarse moduli space $\mathcal{X} \to X$ where X is a separated algebraic space (Keel-Mori Theorem, Theorem 4.3.11).
- If $\mathcal{X}$ is a Deligne–Mumford stack (e.g. algebraic space), there exists a scheme U and a finite morphism $U \to \mathcal{X}$ (Le Lemme de Gabber, Theorem 4.5.1).

3.1.7.4 Properties of algebraic stacks

- If $R \rightrightarrows U$ is a smooth groupoid of schemes, the quotient stack $[U/R]$ is an algebraic stack (Theorem 3.4.11).
- If $\mathcal{X}$ is an algebraic stack of finite type over an algebraically closed field $\Bbbk$ with affine diagonal, every point $x \in \mathcal{X}(k)$ with linearly reductive stabilizer has an affine étale neighborhood $[\operatorname{Spec}(A)/G_x] \to \mathcal{X}$ of x where G is a finite group (Local Structure of Algebraic Stacks).
- Let $\mathcal{X}$ be an algebraic stack of finite type over an algebraically closed field $\Bbbk$ of characteristic 0 with affine diagonal. If $\mathcal{X}$ is S-complete and Θ-complete, there exists a good moduli space $\mathcal{X} \to X$ where X is a separated algebraic space of finite type over $\Bbbk$.

Notes

Deligne–Mumford and algebraic stacks were first introduced in [DM69] and [Art74]—and in both cases referred to as *algebraic stacks*—with conventions slightly different than ours. Namely, [DM69, Def. 4.6] assumed in addition to the existence of an étale presentation that the diagonal is representable by schemes (which is automatic if the diagonal is separated and quasi-compact). On the other hand, [Art74, Def. 5.1] assumed in addition to the existence of a smooth presentation that the stack is locally of finite type over an excellent Dedekind domain. The term *Artin stack*—which we refrain from using—is sometimes reserved for stacks that satisfy Artin's axioms (e.g. algebraic stacks locally of finite type over an excellent scheme with quasi-compact and separated diagonal).

We follow the conventions of [Ols16] and [SP] with the exception that we work over the site $\mathrm{Sch}_{\text{ét}}$ while [SP] works over $\mathrm{Sch}_{\mathrm{fppf}}$. These two sites give equivalent notions of algebraic stacks [SP, Tag 076U].

3.2 Representability of the diagonal

3.2.1 Representability

Theorem 3.2.1 (Representability of the Diagonal)**.**

(1) The diagonal of an algebraic space is representable by schemes.

(2) The diagonal of an algebraic stack is representable.

Proof. Let X be an algebraic space and $U \to X$ be an étale presentation. Define the scheme $R := U \times_X U$. If $T \to X \times X$ is a morphism from a scheme, we need to show that the sheaf $Q_T = X \times_{X \times X} T$ is in fact a scheme. Since $U \to X$ is étale, surjective and representable by schemes, so is $U \times U \to X \times X$. The base change of $T \to X \times X$ by $U \times U \to X \times X$ is a scheme T' which is surjective étale over T. In the cartesian cube

$$\begin{tikzcd}
& Q_{T'} \arrow[rr] \arrow[dl] \arrow[dd] & & T' \arrow[dl] \arrow[dd] \\
R \arrow[rr, crossing over] \arrow[dd] & & U \times U & \\
& Q_T \arrow[rr] \arrow[dl] & & T \arrow[dl] \\
X \arrow[rr] & & X \times X, \arrow[from=uu, crossing over] &
\end{tikzcd} \tag{3.2.1}$$

Q_T is a sheaf on $\mathrm{Sch}_{\text{ét}}$ while $Q_{T'}$ is a scheme. Since $R \to U \times U$ is a separated and locally quasi-finite morphism of schemes, so is $Q_{T'} \to T'$. (If X had quasi-compact diagonal, then by Zariski's main theorem $R \to U \times U$ is quasi-affine and thus so is $Q_{T'} \to T'$.) Since Q_T is a sheaf in the étale topology that pulls back to a scheme $Q_{T'}$ separated and locally quasi-finite over T', we may apply Effective Descent (Proposition 2.2.11) to conclude that Q_T is a scheme.

If $\mathcal{X}$ is an algebraic stack and $U \to \mathcal{X}$ is a smooth presentation, we may imitate the above argument. The fiber product $R := U \times_{\mathcal{X}} U$ is an algebraic space. If $T \to \mathcal{X} \times \mathcal{X}$ is morphism from a scheme, its base change along $U \times U \to \mathcal{X} \times \mathcal{X}$ yields an algebraic space T_1 which is surjective smooth over T. Choose an étale presentation $T_2 \to T_1$. Then $T_2 \to T$ is a surjective smooth morphism of schemes which has a section after an étale cover $T' \to T$ (Proposition A.3.5). The composition $T' \to T_2 \to T_1 \to U \times U$ provides a lift of $T \to \mathcal{X} \times \mathcal{X}$. We obtain a diagram similar to (3.2.1) but where the left and right squares are not necessarily cartesian. The morphism $Q_{T'} \to Q_T$ is étale, surjective and representable by schemes (as $T' \to T$ is). Choosing an étale presentation $V \to Q_{T'}$ of the algebraic space $Q_{T'}$, the composition $V \to Q_{T'} \to Q_T$ yields an étale presentation showing that Q_T is an algebraic space. □

Corollary 3.2.2.

(1) Every morphism from a scheme to an algebraic space is representable by schemes.

(2) Every morphism from a scheme to an algebraic stack is representable.

Proof. This follows directly from Representability of the Diagonal (Theorem 3.2.1) and the cartesian diagram

$$\begin{tikzcd} T_1 \times_{\mathcal{X}} T_2 \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & T_1 \times T_2 \arrow[d] \\ \mathcal{X} \arrow[r] & \mathcal{X} \times \mathcal{X}. \end{tikzcd}$$

associated to any two maps $T_1 \to \mathcal{X}$ and $T_2 \to \mathcal{X}$ from schemes to an algebraic stack. $\square$

Exercise 3.2.3.

(a) If $\mathcal{X} \to \mathcal{Y}$ is a representable morphism of algebraic stacks (e.g. a morphism of algebraic spaces), then $\mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ is representable by schemes.

(b) If $\mathcal{X} \to \mathcal{Y}$ is a morphism of algebraic stacks, then $\mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ is representable.

Exercise 3.2.4. Show that the diagonal of a morphism $\mathcal{X} \to \mathcal{Y}$ of algebraic stacks is locally of finite type.

3.2.2 Stabilizer groups and the inertia stack

Now that we know that the diagonal is representable, we can discuss its properties. One of the most important features of the diagonal is that it encodes the stabilizer groups.

Definition 3.2.5 (Stabilizers)**.** If $\mathcal{X}$ is an algebraic stack and $x\colon \operatorname{Spec} K \to \mathcal{X}$ is a field-valued point, the *stabilizer of* x is defined as the group algebraic space $G_x := \underline{\operatorname{Aut}}_{\mathcal{X}(K)}(x)$.

By Exercise 2.3.30, we can identify G_x with the fiber product

$$\begin{tikzcd} G_x := \underline{\operatorname{Aut}}_{\mathcal{X}(K)}(x) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \operatorname{Spec} K \arrow[d, "{(x,x)}"] \\ \mathcal{X} \arrow[r, "\Delta"] & \mathcal{X} \times \mathcal{X}. \end{tikzcd}$$

The sheaf G_x is representable by an algebraic space over K by Representability of the Diagonal (Theorem 3.2.1). The stabilizer G_x is a group algebraic space, i.e. an algebraic space G_x with multiplication, inverse and identity morphisms satisfying Definition C.1.1 (or equivalently a group object in the category of algebraic spaces). In fact, G_x is actually a group scheme locally of finite type as long as the diagonal of $\mathcal{X}$ is quasi-separated (Corollary 4.4.28).

Remark 3.2.6. Let G be a group scheme over a field $\Bbbk$ acting on a $\Bbbk$-scheme U via $\sigma\colon G \times U \to U$, and let $u \in U(\Bbbk)$. The stabilizer of the image of u in $[U/G]$ is the usual stabilizer group scheme, i.e. the fiber product of $(\sigma, p_2)\colon G \times U \to U \times U$ along $(u, u)\colon \operatorname{Spec} \Bbbk \to U \times U$.

Exercise 3.2.7.

(a) Show that the stabilizer of a field-valued point of a fiber product of algebraic stacks is the fiber product of stabilizers, i.e. for $x' \in (\mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}')(\Bbbk)$, then $G_{x'} = G_x \times_{G_y} G_{y'}$ where x, y and y' are the images of x'.

(b) Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks and $x \in \mathcal{X}(\Bbbk)$ be a field valued point. Show that the fiber of the diagonal $\mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ over the point $(x, x, \mathrm{id}) \in (\mathcal{X} \times_{\mathcal{Y}} \mathcal{X})(\Bbbk)$ is identified with $\ker(G_x \to G_y)$. What is the fiber of the diagonal over an arbitrary field-valued point of $\mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$?

Exercise 3.2.8. Let $\mathcal{X}$ be a (resp. quasi-separated) Deligne–Mumford stack. An algebraic stack is *quasi-separated* if for every morphism $(a, b)\colon S \to \mathcal{X} \times \mathcal{X}$ from a scheme, the fiber product $\underline{\mathrm{Isom}}_{\mathcal{X}(S)}(a, b) \cong \mathcal{X} \times_{\Delta, \mathcal{X} \times \mathcal{X}, (a,b)} \mathcal{X}$ is quasi-compact over S; see also Definition 3.3.10.

(a) For a field-valued point $x \in \mathcal{X}(\Bbbk)$, show that G_x is a separated (resp. finite) étale group scheme over $\Bbbk$.

Hint: First show that G_x is an etale group algebraic space over $\Bbbk$. If $\Bbbk = \overline{\Bbbk}$, use that a section of the structure morphism $G_x \to \operatorname{Spec} \Bbbk$ is an open immersion to give an open covering of G_x by schemes. Apply Proposition C.1.6 to conclude that G_x is separated. For the general case, apply Effective Descent (Proposition 2.2.11).

(b) If $\Bbbk$ is algebraically closed, show that G_x is the discrete and reduced (resp. finite and reduced) group scheme corresponding to the abstract group $G_x(\Bbbk)$.

(c) Show that the diagonal of $\mathcal{X}$ is unramified.

We will see later that these properties characterize Deligne–Mumford stacks; see Theorem 3.6.4.

Varying the point x of $\mathcal{X}$, the stabilizer group varies and naturally forms a family. In fact, we've already seen this: if $a\colon T \to \mathcal{X}$ is an object, then $\mathrm{Isom}_{\mathcal{X}(T)}(a) \to S$ is a group algebraic space such that the fiber over a point $s \in S$ is the stabilizer of the restriction $a|_{\operatorname{Spec} \kappa(s)}$ of a to $\operatorname{Spec} \kappa(s)$. Applying this to the identity map $\mathrm{id}_{\mathcal{X}}\colon \mathcal{X} \to \mathcal{X}$ yields the construction of the inertia stack.

Definition 3.2.9 (Inertia stack)**.** The *inertia stack* of an algebraic stack $\mathcal{X}$ is the fiber product

$$\begin{tikzcd} I_{\mathcal{X}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{X} \arrow[d] \\ \mathcal{X} \arrow[r] & \mathcal{X} \times \mathcal{X}. \end{tikzcd}$$

In the relative setting of a morphism $\mathcal{X} \to \mathcal{Y}$ of algebraic stacks, the *relative inertia stack* is $I_{\mathcal{X}/\mathcal{Y}} := \mathcal{X} \times_{\mathcal{X} \times_{\mathcal{Y}} \mathcal{X}} \mathcal{X}$.

The fiber of $I_{\mathcal{X}} \to \mathcal{X}$ over a field-valued point $x\colon \operatorname{Spec} K \to \mathcal{X}$ is precisely the stabilizer G_x. We can therefore think of $I_{\mathcal{X}}$ as a group scheme (or really group algebraic space) over $\mathcal{X}$ incorporating all of the stabilizers of $\mathcal{X}$. If we let $(\mathrm{Sch}/\mathcal{X})_{\text{ét}}$ be the big étale site of schemes over $\mathcal{X}$, then $I_{\mathcal{X}}$ can be viewed as a sheaf of groups on $(\mathrm{Sch}/\mathcal{X})_{\text{ét}}$ where $I_{\mathcal{X}}(a) = \mathrm{Aut}_S(a)$ for $a \in \mathcal{X}(S)$. If $a' \to a$ is a morphism over $S' \to S$, there is a natural pullback functor $\alpha^*\colon \mathrm{Aut}_S(a) \to \mathrm{Aut}_{S'}(a')$ defined as follows: for $\beta \in \mathrm{Aut}_S(a)$, the image $\alpha^*(\beta)$ is the unique dotted arrow (provided by Axiom (2) of the definition of a prestack (2.3.1)) making the diagram

$$\begin{tikzcd} a' \arrow[r, dashed, "\alpha^*(\beta)"'] \arrow[rr, bend left, "\beta\circ\alpha"] & a' \arrow[r, "\alpha"'] & a \end{tikzcd} \tag{3.2.2}$$

commute. Note that if $\alpha\colon \alpha \to \alpha$ is an isomorphism over the identity, then $\alpha^*(\beta) = \alpha^{-1} \circ \beta \circ \alpha$ is conjugation by α.

Exercise 3.2.10. Let $G \to S$ be a group scheme acting on a scheme $U \to S$, and let $\mathcal{X} = [U/G]$ be the quotient stack. Show that there is a cartesian diagram

$$\begin{tikzcd} S_U \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & U \arrow[d] \\ I_{\mathcal{X}} \arrow[r] & \mathcal{X} \end{tikzcd}$$

where $S_U \to U$ is the stabilizer group scheme, i.e. the fiber product of the action map $G \times U \to U \times U$ and the diagonal $U \to U \times U$.

Example 3.2.11. The inertia class of the classifying stack $\mathbf{B}\mathbb{G}_m$ is $I_{\mathbf{B}\mathbb{G}_m} \cong \mathbb{G}_m \times \mathbf{B}\mathbb{G}_m$. Similarly, if we let $\mathbb{G}_m$ act on $\mathbb{G}_m \times \mathbb{A}^1$ via the product of the trivial and the scaling action and we let $V(x(t-1)) \subset \mathbb{G}_m \times \mathbb{A}^1$ be the $\mathbb{G}_m$-invariant closed subscheme, then $I_{[\mathbb{A}^1/\mathbb{G}_m]} \cong [V(x(t-1))/\mathbb{G}_m]$.

Exercise 3.2.12. If G is a smooth affine algebraic group, show that the inertia stack of $\mathbf{B}G$ is the quotient $[G/G]$ where G acts on itself via conjugation.

Exercise 3.2.13. Let G be a finite group acting on a scheme U, and let $\mathcal{X} = [U/G]$. Show that the inertia stack $I_{\mathcal{X}}$ is isomorphic to

$$I_{\mathcal{X}} = \coprod_{g \in G} [U^g / C(g)]$$

where $C(g)$ is the centralizer of G and $U^g := \{x \in U \mid gx = x\}$ (or alternatively the fiber product of the diagonal $U \to U \times U$ and the map $U \to U \times U$ defined by $x \mapsto (x, gx)$).

Exercise 3.2.14. Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks.

(1) Show that there are morphisms $I_{\mathcal{X}/\mathcal{Y}} \to I_{\mathcal{X}} \to I_{\mathcal{Y}} \times_{\mathcal{Y}} \mathcal{X}$ of algebraic stacks over $\mathcal{X}$ such that the induced morphisms on the fibers over a field-valued point $x \in \mathcal{X}(\Bbbk)$ correspond to a left exact sequence $1 \to K_x \to G_x \to G_{f(x)}$ of algebraic groups.

(2) Show that there is a cartesian diagram

$$\begin{tikzcd} I_{\mathcal{X}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & I_{\mathcal{Y}} \times_{\mathcal{Y}} \mathcal{X} \arrow[d] \\ \mathcal{X} \arrow[r] & \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}. \end{tikzcd}$$

Hint: An object of $I_{\mathcal{Y}} \times_{\mathcal{Y}} \mathcal{X}$ over a scheme S is a quadruple (y, α, x, β) where $y \in \mathcal{X}(S)$, $\alpha\colon y \xrightarrow{\sim} y$, $x \in \mathcal{X}(S)$, and $\beta\colon y \xrightarrow{\sim} f(x)$. On the other hand, an object of $\mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ over S is a triple (x_1, x_2, γ) where $x_1, x_2 \in \mathcal{X}(S)$ and $\gamma\colon f(x_1) \xrightarrow{\sim} f(x_2)$. Define $I_{\mathcal{Y}} \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ on fiber categories by $(y, \alpha, x, \beta) \mapsto (x, x, \beta \circ \alpha \circ \beta^{-1})$. Construct a map $I_{\mathcal{X}}(S)$ to the fiber product of $\mathcal{X}(S)$ and $(I_{\mathcal{Y}} \times_{\mathcal{Y}} \mathcal{X})(S)$ over $(\mathcal{X} \times_{\mathcal{Y}} \mathcal{X})(S)$, and show that it is an equivalence.

3.3 First properties

3.3.1 Properties of morphisms

Recall that a morphism of prestacks $\mathcal{X} \to \mathcal{Y}$ over $\mathrm{Sch}_{\text{ét}}$ is representable by schemes (resp. representable) if for every morphism $T \to \mathcal{Y}$ from a scheme, the base change $\mathcal{X} \times_{\mathcal{Y}} T$ is a scheme (resp. algebraic space); see Definitions 3.1.1 and 3.1.3. Both notions are clearly stable under base change. Morphisms representable by schemes are also clearly stable under composition and the following lemma shows the same for representable morphisms.

Lemma 3.3.1.

(1) If $X \to Y$ is a representable morphism of sheaves on $\mathrm{Sch}_{\text{ét}}$ and Y is an algebraic space, then X is an algebraic space.

(2) The composition of representable morphisms is representable.

Proof. For the first statement, if $V \to Y$ is an étale presentation, then the base change $X_V \to X$ is a morphism of algebraic spaces which is étale, surjective and representable by schemes. Letting $U \to X_V$ be an étale presentation, then the composition $U \to X_V \to X$ is étale, surjective and representable by schemes and thus X is an algebraic space. The second statement follows from the first. □

Definition 3.3.2. Let $\mathcal{P}$ be a property of morphisms of schemes.

(1) If $\mathcal{P}$ is stable under composition and base change and is étale-local (resp. smooth-local) on the source and target, a morphism $\mathcal{X} \to \mathcal{Y}$ of Deligne–Mumford stacks (resp. algebraic stacks) *has property* $\mathcal{P}$ if for all étale (resp. smooth) presentations (equivalently there exist presentations) $V \to \mathcal{Y}$ and $U \to \mathcal{X} \times_{\mathcal{Y}} V$ yielding a diagram

$$\begin{tikzcd} U \arrow[r] & \mathcal{X} \times_{\mathcal{Y}} V \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & V \arrow[d] \\ & \mathcal{X} \arrow[r] & \mathcal{Y}, \end{tikzcd}$$

the composition $U \to V$ has $\mathcal{P}$.

(2) A morphism $\mathcal{X} \to \mathcal{Y}$ of algebraic stacks representable by schemes *has property* $\mathcal{P}$ if for every morphism $T \to Y$ from a scheme, the base change $\mathcal{X} \times_{\mathcal{Y}} T \to T$ has $\mathcal{P}$.

(3) A morphism $\mathcal{X} \to \mathcal{Y}$ of algebraic stacks is an *isomorphism*, *open immersion*, *closed immersion*, *locally closed immersion*, *affine*, or *quasi-affine* if it is representable by schemes and has the corresponding property in the sense of (2).

The properties of flatness, smoothness (resp. smoothness of relative dimension n), surjectivity, locally of finite presentation, and locally of finite type are smooth-local on the source and target. By (1), these properties extend to morphisms of algebraic stacks. Likewise, étaleness and unramifiedness are étale-local on the source and target, and thus extend to morphisms of Deligne–Mumford stacks. These properties are stable under composition and base change.

Representable morphisms and each class of morphisms in (3) are smooth local on the target. They are even fppf local but we won't be able to show this until §6.2.

Proposition 3.3.3. *Let $\mathcal{P}$ be one of the following properties of morphisms of algebraic stacks: representable, isomorphism, open immersion, closed immersion, locally closed immersion, affine, or quasi-affine. Consider a cartesian diagram*

$$\begin{tikzcd} \mathcal{X}' \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{Y}' \arrow[d] \\ \mathcal{X} \arrow[r] & \mathcal{Y} \end{tikzcd}$$

of algebraic stacks where $\mathcal{Y}' \to \mathcal{Y}$ is smooth and surjective. Then $\mathcal{X} \to \mathcal{Y}$ has $\mathcal{P}$ if and only if $\mathcal{X}' \to \mathcal{Y}'$ has $\mathcal{P}$.

Proof. We will show the ($\Leftarrow$) implications as the other directions are clear. For representability, we may assume that $\mathcal{Y}$, $\mathcal{Y}'$ and $\mathcal{X}'$ are schemes and we need to show that $\mathcal{X}$ is an algebraic space. It suffices to show that the ever automorphism $\alpha\colon a \to a$ of an object a over a scheme T is trivial. The base change T' of $a\colon T \to \mathcal{X}$ by $\mathcal{X}' \to \mathcal{X}$ is a scheme since it's also identified with $T \times_{\mathcal{Y}} \mathcal{Y}'$. Since smooth morphisms étale locally have sections (Corollary A.3.6), there is an étale cover $g\colon \widetilde{T} \to T$ that factors through T'. The automorphism α defines a section of $\underline{\mathrm{Aut}}_T(a)$ over T. Since $\underline{\mathrm{Aut}}_T(a)$ is a sheaf on $(\mathrm{Sch}/T)_{\text{ét}}$ and $g^*\alpha = \mathrm{id}$, we have that $\alpha = \mathrm{id}$.

For the other properties, we already know that $\mathcal{X} \to \mathcal{Y}$ is representable and it thus suffices to assume that $\mathcal{Y}$, $\mathcal{Y}'$ and $\mathcal{X}'$ are schemes and that $\mathcal{X}$ is an algebraic space. Fortunately we can apply Effective Descent (Proposition 2.2.11) to conclude that $\mathcal{X}$ is a scheme and that $\mathcal{X} \to \mathcal{Y}$ has property $\mathcal{P}$. □

Example 3.3.4. If $G \to S$ is a smooth affine group scheme acting on an algebraic space $U \to S$, then $[U/G] \to S$ is flat (resp. smooth, surjective, locally of finite presentation, locally of finite type) if and only if $U \to S$ is. In particular, using the quotient stack presentations in the proofs of Theorems 3.1.15 and 3.1.19, we conclude that $\mathscr{M}_g$ is locally of finite type over $\mathbb{Z}$ and $\mathcal{B}\mathrm{un}_{r,d}(C)$ is locally of finite type over $\Bbbk$.

Exercise 3.3.5. Assume that $\mathcal{X} \to \mathcal{Y}$ is a surjective and smooth morphism of algebraic stacks. If $T \to \mathcal{Y}$ is a morphism from a scheme, show that there exists an étale cover $T' \to T$ such that $\mathcal{X}_{T'} \to T'$ has a section.

3.3.2 Properties of algebraic spaces and stacks

Definition 3.3.6 (Properties of algebraic spaces and stacks)**.** Let $\mathcal{P}$ be a property of schemes which is étale (resp. smooth) local. We say that a Deligne–Mumford stack (resp. algebraic stack) $\mathcal{X}$ *has property* $\mathcal{P}$ if for an étale (resp. smooth) presentation (equivalently for all presentations) $U \to \mathcal{X}$, the scheme U has $\mathcal{P}$.

The properties of being locally noetherian, reduced or regular are smooth-local.

Example 3.3.7. Let $G \to S$ be a smooth affine group scheme acting on a scheme U over S. Then $[U/G]$ is locally noetherian, reduced or regular if and only if U is.

Definition 3.3.8 (Substacks)**.** If $\mathcal{X}$ is an algebraic stack, a substack $\mathcal{Z} \subset \mathcal{X}$ is *closed* (resp. *open*, *locally closed*) if the induced morphism $\mathcal{Z} \to \mathcal{X}$ is a closed immersion (resp. open immersion, locally closed immersion).

Exercise 3.3.9. For an action of a smooth affine group scheme $G \to S$ on a scheme U over S, show that there is an equivalence between closed (resp. open) substacks of $[U/G]$ and G-invariant closed (resp. open) subschemes of U.

3.3.3 Separation properties

Separation properties for algebraic stacks are defined in terms of the diagonal.

Definition 3.3.10.

(1) A morphism of algebraic stack $\mathcal{X} \to \mathcal{Y}$ has *affine diagonal* (resp. *quasi-affine diagonal, separated diagonal*) if the diagonal $\mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ is affine (resp. quasi-affine, separated). An algebraic stack $\mathcal{X}$ has *affine diagonal* (resp. *quasi-affine diagonal, separated diagonal*) if $\mathcal{X} \to \operatorname{Spec} \mathbb{Z}$ does.

(2) A morphism of algebraic stack $\mathcal{X} \to \mathcal{Y}$ is *quasi-separated* if the diagonal $\mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ and second diagonal $\mathcal{X} \to \mathcal{X} \times_{\mathcal{X} \times_{\mathcal{Y}} \mathcal{X}} \mathcal{X}$ are quasi-compact. An algebraic stack $\mathcal{X}$ is *quasi-separated* if it is quasi-separated over $\operatorname{Spec} \mathbb{Z}$.

(3) A representable morphism $\mathcal{X} \to \mathcal{Y}$ of algebraic stacks is *separated* if the morphism $\mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$, which is representable by schemes (Exercise 3.2.3), is proper.

Conditions on the diagonal translate to conditions on the Isom sheaves since the base change of $\mathcal{X} \to \mathcal{X} \times \mathcal{X}$ by a morphism $(a, b)\colon S \to \mathcal{X} \times \mathcal{X}$ from a scheme S is identified with $\underline{\mathrm{Isom}}_{\mathcal{X}(S)}(a, b)$ (see Exercise 2.3.30), which is an algebraic space by Representability of the Diagonal (Theorem 3.2.1(2)). In particular, $\mathcal{X}$ has affine diagonal if and only if every algebraic space $\underline{\mathrm{Isom}}_{\mathcal{X}(S)}(a, b)$ is a scheme affine over S. Every algebraic stack with affine or quasi-affine diagonal is necessarily quasi-separated.

Lemma 3.3.11. *Let S be an affine scheme and $G \to S$ be a smooth affine group scheme acting on an algebraic space U over S. If U has affine diagonal (resp. has quasi-affine diagonal), then so does $[U/G]$.*

Proof. Recall that we established that $[U/G]$ is an algebraic stack in Theorem 3.1.9. Representability of the Diagonal (Theorem 3.2.1(2)) implies that $[U/G] \to [U/G] \times_S [U/G]$ is a representable morphism. Using the cartesian diagram

$$\begin{tikzcd} G \times_S U \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & U \times_S U \arrow[d] \\ {[U/G]} \arrow[r] & {[U/G] \times_S [U/G].} \end{tikzcd}$$

Since G is affine, so is the composition $G \times_S U \to U \times_S U \xrightarrow{p_1} U$. The statement follows from the cancellation law and descent. $\square$

The condition of having affine diagonal is satisfied by most moduli problems (except for example $\mathcal{M}_1$).

Example 3.3.12. The moduli stacks $\mathscr{M}_g$ and $\mathcal{B}\mathrm{un}_{r,d}(C)$ have affine diagonal and are thus quasi-separated. The statement for $\mathscr{M}_g$ follows from the above lemma and the quotient presentation $\mathscr{M}_g = [H'/\operatorname{PGL}_{5g-5}]$ in the proof of Theorem 3.1.15 as H' is locally closed subscheme of a projective Hilbert scheme. We will show later that $\mathscr{M}_g$ is separated or in other words that the diagonal of $\mathscr{M}_g$ is a finite morphism.

Similarly in Theorem 3.1.19, we expressed every quasi-compact open substack of $\mathcal{B}\mathrm{un}_{r,d}(C)$ as a quotient stack $[Q'/\operatorname{PGL}_N]$ where Q' is a locally closed subschemes of a projective Quot scheme. To see that $\mathcal{B}\mathrm{un}_{r,d}(C)$ has affine diagonal, it suffices to show that the base change of the along a morphism $\operatorname{Spec} A \to \mathcal{B}\mathrm{un}_{r,d}(C) \times \mathcal{B}\mathrm{un}_{r,d}(C)$ is affine. But such a morphism factors through $\mathcal{U} \times \mathcal{U}$ for some quasi-compact open substack $\mathcal{U} \subset \mathcal{B}\mathrm{un}_{r,d}(C)$ and we know that $\mathcal{U}$ has affine diagonal.

Remark 3.3.13. A quasi-separated Deligne–Mumford stack has *finite* and reduced stabilizer groups (see Exercise 3.2.8).

For morphisms of schemes, the definition of separatedness above agrees with the usual notation as the diagonal of a morphism of schemes is a closed immersion if and only if it is proper. We postpone the definition of separatedness for non-representable morphisms until Definition 3.8.1.

Example 3.3.14. The non-separated union $\mathbb{A}^\infty \bigcup_{\mathbb{A}^\infty \setminus 0} \mathbb{A}^\infty$ is a typical example of a non-quasi-separated scheme. For algebraic spaces and stacks, there are additional pathologies coming from actions of non-quasi-compact group schemes. For instance, $[\mathbb{A}^1/\underline{\mathbb{Z}}]$ is a non-quasi-separated algebraic space (see Example 3.9.22) while $\mathbf{B}\underline{\mathbb{Z}}$ is a non-quasi-separated algebraic stack (see Example 3.9.21).

Exercise 3.3.15. An action of an algebraic group G over a field $\Bbbk$ on an algebraic space U is called *proper* if the action map

$$\Psi \colon G \times U \to U \times U, \quad (g,u) \mapsto (gu, u)$$

is proper.

(a) Show that the action of G on U is proper if and only if $[U/G]$ is separated.

(b) For $u \in U(\Bbbk)$, let $\Psi_u \colon G \to U$ be the map defined by $g \mapsto gu$ (viewing Ψ as a morphism over U via the projections on the second component, then Ψ_u is the fiber of Ψ over u). Show that the following are equivalent:

   (i) $\Psi_u \colon G \to U$ is proper,

   (ii) $u \colon \operatorname{Spec} \Bbbk \to [U/G]$ is proper,

   (iii) $Gu \subset U$ is closed and G_u is proper.

   Hint: To show that (i) or (ii) implies (iii), replace U with the reduced orbit Gu, use Generic Flatness (3.3.30) to show that $\operatorname{Spec} \Bbbk \to [U/G]$ is faithfully flat, and then use fppf descent.

3.3.4 The topological space of a stack

We can associate a topological space $|\mathcal{X}|$ to every algebraic stack $\mathcal{X}$.

Definition 3.3.16 (Topological space of an algebraic stack)**.** If $\mathcal{X}$ is an algebraic stack, we define *the topological space of* $\mathcal{X}$ as the set $|\mathcal{X}|$ consisting of field-valued morphisms $x \colon \operatorname{Spec} K \to \mathcal{X}$. Two morphisms $x_1 \colon \operatorname{Spec} K_1 \to \mathcal{X}$ and $x_2 \colon \operatorname{Spec} K_2 \to \mathcal{X}$ are identified in $|\mathcal{X}|$ if there exists field extensions $K_1 \to K_3$ and $K_2 \to K_3$ such that $x_1|_{\operatorname{Spec} K_3}$ and $x_2|_{\operatorname{Spec} K_3}$ are isomorphic in $\mathcal{X}(K_3)$. A subset $U \subset |\mathcal{X}|$ is open if there exists an open substack $\mathcal{U} \subset \mathcal{X}$ such that $U = |\mathcal{U}|$.

A morphism of stacks $\mathcal{X} \to \mathcal{Y}$ induces a continuous map $|\mathcal{X}| \to |\mathcal{Y}|$.

Exercise 3.3.17. Show that if $\mathcal{X}$ is an algebraic stack and $U \subset |\mathcal{X}|$ is an open subset, then there exists a reduced closed substack $\mathcal{Z} \hookrightarrow \mathcal{X}$ such that $|\mathcal{Z}| = |\mathcal{X}| \setminus U$.

Example 3.3.18. The topological space of the quotient stack $|[\mathbb{A}^1_{\Bbbk}/\mathbb{G}_m]|$ with the standard scaling action consists of two points with representatives $x_0 \colon \operatorname{Spec} k \xrightarrow{0} \mathbb{A}^1 \to [\mathbb{A}^1/\mathbb{G}_m]$ and $x_1 \colon \operatorname{Spec} k \xrightarrow{1} \mathbb{A}^1 \to [\mathbb{A}^1/\mathbb{G}_m]$. In particular, the inclusion of the generic point $\operatorname{Spec} \Bbbk(x) \to \mathbb{A}^1 \to [\mathbb{A}^1/\mathbb{G}_m]$ is equivalent to x_1.

While the stabilizer group $G_{\overline{x}}$ depends on the choice of representative $\overline{x} \colon \operatorname{Spec} \Bbbk \to \mathcal{X}$ of $x \in |\mathcal{X}|$, its dimension—which we denote by $\dim G_x$—is independent of this choice. Similarly, the properties of being smooth, unramified, affine, finite, and reduced are also independent of this choice.

Exercise 3.3.19. Let $x \in |\mathcal{X}|$ be a point of an algebraic stack with two representatives $x_1 \colon \operatorname{Spec} \Bbbk_1 \to \mathcal{X}$ and $x_2 \colon \operatorname{Spec} \Bbbk_2 \to \mathcal{X}$.

(1) Show that the stabilizer group G_{x_1} is smooth (resp. étale, unramified, affine, finite) if and only if G_{x_2} is.

(2) Show that $\dim G_{x_1} = \dim G_{x_2}$.

(3) If $\mathcal{X}$ is Deligne–Mumford and both $\Bbbk_1$ and $\Bbbk_2$ are algebraically closed, show that the abstract discrete groups corresponding to G_{x_1} and G_{x_2} (see Exercise 3.2.8) are isomorphic.

As a consequence of the above exercise, it makes sense to say that $x \in |\mathcal{X}|$ has *smooth* (resp. *étale*, *unramified*, *affine*, *finite*) *stabilizer*. For a Deligne–Mumford stack $\mathcal{X}$, we define the *geometric stabilizer of* x as the discrete group $G = G_{\overline{x}}(\Bbbk)$ where $\overline{x} \colon \operatorname{Spec} \Bbbk \to \mathcal{X}$ is a geometric point representing x.

We can now define topological properties of algebraic stacks and their morphisms.

Definition 3.3.20. We say that an algebraic stack $\mathcal{X}$ is *quasi-compact*, *connected*, or *irreducible* if $|\mathcal{X}|$ is, and we say that $\mathcal{X}$ is *noetherian* if it is locally noetherian, quasi-compact and quasi-separated.

Exercise 3.3.21. Show that an algebraic stack $\mathcal{X}$ is quasi-compact if and only if there exists a smooth presentation $\operatorname{Spec} A \to \mathcal{X}$ and that a quasi-separated algebraic stack $\mathcal{X}$ is noetherian if and only if there exists a smooth presentation $\operatorname{Spec} A \to \mathcal{X}$ where A is a noetherian ring.

Example 3.3.22. The moduli stack $\mathscr{M}_g$ is noetherian and in particular quasi-compact. This follows from the above exercise using the quotient presentation $\mathscr{M}_g = [H' / \operatorname{PGL}_{5g-5}]$ from Theorem 3.1.15. However, $\mathcal{B}\mathrm{un}_{r,d}(C)$ is not quasi-compact.

Exercise 3.3.23.

(a) Show that a morphism $\mathcal{X} \to \mathcal{Y}$ of algebraic stacks is surjective if and only if $|\mathcal{X}| \to |\mathcal{Y}|$ is surjective.

(b) Show that if $\mathcal{X} \to \mathcal{Y}$ and $\mathcal{Y}' \to \mathcal{Y}$ are morphisms of algebraic stacks, then $|\mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}'| \to |\mathcal{X}| \times_{|\mathcal{Y}|} |\mathcal{Y}'|$ is surjective.

Exercise 3.3.24. If $\mathcal{X}$ is a quasi-compact and locally noetherian algebraic stack, show that $|\mathcal{X}|$ is a noetherian topological space.

Exercise 3.3.25. Since the property of being universally open for a morphism of schemes is smooth-local on the source and target, we can define *universally open* morphisms of algebraic stacks using Definition 3.3.2(1). This property includes faithfully flat morphisms locally of finite presentation.

(a) If $f\colon \mathcal{X} \to \mathcal{Y}$ is a universally open morphism of algebraic stacks, show that $f(|\mathcal{X}|) \subset |\mathcal{Y}|$ is open and conclude that for every morphism $\mathcal{Y}' \to \mathcal{Y}$ of algebraic stacks, the map $|\mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}'| \to |\mathcal{Y}'|$ is open.

Hint: Show that the image is identified with the open substack $\mathcal{V} \subset \mathcal{Y}$, whose objects over a scheme T consist of morphisms $T \to \mathcal{Y}$ such that $\mathcal{X}_T \to T$ is surjective.

(b) Show that if $U \to \mathcal{X}$ is a smooth presentation of an algebraic stack, then a set $\Sigma \subset |\mathcal{X}|$ is open (resp. closed) if and only if its preimage in U is.

Definition 3.3.26. A morphism $\mathcal{X} \to \mathcal{Y}$ of algebraic stacks is *quasi-compact* if for every morphism $\operatorname{Spec} B \to \mathcal{Y}$, the fiber product $\mathcal{X} \times_{\mathcal{Y}} \operatorname{Spec} B$ is quasi-compact. We say that $\mathcal{X} \to \mathcal{Y}$ is *of finite type* if $\mathcal{X} \to \mathcal{Y}$ is locally of finite type and quasi-compact.

Example 3.3.27. The moduli stack $\mathscr{M}_g$ is finite type over $\mathbb{Z}$. On the other hand, $\mathcal{B}\mathrm{un}_{r,d}(C)$ is locally of finite type over $\Bbbk$ but not of finite type.

Remark 3.3.28. A quasi-compact morphism $\mathcal{X} \to \mathcal{Y}$ induces a quasi-compact morphism $|\mathcal{X}| \to |\mathcal{Y}|$ on topological spaces. The converse is true if $\mathcal{Y}$ is quasi-separated but not in general, e.g. $\operatorname{Spec} \Bbbk \to B_{\Bbbk}\mathbb{Z}$ (see Example 3.9.21).

Exercise 3.3.29.

(a) Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a quasi-compact morphism of algebraic stacks. For a point $x \in |\mathcal{X}|$, show that $f(\overline{\{x\}}) = \overline{\{f(x)\}}$.

(b) Generalize Chevalley's criterion to algebraic stacks: if $f\colon \mathcal{X} \to \mathcal{Y}$ is a morphism of algebraic stacks locally of finite presentation, then the image $f(|\mathcal{X}|) \subset |\mathcal{Y}|$ is constructible.

(c) Show an open morphism $f\colon \mathcal{X} \to \mathcal{Y}$ of algebraic stacks (i.e. $|\mathcal{X}| \to |\mathcal{Y}|$ is open) satisfies the following lifting property: if $x \in |\mathcal{X}|$ is a point, then every specialization $y' \rightsquigarrow f(x)$ lifts to a specialization $x' \rightsquigarrow x$. Show that the converse is true for morphisms locally of finite presentation.

(d) If $\mathcal{X}$ is a quasi-separated algebraic stack, show that $|\mathcal{X}|$ is a sober topological space, i.e. every irreducible closed subset has a unique generic point.

Exercise 3.3.30 (Generic Flatness)**.** Generalize Theorem A.2.11 to algebraic stacks: if $\mathcal{X} \to \mathcal{Y}$ is a finite type morphism of algebraic stacks with $\mathcal{Y}$ reduced, then there exists a dense open substack $\mathcal{U} \subset \mathcal{Y}$ such that the base change $X_{\mathcal{U}} \to U$ is flat and of presentation.

Exercise 3.3.31. Extend the characterization of locally of finite presentation morphisms given in Proposition A.1.3 to algebraic stacks: a morphism $f\colon \mathcal{X} \to \mathcal{Y}$ of algebraic stacks is locally of finite presentation if and only if for every directed system $\{\operatorname{Spec} A_\lambda\}_{\lambda \in I}$ of affine schemes over $\mathcal{Y}$, the natural map

$$\operatorname{colim}_\lambda \mathrm{MOR}_{\mathcal{Y}}(\operatorname{Spec} A_\lambda, \mathcal{X}) \to \mathrm{MOR}_{\mathcal{Y}}(\operatorname{Spec}(\operatorname{colim}_\lambda A_\lambda), \mathcal{X})$$

is bijective.

3.3.5 Quasi-finite and étale morphisms

A morphism of schemes is *locally quasi-finite* if it is locally of finite type and every fiber is discrete. Since this property is étale-local on the source and target, we can extend this property to morphisms of *algebraic spaces* using Definition 3.3.2.

Definition 3.3.32.

(1) A representable morphism $\mathcal{X} \to \mathcal{Y}$ of algebraic stacks is *locally quasi-finite* if for every morphism $T \to \mathcal{Y}$ from a scheme, the algebraic space $\mathcal{X} \times_{\mathcal{Y}} T$ is locally quasi-finite over T.

(2) A morphism $\mathcal{X} \to \mathcal{Y}$ of algebraic stacks is *locally quasi-finite* if is locally of finite type, the diagonal $\mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ is locally quasi-finite and for every morphism $\operatorname{Spec} \Bbbk \to \mathcal{Y}$ from a field, the topological space $|\mathcal{X} \times_{\mathcal{Y}} \operatorname{Spec} \Bbbk|$ is discrete.

(3) A morphism $\mathcal{X} \to \mathcal{Y}$ of algebraic stacks is *quasi-finite* if it is locally quasi-finite and quasi-compact.

To understand condition (2), recall that the diagonal $\mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ is always a representable morphism (Exercise 3.2.3). The diagonal is quasi-finite (resp. locally quasi-finite) if and only if for every field-valued point $x \in \mathcal{X}(\Bbbk)$ with image $y \in \mathcal{Y}(\Bbbk)$, the kernel $\ker(G_x \to G_y)$ of the induced map of stabilizer groups is finite (resp. discrete); see Exercise 3.2.7. In particular, if $\mathcal{Y}$ is a scheme, the diagonal is quasi-finite if and only if all stabilizers of $\mathcal{Y}$ are finite. For instance, if G is a finite group scheme over a field $\Bbbk$ (e.g. μ_p), then $\mathbf{B}G \to \operatorname{Spec} \Bbbk$ is quasi-finite. On the other hand, $\mathbf{B}\mathbb{G}_m \to \operatorname{Spec} \Bbbk$ is not quasi-finite despite that $|\mathbf{B}\mathbb{G}_m|$ is a single point.

For morphisms of schemes, the property of being étale or unramified is also étale-local on the source and target. We can therefore use Definition 3.3.2 to extend the definition of étale and unramified to morphisms of *Deligne–Mumford stacks*.

Definition 3.3.33. A morphism $\mathcal{X} \to \mathcal{Y}$ of algebraic stacks is *étale* (resp. *unramified*) if for every morphism $T \to \mathcal{Y}$ from a scheme, the fiber product $\mathcal{X} \times_{\mathcal{Y}} T$ is a Deligne–Mumford stack[2] such that $\mathcal{X} \times_{\mathcal{Y}} T \to T$ is étale (resp. unramified).

While étale morphisms are smooth and locally quasi-finite, the converse is not true, e.g. $B_{\Bbbk}\boldsymbol{\mu}_p \to \operatorname{Spec} \Bbbk$ over a characteristic p field $\Bbbk$ (see Exercise 6.2.11). Similarly, étale morphisms are smooth of relative dimension 0, but again the converse doesn't hold, e.g. $[\mathbb{A}^1_{\Bbbk}/\mathbb{G}_m] \to \operatorname{Spec} \Bbbk$ over a field $\Bbbk$. We will later establish that every separated, quasi-finite and representable morphism is quasi-affine (Proposition 4.4.5).

3.4 Equivalence relations and groupoids

Definition 3.4.1. An *étale (resp. smooth) groupoid of schemes* is a pair of schemes U and R together with étale (resp. smooth) morphisms $s\colon R \to U$

[2] A morphism $\mathcal{X} \to \mathcal{Y}$ of algebraic stacks such that each fiber product $\mathcal{X} \times_{\mathcal{Y}} T$ is Deligne–Mumford is called *relatively Deligne–Mumford* or simply *DM*. A morphism $\mathcal{X} \to \mathcal{Y}$ satisfying the weaker condition that the diagonal $\mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ is locally quasi-finite is called *quasi-DM*. See [SP, Tag 04YW].

called the *source* and $t\colon R \to U$ called the *target*, and a *composition* morphism $c\colon R \times_{s,U,t} R \to R$ satisfying:

(1) (associativity) the following diagram commutes

$$\begin{tikzcd} R \times_{s,U,t} R \times_{s,U,t} R \arrow[r, "c\times \mathrm{id}"] \arrow[d, "\mathrm{id}\times c"] & R \times_{s,U,t} R \arrow[d, "c"] \\ R \times_{s,U,t} R \arrow[r, "c"] & R, \end{tikzcd}$$

(2) (identity) there exists a morphism $e\colon U \to R$ (called the *identity*) such that the following diagrams commute

$$\begin{tikzcd} & U \arrow[dl, "\mathrm{id}"'] \arrow[d, "e"] \arrow[dr, "\mathrm{id}"] & \\ U & R \arrow[l, "s"] \arrow[r, "t"'] & U \end{tikzcd} \qquad \begin{tikzcd} R \arrow[r, "e\circ s,\mathrm{id}"] \arrow[dr, "\mathrm{id}"'] & R \times_{s,U,t} R \arrow[d, "c"] & R \arrow[l, "e\circ t,\mathrm{id}"'] \arrow[dl, "\mathrm{id}"] \\ & R, & \end{tikzcd}$$

(3) (inverse) there exists a morphism $i\colon R \to R$ (called the *inverse*) such that the following diagrams commute

$$\begin{tikzcd} R \arrow[r, "i"] \arrow[dr, "s"'] & R \arrow[r, "i"] \arrow[d, "t"] & R \arrow[dl, "s"] \\ & U & \end{tikzcd} \quad \begin{tikzcd} R \arrow[r, "s"] \arrow[d, "{(i,\mathrm{id})}"] & U \arrow[d, "e"] \\ R \times_{s,U,t} R \arrow[r, "c"] & R \end{tikzcd} \quad \begin{tikzcd} R \arrow[r, "t"] \arrow[d, "{(\mathrm{id},i)}"] & U \arrow[d, "e"] \\ R \times_{s,U,t} R \arrow[r, "c"] & R. \end{tikzcd}$$

We will often denote this data as $s,t\colon R \rightrightarrows U$.

If $(s,t)\colon R \to U \times U$ is a monomorphism, then we say that $s,t\colon R \rightrightarrows U$ is an *étale (resp. smooth) equivalence relation.*

If U and R are algebraic spaces, and the source, target and composition are morphisms of algebraic spaces, we obtain the notion of an *étale (resp. smooth) groupoid of algebraic spaces* and similarly an *étale (resp. smooth) equivalence relation of algebraic spaces.*

We can view R as a *scheme of relations on U*: a point $r \in R$ specifies a relation on the points $s(r), t(r) \in U$, which we sometimes write as $s(r) \xrightarrow{r} t(r)$. For every scheme T, the morphisms $R(T) \rightrightarrows U(T)$ define a groupoid of sets, i.e. there is composition morphism $R(T) \times_{s,U(T),t} R(T) \to R(T)$ satisfying axioms analogous to (1)–(3). We can think of an element $r \in R(T)$ as specifying a relation $u \xrightarrow{r} v$ between elements $u, v \in U(T)$. The composition morphism composes relations $u \xrightarrow{r} v$ and $v \xrightarrow{r'} w$ to the relation $u \xrightarrow{r\circ r'} w$ while the identity morphism takes $u \in U(T)$ to $u \xrightarrow{\mathrm{id}} u$ and the inverse morphism takes $u \xrightarrow{r} v$ to $v \xrightarrow{r^{-1}} u$. When $R \rightrightarrows U$ is an equivalence relation, the morphism $R(T) \to U(T) \times U(T)$ is injective and there is thus at most one relation between any two elements of $U(T)$.

Definition 3.4.2 (Orbits and stabilizers)**.** Let $R \rightrightarrows U$ be a smooth groupoid of algebraic spaces, and let $x\colon \operatorname{Spec} \Bbbk \to U$ be a field-valued point. The *stabilizer G_x of x* is defined as the fiber product of $(s,t)\colon R \to U \times U$ by $(x,x)\colon \operatorname{Spec} \Bbbk \to U \times U$. The *orbit* $O_R(x)$ is defined as the *set* $s(t^{-1}(u)) \subset U$.

Remark 3.4.3. Assuming that U is defined over $\Bbbk$ and that $x \in U(\Bbbk)$, then the $\Bbbk$-points of G_x are relations $\rho\colon x \xrightarrow{\sim} x$ in $R(\Bbbk)$ while the orbit $O_R(x)$ consists of points $y \in U$ such that there exists a relation $x \xrightarrow{\sim} y$ in R.

Exercise 3.4.4. Show that the identity and inverse morphism are uniquely determined.

Example 3.4.5. If $G \to S$ is an étale (resp. smooth) group scheme with multiplication $\mu\colon G \times_S G \to G$ acting on a scheme U over S via multiplication $\sigma\colon G \times U \to U$, then

$$p_2, \sigma\colon G \times_S U \rightrightarrows U$$

is an étale (resp. smooth) groupoid of schemes. The inverse $G \times_S U \to G \times_S U$ is given by $(g, u) \mapsto (g^{-1}, gu)$ and the composition is

$$(G \times_S U) \times_{\sigma, U, p_2} (G \times_S U) \to G \times_S U, \quad \big((g', u'), (g, u)\big) \mapsto (g'g, u).$$

where $u' = gu$. Here (g, u) is a T-valued point of $G \times_S U$ and can be viewed as the relation $u \to gu$.

The following identifies projections in the groupoid with maps arising from the group action:

$$\begin{tikzcd}
U \times_{[U/G]} U \times_{[U/G]} U \arrow[r, "\sim"] \arrow[d, shift right=4, "p_{12}"'] \arrow[d, "p_{13}"] \arrow[d, shift left=4, "p_{23}"] & G \times_S G \times_S U \arrow[d, shift right=4, "\mathrm{id} \times \sigma"'] \arrow[d, "\mu \times \mathrm{id}"] \arrow[d, shift left=4, "p_{23}"] \\
U \times_{[U/G]} U \arrow[r, "\sim"] \arrow[d, shift right=2, "p_1"'] \arrow[d, shift left=2, "p_2"] & G \times_S U \arrow[d, shift right=2, "p_2"'] \arrow[d, shift left=2, "\sigma"] \\
U \arrow[r, equal] \arrow[d] & U \\
{[U/G].} &
\end{tikzcd}$$

The identification $U \times_{[U/G]} U \xrightarrow{\sim} G \times_S U$ is given by $u_2 \times_g u_1 \mapsto (g, u_1)$ where $u_2 \times_g u_1$ is shorthand notation for the triple (u_2, u_1, g) (with $u_2 = gu_1$) defining an element of the fiber product. Similarly $U \times_{[U/G]} U \times_{[U/G]} U \xrightarrow{\sim} G \times_S G \times_S U$ is given by $u_3 \times_{g_2} u_2 \times_{g_1} u_1 \mapsto (g_2, g_1, u_1)$.

More generally, the n-fold fiber product $(U/[U/G])^n$ of U over $[U/G]$ is identified with $G^{n-1} \times U$ via $u_n \times_{g_{n-1}} u_{n-1} \cdots \times_{g_1} u_1 \mapsto (g_{n-1}, \dots, g_1, u_1)$. Under these identifications, the projection $p_{\widehat{k}}\colon (U/[U/G])^{n+1} \to (U/[U/G])^n$ forgetting the kth term is identified with that map $G^n \times U \to G^{n-1} \times U$ taking an element $(g_n, \dots, g_1, u_1)$ to $(g_{n-1}, \dots, g_1, u_1)$ for $k = 1$, to $(g_n, \dots, g_{k+1}, g_k g_{k-1}, g_{k-2}, \dots, g_1, u_1)$ for $k = 2, \dots, n$ and to $(g_n, \dots, g_2, g_1 u_1)$ for $k = n + 1$.

Example 3.4.6. Let $\mathcal{X}$ be a Deligne–Mumford stack (resp. algebraic stack) and $U \to \mathcal{X}$ be an étale (resp. smooth) presentation which we assume is not only representable but representable by schemes. Define the scheme $R := U \times_{\mathcal{X}} U$, the source morphism $s = p_1\colon R \to U$, the target morphism $t = p_2\colon R \to U$ and the composition morphism $(s \circ p_1, t \circ p_2)\colon R \times_{s,U,t} R \to R := U \times_{\mathcal{X}} U$. This gives the structure of an étale (resp. smooth) groupoid $R \rightrightarrows U$. If X is an algebraic space, then $R \rightrightarrows U$ is an étale equivalence relation.

Choosing different presentation yields different groupoids which are equivalent under a notion called *Morita equivalence*; we will not use this notion in these notes.

3.4.1 Algebraicity of the quotient of a groupoid

Definition 3.4.7 (Quotient stack of a smooth groupoid)**.** Let $s, t\colon R \rightrightarrows U$ be a smooth groupoid of algebraic spaces. Define $[U/R]^{\mathrm{pre}}$ to be the prestack whose objects are morphisms $T \to U$ from a scheme T. A morphism $(S \xrightarrow{a} U) \to (T \xrightarrow{b} U)$ is the data of a morphism of schemes $f\colon S \to T$ and an element $r \in R(S)$ such that $s(r) = a$ and $t(r) = f \circ b$.

Define $[U/R]$ to be the stackification of $[U/R]^{\mathrm{pre}}$ in the big étale topology $\mathrm{Sch}_{\text{ét}}$.

If in addition $R \rightrightarrows U$ is an equivalence relation, then $[U/R]$ is isomorphic to a sheaf (Exercise 3.4.8) and we denote it as U/R.

The fiber category $[U/R]^{\mathrm{pre}}(T)$ is the groupoid whose objects are $U(T)$ and morphisms are $R(T)$. The identity morphism $\mathrm{id}\colon U \to U$ defines a map $U \to [U/R]^{\mathrm{pre}}$ and therefore a map $p\colon U \to [U/R]$.

Exercise 3.4.8. Let $R \rightrightarrows U$ be a smooth groupoid of algebraic spaces. Show that $[U/R]$ is equivalent to a sheaf if and only if $R \rightrightarrows U$ is an equivalence relation.

Exercise 3.4.9. Extend Exercise 2.3.28 to show that if $s, t\colon R \rightrightarrows U$ is a smooth groupoid of algebraic spaces, the following diagrams are cartesian:

$$\begin{tikzcd} R \arrow[r, "s"] \arrow[d, "t"'] \arrow[dr, phantom, "\square"] & U \arrow[d, "p"] \\ U \arrow[r, "p"] & {[U/R]} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} R \arrow[r, "{(s,t)}"] \arrow[d] \arrow[dr, phantom, "\square"] & U \times U \arrow[d, "p \times p"] \\ {[U/R]} \arrow[r, "\Delta"] & {[U/R] \times [U/R]}. \end{tikzcd}$$

Exercise 3.4.10. Let $R \rightrightarrows U$ be a smooth groupoid of algebraic spaces and $x\colon \operatorname{Spec} \Bbbk \to U$ be a field-valued point. Show that the stabilizer of x as defined in Definition 3.4.2 is identified with the stabilizer of $\operatorname{Spec} \Bbbk \to [U/R]$ as defined in Definition 3.2.5.

Theorem 3.4.11.

(1) If $R \rightrightarrows U$ is an étale (resp. smooth) groupoid of algebraic spaces. Then $[U/R]$ is a Deligne–Mumford stack (resp. algebraic stack) and $U \to [U/R]$ is an étale (resp. smooth) presentation.

(2) If $R \rightrightarrows U$ be an étale equivalence relation of schemes, then U/R is an algebraic space and $U \to U/R$ is an étale presentation.

Remark 3.4.12. In Corollary 4.4.11, we show that in fact the quotient U/R of an étale equivalence relation of *algebraic spaces* is an algebraic space, establishing that one doesn't obtain new algebro-geometric objects by considering sheaves which are étale locally algebraic spaces. This result is delayed until §4.4 as it takes more work to show that the diagonal of U/R is representable by *schemes*.

More generally, if $R \rightrightarrows U$ is an fppf groupoid (resp. fppf equivalence relation) of algebraic spaces, then $[U/R]$ is an algebraic stack [SP, Tag 06FI] (resp. U/R is an algebraic space [SP, Tag 04S6]). See also [Art74, Thm. 6.1] and [LMB, Thm. 10.1].

Proof. For (1), we will show that $U \to \mathcal{X} := [U/R]$ is representable, surjective and smooth. Let $T \to \mathcal{X}$ be a morphism from a scheme T. It follows from the definition of $[U/R]$ as the stackification of $[U/R]^{\text{pre}}$ that there exists an étale cover $T' \to T$ and a commutative diagram

$$\begin{tikzcd}
T' \arrow[r] \arrow[d] & U \arrow[d] \\
T \arrow[r] & \mathcal{X}.
\end{tikzcd}$$

In the commutative cube

$$\begin{tikzcd}
& U_{T'} \arrow[rr] \arrow[dl] \arrow[dd] & & T' \arrow[dl] \arrow[dd] \\
R \arrow[rr, crossing over] \arrow[dd] & & U & \\
& U_T \arrow[rr] \arrow[dl] & & T \arrow[dl] \\
U \arrow[rr] & & \mathcal{X} \arrow[from=uu, crossing over] &
\end{tikzcd} \tag{3.4.1}$$

the front, back, top and bottom squares are cartesian where U_T is a sheaf and $U_{T'}$ is a scheme. Since $T' \to T$ is a surjective étale morphism representable by schemes, so is $U_{T'} \to U_T$. This establishes that U_T is an algebraic space. By descent $U_{T'}$ is surjective and étale over T.

For (2), it suffices to show that the diagonal of the quotient sheaf $X := U/R$ is representable by schemes. Indeed, this implies that $U \to X$ is representable by schemes via the argument of Corollary 3.2.2 and étale descent implies that $U \to X$ is étale and surjective. Let $T \to X \times X$ be a morphism from a scheme and consider the cartesian cube

$$\begin{tikzcd}
& Q_{T'} \arrow[rr] \arrow[dl] \arrow[dd] & & T' \arrow[dl] \arrow[dd] \\
R \arrow[rr, crossing over] \arrow[dd] & & U \times U & \\
& Q_T \arrow[rr] \arrow[dl] & & T \arrow[dl] \\
X \arrow[rr] & & X \times X, \arrow[from=uu, crossing over] &
\end{tikzcd} \tag{3.4.2}$$

as in (3.2.1). Since $R \to U \times U$ is separated and locally quasi-finite, so is $Q_{T'} \to T'$. Effective Descent (Proposition B.3.1) implies that sheaf Q_T is a scheme. $\square$

As a consequence, we see that the affine hypothesis in Theorem 3.1.9 asserting the algebraicity of the quotient stack $[X/G]$ and classifying stack $\mathbf{B}G$ is superfluous.

Exercise 3.4.13. Show that if $\mathcal{X}$ is an algebraic stack (resp. algebraic space) and $U \to \mathcal{X}$ is a smooth presentation, then $\mathcal{X}$ is isomorphic to the quotient stack $[U/R]$ (resp. quotient sheaf U/R) of the étale groupoid (resp. equivalence relation) $R \rightrightarrows U$ where $R = U \times_{\mathcal{X}} U$.

3.4.2 Inducing and slicing presentations

We provide here to useful techniques to build new presentations from given ones.

First, let $\mathcal{X} = [X/H]$ be a quotient stack of a smooth algebraic group H acting on a scheme X over $\Bbbk$ and $H \subset G$ be an inclusion of algebraic groups. Then H acts freely on $G \times X$ via $h \cdot (g, x) = (gh^{-1}, hx)$ and we let $G \times^H X$ be the algebraic space quotient $(G \times X)/H$. When H is finite, this quotient exists by definition of an algebraic space and is affine (resp. quasi-projective, projective) when X is by Theorem 4.3.6 (resp. Exercise 4.2.8). In the non-finite case, it follows from Corollary 3.6.7 $G \times^H X$ is an algebraic space if X is noetherian. There is an action of G on $G \times^H X$ via $g \cdot (g', x) = (gg', x)$.

Exercise 3.4.14. Show that $[X/H] \cong [(G \times^H X)/G]$.

The second method is sometimes referred to as *slicing a groupoid*. Let $U \to \mathcal{X}$ be a smooth presentation of an algebraic stack with the corresponding groupoid $s, t \colon R = U \times_{\mathcal{X}} U \rightrightarrows U$. If $g \colon U' \to U$ is a morphism, we define *the restriction of* $R \rightrightarrows U$ *along* $U' \to U$ to be the groupoid $R|_{U'} \rightrightarrows U'$ defined by the fiber product

$$\begin{tikzcd}
R|_{U'} \arrow[r, "(t', s')"] \arrow[d] \arrow[dr, phantom, "\square"] & U' \times U' \arrow[d] \\
R \arrow[r, "(t, s)"] & U \times U
\end{tikzcd}$$

Exercise 3.4.15.

(a) Show that $R|_{U'}$ fits into a cartesian diagram

$$\begin{tikzcd}
R|_{U'} \arrow[r] \arrow[d] & R \times_{s,U} U' \arrow[r] \arrow[d] & U' \arrow[d, "g"] \\
U' \times_{U,t} R \arrow[r] \arrow[d] & R \arrow[r, "s"] \arrow[d, "t"] & U \arrow[d] \\
U' \arrow[r, "g"] & U \arrow[r] & {[U/R]}
\end{tikzcd}$$

Assume in addition that $U' \times_{U,t} R \to R \xrightarrow{s} U$ is étale (resp. smooth).

(2) Show that $R|_{U'} \rightrightarrows U'$ is an étale (resp. smooth) groupoid.

(3) Show that there is an open immersion $[U'/R|_{U'}] \to [U/R]$.

(4) Show that $[U'/R|_{U'}] \to [U/R]$ is an isomorphism if and only if for every every point $u \in U$, there exists a pont $u' \in U$ and a relation $u \to g(u')$ in R.

3.5 Dimension, tangent spaces, and residual gerbes

3.5.1 Dimension

Recall that the dimension $\dim X$ of a scheme X is the Krull dimension of the underlying topological space while the dimension $\dim_x X$ at a point $x \in X$ is the minimum dimension of open subsets containing x (which is in general distinct from $\dim \mathcal{O}_{X,x}$). We now extend these definitions to algebraic spaces and stacks.

Definition 3.5.1.

(1) Let X be a noetherian algebraic space and $x \in |X|$. We define the *dimension of X at x* to be

$$\dim_x X = \dim_u U \in \mathbb{Z}_{\geq 0} \cup \infty$$

where $U \to X$ is an étale presentation and $u \in U$ is a preimage of x.

(2) Let $\mathcal{X}$ be a noetherian algebraic stack with smooth presentation $U \to \mathcal{X}$ and corresponding smooth groupoid $s, t\colon R \rightrightarrows U$, and let $u \in U$ be a preimage of $x \in |\mathcal{X}|$. We define the *dimension of $\mathcal{X}$ at x* to be

$$\dim_x \mathcal{X} = \dim_u U - \dim_{e(u)} R_u \in \mathbb{Z} \cup \infty$$

where R_u is the fiber of $s\colon R \to U$ over u and $e\colon U \to R$ denotes the identity morphism in the groupoid.

(3) If $\mathcal{X}$ is a noetherian algebraic space or stack, we define the *dimension of $\mathcal{X}$* to be

$$\dim \mathcal{X} = \sup_{x \in |\mathcal{X}|} \dim_x \mathcal{X} \in \mathbb{Z} \cup \infty.$$

Proposition 3.5.2. *The definition of the dimension $\dim_x \mathcal{X}$ of a noetherian algebraic stack $\mathcal{X}$ at a point $x \in |\mathcal{X}|$ is independent of the presentation $U \to \mathcal{X}$ and of the choice of preimage u of x.*

Proof. The definition of the dimension of an algebraic space at a point is clearly well defined as étale morphisms have relative dimension 0.

If $U \to \mathcal{X}$ is a smooth presentation (with U a scheme) and $u \in U$ is a preimage of x with residue field $\kappa(u)$, then the fiber R_u is is identified with the fiber product

$$\begin{tikzcd}
R_u \arrow[r] \arrow[d] & R \arrow[r, "t"] \arrow[d, "s"] & U \arrow[d] \\
\operatorname{Spec} \kappa(u) \arrow[r] & U \arrow[r] & \mathcal{X},
\end{tikzcd}$$

and is a smooth algebraic space over $\kappa(u)$.

If $U' \to \mathcal{X}$ is a second presentation and $u' \in U'$ a preimage of x, then define the algebraic space $U'' := U \times_{\mathcal{X}} U'$. Observe that there is a cartesian diagram

$$\begin{tikzcd}
U''_u \arrow[r] \arrow[d] & U'' \arrow[r] \arrow[d] & U' \arrow[d] \\
\operatorname{Spec} \kappa(u) \arrow[r] & U \arrow[r] & \mathcal{X}
\end{tikzcd} \tag{3.5.1}$$

where the fiber U''_u is identified with $R'_{u'}$. By Exercise 3.5.3 applied to $U'' \to U$, we have the identity

$$\dim_{u''} U'' = \dim_u U + \dim_{u''} U''_u = \dim_u U + \dim_{e'(u')} R'_{u'}. \tag{3.5.2}$$

Choose a representative $\operatorname{Spec} L \to U''$ in $|U''|$ mapping to u and u'. Note that the compositions $\operatorname{Spec} \kappa(u) \to U \to \mathcal{X}$, $\operatorname{Spec} \kappa(u') \to U' \to \mathcal{X}$ and $\operatorname{Spec} L \to U'' \to \mathcal{X}$ all define the same point $x \in |\mathcal{X}|$. Let $R \rightrightarrows U$ and $R' \rightrightarrows U'$ be the corresponding smooth groupoids, and set $R''_{u''} = U'' \times_{\mathcal{X}} \operatorname{Spec} L$.

We need to show that

$$\dim_u U - \dim_{e(u)} R_u = \dim_{u'} U' - \dim_{e'(u')} R'_{u'}$$

and by symmetry between U and U', it suffices to show that

$$\dim_u U - \dim_{e(u)} R_u = \dim_{u''} U'' - \dim_{e''(u'')} R''_{u''}$$

where $e''(u'') \in |R''_{u''}|$ is the image of the map $\operatorname{Spec} L \to R''_{u''} = U'' \times_{\mathcal{X}} \operatorname{Spec} L$ defined by the identity automorphism of u''. By (3.5.2), this is in turn equivalent to

$$\dim_{e''(u'')} R''_{u''} = \dim_{e(u)} R_u + \dim_{e'(u')} R'_{u'}$$

This last fact follows from the cartesian cube

$$\begin{tikzcd}
& R''_{u''} \arrow[rr] \arrow[dl] \arrow[dd] & & R'_{u'} \times_{\kappa(u')} L \arrow[dl] \arrow[dd] \\
U'' \arrow[rr] \arrow[dd] & & U' \arrow[dd] & \\
& R_u \times_{\kappa(u)} L \arrow[rr] \arrow[dl] & & \operatorname{Spec} L \arrow[dl] \\
U \arrow[rr] & & \mathcal{X}. &
\end{tikzcd}$$

and properties of dimension (see Exercise 3.5.3). □

Exercise 3.5.3.

(a) Show that the analogue of Proposition A.3.11 holds for algebraic spaces; that is, if $X \to Y$ is a smooth morphism of noetherian algebraic spaces, and if $x \in |X|$ is a point with image $y \in |Y|$, then

$$\dim_x(X) = \dim_y(Y) + \dim_x(X_y).$$

(b) If X and X' are noetherian algebraic spaces over a field $\Bbbk$ with $\Bbbk$-points x and x', show that

$$\dim_{(x,x')} X \times_{\Bbbk} X' = \dim_x X + \dim_{x'} X'.$$

(c) Let $\mathcal{X}$ be a noetherian algebraic space over a field $\Bbbk$ and $\Bbbk \to L$ be a field extension. Set $\mathcal{X}_L = \mathcal{X} \times_{\Bbbk} L$. If $x' \in |\mathcal{X}_L|$ is a point with image $x \in |\mathcal{X}|$, show that $\dim_{x'} \mathcal{X} \times_{\Bbbk} L = \dim_x \mathcal{X}$.

Example 3.5.4. If U is a scheme of pure dimension with an action of an affine algebraic group G (which is necessarily of pure dimension) over a field $\Bbbk$, then

$$\dim[U/G] = \dim U - \dim G.$$

In particular, the classifying stack has dimension $\dim \mathbf{B}G = -\dim G$ and we see that the dimension may be negative!

3.5.2 Tangent spaces

The *dual numbers* is the ring $\Bbbk[\epsilon] := \Bbbk[\epsilon]/\epsilon^2$ defined over a field $\Bbbk$.

Definition 3.5.5. If $\mathcal{X}$ is an algebraic stack and $x\colon \operatorname{Spec} \Bbbk \to \mathcal{X}$, we define the *Zariski tangent space* or simply the *tangent space* of $\mathcal{X}$ at x as the set

$$T_{\mathcal{X},x} := \left\{ \text{2-commutative diagrams } \begin{tikzcd} \operatorname{Spec} \Bbbk \arrow[d, hook] \arrow[dr, "x"] \arrow[d, phantom, shift left=3ex, "\swarrow\alpha"] & \\ \operatorname{Spec} \Bbbk[\epsilon] \arrow[r, "\tau"] & \mathcal{X} \end{tikzcd} \right\} \Big/ \sim$$

or in other words the set of pairs (τ, α) where $\tau\colon \operatorname{Spec} \Bbbk[\epsilon] \to \mathcal{X}$ and $\alpha\colon x \xrightarrow{\sim} \tau|_{\Bbbk}$. Two pairs are equivalent $(\tau, \alpha) \sim (\tau', \alpha')$ if there is an isomorphism $\beta\colon \tau \xrightarrow{\sim} \tau'$ in $\mathcal{X}(k[\epsilon])$ compatible with α and α', i.e. $\alpha' = \beta|_{\operatorname{Spec} \Bbbk} \circ \alpha$

Proposition 3.5.6. *If $\mathcal{X}$ is an algebraic stack with affine diagonal and $x \in \mathcal{X}(\Bbbk)$, then $T_{\mathcal{X},x}$ is naturally a $\Bbbk$-vector space.*

Proof. Scalar multiplication of $c \in \Bbbk$ on $(\tau, \alpha) \in T_{\mathcal{X},x}$ is defined as the composition $\operatorname{Spec} \Bbbk[\epsilon] \to \operatorname{Spec} \Bbbk[\epsilon] \xrightarrow{\tau} \mathcal{X}$ where the first map is defined by $\epsilon \mapsto c\epsilon$ and with the same 2-isomorphism α.

To define addition, we will show that there is an equivalence of categories

$$\mathcal{X}(\Bbbk[\epsilon_1] \times_{\Bbbk} \Bbbk[\epsilon_2]) \to \mathcal{X}(\Bbbk[\epsilon_1]) \times_{\mathcal{X}(\Bbbk)} \mathcal{X}(\Bbbk[\epsilon_2]) \tag{3.5.3}$$

or in other words that

$$\begin{tikzcd} \operatorname{Spec} \Bbbk \arrow[r, hook] \arrow[d, hook] & \operatorname{Spec} \Bbbk[\epsilon_1] \arrow[d, hook] \\ \operatorname{Spec} \Bbbk[\epsilon_2] \arrow[r, hook] & \operatorname{Spec}(\Bbbk[\epsilon_1] \times_{\Bbbk} \Bbbk[\epsilon_2]) \end{tikzcd}$$

is a pushout among algebraic stacks with affine diagonal (see §A.8). Once this is established, we define addition of (τ_1, α_1) and (τ_2, α_2) by the composition $\operatorname{Spec} \Bbbk[\epsilon] \to \operatorname{Spec}(\Bbbk[\epsilon_1] \times_{\Bbbk} \Bbbk[\epsilon_2]) \to \mathcal{X}$ where the first map is defined sending both $(\epsilon_1, 0)$ and $(0, \epsilon_2)$ to ϵ.

Choose a smooth morphism $(U, u) \to (\mathcal{X}, x)$ from an affine scheme U. Since $\mathcal{X}$ has affine diagonal $U \to \mathcal{X}$ is an affine morphism. Let $\operatorname{Spec} A_0 = \operatorname{Spec} \Bbbk \times_{\mathcal{X}} U$, $\operatorname{Spec} A_1 = \operatorname{Spec} \Bbbk[\epsilon_1] \times_{\mathcal{X}} U$ and $\operatorname{Spec} A_2 = \operatorname{Spec} \Bbbk[\epsilon_2] \times_{\mathcal{X}} U$. Since $\operatorname{Spec}(A_1 \times_A A_2)$ is clearly the pushout of $\operatorname{Spec} A_0 \hookrightarrow \operatorname{Spec} A_1$ and $\operatorname{Spec} A_0 \hookrightarrow \operatorname{Spec} A_2$ in the category of affine schemes, there are unique morphisms $\operatorname{Spec}(A_1 \times_A A_2) \to \operatorname{Spec}(\Bbbk[\epsilon_1] \times_{\Bbbk} \Bbbk[\epsilon_2])$ and $\operatorname{Spec}(A_1 \times_A A_2) \to U$ completing the diagram

$$\begin{tikzcd}
 & \operatorname{Spec} A_0 \arrow[rr] \arrow[dl] \arrow[dd] & & \operatorname{Spec} A_2 \arrow[dl] \arrow[dd] & \\
\operatorname{Spec} \Bbbk \arrow[rr] \arrow[dd] & & \operatorname{Spec} \Bbbk[\epsilon_1] \arrow[dd] \arrow[dddr, bend left, "\tau_1"] & & \\
 & \operatorname{Spec} A_1 \arrow[rr] \arrow[dl] & & \operatorname{Spec}(A_1 \times_A A_2) \arrow[dl] \arrow[dr] & \\
\operatorname{Spec} \Bbbk[\epsilon_2] \arrow[rr] \arrow[drrr, bend right, "\tau_2"'] & & \operatorname{Spec}(\Bbbk[\epsilon_1] \times_{\Bbbk} \Bbbk[\epsilon_2]) \arrow[dr, dashed] & & U \arrow[dl] \\
 & & & \mathcal{X} &
\end{tikzcd}$$

By the Flatness Criterion over Artinian Rings (Proposition A.2.3), we see that the map $\operatorname{Spec}(A_1 \times_A A_2) \to \operatorname{Spec}(\Bbbk[\epsilon_1] \times_{\Bbbk} \Bbbk[\epsilon_2])$ is faithfully flat. By repeating this argument on $U \times_{\mathcal{X}} U$, one argues that the $\operatorname{Spec}(A_1 \times_A A_2) \to U$ descends uniquely providing the desired dotted arrow. $\square$

Exercise 3.5.7. Show that $T_{\mathcal{X},x}$ is naturally a representation of G_x which is given set-theoretically by: $g \cdot (\tau, \alpha) = (\tau, g \circ \alpha)$ for $g \in G_x$ and $(\tau, \alpha) \in T_{\mathcal{X},x}$.

Example 3.5.8. Consider a smooth, connected, and projective curve $[C] \in \mathscr{M}_g(\Bbbk)$ defined over $\Bbbk$ of genus $g \geq 2$. Deformation theory (Proposition D.1.11) implies that $T_{\mathscr{M}_g,[C]} = H^1(C, T_C)$. Since $\deg T_C < 0$, $\mathrm{H}^0(C, T_C) = 0$ and Riemann–Roch implies

$$\dim T_{\mathscr{M}_g,[C]} = \dim \mathrm{H}^1(C, T_C) = -\chi(T_C) = -(\deg T_C + (1-g)) = 3g - 3.$$

Example 3.5.9. Let C be a smooth, connected, and projective curve over $\Bbbk$ and $E \in \mathcal{B}\mathrm{un}_{r,d}(C)(\Bbbk)$ be a vector bundle on C of rank r and degree d. Deformation theory (Proposition D.1.15) implies that $T_{\mathcal{B}\mathrm{un}_{r,d}(C),[E]} = \operatorname{Ext}^1_{\mathcal{O}_C}(E, E) = H^1(C, E \otimes E^\vee)$. By Riemann–Roch, $\chi(E \otimes E^\vee) = r^2(1-g)$. Since $\dim \operatorname{Aut}(E) = \dim_{\Bbbk} \operatorname{Hom}_{\mathcal{O}_C}(E, E) = \mathrm{H}^0(C, E \otimes E^\vee)$, we compute that

$$\dim T_{\mathcal{B}\mathrm{un}_{r,d}(C),[E]} = \dim \operatorname{Ext}^1_{\mathcal{O}_C}(F, F) = \dim \operatorname{Aut}(F) + r^2(g-1).$$

Exercise 3.5.10. Show that Proposition 3.5.6 remains true without the affine diagonal condition.

Remark 3.5.11. Suppose that $A \to A'$ and $A \to A''$ are homomorphisms of artinian local rings such that $A \twoheadrightarrow A'$ is surjective. If $\mathcal{X}$ is an algebraic stack, then the above argument extends to show that

$$\mathcal{X}(A_1 \times_{A_0} A_2) \to \mathcal{X}(A_1) \times_{\mathcal{X}(A_0)} \mathcal{X}(A_2) \tag{3.5.4}$$

is an equivalence of categories. This condition is usually referred as *homogeneity*. The conditions (RS_1)–(RS_2) in Rim–Schlessinger's Criteria (Theorem D.3.11) are weaker versions of homogeneity which ensure the existence of a formal miniversal deformation space, and also appear in Artin's Axioms for Algebraicity (Theorem D.7.4).

More generally, (3.5.4) holds if $A \to A'$ and $A \to A''$ are arbitrary ring homomorphisms with $A \twoheadrightarrow A'$ surjective which shows that the Ferrand pushout $\operatorname{Spec} A' \times_A A''$ (see Section A.8) is a pushout in the category of algebraic stacks.

3.5.3 Residual gerbes

Attached to every point $x \in X$ of a scheme, there is a residue field $\kappa(x)$ and a monomorphism $\operatorname{Spec} \kappa(x) \to X$ with image x. The residual gerbe will provide us with an analogous property for algebraic stacks. Note that the existence of non-trivial stabilizers prevents field-valued points from being monomorphisms (e.g. $\mathbf{B}G$ for a finite group G).

Definition 3.5.12. Let $\mathcal{X}$ be an algebraic stack and $x \in |\mathcal{X}|$ be a point. We say that the *residual gerbe at x exists* if there is a reduced noetherian algebraic stack $\mathcal{G}_x$ and a monomorphism $\mathcal{G}_x \hookrightarrow \mathcal{X}$ such that $|\mathcal{G}_x|$ is a point mapping to x. If it exists, we call $\mathcal{G}_x$ the *residual gerbe at x*.

In Lemma 6.2.49 we show that the residual gerbe $\mathcal{G}_x$ is unique while in Proposition 6.2.50 we show that $\mathcal{G}_x$ is a gerbe over a field $\kappa(x)$ (called the *residue field*). These two results justify the terminology: $\mathcal{G}_x$ is <u>the</u> residual <u>gerbe</u>.

Showing the existence of residual gerbes is fairly straightforward in the case of a *finite type* point. While this result suffices for most of our purposes, we will prove later that residual gerbes exist for any point of a quasi-separated algebraic stack in Proposition 6.2.50: this result is postponed to later as we will utilize the Fppf Criterion for Algebraicity (Theorem 6.2.1).

Definition 3.5.13. A point $x \in |\mathcal{X}|$ in an algebraic stack is *of finite type* if there exists a representative $\operatorname{Spec} \Bbbk \to \mathcal{X}$ locally of finite type.

Remark 3.5.14. If X is a noetherian scheme, a point $x \in X$ is of finite type if and only if $x \in X$ is locally closed. Slightly more generally, a morphism $\operatorname{Spec} \Bbbk \to X$ with image x is of finite type if and only if the image $x \in X$ is locally closed and $\kappa(x)/\Bbbk$ is a finite extension. Indeed, to see the nontrivial ($\Rightarrow$) implication, we replace X with $\overline{\{x\}}$, and since $\operatorname{Spec} \Bbbk \to X$ is of finite type with dense image, Generic Flatness (A.2.11) implies that $\operatorname{Spec} \Bbbk \to X$ is fppf and thus its image is open. Shortly we will establish an analogous property for points of noetherian algebraic stacks (Proposition 3.5.16).

An example of a finite type point of a scheme that is not closed is the generic point of a DVR. However, if X is a scheme of finite type over a field $\Bbbk$, then every finite type point is in fact a closed point. The analogous fact is *not* true for algebraic stacks of finite type over $\Bbbk$, e.g. $\operatorname{Spec} \Bbbk \xrightarrow{1} [\mathbb{A}^1/\mathbb{G}_m]$ is an open finite type point.

Exercise 3.5.15. Let $\mathcal{X}$ be an algebraic stack.

(a) Show that a point $x \in |\mathcal{X}|$ is of finite type if and only if there exists a scheme U, a closed point $u \in U$, and a smooth morphism $(U, u) \to (\mathcal{X}, x)$.

(b) Show that any algebraic stack (resp. quasi-compact algebraic stack) has a finite type point (resp. closed point).

Proposition 3.5.16. *If $\mathcal{X}$ is noetherian and $x \in \mathcal{X}$ is a finite type point, then the residual gerbe $\mathcal{G}_x$ exists at x and is a regular algebraic stack, and the morphism $\mathcal{G}_x \hookrightarrow \mathcal{X}$ is a locally closed immersion.*

If in addition $\mathcal{X}$ is of finite type over a field $\Bbbk$ and $x \in \mathcal{X}(k)$ has an affine smooth stabilizer G_x, then $\mathcal{G}_x = \mathbf{B}G_x$.

Proof. After replacing $\mathcal{X}$ with $\overline{\{x\}}$, we may assume that $\mathcal{X}$ is reduced and $x \in |\mathcal{X}|$ is dense. Let $\operatorname{Spec} \Bbbk \to \mathcal{X}$ be a finitely presented representative of x. By Generic

Flatness (Exercise 3.3.30), $\operatorname{Spec} \Bbbk \to \mathcal{X}$ is flat and therefore its image—which is $x \in |\mathcal{X}|$—is open (Exercise 3.3.25). The corresponding open substack $\mathcal{G}_x \subset \mathcal{X}$ is the residual gerbe. Since $\operatorname{Spec} \Bbbk \to \mathcal{G}_x$ is fppf and the property of being regular descends under fppf morphisms (Proposition B.4.4), $\mathcal{G}_x$ is regular.

For the addendum, there is a monomorphism of prestacks $\mathbf{B}G_x^{\mathrm{pre}} \to \mathcal{X}$: for a $\Bbbk$-scheme T, there is a unique object of $\mathbf{B}G_x^{\mathrm{pre}}$ over T, and this object gets mapped to the composition $T \to \operatorname{Spec} \Bbbk \xrightarrow{x} \mathcal{X}$. Similarly, a morphism over $T' \to T$ corresponds to a map $T' \to G_x$ and this gets mapped to the corresponding morphism in $\mathcal{X}$. Under stackification, this induces a monomorphism $\mathbf{B}G_x \to \mathcal{X}$. By the same argument as above, $\mathbf{B}G_x \hookrightarrow \mathcal{X}$ is locally closed. As $\mathcal{G}_x$ and $\mathbf{B}G_x$ are reduced locally closed substacks, they must be equal. $\square$

Exercise 3.5.17. Show that if $\mathcal{X}$ is an algebraic stack and $x \in \mathcal{X}$ is a finite type point such that the stabilizer is unramified (i.e. the stabilizer group scheme of any representative is unramified), then the residual gerbe $\mathcal{G}_x$ exists and is regular, and $\mathcal{G}_x \hookrightarrow \mathcal{X}$ is locally of finite type.

Corollary 3.5.18. *Let $x \in |\mathcal{X}|$ be a finite type point of a noetherian algebraic stack $\mathcal{X}$. If $(U, u) \to (\mathcal{X}, x)$ is a smooth morphism from a scheme U with $u \in U$ a finite type point, then there is a cartesian diagram*

$$\begin{tikzcd} O(u) \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & U \arrow[d] \\ \mathcal{G}_x \arrow[r, hook] & \mathcal{X} \end{tikzcd} \tag{3.5.5}$$

where $O(u)$ is identified set-theoretically with the orbit $s(t^{-1}(u))$ of the induced groupoid $s, t\colon R := U \times_{\mathcal{X}} U \rightrightarrows U$. $\square$

Remark 3.5.19. If $\mathcal{X} = [U/G]$ is the quotient stack of an affine algebraic group over a field $\Bbbk$ acting on a noetherian $\Bbbk$-scheme U and $u \in U(\Bbbk)$, there is a cartesian diagram

$$\begin{tikzcd} Gu \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & U \arrow[d] \\ \mathbf{B}G_x \arrow[r, hook] & {[U/G].} \end{tikzcd}$$

We recover the familiar fact that orbit $Gu \hookrightarrow U$ is locally closed (Algebraic Group Facts C.3.1(7)).

Corollary 3.5.20. *A finite type point $x \in |X|$ of a noetherian algebraic space has a residue field $\kappa(x)$, i.e. there is a field $\kappa(x)$ and a locally closed immersion* $\operatorname{Spec} \kappa(x) \hookrightarrow X$ *with image x.* $\square$

Exercise 3.5.21. Let $\mathcal{X}$ be a noetherian algebraic stack and $x \in |\mathcal{X}|$ be a finite type point with smooth stabilizer. Let $\overline{x}\colon \operatorname{Spec} \Bbbk \to \mathcal{X}$ be a representative of x. Show that $\dim \mathcal{G}_x = -\dim G_{\overline{x}}$.

3.6 Characterization of Deligne–Mumford stacks

3.6.1 Existence of minimal presentations

Theorem 3.6.1 (Existence of Minimal Presentations). *Let $\mathcal{X}$ be a noetherian algebraic stack and let $x \in |\mathcal{X}|$ be a finite type point with smooth stabilizer G_x. Then there exist a scheme U with a closed point $u \in U$ and a smooth morphism $(U, u) \to (\mathcal{X}, x)$ of relative dimension $\dim G_x$ from a scheme U such that the diagram*

$$\begin{tikzcd} \operatorname{Spec} \kappa(u) \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & U \arrow[d] \\ \mathcal{G}_x \arrow[r, hook] & \mathcal{X} \end{tikzcd}$$

is cartesian.

In particular, if G_x is finite and reduced, there is an étale morphism $(U, u) \to (\mathcal{X}, x)$ from a scheme.

Proof. Let $(U, u) \to (\mathcal{X}, x)$ be a smooth morphism of relative dimension n from a scheme U such that $u \in U$ is a finite type point. From Proposition 3.5.16, the residual gerbe $\mathcal{G}_x$ at x exists and is regular of dimension $-\dim G_x$ (Exercise 3.5.21). We obtain a cartesian diagram

$$\begin{tikzcd} O(u) \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & U \arrow[d] \\ \mathcal{G}_x \arrow[r, hook] & \mathcal{X}. \end{tikzcd}$$

It follows that $O(u)$ is a regular scheme of dimension $c := n - \dim G_x$. Let $f_1, \ldots, f_c \in \mathcal{O}_{O(u),u}$ be a regular sequence generating the maximal ideal at u. After replacing U with an open affine neighborhood of u, we may assume that each f_i is a global function on U. We can consider the closed subscheme $W := V(f_1, \ldots, f_c)$ which by design intersects $O(u)$ transversely at U, i.e. $W \cap O(u) = \operatorname{Spec} \kappa(u)$ scheme-theoretically.

By inductively applying a version of the local criterion for flatness (Corollary A.2.8) to the smooth groupoid $U \times_{\mathcal{X}} U \rightrightarrows U$ at a preimage of u and the applying smooth descent, we conclude that the composition $W \hookrightarrow U \to \mathcal{X}$ is flat at u. Since G_x is smooth, so is $\operatorname{Spec} \kappa(u) \to \mathcal{G}_x$. For flat morphisms, smoothness is a property that can be checked on fibers and thus (again arguing on $R \rightrightarrows U$ and using descent) $W \to \mathcal{X}$ is smooth at u. The statement follows after replacing W with an open neighborhood of u. $\square$

Remark 3.6.2. A smooth presentation $p\colon U \to \mathcal{X}$ is called a *miniversal at* $u \in U(\Bbbk)$ if $T_{U,u} \to T_{\mathcal{X},p(u)}$ is an isomorphism of $\Bbbk$-vector spaces. We will see that the above presentations are miniversal in Proposition 3.7.3.

If the stabilizer G_x is not smooth, there are two candidates for 'minimal presentations'. There still exists a miniversal presentation $(U, u) \to (\mathcal{X}, x)$, but its relative dimension is equal to the dimension of the Lie algebra of G_x (rather than $\dim G_x$) and the fiber product $\mathcal{G}_x \times_{\mathcal{X}} U$ may be positive dimensional. For example, $\mathbf{B}\boldsymbol{\mu}_p$ is an algebraic stack in characteristic p (Proposition 6.2.9) and it can be realized as the quotient of $\mathbb{G}_m$ acting on itself via $t \cdot x = t^p x$; here

$\mathbb{G}_m \to \mathbf{B}\boldsymbol{\mu}_p$ is a a miniversal presentation. On the other hand, there is an fppf (but not smooth) morphism $(U, u) \to (\mathcal{X}, x)$ such that $\mathcal{G}_x \times_{\mathcal{X}} U \cong \operatorname{Spec} \kappa(u)$. In particular, if $\mathcal{X}$ has quasi-finite diagonal, then there is an fppf and quasi-finite morphism $(U, u) \to (\mathcal{X}, x)$. In our example, the map $\operatorname{Spec} \Bbbk \to \mathbf{B}\boldsymbol{\mu}_p$ is such a presentation.

Exercise 3.6.3. If $\mathcal{X}$ is a (possibly non-noetherian) algebraic stack and $x \in \mathcal{X}$ is a finite type point with unramified stabilizer, show that there is an étale morphism $(U, u) \to (\mathcal{X}, x)$ from a scheme U where $u \in U$ is a closed point.

Hint: Replicate the argument above using Exercise 3.5.17.

3.6.2 Equivalent characterizations

Theorem 3.6.4 (Characterization of Deligne–Mumford Stacks)**.** *Let $\mathcal{X}$ be an algebraic stack. The following are equivalent:*

(1) the stack $\mathcal{X}$ is a Deligne–Mumford;

(2) the diagonal $\mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is unramified; and

(3) every point of $\mathcal{X}$ has a finite and reduced stabilizer group.

Proof. The equivalence (2) $\iff$ (3) is essentially the definition of unramified: since the diagonal $\mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is always locally of finite type (Exercise 3.2.4), it is unramified if and only if every geometric fiber (which is either empty or isomorphic to a stabilizer) is discrete and reduced. It is not hard to see that a Deligne–Mumford stack has unramified diagonal (Exercise 3.2.8). For the converse, Existence of Minimal Presentations (Theorem 3.6.1 and Exercise 3.6.3) shows that for every finite type point $x \in \mathcal{X}$, there is an étale morphism $U \to \mathcal{X}$ from a scheme whose image contains x. Thus $\mathcal{X}$ is Deligne–Mumford.

See [LMB, Thm 8.1] and [SP, Tag 06N3]. □

Theorem 3.6.5 (Characterization of Algebraic Spaces)**.** *Let $\mathcal{X}$ be an algebraic stack whose diagonal is representable by schemes. The following are equivalent:*

(1) the stack $\mathcal{X}$ is an algebraic space;

(2) the diagonal $\mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is a monomorphism; and

(3) every point of $\mathcal{X}$ has a trivial stabilizer.

Remark 3.6.6. We will remove the pesky hypothesis that $\Delta_{\mathcal{X}}$ is representable by schemes in Theorem 4.4.10.

Proof. Condition (2) is equivalent to the condition that $\mathcal{X}$ is a sheaf. The implication (1) $\Rightarrow$ (2) follows from the definition of an algebraic space. For the converse, if $\mathcal{X}$ is a sheaf, then Theorem 3.6.1 implies that there exists a surjective, étale, and representable morphism $U \to \mathcal{X}$ from a scheme. Since $\Delta_{\mathcal{X}}$ is representable by schemes, so is $U \to \mathcal{X}$.

The equivalence (2) $\iff$ (3) follows from the fact that a group scheme of finite type is trivial if and only if every fiber is trivial (Proposition C.1.6). □

Corollary 3.6.7. *Let $G \to S$ be a smooth and affine group scheme over a scheme S. Let U be an algebraic space over S with an action of G. Then*

(1) $[U/G]$ is Deligne–Mumford $\iff$ every point of U has a discrete and reduced stabilizer group $\iff$ the action map $G \times U \to U \times U$ is unramified.

(2) $[U/G]$ *is an algebraic space* $\iff$ *every point of* U *has a trivial stabilizer group* $\iff$ *the action map* $G \times U \to U \times U$ *is a monomorphism.*

$\square$

Corollary 3.6.8. *If* $g \geq 2$, $\mathscr{M}_g$ *is a Deligne–Mumford of finite type over* $\mathbb{Z}$ *with affine diagonal.*

Proof. It only remains to show that $\mathscr{M}_g$ is Deligne–Mumford and by Theorem 3.6.4 it suffices to show that for every smooth, connected and proper curve C over $\Bbbk$ that $G = \mathrm{Aut}(C)$ is discrete and reduced, or in other words that the dimension of the Lie algebra $\dim T_{G,e} = 0$. The vector space $T_{G,e}$ is identified with the automorphism group of the trivial first order deformation of C. Deformation theory (Proposition D.2.6) implies that $T_{G,e} = \mathrm{H}^0(C, T_C)$, but this vector space is zero since the degree of $T_C = \Omega_C^\vee$ is $2 - 2g < 0$. $\square$

3.7 Smoothness and the Infinitesimal Lifting Criteria

We state and prove the Infinitesimal Lifting Criteria (Theorem 3.7.1) which provides an extremely useful functorial criteria to check that moduli stacks are smooth. We apply this criteria to establish that the moduli stacks $\mathscr{M}_g$ of smooth curves and $\mathcal{B}\mathrm{un}_{r,d}(C)$ of vector bundles are smooth (Propositions 3.7.4 and 3.7.5)

3.7.1 Infinitesimal Lifting Criteria

Since flatness and smoothness are smooth-local properties on the source and target, we have the notions of smoothness and flatness for arbitrary morphisms of algebraic stacks (Definition 3.3.2). Since étaleness and unramifiedness are étale-local on the source and smooth-local on the target, we can make sense of étale or unramified morphisms of algebraic stacks; see Definition 3.3.33.

The following criteria will be our means for establishing that moduli stacks are smooth.

Theorem 3.7.1 (Infinitesimal Lifting Criteria for Unramifed/Étale/Smooth Morphisms)**.** *Let* $f \colon \mathcal{X} \to \mathcal{Y}$ *be a finite type morphism of noetherian algebraic stacks. Consider 2-commutative diagrams*

$$\begin{tikzcd} \operatorname{Spec} A_0 \arrow[r] \arrow[d, hook] & \mathcal{X} \arrow[d, "f"] \\ \operatorname{Spec} A \arrow[r] \arrow[ur, dashed] & \mathcal{Y}, \end{tikzcd} \tag{3.7.1}$$

of solid arrows where $A \to A_0$ *is a surjection of artinian local rings with residue field* $\Bbbk$ *such that* $\ker(A \to A_0) \cong \Bbbk$ *and* $\operatorname{Spec} \Bbbk \hookrightarrow \operatorname{Spec} A_0 \to \mathcal{X}$ *is a finite type point.*

Then

(1) f *is unramified if and only if for every 2-commutative diagram* (3.7.1)*, any two liftings are isomorphic.*

(2) *f is étale if and only if for every 2-commutative diagram (3.7.1), there exists a lifting which is unique up to unique isomorphism.*

(3) *f is smooth if and only if for every 2-commutative diagram (3.7.1), there exists a lifting.*

Remark 3.7.2. To be explicit, a *lifting* of a 2-commutative diagram

$$\begin{tikzcd} S \arrow[r, "x"] \arrow[d, "g"'] \arrow[dr, phantom, "\swarrow\alpha"] & \mathcal{X} \arrow[d, "f"] \\ T \arrow[r, "y"'] & \mathcal{Y}, \end{tikzcd} \tag{3.7.2}$$

is the data of a morphism $\widetilde{x}\colon T \to \mathcal{X}$ as pictured

$$\begin{tikzcd} S \arrow[r, "x"] \arrow[d, "g"'] \arrow[dr, phantom, "\beta\Uparrow \quad \Downarrow\gamma"] & \mathcal{X} \arrow[d, "f"] \\ T \arrow[r, "y"'] \arrow[ur, "\widetilde{x}"] & \mathcal{Y}, \end{tikzcd}$$

together with 2-morphisms $\beta\colon \widetilde{x} \circ g \xrightarrow{\sim} x$ and $\gamma\colon f \circ \widetilde{x} \xrightarrow{\sim} y$ such that

$$\begin{tikzcd} & f \circ x \arrow[dl, "\alpha"'] & \\ y \circ g & & f \circ \widetilde{x} \circ g \arrow[ul, "f(\beta)"'] \arrow[ll, "g^*\gamma"] \end{tikzcd}$$

commutes. A morphism $(\widetilde{x}, \beta, \gamma) \to (\widetilde{x}', \beta', \gamma')$ of liftings is a 2-morphism $\Theta\colon \widetilde{x} \to \widetilde{x}'$ such that $\beta = \beta' \circ (\Theta \circ g)$ and $\gamma = \gamma' \circ f(\Theta)$.

We can also interpret liftings using the map $\Psi\colon \mathcal{X}(T) \to \mathcal{X}(S) \times_{\mathcal{Y}(S)} \mathcal{Y}(T)$ of groupoids. The 2-commutativity of (3.7.2) defines an object $(x, y, \alpha) \in \mathcal{X}(S) \times_{\mathcal{Y}(S)} \mathcal{Y}(T)$ and the category of liftings is the fiber category over this object, e.g. a lifting is an object $\widetilde{x} \in \mathcal{X}(T)$ together with an isomorphism $\Psi(\widetilde{x}) \to (x, y, \alpha)$. For instance, the existence of a lifting translates to the essential surjectivity of $\mathcal{X}(T) \to \mathcal{X}(S) \times_{\mathcal{Y}(S)} \mathcal{Y}(T)$.

Proof. We handle the smooth case and leave the remaining cases to the reader. We first show that smoothness implies formally smoothness, i.e. every 2-commutative diagram (3.7.1) has a lifting. By replacing $\mathcal{X}$ with $\mathcal{X} \times_{\mathcal{Y}} \operatorname{Spec} A$ and $\mathcal{Y}$ with $\operatorname{Spec} A$, we may assume that $\mathcal{Y}$ is affine and we need to show that a section over $\operatorname{Spec} A_0$

$$\begin{tikzcd} & \mathcal{X} \arrow[d] \\ \operatorname{Spec} A_0 \arrow[ur] \arrow[r, hook] & \operatorname{Spec} A \arrow[u, dashed, bend right] \end{tikzcd}$$

extends to a section over $\operatorname{Spec} A$.

If $\mathcal{X}$ is a scheme, then the existence of a lifting is provided by the Infinitesimal Lifting Criterion for Smoothness (A.3.1) for schemes. If $\mathcal{X} = X$ is an algebraic space, we may choose a étale presentation $U \to X$ from a scheme. Since $U \to X$ is representable by schemes, it is formally smooth and we may lift $\operatorname{Spec} A_0 \to X$

to $\operatorname{Spec} A_0 \to U$. The composition $U \to X \to \operatorname{Spec} A$ is a smooth morphism of schemes, thus formally smooth, and we can lift the section over $\operatorname{Spec} A_0$ to a section over $\operatorname{Spec} A$. In general, if $\mathcal{X}$ is an algebraic stack, we can choose a smooth morphism $U \to \mathcal{X}$ from an algebraic space and a lifting $\operatorname{Spec} \Bbbk \to U$ of $\operatorname{Spec} \Bbbk \hookrightarrow \operatorname{Spec} A_0 \to \mathcal{X}$ (Proposition 4.2.15). Since we've already shown that smooth representable morphisms are formally smooth, there is a lifting $\operatorname{Spec} A_0 \to U$ of $\operatorname{Spec} A_0 \to \mathcal{X}$. Now $U \to \mathcal{X} \to \operatorname{Spec} A$ is a smooth morphism of schemes so we see that there is a section extending $\operatorname{Spec} A_0 \to U$.

Conversely, if $\mathcal{X} \to \mathcal{Y}$ is formally smooth, then choose smooth presentation $V \to \mathcal{Y}$ and $U \to \mathcal{X} \times_{\mathcal{Y}} V$. By the above argument, $U \to \mathcal{X} \times_{\mathcal{Y}} V$ is formally smooth. Since $\mathcal{X} \times_{\mathcal{Y}} V \to V$ is formally smooth, so is the composition $U \to \mathcal{X} \times_{\mathcal{Y}} V$. But as $U \to V$ is a morphism of schemes, it is formally smooth (Smooth Equivalences A.3.1). Since smoothness is a smooth-local property on the source and target, we obtain that $\mathcal{X} \to \mathcal{Y}$ is smooth.

See also [LMB, 4.15(ii)] and [SP, Tag 0DP0]. □

As a first application, we see that the presentations produced by Existence of Minimal Presentations (Theorem 3.6.1) are in fact miniversal, and that the dimension of a smooth algebraic stack can be computed in terms of its tangent space and stabilizer.

Proposition 3.7.3. *Let $\mathcal{X}$ be a noetherian algebraic stack and $x \in |\mathcal{X}|$ be a finite type point with smooth stabilizer. Let $f\colon (U,u) \to (\mathcal{X},x)$ be a smooth morphism from a scheme such that $\mathcal{G}_x \times_{\mathcal{X}} U \cong \operatorname{Spec} \kappa(u)$. Then $U \to \mathcal{X}$ is miniversal at u, i.e. $T_{U,u} \to T_{\mathcal{X},f(u)}$ is an isomorphism of $\kappa(u)$-vector spaces.*

In particular, if $\mathcal{X}$ is a smooth over a field $\Bbbk$ and $x \in \mathcal{X}(\Bbbk)$ is a point with smooth stabilizer. Then

$$\dim_x \mathcal{X} = \dim T_{\mathcal{X},x} - \dim G_x.$$

Proof. Surjectivity of $T_{U,u} \to T_{\mathcal{X},f(u)}$ follows from the Infinitesimal Lifting Criterion (Theorem 3.7.1). Let $\Bbbk = \kappa(u)$. Injectivity follows from the fact that

$$\begin{tikzcd} \operatorname{Spec} \Bbbk \arrow[r, hook] \arrow[d] & U \arrow[d] \\ \mathcal{G}_x \arrow[r, hook] & \mathcal{X} \end{tikzcd}$$

is cartesian. Indeed, if $\tau\colon \operatorname{Spec} \Bbbk[\epsilon] \to U$ is an element of $T_{U,u}$ mapping to $0 \in T_{\mathcal{X},f(u)}$, then by the definition of the residual gerbe, the composition $\operatorname{Spec} \Bbbk[\epsilon] \to U \to \mathcal{X}$ factors through $\mathcal{G}_x$ and therefore also factors through the fiber product $\operatorname{Spec} \Bbbk$. We conclude that $\tau = 0$.

For the last statement, Existence of Minimal Presentations (Theorem 3.6.1) produces a smooth morphism $(U,u) \to (\mathcal{X},x)$ miniversal at u and whose relative dimension is equal to $\dim G_x$. Therefore $\dim_x \mathcal{X} = \dim_u U - \dim G_x$ but since U is smooth at u, we have $\dim_u U = \dim T_{U,u} = \dim T_{\mathcal{X},x}$. □

3.7.2 Smoothness of moduli stacks

The Infinitesimal Lifting Criterion for Smoothness combined with deformation theory allows us to verify the smoothness of a moduli problem and to compute its dimension.

Proposition 3.7.4. *For $g \geq 2$, the Deligne–Mumford stack $\mathscr{M}_g$ is smooth over* $\operatorname{Spec}\mathbb{Z}$ *of relative dimension $3g-3$.*

Proof. Let $\operatorname{Spec}\Bbbk \to \mathscr{M}_g$ be a morphism from a field $\Bbbk$ corresponding to smooth projective and connected curve $C \to \operatorname{Spec}\Bbbk$. Consider a diagram

$$\begin{tikzcd} \operatorname{Spec}\Bbbk \arrow[r] \arrow[rr, bend left, "{[C]}"] & \operatorname{Spec}A_0 \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\not\Downarrow\alpha"] & \mathscr{M}_g \arrow[d, "f"] \\ & \operatorname{Spec}A \arrow[r] & \operatorname{Spec}\mathbb{Z}, \end{tikzcd} \tag{3.7.3}$$

where $A \to A_0$ is surjection of artinian local rings with residue field $\Bbbk$ such that $\Bbbk = \ker(A \to A_0)$. The map $\operatorname{Spec}A_0 \to \mathscr{M}_g$ corresponds to a family of curves $\mathcal{C}_0 \to \operatorname{Spec}A_0$ and a cartesian diagram

$$\begin{tikzcd} C \arrow[r, hook] \arrow[d] & \mathcal{C}_0 \arrow[r, hook, dashed] \arrow[d] & \mathcal{C} \arrow[d, dashed] \\ \operatorname{Spec}\Bbbk \arrow[r, hook] & \operatorname{Spec}A_0 \arrow[r, hook] & \operatorname{Spec}A \end{tikzcd}$$

of solid arrows: a lifting of the diagram (3.7.3) corresponds to a family $\mathcal{C} \to \operatorname{Spec}A$ extending $\mathcal{C}_0 \to \operatorname{Spec}A_0$. By Proposition D.2.6, there is cohomology class $\mathrm{ob}_C \in \mathrm{H}^2(C, T_C)$ such that $\mathrm{ob}_C = 0$ if and only if there exists a lifting. Since C is a curve, $\mathrm{H}^2(C, T_C) = 0$. Finally, deformation theory gives the identification $T_{\mathscr{M}_g,[C]} = \mathrm{H}^1(C, T_C)$ which has dimension $3g-3$ by Riemann–Roch (see Example 3.5.8). Since $\dim \operatorname{Aut}(C) = 0$, we conclude that $\dim_{[C]} \mathscr{M}_g = 3g-3$. □

Proposition 3.7.5. *The algebraic stack $\mathcal{B}\mathrm{un}_{r,d}(C)$ is smooth over* $\operatorname{Spec}\Bbbk$ *of dimension $r^2(g-1)$.*

Proof. Let $[F] \in \mathcal{B}\mathrm{un}_{r,d}(C)(\Bbbk)$ be a vector bundle on C of rank r and degree d. Let $A \to A_0$ be a surjection of artinian local rings with residue field $\Bbbk$ such that $\Bbbk = \ker(A \to A_0)$. We need to check that every vector bundle $\mathcal{F}_0$ on C_{A_0} that restricts to F extends to a vector bundle $\mathcal{F}$ on C_A. By deformation theory (Proposition D.2.15), there is an element $\mathrm{ob}_F \in \operatorname{Ext}^2_{\mathcal{O}_C}(F,F)$ such that $\mathrm{ob}_F = 0$ if and only if there exists an extension. Since C is a smooth curve, $\operatorname{Ext}^2_{\mathcal{O}_C}(F,F) = \mathrm{H}^2(C_\Bbbk, F \otimes F^\vee) = 0$. Deformation theory also provides an identification $T_{\mathcal{B}\mathrm{un}_{r,d}(C),[F]} = \operatorname{Ext}^1_{\mathcal{O}_C}(F,F)$ and a Riemann–Roch calculation yields $\dim \operatorname{Ext}^1_{\mathcal{O}_C}(F,F) = \dim \operatorname{Aut}(F) + r^2(g-1)$ (see Example 3.5.9). Therefore $\dim_{[F]} \mathcal{B}\mathrm{un}_{r,d}(C) = \dim \operatorname{Ext}^1_{\mathcal{O}_C}(F,F) - \dim \operatorname{Aut}(F) = r^2(g-1)$. □

3.8 Properness and the valuative criterion

With some care, we define separatedness and properness for morphisms of algebraic stacks. Recall from Definition 3.3.10 that we say a representable morphism $\mathcal{X} \to \mathcal{Y}$ of algebraic stacks is *separated* if the diagonal $\mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ (which is representable by schemes) is proper.

Definition 3.8.1.

(1) A morphism $\mathcal{X} \to \mathcal{Y}$ of algebraic stacks is *universally closed* if for every morphism $\mathcal{Y}' \to \mathcal{Y}$ of algebraic stacks, the morphism $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}' \to \mathcal{Y}'$ induces a closed map $|\mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}'| \to |\mathcal{Y}'|$.
(2) A representable morphism $\mathcal{X} \to \mathcal{Y}$ of algebraic stacks is *proper* if it is universally closed, separated and of finite type.
(3) A morphism $\mathcal{X} \to \mathcal{Y}$ of algebraic stacks is *separated* if the representable morphism $\mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ is proper.
(4) A morphism $\mathcal{X} \to \mathcal{Y}$ of algebraic stacks is *proper* if it is universally closed, separated and of finite type.

Remark 3.8.2. Notice that we have not defined properness by requiring the diagonal is a closed immersion as with schemes. Indeed, the diagonal of a morphism of algebraic stacks is not a monomorphism. For schemes or algebraic spaces, the diagonal is proper if and only if it is a closed immersion; this follows from the fact that proper monomorphisms of schemes are closed immersions.

Remark 3.8.3. The property of being universally closed is smooth-local on the target. We thus have equivalences: $\mathcal{X} \to \mathcal{Y}$ is universally closed $\iff$ $|\mathcal{X} \times_{\mathcal{Y}} T| \to |T|$ is closed for all maps $T \to \mathcal{Y}$ from affine schemes $\iff$ for a smooth presentation $V \to \mathcal{Y}$, the base change $\mathcal{X} \times_{\mathcal{Y}} V \to V$ is universally closed.

Remark 3.8.4. Recall that the stabilizer G_x of a field-valued point $x\colon \operatorname{Spec} \Bbbk \to \mathcal{X}$ is given by the cartesian diagram

$$\begin{tikzcd} G_x \arrow[r] \arrow[d] & \operatorname{Spec} \Bbbk \arrow[d, "{(x,x)}"] \\ \mathcal{X} \arrow[r] & \mathcal{X} \times \mathcal{X}. \end{tikzcd}$$

If $\mathcal{X}$ is a separated algebraic stack over a scheme S, then G_x is a proper group algebraic space over $\Bbbk$. If an addition $\mathcal{X}$ has affine diagonal, then the stabilizer group G_x is proper and affine, thus finite. Since $\mathcal{B}\mathrm{un}_{r,d}(C)$ has affine diagonal (Example 3.3.12) and infinite automorphism groups, we see that $\mathcal{B}\mathrm{un}_{r,d}(C)$ is *not* separated.

We now state the valuative criteria using the notion of liftings defined formally in Remark 3.7.2. For moduli problems, the valuative criterion translates to the geometric question whether a family of objects over a punctured curve extend to the entire curve. We will apply the valuative criterion later to verify that $\overline{\mathcal{M}}_g$ is proper (Theorem 5.5.3) and that $\mathcal{B}\mathrm{un}^{\mathrm{ss}}_{r,d}$ is universally closed.

Theorem 3.8.5 (Valuative Criteria for Universally Closed/Separated/Proper Morphisms)**.** *Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a finite type morphism of noetherian algebraic stacks with separated diagonals. Consider a 2-commutative diagram*

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] \arrow[dr, phantom, "\not\Downarrow_{\alpha}"] & \mathcal{X} \arrow[d, "f"] \\ \operatorname{Spec} R \arrow[r] & \mathcal{Y} \end{tikzcd} \tag{3.8.1}$$

where R is a DVR with fraction field K. Then

(1) *f is proper if and only if for every diagram (3.8.1), there exists an extension $R \to R'$ of DVRs with the map $K \to K'$ on fraction fields having finite transcendence degree and a lifting unique up to unique isomorphism*

$$\begin{tikzcd} \operatorname{Spec} K' \arrow[r] \arrow[d] & \operatorname{Spec} K \arrow[r] \arrow[d] & \mathcal{X} \arrow[d, "f"] \\ \operatorname{Spec} R' \arrow[r] \arrow[urr, dashed] & \operatorname{Spec} R \arrow[r] & \mathcal{Y}. \end{tikzcd} \tag{3.8.2}$$

(2) *f is separated if and only if every two liftings of a diagram (3.8.1) are uniquely isomorphic.*

(3) *f has separated diagonal if and only if every lifting of a diagram (3.8.1) has no non-trivial automorphisms.*

(4) *f is universally closed if for every diagram (3.8.1), there exists an extension $R \to R'$ of DVRs with the map $K \to K'$ on fraction fields having finite transcendence degree and a lifting as in (3.8.2).*

Remark 3.8.6. See also [LMB, Thm. 7.10], [SP, Tags 0CLV and 0CLY] and [Fal03, §4].

We modify the proof of the valuative criterion for schemes (see §A.4). The starting point is the following lifting criterion for closed morphisms generalizing Lemma A.4.1.

Lemma 3.8.7. *Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a quasi-compact morphism of algebraic stacks. Then f is closed if and only for every point $x \in |\mathcal{X}|$, every specialization $f(x) \rightsquigarrow y_0$ lifts to a specialization $x \rightsquigarrow x_0$.*

Proof. The statement is equivalent to the equality that $f(\overline{\{x\}}) = \overline{\{f(x)\}}$ (Exercise 3.3.29(a)). ☐

To compare specializations to maps from DVRs, we have the following analogue of Proposition A.4.2.

Proposition 3.8.8. *If $f\colon \mathcal{X} \to \mathcal{Y}$ is a finite type morphism of noetherian schemes, $x \in |\mathcal{X}|$ and $f(x) \rightsquigarrow y_0$ is a specialization, then there exists a diagram*

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & \mathcal{X} \arrow[d, "f"] & x \arrow[d, mapsto] & \\ \operatorname{Spec} R \arrow[r] & \mathcal{Y} & f(x) \arrow[r, rightsquigarrow] & y_0. \end{tikzcd}$$

where R is a DVR with fraction field K, the image of $\operatorname{Spec} K \to \mathcal{X}$ is x and $\operatorname{Spec} R \to \mathcal{Y}$ realizes the specialization $f(x) \rightsquigarrow y_0$. In particular, every specialization $x \rightsquigarrow x_0$ in a noetherian algebraic stack is realized by a map $\operatorname{Spec} R \to \mathcal{X}$ from a DVR.

Proof. Let $V \to \mathcal{Y}$ be a smooth presentation and let $v_0 \in V$ be a preimage of y_0. Since $V \to \mathcal{Y}$ is smooth, it is an open morphism (Exercise 3.3.25) and thus there exists a specialization $v \rightsquigarrow v_0$ over $f(x) \rightsquigarrow y_0$ (Exercise 3.3.29(c)). Let $x' \in |\mathcal{X}_V|$ be a preimage of $v \in V$ and $x \in |\mathcal{X}|$. Let $U \to \mathcal{X}_V$ be a smooth presentation and $u \in U$ be a preimage of x'. Applying Proposition A.4.2 to the morphism $U \to V$ of schemes with $u \mapsto v$ and the specialization $v \rightsquigarrow v_0$ gives the desired diagram. ☐

Proof of Theorem 3.8.5. We first show that the universally closed valuative criterion implies the others. The double diagonal $\Delta_{\Delta_f} \colon \mathcal{X} \times_{\mathcal{X} \times_{\mathcal{Y}} \mathcal{X}} \mathcal{X}$ is separated and finite type. Thus f has separated diagonal if and only if Δ_{Δ_f} universally closed, and the existence of a lift for Δ_{Δ_f} translates to the condition that every lift for f has only trivial automorphisms. Assuming f has separated diagonal, then f is separated if and only if the diagonal $\Delta_f \colon \mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ is universally closed (as Δ_f is finite type), and the existence of a lift for Δ_f translates to the condition that any two lifts for f are isomorphic.

Suppose that the valuative criterion for universally closedness holds and that $f \colon \mathcal{X} \to \mathcal{Y}$ is not universally closed. Since the property of a morphism of algebraic stacks being closed can be checked smooth-locally on the target, we can assume that $\mathcal{Y} = Y$ is a scheme and that there exists a morphism $T \to Y$ from a scheme such that $f_T \colon \mathcal{X}_T \to T$ is not closed. We will reduce to the case that $T \to Y$ is a *finite type* morphism. By Lemma 3.8.7, there exists $z \in |\mathcal{X}_T|$ and a specialization $f_T(z) \rightsquigarrow t_0$ which doesn't lift to a specialization $z \rightsquigarrow z_0$. This implies that $\mathcal{Z} = \overline{\{z\}} \subset \mathcal{X}_T$ has trivial intersection with the fiber $(\mathcal{X}_T)_{t_0}$. If $p \colon X \to \mathcal{X}$ is a smooth presentation, then the preimage Z of $\mathcal{Z}$ under $X_T \to \mathcal{X}_T$ does not meet the fiber $(X_T)_{t_0}$. Applying Lemma A.4.6 implies that after replacing T with an open neighborhood of t_0, the morphism $T \to Y$ factors through a finite type morphism $T' \to Y$ via $g \colon T \to T'$ and that there exists a closed subscheme $Z' \subset X_{T'}$ with trivial intersection with the fiber $(X_{T'})_{g(t_0)}$ such that $\operatorname{im}(Z \hookrightarrow X_T \to X_{T'}) \subset Z'$. Letting $z' \in |\mathcal{X}_{T'}|$ be the image of $z \in |\mathcal{X}_T|$, we have that z' maps to $g(f_T(z)) \in T'$ and that there is a specialization $g(f_T(z)) \rightsquigarrow g(t_0)$ which does not lift to a specialization of z'. By Lemma 3.8.7, this shows that $\mathcal{X}_{T'} \to T'$ is also not closed.

For a finite type morphism $T \to \mathcal{Y}$, the base change $\mathcal{X}_T \to T$ is a finite type morphism of noetherian algebraic stacks which also satisfies the valuative criterion. It therefore suffices to show that $f \colon \mathcal{X} \to \mathcal{Y}$ is closed. By Lemma 3.8.7, we need to show that given a point $x \in |\mathcal{X}|$, every specialization $f(x) \rightsquigarrow y_0$ lifts to a specialization $x \rightsquigarrow x_0$. By Proposition 3.8.8, there exists a diagram (3.8.1) such that $\operatorname{Spec} R \to \mathcal{Y}$ realizes $f(x) \rightsquigarrow y_0$ with a lift $\operatorname{Spec} K \to \mathcal{X}$ whose image is x. The valuative criterion implies the existence of a lift $\operatorname{Spec} R \to \mathcal{X}$ which in turn yields a specialization $x \rightsquigarrow x_0$ lifting $f(x) \rightsquigarrow y_0$.

Conversely, assume that $f \colon \mathcal{X} \to \mathcal{Y}$ is universally closed and that we are given a diagram (3.8.1). By replacing $\mathcal{Y}$ with $\operatorname{Spec} R$ and $\mathcal{X}$ with $\mathcal{X} \times_{\mathcal{Y}} \operatorname{Spec} R$, we may assume that $\mathcal{Y} = \operatorname{Spec} R$ and that we have a diagram

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r, "x"] \arrow[d] & \mathcal{X} \arrow[dl] \\ \operatorname{Spec} R & \end{tikzcd}$$

By replacing $\mathcal{X}$ with $\overline{\{x\}}$, we may assume that $\mathcal{X}$ is integral with generic point x. Since $\mathcal{X} \to \operatorname{Spec} R$ is closed, there exists a specialization $x \rightsquigarrow x_0$ mapping to the specialization of the generic point to the closed point in $\operatorname{Spec} R$. As $\operatorname{Spec} K \to \mathcal{X}$ is quasi-compact, Proposition 3.8.8 implies there exists a DVR R' with fraction field K' and a commutative diagram

$$\begin{tikzcd}
\operatorname{Spec} K' \arrow[r] \arrow[d] & \operatorname{Spec} K \arrow[d] \\
\operatorname{Spec} R' \arrow[r] & \mathcal{X}
\end{tikzcd}$$

such that $\operatorname{Spec} R' \to \mathcal{X}$ realizes the specialization $x \rightsquigarrow x_0$. As $\operatorname{Spec} R' \to \operatorname{Spec} R$ is surjective, we see that $R \to R'$ is an extension of DVRs and that $\operatorname{Spec} R' \to \mathcal{X}$ provides a lift of the given diagram. $\square$

Remark 3.8.9. In the valuative criterion for algebraic stacks, it is necessary to allow extensions $R \to R'$. Indeed, consider $\mathcal{X} = \mathbf{B}_{\mathbb{C}}\mathbb{Z}/2$ and the DVR $R = \mathbb{C}[x]_{(x)}$ with fraction field $K = \mathbb{C}(x)$. If we let $\mathbb{Z}/2$ act on $\operatorname{Spec} \mathbb{C}(y)$ via $(-1) \cdot y = -y$, then $\operatorname{Spec} \mathbb{C}(y) \to \operatorname{Spec} \mathbb{C}(x)$ defined by $x \mapsto y^2$ is a $\mathbb{Z}/2$-torsor (corresponding to a morphism $\operatorname{Spec} K \to \mathcal{X}$) that does not extend to a $\mathbb{Z}/2$-torsor over $\operatorname{Spec} R$.

Exercise 3.8.10.

(a) Show that if $\mathcal{X}$ is Deligne–Mumford stack over a field $\Bbbk$ that is not an algebraic space, then there exists a map $\operatorname{Spec} K \to \mathcal{X}$ that does not extend to a map $\operatorname{Spec} R \to \mathcal{X}$, where R is a DVR with fraction field K.

(b) Show that for a *representable* morphism $X \to Y$ of finite type of noetherian algebraic stacks, the valuative criterion for universally closed (resp. separated, proper) holds without requiring an extension of DVRs.

See Remark 5.5.5 for an explicit example illustrating the necessity of extensions in the valuative criterion for $\mathscr{M}_g$. On the other hand, for $\mathcal{B}\mathrm{un}_{r,d}(C)$ it is not necessary to allow for extensions of DVRs.

Exercise 3.8.11.

(a) If G is a finite group, show that $\mathbf{B}_{\mathbb{Z}}G \to \operatorname{Spec} \mathbb{Z}$ is proper.

(b) Show that $\mathbf{B}_{\mathbb{Z}}\mathbb{G}_m \to \operatorname{Spec} \mathbb{Z}$ is universally closed but not separated.

Try to give two arguments for each part—one using the definitions and the other using the valuative criterion.

Exercise 3.8.12. Show that $\mathcal{M}_{1,1}$ is separated over $\operatorname{Spec} \mathbb{Z}$.

We later show that $\mathscr{M}_g$ (and more generally $\overline{\mathcal{M}}_{g,n}$) is proper over $\operatorname{Spec} \mathbb{Z}$ and that $\mathcal{B}\mathrm{un}_{r,d}(C)^{\mathrm{ss}}$ is universally closed over a field $\Bbbk$.

3.9 Further examples

In this section, we provide examples of algebraic spaces, Deligne–Mumford stacks and algebraic stacks.

3.9.1 Examples of algebraic spaces

Example 3.9.1. As discussed in Example 0.5.5, there exists a smooth proper complex 3-fold U with a free of $\mathbb{Z}/2$-action such that there is an orbit not contained in an affine open subscheme. The quotient sheaf $U/(\mathbb{Z}/2)$ is an algebraic space (Corollary 3.1.12) which is not a scheme.

Example 3.9.2 (The bug-eyed cover)**.** Let $\Bbbk$ be field of $\operatorname{char}(k) \neq 2$. Let $\mathbb{Z}/2 = \{\pm 1\}$ act on the non-separated affine line $U = \mathbb{A}^1 \bigcup_{\mathbb{A}^1 \setminus 0} \mathbb{A}^1$ over $\Bbbk$ by swapping the origins and by $(-1) \cdot x = -x$ for $x \neq 0$. Since the orbit of an origin is not contained in an affine, the quotient sheaf $U/(\mathbb{Z}/2)$ is not representable by a scheme; it is however an algebraic space (Corollary 3.1.12).

For an alternative description, let $\mathbb{Z}/2 = \{\pm 1\}$ act on $\mathbb{A}^1$ with multiplication $\sigma \colon \mathbb{Z}/2 \times \mathbb{A}^1 \to \mathbb{A}^1$ defined by $-1 \cdot x = -x$. If we remove the non-identity element of the stabilizer of the origin, we obtain a scheme $R = (\mathbb{Z}/2 \times \mathbb{A}^1) \setminus \{(-1, 0)\}$ and an equivalence relation $\sigma, p_2 \colon R \rightrightarrows \mathbb{A}^1$. The algebraic space quotient $\mathbb{A}^1/R$ is isomorphic to $U/(\mathbb{Z}/2)$ (Exercise 3.9.3(a)) For another way to see that $X = \mathbb{A}^1/R$ is not a scheme, observe that the diagonal $X \to X \times X$ is not a locally closed immersion as there is a cartesian diagram

$$\begin{tikzcd}
(\mathbb{A}^1 \setminus 0) \sqcup \{0\} \arrow[r] \arrow[d] & \mathbb{A}^1 \arrow[d] & x \arrow[d, mapsto] \\
R \arrow[r, "(\sigma, p_2)"] \arrow[d] & \mathbb{A}^1 \times \mathbb{A}^1 \arrow[d] & (x, -x) \\
X \arrow[r] & X \times X. &
\end{tikzcd}$$

Exercise 3.9.3.

(a) Show that $X = \mathbb{A}^1/R$ is isomorphic to $U/(\mathbb{Z}/2)$.

(b) Show that there is a universal homeomorphism $X \to \mathbb{A}^1$ which is ramified over the origin.

(c) Show that every map to a scheme $X \to Z$ factors through $X \to \mathbb{A}^1$. (In other words, while $\mathbb{A}^1$ may be the categorical quotient of U by $\mathbb{Z}/2$ (or equivalently the category quotient of $R \rightrightarrows \mathbb{A}^1$) in the category of schemes, it is distinct from the algebraic space quotient.

(d) Consider the SL_2 action on $V_d = \operatorname{Sym}^d \Bbbk^2$, the space of homogeneous polynomials in x and y of degree d. Let $W \subset V_1 \times V_4$ be the reduced locally closed subscheme defined as the set (L, F) such that $L \neq 0$ and F is the square of a homogeneous quadratic with discriminant 1. Show that the induced SL_2-action on W is free (i.e. $\mathrm{SL}_2 \times W \to W \times W$ is a monomorphism) and that quotient sheaf W/SL_2 is an algebraic space isomorphic to $\mathbb{A}^1/R$ and $U/(\mathbb{Z}/2)$.

While the descriptions of X as $\mathbb{A}^1/R$ and $U/(\mathbb{Z}/2)$ may seem pathological, this exercise shows that in fact this algebraic space arises also as a quotient of a quasi-affine variety by SL_2.

Example 3.9.4. Let $\mathbb{Z}/2 = \{\pm 1\}$ act on $\mathbb{A}^1_{\mathbb{C}}$ via conjugation over $\operatorname{Spec} \mathbb{R}$. Note that the action defined over $\mathbb{R}$ of $\mathbb{Z}/2$ on $\operatorname{Spec} \mathbb{C}$ is free, and therefore the product action of $\mathbb{Z}/2$ on $\mathbb{A}^1_{\mathbb{C}} = \mathbb{A}^1_{\mathbb{R}} \times_{\mathbb{R}} \mathbb{C}$ (which is trivial on the first factor) is also free. Defining $R = (\mathbb{Z}/2 \times \mathbb{A}^1_{\mathbb{C}}) \setminus \{(-1, 0)\}$, show that there is an equivalence relation $\sigma, p_2 \colon R \rightrightarrows U$ such that the algebraic space $X = \mathbb{A}^1_{\mathbb{C}}/R$ is not a scheme. (The quotient X looks like $\mathbb{A}^1_{\mathbb{R}}$ except that the origin has residue field $\mathbb{C}$.)

3.9.2 Examples of stacks with finite stabilizers

In characteristic 0, each of the following examples are Deligne–Mumford stacks.

Example 3.9.5 (Classifying stacks)**.** If G is a abstract finite group scheme over a field $\Bbbk$, then the *classifying stack* $\mathbf{B}G$ *of* G is the stack defined as the category of pairs (T, P) where T is a scheme and $P \to T$ is a G-torsor (Definition 2.3.14). Then $\mathbf{B}G$ is a smooth and proper algebraic stack over $\Bbbk$ of dimension 0. Properness follows from the fact the base change of $\mathbf{B}G \to \mathbf{B}G \times \mathbf{B}G$ by the smooth presentation $\operatorname{Spec} \Bbbk \to \mathbf{B}G \times \mathbf{B}G$ is the finite morphism $G \to \operatorname{Spec} \Bbbk$, and smoothness follows because smoothness is a smooth-local property on the source and $S \to \mathbf{B}G$ is a smooth presentation).

Example 3.9.6 (Weighted projective stacks)**.** For a tuple of positive integers $(d_0, \ldots, d_n)$, let $\mathbb{G}_m$ act on $\mathbb{A}^{n+1}$ via $t \cdot (x_0, \ldots, x_n) = (t^{d_0} x_0, \ldots, t^{d_n} x_n)$. We define the *weighted projective stack* as

$$\mathcal{P}(d_0, \ldots, d_n) = [(\mathbb{A}^{n+1} \setminus 0)/\mathbb{G}_m].$$

If the d_i are all 1, then we recover projective space $\mathbb{P}^n$; otherwise, $\mathcal{P}(d_0, \ldots, d_n)$ is not an algebraic space.

More generally, if R is a finitely generated positively graded $\Bbbk$-algebra, we can define *stacky proj* as $\mathcal{P}\mathrm{roj}\, R = [(\operatorname{Spec}(R) \setminus 0)/\mathbb{G}_m]$, where $\mathbb{G}_m$ acts such that the weight of x_i is the same as its degree.

For example, over $\mathbb{Z}[1/6]$ the stack of stable elliptic curves $\overline{\mathcal{M}}_{1,1}$ is isomorphic to $\mathcal{P}(4, 6)$ by Exercise 3.1.17(c).

Exercise 3.9.7.

(a) If $\Bbbk$ is a field of characteristic p, show that $\mathcal{P}(d_0, \ldots, d_n)$ is a Deligne–Mumford stack if and only if p doesn't divide each d_i.

(b) Classify all the points of $\mathcal{P}(3, 3, 4, 6)$ that have non-trivial stabilizers.

(c) We say that an algebraic stack $\mathcal{X}$ has *generically trivial stabilizer* if there exists a dense open substack $U \subset \mathcal{X}$ which is an algebraic space. Provide conditions for when $\mathcal{P}(d_0, \ldots, d_n)$ has generically trivial stabilizer.

(d) Show that there is a bijective morphism $\mathcal{P}(d_0, \ldots, d_n)$ to weighted projective space $\operatorname{Proj} \Bbbk[x_0, \ldots, x_n]$, where x_i has degree d_i. (This is an example of a coarse moduli space.)

Example 3.9.8. Suppose $\operatorname{char}(k) \neq 2$. Let $\mathbb{Z}/2$ act on $\mathbb{A}^2_{\Bbbk}$ via $-1 \cdot (x, y) = (-x, -y)$. Show that $[\mathbb{A}^2_{\Bbbk}/(\mathbb{Z}/2)]$ is a smooth algebraic stack over a field $\Bbbk$ and that there is a proper and bijective morphism $[\mathbb{A}^2_{\Bbbk}/(\mathbb{Z}/2)] \to Y$ where Y is the singular variety $\operatorname{Spec} \Bbbk[x^2, xy, y^2]$ defined by the $\mathbb{Z}/2$-invariants of $\Gamma(\mathbb{A}^2_{\Bbbk}, \mathcal{O}_{\mathbb{A}^2_{\Bbbk}})$.

Example 3.9.9 (Stacky curves)**.** A *stacky curve* is a one-dimensional Deligne—Mumford stack of finite type over a field $\Bbbk$.

Exercise 3.9.10. If d_1 and d_2 are relatively prime positive integers, show that $\mathbb{P}(d_1, d_2)$ is a smooth and proper stacky curve with generically trivial stabilizer.

Exercise 3.9.11. We say that a stacky curve $\mathcal{X}$ over $\Bbbk$ is *nodal* if there exists a étale presentation $U \to \mathcal{X}$ from a nodal curve (equivalently every étale presentation is a nodal curve); see Definition 5.2.1. Show that a nodal stacky curve has abelian stabilizers.

Example 3.9.12 (Root gerbes)**.** Let X be a scheme and L be a line bundle. This data determines a morphism $[L]\colon X \to \mathbb{B}\mathbb{G}_m$. Let $r\colon \mathbf{B}\mathbb{G}_m \to \mathbf{B}\mathbb{G}_m$ be the morphism induced from the rth power map $r\colon \mathbb{G}_m \to \mathbb{G}_m$, where $t \mapsto t^r$; alternatively $r\colon \mathbf{B}\mathbb{G}_m \to \mathbf{B}\mathbb{G}_m$ is defined functorially on objects by the assignment $L \mapsto L^{\otimes r}$. For a positive integer r, define the *rth root gerbe of X and L* as the fiber product

$$\begin{tikzcd} \sqrt[r]{L/X} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathbf{B}\mathbb{G}_m \arrow[d, "r"] \\ X \arrow[r, "{[L]}"] & \mathbf{B}\mathbb{G}_m. \end{tikzcd}$$

Example 3.9.13 (Root stacks)**.** Let X be a scheme, L be a line bundle, and $s \in \Gamma(X, L)$ be a section. This data determines a morphism $[L, s]\colon X \to [\mathbb{A}^1/\mathbb{G}_m]$ (see Example 3.9.16). Let $r\colon [\mathbb{A}^1/\mathbb{G}_m] \to [\mathbb{A}^1/\mathbb{G}_m]$ be the morphism induced from the map $r\colon \mathbb{A}^1 \to \mathbb{A}^1$, given by $x \mapsto x^r$, which is equivariant under $r\colon \mathbb{G}_m \to \mathbb{G}_m$; alternatively $r\colon [\mathbb{A}^1/\mathbb{G}_m] \to [\mathbb{A}^1/\mathbb{G}_m]$ is defined functorially by $(L, s) \mapsto (L^{\otimes r}, s^r)$. For a positive integer r, define the *rth root stack of X and L along s* as the fiber product

$$\begin{tikzcd} \sqrt[r]{(L,s)/X} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & {[\mathbb{A}^1/\mathbb{G}_m]} \arrow[d, "r"] \\ X \arrow[r, "{[L,s]}"] & {[\mathbb{A}^1/\mathbb{G}_m]}. \end{tikzcd}$$

Exercise 3.9.14. Let S be a scheme and r be an integer invertible in $\Gamma(S, \mathcal{O}_S$. (This ensures that $\boldsymbol{\mu}_{r,S} \to S$ is an étale group scheme, and that the root gerbe $\sqrt[r]{L/X}$ and stack $\sqrt[r]{(L,s)/X}$ are Deligne–Mumford stacks.)

(a) Show that $\sqrt[r]{L/X}$ has the equivalent description as the category of tuples $(T \xrightarrow{f} X, M, \alpha)$ where $f\colon T \to X$ is a morphism from a scheme, M is a line bundle on T and $\alpha\colon M^{\otimes r} \to f^*L$ is an isomorphism.

(b) Show that $\sqrt[r]{(L,s)/X}$ has the equivalent description as the category of triples $(T \xrightarrow{t} X, M, \alpha, t)$ where $f\colon T \to X$ is a morphism from a scheme, M is a line bundle on T, $\alpha\colon M^{\otimes r} \to f^*L$ is an isomorphism, and $t \in \Gamma(T, M)$ is a section such that $\alpha(t^{\otimes r}) = f^*s$.

(c) If $X = \operatorname{Spec} A$ is an affine scheme over S and $L = \mathcal{O}_X$ is trivial, show that

$$\sqrt[r]{L/X} \cong [X/\boldsymbol{\mu}_r] \quad \text{and} \quad \sqrt[r]{(L,s)/X} \cong [\operatorname{Spec}\big(A[x]/(x^r - s)\big)/\boldsymbol{\mu}_r]$$

where $\boldsymbol{\mu}_r$ acts trivially on X and acts on $\operatorname{Spec}\big(A[x]/(x^r - s)\big)$ via $t \cdot x = tx$.

(d) Show that the fiber of $\sqrt[r]{L/X} \to X$ at a point $x \in X$ is isomorphic to $B\boldsymbol{\mu}_{r,\kappa(x)}$.

(e) Show that $\sqrt[r]{(L,s)/X} \to X$ is an isomorphism over $X_s = \{s \neq 0\}$ and that the fiber over a point $x \in X$ is isomorphic to $B\boldsymbol{\mu}_{r,\kappa(x)}$. and a banded $\boldsymbol{\mu}_r$-gerbe over the vanishing $V(s) \subset X$ of s.

(You will show later in Exercise 6.2.37 that $\sqrt[r]{L/X} \to X$ and the restriction of $\sqrt[r]{(L,s)/X} \to X$ along $V(S)$ are banded $\boldsymbol{\mu}_r$-gerbes.))

3.9.3 Examples of algebraic stacks

Example 3.9.15. The classifying stack $\mathbf{B}\,\mathrm{GL}_n$ over $\operatorname{Spec}\mathbb{Z}$ classifies vector bundles of rank n. When $n = 1$, $\mathbf{B}\mathbb{G}_m = \mathbf{B}\,\mathrm{GL}_1$ classifies line bundles. The stack $\mathbf{B}\,\mathrm{GL}_n$ is a universally closed and smooth algebraic stack over $\operatorname{Spec}\mathbb{Z}$ of relative dimension $-n^2$ with affine diagonal. However, $\mathbf{B}\,\mathrm{GL}_n$ is not separated nor Deligne–Mumford.

Example 3.9.16. If $\mathbb{G}_m$ acts on $\mathbb{A}^1$ over $\mathbb{Z}$ via scaling, the quotient stack $[\mathbb{A}^1/\mathbb{G}_m]$ whose objects over a scheme T are pairs (L, s) where L is a line bundle on T and $s \in \Gamma(T, L)$. The stack $[\mathbb{A}^1/\mathbb{G}_m]$ is an algebraic stack universally closed and smooth over $\operatorname{Spec}\mathbb{Z}$ of relative dimension 0 with affine diagonal. The stack $[\mathbb{A}^1/\mathbb{G}_m]$ is not separated nor Deligne–Mumford

Over a field $\Bbbk$, $[\mathbb{A}^1/\mathbb{G}_m]$ has two points—one open and one closed—corresponding to the two $\mathbb{G}_m$-orbits (see Figure 7). There is an open immersion and closed immersion

$$\operatorname{Spec}\Bbbk \hookrightarrow [\mathbb{A}^1/\mathbb{G}_m] \hookleftarrow \mathbf{B}\mathbb{G}_m.$$

The morphism $[\mathbb{A}^1/\mathbb{G}_m] \to \operatorname{Spec}\Bbbk$ identifies the two orbits and is an example of a good moduli space.

Example 3.9.17. Working over a field $\Bbbk$, let $\mathbb{G}_m$ act on $\mathbb{A}^2$ via $t\cdot(x, y) = (tx, t^{-1}y)$. The quotient stack $\mathcal{X} = [\mathbb{A}^2/\mathbb{G}_m]$ is a smooth algebraic stack. An object of $\mathcal{X}$ over a scheme T is a triple (L, s, t) where L is a line bundle on T, $s \in \Gamma(T, L)$ and $t \in \Gamma(T, L^{-1})$. The complement $\mathcal{X} \setminus 0$ of the origin is isomorphic to the non-separated affine line. There is a morphism $\mathcal{X} \to \mathbb{A}^1$ defined by $(x, y) \mapsto xy$, which is an isomorphism over $\mathbb{A}^1 \setminus 0$ and identifies the three orbits defined by $xy = 0$.

Example 3.9.18 (Toric stacks)**.** A fan Σ on a lattice $L = \mathbb{Z}^n$ defines a toric variety $X(\Sigma)$, i.e. a normal separated variety with an action of $\mathbb{G}_m^n$ such that there is a dense orbit with trivial stabilizer; see [Ful93].

Meanwhile a *stacky fan* is a pair (Σ, β) where Σ is a fan on a lattice L and $\beta\colon L \to N$ is a homomorphism of lattices. As L and N are lattices (i.e. finitely generated free abelian groups), the $\mathbb{Z}$-linear duals define tori $T_L := D(L^\vee)$ and $T_N := D(N^\vee)$ (Example C.1.11) where T_L is a torus for the toric variety $X(\Sigma)$. The map β induces a homomorphism $T_\beta\colon T_L \to T_N$, naturally identifying β with the induced map on lattices of 1-parameter subgroups. We can then define $G_\beta = \ker(T_\beta)$ and the *toric stack*

$$X(\Sigma, \beta) := [X(\Sigma)/G_\beta].$$

Example 3.9.19 (Picard schemes and stacks)**.** If X is a scheme over a field $\Bbbk$, the *Picard functor of* X and *Picard stack of* X are defined as the sheaf $\underline{\mathrm{Pic}}(X)$ and stack $\underline{\mathcal{P}\mathrm{ic}}(X)$ on $\mathrm{Sch}_{\text{ét}}$ by

$$\underline{\mathrm{Pic}}(X) = \text{sheafification of } T \mapsto \mathrm{Pic}(X_T)$$
$$\underline{\mathcal{P}\mathrm{ic}}(X)(T) = \{\text{groupoid of line bundles } L \text{ on } X_T\}$$

A morphism $(T, L) \to (T', L')$ in $\underline{\mathcal{P}\mathrm{ic}}(X)$ is the data of a morphism $f\colon T \to T'$ of schemes and an isomorphism $\alpha\colon L \to f^*L'$ (or more precisely a morphism $f_*L \to L'$ whose adjoint is an isomorphism).

If X is proper over a field $\Bbbk$, then $\underline{\mathrm{Pic}}(X)$ is a proper scheme and the tensor product of line bundles provides it with the structure of a group scheme, hence an abelian variety. Moreover, $\underline{\mathcal{P}ic}(X)$ is a smooth algebraic stack over $\Bbbk$ and there morphism $\underline{\mathcal{P}ic}(X) \to \underline{\mathrm{Pic}}(X)$ such that the fiber over a line bundle L is isomorphic to $\mathbf{B}\mathbb{G}_m$. The tensor product of line bundles provides $\underline{\mathcal{P}ic}(X)$ with the structure of a group stack, a notion which we will not spell out precisely.

Gerbes provide another important example of algebraic stacks but we postpone our treatment until §6.2.5.

3.9.4 Pathological examples

Exercise 3.9.20. If $G \to S$ is a smooth and affine group scheme acting on a scheme U over S, then the quotient stack $[U/G]$ is algebraic (Theorem 3.1.9). More generally if $G \to S$ is only assumed smooth, show that $[U/G]$ is algebraic by identifying it with the algebraic stack quotient of the smooth groupoid $G \times U \rightrightarrows U$.

In particular, the classifying stack $\mathbf{B}G = [S/G]$ is algebraic.

Example 3.9.21. Consider the constant group scheme $\underline{\mathbb{Z}}$ over $\operatorname{Spec} \mathbb{Z}$ associated to the abstract discrete group $\mathbb{Z}$. Then $\mathbb{B}\underline{\mathbb{Z}}$ is a non-quasi-separated smooth algebraic stack of dimension 0.

Example 3.9.22. Here we provide an example of a non-quasi-separated algebraic space which is not a scheme. Let $\Bbbk$ be a characteristic 0 field. Let $\underline{\mathbb{Z}}$ act on $\mathbb{A}^1$ over $\Bbbk$ via $n \cdot x = x + n$ for $x \in \mathbb{A}^1$ and $n \in \underline{\mathbb{Z}}$. Then $X = \mathbb{A}^1/\underline{\mathbb{Z}}$ is an algebraic space which is not quasi-separated (as the action map $\underline{\mathbb{Z}} \times \mathbb{A}^1 \to \mathbb{A}^1 \times \mathbb{A}^1$ is not quasi-compact).

If X were a scheme, then there would exist a non-empty open affine subscheme $U = \operatorname{Spec} A \subset X$. Since $p \colon \mathbb{A}^1 \to X$ is an étale presentation, we can compute A as the subring of $\underline{\mathbb{Z}}$-invariants $\Gamma(p^{-1}(U), \mathcal{O}_{\mathbb{A}^1})^{\underline{\mathbb{Z}}}$, which the reader can check consists of only the constant functions, i.e. $A = \Bbbk$. As X is obtained by gluing such affine schemes, it follows that $X = \operatorname{Spec} \Bbbk$, a contradiction.

The algebraic space $X = \mathbb{A}^1/\underline{\mathbb{Z}}$ provides a counterexample to many facts that hold for all schemes and quasi-separated algebraic spaces but fail for all algebraic spaces (e.g. see Exercise 3.9.23).

Similarly, one can consider the algebraic space quotient $\mathbb{A}^1_{\mathbb{C}}/\underline{\mathbb{Z}}^2$ where $(a, b) \cdot x = x + a + ib$. While the analytic quotient $\mathbb{C}/\underline{\mathbb{Z}}^2$ of this action is an elliptic curve over $\mathbb{C}$, the algebraic space quotient is a non-quasi-separated algebraic space that is not a scheme.

Exercise 3.9.23. Let $X = \mathbb{A}^1/\underline{\mathbb{Z}}$ be the algebraic space defined above.

(a) Show that X is locally noetherian and quasi-compact but not noetherian.

(b) Show that the generic point $\operatorname{Spec} \Bbbk(x) \to \mathbb{A}^1 \to X$ is fixed under the $\underline{\mathbb{Z}}$-action.

(c) Show that $\operatorname{Spec} \Bbbk(x) \to X$ does not factor through a monomorphism $\operatorname{Spec} L \to X$ for a field L. (In other words, the generic point of X does not have a residue field.)

Example 3.9.24 (Deligne–Mumford stacks with non-separated diagonal)**.** Let $G \to S$ be a finite group scheme. If $H \subset G$ is a subgroup scheme over S, then G/H is separated if and only if $H \subset G$ is closed. For instance, taking $G = \mathbb{Z}/2 \times \mathbb{A}^1 \to \mathbb{A}^1$ and the subgroup $H = G \setminus \{-1, 0\}$, the quotient $Q = G/H$ is the non-separated affine line and is a group scheme over $\mathbb{A}^1$ which is trivial away from the origin and where the fiber over 0 is $\mathbb{Z}/2$. In this case, $\mathbf{B}_{\mathbb{A}^1} Q$ is a Deligne–Mumford stack with non-separated diagonal; however, $\mathcal{X}$ is quasi-compact and quasi-separated (i.e. $\mathcal{X}$, the first diagonal $\Delta_{\mathcal{X}}$ and second diagonal $\Delta_{\Delta_{\mathcal{X}}}$ are quasi-compact).

Chapter 4

Geometry of Deligne–Mumford stacks

4.1 Quasi-coherent sheaves and cohomology

4.1.1 Sheaves

The small étale site of a Deligne–Mumford stack can be defined analogously to the small étale site of a scheme (Example 2.1.3).

Definition 4.1.1. If $\mathcal{X}$ is a Deligne–Mumford stack, the *small étale site of* $\mathcal{X}$ is the category $\mathcal{X}_{\text{ét}}$ of schemes étale over $\mathcal{X}$. A covering of an $\mathcal{X}$-scheme U is a collection of étale morphisms $\{U_i \to U\}$ over $\mathcal{X}$ such that $\coprod_i U_i \to U$ is surjective.

We can therefore discuss sheaves of abelian groups on $\mathcal{X}_{\text{ét}}$ and their morphisms. We denote $\mathrm{Ab}(\mathcal{X}_{\text{ét}})$ as the category of abelian sheaves on $\mathcal{X}_{\text{ét}}$. For an abelian sheaf F on $\mathcal{X}_{\text{ét}}$, the sections over an étale $\mathcal{X}$-scheme U are denoted by $F(U)$ or $\Gamma(U, F)$; you should remember that this group depends not only on U but the structure morphism $U \to \mathcal{X}$.

Example 4.1.2 (Structure sheaf)**.** The structure sheaf $\mathcal{O}_{\mathcal{X}}$ on a Deligne–Mumford stack is defined by $\mathcal{O}_{\mathcal{X}}(U) = \Gamma(U, \mathcal{O}_U)$ on an étale $\mathcal{X}$-scheme U.

Example 4.1.3 (Differentials)**.** If $\mathcal{X}$ is a Deligne–Mumford stack over a scheme S, the relative sheaf of differentials $\Omega_{\mathcal{X}/S}$ is defined by $\Omega_{\mathcal{X}/S}(U) = \Gamma(U, \Omega_{U/S})$.

Example 4.1.4 (Hodge bundle)**.** Define the sheaf $\mathcal{H}$ on $\mathscr{M}_g$ (for $g \geq 2$) as follows: for every étale morphism $U \to \mathscr{M}_g$ from a scheme corresponding to a family $\mathcal{C} \to U$ of smooth curves, we set $\mathcal{H}(U) = \Gamma(\mathcal{C}, \Omega_{\mathcal{C}/U})$. We will see later that $\mathcal{H}$ is a coherent $\mathcal{O}_{\mathscr{M}_g}$-module which is locally free of rank g, i.e. a vector bundle.

While a sheaf F on $\mathcal{X}_{\text{ét}}$ by definition only has sections defined on étale $\mathcal{X}$-schemes, one can extend the definition to a Deligne–Mumford stack $\mathcal{U}$ étale over $\mathcal{X}$. Choose étale presentations $U \to \mathcal{U}$ and $R \to U \times_{\mathcal{U}} U$ by schemes and define

$$F(\mathcal{U}) := \mathrm{Eq}(F(U) \rightrightarrows F(R)).$$

One checks that this is independent of the choice of presentation. In particular, it makes sense to discuss global sections $\Gamma(\mathcal{X}, F) := F(\mathcal{X})$ over the identity $\mathrm{id}\colon \mathcal{X} \to \mathcal{X}$.

Exercise 4.1.5. If F is an abelian sheaf on a Deligne–Mumford stack $\mathcal{X}$, show that $\Gamma(\mathcal{X}, F) = \mathrm{Hom}_{\mathrm{Ab}(\mathcal{X}_{\text{ét}})}(\underline{\mathbb{Z}}, F)$ where $\underline{\mathbb{Z}}$ is the constant sheaf. If F is an $\mathcal{O}_{\mathcal{X}}$-module, show that $\Gamma(\mathcal{X}, F) = \mathrm{Hom}_{\mathcal{O}_{\mathcal{X}}}(\mathcal{O}_{\mathcal{X}}, F)$.

Given a morphism $f\colon \mathcal{X} \to \mathcal{Y}$ of Deligne–Mumford stacks, there are functors

$$\begin{tikzcd} \mathrm{Ab}(\mathcal{X}_{\text{ét}}) \arrow[r, shift left, "f_*"] & \mathrm{Ab}(\mathcal{Y}_{\text{ét}}) \arrow[l, shift left, "f^{-1}"] \end{tikzcd}$$

where $f_*F(V) := F(V \times_{\mathcal{Y}} \mathcal{X})$ and $f^{-1}G$ is the sheafification of the presheaf

$$U \mapsto \lim_{V \to \mathcal{Y}, U \to V \times_{\mathcal{Y}} \mathcal{X}} G(V),$$

with the limit is taken over the category of pairs of étale morphisms $V \to \mathcal{Y}$ and $U \to V \times_{\mathcal{Y}} \mathcal{X}$ (i.e. étale morphisms $V \to \mathcal{Y}$ and a choice of factorization of $U \to \mathcal{X} \to \mathcal{Y}$ through $V \to \mathcal{Y}$). Note that when $f\colon \mathcal{X} \to \mathcal{Y}$ is étale, then $f^{-1}G(U) = G(U)$ for an étale $\mathcal{X}$-scheme.

Exercise 4.1.6. Show that f^{-1} is left adjoint to f_*.

Exercise 4.1.7. If $\mathcal{X}$ is a Deligne–Mumford stack, define instead the site $\mathcal{X}_{\text{ét}'}$ as the category of *algebraic spaces* over $\mathcal{X}$ where coverings are étale coverings. Show that the categories of sheaves on $\mathcal{X}_{\text{ét}}$ and $\mathcal{X}_{\text{ét}'}$ are equivalent.

4.1.2 $\mathcal{O}_{\mathcal{X}}$-modules

On a Deligne–Mumford stack $\mathcal{X}$, the structure sheaf $\mathcal{O}_{\mathcal{X}}$ is a ring object in $\mathrm{Ab}(\mathcal{X}_{\text{ét}})$ and we define:

Definition 4.1.8. If $\mathcal{X}$ is a Deligne–Mumford stack, a *sheaf of $\mathcal{O}_{\mathcal{X}}$-modules* (or simply an *$\mathcal{O}_{\mathcal{X}}$-module*) is a sheaf F on $\mathcal{X}_{\text{ét}}$ which is a module object for $\mathcal{O}_{\mathcal{X}}$ in the category of sheaves, i.e. for every étale $\mathcal{X}$-scheme U, $F(U)$ is an $\mathcal{O}_{\mathcal{X}}(U)$-module and the module structure is compatible with respect to restriction along étale morphisms $V \to U$ of $\mathcal{X}$-schemes.

We denote $\mathrm{Mod}(\mathcal{O}_{\mathcal{X}})$ for the category of $\mathcal{O}_{\mathcal{X}}$-modules. Given two $\mathcal{O}_{\mathcal{X}}$-modules F and G, we can define the *tensor product* $F \otimes G := F \otimes_{\mathcal{O}_{\mathcal{X}}} G$ as the sheafification of the $\mathcal{O}_{\mathcal{X}}$-module given by $(U \to \mathcal{X}) \mapsto F(U \to \mathcal{X}) \otimes_{\mathcal{O}_{\mathcal{X}}(U \to \mathcal{X})} G(U \to \mathcal{X})$. The *Hom sheaf* $\mathscr{H}om_{\mathcal{O}_{\mathcal{X}}}(F, G)$ has sections $\mathrm{Hom}_{\mathcal{O}_U}(F|_U, G|_U)$ over an étale morphism $f\colon U \to \mathcal{X}$ from scheme, where $F|_U = f^{-1}F$ denotes the restriction of F to $U_{\text{ét}}$.

Given a morphism $f\colon \mathcal{X} \to \mathcal{Y}$ of Deligne–Mumford stacks, there are functors

$$\begin{tikzcd} \mathrm{Mod}(\mathcal{O}_{\mathcal{X}}) \arrow[r, shift left, "f_*"] & \mathrm{Mod}(\mathcal{O}_{\mathcal{Y}}) \arrow[l, shift left, "f^*"] \end{tikzcd}$$

where for an $\mathcal{O}_{\mathcal{X}}$-module F, f_*F is the pushforward as sheaves and is naturally an $\mathcal{O}_{\mathcal{Y}}$-module. For an $\mathcal{O}_{\mathcal{Y}}$-module G, since there is a morphism $f^{-1}\mathcal{O}_{\mathcal{Y}} \to \mathcal{O}_{\mathcal{X}}$ of sheaves of rings in $\mathcal{X}_{\text{ét}}$ and $f^{-1}G$ is a $f^{-1}\mathcal{O}_{\mathcal{Y}}$-module, it makes sense to define the pullback $\mathcal{O}_{\mathcal{X}}$-module

$$f^*G := f^{-1}G \otimes_{f^{-1}\mathcal{O}_{\mathcal{Y}}} \mathcal{O}_{\mathcal{X}}.$$

Exercise 4.1.9. Show that f^* is left adjoint to f_*.

Exercise 4.1.10. Show that $\mathrm{Mod}(\mathcal{O}_{\mathcal{X}})$ is an abelian category.

4.1.3 Quasi-coherent sheaves

Let F be an $\mathcal{O}_{\mathcal{X}}$-module on a Deligne–Mumford stack $\mathcal{X}$. For an étale $\mathcal{X}$-scheme U, we have the restriction $F|_U$ to the étale site of U and the further restriction $F|_{U_{\mathrm{Zar}}}$ restricted to the Zariski topology of U. Note that when X is a scheme, $\mathcal{O}_X$ could refer to the structure sheaf either in $X_{\text{ét}}$ or X_{Zar}. If there is a possibility for confusion, we write either $\mathcal{O}_{X_{\mathrm{Zar}}}$ or $\mathcal{O}_{X_{\text{ét}}}$.

Definition 4.1.11. Let $\mathcal{X}$ be a Deligne–Mumford stack. An $\mathcal{O}_{\mathcal{X}}$-module F is *quasi-coherent* if

(1) for every étale $\mathcal{X}$-scheme U, the restriction $F|_{U_{\mathrm{Zar}}}$ is a quasi-coherent $\mathcal{O}_{U_{\mathrm{Zar}}}$-module, and

(2) for every étale morphism $f: U \to V$ of étale $\mathcal{X}$-schemes, the natural morphism $f^*(F|_{V_{\mathrm{Zar}}}) \to F|_{U_{\mathrm{Zar}}}$ is an isomorphism.

A quasi-coherent F on $\mathcal{X}$ is a *vector bundle* (resp. *vector bundle of rank* r, *line bundle*) if $F|_{U_{\mathrm{Zar}}}$ is for every morphism $U \to \mathcal{X}$ from a scheme.

If in addition $\mathcal{X}$ is locally noetherian, we say F is *coherent* if $F|_{U_{\mathrm{Zar}}}$ is coherent for every morphism $U \to \mathcal{X}$ from a scheme.

We denote by $\mathrm{QCoh}(\mathcal{X})$ and $\mathrm{Coh}(\mathcal{X})$ (in the noetherian setting) the categories of quasi-coherent and coherent sheaves. The condition on F being a vector bundle, line bundle or coherent (in the noetherian setting) are étale local (Proposition B.4.3) and thus it suffices to check the condition on an étale presentation.

Examples 4.1.12. The structure sheaf $\mathcal{O}_{\mathcal{X}}$ is a line bundle which is coherent when $\mathcal{X}$ is locally noetherian. For a Deligne–Mumford stack $\mathcal{X}$ over a scheme S, the relative sheaf of differentials $\Omega_{\mathcal{X}/S}$ of Example 4.1.3 is quasi-coherent since for an étale morphisms $f: U \to V$ of étale $\mathcal{X}$-schemes, $f^*\Omega_{V/S} \to \Omega_{U/S}$ is an isomorphism; it is a vector bundle when $\mathcal{X} \to S$ is smooth.

For $\mathscr{M}_g$ (with $g \geq 2$), the Hodge bundle $\mathcal{H}$ of Example 4.1.4 is a vector bundle of rank g. This follows from Proposition 5.1.9(2): for a smooth family $\pi: \mathcal{C} \to V$ of genus g curves corresponding to a $\mathscr{M}_g$-scheme V, the construction of $\pi_*\Omega_{\mathcal{C}/V}$ commutes with the base change along a map $f: U \to V$, i.e. $f^*(\pi_*\Omega_{\mathcal{C}/V}) \xrightarrow{\sim} \pi_{U,*}\Omega_{\mathcal{C}_U/U})$, which shows quasi-coherence of $\mathcal{H}$. Moreover, $\pi_*\Omega_{\mathcal{C}/V}$ is a vector bundle on V of rank g, which shows that $\mathcal{H}$ is also a vector bundle of rank g.

Example 4.1.13. If G is a finite group viewed as a group scheme over a field $\Bbbk$, a quasi-coherent sheaf on $\mathbf{B}G$ corresponds to a representation V of G. If G acts on an affine $\Bbbk$-scheme $\operatorname{Spec} A$, a quasi-coherent sheaf on $[\operatorname{Spec} A/G]$ is the data of an A-module M equipped with a group homomorphism $G \to \operatorname{End}_A(M)$. These descriptions follow from Exercise 4.1.16(1).

Exercise 4.1.14 (Equivalent definition)**.** There is a general definition of a quasi-coherent module on a site $\mathcal{S}$ with a sheaf of rings $\mathcal{O}$ (see [SGA4$\frac{1}{2}$] and [SP, Tag 03DL]): an $\mathcal{O}$-module F is quasi-coherent if for every object $U \in \mathcal{S}$, there is a covering $\{U_i \to U\}$ such that the restriction $F|_{U_i}$ to the localized site $\mathcal{S}/U_i$ has a free presentation

$$\mathcal{O}_{U_i}^{\oplus J} \to \mathcal{O}_{U_i}^{\oplus I} \to F|_{U_i} \to 0.$$

Show the definition of quasi-coherence above for a Deligne–Mumford stack $\mathcal{X}$ agrees with this general definition on the ringed site $(\mathcal{X}_{\text{ét}}, \mathcal{O}_{\mathcal{X}})$.

The following exercise tells us that the notion of quasi-coherence is consistent when the usual one when $\mathcal{X}$ is a scheme.

Exercise 4.1.15. Let X be a scheme and F be an $\mathcal{O}_{X_{\mathrm{Zar}}}$-module.

(a) Define a presheaf $F_{\text{ét}}$ on $X_{\text{ét}}$ as follows: for an étale map $f\colon U \to \mathcal{X}$ from a scheme, set $F_{\text{ét}}(U) = \Gamma(U, f^*F)$. Show that $F_{\text{ét}}$ is a sheaf of $\mathcal{O}_{X_{\text{ét}}}$-modules and that the assignment $F \mapsto F_{\text{ét}}$ defines an exact functor $\mathrm{Mod}(\mathcal{O}_{X_{\mathrm{Zar}}}) \to \mathrm{Mod}(\mathcal{O}_{X_{\text{ét}}})$.

(b) Show that if F is a quasi-coherent $\mathcal{O}_{X_{\mathrm{Zar}}}$-module, then $F_{\text{ét}}$ is a quasi-coherent $\mathcal{O}_{X_{\text{ét}}}$-module, and that $F \mapsto F_{\text{ét}}$ is an equivalence of categories between quasi-coherent $\mathcal{O}_{X_{\mathrm{Zar}}}$-modules and quasi-coherent $\mathcal{O}_{X_{\text{ét}}}$-modules. See also [SP, Tag 03DX].

Exercise 4.1.16 (Groupoid and functorial perspectives)**.** Let $\mathcal{X}$ be a Deligne–Mumford stack.

(1) Let $U \to \mathcal{X}$ is an étale presentation from a scheme U. If G is a quasi-coherent sheaf on U and $\alpha\colon p_1^*G \xrightarrow{\sim} p_2^*G$ is an isomorphism on $R := U \times_{\mathcal{X}} U$ satisfying the cocycle condition $p_{23}^*\alpha \circ p_{12}^*\alpha = p_{13}^*\alpha$, show that G descends to a unique quasi-coherent sheaf on $\mathcal{X}$.

(2) If F is a quasi-coherent sheaf on $\mathcal{X}$ and $f\colon S \to \mathcal{X}$ is a morphism from a scheme, then show that $(f^*F)|_{S_{\mathrm{Zar}}}$ is a quasi-coherent sheaf on S.

Given a groupoid presentation $R \rightrightarrows U$ of $\mathcal{X}$, (1) gives an equivalence between quasi-coherent sheafs on $\mathcal{X}$ and quasi-coherent sheaves on U with descent datum. Meanwhile, (2) above allows us to think of a quasi-coherent sheaf F on $\mathcal{X}$ as the data of a quasi-coherent sheaf F_S for every map $S \to \mathcal{X}$ and compatible isomorphisms $f^*F_T \to F_S$ for every map $f\colon S \to T$ over $\mathcal{X}$. For instance, the Hodge bundle on $\mathscr{M}_g$ is the data of the sheaf $\pi_*\Omega_{\mathcal{C}/S}$ for every smooth family of curves $\pi\colon \mathcal{C} \to S$

4.1.4 Pushforwards and pullbacks

Exercise 4.1.17 (Pushforward–Pullback Adjunction)**.** Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a morphism of Deligne–Mumford stacks.

(a) Show that if G is a quasi-coherent $\mathcal{O}_{\mathcal{Y}}$-module, then f^*G is quasi-coherent.

Assume in addition that f is quasi-compact and quasi-separated.

(b) Show that if F is a quasi-coherent $\mathcal{O}_{\mathcal{X}}$-module, then f_*F is quasi-coherent.

(c) Show that the functors

$$\mathrm{QCoh}(\mathcal{X}) \underset{f^*}{\overset{f_*}{\rightleftarrows}} \mathrm{QCoh}(\mathcal{Y})$$

are adjoints (with f_* the right adjoint).

Exercise 4.1.18. Let G be a finite group and $\Bbbk$ be a field.

(a) Under the composition $\operatorname{Spec}\Bbbk \xrightarrow{p} \mathbb{B}_{\Bbbk}G \xrightarrow{\pi} \operatorname{Spec}\Bbbk$, show that for a G-representation V, $\pi_*V = V^G$ where V^G is the subspace of G-invariants and $p^*V = V$ forgetting the G-action, and that for a $\Bbbk$-vector space W,

$\pi^* W = W$ with the trivial G-action and $p_* W = W \otimes p_* \Bbbk$ where $p_* \Bbbk$ is the regular representation $\Gamma(G, \mathcal{O}_G)$.

(b) Given an action of G an an affine $\Bbbk$-scheme $\operatorname{Spec} A$, consider the diagram

$$\begin{tikzcd} \operatorname{Spec} A \arrow[r, "p"] & {[\operatorname{Spec} A/G]} \arrow[r, "\pi"] \arrow[d, "q"] & \operatorname{Spec} A^G \\ & \mathbf{B}G & \end{tikzcd}$$

and recall from Example 4.1.13 that a quasi-coherent sheaf on $[\operatorname{Spec} A/G]$ is an A-module M with a group homomorphism $G \to \operatorname{End}_A(M)$. Provide explicit descriptions of the functors $p_*, p^*, \pi_*, \pi^*, q_*$ and q^* on quasi-coherent sheaves.

Exercise 4.1.19 (Flat Base Change)**.** Consider a cartesian diagram

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"] \arrow[dr, phantom, "\square"] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

of Deligne–Mumford stacks, and let F be a quasi-coherent sheaf on X. If $g\colon Y' \to Y$ is flat and $f\colon X \to Y$ is quasi-compact and quasi-separated, the natural adjunction map

$$g^* f_* F \to f'_* g'^* F$$

is an isomorphism.

Exercise 4.1.20. Let $\mathcal{X}$ be a noetherian Deligne–Mumford stack. Prove the following two statements:

(a) Every quasi-coherent sheaf on $\mathcal{X}$ is a directed colimit of its coherent subsheaves.

(b) If $\mathcal{U} \subset \mathcal{X}$ is an open substack, then every coherent sheaf on $\mathcal{U}$ extends to a coherent sheaf on $\mathcal{X}$.

This exercise extends [Har77, Exer II.5.15] from schemes to Deligne–Mumford stacks; see also [LMB, Prop. 15.4], [Ols16, Prop. 7.1.11] and [SP, Tag 01PD].

4.1.5 Quasi-coherent constructions

A *quasi-coherent $\mathcal{O}_{\mathcal{X}}$-algebra* on a Deligne–Mumford stack is a quasi-coherent $\mathcal{O}_{\mathcal{X}}$-module with the compatible structure of a ring object in $\operatorname{Ab}(\mathcal{X}_{\text{ét}})$. We define the *relative spectrum* $\mathcal{S}\mathrm{pec}_{\mathcal{X}} \mathcal{A}$ as the stack whose objects over a scheme S consists of a morphism $f\colon S \to \mathcal{X}$ and a morphism $f^*\mathcal{A} \to \mathcal{O}_S$ of $\mathcal{O}_S$-algebras.

Exercise 4.1.21. Show that $\mathcal{S}\mathrm{pec}_{\mathcal{X}} \mathcal{A}$ is an algebraic stack affine over $\mathcal{X}$.

Example 4.1.22 (Reduction)**.** Let $\mathcal{X}$ be a Deligne-Mumford stack and let $\mathcal{O}_{\mathcal{X}}^{\text{red}}$ be the sheaf of $\mathcal{O}_{\mathcal{X}}$-algebras where $\mathcal{O}_{\mathcal{X}}^{\text{red}}(U) = \Gamma(U, \mathcal{O}_U)_{\text{red}}$ for an étale $\mathcal{X}$-scheme U. Then $\mathcal{O}_{\mathcal{X}}^{\text{red}}$ is a quasi-coherent $\mathcal{O}_{\mathcal{X}}$-algebra and $\mathcal{X}_{\text{red}} := \mathcal{S}\mathrm{pec}_{\mathcal{X}} \mathcal{O}_{\mathcal{X}}^{\text{red}}$ defines the *reduction of* $\mathcal{X}$.

Example 4.1.23 (Normalization)**.** Let $\mathcal{X}$ be an integral Deligne-Mumford stack and let $\mathcal{A}$ be the sheaf of $\mathcal{O}_{\mathcal{X}}$-algebras whose sections over an étale morphism $U \to \mathcal{X}$ from a scheme is the normalization of $\Gamma(U, \mathcal{O}_U)$. Since normalization commutes with étale extensions (Proposition A.5.4), $\mathcal{A}$ is a quasi-coherent $\mathcal{O}_{\mathcal{X}}$-algebra. The *normalization of* $\mathcal{X}$ is defined as $\widetilde{\mathcal{X}} := \mathcal{S}\mathrm{pec}_{\mathcal{X}}\, \mathcal{A}$.

Exercise 4.1.24. Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a quasi-compact and quasi-separated morphism of Deligne–Mumford stacks.

(a) Show that there is factorization $f\colon \mathcal{X} \to \mathcal{S}\mathrm{pec}\, f_*\mathcal{O}_{\mathcal{X}} \to \mathcal{Y}$.

(b) Show that f is affine if and only if $\mathcal{X} \to \mathcal{S}\mathrm{pec}\, f_*\mathcal{O}_{\mathcal{X}}$ is an isomorphism.

(c) Show that f is quasi-affine if and only if $\mathcal{X} \to \mathcal{S}\mathrm{pec}\, f_*\mathcal{O}_{\mathcal{X}}$ is an open immersion.

Exercise 4.1.25. Use Exercise 4.1.20 to show that every quasi-coherent sheaf of algebras on a noetherian Deligne–Mumford stack is a directed colimit of finite type subalgebras.

4.1.6 Cohomology

We develop a cohomology theory for abelian sheaves on Deligne–Mumford stacks. Despite utilizing the cohomology of quasi-coherent sheaves on schemes throughout these notes, we surprisingly have little need for cohomology on algebraic spaces and Deligne–Mumford stacks and many of the results here are included only for completeness.

Existence of enough injective objects is shown analogously to the case of schemes [Har77, Prop. 2.2].

Lemma 4.1.26. *If $\mathcal{X}$ is a Deligne–Mumford stack, the categories* $\mathrm{Ab}(\mathcal{X}_{\text{ét}})$ *and* $\mathrm{Mod}(\mathcal{O}_{\mathcal{X}})$ *have enough injectives. If in addition $\mathcal{X}$ is quasi-separated, then* $\mathrm{QCoh}(\mathcal{X})$ *has enough injectives.*

Proof. Recall that a functor $R\colon \mathcal{A} \to \mathcal{B}$ between abelian categories with an exact left adjoint L preserves injectives: for an injective I in $\mathcal{A}$, we have that $\mathrm{Hom}_{\mathcal{B}}(-, R(I)) = \mathrm{Hom}_{\mathcal{A}}(L(-), I)$ is exact.

By taking Λ to be the constant sheaf $\underline{\mathbb{Z}}$ or the structure sheaf $\mathcal{O}_{\mathcal{X}}$, the first statement will follow if we show that the category $\mathrm{Mod}(\Lambda)$ of Λ-modules has enough injectives for every sheaf of rings Λ on $\mathcal{X}_{\text{ét}}$. Let F be a Λ-module and let $U \to \mathcal{X}$ be an étale presentation. For each $u \in U$, we have a map $j_u\colon \{u\} \hookrightarrow U \to \mathcal{X}$ from a point and the stalk $F_u = j_u^{-1}F$ is an Λ_u-module. Choose an inclusion $F_u \hookrightarrow I_u$ into an injective Λ_u-module. Adjunction gives a map $F \to j_{u,*}I_u$, where $j_{u,*}$ is injective since j_u^{-1} is exact. By taking the product, we obtain an injection $F \to \prod_{u \in U} j_{u,*}I_u$ into an injective Λ-module.

For the final statement, let $F \in \mathrm{QCoh}(\mathcal{X})$ and let $p\colon U = \coprod_i \operatorname{Spec} A_i \to \mathcal{X}$ be an étale presentation. Choose an injection $p^*F \hookrightarrow I$ into an injective quasi-coherent $\mathcal{O}_U$-module. The composition $F \hookrightarrow p_*p^*F \hookrightarrow p_*I$ is injective and since p^* is exact, p_*I is injective. □

Remark 4.1.27. The above argument for the existence of enough injectives in $\mathrm{Mod}(\mathcal{O}_{\mathcal{X}})$ extends to the category of $\mathcal{O}$-modules in any ringed site with enough points (see [Ols16, Thm. 2.3.2]) and is even true in any ringed site [SP, Tag 01DP]. The category of quasi-coherent sheaves on an arbitrary Deligne–Mumford stack (or even algebraic stack) is a Grothendieck abelian category [SP, Tag 0781] and any such category has enough injectives [Gro57b], [SP, Tag 079H].

Definition 4.1.28 (Cohomology)**.** Let $\mathcal{X}$ be a Deligne–Mumford stack and F a sheaf of abelian groups on $\mathcal{X}_{\text{ét}}$. The *cohomology group* $\mathrm{H}^i(\mathcal{X}_{\text{ét}}, F)$ is defined as the ith right derived functor of the global sections functor $\Gamma\colon \mathrm{Ab}(\mathcal{X}_{\text{ét}}) \to \mathrm{Ab}$.

Given a morphism $f\colon \mathcal{X} \to \mathcal{Y}$ of Deligne–Mumford stacks, the *higher direct image* $R^i f_* F$ is defined as the ith right derived functor of $f_*\colon \mathrm{Ab}(\mathcal{X}_{\text{ét}}) \to \mathrm{Ab}(\mathcal{Y}_{\text{ét}})$.

The following is a key input to the development of quasi-coherent cohomology.

Theorem 4.1.29. *For a quasi-coherent $\mathcal{O}_{X_{\text{ét}}}$-module F on an affine scheme X, $\mathrm{H}^i(X_{\text{ét}}, F) = 0$ for all $i > 0$.*

We will prove this using Čech cohomology. Čech cohomology in the étale topology is defined similarly to the case of the Zariski topology [Har77, III.4] replacing intersections $U_{i_0} \cap \cdots \cap U_{i_n}$ with fiber products $U_{i_0} \times_{\mathcal{X}} \cdots \times_{\mathcal{X}} U_{i_n}$ and considering all (possibly non-distinct) indices $i_0, \ldots, i_n$ in any order.

Definition 4.1.30 (Čech cohomology)**.** Given an étale covering $\mathcal{U} = \{U_i \to \mathcal{X}\}_{i \in I}$ of a Deligne–Mumford stack and an abelian sheaf F on $\mathcal{X}_{\text{ét}}$, the *Čech complex of F with respect to $\mathcal{U}$* is $\check{\mathcal{C}}^\bullet(\mathcal{U}, F)$ where

$$\check{\mathcal{C}}^n(\mathcal{U}, F) = \prod_{(i_0, \ldots, i_n) \in I^{n+1}} F(U_{i_0} \times_{\mathcal{X}} \cdots \times_{\mathcal{X}} U_{i_n})$$

with differential

$$d^n\colon \check{\mathcal{C}}^n(\mathcal{U}, F) \to \check{\mathcal{C}}^{n+1}(\mathcal{U}, F), \qquad (s_{i_0, \ldots, i_n}) \mapsto \left(\sum_{k=0}^{n+1} (-1)^k p_{\widehat{k}}^* s_{i_0, \ldots, \widehat{i_k}, \ldots, i_n} \right)_{(i_0, \ldots, i_{n+1})}$$

where $p_{\widehat{k}}\colon U_{i_0} \times_{\mathcal{X}} \cdots \times_{\mathcal{X}} U_{i_n} \to U_{i_0} \times_{\mathcal{X}} \cdots \times_{\mathcal{X}} \widehat{U_{i_k}} \times_{\mathcal{X}} \cdots \times_{\mathcal{X}} U_{i_n}$ is the map forgetting the kth component (with indexing starting at 0). The *Čech cohomology of F with respect to $\mathcal{U}$* is

$$\check{\mathrm{H}}^i(\mathcal{U}, F) := \mathrm{H}^i(\check{\mathcal{C}}^\bullet(\mathcal{U}, F)).$$

The following is a standard result in Čech cohomology whose proof for sites is analogous to topological spaces. It is often referred to as Cartan's criterion; see [God58, II.5.9.2], [Mil80, Prop. 2.12], [SP, Tag 03F9] or [Ols16, Prop. 2.3.15].

Lemma 4.1.31. *Let $\mathcal{X}$ be a Deligne–Mumford stack and let F be an abelian sheaf on $\mathcal{X}_{\text{ét}}$. Suppose $\mathrm{Cov}'(\mathcal{X}) \subset \mathrm{Cov}(\mathcal{X})$ is a subset of coverings of $\mathcal{X}$ such that every covering of $\mathcal{X}$ has a refinement in $\mathrm{Cov}'(\mathcal{X})$. If for every covering $\mathcal{U} \in \mathrm{Cov}'$, $\check{\mathrm{H}}^i(\mathcal{U}, F) = 0$ for $i > 0$, then $\mathrm{H}^i(\mathcal{X}_{\text{ét}}, F) = 0$.*

With these preliminaries, we can prove Theorem 4.1.29.

Proof of Theorem 4.1.29. Let $X = \operatorname{Spec} A$, $F = \widetilde{M}$ be a quasi-coherent $\mathcal{O}_X$-module and $F_{\text{ét}}$ be the corresponding quasi-coherent $\mathcal{O}_{X_{\text{ét}}}$-module (Exercise 4.1.15). The set of étale coverings of the form $\mathcal{U} = \{\operatorname{Spec} B \to \operatorname{Spec} A\}$ is sufficient to refine any other covering. For the covering $\mathcal{U}$, faithful flat descent (Exercise B.1.2) implies that there is a long exact sequence

$$0 \to M \to M \otimes_A B \to M \otimes_A B \otimes_A B \to M \otimes_A B \otimes_A B \otimes_A B \to \cdots,$$

which is identified with the Čech complex $\check{\mathcal{C}}^\bullet(\mathcal{U}, F)$. This shows that $\check{\mathrm{H}}^i(\mathcal{U}, F) = 0$ for $i > 0$ and thus Lemma 4.1.31 implies that $\mathrm{H}^i(X_{\text{ét}}, F_{\text{ét}}) = 0$. $\square$

As with ordinary topological spaces [Har77, Exer. III.4.11], Čech cohomology can be computed using a covering with vanishing cohomology; see for instance [SP, Tag 03F7].

Lemma 4.1.32. *Let F be an abelian sheaf on $\mathcal{X}_{\text{ét}}$ and $(U_i \to \mathcal{X})_{i\in I}$ an étale covering. If $\mathrm{H}^i(U_{j_0} \times_U \cdots \times_U U_{j_n}, F) = 0$ for all $i > 0$, $n \geq 0$ and $j_0, \ldots, j_n \in I$, then $\check{\mathrm{H}}^i(\mathcal{U}, F) = \mathrm{H}^i(\mathcal{X}_{\text{ét}}, F)$.*

On a scheme with affine diagonal, both the étale and Zariski cohomology of a quasi-coherent sheaf can be computed on every affine open covering. We thus obtain:

Proposition 4.1.33. *Let X be a scheme with affine diagonal. Let F be a quasi-coherent $\mathcal{O}_X$-module and let $F_{\text{ét}}$ denote the corresponding quasi-coherent $\mathcal{O}_{X_{\text{ét}}}$-module (see Exercise 4.1.15). Then $\mathrm{H}^i(X, F) = \mathrm{H}^i(X_{\text{ét}}, F_{\text{ét}})$ for all i.*

Remark 4.1.34. The same result holds in the lisse-etale or fppf topology and without the affine diagonal hypothesis; see [SP, Tag 03DW] and [Mil80, Prop. 3.7].

Of course, in addition to being convenient to develop the theory of cohomology, Čech cohomology is also an extremely effective tool to compute cohomology groups. We have the following consequence of Theorem 4.1.29 and Lemma 4.1.32.

Proposition 4.1.35. *Let $\mathcal{X}$ be a Deligne–Mumford stack with affine diagonal and F be a quasi-coherent sheaf. If $\mathcal{U} = \{U_i \to \mathcal{X}\}$ is an étale covering with each U_i affine, then $\mathrm{H}^i(\mathcal{X}_{\text{ét}}, F) = \check{\mathrm{H}}^i(\mathcal{U}, F)$.*

To compare cohomologies computed in $\mathrm{Ab}(\mathcal{X}_{\text{ét}})$, $\mathrm{Mod}(\mathcal{O}_\mathcal{X})$ and $\mathrm{QCoh}(\mathcal{X})$, we have.

Proposition 4.1.36. *Let $\mathcal{X}$ be a Deligne–Mumford stack.*

(1) If F is an $\mathcal{O}_\mathcal{X}$-module, then the cohomology $\mathrm{H}^i(\mathcal{X}_{\text{ét}}, F)$ of F as an abelian sheaf agrees with the ith right derived functor of $\Gamma\colon \mathrm{Mod}(\mathcal{O}_\mathcal{X}) \to \mathrm{Ab}$.

(2) If $\mathcal{X}$ has affine diagonal and F is a quasi-coherent sheaf on $\mathcal{X}$, then the cohomology $\mathrm{H}^i(\mathcal{X}_{\text{ét}}, F)$ of F as an abelian sheaf agrees with the ith right derived functor of $\Gamma\colon \mathrm{QCoh}(\mathcal{X}) \to \mathrm{Ab}$.

For a morphism $f\colon \mathcal{X} \to \mathcal{Y}$ of Deligne–Mumford stacks (resp. quasi-compact morphism of Deligne–Mumford stacks with affine diagonals), then (1) (resp. (2)) holds also for the higher direct images $R^i f_* F$ of an $\mathcal{O}_\mathcal{X}$-module (resp. quasi-coherent sheaf): it can be computed as the ith right derived functor of $f_*\colon \mathrm{Mod}(\mathcal{O}_\mathcal{X}) \to \mathrm{Mod}(\mathcal{O}_\mathcal{Y})$ (resp. $f_*\colon \mathrm{QCoh}(\mathcal{X}) \to \mathrm{QCoh}(\mathcal{Y})$).

Proof. For (1), we need to show that an injective object in $\mathrm{Mod}(\mathcal{O}_\mathcal{X})$ is acyclic in $\mathrm{Ab}(\mathcal{X}_{\text{ét}})$. This uses a standard technique in Čech cohomology. We need some notation: given an étale covering $\mathcal{U} = \{U_i \to \mathcal{X}\}_{i\in I}$, we set $U_{\mathbf{i}} := U_{i_0} \times_\mathcal{X} \cdots \times_\mathcal{X} U_{i_n}$ with structure morphism $j_{\mathbf{i}}\colon U_{\mathbf{i}} \to \mathcal{X}$. There is a chain complex $\underline{\mathbb{Z}}_{\mathcal{U},\bullet}$ of presheaves on $\mathcal{X}$ defined by

$$\underline{\mathbb{Z}}_{\mathcal{U},n} := \bigoplus_{\mathbf{i}\in I^{n+1}} j_{\mathbf{i},!}\underline{\mathbb{Z}}$$

where $\underline{\mathbb{Z}}$ denotes the constant presheaf and $j_{\mathbf{i},!}\underline{\mathbb{Z}}$ is the presheaf whose sections over an $\mathcal{X}$-scheme V are $\bigoplus_{\mathrm{Mor}_\mathcal{X}(V, U_{\mathbf{i}})} \mathbb{Z}$. The differentials of $\underline{\mathbb{Z}}_{\mathcal{U},\bullet}$ are the alternating sums of the natural maps. This complex of presheaves is exact in positive degrees and has the property that for every presheaf F

$$\check{C}(\mathcal{U}, F) = \mathrm{Mor}_{\mathrm{PAb}(\mathcal{X}_{\text{ét}})}(\underline{\mathbb{Z}}_{\mathcal{U},\bullet}, F) = \mathrm{Mor}_{\mathrm{PMod}(\mathcal{O}_{\mathcal{X}})}(\underline{\mathbb{Z}}_{\mathcal{U},\bullet} \otimes_{\underline{\mathbb{Z}}} \mathcal{O}_{\mathcal{X}}, F),$$

where morphisms are computed in the categories $\mathrm{PAb}(\mathcal{X}_{\text{ét}})$ and $\mathrm{PMod}(\mathcal{O}_{\mathcal{X}})$ of presheaves. If $F \in \mathrm{Mod}(\mathcal{O}_{\mathcal{X}})$ is injective, then it is also injective as a presheaf of $\mathcal{O}_{\mathcal{X}}$-modules. It follows that $\check{C}(\mathcal{U}, F)$ is exact in positive degrees and thus $\check{\mathrm{H}}^i(\mathcal{U}, F) = 0$ for $i > 0$. Therefore Lemma 4.1.31 implies that $\mathrm{H}^i(\mathcal{X}_{\text{ét}}, F) = 0$. For more details, see [SP, Tag 03FD] or [Ols16, Cor. 2.3.16].

For (2), let $F \in \mathrm{QCoh}(\mathcal{X})$ be an injective object. Let $p \colon U = \coprod_i \operatorname{Spec} A_i \to \mathcal{X}$ be an étale presentation and choose an injection $p^*F \hookrightarrow G$ into an injective object $G \in \mathrm{QCoh}(U)$. Then pushforward p_*G is injective (as the right adjoint p^* is exact) and we have an inclusion $F \hookrightarrow p_*p^*F \hookrightarrow p_*G$ of injectives which splits. It thus suffices to show that p_*G is acyclic in $\mathrm{Ab}(\mathcal{X}_{\text{ét}})$. Since $\mathcal{X}$ has affine diagonal, $p \colon U \to \mathcal{X}$ is an affine morphism. By descent and Flat Base Change (Exercise 4.1.19), p_* is exact on the category of quasi-coherent sheaves. It follows that $\mathrm{H}^i(\mathcal{X}_{\text{ét}}, p_*G) = \mathrm{H}^i(U_{\text{ét}}, G) = 0$ by Theorem 4.1.29. □

It follows from (2) that for a scheme X with affine diagonal and for a quasi-coherent sheaf F, we have that $\mathrm{H}^i(X, F) = \mathrm{H}^i(X_{\text{ét}}, F_{\text{ét}})$.

Example 4.1.37 (Group cohomology)**.** Let G be a finite group viewed as a group scheme over a field $\Bbbk$, and let V be a G-representation. The *group cohomology* $\mathrm{H}^i(G, V)$ is defined as the ith right derived functor of $\mathrm{Rep}(G) \to \mathrm{Vect}_{\Bbbk}, V \mapsto V^G$. Since $\mathrm{H}^i(\mathbf{B}G_{\text{ét}}, \widetilde{V})$ can be computed in $\mathrm{QCoh}(\mathbf{B}G)$ (Proposition 4.1.36(2)) where $\widetilde{V}$ is the corresponding quasi-coherent sheaf on $\mathbf{B}G$ and there is an equivalence $\mathrm{Rep}_{\Bbbk}(G) \cong \mathrm{QCoh}(\mathbf{B}G)$, we have the identification

$$\mathrm{H}^i(G, V) \cong \mathrm{H}^i(\mathbf{B}G_{\text{ét}}, \widetilde{V}).$$

The Čech complex of $\widetilde{V}$ on $\mathbf{B}G$ corresponding to V with respect to the étale cover $\mathcal{U} = \{\operatorname{Spec} \Bbbk \to \mathbf{B}G\}$ has terms

$$\check{C}^n(\mathcal{U}, V) := \widetilde{V}((\operatorname{Spec} \Bbbk/\mathbf{B}G)^{n+1}) \cong \Gamma(G, \mathcal{O}_G)^{\otimes n} \otimes V.$$

To describe the differentials, let $\mu_k \colon G^{n+1} \to G^n$ for $k = 0, \dots, n$ be defined by sending $(g_1, \dots, g_{n+1})$ to $(g_1, \dots, g_n)$ for $k = 0$ and to $(g_1, \dots, g_{k-1}, g_k g_{k+1}, g_{k+2}, \dots, g_{n+1})$ for $k = 1, \dots, n$. Let $\sigma \colon V \to \Gamma(G, \mathcal{O}_G) \otimes V$ be the coaction map. The projection $p_{\widehat{k}} \colon (\operatorname{Spec} \Bbbk/\mathbf{B}G)^{n+2} \to (\operatorname{Spec} \Bbbk/\mathbf{B}G)^{n+1}$ is identified with $\mu_{n+1-k} \otimes \mathrm{id}$ for $k = 0, \dots n$ and $\mathrm{id} \otimes \sigma$ for $k = n+1$ (see Example 3.4.5). Thus the differentials in $\check{C}^\bullet(\mathcal{U}, V)$ are described by

$$\begin{aligned} d^n \colon \Gamma(G, \mathcal{O}_G)^{\otimes n} \otimes V \to & \Gamma(G, \mathcal{O}_G)^{\otimes (n+1)} \otimes V \\ f \otimes v \mapsto & \sum_{k=0}^{n} (-1)^k \mu_{n+1-k}^*(f) \otimes v + (-1)^{n+1} f \otimes \sigma(v) \end{aligned}$$

In low degrees, we have $d^0(v) = v - \sigma(v)$ and $d^1(f_1, v) = f_1 \otimes 1 \otimes v - \mu^*(f_1) \otimes v + f_1 \otimes \sigma(v)$ where $\mu = \mu_1$ is group multiplication $G \times G \to G$.

Since G is finite, there is an identification $\Gamma(G^n, \mathcal{O}_{G^n}) \otimes V \cong \mathrm{Map}(G^n, V)$ with set-theoretic maps, where a map $\phi \colon G^n \to V$ is identified with $\sum_{g \in G^n} e_g \phi(g)$

where e_g denotes the function which is 1 on g but otherwise 0. Thus the Čech complex $\check{C}^\bullet(\mathcal{U}, V)$ can be equivalently described as

$$0 \to V \xrightarrow{d^0} \operatorname{Map}(G, V) \xrightarrow{d^1} \operatorname{Map}(G^2, V) \xrightarrow{d^2} \cdots \tag{4.1.1}$$

where the differential d^n is defined by the formula

$$(d^n\phi)(g_1, \ldots, g_{n+1}) = \phi(g_1, \ldots, g_n) + \\ \sum_{k=1}^{n} (-1)^{n+1-k} \phi(g_1, \ldots, g_{k-1}, g_k g_{k+1}, \ldots, g_{n+1}) + (-1)^{n+1} g_1 \phi(g_2, \ldots, g_n)$$

for $\phi \in \operatorname{Map}(G^n, V)$. The complex (4.1.1) is sometimes referred to as the bar resolution (except that the differential d^n is usually multiplied by $(-1)^{n+1}$), and is an effective means to compute group cohomology. In low degrees, $d^0(v)(g) = v - gv$ and $d^1(\phi)(g_1, g_2) = \phi(g_1) - \phi(g_1 g_2) + g_1\phi(g_2)$.

Exercise 4.1.38. If $\mathcal{X}$ is a Deligne–Mumford stack and F_i is a directed system of abelian sheaves in $\mathcal{X}_{\text{ét}}$, show that $\operatorname{colim}_i \mathrm{H}^i(\mathcal{X}_{\text{ét}}, F_i) \to \mathrm{H}^i(\mathcal{X}_{\text{ét}}, \operatorname{colim}_i F_i)$ is an isomorphism.

Remark 4.1.39 (Comparison of topologies)**.** One can also define the fppf cohomology groups $\mathrm{H}^i(\mathcal{X}_{\text{fppf}}, F)$ of an abelian sheaf on the small fppf site of $\mathcal{X}$. There are some cases when this agrees with the small étale cohomology. For instance, if $G \to S$ is a *smooth, commutative*, and quasi-projective group scheme, then $\mathrm{H}^i(S_{\text{ét}}, G) = \mathrm{H}^i(S_{\text{fppf}}, G)$ [Mil80, Thm. 3.9]. For $\mathbb{G}_m$, there are identifications $\operatorname{Pic}(X) = \mathrm{H}^1(X_{\text{Zar}}, \mathcal{O}_X^*) = \mathrm{H}^1(X_{\text{ét}}, \mathbb{G}_m) = \mathrm{H}^1(X_{\text{fppf}}, \mathbb{G}_m)$ for a scheme X (Hilbert's Theorem 90, [Mil80, Prop. 4.9]).

On the other hand, if X is a smooth scheme over $\mathbb{C}$ and G is a finite abelian group, then the classical complex cohomology $\mathrm{H}^i(X(\mathbb{C}), G)$ agrees with the étale cohomology $\mathrm{H}^i(X_{\text{ét}}, G)$ of the constant sheaf associated to G [Mil80, Thm. 3.12].

Exercise 4.1.40 (Forms of group schemes)**.** Let G be an algebraic group over a field $\Bbbk$. We say that a group scheme $H \to \operatorname{Spec} \Bbbk$ is a *form of G* if there is an isomorphism $G_{\overline{\Bbbk}} \cong H_{\overline{\Bbbk}}$. We call G the *trivial form of G*.

(a) Show the algebraic group $H = \operatorname{Spec} \mathbb{R}[x, y]/(x^2 + y^2 - 1)$ over $\mathbb{R}$, with the group structure induced from the embedding $H \subset \mathrm{SL}_2$ given by

$$(x, y) \mapsto \begin{pmatrix} x & y \\ -y & x \end{pmatrix},$$

is a non-trivial form of $\mathbb{G}_{m,\mathbb{R}}$.

(b) Assume that $\operatorname{char}(\Bbbk) \neq 2$. Recall the orthogonal groups $O(q)$ defined in Exercise C.2.16 for a non-degenerate quadratic form q on an n-dimensional vector space V. Show that every $O(q)$ is a form of the subgroup $O_n \subset \mathrm{GL}_n$ of orthogonal matrices.

(c) If G is smooth and commutative, show that forms of G are classified by $\mathrm{H}^1((\mathrm{Sch}/\Bbbk)_{\text{ét}}, \operatorname{Aut}(G))$.

Remark 4.1.41 (Other cohomology theories)**.** See §6.1.6 for the development of sheaf cohomology on an algebraic stack. See §6.1.7 for a discussion of the Chow group of an algebraic stack, and §6.1.8 for a discussion of de Rham and singular cohomology.

4.2 Quotients by finite groups and the local structure of Deligne–Mumford stacks

Quotient stacks $[\operatorname{Spec} A/G]$ of an affine scheme by a finite group are particularly nice class of Deligne–Mumford stacks. Their geometry is the G-equivariant geometry of $\operatorname{Spec} A$. In this section, we show that the natural map $[\operatorname{Spec} A/G] \to \operatorname{Spec} A^G$ is universal for maps to algebraic spaces (Theorem 4.3.6) and that every Deligne–Mumford stack is étale locally isomorphic to a quotient stack of the form $[\operatorname{Spec} A/G]$ (Theorem 4.2.11).

4.2.1 Quotients by finite groups

Definition 4.2.1 (Geometric quotients)**.** If G is a finite group acting on an algebraic space U, a G-invariant morphism $U \to X$ is a *geometric quotient* if

(1) for every algebraically closed field $\Bbbk$, the map $U \to X$ induces a bijection $U(\Bbbk)/G \xrightarrow{\sim} X(\Bbbk)$, and

(2) $U \to X$ is universal for G-invariant maps to algebraic spaces, i.e. , every G-invariant map $U \to Y$ to an algebraic space factors uniquely as

$$\begin{tikzcd} U \arrow[d] \arrow[dr] & \\ X \arrow[r, dashed] & Y. \end{tikzcd}$$

If $\pi: U \to X$ is a geometric quotient, we often write $X = U/G$. In the case that G acts freely on U (i.e. the action map $G \times U \to U \times U$ is a monomorphism), then we have already defined the algebraic space quotient U/G and the map $U \to U/G$ is a geometric quotient.

If a finite group G acts on an affine scheme $\operatorname{Spec} A$, then G also acts on the ring A. We define the *invariant ring* as

$$A^G = \{f \in A \mid g \cdot f = f \text{ for all } g \in G\}.$$

We will show shortly that $\operatorname{Spec} A \to \operatorname{Spec} A^G$ is a geometric quotient (Theorem 4.2.6).

Example 4.2.2. Assume $\operatorname{char}(\Bbbk) \neq 2$. Let $G = \mathbb{Z}/2$ acts on $\mathbb{A}^1 = \operatorname{Spec} \Bbbk[x]$ via $-1 \cdot x = -x$, then $\Bbbk[x]^G = \Bbbk[x^2]$. The geometric quotient is the map $\mathbb{A}^1 = \operatorname{Spec} \Bbbk[x] \to \operatorname{Spec} \Bbbk[x^2] = \mathbb{A}^1$ sending p to p^2.

Let $G = \mathbb{Z}/2$ acts on $\mathbb{A}^2 = \operatorname{Spec} \Bbbk[x, y]$ via $-1 \cdot (x, y) = (-x, -y)$. Then $\Bbbk[x, y]^G = \Bbbk[x^2, xy, y^2]$ and the geometric quotient is $\mathbb{A}^2 \to \mathbb{A}^2/G = \operatorname{Spec} \Bbbk[x^2, xy, y^2]$. By setting $A = x^2, B = xy$ and $C = y^2$, the invariant ring can be identified with $\Bbbk[A, B, C]/(B^2 - AC)$ so the quotient $\mathbb{A}^2/G$ is a cone over a conic and in particular singular.

Lemma 4.2.3. *If G is a finite group acting on an affine scheme* $\operatorname{Spec} A$*, then $A^G \to A$ is integral. If A is finitely generated over a noetherian ring R, then $A^G \to A$ is finite and A^G is finitely generated over R.*

Proof. To see that $A^G \to A$ is integral, for every element $a \in A$ the product $\prod_{g \in G}(x - ga) \in A^G[x]$ is polynomial with invariant coefficients which has a as a root. If R is noetherian and $R \to A$ is of finite type, then $A^G \to A$ is also of finite type. As $A^G \to A$ is integral, it is finite (c.f. [AM69, Cor. 5.2]). Since R is noetherian, we may conclude by the Artin–Tate Lemma (c.f. [AM69, Prop. 7.8]) that $R \to A^G$ is of finite type. $\square$

The invariant ring is compatible with flat base change.

Lemma 4.2.4. *Let G be a finite group acting on an affine scheme* $\operatorname{Spec} A$*. If $A^G \to B$ is a flat ring homomorphism, then G acts on the affine scheme* $\operatorname{Spec}(B \otimes_{A^G} A)$ *and* $B = (B \otimes_{A^G} A)^G$.

Proof. By definition, the invariant ring is the equalizer

$$0 \to A^G \to A \underset{p_2}{\overset{p_1}{\rightrightarrows}} \prod_{g \in G} A$$

where $p_1(f) = (f)_{g \in G}$ and $p_2(f) = (gf)_{g \in G}$. Since $A^G \to B$ is flat, we have that

$$0 \to B \to A \otimes_{A^G} B \underset{p_2}{\overset{p_1}{\rightrightarrows}} \prod_{g \in G} A \otimes_{A^G} B$$

is also exact and we conclude that $B = (B \otimes_{A^G} A)^G$. $\square$

Exercise 4.2.5. Let $A^G \to B$ be a ring homomorphism and consider the commutative diagram

$$\begin{tikzcd}
 & \operatorname{Spec} B \otimes_{A^G} A \arrow[r] \arrow[dl] \arrow[d] \arrow[dr, phantom, "\square"] & \operatorname{Spec} A \arrow[d] \\
\operatorname{Spec}(B \otimes_{A^G} A)^G \arrow[r] & \operatorname{Spec} B \arrow[r] & \operatorname{Spec} A^G.
\end{tikzcd}$$

(a) Show that $\operatorname{Spec}(B \otimes_{A^G} A)^G \to \operatorname{Spec} B$ is an integral homeomorphism.

(b) If $|G|$ is invertible in A, show that $B \to (B \otimes_{A^G} A)^G$ is an isomorphism.

(c) Provide an example where $B \to (B \otimes_{A^G} A)^G$ is not an isomorphism.

Theorem 4.2.6. *If G is a finite group acting on an affine scheme* $\operatorname{Spec} A$*, then* $\operatorname{Spec} A \to \operatorname{Spec} A^G$ *is a geometric quotient. If A is finitely generated over a noetherian ring R, then A^G is also finitely generated over R.*

Proof. Consider the commutative diagram

$$\begin{tikzcd}
U = \operatorname{Spec} A \arrow[d] \arrow[dr, "\widetilde{\pi}"] & \\
\mathcal{X} = [U/G] \arrow[r, "\pi"] & X = \operatorname{Spec} A^G.
\end{tikzcd}$$

Since $\widetilde{\pi}$ is integral and dominant, it is surjective. To see that $\widetilde{\pi}$ is injective on G-orbits of geometric points, let $\Bbbk$ be an algebraically closed field and $x, x' \in U(\Bbbk)$ with $\widetilde{\pi}(x) = \widetilde{\pi}(x') \in X(\Bbbk)$. The base change $U \times_X \operatorname{Spec} \Bbbk = \operatorname{Spec}(A \otimes_{A^G} \Bbbk)$ inherits a G-action and the G-orbits $Gx, Gx' \subset U \times_X A^G$ are closed subschemes. If $Gx \neq Gx'$, then the orbits are disjoint and there exists a function $f \in A \otimes_R \Bbbk$ with $f|_{Gx} = 0$ and $f|_{Gx'} = 1$. Then $\widetilde{f} = \prod_{g \in G} gf \in (A \otimes_{A^G} \Bbbk)^G$ is a G-invariant function with $f'(x) = 0$ and $f'(x') = 1$. But this implies that $\widetilde{\pi}(x) \neq \widetilde{\pi}(x') \in X(\Bbbk)$, which is a contradiction.

The map $\widetilde{\pi}\colon U \to X$ is universal for G-invariant maps to algebraic spaces if and only if $\pi\colon \mathcal{X} = [U/G] \to X$ is universal for maps to algebraic spaces. In other words, we need to show that if Y is an algebraic space, then the natural map

$$\operatorname{Map}(X, Y) \to \operatorname{Map}(\mathcal{X}, Y) \tag{4.2.1}$$

is bijective. We note that this is immediate when Y is affine as $\Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) = \Gamma(X, \mathcal{O}_X)$ and the case when Y is a scheme can be reduced to this case without much effort: if $g\colon \mathcal{X} \to Y$ is a map, an affine covering Y_i of Y induces an open covering $X_i = X \setminus \pi(\mathcal{X} \setminus g^{-1}(Y_i))$ of X, and g restricts to a map $\pi^{-1}(X_i) \to Y_i$ which factors uniquely through X_i since $\pi_*\mathcal{O}_{\mathcal{X}} = \mathcal{O}_X$; see also [GIT, §0.6]. We need to work harder to handle the case that Y is an algebraic space.

For the injectivity of (4.2.1), let $h_1, h_2\colon X \to Y$ be two maps such that $h_1 \circ \pi = h_2 \circ \pi$. Let $E \to X$ be the equalizer of h_1 and h_2, i.e. the pullback of the diagonal $Y \to Y \times Y$ along $(h_1, h_2)\colon X \to Y \times Y$. The equalizer $E \to X$ is a monomorphism and locally of finite type. By construction $\pi\colon \mathcal{X} \to X$ factors through $E \to X$ and since π is universally closed and schematically dominant (i.e. $\mathcal{O}_X \to \pi_*\mathcal{O}_{\mathcal{X}}$ is injective), so is $E \to X$. As every universally closed and locally of finite type monomorphism is a closed immersion (see Corollary A.5.5 and Remark A.5.6), we conclude that $E \to X$ is an isomorphism.

For the surjectivity of (4.2.1), let $g\colon \mathcal{X} \to Y$ be a map. We claim that the question is étale-local on X. Indeed, if $V \to X$ is an étale cover and $h\colon V \to Y$ is a morphism such that the two compositions $V \times_X \mathcal{X} \to V \xrightarrow{h} Y$ and $V \times_X \mathcal{X} \to \mathcal{X} \xrightarrow{g} Y$ agree, then by the injectivity of (4.2.1) applied to the good moduli space $V \times_X V \times_X \mathcal{X} \to V \times_X V$, the two compositions $V \times_X V \rightrightarrows V \xrightarrow{h} Y$ agree and thus $h\colon V \to Y$ descends to a morphism $\overline{h}\colon X \to Y$. Étale descent also implies the commutativity of $g = \overline{h} \circ \pi$.

Since $\mathcal{X}$ is quasi-compact, we may assume that Y is quasi-compact as $g\colon \mathcal{X} \to Y$ factors through a quasi-compact open algebraic subspace of Y. Let $Y' \to Y$ be an étale presentation from an affine scheme and let $\mathcal{X}' := \mathcal{X} \times_Y Y'$. We claim that after replacing X with an étale cover $V \to X$ and $\mathcal{X}$ with the base change $\mathcal{X} \times_X V$, there is a section $s\colon \mathcal{X} \to \mathcal{X}'$ of $\mathcal{X}' \to \mathcal{X}$ in the commutative diagram

$$\begin{tikzcd}
\mathcal{X}' \arrow[r] \arrow[d, "g'"] \arrow[dr, phantom, "\square"] & \mathcal{X} \arrow[r, "\pi"] \arrow[d, "g"] \arrow[l, bend right, "s"'] & X \arrow[dl, dashed] \\
Y' \arrow[r] & Y.
\end{tikzcd}$$

The surjectivity of (4.2.1) follows from this claim: since X and Y' are affine, the equality $\Gamma(X, \mathcal{O}_X) = \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ implies that $\mathcal{X} \xrightarrow{s} \mathcal{X}' \xrightarrow{g'} Y'$ factors through $\pi\colon \mathcal{X} \to X$ via a morphism $X \to Y'$. The composition $X \to Y' \to Y$ yields the desired dotted arrow above.

We claim that limit methods allows us to reduce to the case that $X = \operatorname{Spec} A^G$ is the spectrum of a strictly henselian local ring. Indeed, for a closed point u of $U := \operatorname{Spec} A$ over $x \in |\mathcal{X}|$, the strict henselization $X^{\mathrm{sh}} := \mathcal{O}^{\mathrm{sh}}_{X, \pi(x)}$ is the limit $\lim_i X_i$ over all affine étale neighborhoods $X_i \to X$ of $\pi(x)$. The base change $U^{\mathrm{sh}} := U \times_X X^{\mathrm{sh}}$ is the limit of the affine schemes $U_i := U \times_X X_i$. We also set

$\mathcal{X}^{\mathrm{sh}} := \mathcal{X} \times_X X^{\mathrm{sh}} = [U^{\mathrm{sh}}/G]$ and $\mathcal{X}_i := \mathcal{X} \times_X X_i = [U_i/G]$. Since $\mathcal{X}' \to \mathcal{X}$ is locally of finite presentation, the natural map

$$\operatorname{colim}_i \operatorname{Mor}_{\mathcal{X}}(\mathcal{X}_i, \mathcal{X}') \to \operatorname{Mor}_{\mathcal{X}}(\mathcal{X}^{\mathrm{sh}}, \mathcal{X}')$$

is an equivalence; this follows from Exercise 3.3.31 using that $\operatorname{Mor}_{\mathcal{X}}(\mathcal{X}^{\mathrm{sh}}, \mathcal{X}')$ is the equalizer of $\operatorname{Mor}_{\mathcal{X}}(U^{\mathrm{sh}}, \mathcal{X}') \rightrightarrows \operatorname{Mor}_{\mathcal{X}}(G \times U^{\mathrm{sh}}, \mathcal{X}')$ and similarly for the left-hand side. A section of $\mathcal{X}' \times_{\mathcal{X}} \mathcal{X}^{\mathrm{sh}} \to \mathcal{X}^{\mathrm{sh}}$ is determined by a map $\mathcal{X}^{\mathrm{sh}} \to \mathcal{X}'$ over $\mathcal{X}$. This map extends to a morphism $\mathcal{X}_i \to \mathcal{X}'$ for some i which gives us the desired section.

Let κ be the residue field of A^G. As $A^G \to A$ is finite, $A = A_1 \times \cdots \times A_r$ is a product of strictly henselian local rings each finite over A^G (Proposition A.9.6). If $u \in \operatorname{Spec} A_1 \subset \operatorname{Spec} A$ is a closed point, then $\operatorname{Spec} A_1$ is G_u-invariant and the orbit Gu is in bijection with the r connected components of $\operatorname{Spec} A$. There is an isomorphism $\mathcal{X} \cong [\operatorname{Spec} A_1/G_u]$; this can be verified directly by for instance slicing the groupoid $G \times \operatorname{Spec} A \rightrightarrows \operatorname{Spec} A$ by $\operatorname{Spec} A_1 \hookrightarrow \operatorname{Spec} A$ (as in Exercise 3.4.15). We may thus replace $\mathcal{X} = [\operatorname{Spec} A/G]$ with $[\operatorname{Spec} A_1/G_u]$ and we can assume that there is a unique closed point $u \in \operatorname{Spec} A$ which is set-theoretically fixed by G. As $Y' \to Y$ is representable by schemes, we can write $\mathcal{X}' = [U'/G]$ for a scheme U'. Let $u' \in U'$ be a preimage of $u \in \operatorname{Spec} A$. As A is strictly henselian and the G-equivariant morphism $U' \to U$ is the base change of the étale morphism $Y' \to Y$, we see that $\kappa(u') = \kappa(u)$ and $G_{u'} = G_u = G$, and moreover the stabilizers act trivially on the residue fields. Again using that A is strictly henselian, there is a unique section $s\colon \operatorname{Spec} A \to U'$ with $s(u) = u'$ (Proposition A.9.3). This section is G-invariant because for every $g \in G$, both $s \circ g$ and $g \circ s$ are sections of $U' \to \operatorname{Spec} A \xrightarrow{g^{-1}} \operatorname{Spec} A$ with $u' \mapsto u$ and thus the sections agree. It follows that s descends to a section $\mathcal{X} = [\operatorname{Spec} A/G] \to [U'/G] = \mathcal{X}'$ of $\mathcal{X}' \to \mathcal{X}$. This finishes the proof that $\operatorname{Spec} A \to \operatorname{Spec} A^G$ is a geometric quotient.

The final statement follows from Lemma 4.2.3. □

Corollary 4.2.7. *Let G be a finite group acting freely on an affine scheme $U = \operatorname{Spec} A$, then the algebraic space quotient U/G is isomorphic to* $\operatorname{Spec} A^G$. □

Exercise 4.2.8. Let R be a noetherian ring. Let G be a finite group acting on a scheme U projective (resp. quasi-projective, quasi-affine) over a ring R. Show that there exists a geometric quotient $U \to U/G$ such that U/G is a projective (resp. quasi-projective, quasi-affine) scheme over R.

Exercise 4.2.9. Suppose that G is a finite group acting on an affine scheme $\operatorname{Spec} A$ of finite type over a noetherian ring R. If $x \in \operatorname{Spec} A$ is a closed point, show that there is an isomorphism

$$\widehat{A}^{G_x} \cong \widehat{A^G}$$

between the G_x-invariants of the completion at $\operatorname{Spec} A$ at x and the completion of $\operatorname{Spec} A^G$ at the image of x.

The following exercise generalizes Theorem 4.3.6 from quotients of finite groups to quotients of finite flat groupoids.

Exercise 4.2.10. Let $s, t \colon R \rightrightarrows U$ be a finite flat groupoid of affine schemes, and define $A^R \subset A$ as the subring of R-invariants, i.e. the subring of elements $a \in A$ such that $s^*a = t^*a \in \Gamma(R, \mathcal{O}_R)$. Show that $U \to X := \operatorname{Spec} A^R$ induces a bijection $U(\Bbbk)/R(\Bbbk) \xrightarrow{\sim} X(\Bbbk)$ for every algebraically closed field $\Bbbk$ and that $U \to X$ is universal for R-invariant maps to algebraic spaces. Moreover, show that if A is finitely generated over a noetherian ring, then so is A^R.

4.2.2 The Local Structure Theorem

We show that a Deligne–Mumford stack $\mathcal{X}$ near a point x is étale locally the quotient stack $[\operatorname{Spec} A/G_x]$ of an affine scheme by the stabilizer group scheme. Conceptually, this tells us that just as schemes (resp. algebraic spaces) are obtained by gluing affine schemes in the Zariski-topology (resp. étale-topology), Deligne–Mumford stacks are obtained by gluing quotient stacks $[\operatorname{Spec} A/G]$ in the étale topology.[1] Practically, this allows one to reduce many properties of Deligne–Mumford stacks to quotient stacks $[\operatorname{Spec} A/G]$. We will take advantage of this local structure in order to construct a coarse moduli space (Theorem 4.3.11).

The *geometric stabilizer* of a point x of a Deligne–Mumford stack $\mathcal{X}$ is the abstract group defined as the stabilizer of any geometric point $\operatorname{Spec} \Bbbk \to \mathcal{X}$ with image x.

Theorem 4.2.11 (Local Structure Theorem of Deligne–Mumford Stacks)**.** *Let $\mathcal{X}$ be a separated Deligne–Mumford stack and $x \in \mathcal{X}$ be a point with geometric stabilizer G_x. There exists an affine étale morphism*

$$f \colon ([\operatorname{Spec} A/G_x], w) \to (\mathcal{X}, x)$$

where $w \in [\operatorname{Spec} A/G_x]$ such that f induces an isomorphism of geometric stabilizer groups at w.

Proof. Choose a field-valued point $\operatorname{Spec} \Bbbk \to \mathcal{X}$ representing x. Let $(U, u) \to (\mathcal{X}, x)$ be an étale representable morphism from an affine scheme, and let d be the degree over x, i.e. the cardinality of $\operatorname{Spec} \Bbbk \times_{\mathcal{X}} U$. Since $\mathcal{X}$ is separated, $U \to \mathcal{X}$ is affine. Define the affine scheme

$$(U/\mathcal{X})^d := \underbrace{U \times_{\mathcal{X}} \cdots \times_{\mathcal{X}} U}_{d \text{ times}}.$$

For a scheme S, a morphism $S \to (U/\mathcal{X})^d$ corresponds to a morphism $S \to \mathcal{X}$ and d sections $s_1, \ldots, s_d$ of $U_S := U \times_{\mathcal{X}} S \to S$.

Let $(U/\mathcal{X})^d_0$ be the quasi-affine subscheme $(U/\mathcal{X})^d$ which is the complement of all pairwise diagonals, i.e. a map $S \to (U/\mathcal{X})^d_0$ corresponds to $S \to \mathcal{X}$ and n sections $s_1, \ldots, s_n \colon S \to U_S$ which are disjoint (meaning that the intersection of s_i and s_j is empty for $i \neq j$). There is an action of S_d on $(U/\mathcal{X})^d$ by permuting the sections and $(U/\mathcal{X})^d_0 \subset (U/\mathcal{X})^d$ is an S_d-equivariant open subscheme. By the correspondence between principal S_d-bundles and finite étale covers of degree d (Exercise C.2.8), an object of the quotient stack $[(U/\mathcal{X})^d_0/S_d]$ over a scheme S

[1] Of course, Deligne–Mumford stacks are also étale locally schemes but the étale neighborhoods $([\operatorname{Spec} A/G_x], w) \to (\mathcal{X}, x)$ produced by Theorem 4.2.11 preserve the stabilizer group at w.

corresponds to a diagram

$$\begin{tikzcd} Z \arrow[r, hook] \arrow[dr] & U_S \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & U \arrow[d] \\ & S \arrow[r] & \mathcal{X} \end{tikzcd}$$

where $Z \hookrightarrow U_S$ is a closed subscheme and $Z \to S$ is finite étale of degree d. Let $w \in [(U/\mathcal{X})^d_0/S_d](\Bbbk)$ be the point corresponding to $Z = \operatorname{Spec} \Bbbk \times_{\mathcal{X}} U$. There is an induced representable morphism $[(U/\mathcal{X})^d_0/S_d] \to \mathcal{X}$ and a commutative diagram

$$\begin{tikzcd} (U/\mathcal{X})^d_0 \arrow[r, hook] \arrow[d] & (U/\mathcal{X})^d \arrow[d] \\ {[(U/\mathcal{X})^d_0/S_d]} \arrow[dr] & U \arrow[d] \\ & \mathcal{X} \end{tikzcd}$$

Set $W := (U/\mathcal{X})^d_0$. The morphism $[W/S_d] \to \mathcal{X}$ is étale and representable, and induces an isomorphism of stabilizer groups at w.

By quotienting out by $G_x \subset S_d$ instead, the morphism $[W/G_x] \to \mathcal{X}$ which is also étale and representable, and induces an isomorphism of stabilizer groups at w. Letting $W' \subset W$ be an affine open subscheme containing w, we may replace W with the G_x-invariant affine open subscheme $\bigcap_{g \in G_x} g \cdot W'$.

It remains to show that $[W/G_x] \to \mathcal{X}$ is affine. Since $\mathcal{X}$ is separated, its diagonal is affine and the morphism $W \to \mathcal{X}$ from the affine scheme W is affine. The fiber product

$$\begin{tikzcd} {[W/G_x] \times_{\mathcal{X}} W} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & W \arrow[d] \\ {[W/G_x]} \arrow[r] & \mathcal{X} \end{tikzcd}$$

is affine over $[W/G_x]$ and thus isomorphic to a quotient stack $[\operatorname{Spec} B/G_x]$. On the other hand, since $[W/G_x] \to \mathcal{X}$ is representable, the quotient stack $[\operatorname{Spec} B/G_x]$ is an algebraic space and the action of G_x on $\operatorname{Spec} B$ is free. By Corollary 4.2.7, $[\operatorname{Spec} B/G_x]$ is isomorphic to the affine scheme $\operatorname{Spec} B^{G_x}$. By étale descent $[W/G_x] \to \mathcal{X}$ is affine.

See also [LMB, Thm. 6.2]. □

Exercise 4.2.12. Suppose that $\mathcal{X}$ is a Deligne–Mumford stack with quasi-affine diagonal. (In Corollary 4.4.8, we will show that if the diagonal of a Deligne–Mumford stack is separated and quasi-compact, then it is quasi-affine.) Let $x \in \mathcal{X}$ be a point with geometric stabilizer G_x. Modify the above argument to show that there is a quasi-affine and étale morphism $f \colon ([\operatorname{Spec} A/G_x], w) \to (\mathcal{X}, x)$ inducing an isomorphism of geometric stabilizer groups at w.

Exercise 4.2.13. Let $\mathcal{X}$ be a Deligne–Mumford stack. Show that $\mathcal{X}$ is isomorphic to a quotient stack $[U/G]$ where U is an affine scheme (resp. scheme, algebraic space) and G is a finite group if and only if there exists a finite étale morphism $V \to \mathcal{X}$ from an affine scheme (resp. scheme, algebraic space).

Hint: If $V \to \mathcal{X}$ is a finite étale cover of degree d, consider the associated principal S_d-torsor $\underbrace{V \times_{\mathcal{X}} \cdots \times_{\mathcal{X}} V}_{d \text{ times}} \setminus \Delta \to \mathcal{X}$; see Exercise C.2.8.

Proposition 4.2.14. *If $R \rightrightarrows U$ is a finite étale equivalence relation of affine schemes, then the algebraic space quotient U/R is an affine scheme.*

Proof. By Exercise 4.2.13, the algebraic space U/R is isomorphism to V/G for the free action of a finite group G on an affine scheme $V = \operatorname{Spec} B$. Theorem 4.3.6 shows that $V/G \to \operatorname{Spec} B^G$ is universal for maps to algebraic spaces and thus an isomorphism. Alternatively, this follows from Exercise 4.2.10: if $U = \operatorname{Spec} A$, then $U/R \to \operatorname{Spec} A^R$ is universal for maps to algebraic spaces and thus an isomorphism. $\square$

With a similar technique to the proof of Theorem 4.2.11, we can prove the following useful result asserting the existence of presentations with a lift of a given field-valued point.

Proposition 4.2.15. *If $\mathcal{X}$ is an algebraic stack with separated and quasi-compact diagonal and $x \in \mathcal{X}(\Bbbk)$ is a field-valued point, then there exists a smooth morphism $U \to \mathcal{X}$ from an affine scheme and a point $u \in U(\Bbbk)$ over x.*

Proof. Let $U \to \mathcal{X}$ be a smooth presentation and consider the fiber product

$$\begin{tikzcd} U_x \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & U \arrow[d] \\ \operatorname{Spec} \Bbbk \arrow[r, "x"] & \mathcal{X}. \end{tikzcd}$$

Since $\mathcal{X}$ is quasi-separated, so is U_x. If $u \in U_x$ is any closed point, then the inclusion $\operatorname{Spec} \kappa(u) \to U_x$ of the residue field (Proposition 3.5.16) is a closed immersion and $\Bbbk \to \kappa(u)$ is a finite separable extension of fields. Let $d = [\kappa(u) : \Bbbk]$. Following the notation of the proof of Theorem 4.2.11, if we set $V := (U/\mathcal{X})^d_0$ and consider the smooth morphism $[V/S_d] \to \mathcal{X}$. As $\operatorname{Spec} \kappa(u) \to \operatorname{Spec} \Bbbk$ is finite étale of degree d, the closed immersion $\operatorname{Spec} \kappa(u) \hookrightarrow U_x$ defines a $\Bbbk$-point v of $[V/S_d]$. This gives a commutative diagram

$$\begin{tikzcd} & {[V/S_d]} \arrow[d] \\ \operatorname{Spec} \Bbbk \arrow[ur, "v"] \arrow[r, "x"] & \mathcal{X}. \end{tikzcd}$$

By choosing a faithful representation $S_d \subset \mathrm{GL}_n$, we can write $[V/S_d] \cong [V'/\mathrm{GL}_n]$ where $V' = V \times^{S_d} \mathrm{GL}_n$. Then $v\colon \operatorname{Spec} \Bbbk \to [V'/\mathrm{GL}_n]$ corresponds to a principal GL_n-bundle $P \to \operatorname{Spec} \Bbbk$ and a GL_n-equivariant map $P \to V'$. Since principal GL_n-bundles are in bijection to vector bundles (Exercise C.2.11), P is the trivial principal GL_n-bundle and there is a section $\operatorname{Spec} \Bbbk \to P$. The composition $V' \to [V'/\mathrm{GL}_n] \to \mathcal{X}$ is smooth and the composition $\operatorname{Spec} \Bbbk \to P \to V'$ is a lift of x. It remains to show that we can arrange that V' is affine.

To show that we can arrange that V' is affine, we will use the fact that a quasi-separated algebraic space has quasi-affine diagonal; this is proved in Corollary 4.4.8 and relies on only the basic theory of quasi-coherent sheaves and Proposition 4.2.14.

The above argument reduces the claim to the case that $\mathcal{X}$ is an algebraic space. Choose an étale map $U \to \mathcal{X}$ from an affine scheme such that the image contains x. As the diagonal of $\mathcal{X}$ is quasi-affine, the map $U \to \mathcal{X}$ is quasi-affine. Repeating the argument in the proof, we observe that $V = (U/\mathcal{X})_0^d$ is a quasi-affine scheme with a free action of S_d. The quotient V/S_d is a quasi-affine scheme (Exercise 4.2.8) and we can simply choose an affine open neighborhood containing the $\Bbbk$-point v.

See also [LMB, Thm. 6.3]. $\square$

4.3 Coarse moduli spaces and the Keel–Mori Theorem

The goal of this section is to establish the Keel–Mori Theorem: every separated Deligne–Mumford stack $\mathcal{X}$ of finite type over a noetherian scheme admits a separated coarse moduli space $\pi\colon \mathcal{X} \to X$ (see Theorem 4.3.11). One can view this theorem as a way to remove the stackiness of a Deligne–Mumford stack; at the expense of sacrificing universal properties of $\mathcal{X}$ (e.g. existence of a universal family), one can replace $\mathcal{X}$ with an algebraic space without changing the underlying topological space.

We will later apply this theorem to show that the Deligne–Mumford stack $\overline{\mathcal{M}}_g$ parameterizing stable curves admits a coarse moduli space $\pi\colon \overline{\mathcal{M}}_g \to \overline{M}_g$ where $\overline{M}_g$ is a separated algebraic space, which we later show to be proper and then finally projective.

To prove Theorem 4.3.11, we will apply the Local Structure Theorem (4.2.11) to construct étale neighborhoods $[\operatorname{Spec}(A_i)/G] \to \mathcal{X}$ and show that the geometric quotients $\operatorname{Spec}(A_i^G)$ glue in the étale topology to a coarse moduli space of $\mathcal{X}$.

4.3.1 Coarse moduli spaces

We begin with the definition:

Definition 4.3.1. A morphism $\pi\colon \mathcal{X} \to X$ from an algebraic stack to an algebraic space is a *coarse moduli space* if

(1) for every algebraically closed field $\Bbbk$, the induced map $\mathcal{X}(\Bbbk)/{\sim} \to X(\Bbbk)$, from the set of isomorphism classes of objects of $\mathcal{X}$ over $\Bbbk$, is bijective, and

(2) π is universal for maps to algebraic spaces, i.e. every map $\mathcal{X} \to Y$ to an algebraic space factors uniquely as

$$\begin{tikzcd} \mathcal{X} \arrow[d, "\pi"'] \arrow[dr] & \\ X \arrow[r, dashed] & Y. \end{tikzcd}$$

Remark 4.3.2. If G is a finite group acting on an algebraic space U, then $[U/G] \to X$ is a coarse moduli space if and only if $U \to X$ is a geometric quotient (Definition 4.2.1).

Remark 4.3.3. In practice, we desire coarse moduli spaces with additional properties of $\pi\colon \mathcal{X} \to X$ as otherwise it is difficult to work with this notion. For instance, it is not true that this notion is stable under étale base change (or even open immersions) or that $\pi_* \mathcal{O}_{\mathcal{X}} = \mathcal{O}_X$. However, we emphasize that the Keel–Mori Theorem produces a coarse moduli space $\pi\colon \mathcal{X} \to X$ with the additional properties: (a) it is stable under flat base change, (b) $\pi_* \mathcal{O}_{\mathcal{X}} = \mathcal{O}_X$, (c) π is proper (and in particular separated!) and (d) π is a universal homeomorphism.

Lemma 4.3.4. *Let $\pi\colon \mathcal{X} \to X$ be a coarse moduli space such that for every étale morphism $X' \to X$ from an affine scheme, the base change $\mathcal{X} \times_X X' \to X'$ is a coarse moduli space. Then the natural map $\mathcal{O}_X \to \pi_* \mathcal{O}_{\mathcal{X}}$ is an isomorphism.*

Proof. As π is universal for maps to algebraic spaces, we have that $\mathrm{Map}(X, \mathbb{A}^1) \to \mathrm{Map}(\mathcal{X}, \mathbb{A}^1)$ is bijective or in other words $\Gamma(X, \mathcal{O}_X) \cong \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$. For every étale map $X' \to X$, the base change $\mathcal{X}' = \mathcal{X} \times_X X' \to X'$ is also a coarse moduli space and thus $\Gamma(X', \mathcal{O}_{X'}) \cong \Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})$. This shows that $\mathcal{O}_X \to \pi_* \mathcal{O}_{\mathcal{X}}$ is isomorphism. □

The property that a given map is a coarse moduli space can be checked étale locally.

Lemma 4.3.5. *Let $\pi\colon \mathcal{X} \to X$ be a morphism to an algebraic space. Suppose that there is an étale covering $\{X_i \to X\}$ such that $\mathcal{X} \times_X X_i \to X_i$ is a coarse moduli space for each i. Then $\pi\colon \mathcal{X} \to X$ is a coarse moduli space.* □

Proof. Axiom (1) of a coarse moduli space is a condition on geometric fibers and can thus be checked étale locally while Axiom (2) follows from the fact that algebraic spaces are sheaves in the étale topology. □

Theorem 4.3.6. *If G is a finite group acting on an affine scheme $\mathrm{Spec}\, A$, then $\pi\colon [\mathrm{Spec}\, A/G] \to \mathrm{Spec}\, A^G$ is a coarse moduli space. Moreover,*

(1) the base change of π along a flat morphism $X' \to \mathrm{Spec}\, A^G$ of algebraic spaces is a coarse moduli space,

(2) the natural map $X \to \pi_* \mathcal{O}_{\mathcal{X}}$ is an isomorphism, and

(3) if A is finitely generated over a noetherian ring R, then A^G is finitely generated over R and π is a proper universal homeomorphism.

Proof. We've already seen that $\pi\colon [\mathrm{Spec}\, A/G] \to \mathrm{Spec}\, A^G$ is a coarse moduli space (Theorem 4.3.6). To see (3), it suffices by Lemma 4.3.5 to consider flat morphisms $Y' \to Y$ from an affine scheme. But in this case, the base change $\mathcal{X} \times_Y Y'$ is isomorphic to a quotient stack $[\mathrm{Spec}\, B/G]$ and Lemma 4.2.4 implies that $Y' \cong \mathrm{Spec}\, B^G$. It follows that $\mathcal{X} \times_Y Y' \to Y'$ is a coarse moduli space. Part (2) follows directly from (1) by Lemma 4.3.4. For (3), we've already seen that A^G is finitely generated over R and that $A^G \to A$ is finite (Lemma 4.2.3). Since π is bijective and universally closed, its set-theoretic inverse is continuous, and thus π is a homeomorphism. The base change of π along a morphism $\mathrm{Spec}\, B \to \mathrm{Spec}\, A^G$ factors as $[\mathrm{Spec}(B \otimes_{A^G} A)/G] \to \mathrm{Spec}(B \otimes_{A^G} A)^G \to \mathrm{Spec}\, B$ where the first map is a homeomorphism by the above argument and the second is a homeomorphism by Exercise 4.2.5. We conclude that π is a universal homeomorphism. □

4.3.2 Descending étale morphisms to quotients

Proposition 4.3.7. *Let G be a finite group and $f\colon \mathrm{Spec}\, A \to \mathrm{Spec}\, B$ be a G-equivariant morphism of affine schemes of finite type over a noetherian ring R. Let $x \in \mathrm{Spec}\, A$ be a closed point. Assume that* *(a) f is étale at x and*

(b) the induced map $G_x \to G_{f(x)}$ of stabilizer group schemes is bijective.

Then there is an open affine neighborhood $W \subset \operatorname{Spec} A^G$ of the image of x such that $W \to \operatorname{Spec} A^G \to \operatorname{Spec} B^G$ is étale and $\pi_A^{-1}(W) \cong W \times_{\operatorname{Spec} B^G} [\operatorname{Spec} B/G]$, where $\pi_A \colon [\operatorname{Spec} A/G] \to \operatorname{Spec} A^G$.

Remark 4.3.8. In other words, after replacing $\operatorname{Spec} A^G$ with an affine neighborhood W of $\pi_A(x)$ and $\operatorname{Spec} A$ with $\pi_A^{-1}(W)$, it can be arranged that the diagram

$$\begin{tikzcd} {[\operatorname{Spec} A/G]} \arrow[r, "f"] \arrow[d, "\pi_A"] & {[\operatorname{Spec} B/G]} \arrow[d, "\pi_B"] \\ \operatorname{Spec} A^G \arrow[r] & \operatorname{Spec} B^G \end{tikzcd} \tag{4.3.1}$$

is cartesian where both horizontal maps are étale.

Condition (b) can be tested on a field-valued point $\operatorname{Spec} \Bbbk \to \operatorname{Spec} A$ representing x (e.g. the inclusion of the residue field).

In the proof of the Keel–Mori Theorem (Theorem 4.3.11), the above proposition will be applied in the following form.

Corollary 4.3.9. *Let G be a finite group and $f \colon \operatorname{Spec} A \to \operatorname{Spec} B$ be a G-equivariant morphism of affine schemes of finite type over a noetherian ring R. Assume that for every closed point $x \in \operatorname{Spec} A$,*

(a) f is étale at x and

(b) the induced map $G_x \to G_{f(x)}$ of stabilizer group schemes is bijective.

Then $\operatorname{Spec} A^G \to \operatorname{Spec} B^G$ is étale and (4.3.1) is cartesian. □

Proof of Proposition 4.3.7. Set $y = f(x)$. We first claim that the question is étale local around $\pi_B(y) \in \operatorname{Spec} B^G$. Indeed, if $Y' \to Y := \operatorname{Spec} B^G$ is an affine étale neighborhood of $\pi_B(y)$, we let $X', \mathcal{X}'$ and $\mathcal{Y}'$ denote the base changes of $X := \operatorname{Spec} A^G$, $\mathcal{X} := [\operatorname{Spec} A/G]$, and $\mathcal{Y} := [\operatorname{Spec} B/G]$. By Lemma 4.2.4, we know that $\mathcal{Y}' \cong [\operatorname{Spec} B'/G]$ with $Y' \cong \operatorname{Spec} B'^G$ and similarly for $\mathcal{X}'$ and X'. If the result holds after this base change, there is an open neighborhood $W' \subset X'$ containing a preimage of $\pi_A(x)$ such that $W' \hookrightarrow X' \to Y'$ is étale and such that the preimage of W' in $\mathcal{X}'$ is isomorphic to $W' \times_{Y'} \mathcal{Y}'$. Taking W as the image of W' under $X' \to \operatorname{Spec} A^G$ and applying étale descent yields the desired claim.

We now claim that this allows us to assume that B^G is strictly henselian. To see this, let $Y^{\mathrm{sh}} = \operatorname{Spec} \mathcal{O}^{\mathrm{sh}}_{Y,\pi_B(y)}$ and X^{sh}, $\mathcal{X}^{\mathrm{sh}}$ and $\mathcal{Y}^{\mathrm{sh}}$ be the base changes of X, $\mathcal{X}$ and $\mathcal{Y}$ along $Y^{\mathrm{sh}} \to Y$. Suppose $U^{\mathrm{sh}} \subset X^{\mathrm{sh}}$ is an open affine subscheme of the unique point in X^{sh} over x and the closed point of Y^{sh} such that $U^{\mathrm{sh}} \to Y^{\mathrm{sh}}$ is étale with $\pi^{-1}_{\mathcal{X}^{\mathrm{sh}}}(U^{\mathrm{sh}}) \cong U^{\mathrm{sh}} \times_{Y^{\mathrm{sh}}} \mathcal{Y}^{\mathrm{sh}}$. Then $Y = \lim_\lambda Y_\lambda$ is the limit of affine étale neighborhood $Y_\lambda \to Y$ of y and we set X_λ, $\mathcal{X}_\lambda$ and $\mathcal{Y}_\lambda$ to be the base changes of X, $\mathcal{X}$ and $\mathcal{Y}$ along $Y_\lambda \to Y$. By Proposition A.6.4, the morphism $U^{\mathrm{sh}} \to X^{\mathrm{sh}}$ descends to $U_\eta \to X_\eta$ for some η. Setting $U_\lambda = U_\eta \times_{X_\eta} X_\lambda$ for $\lambda > \eta$, it follows from Proposition A.6.7 that for $\lambda \gg 0$ (a) $U_\lambda \to X_\lambda$ is an open immersion, (b) the composition $U_\lambda \to X_\lambda \to Y_\lambda$ is étale, and (c) $\pi^{-1}_{\mathcal{X}_\lambda}(U_\lambda) \cong U_\lambda \times_{Y_\lambda} \mathcal{Y}_\lambda$ (by arguing on the étale presentations of $\mathcal{X}$ and $\mathcal{Y}$).

Finally, As $B^G \to B$ is finite (Lemma 4.2.3), $B = B_1 \times \cdots \times B_r$ is a product of strictly henselian local rings (Proposition A.9.6). As in the proof of Theorem 4.3.6, we may replace $[\operatorname{Spec} B/G]$ with $[\operatorname{Spec} B_1/G_y]$ and $[\operatorname{Spec} A/G]$ with $[f^{-1}(\operatorname{Spec} B_1)/G]$ to assume that G fixes x and y while acting trivially on the residue fields $\kappa(x) = \kappa(y)$. Thus $\operatorname{Spec} A \to \operatorname{Spec} B$ has a unique section $s\colon \operatorname{Spec} B \to \operatorname{Spec} A$ taking y to x. The section s is necessarily G-invariant (just as in the proof Theorem 4.3.6). Thus s descends to section of $\operatorname{Spec} A^G \to \operatorname{Spec} B^G$ which gives our desired open and closed subscheme $W \subset \operatorname{Spec} A^G$. □

Remark 4.3.10. Here's a conceptual reason for why we should expect the induced map of quotients to be étale. For simplicity, assume that $R = \Bbbk$ is an algebraically closed field. Let $\widehat{A}$ and $\widehat{B}$ be the completions of the local rings at x and $f(x)$. The stabilizers G_x and $G_{f(x)}$ act on $\operatorname{Spec} \widehat{A}$ and $\operatorname{Spec} \widehat{B}$, respectively, and the map $\operatorname{Spec} \widehat{A} \to \operatorname{Spec} \widehat{B}$ is equivariant with respect to the map $G_x \to G_{f(x)}$. The completion $\widehat{A^G}$ of A^G at the image of x is isomorphic to $\widehat{A}^{G_x}$ (Exercise 4.2.9) and similarly $\widehat{B^G} = \widehat{B}^{G_{f(x)}}$. Since f is étale at x, $\widehat{B} \to \widehat{A}$ is an isomorphism and since $G_x \to G_{f(x)}$ is bijective, the induced map $\widehat{B^G} \to \widehat{A^G}$ is an isomorphism which shows that $\operatorname{Spec} A^G \to \operatorname{Spec} B^G$ is étale at the image of x.

4.3.3 The Keel–Mori Theorem

We now state and prove the Keel–Mori Theorem.

Theorem 4.3.11. *Let $\mathcal{X}$ be a Deligne-Mumford stack separated and of finite type over a noetherian algebraic space S. Then there exists a coarse moduli space $\pi\colon \mathcal{X} \to X$ with $\mathcal{O}_X = \pi_*\mathcal{O}_{\mathcal{X}}$ such that*

(1) X is separated and of finite type over S,

(2) π is a proper universal homeomorphism, and

(3) for every flat morphism $X' \to X$ of algebraic spaces, the base change $\mathcal{X} \times_X X' \to X'$ is a coarse moduli space.

Remark 4.3.12. The Keel–Mori Theorem [KM97] holds more generally with the 'separated' condition on $\mathcal{X} \to S$ by the finiteness of the inertia $I_{\mathcal{X}} \to \mathcal{X}$; see Remark 4.3.13. In particular, it holds for algebraic stacks with finite but *non-reduced* automorphism groups. The theorem also holds without any noetherian or finiteness conditions; see [Con05b, Ryd13] and [SP, Tag 0DUK].

Proof. We first handle the case when $S = \operatorname{Spec} R$ is affine. The question is Zariski-local on $\mathcal{X}$: if $\{\mathcal{X}_i\}$ is a Zariski open covering of $\mathcal{X}$ with coarse moduli spaces $\mathcal{X}_i \to X_i$, then since coarse moduli spaces are unique (Definition 4.3.1(2)), the X_i's glue to form an algebraic space X and a map $\mathcal{X} \to X$, which is a coarse moduli space by Lemma 4.3.5. It thus suffices to show that every closed point $x \in |\mathcal{X}|$ has an open neighborhood which admits a coarse moduli space.

By the Local Structure Theorem of Deligne–Mumford Stacks (Theorem 4.2.11), there exists an affine étale morphism

$$f\colon (\mathcal{W} = [\operatorname{Spec} A/G_x], w) \to (\mathcal{X}, x)$$

such that f induces an isomorphism of geometric stabilizer groups at w.

We claim that since $\mathcal{X}$ is separated, the locus $\mathcal{U}$ consisting of points $z \in |\mathcal{W}|$, such that f induces an isomorphism of geometric stabilizer groups at z, is open.

To establish this, we will analyze the natural morphism $I_{\mathcal{W}} \to I_{\mathcal{X}} \times_{\mathcal{X}} \mathcal{W}$ of relative group schemes over $\mathcal{W}$ as the fiber of this morphism over $z \in \mathcal{W}(\Bbbk)$ is precisely the morphism $G_z \to G_{f(z)}$ of stabilizers. We will exploit the cartesian diagram

$$\begin{tikzcd} I_{\mathcal{W}} \arrow[r, "\Psi"] \arrow[d] \arrow[dr, phantom, "\square"] & I_{\mathcal{X}} \times_{\mathcal{X}} \mathcal{W} \arrow[d] \\ \mathcal{W} \arrow[r] & \mathcal{W} \times_{\mathcal{X}} \mathcal{W}; \end{tikzcd}$$

see Exercise 3.2.14. Since $\mathcal{W} \to \mathcal{X}$ is representable, étale and separated, the diagonal $\mathcal{W} \to \mathcal{W} \times_{\mathcal{X}} \mathcal{W}$ is an open and closed immersion and thus so is Ψ. Since $I_{\mathcal{X}} \to \mathcal{X}$ is finite, so is $p_2 \colon I_{\mathcal{X}} \times_{\mathcal{X}} \mathcal{W} \to \mathcal{W}$. Thus $p_2(|I_{\mathcal{X}} \times_{\mathcal{X}} \mathcal{W}| \setminus |I_{\mathcal{W}}|) \subset |\mathcal{W}|$ is closed and its complement, which is identified with the locus $\mathcal{U}$, is open.

Let $\pi_{\mathcal{W}} \colon \mathcal{W} \to W = \operatorname{Spec} A^{G_x}$ be the coarse moduli space (Theorem 4.3.6). Choose an affine open subscheme $X_1 \subset W$ containing $\pi_{\mathcal{W}}(w)$. Then $\mathcal{X}_1 = \pi_{\mathcal{W}}^{-1}(X_1)$ is isomorphic to a quotient stack $[\operatorname{Spec} A_1/G_x]$ such that $X_1 = \operatorname{Spec} A_1^{G_x}$. This provides an affine étale morphism

$$g \colon (\mathcal{X}_1 = [\operatorname{Spec} A_1/G_x], w) \to (\mathcal{X}, x)$$

which induces a bijection on all geometric stabilizer groups.

We now show that the open substack $\mathcal{X}_0 := \operatorname{im}(f)$ admits a coarse moduli space. Define $\mathcal{X}_2 := \mathcal{X}_1 \times_{\mathcal{X}} \mathcal{X}_1$ and $\mathcal{X}_3 := \mathcal{X}_1 \times_{\mathcal{X}} \mathcal{X}_1 \times_{\mathcal{X}} \mathcal{X}_1$. Since g is affine, each $\mathcal{X}_i$ is of the form $[\operatorname{Spec} A_i/G_x]$ and there is a coarse moduli space $\pi_i \colon \mathcal{X}_i \to X_i = \operatorname{Spec} A_i^{G_x}$. By universality of coarse moduli spaces, there is a diagram

$$\begin{tikzcd} \mathcal{X}_3 \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] \arrow[d, "\pi_3"] & \mathcal{X}_2 \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "\pi_2"] & \mathcal{X}_1 \arrow[r, "g"] \arrow[d, "\pi_1"] & \mathcal{X}_0 = \operatorname{im}(f) \arrow[d, dashed, "\pi_0"] \\ X_3 \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & X_2 \arrow[r, shift left] \arrow[r, shift right] & X_1 \arrow[r, dashed] & X_0 \end{tikzcd} \tag{4.3.2}$$

where the natural squares commute. Since g induces bijection of geometric stabilizer groups at all points, the same is true for each projection $\mathcal{X}_2 \to \mathcal{X}_1$ and $\mathcal{X}_3 \to \mathcal{X}_2$. Corollary 4.3.9 implies that each map $X_2 \to X_1$ and $X_3 \to X_2$ is étale, and the natural squares of solid arrows in (4.3.2) are cartesian.

The universality of coarse moduli spaces induces an étale groupoid structure $X_2 \rightrightarrows X_1$. To check that this is an étale equivalence relation, it suffices to check that $X_2 \to X_1 \times X_1$ is injective on geometric points but this follows from the observation the $|\mathcal{X}_2| \to |\mathcal{X}_1| \times |\mathcal{X}_1|$ is injective on closed points. Therefore there is an algebraic space quotient $X_0 := X_1/X_2$ and a map $X_1 \to X_0$. By étale descent along $\mathcal{X}_1 \to \mathcal{X}_0$, there is a map $\pi_0 \colon \mathcal{X}_0 \to X_0$ making the right square in (4.3.2) commute.

To argue that $\pi \colon \mathcal{X}_0 \to X_0$ is a coarse moduli space, we will use the commutative cube

$$\begin{tikzcd}[row sep=small, column sep=small]
& \mathcal{X}_2 \arrow[rr] \arrow[dl] \arrow[dd] & & \mathcal{X}_1 \arrow[dl] \arrow[dd] \\
\mathcal{X}_1 \arrow[rr, crossing over] \arrow[dd] & & \mathcal{X}_0 & \\
& X_2 \arrow[rr] \arrow[dl] & & X_1 \arrow[dl] \\
X_1 \arrow[rr] & & X_0, \arrow[from=uu, crossing over] &
\end{tikzcd}$$

where the top, left, and bottom faces are cartesian. It follows from étale descent along $\mathcal{X}_1 \to \mathcal{X}_0$ that the right face is also cartesian and since being a coarse moduli space is étale local on X_0 (Lemma 4.3.5), we conclude that $\mathcal{X}_0 \to X_0$ is a coarse moduli space. Except for the separatedness, the additional properties in the statement are étale-local on X_0 so they follow from the analogous properties of the coarse moduli space $[\operatorname{Spec}(A_1)/G_x] \to \operatorname{Spec}(A_1^{G_x})$ from Theorem 4.3.6. As $\mathcal{X}_0 \to X_0$ is proper, the separatedness of $\mathcal{X}_0$ is equivalent to the separatedness of X_0.

Finally, the case when S is a noetherian algebraic space can be reduced to the affine case by imitating the above argument to étale locally construct the coarse moduli space of $\mathcal{X}$. □

Remark 4.3.13. The more general case when $\mathcal{X}$ is an algebraic stack with finite inertia $I_{\mathcal{X}} \to \mathcal{X}$ (see Remark 4.3.12) is proven in an analogous but more technical manner. Namely, the use of the Local Structure Theorem for Deligne–Mumford stacks (Theorem 4.2.11) is replaced by the existence of an étale neighborhood $\mathcal{W} \to \mathcal{X}$ around every closed point such that $\mathcal{W}$ admits a finite flat presentation $V \to \mathcal{W}$ from an affine scheme and the corresponding groupoid $R := V \times_{\mathcal{W}} V \rightrightarrows V$ is a finite flat groupoid of affine schemes. This in turn is proven in an analogous way to Theorem 4.2.11 where one chooses a quasi-finite and flat surjection $U \to \mathcal{X}$ and one replaces the use of $[(U/\mathcal{X})_0^d/S_d])$ with a Hilbert stack $\mathcal{H}$ whose objects over a scheme S consists of a morphism $S \to \mathcal{X}$ and a closed subscheme $Z \hookrightarrow U_S$ finite and flat (rather than finite and étale) over S. (Aside: it is also possible to prove this without reference to a Hilbert scheme by using étale localization of groupoidsand splitting for groupoids; see [KM97, §4] or [SP, Tags 0DU4 and 04RJ]. Finally, the existence of a coarse moduli space for quotients $[V/R]$ is proven analogously to Theorem 4.3.6 (see Exercise 4.2.10).

The Local Structure Theorem of Deligne–Mumford Stacks (Theorem 4.2.11) can also be formulated étale locally on a coarse moduli space:

Corollary 4.3.14 (Local Structure of Coarse Moduli Spaces)**.** *Let $\mathcal{X}$ be a Deligne–Mumford stack of finite type and separated over a noetherian algebraic space S, and let $\pi \colon \mathcal{X} \to X$ be its coarse moduli space. For every closed point $x \in |\mathcal{X}|$ with geometric stabilizer group G_x, there exists a cartesian diagram*

$$\begin{tikzcd}
{[\operatorname{Spec} A/G_x]} \arrow[r] \arrow[d] & \mathcal{X} \arrow[d, "\pi"] \\
\operatorname{Spec} A^{G_x} \arrow[r] & X
\end{tikzcd}$$

such that $\operatorname{Spec} A^{G_x} \to X$ *is an étale neighborhood of* $\pi(x) \in |X|$.

Proof. This follows from the construction of the coarse moduli space in the proof of Theorem 4.3.11. Alternatively, it follows from the Local Structure Theorem of Deligne–Mumford stacks (Theorem 4.2.11) and Exercise 4.3.15 $\square$

Exercise 4.3.15. Establish the following generalization of Proposition 4.3.7: Let S be a noetherian algebraic space. Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a morphism of Deligne–Mumford stacks separated and of finite type over S and

$$\begin{tikzcd} \mathcal{X} \arrow[r, "f"] \arrow[d, "\pi_{\mathcal{X}}"] & \mathcal{Y} \arrow[d, "\pi_{\mathcal{Y}}"] \\ X \arrow[r] & Y \end{tikzcd}$$

be a commutative diagram where $\pi_{\mathcal{X}}\colon \mathcal{X} \to X$ and $\pi_{\mathcal{Y}}\colon \mathcal{Y} \to Y$ are coarse moduli spaces. Let $x \in |\mathcal{X}|$ be a closed point such that

(1) f is étale at x and

(2) the induced map $G_x \to G_{f(x)}$ of geometric stabilizer groups is bijective.

Then there exists an open neighborhood $U \subset X$ of $\pi_{\mathcal{X}}(x)$ such that $U \to X \to Y$ is étale and $\pi_{\mathcal{X}}(U) \cong U \times_Y \mathcal{Y}$.

Exercise 4.3.16. Let $\mathcal{X}$ be a Deligne–Mumford stack of finite type and separated over a noetherian algebraic space S, and let $\pi\colon \mathcal{X} \to X$ be its coarse moduli space. Assume that the order of the stabilizer of every geometric point of $\mathcal{X}$ is invertible in S.

(a) Show that the functor π_* is exact on quasi-coherent sheaves on $\mathcal{X}$.

(b) Show that for every morphism $X' \to X$ of algebraic spaces, the base change $\mathcal{X} \times_X X' \to X'$ is a coarse moduli space (see Exercise 4.2.5).

Exercise 4.3.17. Let $\mathcal{X}$ be a Deligne–Mumford stack of finite type and separated over a noetherian algebraic space S, and let $\pi\colon \mathcal{X} \to X$ be its coarse moduli space. Show that if $\mathcal{X}$ is normal, then so is X.

4.3.4 Examples

Example 4.3.18. Consider the moduli stack $\mathcal{M}_{1,1}$ of elliptic curves over a field $\Bbbk$ with $\operatorname{char}(\Bbbk) \neq 2, 3$. The Weierstrass form $y^2 = x(x-1)(x-\lambda)$ gives an isomorphism $\mathcal{M}_{1,1} \cong [(\mathbb{A}^1 \setminus \{0,1\})/S_3]$ (see Exercise 3.1.17) where the S_3-orbit of λ is $\{\lambda, 1/\lambda, 1-\lambda, 1/(1-\lambda), \lambda/(\lambda-1), (\lambda-1)/\lambda\}$. The coarse moduli space is given by j-invariant

$$j\colon \mathcal{M}_{1,1} \to \mathbb{A}^1, \quad \lambda \mapsto 2^8 \frac{(\lambda^2-\lambda+1)^3}{\lambda^2(\lambda-1)^3}.$$

Indeed, one can verify $\Bbbk[\lambda]^{S_3}_{\lambda(\lambda-1)} = \Bbbk[j(\lambda)]$.

Alternatively, the Weierstrass form $y^2 = x^3 + ax + b$ gives an isomorphism $\mathcal{M}_{1,1} \cong [\mathbb{A}^2_\Delta/\mathbb{G}_m]$ (see Exercise 3.1.17(b)) where the action is given by $t \cdot (a,b) = (t^4 a, t^6 b)$ and Δ is the discriminant $4a^3 + 27b^2$. As $\Bbbk[a,b]^{\mathbb{G}_m}_\Delta = \Bbbk[a^3/\Delta]$ (noting that $\beta := b^2/\Delta$ is generated by $\alpha := a^3/\Delta$ under the relation $4\alpha + 27\beta = 1$), the coarse moduli space $\mathcal{M}_{1,1} \to \mathbb{A}^1$ is given by $(a,b) \mapsto a^3/\Delta$.

Exercise 4.3.19. Let $\mathrm{char}(\Bbbk) \neq 2$ and $G = \mathbb{Z}/2$.

(a) Let G act on the non-separated union $X = \mathbb{A}^1 \bigcup_{x \neq 0} \mathbb{A}^1$ by exchanging the copies of $\mathbb{A}^1$. The quotient $[X/G]$ is a Deligne–Mumford stack with quasi-finite but not finite inertia, and in particular non-separated. Show nevertheless that there is a coarse moduli space $[X/G] \to \mathbb{A}^1$.

(b) Let X be the non-separated union $\mathbb{A}^2 \bigcup_{x \neq 0} \mathbb{A}^2$. Let $G = \mathbb{Z}/2$ act on X by simultaneously exchanging the copies of $\mathbb{A}^2$ and by acting via the involution $y \mapsto -y$ on each copy. Show that $[X/G]$ does not admit a coarse moduli space.

Example 4.3.20. Consider the action of PGL_2 on the scheme $\mathrm{Sym}^4 \mathbb{P}^1 \cong (\mathbb{P}^1)^4/S_4$ (which is the coarse moduli space of $[\mathbb{P}^1)^4/S_4]$) parameterizing four unordered points in $\mathbb{P}^1$. Let $\mathcal{X} \subset [\mathrm{Sym}^4 \mathbb{P}^1/\mathrm{PGL}_2]$ be the open substack parameterizing tuples (p_1, p_2, p_3, p_4) where at least three points are distinct. Consider the family $(0, 1, \lambda, \infty)$ with $\lambda \in \mathbb{P}^1$. If $\lambda \notin \{0, 1\infty\}$, then we claim that $\mathrm{Aut}(0, 1, \lambda, \infty) = \mathbb{Z}/2 \times \mathbb{Z}/2$. To see this, there is a unique element $\sigma \in \mathrm{PGL}_2$ such that $\sigma(0) = \infty$, $\sigma(\infty) = 0$ and $\sigma(1) = \lambda$ which acts on $\mathbb{P}^1$ via $\sigma([x, y]) = [y, \lambda, x]$ and thus $\sigma(\lambda) = 1$. Similarly, there is an element interchanging 0 with 1 and λ with ∞ and an element interchanging 0 with λ and 1 with ∞. However, if $\lambda \in \{0, 1\infty\}$, then $\mathrm{Aut}(0, 1, \lambda, \infty) = \mathbb{Z}/2$. We therefore see that the inertia $I_{\mathcal{X}} \to \mathcal{X}$ while quasi-finite is not finite and that $\mathcal{X}$ is not separated. Nevertheless, the map $\mathcal{X} \to \mathbb{P}^1$ taking (p_1, p_2, p_3, p_4) to its cross-ratio is a coarse moduli space.

4.3.5 Descending vector bundles to the coarse moduli space

We begin with a Nakayama lemma for coherent sheaves.

Lemma 4.3.21. *Let $\mathcal{X}$ be a Deligne-Mumford stack separated and of finite type over a noetherian algebraic space S, and let $\pi \colon \mathcal{X} \to X$ be its coarse moduli space. Let $x \in |\mathcal{X}|$ be a closed point.*

(1) If F is a coherent sheaf on $\mathcal{X}$ such that $F|_{\mathcal{G}_x} = 0$, then there exists an open neighborhood $U \subset X$ of $\pi(x)$ such that $F|_{\pi^{-1}(U)} = 0$.

(2) If $\phi \colon F \to G$ is a morphism of coherent sheaves (resp. vector bundles of the same rank) on $\mathcal{X}$ such that $\phi|_{\mathcal{G}_x}$ is surjective, then there exists an open neighborhood $U \subset X$ of $\pi(x)$ such that $\phi|_{\pi^{-1}(U)}$ is surjective (resp. an isomorphism).

Proof. For (1), the support $\mathrm{Supp}(F) \subset |\mathcal{X}|$ of F is a closed subset (which follows from using descent along a presentation) and the open set $U = X \setminus \pi(\mathrm{Supp}(F))$ satisfies the conclusion. For (2), apply (1) to the coherent sheaf $\mathrm{coker}(\phi)$ noting that a surjection of vector bundles of the same rank is an isomorphism. □

Definition 4.3.22. A Deligne–Mumford stack $\mathcal{X}$ is *tame* if for every geometric point $x \in \mathcal{X}(K)$, the order of $\mathrm{Aut}_{\mathcal{X}(K)}(x)$ is invertible in $\Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$.

Remark 4.3.23. If $\mathcal{X}$ is defined over a field $\Bbbk$, then this means that the order of every geometric stabilizer group is prime to the characteristic of $\Bbbk$.

Lemma 4.3.24. *Let $\mathcal{X}$ be a Deligne-Mumford stack separated and of finite type over a noetherian algebraic space S, and let $\pi \colon \mathcal{X} \to X$ be its coarse moduli space. If $\mathcal{X}$ is tame, then π_* is exact.*

Proof. The question is étale-local on X: if $g\colon X' \to X$ is an étale cover inducing a cartesian diagram

$$\begin{tikzcd} \mathcal{X}' \arrow[r, "g'"] \arrow[d, "\pi'"] & \mathcal{X} \arrow[d, "\pi"] \\ X \arrow[r, "g"] & X \end{tikzcd}$$

then by flat base change there is an identification $g^*\pi_* = \pi'_* g'^*$ of functors on quasi-coherent sheaves. Since g^* is faithfully exact, we see that π_* is exact if and only π'_* is. We can therefore use Corollary 4.3.14 to reduce to the case that $\mathcal{X} = [\operatorname{Spec} A/G]$ and $X = \operatorname{Spec} A^G$, and this case follows from Exercise 4.2.5. $\square$

We say that a vector bundle F on $\mathcal{X}$ *descends to its coarse moduli space* $\pi\colon \mathcal{X} \to X$ if there exists a vector bundle $\overline{F}$ on X and an isomorphism $F \cong \pi^*\overline{F}$. Observe that one necessary condition is that for every field-valued point $x\colon \operatorname{Spec} \Bbbk \to \mathcal{X}$ which induces a commutative diagram

$$\begin{tikzcd} \mathbf{B}G_x \arrow[r, hook, "i_x"] \arrow[d, "p"'] & \mathcal{X} \arrow[d, "\pi"] \\ \operatorname{Spec} \Bbbk \arrow[r, hook] & X, \end{tikzcd}$$

the pullback $i_x^* F = p^*(\overline{F} \otimes \Bbbk)$ is trivial or in other words G_x acts trivially on the fiber $F \otimes \Bbbk$.

Proposition 4.3.25. *Let $\mathcal{X}$ be a tame Deligne-Mumford stack separated and of finite type over a noetherian algebraic space S, and let $\pi\colon \mathcal{X} \to X$ be its coarse moduli space. A vector bundle F on $\mathcal{X}$ descends to a vector bundle on X if and only if for every field-valued point $x\colon \operatorname{Spec} \Bbbk \to \mathcal{X}$ with closed image, the action of G_x on the fiber $F \otimes \Bbbk$ is trivial. In this case, $\pi_* F$ is a vector bundle and the adjunction map $\pi^*\pi_* F \to F$ is an isomorphism.*

Remark 4.3.26. The above condition is insensitive to field extensions and equivalent to the condition that the restriction of F to the residual gerbe is trivial.

Proof. To see that the condition is sufficient, consider the commutative diagram

$$\begin{tikzcd} \mathcal{G}_x \arrow[r, hook] \arrow[d, "p"'] & \mathcal{X} \arrow[d, "\pi"] \\ \operatorname{Spec} \kappa(x) \arrow[r, hook] & X. \end{tikzcd}$$

We break down the proof into three steps.

Step 1: $\pi^*\pi_* F \to F$ is surjective. It suffices by Lemma 4.3.21 to show that $(\pi^*\pi_* F)|_{\mathcal{G}_x} \to F|_{\mathcal{G}_x}$ is surjective for every closed point $x \in |\mathcal{X}|$. Since $F \to F|_{\mathcal{G}_x}$ is surjective and π_* is exact (Lemma 4.3.24), $(\pi^*\pi_* F)|_{\mathcal{G}_x} \to \pi^*(\pi_*(F|_{\mathcal{G}_x}))|_{\mathcal{G}_x} \cong p^* p_*(F|_{\mathcal{G}_x})$ is surjective. The hypotheses imply that the adjunction $p^* p_*(F|_{\mathcal{G}_x}) \to F|_{\mathcal{G}_x}$ is an isomorphism and it follows that the composition $(\pi^*\pi_* F)|_{\mathcal{G}_x} \to p^* p_*(F|_{\mathcal{G}_x}) \xrightarrow{\sim} F|_{\mathcal{G}_x}$ is surjective.

Step 2: $\pi_* F$ is a vector bundle. We can assume that the rank r of F is constant. Since being a vector bundle is an étale-local property, we can assume that $X = \operatorname{Spec} A$. The surjection $\bigoplus_{s \in \Gamma(X, \pi_* F)} A \to \pi_* F$ pulls back to a surjection $\bigoplus_{s \in \Gamma(\mathcal{X}, F)} \mathcal{O}_{\mathcal{X}} \to \pi^* \pi_* F$ and by Step 1, the composition $\bigoplus_{s \in \Gamma(\mathcal{X}, F)} \mathcal{O}_{\mathcal{X}} \to \pi^* \pi_* F \to F$ is surjective. As $F|_{\mathcal{G}_x} \cong \mathcal{O}_{\mathcal{G}_x}^r$ is trivial, for each closed point $x \in |\mathcal{X}|$, we can find r sections $\phi \colon \mathcal{O}_{\mathcal{X}}^r \to F$ such that $\phi|_{\mathcal{G}_x}$ is an isomorphism. By Lemma 4.3.21, there exists an open neighborhood $U \subset X$ of $\pi(x)$ such that $\phi|_{\pi^{-1}(U)}$ is an isomorphism. Thus $\pi_* \phi \colon \mathcal{O}_X^r \to \pi_* F$ is an isomorphism over U and we conclude that $\pi_* F$ is a vector bundle of the same rank as F.

Step 3: $\pi^* \pi_* F \to F$ is an isomorphism. Since $\pi^* \pi_* F \to F$ is a surjection of vector bundles of the same rank, it is an isomorphism. □

Remark 4.3.27. The analogous statement for coherent sheaves is not true. For example, if the characteristic is not 2, then letting $\mathbb{Z}/2$ on $\mathbb{A}^1$ via $x \mapsto -x$, we have a tame coarse moduli space $[\mathbb{A}^1/(\mathbb{Z}/2)] \to \mathbb{A}^1 = \operatorname{Spec} \operatorname{Spec} k[x^2]$. The inclusion $\mathbf{B}\mathbb{Z}/2 \hookrightarrow [\mathbb{A}^1/(\mathbb{Z}/2)]$ of 0 is a closed substack and $\mathcal{O}_{\mathbf{B}\mathbb{Z}/2}$ is a coherent sheave which does not descend. Observe that in this case, the pullback of the residue field of $0 \in \mathbb{A}^1$ is $k[x]/x^2$. This example also illustrated that the fibers of a coarse moduli space $\mathcal{X} \to X$ can be non-reduced and larger than the residual gerbe.

When $\mathcal{X}$ is not tame, we have the following variant for descending line bundles.

Proposition 4.3.28. *Let $\mathcal{X}$ be a Deligne-Mumford stack separated and of finite type over a noetherian algebraic space S, and let $\pi \colon \mathcal{X} \to X$ be its coarse moduli space. If $\mathcal{L}$ is a line bundle on $\mathcal{X}$, then for N sufficiently divisible $\mathcal{L}^{\otimes N}$ descends to X.*

Proof. To be added. □

Example 4.3.29. Show $\operatorname{Pic}(\mathcal{M}_{1,1}) = \mathbb{Z}/12$ generated by the Hodge bundle (see Example 4.1.4).

4.4 When are algebraic spaces schemes?

We prove various results providing conditions for an algebraic space to be a scheme. We show:

- a quasi-separated algebraic space is a scheme on a dense open subspace (Theorem 4.4.1);
- Zariski's Main Theorem for algebraic spaces (Theorem 4.4.9);
- an algebraic space separated and locally quasi-finite over a scheme is a scheme (Corollary 4.4.7);
- if the diagonal of a Deligne–Mumford stack is separated and quasi-compact diagonal, then the diagonal is quasi-affine (and in particular representable by schemes) (Corollary 4.4.8);
- an algebraic stack with trivial stabilizers is an algebraic space (Theorem 4.4.10) generalizing Theorem 3.6.5;
- Serre's and Chevalley's Criteria for Affineness (Theorems 4.4.15 and 4.4.19) for algebraic spaces;

- if X is a quasi-separated algebraic space locally of finite type over a field $\Bbbk$ such that $X_{\Bbbk}$ has the property that every finite set of points is contained in an affine (e.g. $X_{\Bbbk}$ is quasi-projective), then X is a scheme (Proposition 4.4.24); and
- quasi-separated group algebraic spaces locally of finite type over a field are schemes (Theorem 4.4.25)

We also give applications to the algebraicity of quotients of étale and smooth equivalence relations (Corollary 4.4.11).

4.4.1 Algebraic spaces are schemes over a dense open

Theorem 4.4.1. *Every quasi-separated algebraic space has a dense open subspace which is a scheme.*

Proof. We may assume that X is quasi-compact. Let $f\colon V \to X$ be an étale presentation with V an affine scheme. Since X is quasi-separated, $f\colon V \to X$ is quasi-compact and there exists an open algebraic subspace $U \subset X$ such that $f^{-1}(U) \to U$ is finite. By Exercise 4.2.13, U is isomorphic to a quotient stack $[V/G]$ for the free action of a finite group G on a scheme V. If $V_1 \subset V$ is a dense affine open subscheme, then $V_2 = \bigcap_{g\in G} gV_1$ is a G-invariant quasi-affine open subscheme of V and in particular separated. Repeating this argument, we can choose a dense affine open subscheme $V_3 \subset V_2$ and now $V_4 = \bigcap_{g\in G} gV_3$ is a G-invariant affine open subscheme. Proposition 4.2.14 implies that $V_4/G \cong \operatorname{Spec} A^G$ is a dense affine open algebraic subspace of U. $\square$

Remark 4.4.2. See also [Knu71, II.6.7] and [SP, Tag 06NN]. The above result is not necessarily true if X is not quasi-separated, e.g. $\mathbb{A}^1/\underline{\mathbb{Z}}$ (Example 3.9.22).

Corollary 4.4.3. *An integral quasi-separated algebraic space has a well-defined fraction field.* $\square$

Exercise 4.4.4. Let G be a finite group acting on a quasi-separated algebraic space U. Show that there is a G-invariant affine open subscheme of U.

4.4.2 Zariski's Main Theorem for algebraic spaces

We now prove Zariski's Main Theorem for algebraic spaces and Deligne–Mumford stacks. Its proof relies on the theory of quasi-coherent sheaves. Specifically, we will use the fact that if $f\colon \mathcal{X} \to \mathcal{Y}$ is a quasi-compact and quasi-separated morphism of Deligne–Mumford stacks, then $f_*\mathcal{O}_X$ is a quasi-coherent sheaf of $\mathcal{O}_{\mathcal{Y}}$-algebras (Exercise 4.1.17) and there is a factorization

$$f\colon \mathcal{X} \to \mathcal{S}\mathrm{pec}_{\mathcal{Y}}\, f_*\mathcal{O}_{\mathcal{X}} \to \mathcal{Y}.$$

See §A.5 for a discussion of Zariski's Main Theorem for schemes. In this section, we follow [LMB, Thm. A.2] (see also [SP, Tag 05W7], [Knu71, II.6.15] and [Ols16, Thm. 7.2.10]).

Proposition 4.4.5. *A separated, quasi-finite and representable morphism $f\colon \mathcal{X} \to \mathcal{Y}$ of Deligne–Mumford stacks factors as the composition of an open immersion $\mathcal{X} \hookrightarrow \mathcal{S}\mathrm{pec}_{\mathcal{Y}}\, f_*\mathcal{O}_{\mathcal{X}}$ and an affine morphism $\mathcal{S}\mathrm{pec}_{\mathcal{Y}}\, f_*\mathcal{O}_{\mathcal{X}} \to \mathcal{Y}$. In particular, f is quasi-affine.*

Proof. Since the construction of $f_*\mathcal{O}_{\mathcal{X}}$ commutes with flat base change on $\mathcal{Y}$, so does the formation of the factorization $f\colon \mathcal{X} \to \mathcal{S}\mathrm{pec}_{\mathcal{Y}}\, f_*\mathcal{O}_{\mathcal{X}} \to \mathcal{Y}$. The statement is thus étale-local on $\mathcal{Y}$. In particular, we can assume that $\mathcal{Y} = Y$ is an affine scheme and that $\mathcal{X} = X$ is an algebraic space. After replacing Y with $\mathcal{S}\mathrm{pec}_Y\, f_*\mathcal{O}_X$, we can assume that $f_*\mathcal{O}_X = \mathcal{O}_Y$ and we must show that $f\colon X \to Y$ is an open immersion.

Since X is quasi-compact, there is an étale presentation $\pi\colon U \to X$ from an affine scheme. Since X is separated, $U \to X$ is also separated. As the composition

$$U \xrightarrow{\pi} X \xrightarrow{f} Y$$

is a quasi-finite morphism of schemes, we can apply Étale Localization of Quasi-finite Morphisms (Theorem A.5.1) around every point $y \in Y$: after replacing Y with an étale neighborhood, we can assume that $U = U_1 \sqcup U_2$ with $U_1 \to Y$ finite and $(U_2)_y = \emptyset$. Then $\pi(U_1)$ is open (as π is étale) and closed (as $U_1 \to Y$ is finite and $X \to Y$ is separated). Thus $X = X_1 \sqcup X_2$ with $X_1 = \pi(U_1)$ and $(X_2)_y = \emptyset$. This shows that $\mathcal{O}_Y = f_*\mathcal{O}_X$ is the product $\mathcal{A}_1 \times \mathcal{A}_2$ of quasi-coherent $\mathcal{O}_X$-algebras, and thus we can also decompose Y as $Y_1 \sqcup Y_2$ such that $y \in Y_1$ and $f(Y_i) \subset X_i$ for $i = 1, 2$. After replacing Y with Y_1, the composition $U \to X \to Y$ is finite and Lemma 4.4.6 implies that X is affine. Thus $X = Y = \mathcal{S}\mathrm{pec}_Y\, f_*\mathcal{O}_X$. □

Lemma 4.4.6. *Suppose that $U \to X$ is a surjective étale morphism of algebraic spaces and $X \to Y$ is a separated morphism of algebraic spaces. If the composition $U \to X \to Y$ is finite, so is $X \to Y$.*

Proof. The statement is étale-local on Y so we can assume that Y and U are affine. As $X \to Y$ is separated, $U \to X$ is also finite. Since X is identified with the quotient U/R of the finite étale groupoid $R := U \times_X U \rightrightarrows U$ of affine schemes, Proposition 4.2.14 implies that X is affine. As $U \to Y$ is proper, so is $X \to Y$. As $X \to Y$ is a proper and quasi-finite morphism of schemes, it is finite (Corollary A.5.5).

Alternatively, the properness of $X \to Y$ follows from the properness of $U \to Y$ and we may apply Corollary 4.4.13 to conclude that the proper and quasi-finite morphism $X \to Y$ is finite. □

Corollary 4.4.7. *A morphism of algebraic spaces which is separated and locally quasi-finite is representable by schemes. In particular, an algebraic space separated and locally quasi-finite over a scheme is a scheme.*

Proof. It suffices to show that if $X \to Y = \operatorname{Spec} A$ is a separated and locally quasi-finite, then X is a scheme. Since being a scheme is a Zariski-local property, we can assume that X is quasi-compact. Therefore Proposition 4.4.5 applies. □

Corollary 4.4.8. *The diagonal of a Deligne–Mumford stack with separated and quasi-compact diagonal is quasi-affine. In particular, a quasi-separated algebraic space has quasi-affine diagonal.*

Proof. The diagonal is separated, quasi-finite and representable and we conclude by Proposition 4.4.5. □

As with the case for schemes, we can refine Proposition 4.4.5 to obtain Zariski's Main Theorem.

Theorem 4.4.9 (Zariski's Main Theorem). *A separated, quasi-finite and representable morphism $f: \mathcal{X} \to \mathcal{Y}$ of noetherian Deligne–Mumford stacks factors as the composition of a dense open immersion $\mathcal{X} \hookrightarrow \widetilde{\mathcal{Y}}$ and a finite morphism $\widetilde{\mathcal{Y}} \to \mathcal{X}$.*

Proof. Let $\mathcal{A} \subset f_*\mathcal{O}_{\mathcal{X}}$ be the integral closure of $\mathcal{O}_{\mathcal{Y}} \to f_*\mathcal{O}_{\mathcal{X}}$ where the sections of $\mathcal{A}$ over an étale morphism $T \to \mathcal{Y}$ from a scheme is the integral closure of $\Gamma(T, \mathcal{O}_T) \to \Gamma(\mathcal{X} \times_{\mathcal{Y}} T, \mathcal{O}_{\mathcal{X}\times_{\mathcal{Y}} T})$. Since the integral closure is compatible under étale extensions (Proposition A.5.4), $\mathcal{A}$ is a quasi-coherent sheaf of $\mathcal{O}_{\mathcal{Y}}$-algebras. Using Exercise 4.1.25, write $\mathcal{A} = \operatorname{colim} \mathcal{A}_\lambda$ as the colimit of finite type $\mathcal{O}_{\mathcal{Y}}$-algebras. As $\mathcal{Y}$ is quasi-compact, there exists an étale presentation $p: U \to \mathcal{Y}$ from an affine scheme. Then the base change $\mathcal{X}_U \to U$ is a separated and quasi-finite morphism of algebraic spaces, thus a morphism of schemes by Corollary 4.4.7. We have that $p^*\mathcal{A} = \operatorname{colim} p^*\mathcal{A}_\lambda$ and by Theorem A.5.3 for $\lambda \gg 0$, the morphism $\mathcal{X}_U \to \mathcal{S}\mathrm{pec}_U\, p^*\mathcal{A}_\lambda$ is an open immersion and $\mathcal{S}\mathrm{pec}_U\, p^*\mathcal{A}_\lambda \to U$ is finite. By étale descent, $\mathcal{X} \to \mathcal{S}\mathrm{pec}_U\, \mathcal{A}_\lambda$ is an open immersion and $\mathcal{S}\mathrm{pec}_{\mathcal{Y}}\, \mathcal{A} \to \mathcal{Y}$ is finite. $\square$

4.4.3 Characterization of algebraic spaces

We now can remove the hypothesis in Theorem 3.6.5 that the diagonal is representable by schemes.

Theorem 4.4.10 (Characterization of Algebraic Spaces II). *For an algebraic stack $\mathcal{X}$, the following are equivalent:*

(1) the stack $\mathcal{X}$ is an algebraic space,

(2) the diagonal $\mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is a monomorphism, and

(3) every point of $\mathcal{X}$ has a trivial stabilizer.

Proof. We only need to show (2) $\Rightarrow$ (1). As the diagonal of $\mathcal{X}$ is a monomorphism, it is separated and locally quasi-finite. Corollary 4.4.7 implies that the diagonal $\mathcal{X}$ is representable by schemes and thus Theorem 3.6.5 applies. $\square$

Corollary 4.4.11.

(1) If X is a sheaf on $\mathrm{Sch}_{\text{ét}}$ such that there exists a surjective, étale (resp. smooth), and representable morphism $U \to X$ from an algebraic space, then X is an algebraic space.

(2) If $R \rightrightarrows U$ is an étale (resp. smooth) equivalence relation of algebraic spaces, then the quotient U/R is an algebraic space.

Remark 4.4.12. The above statement holds with with 'étale' replaced with 'fppf'; see Theorem 6.2.1 and corollary 6.2.4.

Proof. We first handle the étale case. For (1), by taking an étale presentation of U by a scheme, we may assume that U is a scheme. Let $T \to X$ be a morphism from a scheme, and we must show that the algebraic space $U \times_X T$ is a scheme. Since $U \times_X T \to U \times T$ is the base change of $X \to X \times X$, it is a monomorphism, thus separated and locally quasi-finite. By Corollary 4.4.7, $U \times_X T$ is a scheme. For (2), let $X = U/R$ be the quotient sheaf. By copying the argument of Theorem 3.4.11(1), we see that $U \to X$ is representable. The statement then follows from (1). Alternatively, Theorem 3.4.11(1) implies that U/R is an algebraic stack and the statement follows from Theorem 4.4.10.

In the noetherian and smooth case, the sheaf X in (1) is an algebraic stack by definiton and the quotient stack $[U/R]$ is an algebraic stack by Theorem 3.4.11. Theorem 4.4.10 implies that X and $[U/R]$ are algebraic spaces. $\square$

Corollary 4.4.13. *A proper and quasi-finite morphism (resp. proper monomorphism) of algebraic spaces is finite (resp. a closed immersion).*

Proof. Proper and quasi-finite morphisms are representable by schemes. Thus the statement follows from the corresponding result for schemes (Corollary A.5.5) and étale descent. $\square$

Exercise 4.4.14. Consider the prestack $\underline{\mathrm{AlgSp}}$ over $\mathrm{Sch}_{\text{ét}}$ whose objects over a scheme T are algebraic spaces over T and where morphisms correspond to cartesian diagrams of algebraic spaces. Show that $\underline{\mathrm{AlgSp}}$ is a stack.

4.4.4 Affineness criteria

Theorem 4.4.15 (Serre's Criterion for Affineness)**.** *Let X be be a quasi-compact and quasi-separated (resp. noetherian) algebraic space. If the functor $\Gamma(X,-)$ is exact on the category of quasi-coherent (resp. coherent) sheaves, then X is an affine scheme.*

Proof. If X is noetherian, then every quasi-coherent sheaf is a colimit of coherent sheaves (Exercise 4.1.20) and $\Gamma(X,-)$ commutes with colimits. Assume that $\Gamma(X,-)$ is exact on coherent sheaves. Given a surjection $p\colon F \twoheadrightarrow G$ of quasi-coherent sheaves on X, write $G = \operatorname{colim}_i G_i$ as a colimit of coherent sheaves and choose coherent subsheaves $F_i \subset p^{-1}(G_i)$ surjecting onto G_i. Then $\Gamma(X,F_i) \twoheadrightarrow \Gamma(X,G_i)$ and the composition $\operatorname{colim}_i \Gamma(X,F_i) \to \Gamma(X,F) \to \Gamma(X,G) = \operatorname{colim}_i \Gamma(X,G_i)$ is surjective. Thus $\Gamma(X,F) \to \Gamma(X,G)$ is surjective and we conclude that $\Gamma(X,-)$ is exact on quasi-coherent sheaves.

We show that the canonical morphism $\pi\colon X \to Y := \operatorname{Spec}\Gamma(X,\mathcal{O}_X)$ is a proper monomorphism. This gives the result as Corollary 4.4.13 implies that $X \to Y$ is a closed immersion so that that X is affine and $X \to Y$ is an isomorphism. As a first step, we establish:

Claim: If $g\colon Y' \to Y$ is a morphism of algebraic spaces, then the base change $\pi'\colon X' := X \times_Y Y' \to Y'$ has the following properties:

- *(a) π'_* induces an equivalence of the categories of quasi-coherent sheaves on X' and Y'.*
- *(b) $\mathcal{O}_{Y'} \to \pi'_*\mathcal{O}_{X'}$ is an isomorphism.*
- *(c) $X' \to Y'$ is a homeomorphism.*

By Flat Base Change (Exercise 4.1.19), properties (a) and (b) are étale local on Y' so we may assume $Y' = \operatorname{Spec} B$. We will show that the adjunction morphisms $G \to \pi'_*\pi'^*G$ and $\pi'^*\pi'_*F \to F$ are isomorphisms for quasi-coherent sheaves G and F and Y' and X', respectively. For the first adjunction, choose a free presentation $\mathcal{O}_Y^{\oplus J} \to \mathcal{O}_Y^{\oplus J} \to g_*G \to 0$ of G as an $\mathcal{O}_Y$-module. As π_* is exact, we have a morphism of right exact sequences

$$\begin{tikzcd}
\mathcal{O}_Y^{\oplus J} \arrow[r] \arrow[d] & \mathcal{O}_Y^{\oplus I} \arrow[r] \arrow[d] & g_*G \arrow[r] \arrow[d] & 0 \\
\pi_*\pi^*(\mathcal{O}_Y^{\oplus J}) \arrow[r] & \pi_*\pi^*(\mathcal{O}_Y^{\oplus I}) \arrow[r] & \pi_*\pi^*g_*G \arrow[r] & 0
\end{tikzcd}$$

The left two vertical arrows are isomorphism since $\pi_*\mathcal{O}_X = \mathcal{O}_Y$. Therefore $g_*G \to g_*\pi_*\pi^*G \cong g_*\pi'_*\pi'^*G$ is an isomorphism. Since g_* is faithfully exact, $G \to \pi'_*\pi'^*G$ is also an isomorphism. We note that property (b) already follows from this fact by taking $G = \mathcal{O}_{Y'}$ and the fact that affine morphisms are faithfully exact on quasi-coherent sheaves.

To see the second adjunction, let K and Q be the kernel and cokernel of $\pi'^*\pi'_*F \to F$. As π_* is exact and g_* is faithfully exact, we see that π'_* is exact. Since $\pi'_*\pi'^*\pi'_*F \to \pi'_*F$ is an isomorphism (using that the first adjunction is an isomorphism), we see that $\pi'_*K = \pi'_*Q = 0$. It thus suffices to show that for a quasi-coherent sheaf F' on X', then $F' \neq 0$ implies $\pi'_*F' \neq 0$. If $x\colon \operatorname{Spec} \Bbbk \to X'$ is a geometric point such that $x^*F \neq 0$, then by base changing by the composition $\pi' \circ x\colon \operatorname{Spec} \Bbbk \to Y'$, we may assume that $Y' = \operatorname{Spec} \Bbbk$ and that $x\colon \operatorname{Spec} \Bbbk \to X'$ is a section of π'. Since every $\Bbbk$-point of an algebraic space defined over $\Bbbk$ is a closed point, $x\colon \operatorname{Spec} \Bbbk \to X'$ is a closed immersion and hence $F \to x_*x^*F = F \otimes \Bbbk$ is surjective. It follows from the exactness of π'_* that $\pi'_*F \to F \otimes \Bbbk$ is surjective and hence $\pi'_*F \neq 0$. This finishes the proof of (a) and (b).

To see (c), if $y\colon \operatorname{Spec} \Bbbk \to Y$ is a geometric point, then by (b) $\Gamma(X_y, \mathcal{O}_{X_y}) = \Bbbk$ as thus the fiber X_y is non-empty. On the other hand, if $x, x' \in X_y(\Bbbk)$ were distinct points each necessarily closed, then $\mathcal{O}_{X_y} \to \mathcal{O}_{\{x,x'\}}$ is surjective. Since π_* is exact, we also get a surjection $\Bbbk = \Gamma(X_y, \mathcal{O}_{X_y}) \to \Bbbk \oplus \Bbbk$, a contradiction. To see that π' is closed, let $Z \subset X'$ be a closed subspace and $q\colon Z \to \operatorname{im}(Z)$ denote the morphism to its scheme-theoretic image. Then $\mathcal{O}_Z \to q_*\mathcal{O}_{\operatorname{im}(Z)}$ is an isomorphism and q_* is exact. Applying the surjectivity result above to q, we see that q is surjective and hence $\pi'(Z)$ is closed.

With the claim established, we now show that $X \to Y$ is a monomorphism and in particular separated. To see that the diagonal $\Delta\colon X \to X \times_Y X$ is an isomorphism, observe that the pushforward of $\mathcal{O}_{X \times_Y X} \to \Delta_*X$ along the first projection $p_1\colon X \times_Y X \to X$ is an isomorphism. Thus (a) applied to p_1 shows that $\mathcal{O}_{X \times_Y X} \to \Delta_*X$ is an isomorphism. Zariski's Main Theorem (A.5.3) implies that Δ is an open immersion. Applying (c) to p_1 shows that $p_1\colon |X \times_Y X| \to |X|$ is bijective. Hence Δ must be an isomorphism.

It remains to show that $X \to Y$ is of finite type. Let $U = \operatorname{Spec} A \to X$ be an étale presentation. Since X is separated, $R := U \times_X U$ is a closed subscheme of $U \times_Y U = \operatorname{Spec} A \otimes_{\Gamma(X, \mathcal{O}_X)} A$. Hence $R = \operatorname{Spec} B$ is affine. Letting s and t

denote the two maps $A \rightrightarrows B$, we have a commutative diagram

$$\begin{tikzcd}
& A \arrow[dr] \arrow[drr, "s"] & & \\
\Gamma(X, \mathcal{O}_X) \arrow[ur] \arrow[dr] & & A \otimes_{\Gamma(X,\mathcal{O}_X)} A \arrow[r] & B. \\
& A \arrow[ur] \arrow[urr, "t"'] & &
\end{tikzcd}$$

Since $U \to X$ is étale, $t\colon A \to B$ is of finite type and there are generators $b_1, \dots, b_n \in B$ over t. For each i, choose a preimage $\sum_j a_{ij} \otimes a'_{ij} \in A \otimes_{\Gamma(X,\mathcal{O}_X)} A$ of b_i. Viewing B as an A-algebra via t, then $\sum_j a'_{ij} s(a_{ij}) = b_i$ and thus we have elements $a_{ij} \in A$ such that $s(a_{ij})$ generate B over t. Then $a_{ij} \in \Gamma(X, p_* \mathcal{O}_U) = A$ define a homomorphism $\mathcal{O}_X[z_{ij}] \to p_* \mathcal{O}_U$ of $\mathcal{O}_X$-algebras taking z_{ij} to a_{ij}. Its pullback via p is identified with $\mathcal{O}_U[z_{ij}] \to p^* p_* \mathcal{O}_U \cong t_* \mathcal{O}_R$, where the last equivalence comes from Flat Base Change (Exercise 4.1.19), and this map is surjective precisely because $s(a_{ij})$ generate B over t. By étale descent, $\mathcal{O}_X[z_{ij}] \to p_* \mathcal{O}_U$ is surjective and therefore so is $\Gamma(X, \mathcal{O}_X)[z_{ij}] \to A$. Thus $\Gamma(X, \mathcal{O}_X) \to A$ is of finite type and by étale descent $X \to Y$ is also of finite type.

See also [Knu71, Thm. III.2.5], [Ryd15, Thm. 8.7] and [SP, Tag 07V6]. □

Corollary 4.4.16. *Let X be be a quasi-compact and quasi-separated (resp. noetherian) algebraic space. Then X is an affine scheme if and only if $\mathrm{H}^i(X, F) = 0$ for every quasi-coherent (resp. coherent) sheaf F and $i > 0$.*

Proof. If X is affine, then Theorem 4.1.29 establishes the vanishing of quasi-coherent cohomology. Conversely, the vanishing of quasi-coherent (resp. coherently) cohomology implies that $\Gamma(X, -)$ is exact on the category of quasi-coherent (resp. coherent) sheaves: if $0 \to F_1 \to F_2 \to F_3 \to 0$ is exact, then $\Gamma(X, F_2) \to \Gamma(X, F_3)$ is surjective as $\mathrm{H}^1(X, F_1) = 0$. □

Remark 4.4.17. Given a quasi-compact and quasi-separated morphism $f\colon \mathcal{X} \to \mathcal{Y}$ of Deligne–Mumford stacks, the condition that $f_*\colon \mathrm{QCoh}(\mathcal{X}) \to \mathrm{QCoh}(\mathcal{Y})$ is exact is fppf local on $\mathcal{Y}$ (see Lemma 6.3.15). Since $R^i f_* F$ can be computed in $\mathrm{QCoh}(\mathcal{X})$, the relative versions of Theorem 4.4.15 and Corollary 4.4.16 also hold: f is affine if and only if $f_*\colon \mathrm{QCoh}(\mathcal{X}) \to \mathrm{QCoh}(\mathcal{Y})$ is exact if and only if $R^i f_* F = 0$ for all $i > 0$ and $F \in \mathrm{QCoh}(\mathcal{X})$.

Proposition 4.4.18. *Let X be a noetherian algebraic space. If X_{red} is a scheme (resp. quasi-affine, affine), then so is X.*

Proof. If X_{red} is affine, then one uses Corollary 4.4.16 to show that X is affine exactly as in [Har77, Exer. III.3.1]: if F is a coherent sheaf on X and $I \subset \mathcal{O}_X$ denotes the nilpotent ideal defining X_{red}, then one shows the vanishing of $\mathrm{H}^i(X, F)$ using the filtration $0 = I^N F \subset I^{N-1} F \subset \cdots \subset IF \subset F$, whose factors $I^k F / I^{k+1} F$ are supported on X_{red}.

If X_{red} is quasi-affine, then $X_{\mathrm{red}} \to \operatorname{Spec} \Gamma(X, \mathcal{O}_X)_{\mathrm{red}}$ is an open immersion. Thus $X \to \operatorname{Spec} \Gamma(X, \mathcal{O}_X)$ is an open immersion and X is quasi-affine. If X_{red} is a scheme, then every point $x \in |X|$ has an open neighborhood U such that U_{red} is affine. Thus U is affine and X is a scheme. □

Theorem 4.4.19 (Chevalley's Criterion for Affineness)**.** *Let Y be a noetherian algebraic space and $X \to Y$ be a finite surjective morphism of algebraic spaces. If X is affine, then so is X.*

Proof. One can argue as in [Har77, Exer. 4.1] using Corollary 4.4.16. □

There is also a cohomological criterion for ampleness generalizing [Har77, Prop. 5.3]:

Exercise 4.4.20. Let X be a proper algebraic space over a noetherian ring. For a line bundle L on X, show that the following are equivalent:

(1) X is a scheme and L is ample;

(2) for every coherent sheaf F on X, there is an integer n_0 such that $\mathrm{H}^i(X, F \otimes L^n) = 0$ for $i > 0$ and $n \geq n_0$.

See also [SP, Tag 0D2W].

The following generalizes [Har77, Exer. III.5.7].

Exercise 4.4.21. Let $f\colon X \to Y$ be a finite surjective morphism of algebraic spaces proper over a noetherian ring. Let L be a line bundle on Y. If X is a ample and f^*L is ample, show that Y is a scheme and L is ample.

See also [SP, Tag 0GFB].

4.4.5 Effective descent along field extensions

Lemma 4.4.22. *Let X be a quasi-separated algebraic space locally of finite type over a field $\Bbbk$. If $X_{\overline{\Bbbk}}$ is an affine scheme, then so is X.*

Proof. By Chevalley's Criterion for Affineness (Theorem 4.4.19), it suffices to show that there is a finite field extension $\Bbbk \to K$ such that X_K is a affine. (Note that the lemma follows directly from the strengthening of Chevelley's Criterion to integral surjective morphisms.)

The algebraic space X is necessarily quasi-compact and we choose an étale presentation $U \to X$ be an affine scheme. We write $\overline{\Bbbk} = \operatorname{colim} \Bbbk_\lambda$ as the colimit of finite field extensions $\Bbbk_\lambda/\Bbbk$. Set $X_\lambda := X_{\Bbbk_\lambda}$ and $U_\lambda = U_{\Bbbk_\lambda}$. By Flat Base Change (Exercise 4.1.19), $\Gamma(X, \mathcal{O}_X) \otimes_{\Bbbk} \Bbbk_\lambda = \Gamma(X_\lambda, \mathcal{O}_{X_\lambda})$ and $\Gamma(X, \mathcal{O}_X) \otimes_{\Bbbk} \overline{\Bbbk} = \Gamma(X_{\overline{\Bbbk}}, \mathcal{O}_{X_{\overline{\Bbbk}}})$. We have a cartesian diagram

$$\begin{tikzcd}
U_{\overline{\Bbbk}} \arrow[r] \arrow[d] & U_\lambda \arrow[r] \arrow[d] & U \arrow[d] \\
X_{\overline{\Bbbk}} \arrow[r] \arrow[d, "\wr"] & X_\lambda \arrow[r] \arrow[d] & X \arrow[d] \\
\operatorname{Spec} \Gamma(X_{\overline{\Bbbk}}, \mathcal{O}_{X_{\overline{\Bbbk}}}) \arrow[r] & \operatorname{Spec} \Gamma(X_\lambda, \mathcal{O}_{X_\lambda}) \arrow[r] & \operatorname{Spec} \Gamma(X, \mathcal{O}_X)
\end{tikzcd}$$

Since $U_{\Bbbk} \to \operatorname{Spec} \Gamma(X_{\overline{\Bbbk}}, \mathcal{O}_{X_{\overline{\Bbbk}}})$ is an étale morphism of schemes, so is $U_\lambda \to \operatorname{Spec} \Gamma(X_\lambda, \mathcal{O}_{X_\lambda})$ for $\lambda \gg 0$ (Proposition A.6.7). Thus $X_\lambda \to \operatorname{Spec} \Gamma(X_\lambda, \mathcal{O}_{X_\lambda})$ is étale for $\lambda \gg 0$. Let $R = U \times_X U$ with base changes $R_\lambda := R_{\Bbbk_\lambda}$ and $R_{\overline{\Bbbk}}$. Since $R_{\overline{\Bbbk}} \to U_{\overline{\Bbbk}} \times_{\overline{\Bbbk}} U_{\overline{\Bbbk}}$ is a closed immersion, so is $R_\lambda \to U_\lambda \times_{\Bbbk_\lambda} U_\lambda$ for $\lambda \gg 0$ (Proposition A.6.7) and in particular X_λ are separated for $\lambda \gg 0$. For $\lambda \gg 0$, since X_λ is étale and separated over a scheme, X_λ is a scheme (Corollary 4.4.7). We may therefore apply Proposition B.4.4 to X (or Proposition A.6.7 to $X \to \operatorname{Spec}\Gamma(X, \mathcal{O}_X)$) to conclude that X_λ is affine for $\lambda \gg 0$. $\square$

Proposition 4.4.23. *Let X be a quasi-separated algebraic space of finite type over a field $\Bbbk$. If $X_{\overline{\Bbbk}}$ is a scheme, then there exists a finite separable field extension $\Bbbk \to K$ such that X_K is a scheme.*

Proof. Choose an étale presentation $U \to X$ be an affine scheme and set $R = U \times_X U$. As in the proof of the previous lemma, we write $\overline{\Bbbk} = \operatorname{colim} \Bbbk_\lambda$ with $\Bbbk_\lambda/\Bbbk$ finite, and set $X_\lambda := X_{\Bbbk_\lambda}$, $U_\lambda = U_{\Bbbk_\lambda}$ and $R_\lambda = R_{\Bbbk_\lambda}$.

Let $\overline{V} \subset X_{\overline{\Bbbk}}$ be an open affine subscheme. We claim that for $\lambda \gg 0$, there exists an open subscheme $V_\lambda \subset X_\lambda$ such that $\overline{Z} = U_\lambda \times_{\Bbbk_\lambda} \overline{\Bbbk}$. Indeed, the preimage $V' \subset U_{\overline{\Bbbk}}$ of $\overline{V}$ has the property that its two preimages in $R_{\overline{\Bbbk}}$ are equal. Using Proposition A.6.4 and Proposition A.6.7, for $\lambda \gg 0$ there is an open subscheme $V'_\lambda \subset U_\lambda$ with $V' = V'_\lambda \times_{\Bbbk_\lambda} \overline{\Bbbk}$ such that the two preimages of V'_λ in R_λ are equal. By étale descent, V'_λ descends to the desired closed subscheme $V_\lambda \subset U_\lambda$.

Lemma 4.4.22 implies that V_λ is a scheme. By covering $X_{\overline{\Bbbk}}$ with finitely many affines and choosing λ sufficiently large, we obtain a finite field extension $K = \Bbbk_\lambda$ of $\Bbbk$ such that X_λ is a scheme. If $\Bbbk^s \subset K$ be the separable closure of $\Bbbk$, then $X_K \to X_{\Bbbk^s}$ is a finite universal homeomorphism and by Chevalley's Theorem for Affineness (Theorem 4.4.19), the image of an affine subscheme X_K in $X_{\Bbbk^s}$ is also affine. We conclude that $X_{\Bbbk^s}$ is a scheme. $\square$

With an additional condition on $X_{\overline{\Bbbk}}$, we can conclude that X is a scheme.

Proposition 4.4.24. *Let X be a quasi-separated algebraic space locally of finite type over a field $\Bbbk$. If $X_{\overline{\Bbbk}}$ is a scheme such that every finite set of $\overline{\Bbbk}$-points is contained in an affine (e.g. $X_{\overline{\Bbbk}}$ is quasi-projective), then X is a scheme.*

Proof. We may assume that X is quasi-compact. We will show that every closed point $x \in X$ has an affine open neighborhood. Let $\operatorname{Spec} l \hookrightarrow X$ be the inclusion of the residue field of x (Corollary 3.5.20) and let $\Bbbk^s$ be the separable closure of the finite field extension $\Bbbk \to l$. We have a cartesian diagram

$$\begin{tikzcd}
\coprod_{i=1}^n \operatorname{Spec} A_i \arrow[r, hook] \arrow[d] & X_{\overline{\Bbbk}} \arrow[r] \arrow[d] & \operatorname{Spec} \overline{\Bbbk} \arrow[d] \\
\operatorname{Spec} l \arrow[r, hook, "x"] & X \arrow[r] & \operatorname{Spec} \Bbbk
\end{tikzcd}$$

where A_i is a artinian local $\Bbbk$-algebra where n is the degree of the separable closure $\kappa^s \subset l$ of $\Bbbk$; here we are using that $\Bbbk^s \otimes_\Bbbk \overline{\Bbbk} = \prod_{i=1}^n \overline{\Bbbk}$ and that $\operatorname{Spec} l \to \operatorname{Spec} \Bbbk^s$ is a finite universal homeomorphism. The hypotheses on $X_{\overline{\Bbbk}}$ ensure that there is an affine open subscheme $\overline{U} \subset X_{\overline{\Bbbk}}$ containing the images of each $\operatorname{Spec} A_i$.

By Proposition 4.4.23, there is a finite field extension $\Bbbk \to K$ such that X_K is a scheme. After enlarging K, we can arrange that $\overline{U}$ descends to an affine open subscheme $U' \subset U_K$ by using Proposition A.6.4 to descend the morphism $\overline{U} \to X$, Proposition A.6.7 to arrange that it is an open immersion and Proposition B.4.4 to arrange affineness. Observe that U' contains all preimages of x under $X_K \to X$. By taking the normal closure of K, we can assume K is normal over $\Bbbk$. Let $G = \operatorname{Aut}(K/\Bbbk)$ so that K^G is a purely inseparable field extension of $\Bbbk$. Then G acts on X_K freely such that $X_K/G = X_{K^G}$.

The intersection of the translates of U' by elements of G is a G-invariant quasi-affine variety U''. Choosing an affine in U'' containing all of the preimages of x and intersecting again the translates of G, we obtain a G-invariant affine $V \subset X_K$ containing the preimages of x. Then the quotient V/G is an affine subscheme of X_{K^G} containing the unique preimage of x (Theorem 4.3.6). Letting W be the image of V/H under the finite universal homeomorphism $X_{K^G} \to X$, Chevelley's Criterion for Affineness (Theorem 4.4.19) implies that W is an affine neighborhood of x. □

4.4.6 Group algebraic spaces are schemes

Every quasi-separated group algebraic space over a field $\Bbbk$ is a scheme. When $\Bbbk$ is algebraically closed, this follows easily from Theorem 4.4.1 as we know there is a dense open that is a scheme and we can translate this around by rational points. The general case relies on Proposition 4.4.24.

Theorem 4.4.25. *A quasi-separated group algebraic space G locally of finite type over a field $\Bbbk$ is a scheme.*

Remark 4.4.26. If G is not quasi-separated, then the above corollary does not hold, e.g. $G = \mathbb{G}_a/\underline{\mathbb{Z}}$ over $\Bbbk$ (Example 3.9.22).

Note that Proposition 4.4.18 implies that the result also holds over an Artinian base. Over a general base scheme, the statement is not true; see [Ray70, Lem. X.14].

Proof. Assume first that $\Bbbk$ is algebraically closed. There is a non-empty open subscheme U of G (Theorem 4.4.1) with a point $h \in U(\Bbbk)$. For every $g \in G(\Bbbk)$, left multiplication by gh^{-1} defines an isomorphism $G \xrightarrow{\sim} G$ and the image $gh^{-1}U$ of U is a scheme containing g.

The general case follows from Proposition 4.4.24 using that $G_{\overline{\Bbbk}}$ is a scheme with the property that every finite set of points is contained in an affine (Lemma 4.4.27).

See also [Art69b, Lem. 4.2] and [SP, Tag 0B8D]. □

Lemma 4.4.27. *Every group scheme G locally of finite type over an algebraically closed field $\Bbbk$ has the property that every finite set of $\Bbbk$-points is contained in an affine open subscheme.*

Proof. Let $g_1, \ldots, g_n \in G(\Bbbk)$. We first use induction on n to assume that all of the elements g_i are in the same connected component. If not, we can write $G = W_1 \sqcup W_2$ with r points in W_1 and $n-r$ points in W_2 for $0 < r < n$. By induction, there are affine opens $U_1 \subset W_1$ and $U_2 \subset W_2$ containing the r and $n-r$ points, respectively. Then $U_1 \sqcup U_2$ is an affine containing each g_i.

By translating by g_1^{-1}, we may assume that $g_1, \ldots, g_n \in G^0(\Bbbk)$. Let $U \subset G^0$ be affine open neighborhood of the identity. Since G^0 is irreducible (C.3.1(5)), $Ug_1^{-1} \cap \cdots \cap Ug_n^{-1}$ is non-empty and contains a closed point h. Since $h \in Ug_i^{-1}$, each g_i is contained in the affine open $h^{-1}U$.

See also [SP, Tag 0B7S]. It is also true that every group scheme of *finite type* over a field is quasi-projective [SP, Tag 0BF7]. □

Corollary 4.4.28. *Let $\mathcal{X}$ be an algebraic stack with quasi-separated diagonal. Then the stabilizer of every field-valued point is a group scheme locally of finite type.*

Proof. By Exercise 3.2.4 the diagonal of $\mathcal{X}$ is locally of finite type. As the stabilizer is the base change of the diagonal, the statement follows from Theorem 4.4.25. □

4.5 Finite covers of Deligne–Mumford stacks

The goal of this section is to prove the following theorem asserting that Deligne–Mumford stacks have finite covers by schemes.

Theorem 4.5.1 (Le Lemme de Gabber)**.** *Let $\mathcal{X}$ be a Deligne-Mumford stack separated and of finite type over a noetherian scheme S. Then there exists a finite, generically étale, and surjective morphism $Z \to \mathcal{X}$ from a scheme Z.*

By applying Chow's Lemma (c.f [Har77, Exer. II.4.10]) to Z, we obtain:

Corollary 4.5.2. *There exists a projective, generically étale, and surjective morphism $Z \to \mathcal{X}$ from a scheme Z quasi-projective over S.* □

See also [LMB, Thm. 16.6 and Cor. 16.6.1], [Del85], [Vis89, Prop. 2.6], [Ols16, Thm. 11.4.1] and [SP, Tag 09YC] (for the case of algebraic spaces). More generally, every separated algebraic stack of finite type over S has a proper cover by a quasi-projective scheme [Ols05].

We provide two arguments, each which uses normalization to construct a finite cover.

Proof 1 (following [LMB, Thm. 16.6]):

By replacing $\mathcal{X}$ with the disjoint union of the irreducible components with their reduced stack-structure, we may assume that $\mathcal{X}$ is irreducible and reduced. Every étale presentation $U \to \mathcal{X}$ is separated, quasi-finite and representable and thus factors as the composition of an open immersion $U \hookrightarrow \widetilde{\mathcal{X}}$ and a finite morphism $\widetilde{\mathcal{X}} \to \mathcal{X}$ by Zariski's Main Theorem (4.4.9). After replacing $\mathcal{X}$ with $\widetilde{\mathcal{X}}$, we may assume that $\mathcal{X}$ has a dense open subscheme. If $p\colon U \to \mathcal{X}$ is an étale presentation, there is therefore a dense open subscheme $V \subset \mathcal{X}$ such that $p^{-1}(V) \to V$ is finite étale of degree d. We may choose a finite étale covering $V' \to V$ such that $p^{-1}(V) \times_V V' \to V'$ is a trivial étale covering; indeed as in Proposition A.3.12, we may take V' to be the complement of all pairwise diagonals in $(V'/V)^d = \underbrace{V' \times_V \cdots \times_V V'}_{d}$. Applying Zariski's Main Theorem (Theorem 4.4.9) to the composition $V' \to V \hookrightarrow \mathcal{X}$ gives a finite surjective morphism $\widetilde{\mathcal{X}} \to \mathcal{X}$ restricting to $V' \to V$. Thus after replacing $\mathcal{X}$ with $\widetilde{\mathcal{X}}$, we may assume that there is an étale presentation $U \to \mathcal{X}$ which over a dense open subscheme $j\colon V \hookrightarrow \mathcal{X}$ is a trivial étale covering, i.e. there is a cartesian diagram

$$\begin{tikzcd} \coprod_{i=1}^d V \arrow[r, hook, "j'"] \arrow[d] & U \arrow[d, "p"] \\ V \arrow[r, hook, "j"] & \mathcal{X}. \end{tikzcd}$$

We will construct a finite surjective morphism $Z \to \mathcal{X}$ from a scheme which is an isomorphism over V. Let $\mathcal{A} \subset j_* \mathcal{O}_{\mathcal{X}}$ be integral closure of $\mathcal{O}_{\mathcal{X}} \to j_* \mathcal{O}_{\mathcal{X}}$. Then $\pi^* \mathcal{A}$ is the integral closure of $\mathcal{O}_U$ in $j'_* \mathcal{O}_{\sqcup V} = p^* j_* \mathcal{O}_V$ (Proposition A.5.4). The idempotent $e_i \in \Gamma(U, j'_* \mathcal{O}_{\sqcup V}) = \Gamma(\sqcup V, \mathcal{O}_{\sqcup V})$, defining the ith copy of V, is integral over $\mathcal{O}_U$ and thus defines a global section $e_i \in \Gamma(U, p^* \mathcal{A})$. Now write $\mathcal{A} = \mathrm{colim}_\lambda \mathcal{C}_\lambda$ as a filtered colimit of finite type $\mathcal{O}_{\mathcal{X}}$ algebra (Exercise 4.1.25). Since $\mathcal{A}$ is integral over $\mathcal{O}_{\mathcal{X}}$, each $\mathcal{C}_\lambda$ is a finite $\mathcal{O}_{\mathcal{X}}$-algebra. For $\lambda \gg 0$, we have

$e_i \in \Gamma(U, p^*\mathcal{A}_\lambda)$. The Deligne–Mumford stack $Z := \mathcal{S}\mathrm{pec}_{\mathcal{X}}\, \mathcal{A}_\lambda$ is finite over $\mathcal{X}$ and we claim that Z is a scheme. To see this, consider the cartesian diagram

$$\begin{tikzcd} Z' \arrow[r] \arrow[d] & U \arrow[d, "p"] \\ Z \arrow[r] & \mathcal{X} \end{tikzcd}$$

noting that Z' is a scheme since it is finite over U. Each idempotent e_i defines a global sections of Z' and thus yield a decomposition $Z' = \coprod_{i=1}^d Z'_i$. Each morphism $Z'_i \to Z$ is étale, separated and birational, thus an open immersion. Since $Z' \to Z$ is surjective, the collection of Z'_i defines an open covering of Z and it follows that Z is a scheme.

Proof 2 (following [Vis89, Prop. 2.6]):

We first use limit methods to reduce to the case that S is of finite type over $\mathbb{Z}$ in order to ensure that normalizations are finite. By Noetherian Approximation (Proposition A.6.2), we may write $S = \lim_\lambda S_\lambda$ as the limit of schemes with affine transition maps where each S_i is of finite type over $\mathbb{Z}$. Let $U \to \mathcal{X}$ be an étale presentation and set $R = U \times_{\mathcal{X}} U \rightrightarrows U$ be the corresponding étale groupoid equipped with source, target, identity and compositions morphisms s,t,i and c. There exists an index 0 and schemes U_0 and R_0 of finite type over S_0 such that $U = U_0 \times_{S_0} S$ and $R = R_0 \times_{R_0} S$ (Proposition A.6.4(2)). For $\lambda \geq 0$, set $U_\lambda = U_0 \times_{S_0} S_\lambda$ and $R_\lambda = R_0 \times_{S_0} S_\lambda$. For $\lambda \gg 0$, there are morphisms $s_\lambda, t_\lambda \colon R_\lambda \to U_\lambda$, $i_\lambda \colon R_\lambda \to R_\lambda$ and $c_\lambda \colon R_\lambda \times_{t_\lambda, U_\lambda, s_\lambda} R_\lambda \to R_\lambda$ that base change to s,t,i and c (Proposition A.6.4(1)). Finally, for $\lambda \gg 0$, the morphisms s_λ and t_λ are étale, and $R_\lambda \to U_\lambda \times_{S_\lambda} R_\lambda$ is finite (Proposition A.6.7). It follows that $R_\lambda \rightrightarrows U_\lambda$ defines an étale groupoid of schemes and that the quotient stack $\mathcal{X}_\lambda := [U_\lambda/R_\lambda]$ is a Deligne–Mumford stack separated and of finite type over S_λ such that $\mathcal{X} \cong \mathcal{X}_\lambda \times_{S_\lambda} S$. A finite, generically étale cover of $\mathcal{X}_\lambda$ by scheme will pullback to a finite, generically étale cover of $\mathcal{X}$ by a scheme. This finishes the reduction.

By replacing $\mathcal{X}$ with the disjoint union of the irreducible components with their reduced stack-structure, we may assume that $\mathcal{X}$ is irreducible and reduced. Let $\widetilde{\mathcal{X}}$ be the normalization of $\mathcal{X}$ (Example 4.1.23). Then $\widetilde{\mathcal{X}} \to \mathcal{X}$ is finite and so after replacing $\mathcal{X}$ with $\widetilde{\mathcal{X}}$, we may assume that $\mathcal{X}$ is also normal.

Let $\mathcal{X} \to X$ be the coarse moduli space (Theorem 4.3.11) and let $U \to \mathcal{X}$ be an étale presentation. As $\mathcal{X}$ is normal, so is X (Exercise 4.3.17). We can write $U = \coprod_i U_i$ as the disjoint union of integral affine schemes U_i; each morphism $U_i \to \mathcal{X}$ is étale and in particular quasi-finite and dominant.

Each field extension $\mathrm{Frac}(X) \to \mathrm{Frac}(U_i)$ of fraction fields is finite, and we let F be a finite normal extension of $\mathrm{Frac}(X)$ containing each $\mathrm{Frac}(U_i)$. The normalization $Y \to X$ of X in F is finite; here X is an algebraic space and the normalization is well-defined by Proposition A.5.4. Meanwhile, by the universal property of the normalization $Y \to X$, the normalization Y_i of U_i in F admits a morphism $Y_i \to Y$ over X. As $Y_i \to Y$ is separated, quasi-finite and birational, it is an open immersion.

The automorphism group $G = \mathrm{Aut}(F/\mathrm{Frac}(X))$ acts on Y over X and for each pair $\alpha = (i, \sigma)$ of an integer i and $\sigma \in G$, we set $Y_\alpha = \sigma(Y_i)$. We claim that $Y = \bigcup_\alpha Y_\alpha$. To see this, we first show that G acts transitively on the fibers of $Y \to X$. The fixed field F^G is a purely inseparable field extension of $\mathrm{Frac}(X)$

and the normalization $X' \to X$ of X in F^G is a universal homeomorphism. Thus to see that G acts transitively on the fibers, we may assume that $\mathrm{Frac}(X) \to F$ is a Galois extension. We may also assume that $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$ with B the integral closure of A in F. Then G acts on B and we have inclusions $A \subset B^G \subset F^G = \mathrm{Frac}(X)$. Since A is normal and B^G is integral over A, we see that $A = B^G$. By Theorem 4.3.6, $[\operatorname{Spec} A/G] \to \operatorname{Spec} B$ is a coarse moduli space and it follows that G acts transitively on the fibers of $\operatorname{Spec} A \to \operatorname{Spec} B$. To prove the claim, observe that since $\coprod_i Y_i \to \coprod_i U_i \to X$ is surjective, each point $x \in X$ has a preimage $y \in Y_i$ for some i. Since G acts transitively on the fibers, $\bigcup Y_\alpha$ contains the fiber of $Y \to X$ over x.

The claim implies that Y is a scheme and that $Y \to X$ factors through $\mathcal{X}$ Zariski-locally on Y. Indeed, each Y_α is separated and quasi-finite over U_i and thus a scheme by Corollary 4.4.7. Each $Y_\alpha \to X$ factors via $s_\alpha \colon Y_\alpha \to U_i \to \mathcal{X}$. After replacing X with Y and $\mathcal{X}$ with $\mathcal{X} \times_X Y$, we may assume that we have a coarse moduli space $\mathcal{X} \to X$ with X a scheme and an open covering $X = \bigcup X_\alpha$ together with a commutative diagram

$$\begin{tikzcd} & \mathcal{X} \arrow[d] \\ X_\alpha \arrow[ur, "s_\alpha"] \arrow[r, hook] & X \end{tikzcd}$$

for each α. We will show that after replacing X with a finite cover, the sections s_α glue to a global section s. Such a section is necessarily finite since $\mathcal{X}$ is Deligne–Mumford and this then finishes the proof as $X \to \mathcal{X}$ is a finite surjective morphism from a scheme.

To show that the sections glue, we first claim that the diagonal $\Delta_{\mathcal{X}} \colon \mathcal{X} \to \mathcal{X} \times_X \mathcal{X}$ is étale. This is a Zariski local question on X so we may assume that there is a section $s \colon X \to \mathcal{X}$ of $\pi \colon \mathcal{X} \to X$. Then $s \colon X \to \mathcal{X}$ is a dominant and unramified (since $\Delta_{\mathcal{X}}$ is unramified) morphism of normal Deligne–Mumford stacks and thus étale (Proposition A.3.13). It follows that $\Delta_{\mathcal{X}} \colon \mathcal{X} \to \mathcal{X} \times_X \mathcal{X}$ is étale (and also that $\pi \colon \mathcal{X} \to X$ is étale).

Since the diagonal $\mathcal{X} \to \mathcal{X} \times_X \mathcal{X}$ is finite and étale, the scheme $J_{\alpha,\beta} := \underline{\mathrm{Isom}}_{X_{\alpha,\beta}}(s_\alpha|_{X_{\alpha,\beta}}, s_\alpha|_{X_{\alpha,\beta}})$ of isomorphisms is finite and étale over $X_{\alpha,\beta} := X_\alpha \cap X_\beta$. We may choose a finite étale cover $V_{\alpha,\beta} \to X_{\alpha,\beta}$ trivializing $J_{\alpha,\beta} \to X_{\alpha,\beta}$ (see Proposition A.3.12). By Zariski's Main Theorem (A.5.3), $V_{\alpha,\beta} \to X$ factors as an open immersion $V_{\alpha,\beta} \hookrightarrow \widetilde{X}$ and a finite morphism $\widetilde{X} \to X$. After replacing X with $\widetilde{X}$, we may assume that $J_{\alpha,\beta} \to X_{\alpha,\beta}$ is trivial.

The intersection $\bigcap_\alpha X_\alpha$ is non-empty and we may choose a geometric point $x \colon \operatorname{Spec} \Bbbk \to \bigcap_\alpha X_\alpha$. All objects in the fiber $\mathcal{X}_x(\Bbbk)$ of $\mathcal{X} \to X$ over x are isomorphic. We may therefore choose an object $t \in \mathcal{X}_x(\Bbbk)$ and isomorphisms $\mu_\alpha \colon t \xrightarrow{\sim} x^* s_\alpha$ for each α. This allows us to define isomorphisms $\phi_{\alpha,\beta} \colon x^* s_\alpha \xrightarrow{\mu_\alpha^{-1}} t \xrightarrow{\mu_\beta} x^* s_\beta$. It is readily checked that the isomorphisms $\phi_{\alpha,\beta}$ satisfy the cocycle $\phi_{\alpha,\gamma} = \phi_{\beta,\gamma} \circ \phi_{\alpha,\beta}$. Each $\phi_{\alpha,\beta}$ defines a lift $\operatorname{Spec} \Bbbk \to J_{\alpha,\beta}$ of $x \colon \operatorname{Spec} \Bbbk \to X_{\alpha,\beta}$ which extends uniquely to a section $\lambda_{\alpha,\beta} \colon X_{\alpha,\beta} \to J_{\alpha,\beta}$. The triple intersections $X_\alpha \cap X_\beta \cap X_\gamma$ are connected and since the $\phi_{\alpha,\beta}$ satisfy the cocycle condition, so does $\lambda_{\alpha,\beta}$. The isomorphisms $\lambda_{\alpha,\beta}$ between $s_\alpha|_{X_{\alpha,\beta}}$ and $s_\alpha|_{X_{\alpha,\beta}}$ therefore glue to a global section of $\mathcal{X} \to X$. □

Exercise 4.5.3. Let X be a normal algebraic space of finite type over a noetherian scheme S. Show that there is a normal scheme U with an action of a finite group G such that X is the quotient of U by G, i.e. $[U/G] \to X$ is a coarse moduli space.

Hint: After reducing to the case that X is integral, choose a finite, generically étale and surjective morphism $U \to X$ from a scheme. Let K be the Galois closure of the finite separable field extension $\operatorname{Frac}(U)/\operatorname{Frac}(X)$. Then take U to be the integral closure of X in K (which is finite over X as $K/\operatorname{Frac}(X)$ is separable) and take $G = \operatorname{Gal}(K/\operatorname{Frac}(X))$. See also [LMB, Cor. 16.6.2].

Chapter 5

Moduli of stable curves

5.1 Review of smooth curves

5.1.1 Curves

A *curve* over a field $\Bbbk$ is a one-dimensional scheme C of finite type over $\Bbbk$. Proper curves are projective; this can be reduced to the case of smooth curves [Har77, Prop. I.6.7]. More generally, every one-dimensional separated algebraic space is a quasi-projective scheme; see [SP, Tags 0ADD and 09NZ].

If C is a proper curve over a field $\Bbbk$, we define the *arithmetic genus of* C or simply the *genus of* C as

$$g(C) = 1 - \chi(C, \mathcal{O}_C),$$

which is equal to $\mathrm{h}^1(C, \mathcal{O}_C)$ if C is geometrically connected and reduced.

For a connected, reduced, and projective curve C over an algebraically closed field $\Bbbk$, the degree of a *very ample* line bundle L on C is defined as the number of zeros (counted with multiplicity) of any section of L. In other words, if $C \hookrightarrow \mathbb{P}^n$ is the projective embedding defined by L, then $\deg L = \dim_{\Bbbk} \Gamma(C \cap H, \mathcal{O}_{C \cap H})$, where H is any hyperplane and $C \cap H$ is the scheme-theoretic intersection. Any line bundle on C can be written as difference of two very ample line bundles: if M is very ample on C, then $M' := L \otimes M^n$ is very ample for $n \gg 0$, and $L \cong M' \otimes (M^{\otimes n})^\vee$. In this way, we also see that $L = \mathcal{O}_C(D)$ for a divisor $D = \sum_{n_i} p_i$ supported on the smooth locus of C, i.e. each $p_i \in C$ is a smooth point. Note that $\deg(L \otimes M) = \deg L + \deg M$, and that if $C = \bigcup_i C_i$ denotes the irreducible decomposition, then $\deg L = \sum_i \deg L|_{C_i}$.

Theorem 5.1.1 (Riemann–Roch)**.** *Let C be a connected, reduced, and projective curve of genus g over an algebraically closed field $\Bbbk$. If L is a line bundle on C, then*

$$\chi(C, L) = \deg L + 1 - g.$$

Proof. We can write $L = \mathcal{O}_C(D)$ for a divisor D supported on the smooth locus. Since Riemann–Roch holds for $\mathcal{O}_C$, it suffices by adding and subtracting points to show that Riemann–Roch holds for $\mathcal{O}_C(D)$ if and only if it holds for $\mathcal{O}_C(D+p)$ for a smooth point $p \in C(\Bbbk)$. This follows by consider the short exact sequence

$$0 \to \mathcal{O}_C(D) \to \mathcal{O}_C(D+p) \to \kappa(p) \to 0$$

and the identity $\chi(C, \mathcal{O}_C(D+p)) = \chi(C, \mathcal{O}_C(D)) + 1$. See also [Har77, Thm IV.1.3, Exer. IV.1.9] and [Vak17, Exers. 18.4.B and S]. □

5.1.2 Smooth curves

We review some basic properties of smooth curves which we will later generalize to nodal curves. If C is a smooth curve, then the sheaf of differentials Ω_C is a line bundle. Serre Duality states Ω_C is in fact a dualizing sheaf on C; this is a deep result that is in indispensable in the study of curves.

Theorem 5.1.2 (Serre Duality for Smooth Curves)**.** *If C is a smooth projective curve over a field $\Bbbk$, then Ω_C is a dualizing sheaf, i.e. there is a linear map $\mathrm{tr}\colon H^1(C, \Omega_C) \to \Bbbk$ such that for every coherent sheaf $\mathcal{F}$, the natural pairing*

$$\mathrm{Hom}_{\mathcal{O}_C}(\mathcal{F}, \Omega_C) \times H^1(C, \mathcal{F}) \to H^1(C, \Omega_C) \xrightarrow{\mathrm{tr}} \Bbbk$$

is perfect.

Proof. See [Har77, Cor. III.7.12]. □

Remark 5.1.3. The pairing being perfect means that the $\mathrm{Hom}_{\mathcal{O}_C}(\mathcal{F}, \Omega_C)$ is identified with the dual $H^1(C, \mathcal{F})^\vee$. If $\mathcal{F}$ is a vector bundle, $\mathrm{Hom}_{\mathcal{O}_C}(\mathcal{F}, \Omega_C) \cong H^0(C, \mathcal{F}^\vee \otimes \Omega_C)$ and Serre Duality gives an isomorphism

$$H^0(C, \mathcal{F}^\vee \otimes \Omega_C) \cong H^1(C, \mathcal{F})^\vee.$$

Taking $\mathcal{F} = \Omega_C$, we see that $H^1(C, \Omega_C) \cong H^0(C, \mathcal{O}_C)^\vee$ and in particular that the trace map $\mathrm{tr}\colon H^1(C, \Omega_C) \to \Bbbk$ is an isomorphism if C is geometrically connected and reduced.

Combining the above version of Riemann–Roch (5.1.1) with Serre Duality leads to the more powerful version of Riemann–Roch.

Theorem 5.1.4 (Riemann–Roch II)**.** *Let C be a smooth, connected, and projective curve of genus g over an algebraically closed field. If L is a line bundle on C, then*

$$\mathrm{h}^0(C, L) - \mathrm{h}^0(C, \Omega_C \otimes L^\vee) = \deg L + 1 - g.$$

Remark 5.1.5. This is often written in divisor form as $\mathrm{h}^0(C, L) - \mathrm{h}^0(C, K - L) = \deg L + 1 - g$ where K denotes a canonical divisor, i.e. $\Omega_C = \mathcal{O}_C(K)$.

Like Riemann–Roch, Riemann–Hurwitz (5.7.2) plays an essential role in the study of smooth curves. Riemann–Hurwitz informs us on how the sheaf of differentials behaves under finite morphisms of smooth curves; the statement is postponed until our discussion of branched covers.

5.1.3 Positivity of divisors on smooth curves

The following consequence of Riemann–Roch provides a useful criteria to determine whether a given line bundle is base point free (equivalently globally generated), ample, or very ample.

Corollary 5.1.6. *Let C be a connected, smooth, and projective curve over an algebraically closed field $\Bbbk$, and let L be a line bundle on C.*

(1) if $\deg L < 0$*, then* $\mathrm{h}^0(C, L) = 0$*;*

(2) if $\deg L > 0$*, then* L *is ample;*

(3) if $\deg L \geq 2g$*, then* L *is base point free; and*

(4) if $\deg L \geq 2g + 1$*, then* L *is very ample.*

Proof. See [Har77, Cor. IV.3.2]. $\square$

Remark 5.1.7. If $g > 1$, we can use Riemann–Roch and Serre Duality to compute that: (a) $\mathrm{h}^0(C, \Omega_C) = \mathrm{h}^1(C, \mathcal{O}_C) = g$, (b) $\mathrm{h}^1(C, \Omega_C) = \mathrm{h}^0(C, \mathcal{O}_C) = 1$ and (c) Ω_C has degree $2g - 2$ and is thus ample on C. Similarly, if $k > 1$, we have: (a) $\mathrm{h}^0(C, \Omega_C^{\otimes k}) = (2k-1)(g-1)$, (b) $\mathrm{h}^1(C, \Omega_C^{\otimes k}) = 0$ and (c) $\Omega_C^{\otimes k}$ has degree $2k(g-1)$ and is very ample if $k \geq 3$. Note that Ω_C is not very ample precisely when C is *hyperelliptic*. On the other hand, if $g = 1$ then $\Omega_C \cong \mathcal{O}_C$, and if $g = 0$ then $C = \mathbb{P}^1$ and $\Omega_C = \mathcal{O}(-2)$.

5.1.4 Families of smooth curves

Definition 5.1.8. A *family of smooth curves (of genus g)* over a scheme S is a smooth and proper morphism $\mathcal{C} \to S$ of schemes such that every geometric fiber is a connected curve (of genus g).

Recall that the relative sheaf of differentials $\Omega_{\mathcal{C}/S}$ is a line bundle on $\mathcal{C}$ such that for every geometric point $s\colon \operatorname{Spec} \Bbbk \to S$, the restriction $\Omega_{\mathcal{C}/S}|_{\mathcal{C}_s}$ is identified with $\Omega_{\mathcal{C}_s}$. More generally, for every morphism $T \to S$ of schemes, the pullback of $\Omega_{\mathcal{C}/S}$ to $\mathcal{C} \times_S T$ is canonically isomorphic to $\Omega_{\mathcal{C} \times_S T/T}$. We now show that for $k \geq 3$, the kth relative pluricanonical sheaf $\Omega_{\mathcal{C}/S}^{\otimes k}$ is relatively very ample, and that its pushforward is a vector bundle on S.

Proposition 5.1.9 (Properties of Families of Smooth Curves)**.** *Let* $\pi\colon \mathcal{C} \to S$ *be a family of smooth curves of genus* $g \geq 2$*.*

(1) $\pi_* \mathcal{O}_{\mathcal{C}} = \mathcal{O}_S$*;*

(2) The pushforward $\pi_*(\Omega_{\mathcal{C}/S}^{\otimes k})$ *is a vector bundle of rank*

$$r(k) := \begin{cases} g & \text{if } k = 1 \\ (2k-1)(g-1) & \text{if } k > 1. \end{cases}$$

whose construction commutes with base change (i.e. for a morphism $f\colon T \to S$ *of schemes,* $f^* \pi_*(\Omega_{\mathcal{C}/S}^{\otimes k}) \cong \pi_{T,*}(\Omega_{\mathcal{C}_T/T}^{\otimes k})$*).*

(3) $R^1 \pi_* \Omega_{\mathcal{C}/S}^{\otimes k}$ *is isomorphic to* $\mathcal{O}_S$ *if* $k = 1$ *and zero otherwise.*

(4) For $k \geq 3$*,* $\Omega_{\mathcal{C}/S}^{\otimes k}$ *is relatively very ample.*

Proof. Items (1)–(3) follows from Cohomology and Base Change (A.7.5) as detailed in Proposition A.7.8. For (4), observe that for every point $s \in S$, the fiber $\Omega_{\mathcal{C}/S}^{\otimes k} \otimes \kappa(s) = \Omega_{\mathcal{C}_s}^{\otimes k}$ is very ample by Corollary 5.1.6 as $\deg \Omega_{\mathcal{C}_s}^{\otimes k} = k(2g-2) > 0$. Since $\mathrm{H}^1(\mathcal{C}_s, \Omega_{\mathcal{C}_s}^{\otimes k}) = 0$, we may apply Proposition E.2.1 to conclude that $\Omega_{\mathcal{C}/S}^{\otimes k}$ is relatively very ample. $\square$

Remark 5.1.10. In particular, (4) above implies that every family of smooth curves is projective.

It is also true that the relative sheaf of differentials $\Omega_{\mathcal{C}/S}$ is a relative dualizing sheaf, i.e. satisfies a relative version of Serre Duality; see [Liu02, §6.4].

5.2 Nodal curves

5.2.1 Nodes

Definition 5.2.1 (Nodes)**.** Let C be a curve over a field $\Bbbk$.

- If $\Bbbk$ is algebraically closed, we say that $p \in C(\Bbbk)$ is a *node* if there is an isomorphism $\widehat{\mathcal{O}}_{C,p} \cong \Bbbk[\![x,y]\!]/(xy)$.
- If $\Bbbk$ is an arbitrary field, we say that a closed point $p \in C$ is a *node* if there exists a node $\overline{p} \in C_{\overline{\Bbbk}}$ over p.

We say that C is a *nodal curve* (or has *at-worst nodal singularities*) if C has pure dimension one and every closed point is either smooth or nodal.

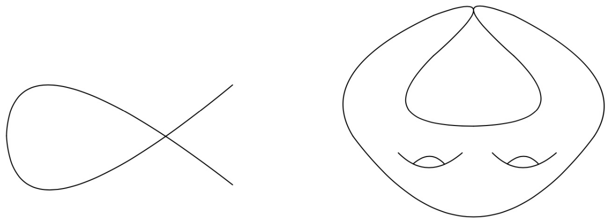


Figure 5.1: A node of a curve over $\mathbb{C}$ viewed algebraically (left hand side) or analytically (right hand side).

Example 5.2.2.

(1) The curves $\operatorname{Spec} \Bbbk[x,y]/(xy)$ and $\operatorname{Spec} \Bbbk[x,y]/(y^2 - x^2(x+1))$ have nodes at 0.

(2) The curve $C = \operatorname{Spec} \mathbb{R}[x,y]/(x^2+y^2)$ has a node at 0. Since the quadratic form x^2+y^2 does not split into linear factors, the completion $\widehat{\mathcal{O}}_{C,0}$ is not isomorphic to $\mathbb{R}[\![s,t]\!]/(st)$.

(3) The curve $\operatorname{Spec} \mathbb{Q}[x,y]/(x^2-2)(y^2-3)$ has a node at the point p defined by the maximal ideal (x^2-2, y^2-3). Note that unlike the previous example where the node 0 is a rational point, the node p in this example is not a rational point and the field extension $\mathbb{Q} \to \kappa(p)$ has degree 4.

5.2.2 Equivalent characterizations of nodes

Recall that the singular locus $\operatorname{Sing}(C)$ of C is defined scheme-theoretically as the first fitting ideal of Ω_C (see §A.3.6): locally if $C = V(f_1, \ldots, f_m) \subset \mathbb{A}^n$, then $\operatorname{Sing}(C)$ is defined by the vanishing of all $(n-1)\times(n-1)$ minors of the Jacobian matrix $J = (\frac{\partial f_j}{\partial x_i})$; note that if $C = V(f) \subset \mathbb{A}^2$ is a plane affine curve, then $\operatorname{Sing}(C) = V(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$. We will also use properties of local complete intersections as discussed in §A.3.8.

Proposition 5.2.3. *Let C be a pure dimension one curve over a field $\Bbbk$, and let $p \in C$ be a closed point. The following are equivalent:*

(1) $p \in C$ is a node;

(2) C is a local complete intersection at p and $\operatorname{Sing}(C)$ is unramified over $\Bbbk$ at p;

(3) $\Bbbk \to \kappa(p)$ is separable, $\mathcal{O}_{C,p}$ is reduced, $\dim \mathfrak{m}_p/\mathfrak{m}_p^2 = 2$, and there is a nondegenerate quadratic form $q \in \operatorname{Sym}^2 \mathfrak{m}_p/\mathfrak{m}_p^2$ mapping to 0 in $\mathfrak{m}_p^2/\mathfrak{m}_p^3$;

(4) $\Bbbk \to \kappa(p)$ is separable and $\widehat{\mathcal{O}}_{C,p} \cong \kappa(p)[\![x,y]\!]/(q)$ where q is a nondegenerate quadratic form; and

(5) there is a separable finite field extension $\Bbbk \to \Bbbk'$ such that $\widehat{\mathcal{O}}_{C,p} \otimes_{\Bbbk} \Bbbk' \cong \Bbbk'[\![x,y]\!]/(xy)$.

Proof. Assuming (1), let $\overline{p} \in C_{\overline{\Bbbk}}$ be a node over p and let $\operatorname{Sing}(C) \subset C$ be the scheme-theoretic singular locus. Then $\operatorname{Sing}(C) \times_{\Bbbk} \overline{\Bbbk} = \operatorname{Sing}(C_{\overline{\Bbbk}})$ and the preimage of $\operatorname{Sing}(C_{\overline{\Bbbk}})$ under $\operatorname{Spec} \widehat{\mathcal{O}}_{C_{\overline{\Bbbk}},\overline{p}} \to C_{\overline{\Bbbk}}$ is $\operatorname{Sing}(\operatorname{Spec} \widehat{\mathcal{O}}_{C_{\overline{\Bbbk}},\overline{p}})$ by properties of fitting ideals (see §A.3.6). Since $\widehat{\mathcal{O}}_{C_{\overline{\Bbbk}},\overline{p}} \cong \overline{\Bbbk}[\![x,y]\!]/(xy)$, $\operatorname{Sing}(\operatorname{Spec} \widehat{\mathcal{O}}_{C_{\overline{\Bbbk}},\overline{p}}) = V(x,y) = \operatorname{Spec} \overline{\Bbbk}$. Therefore $\operatorname{Sing}(C) \to \operatorname{Spec} \Bbbk$ is unramified at p. Since $\widehat{\mathcal{O}}_{C_{\overline{\Bbbk}},\overline{p}}$ is a complete intersection, C is a local complete intersection at p (Proposition A.3.15). This gives (2).

Assuming (2), since $\operatorname{Sing}(C)$ is unramified at p, the field extension $\Bbbk \to \kappa(p)$ is separable and there is an open neighborhood $U \subset C$ of p such that $\operatorname{Sing}(U) = \operatorname{Sing}(C) \cap U = \{p\}$. In particular, C and $\mathcal{O}_{C,p}$ are generically reduced. On the other hand, since C is a local complete intersection, $\mathcal{O}_{C,p}$ is a dimension 1 Cohen–Macaulay local ring and thus has no embedded primes. It follows that $\mathcal{O}_{C,p}$ is reduced. Using that C is a local complete intersection, we can write $\widehat{\mathcal{O}}_{C,p} = R/(f_1, \ldots, f_{n-1})$ where $R = \Bbbk[\![x_1, \ldots, x_n]\!]$. Since $\operatorname{Sing}(C)$ is unramified at p, the $(n-1) \times (n-1)$ minors of the Jacobian matrix $\left(\frac{\partial f_j}{\partial x_i}\right)_{i,j}$ generate the maximal ideal $\mathfrak{m} = (x_1, \ldots, x_n) \subset \widehat{\mathcal{O}}_{C,p}$. If $\frac{\partial f_j}{\partial x_i} \in R$ is a unit for some i and j, then the sequence $x_1, \ldots, \widehat{x_i}, x_n, f_j$ also generates $\mathfrak{m}/\mathfrak{m}^2$. We may use Lemma A.10.15 to change coordinates by replacing the generators $x_1, \ldots, x_n$ with $x_1, \ldots, \widehat{x_i}, x_n, f_j$.By eliminating f_j, this allows us to write $R = \Bbbk[\![x_1, \ldots, \widehat{x_i}, x_n]\!]/(f_1, \ldots, \widehat{f_j}, \ldots, f_{n-1})$. After finitely many such replacements, we can assume that $\frac{\partial f_j}{\partial x_i} \in \mathfrak{m}$ for every i, j. This implies that every $(n-1) \times (n-1)$ minor is in $\mathfrak{m}^{n-1}$, but since these minors generate $\mathfrak{m}$, we must have that $n = 2$. Therefore, $\widehat{\mathcal{O}}_{C,p} = \Bbbk[\![x,y]\!]/(f)$ with $f = f_2 + f_3 + \cdots$ and each f_i homogeneous of degree i. Since the partials f_x and f_y generate (x,y), the quadratic form $q := f_2 \in \operatorname{Sym}^2 \mathfrak{m}/\mathfrak{m}^2$ must be nondegenerate. This gives (3).

Assuming (3), we have that $\dim_{\kappa(p)} \mathfrak{m}^d/\mathfrak{m}^{d+1} = 2$ for every $d \geq 1$ since q maps to 0 in $\mathfrak{m}^2/\mathfrak{m}^3$. A choice of elements $x_0, y_0 \in \mathfrak{m}$ mapping to a basis in $\mathfrak{m}/\mathfrak{m}^2$ induces a surjection $\Bbbk[\![x,y]\!] \to \widehat{\mathcal{O}}_{C,p}$ (Lemma A.10.15). Since $\mathcal{O}_{C,p}$ is reduced, so is $\widehat{\mathcal{O}}_{C,p}$ (see Remark B.4.5). Therefore, we may use that $\Bbbk[\![x,y]\!]$ is a UFD to conclude that the kernel $\Bbbk[\![x,y]\!] \to \widehat{\mathcal{O}}_{C,p}$ is generated by an element f expressed as a product of distinct irreducible elements. Thus $\widehat{\mathcal{O}}_{C,p} \cong \Bbbk[\![x,y]\!]/(f)$ where the quadratic component $q = ax^2 + bxy + cy^2$ of f is a nondegenerate quadratic form. We claim that we can modify our choice of coordinates $x_0, y_0 \in \mathfrak{m}$ so that $q(x_0, y_0) = 0 \in \widehat{\mathcal{O}}_{C,p}$, or in other words that $f = q$. We will show inductively that for each N, there exists elements $x_i, y_i \in \mathfrak{m}^{i+1}$ for $i = 0, \ldots, N$ such that $q(x_0 + \cdots + x_N, y_0 + \cdots + y_N) \in \mathfrak{m}^{N+3}$. Since $\widehat{\mathcal{O}}_{C,p}$ is complete, this would enable us to replace x_0 and y_0 with $\sum_i x_i$ and $\sum_i y_i$ and conclude that $\widehat{\mathcal{O}}_{C,p} \cong \Bbbk[\![x,y]\!]/(q)$.

Supposing that we've already chosen $x' = x_0 + \cdots + x_{N-1}$ and $y' = y_0 + \cdots + y_{N-1}$, then for every x_N and $y_N \in \mathfrak{m}^{N+1}$, we have that

$$q(x' + x_N, y' + y_N) = q(x', y') + (2ax_0 + by_0)x_N + (bx_0 + 2cy_0)y_N \mod \mathfrak{m}^{N+3}$$

The nondegeneracy of $q = ax^2 + bxy + y^2$ implies that $2ax_0 + by_0$ and $bx_0 + 2cy_0$ are linearly independent. Since $\dim_{\kappa(p)} \mathfrak{m}^{N+2}/\mathfrak{m}^{N+3} = 2$, we may choose x_N and y_N such that $Q(x' + x_N, y' + y_N) \in \mathfrak{m}^{N+3}$. This completes (4).

Assuming (4) and using that q is nondegenerate, we may choose a degree 2 separable field extension $\kappa(p) \to \Bbbk'$ such that q splits as a product of a linear forms. Thus $\widehat{\mathcal{O}}_{C,p} \otimes_{\Bbbk} \Bbbk' \cong \Bbbk'[\![x, y]\!]/(xy)$, yielding (5). Finally, (5) clearly implies that p is a node.

See also [SP, Tags 0C49, 0C4D and 0C4E]. □

Exercise 5.2.4. Show that $\operatorname{Spec} \Bbbk[x, y]/(f)$ has a node at 0 if and only if $f(0) = f_x(0) = f_y(0) = 0$ and the Hessian $\det \begin{pmatrix} f_{xx}(0) & f_{xy}(0) \\ f_{yx}(0) & f_{yy}(0) \end{pmatrix}$ is nonzero.

Exercise 5.2.5. Let C be a pure dimension 1 reduced curve over a field $\Bbbk$ with normalization $\pi \colon \widetilde{C} \to C$. Show that $p \in C$ is a node if and only if $\Bbbk \to \kappa(p)$ is separable, $(\pi_* \mathcal{O}_{\widetilde{C}}/\mathcal{O}_C) \otimes \kappa(p) = 1$, and $\sum_{\pi(q)=p} [\kappa(q) \colon \kappa(p)]_{\mathrm{sep}} = 2$.

Hint: Identify $(\pi_* \mathcal{O}_{\widetilde{C}}/\mathcal{O}_C) \otimes \kappa(p)$ with the quotient $\widetilde{A}/A$ where $A = \widehat{\mathcal{O}}_{C,p}$ and $\widetilde{A}$ is its normalization, using that normalization commutes with completion (see Remark B.4.5). To show ($\Leftarrow$), use that $\widetilde{A}$ is a product of complete DVRs to derive the structure of A. See also [SP, Tag 0C4A].

Remark 5.2.6. The quantity $(\pi_* \mathcal{O}_{\widetilde{C}}/\mathcal{O}_C) \otimes \kappa(p)$ is referred to as the *δ-invariant of p*, and the sum $\sum_{\pi(q)=p} [\kappa(q) \colon \kappa(p)]_{\mathrm{sep}}$ is referred to as the *number of geometric branches over p*. A cusp $\Bbbk[\![x, y]\!]/(y^2 - x^3)$ has δ-invariant 1 but has only one geometric branch.

Proposition 5.2.7 (Local Structure of Nodes)**.** *Let C be a curve over a field $\Bbbk$. If $p \in C$ is a node, then there exist a finite separable field extension $\Bbbk \to \Bbbk'$ and étale neighborhoods*

$$\begin{tikzcd} & (U, u) \arrow[dl] \arrow[dr] & \\ (C, p) & & (\operatorname{Spec} \Bbbk'[x, y]/(xy), 0). \end{tikzcd} \tag{5.2.1}$$

Proof. Using the characterization of nodes from Proposition 5.2.3(5), there is a finite separable field extension $\Bbbk \to \Bbbk'$ such that $\widehat{\mathcal{O}}_{C,p} \otimes_{\Bbbk} \Bbbk' \cong \Bbbk'[\![x, y]\!]/(xy)$. The result is now a consequence of Artin Approximation (Corollary A.10.13). □

We will prove a more general statement in Theorem 5.2.18 regarding the local structure of families of nodal curves.

Exercise 5.2.8. Provide a proof of the Local Structure of Nodes (5.2.7) without appealing to Artin Approximation.

Hint: Use that the normalization of a strict henselization $\mathcal{O}^{\mathrm{sh}}_{C,p}$ has two components to find an affine étale neighborhood $(\operatorname{Spec} R, u) \to (C, p)$ of p with $\widetilde{R} = R_1 \times R_2$.

Use the exact sequence $0 \to R \to R_1 \times R_2 \to \kappa(u) \to 0$ to construct elements $x, y \in R$ mapping to $(1,0)$, $(0,1) \in R_1 \times R_2$, and argue that $\kappa(u)[x,y]/(xy) \to R$ is étale.

5.2.3 Genus formula

Proposition 5.2.9 (Genus Formula)**.** *Let C be a connected, nodal, and projective curve over an algebraically closed field $\Bbbk$ with δ nodes $p_1, \ldots, p_\delta \in C$ and ν irreducible components $C_1, \ldots, C_\nu$. Let $g(\widetilde{C}_i)$ be the genus of the normalization $\widetilde{C}_i$ of C_i, i.e. the geometric genus of C_i. The genus g of C satisfies*

$$g = \sum_{i=1}^{\nu} g(\widetilde{C}_i) + \delta - \nu + 1.$$

Proof. We claim that the normalization $\pi \colon \widetilde{C} \to C$ induces a short exact sequence

$$0 \to \mathcal{O}_C \to \pi_* \mathcal{O}_{\widetilde{C}} \to \bigoplus_i \kappa(p_i) \to 0.$$

It suffices to verify this étale-locally around a node $p_i \in C$, and so by the Local Structure of Nodes (5.2.7), we can assume that $C = \operatorname{Spec} \Bbbk[x,y]/(xy)$. In this case, $\widetilde{C} = \operatorname{Spec}(\Bbbk[x] \times \Bbbk[y])$ and the sequence above corresponds to $0 \to \Bbbk[x,y]/(xy) \to \Bbbk[x] \times \Bbbk[y] \to \Bbbk \to 0$. Alternatively, normalization commutes with completion and a direct calculation as above shows that if $A := \widehat{\mathcal{O}}_{C,p} \cong \Bbbk[\![x,y]\!]/(xy)$, then $\widetilde{A}/A \cong \Bbbk$; see also Exercise 5.2.5.

The short exact sequence induces a long exact sequence on cohomology

$$0 \to \underbrace{H^0(C, \mathcal{O}_C)}_{1} \to \underbrace{H^0(\widetilde{C}, \mathcal{O}_{\widetilde{C}})}_{\nu} \to \underbrace{\oplus_i \kappa(p_i)}_{\delta} \to \underbrace{H^1(C, \mathcal{O}_C)}_{g} \to \underbrace{H^1(\widetilde{C}, \mathcal{O}_{\widetilde{C}})}_{\sum_i g(\widetilde{C}_i)} \to 0$$

where the labels underneath indicate the dimension. The statement follows. $\square$

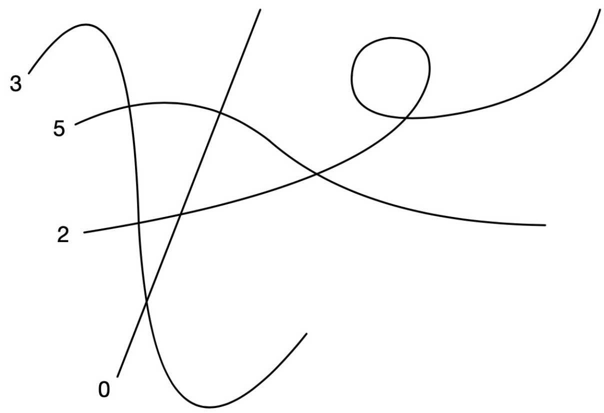


Figure 5.2: An example of a nodal curve of genus 14.

Remark 5.2.10. Notice that $\delta - \nu + 1$ is precisely the number of connected regions bounded by the curve C as in Figure 5.2. Thus, the genus of a nodal curve can be easily computed from the picture by summing the geometric genera of the irreducible components and adding the number of bounded regions.

5.2.4 The dualizing sheaf

If C is a connected, nodal, and projective curve over an algebraically closed field $\Bbbk$, then C is locally a complete intersection and therefore C has a *dualizing sheaf* ω_C with a trace map $\mathrm{tr}_C\colon H^1(C,\omega_C) \xrightarrow{\sim} \Bbbk$; see [Har77, III.7.11] or [Ser88, §IV]. In other words, for every coherent sheaf $\mathcal{F}$, the natural pairing

$$\mathrm{Hom}_{\mathcal{O}_C}(\mathcal{F},\omega_C) \times H^1(C,\mathcal{F}) \to H^1(C,\omega_C) \xrightarrow{\mathrm{tr}} \Bbbk$$

is perfect.

Due to its importance in the study of stable curves, we will provide an explicit description of ω_C below and show that it is a line bundle. Let $\Sigma := C^{\mathrm{sing}}$ be the singular locus and $U = C \setminus \Sigma$. Let $\pi\colon \widetilde{C} \to C$ be the normalization of C, and let $\widetilde{\Sigma}$ and $\widetilde{U}$ be the preimages of Σ and U as in the diagram

$$\begin{tikzcd}
\widetilde{U} \arrow[r, hook, "\widetilde{j}"] \arrow[d] & \widetilde{C} \arrow[d, "\pi"] & \widetilde{\Sigma} \arrow[l, hook'] \arrow[d] \\
U \arrow[r, hook, "j"] & C & \Sigma \arrow[l, hook']
\end{tikzcd}$$

Let $\Sigma = \{z_1,\dots,z_n\}$ be an ordering of the points and $\pi^{-1}(z_i) = \{p_i,q_i\}$. Since $\widetilde{C}$ is smooth, the sheaf of differentials $\Omega_{\widetilde{C}}$ is a dualizing sheaf and is a line bundle. There is a short exact sequence

$$0 \to \Omega_{\widetilde{C}} \to \Omega_{\widetilde{C}}(\widetilde{\Sigma}) \to \mathcal{O}_{\widetilde{\Sigma}} \to 0 \tag{5.2.2}$$

obtained by tensoring the sequence $0 \to \mathcal{O}_{\widetilde{C}}(-\widetilde{\Sigma}) \to \mathcal{O}_{\widetilde{C}} \to \mathcal{O}_{\widetilde{\Sigma}} \to 0$ with $\Omega_{\widetilde{C}}(\widetilde{\Sigma})$. As $\Omega_{\widetilde{C}}(\widetilde{\Sigma})|_{\widetilde{U}} = \Omega_{\widetilde{U}}$, we can interpret sections of $\Omega_{\widetilde{C}}(\widetilde{\Sigma})$ as rational sections of $\Omega_{\widetilde{C}}$ with at worst simple poles along $\widetilde{\Sigma}$. Evaluating (5.2.2) on an open $\widetilde{V} \subset \widetilde{C}$ yields

$$\begin{aligned} 0 \longrightarrow \Gamma(\widetilde{V},\Omega_{\widetilde{C}}) \longrightarrow \Gamma(\widetilde{V},\Omega_{\widetilde{C}}(\widetilde{\Sigma})) &\longrightarrow \textstyle\bigoplus_{y\in \widetilde{V}\cap\widetilde{\Sigma}} \kappa(y), \\ s &\mapsto (\mathrm{res}_y(s)) \end{aligned}$$

where the last map takes a rational section $s \in \Gamma(\widetilde{V}\cap\widetilde{U},\Omega_{\widetilde{C}})$ to the tuple whose coordinate at $y \in \widetilde{V}\cap\widetilde{\Sigma}$ is the *residue* $\mathrm{res}_y(s)$ *of* s *at* y.

Definition 5.2.11. We define the subsheaf $\omega_C \subset \pi_*\Omega_{\widetilde{C}}(\widetilde{\Sigma})$ by declaring that sections along $V \subset C$ consist of rational sections s of $\Omega_{\widetilde{C}}$ along $\pi^{-1}(V)$ (with at worst simple poles along $\widetilde{\Sigma}$) such that for all nodes $z_i \in V\cap\Sigma$, the $\mathrm{res}_{p_i}(s)+\mathrm{res}_{q_i}(s)$, i.e. the sum of the residues at the two points p_i, q_i above x_i is zero.

The definition implies that ω_C sits in the following two exact sequences:

$$\begin{aligned} 0 \longrightarrow \omega_C \longrightarrow \pi_*\Omega_{\widetilde{C}}(\widetilde{\Sigma}) &\longrightarrow \textstyle\bigoplus_{z_i\in\Sigma} \Bbbk \longrightarrow 0 \\ s &\mapsto (\mathrm{res}_{p_i}(s) - \mathrm{res}_{q_i}(s)) \end{aligned} \tag{5.2.3}$$

$$\begin{aligned} 0 \longrightarrow \pi_*\Omega_{\widetilde{C}} \longrightarrow \omega_C &\longrightarrow \textstyle\bigoplus_{z_i\in\Sigma} \Bbbk \longrightarrow 0 \\ s &\mapsto (\mathrm{res}_{p_i}(s)) \end{aligned} \tag{5.2.4}$$

Example 5.2.12 (Local calculation)**.** Let $C = \operatorname{Spec} \Bbbk[x,y]/(xy)$. Then $\widetilde{C} = \mathbb{A}^1 \sqcup \mathbb{A}^1$ with coordinates x and y respectively. The singular locus of C is $\Sigma = \{0\}$ with preimage $\widetilde{\Sigma} = \{p, q\}$ consisting of the two origins. Then $\Gamma(\widetilde{C}, \Omega_{\widetilde{C}}) = \Gamma(\mathbb{A}^1, \omega_{\mathbb{A}^1}) \times \Gamma(\mathbb{A}^1, \omega_{\mathbb{A}^1})$ and $(\frac{dx}{x}, -\frac{dy}{y})$ is a rational section with opposite residues at p and q. In fact, every section of $\Gamma(C, \omega_C)$ is of the form

$$\left(f(x)\frac{dx}{x}, g(y)\frac{-dy}{y} \right) = (f(x) + g(y) - f(0)) \cdot (\frac{dx}{x}, \frac{-dy}{y})$$

for polynomials $f(x)$ and $g(y)$ such that $f(0) = g(0)$, which is precisely the condition for $(f, g) \in \Gamma(\widetilde{C}, \mathcal{O}_{\widetilde{C}})$ to descend to a global function on C. In other words, $\omega_C \cong \mathcal{O}_C$ with generator $(\frac{dx}{x}, -\frac{dy}{y})$.

Example 5.2.13. Let C be the nodal projective plane cubic and $\mathbb{P}^1 \to C$ be the normalization with coordinates $[x : y]$ such that 0 and ∞ are the fibers of the node. Observe that the rational differential $\eta := \frac{dx}{x} = -\frac{dy}{y}$ on $\mathbb{P}^1$ satisfies $\operatorname{res}_0 \eta + \operatorname{res}_\infty \eta = 0$. It is easy to see that every local section of ω_C is a multiple of η or in other words that $\eta \colon \mathcal{O}_C \to \omega_C$ is an isomorphism.

Exercise 5.2.14. Let C be a connected, nodal, and projective curve over an algebraically closed field $\Bbbk$.

(a) Show that if $\pi \colon C' \to C$ is an étale morphism, then $\pi^* \omega_C \cong \omega_{C'}$.

Hint: Use the fact that normalization commutes with étale base change.

(b) Conclude that ω_C is a line bundle.

(c) Show that ω_C is a dualizing sheaf.

Hint: Reduce to the case of a smooth curve by considering the normalization.

(d) If $T \subset C$ is a subcurve with complement $T^c := \overline{C \setminus T}$, show that

$$\omega_C|_T = \omega_T(T \cap T^c).$$

Exercise 5.2.15. Let C be a connected, nodal, and projective curve over an algebraically closed field $\Bbbk$. Let $\widetilde{C} \to C$ be the normalization and $\widetilde{\Sigma} \subset \widetilde{C}$ the set of preimages of nodes. Show that there is an identification

$$\operatorname{Hom}_{\mathcal{O}_{\widetilde{C}}}(\Omega_{\widetilde{C}}(\widetilde{\Sigma}), \mathcal{O}_{\widetilde{C}}) \cong \operatorname{Hom}_{\mathcal{O}_C}(\Omega_C, \mathcal{O}_C),$$

or in other words that regular vector fields on C correspond to regular vector fields on $\widetilde{C}$ vanishing at the preimages of nodes.

5.2.5 Nodal families

Recall that the relative singular locus $\operatorname{Sing}(\mathcal{C}/S)$ of a morphism $\mathcal{C} \to S$ with dimension one fibers is defined as the first fitting ideal of $\Omega_{\mathcal{C}/S}$; see Definition A.3.14. Syntomic morphisms are fppf morphisms whose fibers are local complete intersections; see §A.3.8.

Proposition 5.2.16. *Let $\mathcal{C} \to S$ be an fppf morphism of schemes and $s \in S$ a point such that the fiber $\mathcal{C}_s$ has pure dimension one. A point $p \in \mathcal{C}_s$ is a node if and only if $\mathcal{C} \to S$ is syntomic at p and the relative singular locus $\operatorname{Sing}(\mathcal{C}/S) \to S$ is unramified at p.*

Proof. The conditions that $\mathcal{C} \to S$ is syntomic at p and $\mathrm{Sing}(\mathcal{C}) \to S$ is unramified at p are conditions on the fibers over s. Since $\mathrm{Sing}(\mathcal{C}/S)_s = \mathrm{Sing}(\mathcal{C}_s)$, the result follows from the equivalence of (1)-(4) of Proposition 5.2.3. □

The above characterization allows us to show that the property of being a nodal family descends under limits (Definition A.6.6).

Lemma 5.2.17. *The following property of morphisms of schemes descends under limits: an fppf morphism such that every fiber is a pure dimension one nodal curve.*

Proof. From Descending Properties of Morphisms under Limits (A.6.7), we know that the properties of being fppf, syntomic, unramified, and having connected pure one-dimensional fibers descend under limits. Since the relative singular locus commutes with base change, the result follows from Proposition 5.2.16. □

A *family of nodal curves* is a proper fppf morphism $\mathcal{C} \to S$ of schemes such that every geometric fiber $\mathcal{C}_s$ is a connected nodal curve.

5.2.6 Local structure of nodal families

Recall that if $\mathcal{C} \to S$ is a family of smooth curves, then every point $p \in \mathcal{C}$ over $s \in S$ is étale locally isomorphic to relative affine space of dimension one. More precisely, there are étale neighborhoods

$$\begin{tikzcd}
(\mathcal{C}, p) \arrow[d] & (U, u) \arrow[l, "\text{ét}"'] \arrow[r, "\text{ét}"] \arrow[d] & (S' \times_{\mathbb{Z}} \mathbb{A}^1_{\mathbb{Z}}, (s', 0)) \arrow[dl] \\
(S, s) & (S', s') \arrow[l, "\text{ét}"'] &
\end{tikzcd}$$

where $S' \times_{\mathbb{Z}} \mathbb{A}^1_{\mathbb{Z}} = \mathbb{A}^1_{S'} \to S'$ is the base change of $\mathbb{A}^1_{\mathbb{Z}} \to \operatorname{Spec} \mathbb{Z}$; see **??**. We now give a local structure of a family of nodal curves generalizing the Local Structure of Nodes (5.2.7).

Theorem 5.2.18 (Local Structure of Nodal Families)**.** *Let $\pi \colon \mathcal{C} \to S$ be an fppf morphism such that every geometric fiber is a curve. Let $p \in \mathcal{C}$ be a node in a fiber $\mathcal{C}_s$. There is a commutative diagram*

$$\begin{tikzcd}
(\mathcal{C}, p) \arrow[d] & (U, u) \arrow[l, "ét"'] \arrow[r, "ét"] \arrow[d] & (\operatorname{Spec} A[x,y]/(xy - f), (s', 0)) \arrow[dl] \\
(S, s) & (\operatorname{Spec} A, s') \arrow[l, "ét"'] &
\end{tikzcd} \tag{5.2.5}$$

where each horizontal map is étale and $f \in A$ is a function vanishing at s'.

Remark 5.2.19. In other words, every family of nodal curves is étale locally on the source and target the base change of the morphism

$$\operatorname{Spec} \mathbb{Z}[x,y,t]/(xy - t) \to \operatorname{Spec} \mathbb{Z}[t]$$

by a map $\operatorname{Spec} A \to \operatorname{Spec} \mathbb{Z}[t]$ induced by a function $f \in A$.

Proof 1 (local-to-global).

Step 1: Reduce to the case where S is of finite type over $\mathbb{Z}$. Use limit methods and Lemma 5.2.17.

Step 2: Reduce to the case where $\widehat{\mathcal{O}}_{\mathcal{C}_s,p} \cong \kappa(s)[\![x,y]\!]/(xy)$. By Proposition 5.2.3, there is a finite separable field extension $\kappa(s) \to \Bbbk'$ and a point $p' \in \mathcal{C}_s \times_{\kappa(s)} \Bbbk'$ whose completion is isomorphic to $\Bbbk'[\![x,y]\!]/(xy)$. Letting $(S',s') \to (S,s)$ be an étale morphism such that there is an isomorphism $\kappa(s') \cong \Bbbk'$ over $\kappa(s)$, we replace S with S'.

Step 3: Show that $\widehat{\mathcal{O}}_{\mathcal{C},p} \cong \widehat{\mathcal{O}}_{S,s}[\![x,y]\!]/(xy-\widehat{f})$ where $\widehat{f} \in \widehat{\mathfrak{m}}_s \subset \widehat{\mathcal{O}}_{S,s}$. We claim that there exists elements $x_n, y_n \in \widehat{\mathcal{O}}_{\mathcal{C},p}$ and $f_n \in \widehat{\mathcal{O}}_{S,s}$ for $n \geq 0$ which are compatible (i.e. $x_{n+1} \equiv x_n (\mathrm{mod}\ \mathfrak{m}_p^{n+1})$, $y_{n+1} \equiv y_n (\mathrm{mod}\ \mathfrak{m}_p^{n+1})$, and $f_{n+1} \equiv f_n (\mathrm{mod}\ \mathfrak{m}_s^{n+1})$) and such that there is an isomorphism

$$(\mathcal{O}_{S,s}/\mathfrak{m}_s^{n+1})[\![x,y]\!]/(xy-f_n) \xrightarrow{\sim} \mathcal{O}_{\mathcal{C},p}/\mathfrak{m}_s^{n+1}\mathcal{O}_{\mathcal{C},p} \tag{5.2.6}$$

induced by the map sending x and y to the images of x_n and y_n. The condition that the map (5.2.6) is an isomorphism is equivalent to $x_n y_n - f_n \in \widehat{\mathfrak{m}}_s^{n+1}\widehat{\mathcal{O}}_{\mathcal{C},p}$.

We will prove this by induction. The base case $n=0$ is handled by Step 2. Assuming the claim holds for n, write

$$x_n y_n - f_n = \sum_i a_i b_i \quad \text{with } a_i \in \widehat{\mathfrak{m}}_s^{n+1} \text{ and } b_i \in \widehat{\mathcal{O}}_{\mathcal{C},p}.$$

Since x_n and y_n generate the maximal ideal of p in the fiber $\mathcal{C}_s$, and since $\kappa(s) = \kappa(p)$, we may find $a_i' \in \widehat{\mathcal{O}}_{S,s}$ and $b_i', b_i'' \in \widehat{\mathcal{O}}_{\mathcal{C},p}$ such that

$$b_i - (x_n b_i' + y_n b_i'' + a_i') \in \widehat{\mathfrak{m}}_s \widehat{\mathcal{O}}_{\mathcal{C},p}.$$

We then define

$$x_{n+1} = x_n - \sum_i a_i b_i'', \quad y_{n+1} = y_n - \sum_i a_i b_i', \quad f_{n+1} = f_n + \sum_i a_i a_i',$$

and check that

$$\begin{aligned} x_{n+1}y_{n+1} - f_{n+1} &= (x_n - \sum_i a_i b_i'')(y_n - \sum_i a_i b_i') - (f_n + \sum_i a_i a_i') \\ &= (x_n y_n - f_n) - x_n \sum_i a_i b_i' - y_n \sum_i a_i b_i'' - \sum_i a_i a_i' + \sum_{i,j} a_i a_j b_i'' b_j' \\ &= \sum_i a_i b_i - x_n \sum_i a_i b_i' - y_n \sum_i a_i b_i'' - \sum_i a_i a_i' + \sum_{i,j} a_i a_j b_i'' b_j' \\ &= \sum_i \underbrace{a_i}_{\widehat{\mathfrak{m}}_s^{n+1}} \underbrace{(b_i - x_n b_i' - y_n b_i'' - a_i')}_{\widehat{\mathfrak{m}}_s \widehat{\mathcal{O}}_{C,p}} + \sum_{i,j} \underbrace{a_i a_j}_{\widehat{\mathfrak{m}}_s^{2(n+1)}} b_i'' b_j' \end{aligned}$$

is an element of $\widehat{\mathfrak{m}}_s^{n+2}\widehat{\mathcal{O}}_{C,p}$. Setting $\widehat{x} = \lim_n x_n, \widehat{y} = \lim_n y_n \in \widehat{\mathcal{O}}_{C,p}$, and $\widehat{f} = \lim_n f_n \in \widehat{\mathfrak{m}}_s$, we see that the map $\widehat{\mathcal{O}}_{S,s}[\![x,y]\!]/(xy-\widehat{f}) \to \widehat{\mathcal{O}}_{C,p}$, defined by $x \mapsto \widehat{x}$ and $y \mapsto \widehat{y}$, is an isomorphism.

Step 4: Construct the desired étale neighborhoods. Step 3 provides a diagram

$$\begin{tikzcd}
\operatorname{Spec} \widehat{\mathcal{O}}_{S,s} \times_S \mathcal{C} \arrow[dr] & & \operatorname{Spec} \widehat{\mathcal{O}}_{S,s}[x,y]/(xy - \widehat{f}) \arrow[dl] \\
& \operatorname{Spec} \widehat{\mathcal{O}}_{S,s}. &
\end{tikzcd}$$

such that the points $(s,p) \in \operatorname{Spec} \widehat{\mathcal{O}}_{S,s} \times_S \mathcal{C}$ and $(s,0) \in \operatorname{Spec} \widehat{\mathcal{O}}_{S,s}[x,y]/(xy - \widehat{f})$ have isomorphic completion, where s denotes also the closed point of $\operatorname{Spec} \widehat{\mathcal{O}}_{S,s}$. A consequence of Artin Approximation (Corollary A.10.13) implies that there are étale morphisms

$$\begin{tikzcd}
& (U,u) \arrow[dl] \arrow[dr] & \\
(\operatorname{Spec} \widehat{\mathcal{O}}_{S,s} \times_S \mathcal{C}, (s,p)) & & (\operatorname{Spec} \widehat{\mathcal{O}}_{S,s}[x,y]/(xy - \widehat{f}), (s,0))
\end{tikzcd} \tag{5.2.7}$$

defined over $\operatorname{Spec} \widehat{\mathcal{O}}_{S,s}$. After replacing S with an open affine neighborhood of s, we can assume that $S = \operatorname{Spec} A$ is affine. By Neron–Popescu (A.10.4), we may write $\widehat{\mathcal{O}}_{S,s} = \operatorname{colim} B_\lambda$ as a directed colimit of smooth A-algebras. Set $S_\lambda = \operatorname{Spec} B_\lambda$, $\mathcal{C}_\lambda = \mathcal{C} \times_S S_\lambda$, and $U_\lambda = U \times_S S_\lambda$. For $\lambda \gg 0$, $\widehat{f} \in \widehat{\mathcal{O}}_{S,s}$ is the image of an element $f_\lambda \in B_\lambda$, and the pullbacks of x and y to $\Gamma(U, \mathcal{O}_U)$ are the pullbacks of elements in $\Gamma(U_\lambda, \mathcal{O}_{U_\lambda})$ under $U \to U_\lambda$. This yields a commutative diagram

$$\begin{tikzcd}
& U_\lambda \arrow[dl] \arrow[dr] & \\
\mathcal{C}_\lambda \arrow[dr] & & \operatorname{Spec} B_\lambda[x,y]/(xy - f_\lambda) \arrow[dl] \\
& S_\lambda = \operatorname{Spec} B_\lambda &
\end{tikzcd}$$

which base changes to (5.2.7) under $\operatorname{Spec} \widehat{\mathcal{O}}_{S,s} \to S_\lambda$. Since étaleness descends under limits (A.6.7), the maps $U_\lambda \to \mathcal{C}_\lambda$ and $U_\lambda \to \operatorname{Spec} B_\lambda[x,y]/(xy - f_\lambda)$ are étale for $\lambda \gg 0$. Letting $u_\lambda = (u, s_\lambda) \in U_\lambda$, we have a commutative diagram

$$\begin{tikzcd}
(\mathcal{C}, p) \arrow[d] & (U_\lambda, u_\lambda) \arrow[l, "\mathrm{sm}"'] \arrow[r, "\text{ét}"] \arrow[d] & (\operatorname{Spec} B_\lambda[x,y]/(xy - f_\lambda), (s_\lambda, 0)) \arrow[dl] \\
(S, s) & (\operatorname{Spec} B_\lambda, s_\lambda) \arrow[l, "\mathrm{sm}"'] &
\end{tikzcd}$$

This gives our desired diagram (5.2.5) except that the left horizontal arrows are smooth rather than étale. Since smooth maps étale locally have sections (Corollary A.3.6), there is an étale map $(\operatorname{Spec} A, s') \to (S, s)$ and a map $(\operatorname{Spec} A, s') \to (S_\lambda, s_\lambda)$ over S. The result follows from setting $U = U_\lambda \times_{S_\lambda} \operatorname{Spec} A$ and $u = (u_\lambda, s')$.

See also [SP, Tag 0CBY] $\square$

Proof 2 (avoiding Artin Approximation/Neron–Popescu). We can reduce to the case where S is of finite type over $\mathbb{Z}$ by Lemma 5.2.17. By Proposition 5.2.16, we may replace $\mathcal{C}$ with an open neighborhood of p such that $\mathcal{C} \to S$ is syntomic and $\mathrm{Sing}(\mathcal{C}/S) \to S$ is unramified. After replacing $\mathcal{C}$ and S with open neighborhoods, we may also assume that $\mathcal{C}$ and S are affine, and that the geometric fibers of $\mathcal{C} \to S$ are connected with at most two irreducible components. We may choose an closed immersion $S \hookrightarrow \mathbb{A}^n_{\mathbb{Z}}$ and apply Proposition A.3.16 to find a syntomic morphism $\mathcal{C}' \to \mathbb{A}^n_{\mathbb{Z}}$ extending $\mathcal{C} \to S$. The fiber $\mathcal{C}'_s$ has a node at p and after replacing $\mathcal{C} \to S$ with $\mathcal{C}' \to \mathbb{A}^n_{\mathbb{Z}}$, we may assume that the base S is regular. By the étale local structure of unramified morphisms (Proposition A.3.7), after replacing $\mathcal{C}$ and S with étale neighborhoods, we can arrange that $\mathrm{Sing}(\mathcal{C}/S) \hookrightarrow S$ is a closed immersion.

We claim that after replacing S with an open neighborhood of s, we can arrange that $\mathrm{Sing}(\mathcal{C}/S) = S$ or $\mathrm{Sing}(\mathcal{C}/S)$ is defined by a nonzerodivisor $f \in \Gamma(S, \mathcal{O}_S)$. This holds over the completion of S at s by Step 3 in the first proof above: since $\widehat{\mathcal{O}}_{\mathcal{C},p} \cong \widehat{\mathcal{O}}_{S,s}[\![x,y]\!]/(xy - \widehat{f})$ where $\widehat{f} \in \widehat{\mathfrak{m}}_s$, $\mathrm{Sing}(\mathcal{C}/S) \times_S \mathrm{Spec}\, \widehat{\mathcal{O}}_{S,s} = V(\widehat{f})$. The claim then follows from using fppf descent along $\mathrm{Spec}\, \widehat{\mathcal{O}}_{S,s} \to \mathrm{Spec}\, \mathcal{O}_{S,s}$ and properties of the ideal sheaf $\mathcal{I}$ defining $\mathrm{Sing}(\mathcal{C}/S)$. Indeed, if $\widehat{f} = 0$, then $\mathcal{I}_s = 0$ and hence $\mathcal{I}$ is zero in an open neighborhood of s. If $\widehat{f}$ is a nonzerodivisor, then $\mathcal{I}_s$ is a line bundle (by Proposition B.4.3) and hence $\mathcal{I}$ is defined by a nonzerodivisor in an open neighborhood of s.

If $\mathrm{Sing}(\mathcal{C}/S) = S$, we first claim that after replacing $\mathcal{C}$ with an étale neighborhood, we can arrange that $\mathcal{C}$ is the scheme-theoretic union $\mathcal{C}_1 \cup \mathcal{C}_1$ of closed subschemes such that $\mathrm{Sing}(\mathcal{C}/S) = \mathcal{C}_1 \cap \mathcal{C}_2$. The normalization $\widetilde{Z} \to \mathrm{Spec}\, \mathcal{O}^{\mathrm{h}}_{\mathcal{C},p}$ of the henselization is a finite morphism, and since normalization commutes with completion (see Remark B.4.5), there are two preimages in $\widetilde{Z}$ of the unique closed point. By properties of the henselization (Proposition A.9.6), $\widetilde{Z}$ is the disjoint union $\widetilde{Z} = \widetilde{Z}_1 \amalg \widetilde{Z}_2$. Therefore $\mathrm{Spec}\, \mathcal{O}^h_{\mathcal{C},p}$ is the union of the (closed) images of $\widetilde{Z}_1$ and $\widetilde{Z}_2$. This establishes the claim. After replacing $\mathcal{C}$ with an open neighborhood, we can arrange that $\mathcal{C}_1$ and $\mathcal{C}_2$ are defined by global functions $g_1, g_2 \in B := \Gamma(\mathcal{C}, \mathcal{O}_{\mathcal{C}})$ on $\mathcal{C}$ with $g_1 g_2 = 0$. Letting $S = \mathrm{Spec}\, A$, the ring map $A[x,y]/(xy) \to B$, defined by $x \mapsto g_1$ and $y \mapsto g_2$, induces a morphism $\mathcal{C} \to \mathrm{Spec}\, A[x,y]/(xy)$ over S. This map is étale at p since it induces an isomorphism of completions at p.

If $\mathrm{Sing}(\mathcal{C}/S) = V(f)$ with $f \in A := \Gamma(S, \mathcal{O}_S)$, then the argument above shows that $\mathcal{C} \times_A (A/f)$ is the scheme-theoretic union $Z_1 \cup Z_1$ of effective Cartier divisors such that $\mathrm{Sing}(\mathcal{C}/S) = Z_1 \cap Z_2$. After replacing $\mathcal{C}$ with an open neighborhood, we can write each $Z_i = V(g_i)$ for global functions $g_i \in B := \Gamma(\mathcal{C}, \mathcal{O}_{\mathcal{C}})$. As the restrictions of $g_1 g_2$ and f define the same closed subscheme of $\mathrm{Spec}\, \widehat{\mathcal{O}}_{\mathcal{C},p}$, we have that $f = u g_1 g_2$ for a unit $u \in B$ after replacing $\mathcal{C}$ with an open neighborhood. The ring map $A[x,y]/(xy - f) \to B$, defined by $x \mapsto u g_1$ and $y \mapsto g_2$, induces a morphism $\mathcal{C} \to \mathrm{Spec}\, A[x,y]/(xy)$ over S; this map is étale at p since it induces an isomorphism of completions at p. □

One direct consequence of this local structure theorem is that if $\mathcal{C} \to S$ is an fppf morphism such every fiber is a pure dimension one curve, then the locus $\mathcal{C}^{\leq \mathrm{nod}} \subset \mathcal{C}$ of points which are smooth or nodal is open. And if we add a properness condition on $\mathcal{C} \to S$, then $\pi(\mathcal{C} \setminus \mathcal{C}^{\leq \mathrm{nod}}) \subset S$ is closed and therefore the locus of points $s \in S$ such that $\mathcal{C}_s$ is a nodal curve is the *open* subscheme $S \setminus \pi(\mathcal{C} \setminus \mathcal{C}^{\leq \mathrm{nod}})$. This will be applied later to conclude that the stack parameterizing families of nodal curves is an open substack of the stack of all curves.

Corollary 5.2.20. *If $\mathcal{C} \to S$ is a proper fppf morphism of schemes such that every geometric fiber is a curve, then the locus of points $s \in S$ such that $\mathcal{C}_s$ is nodal is open.* □

5.3 Stable curves

Stable curves were introduced in unpublished joint work by Mayer and Mumford [MM64].

5.3.1 Definition and equivalences

An *n-pointed curve* is a curve C over a field $\Bbbk$ together with an ordered collection of $\Bbbk$-points $p_1, \ldots, p_n \in C$; we call the $p_i \in C$ *marked points*. A point $q \in C$ of an n-pointed curve is called *special* if q is a node or a marked point.

Definition 5.3.1 (Stable curves)**.** An n-pointed curve $(C, p_1, \ldots, p_n)$ over a field $\Bbbk$ is *stable* if C is geometrically connected, nodal, and projective, and $p_1, \ldots, p_n \in C$ are distinct smooth points such that

(1) every smooth rational subcurve $\mathbb{P}^1 \subset C$ contains at least 3 special points, and

(2) C is not of genus 1 without marked points.

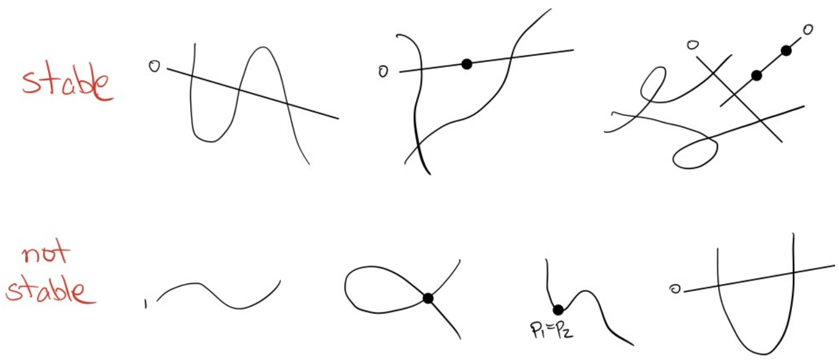


Figure 5.3: The curves in the top row are stable while those in the second row are not.

Definition 5.3.2 (Semistable and prestable curves)**.** An n-pointed curve $(C, p_1, \ldots, p_n)$ is *semistable* if we take the same conditions as Definition 5.3.1 except that (1) is replaced with the condition that every smooth rational subcurve $\mathbb{P}^1 \subset C$ contains at least 2 (rather than 3) special points. We define $(C, p_1, \ldots, p_n)$ to be *prestable* by dropping both condition (1) and (2), i.e. C is a connected, nodal, and projective curve and the points p_i are distinct smooth points of C.

An unpointed connected projective curves is prestable if and only if it is nodal.

Remark 5.3.3. Note that there are no n-pointed stable curve of genus g if $(g,n) \in \{(0,0),(0,1),(0,2),(1,0)\}$ or equivalently $2g-2+n \leq 0$. We will often impose the condition that $2g-2+n > 0$ in order to exclude these special cases.

An automorphism of a stable curve $(C, p_1, \ldots, p_n)$ is an automorphism $\alpha \colon C \xrightarrow{\sim} C$ such that $\alpha(p_i) = p_i$. We denote by $\mathrm{Aut}(C, p_1, \ldots, p_n)$ the (abstract) group of automorphisms. Recall also that if C is a connected, smooth, and projective curve of genus $g \geq 2$, then $\mathrm{Aut}(C)$ is finite [Har77, Exer. III.2.5].

Proposition 5.3.4. *Let $(C, p_1, \ldots, p_n)$ be an n-pointed prestable curve. The following are equivalent:*

(1) $(C, p_1, \ldots, p_n)$ *is stable,*

(2) $\mathrm{Aut}(C, p_1, \ldots, p_n)$ *is finite, and*

(3) $\omega_C(p_1 + \cdots + p_n)$ *is ample.*

Proof. The equivalence (1) $\iff$ (2) follows from Exercise 5.3.5 and the observation that the only way a smooth, connected, and projective n-pointed curves (C, p_i) can have a positive dimensional automorphism group is if $C = \mathbb{P}^1$ with $n \leq 2$ or if C is a genus 1 curve with $n = 0$.

To see the equivalence with (3), we will use the fact that for a subcurve $T \subset C$, we have $\omega_C|_T = \omega_T(T \cap T^c)$ (Exercise 5.2.14). If $\pi \colon \widetilde{C} \to C$ is the normalization, then$\omega_C(p_1 + \cdots + p_n)$ is ample if and only if $\pi^*\big(\omega_C(p_1 + \cdots + p_n)\big)$ is ample if and only if for each irreducible component $T \subset C$,

$$\omega_C(p_1 + \cdots + p_n)|_T = \omega_T\Big(\sum_{p_i \in T} p_i + (T \cap T^c)\Big)$$

is ample. This latter condition holds precisely if each $\mathbb{P}^1 \subset \widetilde{C}$ contains at least three points that lie over nodes or marked points. □

Exercise 5.3.5. Let $(C, p_1, \ldots, p_n)$ be an n-pointed nodal projective curve such that the points p_i are distinct and smooth. Let $\pi \colon \widetilde{C} \to C$ be the normalization of C, $\widetilde{p}_i \in \widetilde{C}$ be the unique preimage of p_i, and $\widetilde{q}_1, \ldots, \widetilde{q}_m \in \widetilde{C}$ be an ordering of the preimages of nodes.

(a) Show that (C, p_i) is stable if and only if every connected component of $(\widetilde{C}, \{\widetilde{p}_i\}, \{\widetilde{q}_i\})$ is stable.

(b) Show that the automorphism group scheme $\underline{\mathrm{Aut}}(C, p_i)$ is an algebraic group.

(c) Show that $\underline{\mathrm{Aut}}(C, p_i)$ is naturally a closed subgroup of $\underline{\mathrm{Aut}}(\widetilde{C}, \{\widetilde{p}_i\}, \{\widetilde{q}_i\})$ with the same connected component of the identity (i.e. $\underline{\mathrm{Aut}}(C, p_i)^0 = \underline{\mathrm{Aut}}(\widetilde{C}, \{\widetilde{p}_i\}, \{\widetilde{q}_i\})^0$).

(d) Provide an example where $\underline{\mathrm{Aut}}(C, p_i) \neq \underline{\mathrm{Aut}}(\widetilde{C}, \{\widetilde{p}_i\}, \{\widetilde{q}_i\})$.

5.3.2 Positivity of ω_C

Exercise 5.3.6. If $(C, p_1, \ldots, p_n)$ is an n-pointed prestable curve of genus g, and let $L := \omega_C(p_1 + \cdots + p_n)$.

(a) If (C, p_i) is semistable, show that $L^{\otimes k}$ is base point free for $k \geq 2$,

(b) If (C, p_i) is stable, show that $L^{\otimes k}$ is very ample for $k \geq 3$ and that $\mathrm{H}^1(C, (\omega_C(p_1 + \cdots + p_n))^{\otimes k}) = 0$ for $k \geq 2$.

Hint: For (b), show that the global sections of $L^{\otimes k}$ separate points and tangent vectors. In other words, show that the maps

$$H^0(C, L^{\otimes k}) \to \big(L^{\otimes k} \otimes \kappa(x)\big) \oplus \big(L^{\otimes k} \otimes \kappa(y)\big) \qquad H^0(C, L^{\otimes k}) \to L^{\otimes k} \otimes \mathcal{O}_{C,x}/\mathfrak{m}_x^2$$

are surjective. Establish this by using Serre Duality and a case analysis on whether x, y are smooth or nodal. See also [DM69, Thm. 2], [ACG11, Lem. 10.6.1], [SP, Tag 0E8X], and [Ols16, Prop. 13.2.17].

Exercise 5.3.7.

(a) If C is the nodal union $C_1 \cup C_2$ of genus i and $g - i$ curves along a single node $p = C_1 \cap C_2 \in C$, show that ω_C has a base point at p.

(b) If C the nodal union $C_1 \cup E \cup C_2$ of curves of genus i, 1, and $g - i - 1$ along nodes at $C_1 \cap C_2 = p_1$ and $C_1 \cap C_2 = p_2$, show that $\omega_C^{\otimes 2}$ is not ample.

5.3.3 Families of stable curves

Definition 5.3.8.

(1) A *family of n-pointed nodal curves* is a flat, proper, and finitely presented morphism $\mathcal{C} \to S$ of schemes with n sections $\sigma_1, \dots, \sigma_n \colon S \to \mathcal{C}$ such that every geometric fiber $\mathcal{C}_s$ is a (reduced) connected nodal curve.

(2) A *family of n-pointed stable curves* (resp. *semistable curves, prestable curves*) is a family $\mathcal{C} \to S$ of n-pointed nodal curves such that every geometric fiber $(\mathcal{C}_s, \sigma_1(s), \dots, \sigma_n(s))$ is stable (resp. semistable, prestable).

In a family of n-pointed nodal curves, marked points may lie at the nodes; this is the not the case for prestable (and thus also for semistable and stable) curves.

If $\mathcal{C} \to S$ is a family of prestable curves, then $\mathcal{C} \to S$ is locally a complete intersection morphism and thus there is a relative dualizing line bundle $\omega_{\mathcal{C}/S}$ that is compatible with base change $T \to S$ and in particular restricts to the dualizing line bundle $\omega_{\mathcal{C}_s}$ on every fiber of $\mathcal{C} \to S$; see [Har66c] or [Liu02, §6.4]. Note also that since the geometric fibers are stable curves, the image of each σ_i is a divisor contained in the smooth locus and we can form the line bundle $\omega_{\mathcal{C}/S}(\sigma_1 + \cdots \sigma_n)$.

We have the following generalization of Proposition 5.1.9 which is proven in the same way but using the very ampleness of third tensor power of $\omega_C(p_1 + \cdots + p_n)$ in Exercise 5.3.6.

Proposition 5.3.9 (Properties of Families of Stable Curves)**.** *Let $(\mathcal{C} \to S, \{\sigma_i\})$ be a family of n-pointed stable curves of genus g, and set $L := \omega_{\mathcal{C}/S}(\sum_i \sigma_i)$. If $k \geq 3$, then $L^{\otimes k}$ is relatively very ample and $\pi_* L^{\otimes k}$ is a vector bundle of rank $(2k-1)(g-1) + kn$.* □

In particular, stable n-pointed families are projective morphisms.

Proposition 5.3.10 (Openness of Stability)**.** *Let $(\mathcal{C} \to S, \{\sigma_i\})$ be a family of n-pointed nodal curves. The locus of points $s \in S$ such that $(\mathcal{C}_s, \{\sigma_i(s)\})$ is stable is open.*

Proof. The locus in S where $\sigma_1(s), \dots, \sigma_n(s)$ are distinct and smooth is open. We may thus assume that $(\mathcal{C} \to S, \{\sigma_i\})$ is a family of prestable n-pointed curves.

Argument 1: since $\underline{\mathrm{Aut}}(\mathcal{C}/S, \sigma_1, \dots, \sigma_n) \to S$ is a group scheme of finite presentation, upper semicontinuity implies that the locus of points $s \in S$ such that $\mathrm{Aut}(\mathcal{C}_s, \sigma_1(s), \dots, \sigma_n(s))$ is finite is open. By the equivalence Proposition 5.3.4(2), this open subset is identified with the stable locus.

Argument 2: the locus of points $s \in S$ such that $\omega_{\mathcal{C}/S}(\sum_i \sigma_i)|_{\mathcal{C}_s} \cong \omega_{\mathcal{C}_s}(\sum_i \sigma_i(s))$ is ample is open (Proposition E.2.1). By the equivalence Proposition 5.3.4(3), this open subset is identified with the stable locus. □

5.3.4 Deformation theory of stable curves

If C is smooth curve over a field $\Bbbk$, then every first order deformation is locally trivial (Proposition D.1.8) and the set $\mathrm{Def}(C)$ of isomorphism classes of first order deformations is naturally in bijection with $\mathrm{H}^1(C, T_C)$ (Proposition D.1.11). Moreover, automorphisms, deformations, and obstructions of higher order deformations are classified by $\mathrm{H}^i(C, T_C)$ for $i = 0, 1, 2$ (Proposition D.2.6).

Nodal singularities on the other hand have first order deformations that are not locally trivial, e.g. $\operatorname{Spec} \Bbbk[x, y, \epsilon]/(xy - \epsilon) \to \operatorname{Spec} \Bbbk[\epsilon]$ is non-locally trivial deformation of $C = \operatorname{Spec} \Bbbk[x, y]/(xy)$. In this section, we classify automorphisms, deformations and obstructions of nodal curves (Proposition 5.3.11), describe first order deformations of a nodal curve in terms of the pointed normalization and the singularities (Proposition 5.3.13), and then compute the dimensions of the groups classifying automorphisms, deformations, and obstructions of a stable curve (Proposition 5.3.14).

Proposition 5.3.11. *Let (C, p_i) be a prestable curve over a field $\Bbbk$. Let $A' \twoheadrightarrow A$ be a surjection of artinian local $\Bbbk$-algebras with residue field $\Bbbk$. Suppose that $J = \ker(A' \to A)$ satisfies $\mathfrak{m}_{A'} J = 0$. If $\mathcal{C} \to \operatorname{Spec} A$ is a family of nodal curves such that $C \cong \mathcal{C} \times_A \Bbbk$, then*

1. *The group of automorphisms of a deformation $\mathcal{C}' \to \operatorname{Spec} A'$ of $\mathcal{C} \to \operatorname{Spec} A$ over A' is bijective to $\mathrm{Ext}^0_{\mathcal{O}_C}(\Omega_C(\sum_i p_i), \mathcal{O}_C \otimes_\Bbbk J)$.*
2. *If there exists a deformation of $\mathcal{C} \to \operatorname{Spec} A$ over A', then the set of isomorphism classes of all such deformations is a torsor under $\mathrm{Ext}^1_{\mathcal{O}_C}(\Omega_C(\sum_i p_i), \mathcal{O}_C \otimes_\Bbbk J)$.*
3. *There is an element $\mathrm{ob}_\mathcal{C} \in \mathrm{Ext}^2_{\mathcal{O}_C}(\Omega_C(\sum_i p_i), \mathcal{O}_C \otimes_\Bbbk J)$ with the property that there exists a deformation of $\mathcal{C} \to \operatorname{Spec} A$ over A' if and only if $\mathrm{ob}_\mathcal{C} = 0$.*

Proof. Since nodal curves are generically smooth and local complete intersections, the unpointed case follows from Proposition D.2.11. We leave the generalization to n-pointed curves to the reader. □

Lemma 5.3.12. *Let (C, p_i) be a prestable curve over a field $\Bbbk$. Let $q_1, \dots, q_s \in C$ be the nodes of C. Let $(\widetilde{C}, p_i, q'_j, q''_j)$ be the pointed normalization where $\pi \colon \widetilde{C} \to C$ is the normalization and $\pi^{-1}(q_j) = \{q'_j, q''_j\}$. There is a convergent spectral sequence*

$$H^p(C, \mathscr{E}xt^q_{\mathcal{O}_C}(\Omega_C(\textstyle\sum_i p_i), \mathcal{O}_C)) \Rightarrow \mathrm{Ext}^{p+q}_{\mathcal{O}_C}(\Omega_C(\textstyle\sum_i p_i), \mathcal{O}_C).$$

such that the induced exact sequence of low-degree terms is identified with

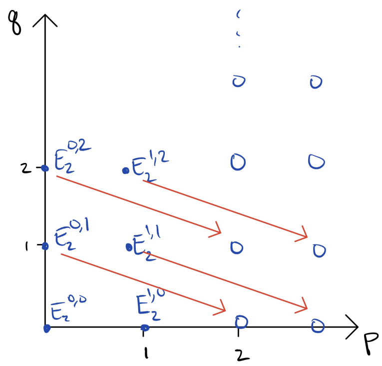
$$0 \to \mathrm{H}^1(C, \mathscr{H}om_{\mathcal{O}_C}(\Omega_C(\textstyle\sum_i p_i), \mathcal{O}_C)) \to \mathrm{Ext}^1_{\mathcal{O}_C}(\Omega_C(\textstyle\sum_i p_i), \mathcal{O}_C) \to \bigoplus_j \mathrm{Ext}^1_{\widehat{\mathcal{O}}_{C,q_j}}(\Omega_{\widehat{\mathcal{O}}_{C,q_j}}, \widehat{\mathcal{O}}_{C,q_j}) \to 0. \quad (5.3.1)$$

Moreover, $\mathrm{Ext}^1_{\widehat{\mathcal{O}}_{C,q_j}}(\Omega_{\widehat{\mathcal{O}}_{C,q_j}}, \widehat{\mathcal{O}}_{C,q_j}) = \Bbbk$ *for each* i *and* $\mathrm{Ext}^2_{\mathcal{O}_C}(\Omega_C(\sum_i p_i), \mathcal{O}_C) = 0$.

Proof. For simplicity, we handle only the case without marked points, i.e. $n = 0$. As $\mathrm{Hom}_{\mathcal{O}_C}(\Omega_C, -)$ is the composition $\Gamma \circ \mathscr{H}om_{\mathcal{O}_C}(\Omega_C, -)$ of left exact functors, there is a Grothendieck spectral sequence with E_2-page

$$E_2^{p,q} = \mathrm{H}^p(C, \mathscr{E}xt^q_{\mathcal{O}_C}(\Omega_C, \mathcal{O}_C))$$

which converges to $\mathrm{Ext}^{p+q}_{\mathcal{O}_C}(\Omega_C, \mathcal{O}_C)$ (c.f [Wei94, Thm. 5.8.3]). Since $\dim C = 1$, we have that $E_2^{p,q} = 0$ if $p \geq 2$. We can thus draw the E_2 page as:



The associated exact sequence of low-degree terms is

$$0 \to E_2^{1,0} \to \mathrm{Ext}^1_{\mathcal{O}_C}(\Omega_C, \mathcal{O}_C) \to E_2^{0,1} \to E_2^{2,0} = 0.$$

As Ω_C is locally free away from the nodes, $\mathscr{E}xt^1_{\mathcal{O}_C}(\Omega_C, \mathcal{O}_C)$ is a zero-dimensional sheaf supported only at the nodes of C. This shows that $E_2^{1,1} = \mathrm{H}^1(C, \mathscr{E}xt^1_{\mathcal{O}_C}(\Omega_C, \mathcal{O}_C)) = 0$ and

$$E_2^{0,1} = \mathrm{H}^0(C, \mathscr{E}xt^1_{\mathcal{O}_C}(\Omega_C, \mathcal{O}_C)) = \bigoplus_j \mathrm{Ext}^1_{\mathcal{O}_{C,q_j}}(\Omega_{C,q_j}, \mathcal{O}_{C,q_j}) = \bigoplus_j \mathrm{Ext}^1_{\widehat{\mathcal{O}}_{C,q_j}}(\Omega_{\widehat{\mathcal{O}}_{C,q_j}}, \widehat{\mathcal{O}}_{C,q_j})$$

where we've used that $\widehat{\Omega}_{C,q_j} \cong \Omega_{\widehat{\mathcal{O}}_{C,q_j}}$. This gives the exact sequence (5.3.1).

Similarly, $\mathscr{E}xt^2_{\mathcal{O}_C}(\Omega_C, \mathcal{O}_C)$ is a zero-dimensional sheaf supported only at the nodes and we have identifications

$$E_2^{0,2} = \mathrm{H}^0(C, \mathscr{E}xt^2_{\mathcal{O}_C}(\Omega_C, \mathcal{O}_C)) = \bigoplus_j \mathrm{Ext}^2_{\widehat{\mathcal{O}}_{C,q_j}}(\Omega_{\widehat{\mathcal{O}}_{C,q_j}}, \widehat{\mathcal{O}}_{C,q_j})$$

Write $\widehat{\mathcal{O}}_{C,q_j} = \Bbbk[\![x, y]\!]/(xy)$ and consider the locally free resolution

$$0 \to \widehat{\mathcal{O}}_{C,q_j} \xrightarrow{\binom{y}{x}} \widehat{\mathcal{O}}^{\oplus 2}_{C,q_j} \xrightarrow{(dx,dy)} \Omega_{\widehat{\mathcal{O}}_{C,q_j}} \to 0.$$

This allows us to compute that $\mathrm{Ext}^1_{\widehat{\mathcal{O}}_{C,q_j}}(\Omega_{\widehat{\mathcal{O}}_{C,q_j}}, \widehat{\mathcal{O}}_{C,q_j}) = \mathrm{coker}(\widehat{\mathcal{O}}_{C,q_j}^{\oplus 2} \xrightarrow{(x,y)} \widehat{\mathcal{O}}_{C,q_j}) = \Bbbk$ and $\mathrm{Ext}^2_{\widehat{\mathcal{O}}_{C,q_j}}(\Omega_{\widehat{\mathcal{O}}_{C,q_j}}, \widehat{\mathcal{O}}_{C,q_j}) = 0$. As $E_2^{0,2} = E_2^{1,1} = E_2^{2,0} = 0$, we have $\mathrm{Ext}^2_{\mathcal{O}_C}(\Omega_C, \mathcal{O}_C) = 0$. □

Proposition 5.3.13. *Let (C, p_i) be an n-pointed prestable curve. Let $q_1, \ldots, q_s \in C$ be the nodes of C. Let $(\widetilde{C}, p_i, q_j', q_j'')$ be the pointed normalization where $\pi \colon \widetilde{C} \to C$ is the normalization and $\pi^{-1}(q_j) = \{q_j', q_j''\}$. There is an exact sequence*

$$0 \to \mathrm{Def}^{\mathrm{lt}}(C, p_i) \to \mathrm{Def}(C, p_i) \to \bigoplus_j \mathrm{Def}(\widehat{\mathcal{O}}_{C,q_j}) \to 0 \tag{5.3.2}$$

and identifications

$$\begin{aligned} \mathrm{Def}^{\mathrm{lt}}(C, p_i) &\cong \mathrm{Def}(\widetilde{C}, p_i, q_j', q_j'') \cong \mathrm{H}^1(\widetilde{C}, T_{\widetilde{C}}(-\textstyle\sum_i p_i - \sum_j (q_j' + q_j''))) \\ \mathrm{Def}(C, p_i) &\cong \mathrm{Ext}^1_{\mathcal{O}_C}(\Omega_C(\textstyle\sum_i p_i), \mathcal{O}_C) \\ \mathrm{Def}(\widehat{\mathcal{O}}_{C,q_j}) &\cong \mathrm{Ext}^1_{\widehat{\mathcal{O}}_{C,q_j}}(\Omega^1_{\widehat{\mathcal{O}}_{C,q_j}}, \widehat{\mathcal{O}}_{C,q_j}) \cong \Bbbk. \end{aligned}$$

Under these identifications, (5.3.2) corresponds to short exact sequence (5.3.1).

Proof. For simplicity, we handle again the case without marked points. The identification $\mathrm{Def}^{\mathrm{lt}}(C) \cong \mathrm{H}^1(C, T_C)$ was given in Proposition D.1.11. If $C' \to \operatorname{Spec} \Bbbk[\epsilon]$ is a locally trivial first order deformation of C, each node $q_j \colon \operatorname{Spec} \Bbbk \to C$ extends to a section $\widetilde{q}_j \colon \operatorname{Spec} \Bbbk[\epsilon] \to C$ whose image is contained in the relative singular locus of C over $\Bbbk[\epsilon]$. The pointed normalization of C along the sections $\widetilde{q}_j$ is a first order deformation of the (possibly disconnected) pointed normalization $(\widetilde{C}, q_j', q_j'')$. This gives a map $\mathrm{Def}^{\mathrm{lt}}(C) \to \mathrm{Def}(\widetilde{C}, q_j', q_j'')$. The inverse is provided by gluing the sections of a first order deformation $(\widetilde{C}', \widetilde{\sigma}_j', \widetilde{\sigma}_j'')$ of $(\widetilde{C}, q_j', q_j'')$ along nodes; more precisely, the deformation C' is obtained as the pushout (see Theorem A.8.1)

$$\begin{array}{ccc} \coprod_j (\operatorname{Spec} \Bbbk[\epsilon] \amalg \operatorname{Spec} \Bbbk[\epsilon]) & \xrightarrow{q_j' \amalg q_j''} & \widetilde{C}' \\ \downarrow & & \downarrow \\ \coprod_j \operatorname{Spec} \Bbbk[\epsilon] & \longrightarrow & C'. \end{array}$$

For the second bijection, if $C' \to \operatorname{Spec} \Bbbk[\epsilon]$ is a first order deformation, then the ideal sheaf I defining $C \hookrightarrow C'$ is the pullback of the ideal $(\epsilon) \subset \Bbbk[\epsilon]$. Since $(\epsilon) \cong \Bbbk$ as $\Bbbk[\epsilon]$-modules, we see that $I/I^2 = I \cong \mathcal{O}_C$. The right exact sequence

$$I/I^2 \to \Omega^1_{C'} \to \Omega^1_C \to 0 \tag{5.3.3}$$

is left exact at every smooth point of C. Since C is generically smooth, the map $\mathcal{O}_C \cong I/I^2 \to \Omega^1_{C'}$ is generically injective, and since $\mathcal{O}_C$ is a line bundle, the map is in fact injective. The sequence (5.3.3) is therefore exact and defines an extension class in $\mathrm{Ext}^1_{\mathcal{O}_C}(\Omega_C, \mathcal{O}_C)$ (see [Har77, Exer. III.6.1]). The reader is left to verify that this defines a bijection. A similar argument gives the bijection $\mathrm{Def}(\widehat{\mathcal{O}}_{C,q_j}) \cong \mathrm{Ext}^1_{\widehat{\mathcal{O}}_{C,q_j}}(\Omega^1_{\widehat{\mathcal{O}}_{C,q_j}}, \widehat{\mathcal{O}}_{C,q_j})$, and Lemma 5.3.12 gives the identification with $\Bbbk$.

See also [DM69, Prop. 1.5] and [ACG11, §11.3]. □

Recall that automorphisms, deformations, and obstructions of a nodal curve (C, p_i) are classified by $\mathrm{Ext}^i_{\mathcal{O}_C}(\Omega_C(\sum_i p_i), \mathcal{O}_C)$ for $i = 0, 1, 2$ (Proposition 5.3.11).

Proposition 5.3.14. *Let $(C, p_1, \ldots, p_n)$ be an n-pointed stable curve of genus g over $\Bbbk$. Then*

$$\dim_{\Bbbk} \mathrm{Ext}^i_{\mathcal{O}_C}(\Omega_C(\textstyle\sum_i p_i), \mathcal{O}_C) = \begin{cases} 0 & \text{if } i = 0, 2 \\ 3g - 3 + n & \text{if } i = 1 \end{cases}$$

Proof. We may assume $\Bbbk = \overline{\Bbbk}$ and for simplicity we handle the case that there are no marked points. Let $\pi \colon \widetilde{C} \to C$ be the normalization, $\Sigma \subset C$ be the set of nodes of C, and $\widetilde{\Sigma} = \pi^{-1}(\Sigma)$. The vanishing of Ext^2 was established in Lemma 5.3.12. For Ext^0, we will use the identification $\mathrm{Hom}_{\mathcal{O}_{\widetilde{C}}}(\Omega_{\widetilde{C}}(\widetilde{\Sigma}), \mathcal{O}_{\widetilde{C}}) \cong \mathrm{Hom}_{\mathcal{O}_C}(\Omega_C, \mathcal{O}_C)$ of Exercise 5.2.15. Since the pointed normalization $(\widetilde{C}, \widetilde{\Sigma})$ is smooth and each connected component is stable (Exercise 5.3.5), the degree of the restriction of $T_{\widetilde{C}}(-\widetilde{\Sigma})$ to each connected component of $\widetilde{C}$ is strictly negative. Thus, $\mathrm{Hom}_{\mathcal{O}_{\widetilde{C}}}(\Omega_{\widetilde{C}}(\widetilde{\Sigma}), \mathcal{O}_{\widetilde{C}}) = H^0(\widetilde{C}, T_{\widetilde{C}}(-\widetilde{\Sigma})) = 0$.

To compute Ext^1, Proposition 5.3.13 implies that there is an exact sequence

$$0 \to \mathrm{H}^1(\widetilde{C}, T_C(-\widetilde{\Sigma}) \to \mathrm{Ext}^1_{\mathcal{O}_C}(\Omega_C, \mathcal{O}_C) \to \bigoplus_j \mathrm{Ext}^1_{\widehat{\mathcal{O}}_{C,q_j}}(\Omega_{\widehat{\mathcal{O}}_{C,q_j}}, \widehat{\mathcal{O}}_{C,q_j}) \to 0$$

and the identification $\mathrm{Ext}^1_{\widehat{\mathcal{O}}_{C,q_j}}(\Omega_{\widehat{\mathcal{O}}_{C,q_j}}, \widehat{\mathcal{O}}_{C,q_j}) = \Bbbk$. We write $\widetilde{C} = \coprod_{i=1}^{\nu} \widetilde{C}_i$ as a union of its connected components and define $\widetilde{\Sigma}_i = \widetilde{C}_i \cap \widetilde{\Sigma}$. Using that $\Omega_{\widetilde{C}_i}$ is a line bundle, we compute using Serre Duality and Riemann–Roch that

$$\mathrm{h}^1(\widetilde{C}_i, T_{\widetilde{C}_i}(-\widetilde{\Sigma}_i)) = \mathrm{h}^0(\widetilde{C}_i, \Omega^{\otimes 2}_{\widetilde{C}_i}(\widetilde{\Sigma}_i)) = 3g(\widetilde{C}_i) - 3 + |\widetilde{\Sigma}_i|.$$

Thus

$$\begin{aligned} \dim_{\Bbbk} \mathrm{Ext}^1_{\mathcal{O}_C}(\Omega_C, \mathcal{O}_C) &= \mathrm{h}^1(\widetilde{C}, T_{\widetilde{C}}(-\widetilde{\Sigma})) + |\Sigma| \\ &= \sum_{i=1}^{\nu} \big(3g(\widetilde{C}_i) - 3 + |\widetilde{\Sigma}_i|\big) + |\Sigma| \\ &= 3\big(\sum_{i=1}^{\nu} g(\widetilde{C}_i) - \nu + |\Sigma|\big) \\ &= 3g - 3 \end{aligned},$$

where we've used the Genus Formula (5.2.9) $g = \sum_{i=1}^{\nu} g(\widetilde{C}_i) - \nu + |\Sigma| + 1$. ☐

Remark 5.3.15 (Consequences of deformation theory)**.** In Theorem 5.4.8, we will use deformation theory to argue that $\overline{\mathcal{M}}_{g,n}$ is a smooth Deligne–Mumford stack of dimension $3g - 3 + n$. Here's the central idea:

- Ext^0: We've already seen that a stable curve (C, p_i) has finitely many automorphism (Proposition 5.3.4). The vanishing of Ext^0 implies that an n-pointed stable curve (C, p_i) has no infinitesimal automorphisms, i.e. that the Lie algebra of the automorphism group scheme $\underline{\mathrm{Aut}}(C, p_i)$ is trivial. Since $\underline{\mathrm{Aut}}(C, p_i)$ is of finite type, it must be finite and discrete. Once we know that the algebraicity of the stack $\overline{\mathcal{M}}_{g,n}$, we use the Characterization of Deligne–Mumford Stacks (3.6.4) to conclude that $\overline{\mathcal{M}}_{g,n}$ is Deligne–Mumford.

- Ext^1: Since Ext^1 parametrizes isomorphism classes of deformations of (C, p_i), it is identified with the Zariski tangent space of $\overline{\mathcal{M}}_{g,n}$ at the point corresponding to (C, p_i). The computation of Ext^1 therefore implies that $\overline{\mathcal{M}}_{g,n}$ has relative dimension $3g - 3 + n$ over $\mathrm{Spec}\, \mathbb{Z}$.
- Ext^2: The vanishing of Ext^2 implies that there are no obstructions to deforming C. The Infinitesimal Lifting Criterion (Theorem 3.7.1) implies that $\overline{\mathcal{M}}_{g,n}$ is smooth over $\mathrm{Spec}\, \mathbb{Z}$.

5.3.5 Stabilization of rational tails and bridges

Definition 5.3.16 (Rational tails and bridges)**.** Let $(C, p_1, \ldots, p_n)$ be an n-pointed prestable curve over an algebraically closed field $\Bbbk$. We say that a smooth genus 0 subcurve $E \subset C$ is

- a *rational tail* if $E \cap E^c = 1$ (where $E^c = \overline{C \setminus E}$), and E contains no marked points;
- a *rational bridge* if either $E \cap E^c = 2$ and E contains no marked points, or $E \cap E^c = 1$ and E contains one marked point.

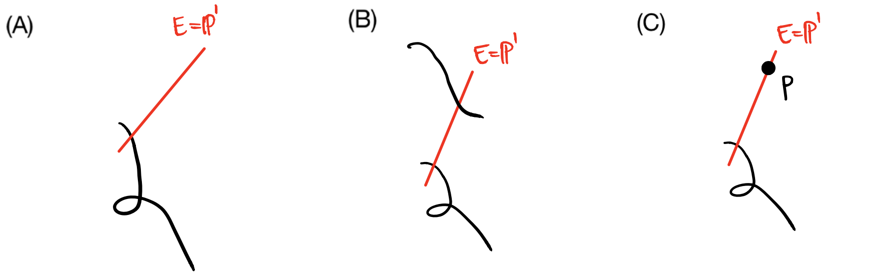


Figure 5.4: (A) features a rational tail while (B) and (C) feature rational bridges.

From the definition of stability (Definition 5.3.1), we see that if $(C, p_1, \ldots, p_n)$ is not stable and $(g, n) \neq (1, 0)$, then C necessarily contains a rational tail or bridge. Note that C can also contain a chain of rational tails or bridges of arbitrary length.

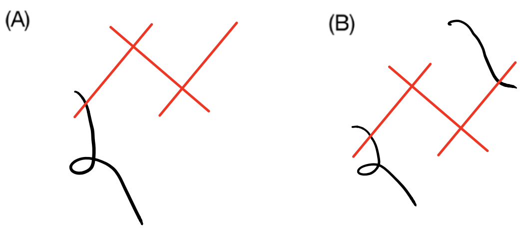


Figure 5.5: Examples of chains of rational tails and bridges

Suppose that $\mathcal{C} \to \Delta = \operatorname{Spec} R$ is a family of nodal curves over a DVR R with algebraically closed residue field $\Bbbk$ such that the generic fiber $\mathcal{C}^*$ is smooth. If $E \cong \mathbb{P}^1 \subset \mathcal{C}'_0$ is a smooth rational subcurve in the central fiber, then $E^2 = -E \cdot E^c$; indeed this follows from $0 = E \cdot \mathcal{C}_0 = E \cdot E + E \cdot E^c$. Thus if E is rational tail (resp. rational bridge without a marked point), then $E^2 = -1$ (resp. $E^2 = -2$).

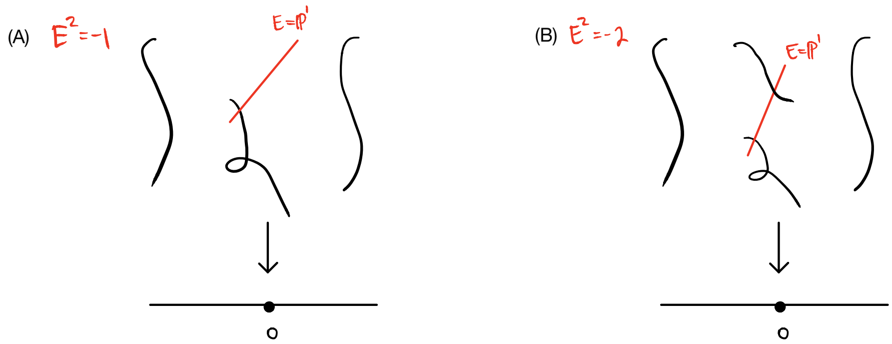


Figure 5.6: In (A) (resp. (B)), the exceptional component E meets the rest of the curve at one point (resp. two points) and $E^2 = -1$ (resp. $E^2 = -2$).

Definition 5.3.17. The *stabilization* of an n-pointed prestable curve $(C, p_1, \ldots, p_n)$ over afield $\Bbbk$ is the curve $(C^{\mathrm{st}}, p'_1, \ldots, p'_n)$ where C^{st} is the curve obtained by contracting all rational bridges and tails E_i and p'_i are the images of p_i under the contraction morphism $C \to C^{\mathrm{st}}$.

Remark 5.3.18. The contraction of a rational tail $E \subset C$ or a rational bridge with a marked point is the curve $E^c = \overline{C \setminus E}$. The contraction of a rational bridge $E \subset C$ without a marked point can be constructed as follows: since $E \cdot E^c = \dim_{\Bbbk} \Gamma(E \cap E^c, \mathcal{O}_{E \cap E^c} = 2$, the scheme-theoretic intersection $E \cdot E^c$ is isomorphic to either $\operatorname{Spec} \Bbbk \times \Bbbk$ or $\operatorname{Spec} \Bbbk'$ for a degree 2 separable field extension. The contraction C' is defined as the pushout $E^c \amalg_{E \cap E^c} \operatorname{Spec} \Bbbk$ (see §A.8). A local calculation shows that the contraction C' has a node at the image of $\operatorname{Spec} \Bbbk \to C'$.

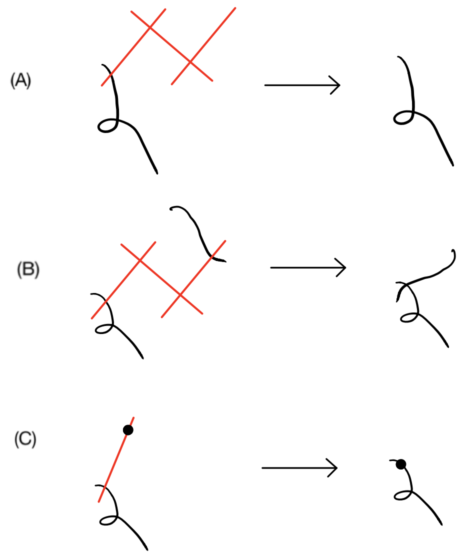


Figure 5.7: Examples of stabilizations

Exercise 5.3.19. Let $(C, p_1, \ldots, p_n)$ be an n-pointed prestable curve over a field $\Bbbk$.

(a) Show that the stabilization morphism $\pi \colon C \to C^{\mathrm{st}}$ is the unique morphism such that $(C^{\mathrm{st}}, \pi(p_1), \ldots, \pi(p_n))$ is stable, $\pi_* \mathcal{O}_C = \mathcal{O}_{C^{\mathrm{st}}}$, and $\mathrm{R}^1 \pi_* \mathcal{O}_C = 0$. See also [SP, Tag 0E7Q].

(b) If (C, p_i) is semistable and $L := \omega_C(\sum_i p_i)$, show that $C^{\mathrm{st}} \cong \operatorname{Proj} \bigoplus_{d \geq 0} \mathrm{H}^0(C, L^{\otimes 4d}))$ and that $\omega_C(\sum_i p_i) \cong \pi^* \omega_{C^{\mathrm{st}}}(\sum_i p_i')$.

Hint: Use Exercise 5.3.6 to show that $L^{\otimes 4}$ is base point free. Show that the multiplication map

$$\mathrm{H}^0(C, L^{\otimes 2}) \otimes \mathrm{H}^0(C, L^{\otimes d}) \to \mathrm{H}^0(C, L^{\otimes (d+2)})$$

is surjective for $d \geq 4$ to conclude that $\bigoplus_{d \geq 0} \mathrm{H}^0(C, L^{\otimes 4d})$ is finitely generated. See also [ACG11, Cor. 10.6.4].

The construction of the stabilization extends to families of nodal curves.

Proposition 5.3.20 (Stabilization of a Prestable Family)**.** *Let $(\mathcal{C} \to S, \sigma_1, \ldots, \sigma_n)$ be a family of n-pointed prestable curves of genus g. Assume $2g - 2 + n > 0$. Then there exists a unique morphism $\pi \colon \mathcal{C} \to \mathcal{C}^{\mathrm{st}}$ over S such that*

(1) $(\mathcal{C}^{\mathrm{st}} \to S, \{\sigma_i'\})$ is an n-pointed family of stable curves of genus g where $\sigma_i' = \pi \circ \sigma_i$;

(2) for each $s \in S$, $(\mathcal{C}_s, \{\sigma_i(s)\}) \to (\mathcal{C}_s^{\mathrm{st}}, \{\sigma_i'(s)\})$ contracts all rational bridges and tails; and

(3) $\mathcal{O}_{\mathcal{C}^{\mathrm{st}}} = \pi_* \mathcal{O}_{\mathcal{C}}$ and $R^1 \pi_* \mathcal{O}_{\mathcal{C}} = 0$ and this remains true after base change by a morphism $S' \to S$ of schemes;

(4) If $\mathcal{C} \to S$ is a family of semistable curves, then $\omega_{\mathcal{C}/S}(\sum_i \sigma_i)$ is the pullback of the relatively ample line bundle $\omega_{\mathcal{C}^{\mathrm{st}}/S}(\sum_i \sigma_i')$.

Proof. TO ADD. See [SP, Tag 0E7B] or [ACG11, Prop. 10.6.7]. □

5.4 The stack of all curves

5.4.1 Families of arbitrary curves

In this subsection, we redefine a *curve over a field* k to mean a scheme C of finite type over k of dimension 1 (rather than pure dimension 1). The genus of C is defined as $g(C) = 1 - \chi(C, \mathcal{O}_C)$.

Remark 5.4.1. The reason we allow for non-pure dimensional and non-connected curves is that they may arise as deformations of connected pure one-dimensional curves; without this relaxation, the stack of all curves would fail to be algebraic. For instance, consider a rational normal curve $\mathbb{P}^1 \hookrightarrow \mathbb{P}^3$ embedded via $[x, y] \mapsto [x^3, x^2y, xy^2, ty^3]$ for every $t \neq 0$. As $t \to 0$, these curves degenerate in a flat family to a non-reduced curve C_0 which is supported along a plane nodal cubic and has an embedded point at the node; see [Har77, Ex. 9.8.4]. On the other hand, the curve C_0 deforms to the disjoint union of a plane nodal cubic and a point in $\mathbb{P}^3$.

A *family of curves* over a scheme S is a flat, proper and finitely presented morphism $\mathcal{C} \to S$ of algebraic spaces such that every fiber is a curve.

A *family of n-pointed curves* is a family of curves $\mathcal{C} \to S$ together with n sections $\sigma_1, \ldots, \sigma_n : S \to \mathcal{C}$ (with no condition on whether they are distinct or land in the relative smooth locus of $\mathcal{C}$ over S).

Remark 5.4.2. While every pure one-dimensional separated algebraic space over a field is in fact a scheme, in the relative setting the total family $\mathcal{C}$ may not be a scheme. There are examples of a family of prestable genus 0 curves [Ful10, Ex. 2.3] and a family of smooth genus 1 curves [Ray70, XIII 3.2] where the total family is not a scheme. Therefore, if we wish define a *stack* of all curves, then in order to satisfy the decent condition, we better allow for the case that the total family is not a scheme. In the stable case however there is no difference: if $\mathcal{C} \to S$ is a family of curves (with $\mathcal{C}$ an algebraic space) such that every geometric fiber is stable, then $\omega_{\mathcal{C}/S}$ is relatively ample (Proposition 5.3.9) and $\mathcal{C} \to S$ is projective; in particular, $\mathcal{C}$ is a scheme.

Proposition 5.4.3. *If $\mathcal{C} \to S$ is a family of curves over a scheme S, there exists an étale cover $S' \to S$ such that $\mathcal{C}_{S'} \to S'$ is projective.*

Vague sketch. Approach 1: Local to global

For a point $s \in S$, define $S_n = \operatorname{Spec} \mathcal{O}_{S,s}/\mathfrak{m}_s^{n+1}$ and $\widehat{S} = \operatorname{Spec} \widehat{\mathcal{O}}_{S,s}$. Consider the cartesian diagram

$$\begin{tikzcd} \mathcal{C}_s = \mathcal{C}_0 \arrow[r, hook] \arrow[d] & \mathcal{C}_1 \arrow[r, hook] \arrow[d] & \cdots & \widehat{\mathcal{C}} \arrow[r] \arrow[d] & \mathcal{C} \arrow[d] \\ \operatorname{Spec} \kappa(s) = S_0 \arrow[r, hook] & S_1 \arrow[r, hook] & \cdots & \widehat{S} \arrow[r] & S \end{tikzcd}$$

Case 1: $\mathcal{C}_s \to \operatorname{Spec} \kappa(s)$. Since separated one-dimensional algebraic spaces are schemes and that proper one-dimensional schemes are projective, there exists an ample line bundle L_0 on $\mathcal{C}_0$.

Case 2: $\mathcal{C}_n \to S_n$. The obstruction to deforming a line bundle L_n on $\mathcal{C}_n$ to L_{n+1} on $\mathcal{C}_{n+1}$ lives in $H^2(\mathcal{C}_0, \mathcal{O}_{\mathcal{C}_0})$ and thus vanishes as $\dim \mathcal{C}_0 = 1$. Thus there exists a compatible sequence of line bundles L_n on $\mathcal{C}_n$. Since ampleness is an open condition in families and L_0 is ample, L_n is also ample.

Case 3: $\widehat{\mathcal{C}} \to \widehat{S}$ with $\widehat{S}$ noetherian. Use Grothendieck's Existence Theorem: $\mathrm{Coh}(\widehat{cC}) \to \varprojlim \mathrm{Coh}(\mathcal{C}_n)$ is an equivalence of categories. The classical case is when $\widehat{\mathcal{C}} \to \widehat{S}$ is a proper morphism of schemes. Chow's Lemma for Algebraic Spaces implies that there exists a projective birational morphism $\mathcal{C}' \to \widehat{\mathcal{C}}$ of algebraic spaces such that $\mathcal{C}' \to \widehat{S}$ is projective. This allows one to reduce Grothendieck's Existence Theorem for $\widehat{\mathcal{C}} \to \widehat{S}$ to $\mathcal{C}' \to \widehat{S}$ using devissage similar to how the proper case of schemes is reduced to the projective case.

As a result, we can extend the sequence of line bundle L_n to a line bundle $\widehat{L}$ on $\widehat{\mathcal{C}}$ which is ample (using again that ampleness is an open condition in families).

Case 4: S is of finite type over $\mathbb{Z}$. For every closed point $s \in S$, apply Artin Approximation to the functor

$$\mathrm{Sch}/S \to \mathrm{Sets}, \qquad (T \to S) \mapsto \mathrm{Pic}(\mathcal{C}_T)$$

to obtain an étale neighborhood $(S', s') \to (S, s)$ of s and a line bundle L' on $\mathcal{C}_{S'}$ extending L_0. By openness of ampleness, we can replace S' with an open neighborhood of s' such that L' is relatively ample over S'.

Case 5: S is an arbitrary scheme. Apply Noetherian Approximation.

Approach 2: Explicitly extend an ample line bundle

The idea here is to use geometric methods to extend a line bundle L_s on $\mathcal{C}_s$ to a line bundle on $\mathcal{C}$. If we assume in addition that every fiber of $\mathcal{C} \to S$ is generically reduced (and thus also generically smooth), then we may follow the argument of [Ols16, Cor. 13.2.5]. Choose smooth points $p_1, \ldots, p_n \in \mathcal{C}_s$ such that every irreducible one-dimensional component of $\mathcal{C}_s$ contains at least one of the p_i's. Our hypothesis imply that the relative smooth locus $\mathcal{C}^0$ of $\mathcal{C} \to S$ surjects onto S. As smooth morphisms étale locally have sections, there is an étale neighborhood $S' \to S$ of s and sections $\sigma_i \colon S' \to \mathcal{C}^0$ extending p_i. The line bundle $L' := \mathcal{O}_{\mathcal{C}_{S'}}(\sigma_1 + \cdots + \sigma_n)$ extends the ample line bundle $L_s := \mathcal{O}_{\mathcal{C}_s}(p_1 + \cdots + p_n)$. By openness of ampleness in families, L' is relatively ample over S' in an open neighborhood of s'.

(An alternative argument that works without any restrictions is presented in [Hal13, Lem. 1.2] (based on ideas in [SGA4$\frac{1}{2}$, IV.4.1]) where one first uses Noetherian approximation and étale localization to reduce to $S = \operatorname{Spec} R$ where R is an excellent strictly henselian local ring. One can then reduce to the case where $\mathcal{C}$ is a scheme by appealing to the fact that there exists a finite surjection $\mathcal{C}' \to \mathcal{C}$ from a scheme and the fact that $\mathcal{C}$ satisfies the Chevalley-Kleiman property (i.e. every finite set of points is contained in an open affine) if and only if $\mathcal{C}'$ does. Using deformation theory as above, one can further reduce to the case where $\mathcal{C}$ is reduced. Finally, one attempts to explicitly extend an ample line bundle on $\mathcal{C}_s$ by extending a function $f \in \Gamma(U, \mathcal{O}_{\mathcal{C}_s})$ to a function defined on an open neighborhood of $s \in \mathcal{C}$ so that it defines an effective Cartier divisor.) $\square$

Remark 5.4.4. Raynaud gives an example of a family of smooth $g = 1$ curves over an affine curve which is Zariski-locally projective but not projective [Ray70, XIII 3.1]. The examples in Remark 5.4.2 are not even Zariski-locally projective.

5.4.2 Algebraicity of the stack of all curves

Definition 5.4.5. Let $\mathcal{M}_{g,n}^{\mathrm{all}}$ denote the category over $\mathrm{Sch}_{\text{ét}}$ whose objects over a scheme S consists of families of curves $\mathcal{C} \to S$ and n sections $\sigma_1, \dots, \sigma_n \colon S \to \mathcal{C}$. A morphism $(\mathcal{C}' \to S', \sigma_1', \dots, \sigma_n') \to (\mathcal{C} \to S, \sigma_1, \dots, \sigma_n)$ is the data of a cartesian diagram

$$\begin{tikzcd} \mathcal{C}' \arrow[r, "g"] \arrow[d] & \mathcal{C} \arrow[d] \\ S' \arrow[r, "f"] \arrow[u, bend left, "\sigma_i'"] & S \arrow[u, bend left, "\sigma_i"] \end{tikzcd}$$

such that $g \circ \sigma_i' = \sigma_i \circ f$.

As a stepping stone to the algebraicity of $\mathcal{M}_{g,n}^{\mathrm{all}}$, we first show that the diagonal is representable.

Lemma 5.4.6. *The diagonal $\mathcal{M}_{g,n}^{\mathrm{all}} \to \mathcal{M}_{g,n}^{\mathrm{all}} \times \mathcal{M}_{g,n}^{\mathrm{all}}$ is representable.*

Proof. For simplicity, we handle the case when $n = 0$. Let S be a scheme and $S \to \mathcal{M}_g^{\mathrm{all}} \times \mathcal{M}_g^{\mathrm{all}}$ be a morphism corresponding to families of curves $\mathcal{C}_1 \to S$ and $\mathcal{C}_2 \to S$. Considering the cartesian diagram

$$\begin{tikzcd} \underline{\mathrm{Isom}}_S(\mathcal{C}_1, \mathcal{C}_2) \arrow[r] \arrow[d] & S \arrow[d] \\ \mathcal{M}_g^{\mathrm{all}} \arrow[r] & \mathcal{M}_g^{\mathrm{all}} \times \mathcal{M}_g^{\mathrm{all}}, \end{tikzcd}$$

we need to show that $\underline{\mathrm{Isom}}_S(\mathcal{C}_1, \mathcal{C}_2)$ is an algebraic space. By Proposition 5.4.3, there exists an étale cover $S' \to S$ such that $\mathcal{C}_{S'} \to S'$ is projective. Since $\underline{\mathrm{Isom}}_S(\mathcal{C}_1, \mathcal{C}_2) \times_S S' = \underline{\mathrm{Isom}}_{S'}(\mathcal{C}_{1,S'}, \mathcal{C}_{2,S'})$, the morphism $\underline{\mathrm{Isom}}_{S'}(\mathcal{C}_{1,S'}, \mathcal{C}_{2,S'}) \to \underline{\mathrm{Isom}}_S(\mathcal{C}_1, \mathcal{C}_2)$ is representable, surjective and étale. We may thus assume that $\mathcal{C}_1$ and $\mathcal{C}_2$ are projective over S.

We will use the following fact from scheme theory: if $X \to Y$ is a morphism of schemes each proper over S, there exists an open subscheme $S^0 \subset S$ such that for every map $T \to S$ of schemes $X_T \xrightarrow{\sim} Y_T$ is an isomorphism if and only if $T \to S$ factors through $S_0 \subset S$.

Consider the inclusion of functors:

$$\underline{\mathrm{Isom}}_S(\mathcal{C}_1, \mathcal{C}_2) \subset \underline{\mathrm{Mor}}_S(\mathcal{C}_1, \mathcal{C}_2) \subset \mathrm{Hilb}_S(\mathcal{C}_1 \times_S \mathcal{C}_2)$$

where the second inclusion assigns to a morphism $\mathcal{C}_1 \xrightarrow{\alpha} \mathcal{C}_2$ the graph $\mathcal{C}_1 \xhookrightarrow{\Gamma_\alpha} \mathcal{C}_1 \times_S \mathcal{C}_2$ (and is similarly defined on T-valued points). The first inclusion is a representable open immersion by the above fact. Analyzing the second inclusion, we see that a subscheme $[\mathcal{Z} \subset \mathcal{C}_1 \times_S \mathcal{C}_2] \in \mathrm{Hilb}_S(\mathcal{C}_1 \times_S \mathcal{C}_2)(S)$ is in the image of element of $\underline{\mathrm{Mor}}(\mathcal{C}_1, \mathcal{C}_2)(S)$ if and only if the composition $Z \hookrightarrow \mathcal{C}_1 \times_S \mathcal{C}_2 \xrightarrow{p_1} \mathcal{C}_1$ is an isomorphism (and similarly for T-valued points). Therefore, the above fact also establishes that the second inclusion is a representable open immersion. $\square$

Theorem 5.4.7. *$\mathcal{M}_{g,n}^{\mathrm{all}}$ is an algebraic stack locally of finite type over* $\mathrm{Spec}\,\mathbb{Z}$.

Sketch.

- Suffices to show the $n=0$ case: $\mathcal{M}_{g,1}^{\mathrm{all}}$ is the universal family over $\mathcal{M}_g^{\mathrm{all}}$ and more generally $\mathcal{M}_{g,n+1}^{\mathrm{all}}$ is the universal family over $\mathcal{M}_{g,n}^{\mathrm{all}}$. (We will see that the same holds for $\overline{\mathcal{M}}_g$ but this is a more remarkable fact since an n-pointed stable curve can become unstable if a marked point is forgotten.)

- $\mathcal{M}_g^{\mathrm{all}}$ is a stack over $\mathrm{Sch}_{\text{ét}}$: Suppose $S' \to S$ is an étale cover of schemes, $\mathcal{C}' \to S'$ is a family of curves, and $\alpha \colon p_1^*\mathcal{C}' \to p_2^*\mathcal{C}'$ is an isomorphism over $S' \times_S S'$ satisfying the cocycle condition. The quotient of the étale equivalence relation

$$R' := p_1^*\mathcal{C}' \underset{p_2 \circ \alpha}{\overset{p_1}{\rightrightarrows}} \mathcal{C}'$$

is an algebraic space $\mathcal{C} := \mathcal{C}'/R$ and $\mathcal{C} \to S$ is a family of curves such that $\mathcal{C}_{S'} \cong \mathcal{C}'$.

- It to suffices show that for all projective curves C_0 over a field $\Bbbk$, there exists a representable, smooth morphism $U \to \mathcal{M}_g^{\mathrm{all}}$ from a scheme with $[C_0]$ in the image. Choose an embedding $C_0 \hookrightarrow \mathbb{P}^N$ such that $\mathrm{h}^1(C_0, \mathcal{O}_{C_0}(1)) = 0$, and let $P(t)$ be its Hilbert polynomial.

- Let $H := \mathrm{Hilb}^P(\mathbb{P}^N_{\mathbb{Z}}/\mathbb{Z})$ be the Hilbert scheme, which is projective over $\mathbb{Z}$ by Theorem 1.1.2. Considering the universal family

$$\begin{tikzcd} \mathcal{C} \arrow[r, hook] \arrow[d] & \mathbb{P}^N_H \arrow[dl] \\ H, & \end{tikzcd}$$

there is a point $h_0 \in H(\Bbbk)$ such that $\mathcal{C}_{h_0} = C_0$ as closed subschemes of $\mathbb{P}^N_{\Bbbk}$. Cohomology and Base Change implies that there exists an open neighborhood $H' \subset H$ of h_0 such that for all $s \in H'$, $\mathrm{h}^1(\mathcal{C}_s, \mathcal{O}_{\mathcal{C}_s}(1)) = 0$.

- Consider the morphism

$$H' \to \mathcal{M}_g^{\mathrm{all}}, \qquad [C \hookrightarrow \mathbb{P}^n] \mapsto [C],$$

which is representable by Lemma 5.4.6 and the fact that representability of the diagonal implies that every morphism from a scheme is representable (see the argument of Corollary 3.2.2).

- Claim: $H' \to \mathcal{M}_g^{\mathrm{all}}$ is smooth.

  We will use the Infinitesimal Lifting Criterion (Theorem 3.7.1)—even though we don't yet know $\mathcal{M}_g^{\mathrm{all}}$ is algebraic, we may still use this criterion as we know that $H' \to \mathcal{M}_g^{\mathrm{all}}$ is representable so it suffices to show for all maps $S \to \mathcal{M}_g^{\mathrm{all}}$ from a scheme, the induced morphism $H'_S \to S$ is a smooth morphism of algebraic spaces. We need to check that for all surjections $A \to A_0$ of artinian local rings with residue field $\Bbbk$ such that $\Bbbk = \ker(A \to A_0)$ and for all diagrams

$$\begin{tikzcd}
\operatorname{Spec} \Bbbk \arrow[dr] \arrow[drr, bend left, "{[C\subset \mathbb{P}^N_{\Bbbk}]}"] & & \\
& \operatorname{Spec} A_0 \arrow[r, "{[\mathcal{C}_0\subset \mathbb{P}^N_{A_0}]}"] \arrow[d, hook] & H' \arrow[d, "f"] \\
& \operatorname{Spec} A \arrow[r, "{[\mathcal{C}]}"] \arrow[ur, dashed, "{[\mathcal{C}\subset \mathbb{P}^N_A]}"] & \mathcal{M}_g^{\mathrm{all}}
\end{tikzcd} \tag{5.4.1}$$

of solid arrows, there exists a dotted arrow. The existence of a dotted arrow in the above diagram is equivalent to the existence of a dotted arrow in the below diagram

$$\begin{tikzcd}
& \mathbb{P}^N_{\Bbbk} & & \mathbb{P}^N_{A_0} & & \mathbb{P}^N_A \arrow[dl] \\
C \arrow[ur, hook] \arrow[rr] \arrow[d] & & \mathcal{C}_0 \arrow[ur, hook] \arrow[rr] \arrow[d] & & \mathcal{C} \arrow[ur, hook, dashed] \arrow[d] & \\
\operatorname{Spec} \Bbbk \arrow[rr, hook] & & \operatorname{Spec} A_0 \arrow[rr, hook] & & \operatorname{Spec} A &
\end{tikzcd}$$

of solid arrows: a lifting of the diagram (3.7.3) corresponds to a family $\mathcal{C} \to \operatorname{Spec} A$ extending $\mathcal{C}_0 \to \operatorname{Spec} A_0$. By Proposition D.2.6, there is a cohomology class $\mathrm{ob} \in H^2(C, T_C)$ such that $\mathrm{ob} = 0$ if and only if there exists a lifting. Since C is a curve, $H^2(C, T_C) = 0$.

- Use deformation theory to extend $\mathcal{C}_0 \hookrightarrow \mathbb{P}^N_{A_0}$ to $\mathcal{C} \hookrightarrow \mathbb{P}^N_A$. We will use the simplifying assumption that C is a complete local intersection; the general case is handled by more advanced deformation theory (see [Hal13, Prop. 4.2]). This implies that the ideal sheaf $\mathcal{I}$ defining $C \hookrightarrow \mathbb{P}^N_{\Bbbk}$ is cut out locally by a regular sequence and that $\mathcal{I}/\mathcal{I}^2$ is a vector bundle on C fitting into an exact sequence
$$0 \to \mathcal{I}/\mathcal{I}^2 \to \Omega_{\mathbb{P}^N_{\Bbbk}}|_C \to \Omega_C \to 0.$$
Applying $\operatorname{Hom}_{\mathcal{O}_C}(-, \mathcal{O}_C)$ gives a long exact sequence where the relevant terms for us are
$$\operatorname{Hom}_{\mathcal{O}_C}(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_C) \to \operatorname{Ext}^1_{\mathcal{O}_C}(\Omega_C, \mathcal{O}_C) \to \operatorname{Ext}^1_{\mathcal{O}_C}(\Omega_{\mathbb{P}^N_{\Bbbk}}|_C, \mathcal{O}_C) = H^1(C, T_{\mathbb{P}^N_{\Bbbk}}|_C).$$
The first term classifies embedded deformations of $\mathcal{C}_0 \hookrightarrow \mathbb{P}^N_{A_0}$ over A_0 to $\mathcal{C}' \hookrightarrow \mathbb{P}^N_A$ over A while the second term classifies deformations of $\mathcal{C}_0$ over A_0 to $\mathcal{C}'$ over A. The boundary map $\operatorname{Hom}_{\mathcal{O}_C}(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_C) \to \operatorname{Ext}^1_{\mathcal{O}_C}(\Omega_C, \mathcal{O}_C)$ assigns an embedded deformation $[\mathcal{C}' \hookrightarrow \mathbb{P}^N_A]$ to $[\mathcal{C}']$.
Finally, we have the restriction of the Euler sequence to C
$$0 \to \mathcal{O}_C \to \mathcal{O}_C(1)^{\oplus(N+1)} \to T_{\mathbb{P}^N}|C \to 0.$$
Since $H^2(C, \mathcal{O}_C) = 0$ (as $\dim C = 1$) and $H^1(C, \mathcal{O}_C(1)) = 0$ (as $[C] \in H'$), we conclude that $H^1(C, T_{\mathbb{P}^N}|C) = 0$. Thus, our given deformation $[\mathcal{C}] \in \operatorname{Ext}^1_{\mathcal{O}_C}(\Omega_C, \mathcal{O}_C)$ maps to 0 in $H^1(C, \mathcal{O}_C(1))$, and thus is the image of an embedded deformation $[\mathcal{C} \hookrightarrow \mathbb{P}^N_A] \in \operatorname{Hom}_{\mathcal{O}_C}(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_C)$.

$\square$

5.4.3 Algebraicity of $\overline{\mathcal{M}}_{g,n}$: openness and boundedness of stable curves

Consider the inclusions of prestacks

$$\mathcal{M}_{g,n} \subset \overline{\mathcal{M}}_{g,n} \subset \mathcal{M}_{g,n}^{\mathrm{ss}} \subset \mathcal{M}_{g,n}^{\mathrm{pre}} \subset \mathcal{M}_{g,n}^{\leq \mathrm{nodal}} \subset \mathcal{M}_{g,n}^{\mathrm{all}} \tag{5.4.2}$$

where $\overline{\mathcal{M}}_{g,n}$ (resp. $\mathcal{M}_{g,n}^{\mathrm{ss}}, \mathcal{M}_{g,n}^{\mathrm{pre}}, \mathcal{M}_{g,n}^{\leq \mathrm{nodal}}$) denotes the full subcategory of $\mathcal{M}_{g,n}^{\mathrm{all}}$ consisting of n-pointed families $(\mathcal{C} \to S, \sigma_1, \dots, \sigma_n)$ of stable curves (resp. semistable, prestable, and nodal curves).

- By Theorem 5.4.7, $\mathcal{M}_{g,n}^{\mathrm{all}}$ is an algebraic stack locally of finite type over $\operatorname{Spec} \mathbb{Z}$.
- $\mathcal{M}_{g,n}^{\leq \mathrm{nodal}} \subset \mathcal{M}_{g,n}^{\mathrm{all}}$ is an open substack: this is equivalent to showing that $\mathcal{C} \xrightarrow{\pi} S$ is a family of curves (with $\mathcal{C}$ possibly an algebraic space) then the locus $\{s \in S \mid \mathcal{C}_s \text{ is nodal}\} \subset S$ is open. This is established in Corollary 5.2.20 when $\mathcal{C}$ is a scheme by relying on the Local Structure of Nodes (Theorem 5.2.18) and can be established in general by choosing an étale cover $\mathcal{C}' \xrightarrow{g} \mathcal{C}$ by a scheme and using the observation that a point $p \in \mathcal{C}'$ is node in its fiber $\mathcal{C}'_{\pi(p)}$ if and only if $g(p) \in \mathcal{C}_{\pi(p)}$ is a node.
- $\mathcal{M}_{g,n}^{\mathrm{pre}} \subset \mathcal{M}_{g,n}^{\leq \mathrm{nodal}}$ is an open substack: for a family $(\mathcal{C} \to S, \{\sigma_i\})$ of nodal curves, the locus $\{s \in S \mid \sigma_i(s) \text{ are disjont and smooth}\}$ is open.
- $\mathcal{M}_{g,n}^{\mathrm{ss}} \subset \mathcal{M}_{g,n}^{\mathrm{pre}}$ is an open substack: the condition that a prestable curve $(\mathcal{C} \to S, \{\sigma_i\})$ is semistable is equivalent to the nefness of $\omega_{\mathcal{C}/S}(\sigma_1 + \cdots \sigma_n)$ and nefness is an open condition in flat families.
- $\mathcal{M}_{g,n} \subset \mathcal{M}_{g,n}^{\mathrm{ss}}$ is an open substack: the condition that a semistable curve $(\mathcal{C} \to S, \{\sigma_i\})$ is stable is equivalent to the ampleness of $\omega_{\mathcal{C}/S}(\sigma_1 + \cdots \sigma_n)$ and ampleness is an open condition in families. See also Proposition 5.3.10.

It follows that each prestack featured in (5.4.2) is an algebraic stack locally of finite type over $\operatorname{Spec} \mathbb{Z}$.

To show the boundedness (i.e. finite typeness or equivalently quasi-compactness) of $\overline{\mathcal{M}}_{g,n}$, we will appeal to the fact that if $(C, p_1, \dots, p_n)$ is an n-pointed stable curve over a field $\Bbbk$, then the third power of the twist of the dualizing sheaf $(\omega_{C/\Bbbk}(p_1 + \cdots p_n))^{\otimes 3}$ is very ample (Exercise 5.3.6). Let $P(t)$ be the Hilbert polynomial of $C \hookrightarrow \mathbb{P}^N_{base}$ embedded via $(\omega_{C/\Bbbk}(p_1 + \cdots p_n))^{\otimes 3}$; this is independent of $[C, \{p_i\}] \in \overline{\mathcal{M}}_{g,n}$. Consider the closed subscheme

$$H \subset \operatorname{Hilb}^P(\mathbb{P}^N_{\mathbb{Z}}/\mathbb{Z}) \times (\mathbb{P}^N)^n$$

of an embedded curve and n points $(C \hookrightarrow \mathbb{P}^N, p_1, \dots, p_n)$ such that $p_i \in C$. There is a forgetful functor

$$H \to \mathcal{M}_{g,n}^{\mathrm{all}} \qquad [C \hookrightarrow \mathbb{P}^N, p_1, \dots, p_n] \mapsto (C, p_1, \dots, p_n).$$

Since $\operatorname{Hilb}^P(\mathbb{P}^N_{\mathbb{Z}}/\mathbb{Z})$ is a projective scheme (Theorem 1.1.2) and in particular quasi-compact and the image of $|H| \to |\mathcal{M}_{g,n}^{\mathrm{all}}|$ contains $\overline{\mathcal{M}}_{g,n}$, we conclude that $\overline{\mathcal{M}}_{g,n}$ is quasi-compact.

At this point, we've shown that $\overline{\mathcal{M}}_{g,n}$ is an algebraic stack of finite type over $\operatorname{Spec}\mathbb{Z}$. We now invoke each part of Proposition 5.3.14 characterizing automorphisms, deformations and obstructions of stable curve exactly as in the proof of the analogous fact for $\overline{\mathcal{M}}_g$ (Proposition 3.7.4). Indeed, $\operatorname{Ext}^0_{\mathcal{O}_C}(\Omega_C(\sum_i p_i), \mathcal{O}_C) = 0$ implies that the Lie algebra of $\operatorname{Aut}(C, \{p_i\})$ is trivial and thus that $\operatorname{Aut}(C, \{p_i\})$ is a finite and reduced group scheme. By the Characterization of Deligne–Mumford stacks (Theorem 3.6.4), we conclude that $\overline{\mathcal{M}}_{g,n}$ is Deligne–Mumford. Since $\operatorname{Ext}^2_{\mathcal{O}_C}(\Omega_C(\sum_i p_i), \mathcal{O}_C) = 0$, there are no obstructions to deforming stable curves and the Infinitesimal Lifting Criterion (Theorem 3.7.1) implies that $\overline{\mathcal{M}}_{g,n}$ is smooth over $\operatorname{Spec}\mathbb{Z}$. Finally, since $\dim_{\Bbbk} \operatorname{Ext}^1_{\mathcal{O}_C}(\Omega_C(\sum_i p_i), \mathcal{O}_C) = 3g-3+n$ and there is a bijection of this Ext group with the Zariski tangent space of $[C, \{p_i\}] \in \overline{\mathcal{M}}_{g,n} \times_{\mathbb{Z}} K$, we see that $\overline{\mathcal{M}}_{g,n} \to \operatorname{Spec}\mathbb{Z}$ has relative dimension $3g-3+n$.

Putting everything together, we've proved:

Theorem 5.4.8. *If $2g-2+n > 0$, then $\overline{\mathcal{M}}_{g,n}$ is a quasi-compact Deligne–Mumford stack smooth over $\operatorname{Spec}\mathbb{Z}$ of relative dimension $3g-3+n$.* □

Exercise 5.4.9. Show that $\overline{\mathcal{M}}_{g,n}$ is algebraic by following the proof of Theorem 3.1.15.

5.5 Stable reduction: properness of $\overline{\mathcal{M}}_{g,n}$

In the section, we discuss stable reduction of curves. Following the exposition of [HM98, §3.C], we give a complete proof in characteristic 0 relying on the birational geometry of surfaces and specifically the existence of embedded resolutions for curves on surfaces (see §E.1).

Theorem 5.5.1 (Stable Reduction)**.** *Let R be a DVR with fraction field K, and set $\Delta = \operatorname{Spec} R$ and $\Delta^* = \operatorname{Spec} K$. If $(\mathcal{C}^* \to \Delta^*, s_1^*, \ldots, s_n^*)$ is a family of n-pointed stable curves of genus g, then there exists a finite cover $\Delta' \to \Delta$ of spectrums of DVRs and a family $(\mathcal{C}' \to \Delta', s_1', \ldots, s_n')$ of stable curves extending $\mathcal{C}^* \times_{\Delta^*} \Delta'^* \to \Delta'^*$.*

Remark 5.5.2. This theorem was first established in [DM69] by embedding the generic fiber into its Jacobian and reducing the statement to semistable reduction for abelian varieties, which had been established in [SGA7-I, SGA7-II]. Interestingly, Gieseker also established this theorem by using GIT rather than the geometry of families of curves over a DVR [Gie82]. Later arguments due to Artin–Winters [AW71] and Saito [Sai87] follow essentially the strategy outlined below. See [SP, Tag 0C2Q] or Remark 5.5.8 for more background.

After introducing the basic strategy to establish Stable Reduction in Section 5.5.1, we prove Stable Reduction (Theorem 5.5.1) in characteristic 0 in Section 5.5.3. We also illustrate in Sections 5.5.4 and 5.5.5 how one can explicitly compute the stable limit of a given family $\mathcal{C}^* \to \Delta^*$ of stable curves: while the proof of Stable Reduction offers a strategy, additional care and techniques are needed to get an explicit handle on the stable limit. Finally, in Section 5.5.6, we prove the uniqueness of the stable limit (Proposition 5.5.15) in arbitrary (possibly mixed) characteristic. This implies the properness of $\overline{\mathcal{M}}_{g,n}$ via the Valuative Criterion for Properness (Theorem 3.8.5).

Theorem 5.5.3. *If $2g-2+n>0$, the Deligne–Mumford stack $\overline{\mathcal{M}}_{g,n}$ is proper over* $\operatorname{Spec}\mathbb{Z}$. □

By applying the Keel–Mori Theorem (4.3.11), we obtain:

Corollary 5.5.4. *If $2g-2+n>0$, there exists a coarse moduli space $\overline{\mathcal{M}}_{g,n} \to \overline{M}_{g,n}$ where $\overline{M}_{g,n}$ is an algebraic space proper over* $\operatorname{Spec}\mathbb{Z}$. □

5.5.1 Basic strategy

We provide the basic strategy to exhibit the existence of stable reduction for a given family $\mathcal{C}^* \to \Delta^*$ of stable curves. For simplicity of notation, we assume that there are no marked points, i.e. $n=0$.

Throughout, we use the notation: $\Delta = \operatorname{Spec} R$ for a DVR R, $\Delta^* = \operatorname{Spec} K$ with K the fraction field of R, $t \in R$ uniformizer, and $0 = (t) \in \operatorname{Spec} R$ the unique closed point.

Step 0: Reduce to the case where $\mathcal{C}^* \to \Delta^*$ is smooth. If $\mathcal{C}^*$ has k nodes, then possibly after a finite extension of K we can arrange that each node is given by a K-point $p_i \in \mathcal{C}^*(K)$. Let $(\widetilde{\mathcal{C}}^*, \widetilde{p}_1, \ldots, \widetilde{p}_{2k})$ be the pointed normalization. By induction on the genus g (relying on stable reduction for $2k$*-pointed curves* of genus $< g$), we perform stable reduction on each connected component and then take the nodal union along sections. After possibly an extension of K (and R), this produces a family of curves $\mathcal{C} \to \Delta$ extending $\mathcal{C}^* \to \Delta^*$.

Step 1: Find some flat extension $\mathcal{C} \to \Delta$.

Using that $\omega_{\mathcal{C}^*/\Delta^*}^{\otimes 3}$ is very ample (Proposition 5.3.9), we may embed $\mathcal{C}^*$ as a closed subscheme of $\mathbb{P}^{5g-6} \times \Delta^*$. The scheme-theoretic image $\mathcal{C}$ of $\mathcal{C}^* \hookrightarrow \mathbb{P}^{5g-6} \times \Delta$ is flat over Δ using the Flatness Criterion over Smooth Curves (Proposition A.2.2) and the fact the closure doesn't introduce any embedded points in the central fiber. Thus we have a proper flat family of curves $\mathcal{C} \to \Delta$ extending $\mathcal{C}^* \to \Delta^*$. (This is the same argument that establishes the valuative criterion for properness of the Hilbert scheme.)

Step 2: Use Embedded Resolutions to find a resolution of singularities $\widetilde{\mathcal{C}} \to \mathcal{C}$ so that the reduced central fiber $(\widetilde{\mathcal{C}}_0)_{\mathrm{red}}$ is nodal.

Applying Embedded Resolutions (Theorem E.1.2), there is a finite sequence of blow-ups at closed points of $\mathcal{C}_0$ yielding a projective birational morphism

$$\begin{tikzcd} \widetilde{\mathcal{C}} \arrow[r] \arrow[dr] & \mathcal{C} \arrow[d] \\ & \Delta \end{tikzcd}$$

such that $\widetilde{\mathcal{C}}$ is regular, $\widetilde{\mathcal{C}} \to \Delta$ is a (flat) family of curves and such that the preimage $\widetilde{\mathcal{C}}_0$ of $\mathcal{C}_0$ has set-theoretic normal crossings, i.e. $(\widetilde{\mathcal{C}}_0)_{\mathrm{red}}$ is nodal. Replace $\mathcal{C}$ with $\widetilde{\mathcal{C}}$.

Step 3: Take a ramified base extension $\Delta' = \operatorname{Spec} R \to \operatorname{Spec} R = \Delta$ by $t \mapsto t^m$ such that the central fiber of the normalization of $\mathcal{C} \times_\Delta \Delta'$ becomes reduced and nodal.

We will explain the details of this step in Section 5.5.3. This step is where we will use the characteristic 0 assumption. Replacing $\mathcal{C}$ with the normalization $\widetilde{\mathcal{C}'}$ of $\widetilde{\mathcal{C}'} = \mathcal{C} \times_\Delta \Delta'$, we may assume that $\mathcal{C} \to \Delta$ is a prestable family (i.e. nodal family) of curves with $\mathcal{C}$ regular.

Step 4: After taking the minimal model $\widetilde{\mathcal{C}}_{\min} \to \mathcal{C}$, contract all rational tails and bridges in the central fiber.

In other words, we take the stable model of the family $\widetilde{\mathcal{C}}_{\min} \to \Delta$ as in Proposition 5.3.20. Alternatively as we argue in Section 5.5.3, one can explicitly contract the rational tails (smooth rational -1 curves) and rational bridges (smooth rational -2) curves.

Remark 5.5.5. Add example showing why we must allow for extensions of DVRs.

5.5.2 Semistable reduction

In Step 4 above, if we stop after contracting only rational tails (and not the rational bridges), i.e. the smooth rational -1 curves, then we obtain a family $\mathcal{C} \to \Delta$ of semistable curves such that $\mathcal{C}$ is regular (by Theorem E.1.5). This is called Semistable Reduction, an important variant of Stable Reduction.

Theorem 5.5.6 (Semistable Reduction)**.** *Let R be a DVR with fraction field K, and set $\Delta = \operatorname{Spec} R$ and $\Delta^* = \operatorname{Spec} K$. If $\mathcal{C}^*$ is a smooth projective curve over Δ^*, there exists a cover $\Delta' \to \Delta$ of spectrums of DVRs and a family $\mathcal{C}' \to \Delta'$ of semistable curves extending $\mathcal{C}^* \times_{\Delta^*} \Delta'^* \to \Delta'^*$ such that $\mathcal{C}'$ is regular.*

5.5.3 Proof of stable reduction in characteristic 0

Proof of Theorem 5.5.1 in characteristic 0. Following Steps 0-2 in the basic strategy discussed in Section 5.5.1, we may assume that $\mathcal{C} \to \Delta$ is a generically smooth family of stable curves such that the reduced central fiber $(\mathcal{C}_0)_{\mathrm{red}}$ is nodal and $\mathcal{C}$ is regular.

Step 3: Perform a base change $\Delta' \to \Delta$ such that the normalization of the total family $\mathcal{C} \times_\Delta \Delta'$ has a reduced and nodal central fiber. Around every point $p \in \mathcal{C}_0$, we can choose local coordinates x, y (either étale locally or formally locally at p) such that the morphism $\mathcal{C} \to \Delta$ can be described explicitly as follows (**??**):

- If $p \in (\mathcal{C}_0)_{\mathrm{red}}$ is a smooth point, then $(x, y) \mapsto x^a$ and the multiplicity of the irreducible component of $\mathcal{C}_0$ containing p is a.
- If $p \in (\mathcal{C}_0)_{\mathrm{red}}$ is a separating node (i.e. $\mathcal{C}_0 \setminus p$ is disconnected), then $(x, y) \mapsto x^a y^b$ and the two components of $\mathcal{C}_0$ containing p have multiplicities a and b.
- If $p \in (\mathcal{C}_0)_{\mathrm{red}}$ is a non-separating node, then $(x, y) \mapsto x^a y^a$ and the components of $\mathcal{C}_0$ containing p has multiplicity a.

Let m be the least common multiple of the multiplicities of the irreducible components of $\mathcal{C}_0$. Let $\Delta' = \operatorname{Spec} R \to \operatorname{Spec} R = \Delta$ be defined by $t \mapsto t^m$ where t denotes a uniformizing parameter. Let $\mathcal{C}' := \mathcal{C} \times_\Delta \Delta'$ and $\widetilde{\mathcal{C}'}$ be its normalization

Let ρ be a primitive mth root of unity. If $p \in (\mathcal{C}_0)_{\text{red}}$ is a smooth point, then $\mathcal{C}'$ locally around the unique preimage of p is defined by $x^a = t^m$ which factors as $\prod_{i=0}^{a-1}(x - \rho^i t^{m/a})$. Thus $p \in \mathcal{C}$ has a preimages in $\widetilde{\mathcal{C}}'$ and each preimage is locally defined by $x = \rho^i t^{m/a}$ and is thus a smooth point in the central fiber $\widetilde{\mathcal{C}}'_0$. If $p \in (\mathcal{C}_0)_{\text{red}}$ is a node defined by $x^a y^b$, then one computes that each preimage of p is locally defined by $t^k = xy$ (see Exercise 5.5.7) and thus is a reduced and nodal point in $\widetilde{\mathcal{C}}'_0$. Note that if $k > 1$, then $\widetilde{\mathcal{C}}'$ has an A_{k-1}-singularity at the preimage.

We now replace $\mathcal{C}$ with $\widetilde{\mathcal{C}}'$. At the expense of introducing singularities into the total family, we have arranged the central fiber to be reduced and nodal.

Step 4: Take a minimal resolution of $\mathcal{C}$ and contract curves with negative self-intersection. Let $\mathcal{C}' \to \mathcal{C}$ be a Minimal Resolution (Theorem E.1.1) which replaces each A_k-singular with a chain of $\lfloor \frac{k}{2} \rfloor$ rational curves. At this stage $\mathcal{C}' \to \Delta$ is a prestable family of curves, i.e. a proper flat family of reduced nodal curves, such that the total family $\mathcal{C}'$ is regular. The central fiber $\mathcal{C}'_0$ however may not be stable.

If $\mathcal{C}'_0$ is not stable, then it contains either a rational tail or bridge as in Figure 5.6. Each rational tail E has self-intersection -1 can be blown down by Castelnuovo's Contraction Theorem (E.1.5). Contracting all rational tails yields a projective birational morphism $\mathcal{C}' \to \mathcal{C}'_{\min}$, which is the Minimal Model (Corollary E.1.7). Replacing $\mathcal{C}$ with $\mathcal{C}'_{\min}$, we obtain a semistable family $\mathcal{C} \to \Delta$ of curves such that the total family $\mathcal{C}$ is regular.

Finally, we apply the stabilization construction (Proposition 5.3.20) to obtain a morphism $\mathcal{C} \to \mathcal{C}^{\text{st}}$ contracting each rational bridge and where $\mathcal{C}^{\text{st}} \to \Delta$ is a stable family of curves. We note that $\mathcal{C}^{\text{st}}$ is precisely the relative canonical model of $\mathcal{C}$ (Proposition 5.3.20(4)). Alternatively, one can realize this final step by iteratively contracting each rational bridge E since each such subcurve satisfies $E^2 = -2$. Indeed, a version of Castelnuovo's Contraction Theorem is valid even if $E^2 < -1$ (the only difference is that the contracted surface may not be regular) and the contraction yields a family of stable curves. □

Exercise 5.5.7. Let a, b, m be positive integers such that both a and b divide m. Let $X = \operatorname{Spec} \Bbbk[x, y, t]/(t^m - x^a y^b)$ and $\widetilde{X} \to X$ be its normalization. Show that each preimage of the origin is locally defined by $t^k = xy$, and in particular is a reduced and nodal point in the fiber over $t = 0$.

Remark 5.5.8. The above argument fails if the residue field of R has positive characteristic p. Indeed, in Step 3, if any of the multiplicities of the components of the central fiber are divisible by p, then the extension $\operatorname{Spec} R \to \operatorname{Spec} R$ given by $t \mapsto t^m$ is not tamely ramified and the base change $\mathcal{C} \times_\Delta \Delta'$ may remain non-reduced.

A different approach is therefore needed in positive characteristic. The approach of [AW71] starts as above by taking a resolution of singularities $\mathcal{C}$ of some family of curves over R extending $\mathcal{C}^*$. One then chooses an extension $K \to K'$ (and a corresponding extension $R \to R'$ of DVRs) such that $\mathcal{C}^*$ has a K'-point and such that the l-torsion $\operatorname{Pic}(\mathcal{C}^*_{K'})[l] \cong (\mathbb{Z}/l\mathbb{Z})^{2g}$ for a sufficiently large prime $l \neq p$. This magically forces the central fiber of $\mathcal{C} \times_R R'$ to be reduced and nodal! See [AW71] or [SP, Tag 0E8C].

5.5.4 First examples

In these examples $\Delta = \operatorname{Spec} R$ where R is a DVR with uniformizing parameter t.

Example 5.5.9 (Nodal elliptic curves)**.** Consider the family of elliptic curves $(\mathcal{C}^* \to \Delta^*, \sigma)$ defined by the equation $y^2z = x(x-z)(x-tz)$ in $\mathbb{P}^2 \times \Delta$ and the section $\sigma(t) = [0,1,0]$. The stable limit in $\overline{\mathcal{M}}_{1,1}$ as $t \to 0$ is the nodal cubic $y^2z = x^2(x-z)$; see Figure 18.

Example 5.5.10 (Colliding marked points)**.** Let C be a smooth curve and consider the constant family $\mathcal{C} = C \times \Delta$. Let $p \in C$ be a $\mathbb{k}$-point and $\sigma_1 \colon \Delta \to \mathcal{C}$ be the constant section $t \mapsto p$. Suppose that $\sigma_2 \colon \Delta \to \mathcal{C}$ is another section meeting σ_1 transversely at $(p,0) \in \mathcal{C}$ as shown below:

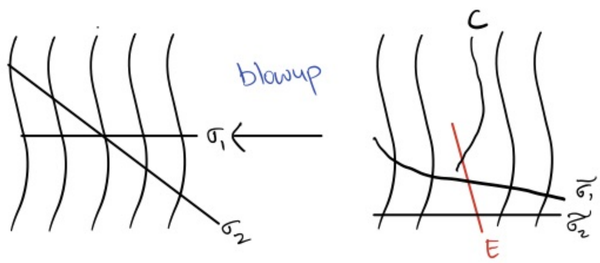


Figure 5.8:

To obtain the stable limit, we simply blowing up the surface at $(p,0)$. The stable limit is the nodal union of C and $\mathbb{P}^1$ at p.

For a more involved example of colliding points, consider again the constant family $C \times \Delta$ with sections locally defined by $(\sigma_1, \sigma_2, \sigma_3) = (t^2, -t^2, 4t)$.

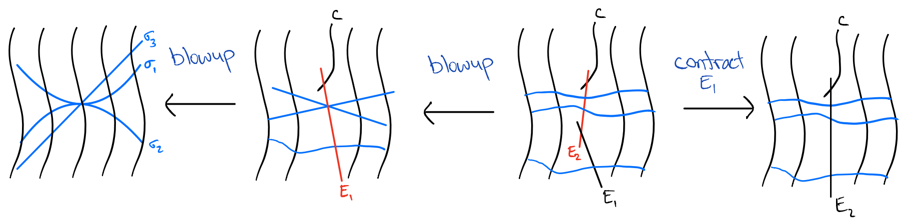


Figure 5.9:

After blowing up twice, the sections become disjoint but the central fiber is unstable as the exceptional component $E_1 \cong \mathbb{P}^1$ only has one node and one marked point. The stable limit is obtained by contracting E_1.

Example 5.5.11 (A node degenerating to a cusp)**.** Consider a smooth curve C with two points $p, q \in C$. Gluing p and q yields a nodal curve. Now if we fix p and slide q toward p, we have a family of nodal curves $\mathcal{C}^* \to C \setminus p$ as in Figure 5.10. For instance, this family could be defined locally by $y^2 = x^3 + tx^2$ in which case we have an extension $\mathcal{C} \to C$ where the central fiber $\mathcal{C}_p$ (given by $t = 0$) has a cusp. We would like to compute the stable limit.

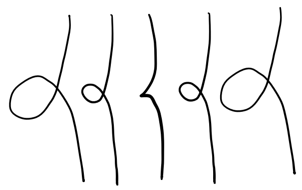


Figure 5.10: What is the stable limit of the above nodal degeneration?

In this case, the base curve is C itself but it would be no different to work over $\operatorname{Spec} \mathcal{O}_{C,p}$. The pointed normalization of the family $\mathcal{C}^*$ extends to a family $C \times C \to C$ with the diagonal section Δ and the constant section $\Gamma_p = \{p\} \times C$.

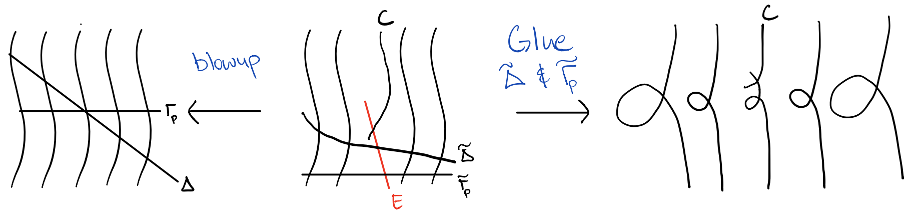


Figure 5.11: Recipe for computing the stable reduction

We first find the stable limit of the pointed normalization exactly as in Example 5.5.10: we blowup so that the strict transforms $\widetilde{\Delta}$ and $\widetilde{\Gamma}_p$ become disjoint. We then glue the sections $\widetilde{\Delta}$ and $\widetilde{\Gamma}_p$ to obtain a family $\mathcal{C} \to C$ of nodal curves where the central fiber is the nodal union of C and a rational nodal curve at the point $p \in C$.

The above examples are too simple to reveal the general stable reduction procedure as no base changes were needed.

5.5.5 Explicit stable reduction

The biggest challenge in explicitly computing the stable limit of a family $\mathcal{C}^* \to \Delta^*$ following the basic strategy of Section 5.5.1 is in Step 3: computing the normalization $\widetilde{\mathcal{C}}'$ of the family $\mathcal{C}' = \mathcal{C} \times_\Delta \Delta'$ obtained by base changing $\mathcal{C} \to \Delta$ along a ramified cover $\Delta' \to \Delta$ defined by $t \mapsto t^m$. It is often simpler to factor $\Delta' \to \Delta$ as a composition of prime order base changes and use the following observation.

Proposition 5.5.12. *Let $\mathcal{C} \to \Delta$ be a generically smooth, proper and flat family such that $(\mathcal{C}_0)_{\mathrm{red}}$ is nodal. As a divisor on $\mathcal{C}$, we may write $\mathcal{C}_0 = \sum a_i D_i$ where a_i is the multiplicity of the irreducible component D_i. Let $\Delta' \to \Delta$ be defined by $t \mapsto t^p$ where p is prime, and set $\mathcal{C}' := \mathcal{C} \times_\Delta \Delta'$ with normalization $\widetilde{\mathcal{C}}'$. Then $\widetilde{\mathcal{C}}' \to \mathcal{C}$ is a branched cover ramified over $\sum (a_i \mod p) D_i$.*

Example 5.5.13 (Stable Reduction of an A_{2k+1}-singularity)**.** Suppose $\mathcal{C} \to \Delta$ is a generically smooth family degenerating to a A_{2k+1}-singularity in the central fiber such that the local equation around the singular point is $y^2 = x^{2k+1} + t$. In particular, the total family $\mathcal{C}$ is smooth. Figure 5.12 provides a pictorial representation of the stable reduction procedure.

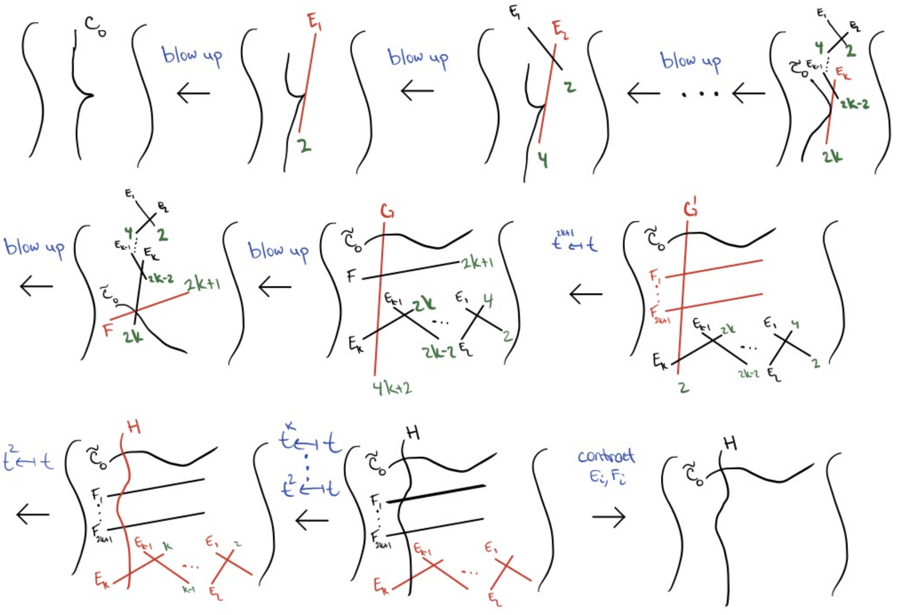


Figure 5.12: Recipe for computing the stable limit of a A_{2k+1}-singularity. The altered components in each step are colored in red while the green numbers indicate the multiplicity of the component.

We are already given a flat limit $\mathcal{C} \to \Delta$ so may begin with Step 2.

Step 2: Repeatedly blow-up to find a resolution of singularities $\widetilde{\mathcal{C}} \to \mathcal{C}$ so that the reduced central fiber $(\widetilde{\mathcal{C}}_0)_{\mathrm{red}}$ is nodal.

We repeatedly blow up the (reduced) singular point in the central fiber. To keep track of the local equations, we will always use local coordinates x, y on the original surface and $\widetilde{x}, \widetilde{y}$ on the new surface. In one chart of the blowup, $\widetilde{x} = x, \widetilde{y} = y/x$ with exceptional divisor $\widetilde{x} = 0$ while in the other chart, $\widetilde{x} = x/y, \widetilde{y} = y$ with exceptional divisor $\widetilde{y} = 0$.

For the first blow up, the preimage of $y^2 - x^{2k+1}$ in the chart $\widetilde{x} = x, \widetilde{y} = y/x$ is given by $\widetilde{x}^2(\widetilde{y}^2 - \widetilde{x}^{2k-1})$ and in the other chart by $\widetilde{y}^2(1 - \widetilde{x}^{2k+1}\widetilde{y}^{2k-1})$. The exceptional divisor E_1 has multiplicity 2.

For the second blow up, the preimage of $x^2(y^2 - x^{2k-1})$ in the chart $\widetilde{x} = x, \widetilde{y} = y/x$ is given by $\widetilde{x}^4(\widetilde{y}^2 - \widetilde{x}^{2k-3})$ and in the other chart by $\widetilde{x}^2\widetilde{y}^4(1 - \widetilde{x}^{2k-1}\widetilde{y}^{2k-3})$ (where $\widetilde{x}$ defines E_1 and $\widetilde{y}$ defines E_2). The new exceptional divisor E_2 has multiplicity 4.

After k blow ups, one obtains a surface with local equation $x^{2k}(y^2 - x)$ at the singular point in the central fiber. The equation $y^2 - x$ defines the normalization $\widetilde{\mathcal{C}}_0$ of the original central fiber and x defines the exceptional divisor E_k which has multiplicity $2k$. There is a chain of nodally attached exceptional divisors $E_k, \ldots, E_1$ such that the multiplicity of E_i is $2i$.

Blowing up again, the strict transform of $x^{2k}(y^2 - x)$ in the chart $\widetilde{x} = x/y, \widetilde{y} = y$ becomes $\widetilde{x}^{2k}\widetilde{y}^{2k+1}(\widetilde{y} - \widetilde{x})$ where $\widetilde{x}$ defines E_k, $\widetilde{y}$ defines the new exceptional divisor F which has multiplicity $2k+1$, and $\widetilde{y} - \widetilde{x}$ defines $\widetilde{\mathcal{C}}_0$.

Blowing up one final time, the strict transform of $x^{2k}y^{2k+1}(y - x)$ in the chart $\widetilde{x} = x, \widetilde{y} = y/x$ becomes $\widetilde{x}^{4k+2}\widetilde{y}^{2k+1}(\widetilde{y} - 1)$ where $\widetilde{x}$ defines the new exceptional divisor G which has multiplicity $4k+2$, $\widetilde{y}$ defines F and $\widetilde{y}-1$ defines $\widetilde{\mathcal{C}}_0$. In particular, the (non-reduced) central fiber is set-theoretically nodal.

Step 3: Perform a base change $\Delta' \to \Delta$ such that the normalization of the total family $\mathcal{C} \times_\Delta \Delta'$ has a reduced and nodal central fiber.

We begin by base changing by $\Delta' \to \Delta, t \mapsto t^{2k+1}$ and normalizing. For this analysis, we assume that $2k+1$ is prime but one can inductively apply the same process to a prime factorization of $2k+1$ and obtain the same result in the end; the only difference is in the numerics of the multiplicities of the exceptional components E_i but these can be resolved in the last step in the same way.

By applying Proposition 5.5.12, the new surface is a degree $2k+1$ cover ramified over $\widetilde{\mathcal{C}}_0 + \sum_i E_i$ as the other components of the central fiber have multiplicities divisible by $2k+1$. The preimage G' of G is $2k+1$ degree cover of $\mathbb{P}^1$ ramified over two points, each with ramification index $2k$. By Riemann–Hurwitz, the genus $g(G')$ of G' satisfies $2g(G')-2 = (2k+1)(g(\mathbb{P}^1)-2)+R$ and since the ramification divisor R has degree $2(2k)$, we see that $g(G')=0$. Meanwhile, the preimage of F is the disjoint union of $2k+1$ smooth rational curves $F_1, \dots, F_{2k+1}$. Over Δ, the new special fiber is

$$(2k+1)\widetilde{\mathcal{C}}_0 + (4k+2)G' + (2k+1)\sum_i F_i + (2k+1)\sum_i 2iE_i$$

which over Δ' becomes $\widetilde{\mathcal{C}}_0 + 2G' + \sum_i F_i + \sum_i 2iE_i$

We now base change by $\Delta' \to \Delta, t \mapsto t^2$ and normalize. By Proposition 5.5.12, the new surface is a $2:1$ cover ramified over $\widetilde{\mathcal{C}}_0 + \sum_i F_i$. The preimage H of $G' \cong \mathbb{P}^1$ is a $2:1$ cover ramified over $2k+2$ points, with one of those points being the node $H \cap \widetilde{\mathcal{C}}_0$. Thus G' is a hyperelliptic curve of genus g attached to $\widetilde{\mathcal{C}}_0$ at a ramification point (otherwise known as a Weierstrass point). The new central fiber over Δ' becomes reduced except for the components E_i which have multiplicity i.

Finally, we inductively base change and normalize by the ramified covers defined by $t \mapsto t^k, \dots, t \mapsto t^2$ so that the central fiber becomes reduced and nodal.

Step 4: Contract rational tails in the central fiber.

The exceptional components F_i are smooth rational -1 curves which we can contract. We then inductive contract $E_1, E_2, \dots, E_k$ (note that while E_1 is a -1 curve, E_2 is a -2 curve but becomes a -1 curve once E_1 is contracted). In the end, we obtain a reduced central fiber which is the nodal union of the normalization $\widetilde{\mathcal{C}}_0$ of the original central fiber and a hyperelliptic genus k curve H. The node in H is a ramification point of the $2:1$ cover $H \to \mathbb{P}^1$ while the node in $\widetilde{\mathcal{C}}_0$ is the preimage of the singular point of $\mathcal{C}_0$.

The above example begs the following questions:

- Precisely which hyperelliptic curve H appears in the stable limit?
- How does the stable limit depend on the choice of degeneration? By calculating the deformation space of a A_{2k+1}-singularity, one sees that every degeneration can be written locally as $y^2 = x^{2k+1} + a_{2k-1}(t)x^{2k-1} + \cdots + a_0(t)$ for polynomials $a_{2k-1}, \dots, a_0$. In other words, we are asking how does the stable limit depend on $a_i(t)$. In particular, what happens when the total family of the surface is singular (e.g. $y^2 = x^{2k+1} + t^2$)?

These questions are addressed in detail in [HM98, §3.C] in the case of a cusp $y^2 = x^3$ (i.e. $k = 1$). The reader is also encouraged to refer to *loc. cit.* for additional examples of stable reduction and other aspects of this story.

Exercise 5.5.14. Work out the stable reduction of a smooth family of curves degenerating to an A_{2k+2}-singularity with local equation $y^2 = x^{2k+2} + t$.

5.5.6 Separatedness of $\overline{\mathcal{M}}_{g,n}$

We now show that the stable limit is unique. The following proposition establishes via the Valuative Criterion for Separatedness (Theorem 3.8.5) that $\overline{\mathcal{M}}_{g,n}$ is separated.

Proposition 5.5.15. *Let R be a DVR with fraction field K, and set $\Delta = \operatorname{Spec} R$ and $\Delta^* = \operatorname{Spec} K$. If $(\mathcal{C} \to \Delta, \sigma_1^*, \ldots, \sigma_n^*)$ and $(\mathcal{D} \to \Delta, \tau_1^*, \ldots, \tau_n^*)$ are families of n-pointed stable curves, then every isomorphism $\alpha^* \colon \mathcal{C}^* \to \mathcal{D}^*$ over Δ^* with $\tau_i^* = \alpha^* \circ \sigma_i^*$ of the generic fibers as pictured*

$$\begin{tikzcd} \mathcal{C}^* \arrow[r, "\alpha^*"] \arrow[dr] & \mathcal{D}^* \arrow[d] & \mathcal{C} \arrow[r, dashed, "\alpha"] \arrow[dr] & \mathcal{D} \arrow[d] \\ & \Delta^* \arrow[rr, hook] & & \Delta \end{tikzcd}$$

extends to a unique isomorphism $\alpha \colon \mathcal{C} \to \mathcal{D}$ over Δ with $\tau_i = \alpha \circ \sigma_i$.

Proof. We will prove the case when there are no marked points ($n = 0$) and the generic fiber $\mathcal{C}^* \cong \mathcal{D}^*$ is smooth over Δ^*. We leave the general case to the reader.

Let $\widetilde{\mathcal{C}} \to \mathcal{C}$ and $\widetilde{\mathcal{D}} \to \mathcal{D}$ be the minimal resolutions (Theorem E.1.1). Let $\Gamma \subset \widetilde{\mathcal{C}} \times_\Delta \widetilde{\mathcal{D}}$ be the closure of the graph $\mathcal{C}^* \xrightarrow{(\mathrm{id}, \alpha^*)} \mathcal{C}^* \times_{\Delta^*} \mathcal{D}^*$ of α^* and let $\widetilde{\Gamma} \to \Gamma$ be the minimal resolution. We have a commutative diagram

$$\begin{tikzcd} & \widetilde{\Gamma} \arrow[dl] \arrow[dr] & \\ \widetilde{\mathcal{C}} \arrow[d] & & \widetilde{\mathcal{D}} \arrow[d] \\ \mathcal{C} \arrow[dr] & & \mathcal{D} \arrow[dl] \\ & \Delta. & \end{tikzcd} \tag{5.5.1}$$

Since $\widetilde{\Gamma} \to \widetilde{\mathcal{C}}$ and $\widetilde{\Gamma} \to \widetilde{\mathcal{C}}$ are birational morphisms of smooth projective surfaces over Δ and the relative dualizing sheaves are line bundles, we have identifications of the pluricanonical sections

$$\Gamma(\widetilde{\mathcal{C}}, \omega_{\widetilde{\mathcal{C}}/\Delta}^{\otimes k}) \cong \Gamma(\widetilde{\Gamma}, \omega_{\widetilde{\Gamma}/\Delta}^{\otimes k}) \cong \Gamma(\widetilde{\mathcal{D}}, \omega_{\widetilde{\mathcal{D}}/\Delta}^{\otimes k})$$

for each non-negative integer k; see [Har77, Thm. II.8.19]. Furthermore, we know that $\mathcal{C}$ and $\mathcal{D}$ are the stable models of $\widetilde{\mathcal{C}}$ and $\widetilde{\mathcal{C}}$ obtained by contracting rational tails and bridges (Proposition 5.3.20). Thus we have an isomorphism

$$\mathcal{C} \cong \operatorname{Proj} \bigoplus_k \Gamma(\widetilde{\mathcal{C}}, \omega_{\widetilde{\mathcal{C}}/\Delta}^{\otimes k}) \cong \operatorname{Proj} \bigoplus_k \Gamma(\widetilde{\mathcal{D}}, \omega_{\widetilde{\mathcal{D}}/\Delta}^{\otimes k}) \cong \mathcal{D}$$

extending $\alpha^* \colon \mathcal{C}^* \to \mathcal{D}^*$. □

Remark 5.5.16. We can also argue more explicitly using our understanding of the birational geometry of surfaces. First, notice that the local structure of the surface $\mathcal{C}$ or $\mathcal{D}$ around a node z in the central fiber is of the form $xy = t^{n+1}$, where $t \in R$ is a uniformizer (Theorem 5.2.18). This is an A_n-surface singularity and in particular normal, and its preimage under the resolution $\widetilde{\mathcal{C}} \to \mathcal{C}$ is a chain $E_1 \cup \cdots \cup E_n$ of rational bridges with $E_i^2 = -2$. By construction, there are no smooth rational -1 curves in the fibers of $\widetilde{\mathcal{C}} \to \mathcal{C}$ and $\widetilde{\mathcal{D}} \to \mathcal{D}$, and since $\mathcal{C}$ and $\mathcal{D}$ are families of stable curves, they have no rational tails and thus no smooth rational -1 curves. We conclude that $\widetilde{\mathcal{C}}$ and $\widetilde{\mathcal{D}}$ are birational smooth surfaces over Δ with no smooth rational -1 curves whose generic fibers $\mathcal{C}^*$ and $\mathcal{D}^*$ are isomorphic.

By the Structure Theorem of Birational Morphisms of Surfaces (Theorem E.1.3), both $\widetilde{\Gamma} \to \widetilde{\mathcal{C}}$ and $\widetilde{\Gamma} \to \widetilde{\mathcal{D}}$ are the compositions of finite sequences of blow-ups at closed points. Since $\widetilde{\Gamma}$ is minimal over Γ, there are no smooth rational -1 curves in $\widetilde{\Gamma}$ that get contracted under both $\widetilde{\Gamma} \to \widetilde{\mathcal{C}}$ and $\widetilde{\Gamma} \to \widetilde{\mathcal{D}}$.

We now claim that $\widetilde{\Gamma} \to \widetilde{\mathcal{C}}$ and $\widetilde{\Gamma} \to \widetilde{\mathcal{D}}$ are isomorphism. Suppose for instance that $\widetilde{\Gamma} \to \widetilde{\mathcal{C}}$ is not an isomorphism. Then there is a smooth rational -1 curve $E \subset \widetilde{\Gamma}$ not contracted under $\widetilde{\Gamma} \to \widetilde{\mathcal{D}}$ and let $E_{\widetilde{\mathcal{D}}} \subset \widetilde{\mathcal{D}}$ be its image. On the one hand, since blowing up only decreases the self-intersection number (indeed, if we write the pre-image of $E_{\widetilde{\mathcal{D}}}$ in $\widetilde{\Gamma}$ as $E + F$, then the projection formula implies that $E_{\widetilde{\mathcal{D}}}^2 = E \cdot (E + F) = E^2 + E \cdot F$), we have that $E_{\widetilde{\mathcal{D}}}^2 \geq E^2 = -1$. The Hodge Index Theorem for Exceptional Curves (Theorem E.1.4) implies however that the self-intersection of $E_{\widetilde{\mathcal{D}}}$ must be negative, and we conclude that $E_{\widetilde{\mathcal{D}}}^2 = -1$. On the other hand, since $E_{\widetilde{\mathcal{D}}}$ is not a smooth rational -1 curve, $E_{\widetilde{\mathcal{D}}}$ must be a singular curve and one of the blow-ups in the composition $\widetilde{\Gamma} \to \widetilde{\mathcal{D}}$ must be along a singular point of $E_{\widetilde{\mathcal{D}}}$. But this implies that exceptional locus F of $\widetilde{\Gamma} \to \widetilde{\mathcal{D}}$ intersects E with multiplicity at least 2 so that $E_{\widetilde{\mathcal{D}}}^2 \geq E^2 + 2$, a contradiction.

We finish the proof as before by observing that both $\mathcal{C}$ and $\mathcal{D}$ are the stable models of $\widetilde{\mathcal{C}} \cong \widetilde{\mathcal{D}}$. Since the stable model is unique (Proposition 5.3.20), there is an isomorphism $\mathcal{C} \xrightarrow{\sim} \mathcal{D}$ extending $\mathcal{C}^* \xrightarrow{\sim} \mathcal{D}^*$.

5.6 Gluing and forgetful morphisms

5.6.1 Gluing morphisms

Proposition 5.6.1. *There are finite morphisms of algebraic stacks*

$$\begin{aligned} \overline{\mathcal{M}}_{i,n} \times \overline{\mathcal{M}}_{g-i,m} &\to \overline{\mathcal{M}}_{g,n+m-2} \\ \big((C, p_1, \ldots, p_n), (C', p'_1, \ldots, p'_m)\big) &\mapsto (C \cup C', p_1, \ldots, p_{n-1}, p'_1, \ldots, p'_m). \end{aligned} \tag{5.6.1}$$

and

$$\begin{aligned} \overline{\mathcal{M}}_{g-1,n} &\to \overline{\mathcal{M}}_{g,n-2} \\ (C, p_1, \ldots, p_n) &\mapsto (C/_{p_{n-1} \sim p_n}, p_1, \ldots, p_{n-2}). \end{aligned} \tag{5.6.2}$$

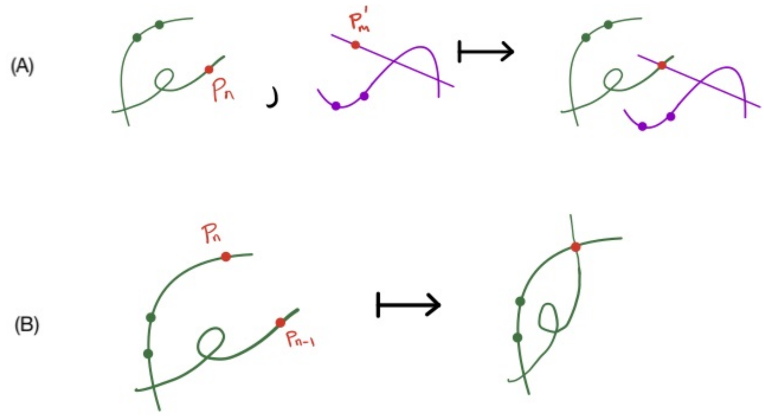


Figure 5.13: (A) is an example of (5.6.1) while (B) is an example of (5.6.2)

Remark 5.6.2. To simplify the notation, we chose to write only the case of gluing the nth marked point p_n and the mth marked point p'_m curve in (5.6.1), and likewise only case of gluing the p_{n-1} and p_n in (5.6.2). Clearly the same holds for the gluing of any two marked points.

Sketch. To simplify the notation, we will establish the proposition in the following two cases:

(a) In (5.6.1), we assume $n = m = 1$.

(b) In (5.6.2), we assume $n = 2$.

Note that once we establish the existence of the morphisms of algebraic stacks, it follows from Stable Reduction (Theorem 5.5.1) that the morphisms are proper. By inspection, they are clearly representable and have finite fibers and thus follows that the morphisms are finite.

Case (a): Let $(\mathcal{C} \xrightarrow{\pi} S, \sigma)$ and $(\mathcal{C}' \xrightarrow{\pi'} S, \sigma')$ be two families of 1-pointed stable curves over a scheme S.

Argument 1 (pushout construction): Consider the pushout diagram

$$\begin{array}{ccc} S & \xrightarrow{\sigma} & \mathcal{C} \\ \big\downarrow{\scriptstyle \sigma'} & & \big\downarrow \\ \mathcal{C}' & \longrightarrow & \widetilde{\mathcal{C}} \end{array}$$

which exists by Ferrand's Theorem on the Existence of Pushouts (Theorem A.8.1). We claim that $\widetilde{\mathcal{C}} \to S$ is a family of stable curves. First, note that $\widetilde{\mathcal{C}} \to S$ is proper as there is a finite cover $\mathcal{C} \sqcup \mathcal{C}' \to \mathcal{C}$ with $\mathcal{C} \sqcup \mathcal{C}'$ proper over S. One can use properties of pushouts to show that $\widetilde{\mathcal{C}} \to S$ is flat (missing details). It remains to chose that the geometric fibers of $\widetilde{\mathcal{C}} \to S$ are stable curves and in particular nodal.

For every point $s \in S$, since $\sigma(s)$ is a smooth point of $\mathcal{C}$, there is an étale neighborhood $\operatorname{Spec} A[x] \to \mathcal{C}$ of $\sigma(s)$ which pulls back to an étale neighborhood $\operatorname{Spec} A \to S$ of s. Since an étale morphism from an affine scheme extend over closed immersions (missing reference), there is an étale neighborhood $\operatorname{Spec} A[y] \to \mathcal{C}'$ of

$\sigma'(s)$ which also pulls back to $\operatorname{Spec} A \to S$. The geometric pushout of $[\operatorname{Spec} A[x] \leftarrow \operatorname{Spec} A \to \operatorname{Spec} A[y]]$ is $\operatorname{Spec} A[x,y]/(xy)$, and we have a commutative cube

$$\begin{tikzcd}
& \operatorname{Spec} A \arrow[rr, hook] \arrow[dl] \arrow[dd] & & \operatorname{Spec} A[y] \arrow[dl] \arrow[dd] \\
\operatorname{Spec} A[x] \arrow[rr, hook] \arrow[dd] & & \operatorname{Spec} A[x,y]/(xy) \arrow[dd] & \\
& S \arrow[rr, hook] \arrow[dl] & & \mathcal{C} \arrow[dl] \\
\mathcal{C}' \arrow[rr, hook] & & \widetilde{\mathcal{C}} &
\end{tikzcd}$$

We see by Proposition A.8.5 that $\operatorname{Spec} A[x,y]/(xy) \to \widetilde{\mathcal{C}}$ is an étale neighborhood of the image of s. This shows that $\widetilde{\mathcal{C}} \to S$ is nodal along S and since $\widetilde{\mathcal{C}}$ is either isomorphic to $\mathcal{C}$ or $\mathcal{C}'$ outside S, we see that $\widetilde{\mathcal{C}} \to S$ is a nodal family of curves. Finally one checks (missing details) that $\widetilde{\mathcal{C}}_s$ is identified with the nodal union $\mathcal{C}_s$ and $\mathcal{C}_{s'}$, which is stable.

*Argument 2 (*Proj *construction):* We know that $\omega_{\mathcal{C}}(\sigma)$ is ample. There is a surjection $\omega_{\mathcal{C}}(\sigma) \to \mathcal{O}_{\sigma_1}$ and for each $k \geq 0$, the pushforward of the surjection $(\omega_{\mathcal{C}}(\sigma))^{\otimes k} \to \mathcal{O}_{\sigma_1}$ under $\pi \colon \mathcal{C} \to S$ is $\pi_*(\omega_{\mathcal{C}}(\sigma)^{\otimes k}) \to \mathcal{O}_S$. We have a similar construction for $\pi' \colon \mathcal{C}' \to S$, and we can consider the fiber product of quasi-coherent $\mathcal{O}_S$-modules

$$\begin{tikzcd}
\mathcal{A}_k \arrow[r] \arrow[d] & \pi_*(\omega_{\mathcal{C}}(\sigma)^{\otimes k}) \arrow[d] \\
\pi_*(\omega_{\mathcal{C}'}(\sigma')^{\otimes k}) \arrow[r] & \mathcal{O}_S
\end{tikzcd}$$

One checks that $\mathcal{A} := \bigoplus_{k \geq 0} \mathcal{A}_k$ is a finitely generated quasi-coherent $\mathcal{O}_S$-algebra and that $\widetilde{\mathcal{C}} := \mathcal{P}\mathrm{roj}_S\, \mathcal{A}$ is a family of stable curves over S such that $\widetilde{\mathcal{C}}_s$ is the nodal union $\mathcal{C}_s$ of $\mathcal{C}_{s'}$.

Case (b): Let $(\mathcal{C} \to S, \sigma_1, \sigma_2)$ be a 2-pointed family of stable curves over a scheme S.

Argument 1 (pushout construction): We use the pushout diagram

$$\begin{tikzcd}
S \sqcup S \arrow[r, "\sigma_1 \sqcup \sigma_2"] \arrow[d] & \mathcal{C} \arrow[d] \\
S \arrow[r] & \widetilde{\mathcal{C}}
\end{tikzcd}$$

By the étale local properties of pushouts (Proposition A.8.5), the local structure of $\widetilde{\mathcal{C}}$ is determined by the pushout diagram

$$\begin{tikzcd}
\operatorname{Spec} A \times A \arrow[r, "{(0,1)}"] \arrow[d] & \operatorname{Spec} A[t] \arrow[d] \\
\operatorname{Spec} A \arrow[r] & \operatorname{Spec} A \times_{A \times A} A[t].
\end{tikzcd}$$

The subalgebra $A \times_{A \times A} A[t] \subset A[t]$ consists of functions $f \in A[t]$ such that $f(0) = f(1) \in A$. The elements $x := t^2 - 1$ and $y := t^3 - t$ generate $A \times_{A \times A} A[t]$ as an A-algebra and since x and y satisfy $y^2 = x^2(x+1)$, we see that $A \times_{A \times A} A[t] \cong A[x,y]/(y^2 - x^2(x+1))$.

Argument 2 (Proj construction): One defines $\widetilde{\mathcal{C}} := \mathcal{P}\mathrm{roj}_S \bigoplus_{k \geq 0} \mathcal{A}_k$ where $\mathcal{A}_k$ is defined as the fiber product

$$\begin{tikzcd}
\mathcal{A}_k \arrow[r] \arrow[d] & \mathcal{O}_S \arrow[d, "\Delta"] \\
\pi_*(\omega_{\mathcal{C}}(\sigma_1)^{\otimes k}) \sqcup \pi_*(\omega_{\mathcal{C}}(\sigma_2)^{\otimes k}) \arrow[r] & \mathcal{O}_S \sqcup \mathcal{O}_S.
\end{tikzcd}$$

□

5.6.2 Boundary divisors of $\overline{\mathcal{M}}_g$

Define the closed substacks

$$\delta_0 = \mathrm{im}(\overline{\mathcal{M}}_{g-1,2} \to \overline{\mathcal{M}}_g)$$
$$\delta_i = \mathrm{im}(\overline{\mathcal{M}}_{i,1} \times \overline{\mathcal{M}}_{g-i,1} \to \overline{\mathcal{M}}_g)$$

where $i = 1, \dots, \lfloor g/2 \rfloor$.

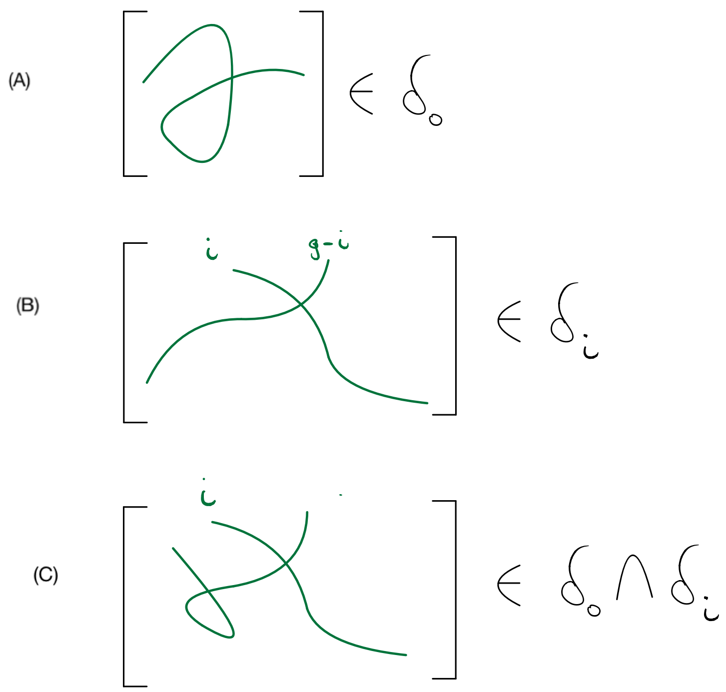


Figure 5.14: Examples of stable curves in the boundary.

Once we show that $\mathscr{M}_g$ is dense in $\overline{\mathcal{M}}_g$, it will follow that δ_0 and δ_i are the closure of the locus of curves with a single node as featured in (A) and (B) of Figure 5.14.

To see that δ_0 and δ_i are divisors in $\overline{\mathcal{M}}_g$, we can do a simple dimension count. As $\overline{\mathcal{M}}_{g-1,2} \to \overline{\mathcal{M}}_g$ and $\overline{\mathcal{M}}_{i,1} \times \overline{\mathcal{M}}_{g-i,1} \to \overline{\mathcal{M}}_g$ are finite morphisms, we compute that $\dim \delta_0 = \dim \overline{\mathcal{M}}_{g-1,2} = 3(g-1) - 3 + 2 = 3g - 4$ and that $\dim \delta_i = \dim \overline{\mathcal{M}}_{i,1} + \dim \overline{\mathcal{M}}_{g-i,1} = (3i - 3 + 1) + (3(g-i) - 3 + 1) = 3g - 4$.

By analyzing the formal deformation space of a stable curve, one can show that more is true: $\delta = \delta_0 \cup \cdots \cup \delta_{\lfloor \frac{g}{2} \rfloor}$ is a normal crossings divisor.

5.6.3 The forgetful morphism

Proposition 5.6.3. *There is a morphism of algebraic stacks*

$$\begin{aligned} \overline{\mathcal{M}}_{g,n} &\to \overline{\mathcal{M}}_{g,n-1} \\ (C, p_1, \ldots, p_n)) &\mapsto (C^{\mathrm{st}}, p_1, \ldots, p_{n-1}). \end{aligned}$$

where $(C^{\mathrm{st}}, p_1, \ldots, p_{n-1})$ *is the stable model of* $(C, p_1, \ldots, p_{n-1})$.

(A)

(B)

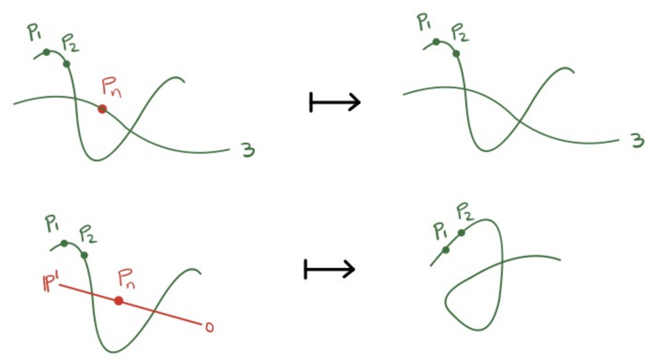


Figure 5.15: In (A), the nth point is simply forgotten. In (B), if p_n is forgotten, then the curve is no longer stable and we must contract the rational bridge.

Proof. If $(\mathcal{C} \to S, \sigma_1, \ldots, \sigma_n)$ is an n-pointed family of stable curves, then if we forget the nth section, the $(n-1)$-pointed family $(\mathcal{C} \to S, \sigma_1, \ldots, \sigma_{n-1})$ may not be stable. However, we have already constructed the stable model $(\mathcal{C}^{\mathrm{st}} \to S, \sigma_1, \ldots, \sigma_{n-1})$ in Proposition 5.3.20. $\square$

5.6.4 The universal family $\overline{\mathcal{M}}_{g,1} \to \overline{\mathcal{M}}_g$

Let $\mathcal{U}_g \to \overline{\mathcal{M}}_g$ be the universal family: this is a proper and flat morphism of algebraic stacks whose geometric fibers are genus g curves. (The existence of the universal family follows from applying descent and the 2-Yoneda Lemma (Lemma 2.3.20) to the identity morphism $\mathrm{id}\colon \overline{\mathcal{M}}_g \to \overline{\mathcal{M}}_g$.) Objects of $\mathcal{U}_g$ over a scheme S correspond to a family of stable curves $\mathcal{C} \to S$ and a section $\sigma\colon S \to \mathcal{C}$ (that may land in the relative singular locus).

There is a morphism of algebraic stacks

$$\overline{\mathcal{M}}_{g,1} \to \mathcal{U}_g$$

sending $(\mathcal{C} \to S, \sigma)$ to $(\mathcal{C}^{\mathrm{st}} \to S, \sigma^{\mathrm{st}})$ where $\pi\colon \mathcal{C} \to \mathcal{C}^{\mathrm{st}}$ is the stabilization of $\mathcal{C} \to S$ (see Proposition 5.3.20) and $\sigma^{\mathrm{st}} = \pi \circ \sigma$. This yields a commutative diagram

$$\begin{tikzcd} \overline{\mathcal{M}}_{g,1} \arrow[r] \arrow[dr] & \mathcal{U}_g \arrow[d] \\ & \overline{\mathcal{M}}_g, \end{tikzcd}$$

where $\overline{\mathcal{M}}_{g,1} \to \overline{\mathcal{M}}_g$ is the forgetful morphism of Proposition 5.6.3.

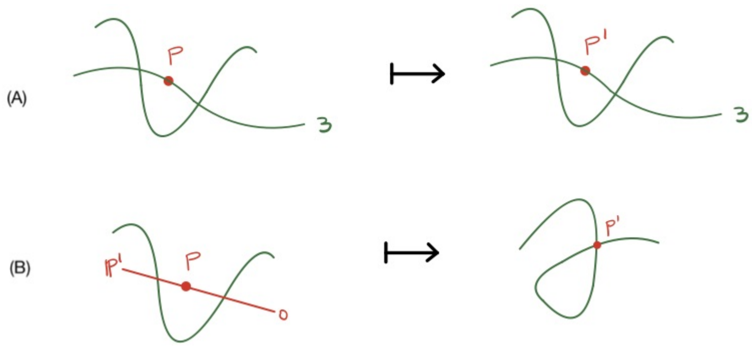


Figure 5.16: In Example (A), $\overline{\mathcal{M}}_{g,1} \to \mathcal{U}_g$ sends (C,p) to itself while in Example (B), the morphism sends (C,p) to the curve (C',p') obtained by contracting the rational bridge.

Proposition 5.6.4. *The morphism $\overline{\mathcal{M}}_{g,1} \to \mathcal{U}_g$ is an isomorphism over $\overline{\mathcal{M}}_g$. In other words, the morphism $\overline{\mathcal{M}}_{g,1} \to \overline{\mathcal{M}}_g$, which forgets the marked point and stabilizes the curve, is the universal family.*

Proof. TO ADD $\square$

Exercise 5.6.5. Show that the above arguments can be modified to show that $\overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ is a universal family.

5.7 Irreducibility

In this section, we show that the algebraic stack $\overline{\mathcal{M}}_{g,n}$ is irreducible over an algebraically closed field $\Bbbk$. After reviewing properties of branched coverings in §5.7.1, we provide the classical topological argument due to Clebsch and Hurwitz in the late 19th century establishing irreducibility of M_g in characteristic 0 (Theorem 5.7.12). We then provide a purely algebraic argument for the irreducibility of $\overline{\mathcal{M}}_{g,n}$ (Theorem 5.7.15) by using admissible covers to show that every smooth curves degenerates to a singular stable curve and induction on the genus to show that the boundary $\delta = \overline{\mathcal{M}}_g \setminus \mathcal{M}_g$ is connected. Finally, in §5.7.4, we provide the arguments from the seminal papers from 1969 of Deligne and Mumford [DM69] and Fulton [Ful69] which establish the irreducibility of $\overline{M}_{g,n}$ in positive characteristic (where Fulton's argument has the restriction $p > g+1$) by reduction to characteristic 0.

We begin with a few remarks regarding the relations between the connectedness/irreducibility of $\overline{\mathcal{M}}_{g,n}$, $\mathcal{M}_g$, and their coarse moduli spaces. Since $\overline{\mathcal{M}}_{g,n}$ is a smooth algebraic stack, its irreducibility is equivalent to its connectedness. Moreover, since $\overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ is the universal family (Proposition 5.6.4) and in particular has connected fibers, it suffices to verify the connectedness of $\overline{\mathcal{M}}_g$. We thus have equivalences

$$\begin{aligned} \overline{\mathcal{M}}_{g,n} \text{ irreducible } &\iff \overline{\mathcal{M}}_{g,n} \text{ connected} \\ &\iff \overline{\mathcal{M}}_g \text{ connected} \\ &\iff \mathcal{M}_g \text{ connected and dense in } \overline{\mathcal{M}}_g \end{aligned}$$

Finally, we note that since the coarse moduli space $\overline{\mathcal{M}}_{g,n} \to \overline{M}_{g,n}$ induces a homeomorphism $|\overline{\mathcal{M}}_{g,n}| \xrightarrow{\sim} |\overline{M}_{g,n}|$ on topological spaces, each statement above can be equivalently stated in terms of the coarse moduli space.

5.7.1 Branched coverings

If $f\colon C \to D$ is a finite separable morphism of smooth connected curves and $P \in C(\Bbbk)$ with image Q, then the *ramification index* e_P is the integer e such that under the map $\mathcal{O}_{D,Q} \to \mathcal{O}_{C,P}$ a uniformizer $t \mapsto us^e$ maps to a unit times the eth power of a uniformizer. We say that f is *ramified at* P if $e_P > 1$, *tamely ramified at* P if $\operatorname{char} k = 0$ or $\operatorname{char}(k) \nmid e_p$, and *unramified at* P if $e_P = 1$.

There is a short exact sequence on differentials

$$0 \to f^*\Omega_D \to \Omega_C \to \Omega_{C/D} \to 0. \tag{5.7.1}$$

Indeed, the sequence above is always right exact. Since $f^*\Omega_D$ and Ω_C are line bundles, injectivity for the left map is equivalent to the map being non-zero. However, $K(D) \to K(C)$ is separable so $\Omega_{C/D} \otimes K(C) = \Omega_{K(C)/K(D)} = 0$, and thus $f^*\Omega_D \to \Omega_C$ is non-zero at the generic point. Examining the sequence above at the stalks at a point $P \in C(\Bbbk)$, the differential dt maps to $d(us^{e_P}) = eus^{e_P-1}ds + s^{e_P}du$. If f is tamely ramified at p, then $(\Omega_{C/D})_P \cong \mathcal{O}_{C,p}\langle ds\rangle/(s^{e_P-1}ds)$ and $\operatorname{length}(\Omega_{C/D})_P = \dim \Omega_{C/D} \otimes \kappa(p) = e_P - 1$.

If f is ramified at P, then the scheme-theoretic fiber over $f(P)$ at P is isomorphic to $\operatorname{Spec} \kappa(P)$, and thus this agrees with the definition of unramified in Unramified Equivalences A.3.4. Moreover, since f is flat, f is unramified at P if and only if f is étale at P.

Definition 5.7.1. Let $\Bbbk$ be an algebraically closed field.

(1) A *branched covering* is a finite separable morphism $f\colon C \to D$ of smooth connected curves over $\Bbbk$.

(2) A *simply branched covering* is a branched covering such that

- there is at most one ramification point in every fiber, and
- every ramification point $P \in C(\Bbbk)$ is tamely ramified with ramification index $e_P = 2$.

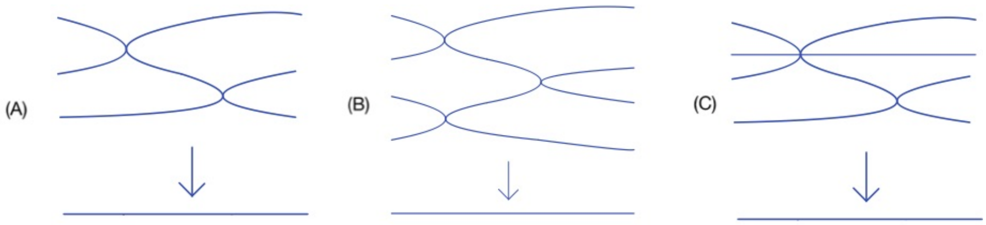


Figure 5.17: Examples of branched coverings over $\mathbb{P}^1$: (A) is simply branched while (B) and (C) are not. While the picture may suggest that the source curve C is not smooth, C is in fact smooth over the base field $\Bbbk$. However, the map $C \to \mathbb{P}^1$ is not smooth and the pictures above are designed to reflect the singularities of C over $\mathbb{P}^1$.

Theorem 5.7.2 (Riemann–Hurwtiz)**.** *If $f\colon C \to D$ is a branched covering and $R = \sum_{P \in C(\Bbbk)} \operatorname{length}(\Omega_{C/D})_P \cdot P$ is the ramification divisor on C, then $\Omega_C \cong f^*\Omega_D \otimes \mathcal{O}_C(R)$ and*

$$2g(C) - 2 = \deg(f)(2g(D) - 2) + \deg R.$$

In particular, $f\colon C \to \mathbb{P}^1$ is simply branched, then it is ramified over $2g + 2d - 2$ distinct points.

Proof. This follows directly from the exact sequence (5.7.1). See also [Har77, Prop. IV.2.3] □

Example 5.7.3. For a local model of a branched cover, consider the map $f\colon \mathbb{A}^1 \to \mathbb{A}^1$ defined by $x \mapsto x^n$. The relative sheaf of differentials is $\Omega_{\mathbb{A}^1/\mathbb{A}^1} = k[x]\langle dx\rangle/(nx^{n-1}dx)$ and thus if $\operatorname{char}(\Bbbk)$ does not divide n, then $f\colon \mathbb{A}^1 \to \mathbb{A}^1$ is étale over $\mathbb{A}^1$ and ramified at 0 with index $n-1$.

Exercise 5.7.4. Show that every branched covering is étale locally isomorphic to $\mathbb{A}^1 \to \mathbb{A}^1, x \mapsto x^n$ around a branched point of index $n-1$.

Lemma 5.7.5. *Let C be a smooth, connected, and projective curve of genus g over an algebraically closed field $\Bbbk$ of characteristic 0. If L is a line bundle of degree $d \geq g+1$, then for a general linear series $V \subset \mathrm{H}^0(C, L)$ of dimension 2, $C \xrightarrow{V} \mathbb{P}^1$ is simply branched.*

Proof. We proceed with a dimension count. Since $\mathrm{h}^0(C, L) = d + 1 - g$, the dimension of the Grassmanian $\mathrm{Gr}(2, \mathrm{H}^0(L))$ of 2-dimensional subspaces is $2(d - g - 1)$. Since $\operatorname{char}(\Bbbk) = 0$, every finite morphism $C \to \mathbb{P}^1$ is automatically separable. Thus, if $C \xrightarrow{V} \mathbb{P}^1$ is not simply branched, then one of the following three conditions must hold:

(a) V has a base point;

(b) there exists a ramification point with index > 2; or

(c) there exists 2 ramification points in the same fiber.

We handle only case (b) and leave the other cases to the reader. There must exist a section $s \in V$ vanishing to order 3 at a point $p \in C$, i.e. $s \in \mathrm{H}^0(C, L(-3p))$.

The dimension of $V \in \mathrm{Gr}(2, \mathrm{H}^0(L))$ having a branched point at $p \in C$ with index at least 3 can be calculated as

$$\dim \mathbb{P}\mathrm{H}^0(L(-3p)) + \dim \mathbb{P}(\mathrm{H}^0(L)/\langle s\rangle) = 2d - 2g - 4.$$

Varying $p \in C$, the locus of all $V \in \mathrm{Gr}(2, \mathrm{H}^0(L))$ failing condition (b) is thus $2d - 2g - 3 = \dim \mathrm{Gr}(2, \mathrm{H}^0(L)) - 1$. □

For a branched cover $C \to \mathbb{P}^1$, we denote by $\mathrm{Aut}(C/\mathbb{P}^1) = 1$ the group of automorphisms $C \to C$ over $\mathbb{P}^1$.

Lemma 5.7.6. *If $C \to \mathbb{P}^1$ is a simply branched cover of degree $d > 2$ in characteristic 0, then $\mathrm{Aut}(C/\mathbb{P}^1)$ is trivial.*

Proof. Every automorphism $C \to C$ over $\mathbb{P}^1$ must fix the $2g + 2d - 2$ branched points but this contradicts the classical result of Mayer which asserts that there are no non-trivial automorphisms of a smooth curve fixing more than $2g + 2$ points. □

The above lemma shows that there are no stacky issues that arises when defining moduli spaces of simply branched covering. We define

$$H_{d,b} := \{C \to \mathbb{P}^1 \text{ simply branched covering of degree } d \text{ over } b \text{ points}\}$$

as the moduli space of simply branched coverings where

$$b = 2g + 2d - 2.$$

The moduli space $H_{d,b}$ can be defined either as a topological space (if $\Bbbk = \mathbb{C}$) or as an algebraic space; we leave the details to the reader. Denoting $\mathrm{Sym}^b \mathbb{P}^1 \setminus \Delta$ as the variety of b unordered distinct points in $\mathbb{P}^1$ (which can also be written as the complement $\mathbb{P}^b \setminus \Delta$ of the discriminant hypersurface), we have a diagram

$$\begin{tikzcd} & H_{d,b} \arrow[dl] \arrow[dr] & \\ \mathcal{M}_g & & \mathrm{Sym}^b \mathbb{P}^1 \setminus \Delta \end{tikzcd} \tag{5.7.2}$$

where a simply branched covering $[C \to \mathbb{P}^1]$ gets mapped to $[C]$ under $H_{d,b} \to \mathcal{M}_g$ and the b branched points under $H_{d,b} \to \mathrm{Sym}^b \mathbb{P}^1 \setminus \Delta$.

Lemma 5.7.7. *In characteristic 0, the morphism $H_{d,b} \to \mathrm{Sym}^b \mathbb{P}^1 \setminus \Delta$ is finite and étale.*

Proof. We only establish étaleness. It is straightforward to see that $H_{d,b} \to \mathrm{Sym}^b \mathbb{P}^1 \setminus \Delta$ is a topological covering space. Consulting Figure 5.18, given a branched covering $f\colon C \to \mathbb{P}^1$ and a branched point $p \in C$, we can choose an analytic open neighborhood $U \subset \mathbb{P}^1$ around $f(p)$ such that $f^{-1}(U) \to U$ is isomorphic to an open neighborhood of $\mathbb{C} \to \mathbb{C}, x \mapsto x^n$. For every other point $q' \in U$, we can construct a branched cover $C' \to \mathbb{P}^1$ which outside U is the same as $C \to \mathbb{P}^1$ and over U is locally isomorphic to $x \mapsto x^n$ but centered over q' (rather than $f(p)$).

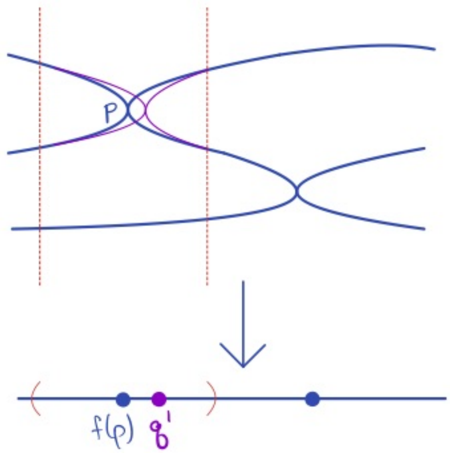


Figure 5.18:

For an algebraic argument, it suffices to show that for a covering $f\colon C \to \mathbb{P}^1$ simply branched over $p_1, \ldots, p_b$, the map

$$\operatorname{Def}(C \xrightarrow{f} \mathbb{P}^1) \to \operatorname{Def}(\{p_i\}_{i=1}^b \subset \mathbb{P}^1)$$

on first order deformation spaces is bijective. There is an identification $\operatorname{Def}(C \xrightarrow{f} \mathbb{P}^1) = \mathrm{H}^0(C, N_f)$ where N_f sits in a short exact sequence

$$0 \to T_C \to f^*T_{\mathbb{P}^1} \to N_f \to 0.$$

On cohomology, this induces a short exact sequence

$$0 \to \mathrm{H}^0(C, f^*T_{\mathbb{P}^1}) \to \mathrm{H}^0(C, N_f) \to \mathrm{H}^1(C, T_C) \to 0.$$

Riemann–Roch allows us to compute $\mathrm{h}^0(C, f^*T_{\mathbb{P}^1}) = 2d+1-g$ and $\mathrm{h}^1(T_C) = 3g-3$, and thus $\dim \operatorname{Def}(C \xrightarrow{f} \mathbb{P}^1) = \mathrm{h}^0(C, N_f) = 2d + 2g - 2 = b$ is the same as the dimension of $\operatorname{Def}(\{p_i\}_{i=1}^b \subset \mathbb{P}^1)$. We leave the remaining details to the reader. $\square$

5.7.1.1 Relation between algebraic and topological branched coverings

The Clebsch–Hurwitz argument below relies on the following correspondence between topological, analytic, and algebraic branched coverings. (Topological and analytic coverings can be defined analogously to algebraic coverings—to be added.) This can be viewed as a version of the Riemann Existence Theorem.

Proposition 5.7.8. *Over $\mathbb{C}$, there are natural bijections*

$$\{C \to \mathbb{P}^1 \text{ algebraic branched coverings}\} \longleftrightarrow \{C \to \mathbb{P}^1 \text{ topological branched coverings}\}$$
$$\longleftrightarrow \{C \to \mathbb{P}^1 \text{ analytic branched coverings}\}$$

Proof. An algebraic branched covering is clearly topological and if $C \to \mathbb{P}^1$ is a topological covering, then the holomorphic structure on $\mathbb{P}^1$ induces naturally a holomorphic structure on C such that $C \to \mathbb{P}^1$ is analytic. The Riemann Existence Theorem implies than every holomorphic branched covering is in fact algebraic. $\square$

5.7.1.2 Monodromy actions

Let $C \to \mathbb{P}^1$ be a (topological) branched covering over $\mathbb{C}$ and $B \subset \mathbb{P}^1$ its ramification locus. Choose a base point $p \in \mathbb{P}^1 \setminus B$. The *monodromy action of* $\pi_1(\mathbb{P}^1 \setminus B, p)$ *on the fiber* $\pi^{-1}(p)$ is defined as follows: for $\gamma \in \pi_1(\mathbb{P}^1 \setminus B, p)$ and $q \in \pi^{-1}(p)$, then the path $\gamma\colon [0,1] \to \mathbb{P}^1$ lifts uniquely to a path $\widetilde{\gamma}\colon [0,1] \to C$ such that $\widetilde{\gamma}(0) = q$ and the action is defined by $\gamma \cdot q = \widetilde{\gamma}(1)$.

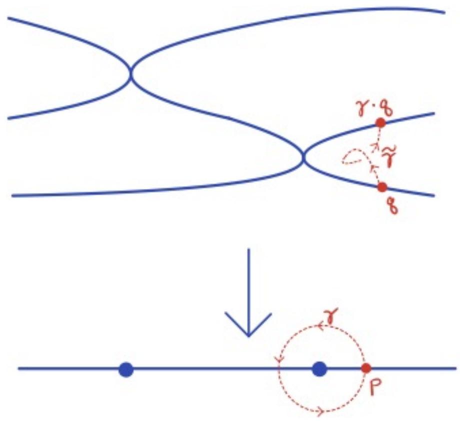


Figure 5.19:

We now summarize some of the key properties of the monodromy action.

Proposition 5.7.9. *Let $B \subset \mathbb{P}^1$ be a finite subset, $p \in \mathbb{P}^1 \setminus B$ be a point, and $d > 0$ a positive integer. There is a natural bijection between topological branched coverings $C \to \mathbb{P}^1$ of degree d and group homomorphisms $\rho\colon \pi_1(X \setminus B, x) \to S_d$ such that $\mathrm{im}(\rho) \subset S_d$ is a transitive subgroup. Here two branched covers $C \to \mathbb{P}^1$ and $C' \to \mathbb{P}^1$ are equivalent if there is an isomorphism $C \to C'$ over $\mathbb{P}^1$, and two homomorphisms $\rho, \rho'\colon \pi_1(X \setminus B, x) \to S_d$ are equivalent if they differ by an inner automorphism of S_d, i.e. $\exists h \in S_d$ such that $\rho' = \mathrm{h}^{-1}\rho h$.*

Moreover, if we let $\sigma_1, \ldots, \sigma_b$ be simple loops around the b distinct points of B, then $\pi_1(\mathbb{P}^1 \setminus B, x) = \langle \sigma_i | \sigma_1 \cdots \sigma_b = 1\rangle$, and under this correspondence a simply branched cover corresponds to a homomorphism $\pi_1(X \setminus B, p) \to S_d$ such that each σ_i maps to a transposition.

Remark 5.7.10. Recall that by definition in a branched covering $C \to \mathbb{P}^1$, the curve C is necessarily connected. This is the reason for the condition above that $\mathrm{im}(\rho) \subset S_d$ is transitive: every group homomorphism $\rho\colon \pi_1(X \setminus B, x) \to S_d$ corresponds to a possibly non-connected branched covering $C \to \mathbb{P}^1$, and C is connected if and only if $\mathrm{im}(\rho) \subset S_d$ is transitive.

Remark 5.7.11. Like with Riemann–Hurwitz, the fact that the base is $\mathbb{P}^1$ plays no role: the above proposition holds for arbitrary branched covers of smooth curves (except for the explicit description of π_1).

5.7.2 The Clebsch–Hurwitz argument

We now provide the classical argument due to Clebsch [Cle73] and Hurwitz [Hur91] that $\mathcal{M}_g$ is connected over $\mathbb{C}$. For a modern treatment, see [Ful69, §1].

This argument uses a single non-algebraic input, namely Riemann's Existence Theorem in the form of Proposition 5.7.8. There are of course other non-algebraic approaches, e.g. using Teichmüller theory.

By taking $d \geq g+1$, we know that every smooth, connected, and projective complex curve C of genus g admits a map $C \to \mathbb{P}^1$ which is a covering simply branched over $b = 2d+2g-2$ points (Lemma 5.7.5). This shows that the map

$$H_{d,b} \to \mathcal{M}_g, \qquad [C \to \mathbb{P}^1] \mapsto [C]$$

is surjective, where $H_{d,b}$ is the moduli space of coverings $C \to \mathbb{P}^1$ simply branched over b points. The connectedness of $H_{d,b}$ thus implies the connectedness of $\mathcal{M}_g$.

Theorem 5.7.12 (Clebsch, Hurwitz)**.** *$H_{d,b}$ is connected.*

Proof. We will use the diagram

$$\begin{tikzcd} & H_{d,b} \arrow[dl] \arrow[dr, "\beta"] & \\ \mathcal{M}_g & & \operatorname{Sym}^b \mathbb{P}^1 \setminus \Delta \end{tikzcd}$$

where $H_{d,b} \to \mathcal{M}_g$ is surjective (Lemma 5.7.5) and $\beta\colon H_{d,b} \to \operatorname{Sym}^b \mathbb{P}^1 \setminus \Delta$ is finite and étale (Lemma 5.7.7).

For every finite set $B = \{p_1, \ldots, p_b\} \subset \mathbb{P}^1$ of $b = 2d+2g-2$ points and $p \in \mathbb{P}^1 \setminus B$, the fundamental group $\pi_1(\mathbb{P}^1 \setminus B, p) = \langle \sigma_i | \sigma_1 \cdots \sigma_b = 1\rangle$ acts on the fiber $\pi^{-1}(p)$ of a simply branched covering $\pi\colon C \to \mathbb{P}^1$. Similarly, $\pi_1(\operatorname{Sym}^b \mathbb{P}^1 \setminus \Delta, B)$ acts on the fiber $H_{d,B} := \beta^{-1}(B)$ of $\beta\colon H_{d,b} \to \operatorname{Sym}^b \mathbb{P}^1 \setminus \Delta$. Using Proposition 5.7.9, we have bijections

$$\begin{aligned} H_{d,b} = \beta^{-1}(B) &= \{\text{coverings } C \to \mathbb{P}^1 \text{ simply branched over } B\} \\ &= \{\text{group homomorphisms } \pi_1(\mathbb{P}^1 \setminus B, p) \xrightarrow{\rho} S_d \text{ such that} \\ &\qquad \operatorname{im}(\rho) \subset S_d \text{ is transitive and each } \rho(\sigma_i) \text{ is a transposition}\} \\ &= \{(\tau_1, \ldots, \tau_b) \in (S_d)^b \mid \text{each } \tau_i \text{ is a transposition and } \tau_1 \cdots \tau_b = 1\}. \end{aligned}$$

The connectedness of $H_{d,b}$ is equivalent to the transitivity of the action of $\pi_1(\operatorname{Sym}^b \mathbb{P}^1 \setminus \Delta, B)$ on the fiber $H_{d,B}$. The strategy of proof is to find loops in $\operatorname{Sym}^b \mathbb{P}^1 \setminus \Delta$ that act on $(\tau_1, \ldots, \tau_b) \in H_{d,B}$ in a prescribed way and to find enough loops so that we can show that each orbit contains the element

$$\boldsymbol{\tau}^* := \big(\underbrace{(12),(12),(13),(13),\ldots,(1\,d-1),(1\,d-1)}_{2(d-2)}, \underbrace{(1d),(1d),\cdots,(1d)}_{2g+2}\big).$$

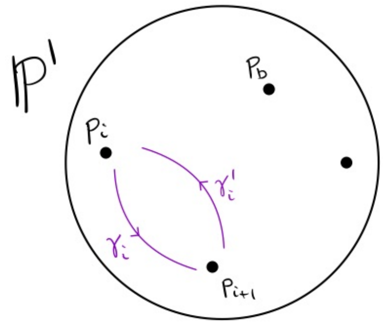


Figure 5.20:

Referring to Figure 5.20, we define the loop

$$\begin{aligned}\Gamma_i \colon [0,1] &\to \operatorname{Sym}^b \mathbb{P}^1 \setminus \Delta \\ t &\mapsto (p_1, \ldots, p_{i-1}, \gamma_i(t), \gamma_i'(t), p_{i+2}, \ldots, p_b).\end{aligned}$$

One checks that

$$\Gamma_i \cdot (\tau_1, \ldots, \tau_b) = (\tau_1, \ldots, \tau_{i-1}, \tau_i^{-1}\tau_{i+1}\tau_i, \tau_i, \tau_{i+2}, \ldots, \tau_b)$$

and that for every element $(\tau_1, \ldots, \tau_b) \in H_{d,B}$, there exists a sequence $\Gamma_{i_1}, \ldots, \Gamma_{i_k}$ of loops such that $\boldsymbol{\tau}^* = \Gamma_{i_1}\Gamma_{i_2}\cdots\Gamma_{i_k} \cdot (\tau_1, \ldots, \tau_b)$. We leave the details of this combinatorial problem to the reader.

□

5.7.3 Irreducibility using admissible covers

We now give a completely algebraic argument of the irreducibility of $\overline{\mathcal{M}}_g$ in characteristic 0. The main idea is to show that every smooth curve of genus g degenerates in a one-dimensional family to a singular stable curve (Proposition 5.7.13) and to show the connectedness of $\delta = \overline{\mathcal{M}}_g \setminus \mathcal{M}_g$ using the inductive structure of the boundary and explicitly the gluing maps of Proposition 5.6.1. The most challenging aspect of this argument is in degenerating a smooth curve to a singular stable curve. To achieve this, we will use the theory of admissible covers. We follow the treatment in Fulton's appendix of the paper [HM82] by Harris and Mumford that introduced admissible covers as a means to compute the Kodaira dimension of $\overline{M}_g$.

Proposition 5.7.13. *Let C be a smooth, connected, and projective curve of genus g over an algebraically closed field $\Bbbk$ of characteristic 0. There exists a connected curve T with points $t_1, t_2 \in T$ and a family $\mathcal{C} \to T$ of stable curves such that $\mathcal{C}_{t_1} \cong C$ and $\mathcal{C}_{t_2}$ is a singular stable curve.*

Proof. By Lemma 5.7.5, for $d \gg 0$ there exists a finite covering $C \to \mathbb{P}^1$ of degree d simply branched over $b = 2g + 2d - 2$ distinct points $p_1, \ldots, p_b \in \mathbb{P}^1$. This defines a b-pointed stable curve $G = [\mathbb{P}^1, \{p_i\}] \in M_{0,n}$. By Lemma 5.7.7, we may assume that $G \in M_{0,n}$ is general. Since $\overline{M}_{0,n}$ is connected, G degenerates to the b-pointed rational curve $(D_0, q_1, \ldots, q_b)$ which is the nodal union of a chain of $b-2$ $\mathbb{P}^1$'s where q_1, q_2 lie on the first $\mathbb{P}^1$, q_3 on the second $\mathbb{P}^1$, and so on with q_{b-1}, q_b lying on the last $\mathbb{P}^1$; see Figure 5.21.

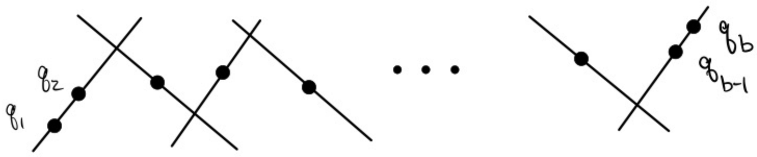


Figure 5.21:

In other words, there is a DVR R with fraction field K and a map $\Delta = \operatorname{Spec} R \to \overline{M}_{0,n}$ corresponding to a b-pointed stable family $(\mathcal{D} \to \Delta, \sigma_i)$ such that the generic fiber $(\mathcal{D}^*, \sigma_i^*)$ is isomorphic to $G = (\mathbb{P}^1, \{g_i\})$ and the special fiber to $(D_0, \{q_i\})$. We have a simply branched covering $\mathcal{C}^* \to \Delta^*$ which fits into a diagram

$$\begin{tikzcd}
\mathcal{C}^* \arrow[r, hook, dashed] \arrow[d] & \mathcal{C} \arrow[d, dashed] \\
\mathcal{D}^* \arrow[r, hook] \arrow[d] & \mathcal{D} \arrow[d] \\
\Delta^* \arrow[r, hook] & \Delta
\end{tikzcd}$$

and extends to a finite morphism $\mathcal{C} \to \mathcal{D}$ by taking $\mathcal{C}$ as the integral closure of $\mathcal{O}_{\mathcal{D}}$ in $K(\mathcal{C}^*)$.

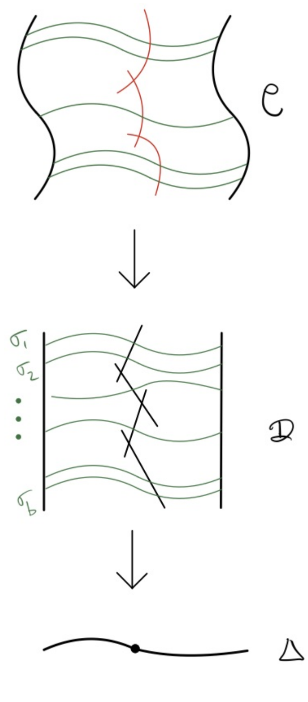


Figure 5.22:

Purity of the branch locus implies that the ramification of $\mathcal{C} \to \mathcal{D}$ is a divisor when restricted to the relative smooth locus of $\mathcal{C} \to \mathcal{D}$. Therefore, the central fiber

$\mathcal{C}_0 \to \mathcal{D}_0$ is ramified over $\sigma_1(0), \ldots, \sigma_b(0)$ and possibly over irreducible components of $\mathcal{D}_0$ (where $\mathcal{C}_0$ may be non-reduced). As in the proof of stable reduction, after a suitable base change $\Delta \to \Delta, t \mapsto t^m$ and replacing $\mathcal{C}$ with the normalization $\mathcal{C} \times_\Delta \Delta$, we can arrange that $\mathcal{C}_0 \to \mathcal{D}_0$ is ramified only over $\sigma_i(0)$ and possibly over nodes of $\mathcal{D}_0$. By an analysis of possible extensions $\mathcal{C} \to \mathcal{D}$, one can show that $\mathcal{C}_0$ is a nodal curve (missing details). Therefore $\mathcal{C} \to \Delta$ is a family of nodal curves.

Since $\mathcal{C}_0$ necessarily has nodes, we are done if $\mathcal{C}_0$ is a stable curve! Otherwise, we can contract rational tails and bridges to obtain the stable model $\mathcal{C}^{\mathrm{st}} \to \Delta$ (Proposition 5.3.20). We must check that $\mathcal{C}_0^{\mathrm{st}}$ is not smooth. Let $T \subset \mathcal{C}_0^{\mathrm{st}}$ be a smooth irreducible component. Applying Riemann–Hurwitz to the induced morphism $T \to \mathbb{P}^1 \subset \mathcal{D}_0$ shows that $2g(T) - 2 = -2d + R$ where R is the degree of the ramification divisor on T. If the component $\mathbb{P}^1 \subset \mathcal{D}_0$ is a rational tail (i.e. is either the first or last $\mathbb{P}^1$ in the chain), then $R \leq 2 + (d-1)$ as $T \to \mathbb{P}^1$ is simply ramified over the two marked points and has index at worst $d-1$ over the node. On the other hand, if $\mathbb{P}^1 \subset \mathcal{D}_0$ is a rational bridge, then $R \leq 1 + 2(d-1)$. In either case, we have $R \leq 2d - 1$ and $2g(T) - 2 \leq -2 + (2d-1) = 1$ which establishes that $g(T) = 0$. We've shown every smooth irreducible component of $\mathcal{C}_0^{\mathrm{st}}$ is rational which immediately implies that $\mathcal{C}_0^{\mathrm{st}}$ is singular. □

Proposition 5.7.14. *If we assume that $\overline{\mathcal{M}}_{g',n'}$ is irreducible for all $g' < g$, then the boundary $\delta = \overline{\mathcal{M}}_g \setminus \mathcal{M}_g$ is connected.*

Proof. We write $\delta = \delta_0 \cup \cdots \cup \delta_{\lfloor g/2 \rfloor}$ where $\delta_0 = \mathrm{im}(\overline{\mathcal{M}}_{g-1,2} \to \overline{\mathcal{M}}_g)$ and $\delta_i = \mathrm{im}(\overline{\mathcal{M}}_{i,1} \times \overline{\mathcal{M}}_{g-i,1} \to \overline{\mathcal{M}}_g)$ as defined in §5.6.2 using the gluing maps from Proposition 5.6.1. The hypotheses imply that δ_0 and δ_i are connected (and even irreducible). But on the other hand, the boundary divisors δ_i intersect! Namely, for every $i, j = 0, \ldots, \lfloor g/2 \rfloor$, the intersection $\delta_i \cap \delta_j$ contains curves as in Figure 5.23.

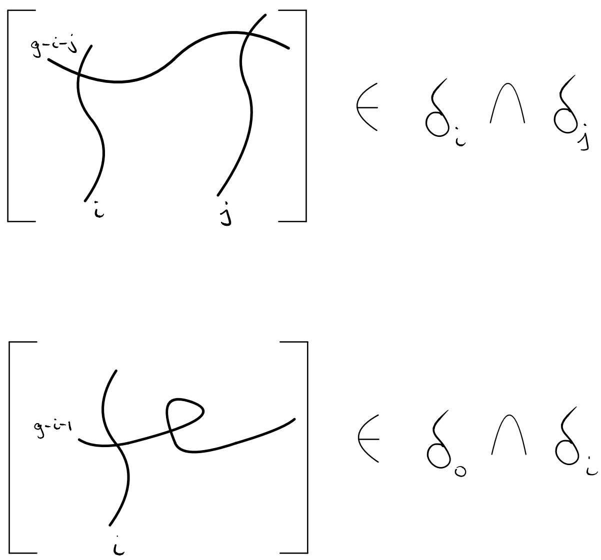


Figure 5.23:

□

Theorem 5.7.15. *$\overline{\mathcal{M}}_{g,n}$ is irreducible.*

Proof. Since $\overline{\mathcal{M}}_{g,n}$ is smooth (Theorem 5.4.8), the irreducibility of $\overline{\mathcal{M}}_{g,n}$ is equivalent to its connectedness. Since $\overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ is the universal family (Proposition 5.6.4) and in particular has connected fibers, it suffices to verify the connectedness of $\overline{\mathcal{M}}_g$. Since every smooth curve degenerates to a stable singular curve in the boundary $\delta = \overline{\mathcal{M}}_g \setminus \mathcal{M}_g$ (Proposition 5.7.13) and the boundary δ itself is connected (Proposition 5.7.14) by induction on g, we obtain that $\overline{\mathcal{M}}_g$ is connected. $\square$

Remark 5.7.16 (Admissible Covers)**.** The above argument was motivated by the theory of admissible covers as introduced by Harris and Mumford [HM82]. Admissible covers are a generalization of simply branched covers $C \to \mathbb{P}^1$ where the source *and* target curve are allowed to have nodal singularities. The main goal is to extend the map $\mathcal{H}_{d,b} \to \mathcal{M}_g$ taking $[C \to \mathbb{P}^1] \to [C]$ to a map $\overline{\mathcal{H}}_{d,b} \to \overline{\mathcal{M}}_g$ over the boundary where $\overline{\mathcal{H}}_{d,b}$ also has a moduli interpretation.

An *admissible cover of degree d* over a stable b-pointed genus 0 curve $(B, p_1, \ldots, p_b)$ is a morphism $f\colon C \to B$ such that

(a) $f^{-1}(B^{\mathrm{sm}}) = C^{\mathrm{sm}}$ and $C^{\mathrm{sm}} \to B^{\mathrm{sm}}$ is simply branched of degree d over the points p_i, i.e. each ramification index is 2 and there is at most one ramification point in every fiber; and

(b) for every node $q \in B$ and every node $r \in C$ over q, the local structure (either formally or étale) of $C \to B$ at r is of the form $\Bbbk[x,y]/(xy) \to \Bbbk[x,y]/(xy)$ defined by $(x,y) \mapsto (x^m, y^m)$ for some m.

This definition extends to families of admissible covers and the stack $\overline{\mathcal{H}}_{d,b}$ parameterizing admissible covers of degree d branched over b points is a proper Deligne–Mumford stack.

The total space C of an admissible cover need not be stable. Nevertheless, using the contraction morphism (Proposition 5.3.20), there is a morphism $\overline{\mathcal{H}}_{d,b} \to \overline{\mathcal{M}}_g$ sending an admissible cover $[C \to B]$ to the stable model C^{st} of C. There is also a *finite* morphism $\overline{\mathcal{H}}_{d,b} \to \overline{M}_{0,n}$ sending $[C \to B]$ to $(B, \{p_i\})$ where $p_i \in B$ are the branched points. To summarize, there is a diagram

$$\begin{tikzcd} & \overline{\mathcal{H}}_{d,b} \arrow[dl] \arrow[dr] & \\ \overline{\mathcal{M}}_g & & \overline{M}_{0,n} \end{tikzcd}$$

extending the uncompactified diagram (5.7.2).

The argument of Proposition 5.7.13 can be rewritten in this language. For $d \gg 0$, given a smooth curve $[C] \in \mathcal{M}_g$, we choose a preimage $[C \to \mathbb{P}^1] \in \overline{\mathcal{H}}_{d,b}$ (Lemma 5.7.5). By Lemma 5.7.7, we can assume that the branched points $g_1, \ldots, g_b \in \mathbb{P}^1$ are general. Since $\overline{M}_{0,n}$ is connected, there is a map $\Delta = \operatorname{Spec} R \to \overline{M}_{0,n}$ (where R is a DVR) such that the generic point maps to $(\mathbb{P}^1, \{g_i\})$ and the closed points maps to the b-pointed stable curve $(D_0, q_1, \ldots, q_b)$ of Figure 5.21. Since $\overline{\mathcal{H}}_{d,b} \to \overline{M}_{0,n}$ is finite, we may use the valuative criterion to lift $\Delta \to \overline{M}_{0,n}$ to $\Delta \to \overline{\mathcal{H}}_{d,b}$ such that the image of the generic point is $[C \to \mathbb{P}^1]$. The composition $\Delta \to \overline{\mathcal{H}}_{d,b} \to \overline{\mathcal{M}}_g$ gives the desired degeneration.

5.7.4 Irreducibility in positive characteristic: Deligne–Mumford and Fulton's arguments

The year 1969 was a remarkable year for mathematics in part due to the seminal contributions of Deligne and Mumford's paper [DM69] and Fulton's paper [Ful69]. The papers provided independent arguments for the irreducibility of $\overline{M}_g$ in positive characteristic (where Fulton's argument has the restriction that $p > g+1$). Both papers relied on the connectedness of M_g over $\mathbb{C}$ and the time, there was no purely algebraic argument; the algebraic argument establishing Theorem 5.7.15 used admissible covers and became available only in 1982. The connectedness of M_g over $\mathbb{C}$ is a classical result. Clebsch and Hurwitz's arguments in the 19th century (featured in Theorem 5.7.12) used the Hurwitz space of branched covers and used on a single non-algebraic input, namely the Riemann's Existence Theorem. There are of course other non-algebraic arguments, e.g. using the Teichmüller space.

5.7.4.1 Deligne–Mumford's first argument

The first argument appearing [DM69] is very similar in spirit to the argument in §5.7.3. As with most results, there are many approaches to construct a proof and the first approach in [DM69, §3] reflects the state of technology at the time.

For a field $\Bbbk$ of characteristic p, the argument for irreducibility of $M_g \times_{\mathbb{Z}} \Bbbk$ proceeds along three steps:

Step 1: There is no proper connected component of $M_g \times_{\mathbb{Z}} \Bbbk$.

Let $W(\Bbbk)$ be the Witt vectors for $\Bbbk$; $W(\Bbbk)$ is a noetherian complete local ring whose generic point η has characteristic 0 and whose closed point 0 has residue field is $\Bbbk$. (For example, $W(\mathbb{F}_p) = \mathbb{Z}_p$ is the ring of p-adics.) We now use the existence of a quasi-projective coarse moduli space $\mathcal{M}_g \to M_g$ over $W(\Bbbk)$ as established in [GIT]. (Although appearing in the definitive book on GIT, this would not be viewed as a "GIT construction" today as it relies on some ad hoc techniques and doesn't use the Hilbert–Mumford criterion. Indeed, the standard GIT toolkit only became available in positive characteristic in 1975 after Haboush resolved Mumford's conjecture [Hab75] and in the relative setting in 1977 after Seshadri's paper [Ses77].)

Choosing a projective compactification $M_g \subset X$ over $W(\Bbbk)$, the connectedness of the generic fiber of $M_g \to \operatorname{Spec} W(\Bbbk)$ ensures that the generic fiber of X_η is also connected. The scheme M_g is normal as GIT quotients (or alternatively coarse moduli spaces) preserve normality. By taking the normalization of X, we can assume that X is also normal. Zariski's connectedness theorem implies that the number of connected components in a fiber X_w is independent of $w \in W(\Bbbk)$. Thus, X_0 is also connected.

Suppose $Y \subset M_g \times_{W(\Bbbk)} \Bbbk$ is a proper connected component. Then $Y \subset M_g \times_{W(\Bbbk)} \Bbbk \subset X_0$ is an open subscheme; but it's also a closed subscheme since Y is proper. Since X_0 is connected, we conclude that $Y = M_g \times_{W(\Bbbk)} \Bbbk$ is proper and irreducible. To obtain a contradiction, denote by $A_{g,\Bbbk}$ the moduli of principally polarized g-dimensional abelian varieties over $\Bbbk$ and consider the morphism

$$\Theta\colon M_g \times_{W(\Bbbk)} \Bbbk \to A_{g,\Bbbk}, \qquad C \mapsto \operatorname{Jac}(C)$$

assigning to a smooth curve C its Jacobian $\operatorname{Jac}(C)$. The properness of $M_g \times_{W(\Bbbk)} \Bbbk$ implies that the image would be a closed but there are explicit examples where the closure of the image of Θ contains products of lower dimensional Jacobians.

Step 2: There is no connected component of $\overline{M}_g \times_{\mathbb{Z}} \Bbbk$ consisting entirely of smooth curves.

Let $\overline{M}_{g,1}, \ldots, \overline{M}_{g,r}$ be the connected components of $\overline{M}_g$. For each i, Step 1 implies that $\overline{M}_{g,i}$ is not proper. Let $\Delta = \operatorname{Spec} \Bbbk[\![t]\!]$ and $\Delta^* = \operatorname{Spec} \Bbbk(\!(t)\!) \to M_{g,i} := M_g \cap \overline{M}_{g,i}$ be a morphism that does not extend to Δ. By Stable reduction, after possibly replacing Δ with a finite extension, $\Delta^* \to M_{g,i}$ extends to a morphism $\Delta \to \overline{M}_g$. This shows that $\overline{M}_{g,i} \setminus M_{g,i}$ is non-empty.

Step 3: The boundary $\delta = \overline{M}_g \setminus M_g$ is connected.

Note that Steps 1 and 2 show that every smooth curve degenerates to a singular stable curve (Proposition 5.7.13). This step proceeds precisely as in Proposition 5.7.14 but without using the formalism of the moduli $\overline{\mathcal{M}}_{g,n}$ of n-pointed stable curves and the gluing morphisms.

5.7.4.2 Deligne–Mumford's second argument

The stack $\overline{\mathcal{M}}_g$ of stable curves is smooth and proper over $\operatorname{Spec} \mathbb{Z}$. Zariski's connectedness theorem implies that for every smooth and proper morphism $X \to Y$ of schemes, the number of connected components of a geometric fiber is a locally constant function on Y. (In fact, for a flat and proper morphism $X \to Y$, this function is lower semi-continuous and it is enough for the fibers of $X \to Y$ to be geometrically normal in order to show constancy.) This fact extends to morphisms of algebraic stacks. Applying this fact to the morphism $\overline{\mathcal{M}}_g \to \operatorname{Spec} \mathbb{Z}$, we see that the connectedness $\overline{\mathcal{M}}_g \times_{\mathbb{Z}} \mathbb{C}$ implies the connectedness of every geometric fiber. In [DM69, §5], the connectedness of $\overline{\mathcal{M}}_g \times_{\mathbb{Z}} \mathbb{C}$ is argued by relating it to the moduli of Teichmüller structures of level n and the connectedness of the Teichmüller space [Man39].

5.7.4.3 Fulton's argument

In [Ful69], Fulton defines the Hurwitz scheme $H_{d,b}$ of simply branched covers over $\mathbb{Z}$ and shows that there is a diagram

$$\begin{tikzcd} & H_{d,b} \arrow[dl] \arrow[dr] & \\ \mathcal{M}_g & & \operatorname{Sym}^b \mathbb{P}^1 \setminus \Delta \end{tikzcd}$$

defined over $\mathbb{Z}$. He shows that the map $H_{d,b} \to \operatorname{Sym}^d \mathbb{P}^1 \setminus \Delta$, taking a simply branched cover to its branch locus, is étale. Moreover, if all primes $p \leq g+1$ are inverted, then $H_{d,b} \to \operatorname{Sym}^d \mathbb{P}^1 \setminus \mathbb{P}^1$ is finite; examples are given where is not finite over primes $p \leq g+1$. Fulton then establishes a "reduction theorem" allowing him to deduce the connectedness of $H_{d,b} \times_{\mathbb{Z}} \overline{\mathbb{F}}_p$ from $H_{d,b} \times_{\mathbb{Z}} \mathbb{C}$ for primes $p > g+1$.

5.8 Projectivity

In this section, we prove that the coarse moduli space $\overline{M}_{g,n}$ is projective (Theorem 5.8.14). We follow the approach introduced by Kollár in [Kol90] partially building on ideas of Viehweg (see [Vie95]). We will primarily focus on the unpointed coarse moduli space $\overline{M}_g$ as this will be enough to deduce the projectivity of $\overline{M}_{g,n}$.

To introduce the general strategy to establish projectivity, we need to introduce some terminology. Let $\pi\colon \mathcal{U}_g \to \overline{\mathcal{M}}_g$ be the universal family and for each integer $k \geq 1$ define the *kth pluri-canonical bundle* as the vector bundle

$$\pi_*(\omega_{\mathcal{U}_g/\overline{\mathcal{M}}_g}^{\otimes k}) \tag{5.8.1}$$

on $\overline{\mathcal{M}}_g$. Its rank $r(k)$ can be computed via Riemann–Roch:

$$r(k) := \begin{cases} g & \text{if } k = 1 \\ (2k-1)(g-1) & \text{if } k > 1. \end{cases} \tag{5.8.2}$$

We obtain line bundles on $\overline{\mathcal{M}}_g$ by taking the determinant

$$\lambda_k := \det \pi_*(\omega_{\mathcal{U}_g/\overline{\mathcal{M}}_g}^{\otimes k}).$$

These provide natural candidates of line bundles on $\overline{\mathcal{M}}_g$ that descend to ample line bundles on $\overline{M}_g$.

Strategy for projectivity: Show that for $k \gg 0$, a positive power of λ_k descends to an *ample* line bundle on the coarse moduli space $\overline{M}_g$.

Outline of this section: In §5.8.1, we prove Kollar's Criterion for ampleness (Theorem 5.8.5). In §5.8.2, we setup the application of Kollár's Criterion to $\overline{M}_g$ by establishing Proposition 5.8.13: projectivity of $\overline{M}_g$ follows from (a) Stable Reduction (Theorem 5.5.1) and (b) the nefness of $\pi_*(\omega_{\mathcal{C}/T}^{\otimes k})$ for a family of stable curves $\mathcal{C} \to T$ over a smooth projective curve and for $k \gg 0$ (Theorem 5.8.17). In §5.8.3, we prove this nefness statement which finishes the proof of projectivity. Finally, in §5.8.4, we compare this argument to the GIT construction of $\overline{M}_g$.

5.8.1 Kollár's criteria

In this section, we prove Kollár's Criterion for projectivity (Theorem 5.8.5), which we will apply to show that λ_k is ample on $\overline{M}_g$ for $k \gg 0$. We first extend ampleness criteria of §E.2.5 to proper algebraic spaces and in particular establish that the Nakai–Moishezon criterion still holds (Theorem 5.8.4).

Lemma 5.8.1. *Let $\mathcal{X}$ be a proper Deligne-Mumford stack with coarse moduli space $\mathcal{X} \to X$. Suppose that L is a line bundle on $\mathcal{X}$ satisfying*

(a) L is semiample (i.e. L^N is base point free for some $N > 0$); and

(b) for every map $f\colon T \to \mathcal{X}$ from a proper integral curve such that $f(T) \subset |\mathcal{X}|$ is not a single point, $\deg L|_T > 0$.

Then for some $N > 0$, $L^{\otimes N}$ descends to an ample line bundle. In particular, X is projective.

Remark 5.8.2. Lemma E.2.16 handles the case when $\mathcal{X}$ is a scheme. Even though we won't actually quote this lemma, it provides a basic technique which underlies many ampleness arguments, e.g. the Nakai-Moishezon criterion.

Proof. For N sufficiently divisible, consider the diagram

$$\begin{tikzcd} \mathcal{X} \arrow[d] \arrow[dr] & \\ X \arrow[r] & \mathbb{P}(H^0(\mathcal{X}, L^N)). \end{tikzcd}$$

Property (a) implies that $\mathcal{X} \to \mathbb{P}(H^0(\mathcal{X}, L^N))$ is well-defined and (b) implies that it doesn't contract curves. The universal property for coarse moduli spaces gives the existence of the factorization $X \to \mathbb{P}(H^0(\mathcal{X}, L^N))$, which also doesn't contract curves. Thus $X \to \mathbb{P}(H^0(\mathcal{X}, L^N))$ is quasi-finite and proper (as both X and projective space are proper), and thus finite by Zariski's Main Theorem. It follows that the pullback M of $\mathcal{O}(1)$ under $X \to \mathbb{P}(H^0(\mathcal{X}, L^N))$ is ample; moreover, the pullback of M under $\mathcal{X} \to X$ is $L^{\otimes N}$. □

Remark 5.8.3. The semiampleness condition in (a) can be very challenging to verify in practice. Keep in mind that in the GIT approach, semiampleness is hard-coded into the definition of semistability (see Remark 5.8.22). If G is a reductive group acting linearly on projective space $\mathbb{P}(V)$ and $L = \mathcal{O}(1)$ is the corresponding G-line bundle on $[\mathbb{P}(V)/G]$, then a nonzero vector $v \in V$ is in the stable base locus of L (i.e. $s(v) = 0$ for all $s \in \Gamma([\mathbb{P}(V)/G], L^{\otimes d})$ and $d > 0$) if and only if $0 \in \overline{Gv} \subset V$ if and only if there exists a one-parameter subgroup $\lambda\colon \mathbb{G}_m \to G$ such that $\lim_{t\to 0} \lambda(t) \cdot v = 0$. This latter equivalence is the Hilbert–Mumford criterion and can sometimes be verified combinatorially.

On the other hand, in Kollár's Criterion, the existence of sufficient sections of the line bundle follows from the bigness of suitable vector bundles.

Theorem 5.8.4 (Nakai–Moishezon Criterion)**.** *If X is a proper algebraic space, a line bundle L is ample if and only if for all irreducible closed subvarieties $Z \subset X$, $c_1(L)^{\dim Z} \cdot Z > 0$*

Proof. By Le Lemme de Gabber (Theorem 4.5.1), there exists a finite surjection $f\colon X' \to X$ from a scheme X, and L is ample if and only if f^*L is ample (Exercise 4.4.21). The statement then follows for the Nakai–Moishezon Criterion for schemes (Theorem E.2.18). □

Let X be a proper algebraic space over $\Bbbk$. Let $W \to Q$ be a surjection of vector bundles of rank w and q. Suppose that W has structure group $G \to \mathrm{GL}_w$. There is a *classifying map*

$$\begin{array}{rcl} X & \to & [\mathrm{Gr}(q, \Bbbk^w)/G] \\ x & \mapsto & [W \otimes \kappa(x) \twoheadrightarrow Q \otimes \kappa(x)] \end{array}$$

which is well-defined because a choice of isomorphism $W \otimes \kappa(x) \cong \kappa(x)^w$ of the fiber of W over x is well-defined up to the structure group G. Thus, the image of x is identified with the quotient $[\kappa(x)^w \cong W \otimes \kappa(x) \twoheadrightarrow Q \otimes \kappa(x)] \in \mathrm{Gr}(q, \Bbbk^w)$.

For simplicity, we state the following criteria in characteristic 0. The criteria first appears in [Kol90, Lem. 3.9] with improvements from [KP17, Thm. 4.1].

Theorem 5.8.5 (Kollár's Criterion)**.** *Let X be a proper algebraic space over a field $\Bbbk$ of characteristic 0. Let $W \twoheadrightarrow Q$ be a surjection of vector bundles of rank w and q, where W has structure group $G \to \mathrm{GL}_w$. Suppose that*

(a) The classifying map $X(\Bbbk) \to \mathrm{Gr}(q, \Bbbk^w)(\Bbbk)/G(\Bbbk)$ has finite fibers; and

(b) W is nef.

Then $\det Q$ is ample.

Remark 5.8.6. Condition (a) is equivalent to the map $|X| \to |[\mathrm{Gr}(q, \Bbbk^w)/G]|$ on topological spaces having finite fibers. This set-theoretic condition is weaker than the quasi-finiteness[1] of $X \to [\mathrm{Gr}(q, \Bbbk^w)/G]$, as the latter condition also requires that for every $x \in X(\Bbbk)$ only finitely many elements of $G(\Bbbk)$ leave $\ker(W \otimes \kappa(x) \to Q \otimes \kappa(x))$ invariant (or equivalently that the image of x in $[\mathrm{Gr}(q, \Bbbk^w)/G]$ has finite stabilizer.) In fact, Condition (a) is equivalent to quasi-finiteness of the projection morphism $\mathrm{im}(X \to X \times [\mathrm{Gr}(q, \Bbbk^w)/G]) \to [\mathrm{Gr}(q, \Bbbk^w)/G]$ from the scheme-theoretic image of the graph of the classifying map; it is this property that we will use in the proof.

Remark 5.8.7. An easy case of this theorem is when W is the trivial vector bundle so that there is a reduction of structure group to the trivial group $G = \{1\}$. In this case, the classifying map $X \to \mathrm{Gr}(q, \Bbbk^w)$ is quasi-finite by condition (a) and proper since both X and $\mathrm{Gr}(q, \Bbbk^w)$ are proper. Thus $X \to \mathrm{Gr}(q, \Bbbk^w)$ is finite and $\det(Q)$ is ample as its the pullback of the ample line bundle on $\mathrm{Gr}(q, \Bbbk^w)$ defining the Plücker embedding.

Note that in the above theorem, we do not require that the image of X lands in the G-stable locus of $\mathrm{Gr}(q, \Bbbk^w)$. However, if this is true, then we have a commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[dr] & {[\mathrm{Gr}(q, \Bbbk^w)^{\mathrm{ss}}/G]} \arrow[d] \\ & \mathrm{Gr}(q, \Bbbk^w)/\!\!/G \end{tikzcd}$$

where $\mathrm{Gr}(q, \Bbbk^w)/\!\!/G$ denotes the projective GIT quotient. Since the image of X lands in the *stable* locus, $X \to \mathrm{Gr}(q, \Bbbk^w)/\!\!/G$ is quasi-finite; as it's also proper, we conclude that it's finite. Moreover, we obtain ampleness of $(\det Q)^w \otimes (\det W)^{-q}$, the pullback of the ample line bundle $\mathrm{Gr}(q, \Bbbk^w)/\!\!/G$ coming from GIT. This is a stronger ampleness statement than merely the ampleness of $\det Q$.

Remark 5.8.8. The nefness of $\det(Q)$ is an immediate consequence of the nefness of W as $\det(Q) = \bigwedge^q Q$ is a quotient of $\bigwedge^q W$, which is nef by Proposition E.2.27. The proof will proceed by reducing the ampleness of $\det Q$ to its bigness, which in turn is established by using the quasi-finiteness and nefness to express $\det Q$ as the sum of an effective line bundle and a big and globally generated line bundle.

Proof of Theorem 5.8.5. We will verify the Nakai–Moishezon criterion: for each irreducible subvariety $Z \subset X$, we verify that $\det(Q)|_Z$ is big. Since both conditions (a) and (b) also hold for Z and the restrictions $W|_Z \twoheadrightarrow Q|_Z$, it suffices to verify that if X is an integral scheme with $W \twoheadrightarrow Q$ satisfying (a) and (b), then $\det(Q)$ is big.

The property of bigness (unlike ampleness) is conveniently invariant under birational maps (and we desire this flexibility because in the proof of Proposition 5.8.9 below, we will make a series of reductions where we perform blowups to resolve the indeterminacy locus of certain rational maps). In fact, for a generically quasi-finite and proper morphism $f \colon Y \to X$ of integral schemes, the projection formula implies that $\det(f^*Q)^{\dim Y} = \deg(f) \det(Q)^{\dim X} > 0$ and thus $\det(Q)$ is big if and only if $f^*(\det Q)$ is big. By Le Lemme de Gabber (Corollary 4.5.2), there exists a projective, generically quasi-finite and surjective morphism $f \colon Y \to X$ from a projective integral scheme. By taking the normalization, we can assume

[1] Recall that a morphism $f \colon \mathcal{X} \to \mathcal{Y}$ of algebraic stacks is *quasi-finite* if $|\mathcal{X}| \to |\mathcal{Y}|$ has finite fibers and the relative inertia $I_{\mathcal{X}/\mathcal{Y}}$ is quasi-finite (or equivalently for every field-valued point $x \in \mathcal{X}(K)$ the morphism $\mathrm{Aut}_{\mathcal{X}(K)}(x) \to \mathrm{Aut}_{\mathcal{Y}(K)}(f(x))$ has finite cokernel).

that Y is normal. The theorem therefore follows from the bigness of $f^*(\det Q)$, which is the conclusion of the following proposition. $\square$

Proposition 5.8.9. *Let Y be a normal projective integral scheme over a field $\Bbbk$ of characteristic 0. Let $W \twoheadrightarrow Q$ be a surjection of vector bundles of rank w and q, where W has structure group $G \to \mathrm{GL}_w$. Suppose that*

(a′) The classifying map $Y(\Bbbk) \to \mathrm{Gr}(q, \Bbbk^w)(\Bbbk)/G(\Bbbk)$ generically has finite fibers;

(b) W is nef.

Then $\det Q$ is big.

Remark 5.8.10. Condition (a′) means that there is a non-empty open subscheme $U \subset Y$ such that $U(\Bbbk) \to \mathrm{Gr}(q, \Bbbk^w)(\Bbbk)/G(\Bbbk)$ has finite fibers.

Note that the difference in the hypotheses between Theorem 5.8.5 and Proposition 5.8.9 is that we relaxed the condition on the classifying map from having finite fibers to generically having finite fibers but now we assume that Y is already projective (in addition to being normal and integral). Also the conclusion is weaker in that it asserts the bigness of $\det(Q)$ rather than the ampleness.

Proof.

Step 1: Use the universal basis map to lift the classifying map to a morphism $\mathbb{P} \setminus \Delta \to \mathrm{Gr}(q, \Bbbk^w)$ where $\mathbb{P} \subset \mathbb{P}_Y((W^\vee)^{\oplus w})$ is a closed subscheme and $\Delta \subset \mathbb{P}$ is a divisor.

Define $\widetilde{\mathbb{P}} := \mathbb{P}_Y((W^\vee)^{\oplus w})$ as the projective space of matrices whose columns belong to W, and let $\widetilde{\pi}\colon \widetilde{\mathbb{P}} \to Y$ denote the projection. There is a *universal basis map*

$$\mathcal{O}_{\widetilde{\mathbb{P}}}^{\oplus w} \to \widetilde{\pi}^* W \otimes \mathcal{O}_{\widetilde{\mathbb{P}}}(1) \tag{5.8.3}$$

defined by the isomorphisms

$$H^0(\widetilde{\mathbb{P}}, \widetilde{\pi}^* W \otimes \mathcal{O}_{\widetilde{\mathbb{P}}}(1)) \cong H^0(Y, \widetilde{\pi}_*(\widetilde{\pi}^* W \otimes \mathcal{O}_{\widetilde{\mathbb{P}}}(1))) \cong H^0(Y, W \otimes (W^\vee)^{\oplus w}).$$

The universal basis map (5.8.3) restricts to an isomorphism on the complement $\widetilde{\mathbb{P}} \setminus \Delta$ where $\Delta \subset \widetilde{\mathbb{P}}$ is the divisor of matrices with determinant 0, and thus provides a trivialization of $(\widetilde{\pi}^* W \otimes \mathcal{O}_{\widetilde{\mathbb{P}}}(1))|_{\widetilde{\mathbb{P}} \setminus \Delta}$. Note also that there is a natural PGL_w action on $\widetilde{\mathbb{P}}$ which is free on $\widetilde{\mathbb{P}} \setminus \Delta$ and such that $\widetilde{\pi}\colon \widetilde{\mathbb{P}} \setminus \Delta \to Y$ is a PGL_w-torsor and fits into the cartesian diagram

$$\begin{tikzcd}
\widetilde{\mathbb{P}} \setminus \Delta \arrow[r] \arrow[d] & \mathrm{Gr}(q, \Bbbk^w) \arrow[r] \arrow[d] & \operatorname{Spec} \Bbbk \arrow[d] \\
Y \arrow[r] & {[\mathrm{Gr}(q, \Bbbk^w)/\mathrm{PGL}_w]} \arrow[r] & B\,\mathrm{PGL}_w .
\end{tikzcd}$$

We can also consider the fiber product with respect to the G-action

$$\mathbb{P} \setminus \Delta := Y \times_{[\mathrm{Gr}(q, \Bbbk^w)/G]} \mathrm{Gr}(q, \Bbbk^w).$$

The inclusion $\mathbb{P} \setminus \Delta \hookrightarrow \widetilde{\mathbb{P}} \setminus \Delta$ is a closed immersion and we define $\mathbb{P} \subset \widetilde{\mathbb{P}}$ to be the closure of $\mathbb{P} \setminus \Delta$, where we abuse notation by using the same symbol Δ for the divisor in $\widetilde{\mathbb{P}}$ and its intersection in $\mathbb{P}$. One way to see that $\mathbb{P} \setminus \Delta = Y \times_{BG} \operatorname{Spec} \Bbbk \hookrightarrow Y \times_{B\,\mathrm{PGL}_w} \operatorname{Spec} \Bbbk = \widetilde{\mathbb{P}} \setminus \Delta$ is a closed immersion is to realize it as the base change of the diagonal $BG \to BG \times_{B\,\mathrm{PGL}_w} BG$; here we use that $BG \to B\,\mathrm{PGL}_w$ is separated (it is in fact even affine since G is reductive). Alternatively, one can view $\mathbb{P} \subset \widetilde{\mathbb{P}}$ as the closure of a generic G-orbit in $\widetilde{\mathbb{P}}$.

In summary, we have a cartesian diagram

$$\begin{tikzcd} \mathbb{P} \setminus \Delta \arrow[r] \arrow[d, "\pi"] & \mathrm{Gr}(q, \Bbbk^w) \arrow[r, hook] \arrow[d] & \mathbb{P}(\bigwedge^q \Bbbk^w) \arrow[d] \\ Y \arrow[r] & {[\mathrm{Gr}(q, \Bbbk^w)/G]} \arrow[r, hook] & {[\mathbb{P}(\bigwedge^q \Bbbk^w)/G]} \end{tikzcd}$$

where the right hand square is given by the Plücker embedding. The map $\mathbb{P} \setminus \Delta \to Y$ extends to a map $\pi \colon \mathbb{P} \to Y$ (i.e. the composition $\mathbb{P} \hookrightarrow \widetilde{\mathbb{P}} \xrightarrow{\widetilde{\pi}} Y$). The map $\mathbb{P} \setminus \Delta \to \mathrm{Gr}(q, \Bbbk^w)$ is defined by the restriction of the composition

$$\mathcal{O}_{\mathbb{P}}^{\oplus w} \to \pi^* W \otimes \mathcal{O}_{\mathbb{P}}(1) \to \pi^* Q \otimes \mathcal{O}_{\mathbb{P}}(1) \tag{5.8.4}$$

of the universal basis map (5.8.3) with the quotient $\pi^* W \to \pi^* Q$. The image of (5.8.4) may not be locally free and thus the rational map $\mathbb{P} \dashrightarrow \mathrm{Gr}(q, \Bbbk^w)$ may not be defined everywhere.

Step 2: Blowup $\mathbb{P}$ in order to extend the map $\mathbb{P} \setminus \Delta \to \mathrm{Gr}(q, \Bbbk^w)$.

(Note that if (5.8.4) is surjective, then $\mathbb{P} \setminus \Delta \to \mathrm{Gr}(q, \Bbbk^w)$ extends to a morphism $\mathbb{P} \to \mathrm{Gr}(q, \Bbbk^w)$ such that the pullback of the Plücker line bundle is $\pi^*(\det Q) \otimes \mathcal{O}_{\mathbb{P}}(q)$.)

We blowup the image ideal sheaf of (5.8.4) (more precisely, if $I \subset \pi^*(\det Q) \otimes \mathcal{O}_{\mathbb{P}}(q)$ denotes the image subsheaf of (5.8.4), we blowup $I \otimes (\pi^*(\det Q) \otimes \mathcal{O}_{\mathbb{P}}(q))^\vee \subset \mathcal{O}_{\mathbb{P}}$). This yields a map $g \colon \mathbb{P}' \to \mathbb{P}$ which is an isomorphism over $\mathbb{P} \setminus \Delta$ and such that $\mathbb{P} \setminus \Delta \to \mathrm{Gr}(q, \Bbbk^w)$ extends to a morphism $\gamma \colon \mathbb{P}' \to \mathrm{Gr}(q, \Bbbk^w)$. This yields a commutative diagram

$$\begin{tikzcd} \mathbb{P}' \arrow[d, "g"] \arrow[dr, "\gamma"] \arrow[dd, bend right, "\pi'"'] & \\ \mathbb{P} \arrow[r, dashed] \arrow[d, "\pi"] & \mathrm{Gr}(q, \Bbbk^w) \arrow[d] \\ Y \arrow[r] & {[\mathrm{Gr}(q, \Bbbk^w)/G].} \end{tikzcd}$$

The effective divisor $E \subset \mathbb{P}'$ satisfies

$$g^*(\pi^*(\det Q) \otimes \mathcal{O}_{\mathbb{P}}(q)) \cong \gamma^* \mathcal{O}_{\mathrm{Gr}(q, \Bbbk^w)}(1) \otimes \mathcal{O}_{\mathbb{P}'}(E). \tag{5.8.5}$$

where $\mathcal{O}_{\mathrm{Gr}(q, \Bbbk^w)}(1)$ denotes the Plücker line bundle.

Step 3: Use the generic quasi-finiteness to show that $\gamma^*(\mathcal{O}_{\mathrm{Gr}(q, \Bbbk^w)}(m)) \otimes \pi'^* H^\vee$ is effective for some $m > 0$, where H is a ample line bundle on Y.

(Note that under the stronger assumption that the classifying map $Y \to [\mathrm{Gr}(q, \Bbbk^w)/G]$ is generically quasi-finite, then $\gamma \colon \mathbb{P}' \to \mathrm{Gr}(q, \Bbbk^w)$ is also generically quasi-finite. Thus $\gamma^* \mathcal{O}_{\mathrm{Gr}(q, \Bbbk^w)}(1)$ is big and Kodaira's Lemma (Proposition E.2.9) immediately gives the desired statement.)

Let Z be the scheme-theoretic image of the graph $Y \to Y \times [\mathrm{Gr}(q, \Bbbk^w)/G]$ of the classifying map. The hypothesis that $Y(\Bbbk) \to \mathrm{Gr}(q, \Bbbk^w)(\Bbbk)/G(\Bbbk)$ is generically quasi-finite implies that $Z \to [\mathrm{Gr}(q, \Bbbk^w)/G]$ is generically quasi-finite. Consider the commutative diagram

$$\begin{tikzcd}
\mathbb{P}' \arrow[dr, "p"] \arrow[rrrr, bend left, "\gamma"] & & & & \\
\mathbb{P} \setminus \Delta \arrow[u, hook] \arrow[r] \arrow[d, "\pi"] & Z' \arrow[r, hook] \arrow[d] \arrow[rrr, bend left, "\gamma'"] & Y \times \mathrm{Gr}(q, \Bbbk^w) \arrow[rr] \arrow[d] & & \mathrm{Gr}(q, \Bbbk^w) \arrow[d] \\
Y \arrow[r] & Z \arrow[r, hook] & Y \times [\mathrm{Gr}(q, \Bbbk^w)/G] \arrow[rr] & & {[\mathrm{Gr}(q, \Bbbk^w)/G]}
\end{tikzcd}$$

where the squares are cartesian and where Z' is the scheme-theoretic image of $\mathbb{P} \setminus \Delta \to Y \times \mathrm{Gr}(q, \Bbbk^w)$ (and also of $\mathbb{P}' \to Y \times \mathrm{Gr}(q, \Bbbk^w)$). We see that $\gamma' \colon Z' \to \mathrm{Gr}(q, \Bbbk^w)$ is also generically quasi-finite and it follows that $\gamma'^*(\mathcal{O}_{\mathrm{Gr}(q,\Bbbk^w)}(1))$ is big. If we denote by H' the pullback of H to Z', then by Kodaira's Lemma (Proposition E.2.9), $\gamma'^*(\mathcal{O}_{\mathrm{Gr}(q,\Bbbk^w)}(m)) \otimes H'^\vee$ is effective on Z for some $m > 0$. Its pullback $p^*(\gamma'^*(\mathcal{O}_{\mathrm{Gr}(q,\Bbbk^w)}(m)) \otimes H'^\vee) \cong \gamma^*(\mathcal{O}_{\mathrm{Gr}(q,\Bbbk^w)}(m)) \otimes \pi'^* H^\vee$ is also effective.

Step 4: Pushforward a section to construct a map $\pi_* \mathcal{O}_{\mathbb{P}}(mq)^\vee \to (\det Q)^{\otimes m} \otimes H^\vee$.

Using (5.8.5), we see that

$$\begin{aligned}
\gamma^*(\mathcal{O}_{\mathrm{Gr}(q,\Bbbk^w)}(m)) \otimes \pi'^* H^\vee &\cong \pi'^*((\det Q)^{\otimes m} \otimes H^\vee) \otimes g^* \mathcal{O}_{\mathbb{P}}(mq) \otimes \mathcal{O}_{\mathbb{P}'}(-mE) \\
&\subset \pi'^*((\det Q)^{\otimes m} \otimes H^\vee) \otimes g^* \mathcal{O}_{\mathbb{P}}(mq)) \\
&\cong g^*\big(\pi^*((\det Q)^{\otimes m} \otimes H^\vee) \otimes \mathcal{O}_{\mathbb{P}}(mq)\big)
\end{aligned}$$

and therefore we may choose a non-zero section

$$\mathcal{O}_{\mathbb{P}'} \to g^*\big(\pi^*((\det Q)^{\otimes m} \otimes H^\vee) \otimes \mathcal{O}_{\mathbb{P}}(mq)\big).$$

Pushing forward under $g \colon \mathbb{P}' \to \mathbb{P}$ and using the projection formula gives a non-zero section

$$\mathcal{O}_{\mathbb{P}} \to \pi^*((\det Q)^{\otimes m} \otimes H^\vee) \otimes \mathcal{O}_{\mathbb{P}}(mq)$$

and pushing forward again under $\pi \colon \mathbb{P} \to Y$ gives a non-zero section

$$\mathcal{O}_Y \to (\det Q)^{\otimes m} \otimes H^\vee \otimes \pi_* \mathcal{O}_{\mathbb{P}}(mq)$$

which we rearrange as

$$\pi_* \mathcal{O}_{\mathbb{P}}(mq)^\vee \to (\det Q)^{\otimes m} \otimes H^\vee. \tag{5.8.6}$$

Step 5: Show that the nefness of W implies the nefness of $\pi_* \mathcal{O}_{\mathbb{P}}(mq)^\vee$.

We compare $\pi_* \mathcal{O}_{\mathbb{P}}(mq)$ to $\pi_* \mathcal{O}_{\widetilde{\mathbb{P}}}(mq) \cong \mathrm{Sym}^{mq}((W^\vee)^{\oplus w})$ (and their duals) under the closed immersion $\mathbb{P} \hookrightarrow \widetilde{\mathbb{P}}$ (where we are using π to denote both projections $\mathbb{P} \to Y$ and $\widetilde{\mathbb{P}} \to Y$). For $m \gg 0$, the map $\pi_* \mathcal{O}_{\widetilde{\mathbb{P}}}(mq) \to \pi_* \mathcal{O}_{\mathbb{P}}(mq)$ is surjective and dualizes to an inclusion

$$(\pi_* \mathcal{O}_{\mathbb{P}}(mq))^\vee \hookrightarrow (\pi_* \mathcal{O}_{\widetilde{\mathbb{P}}}(mq))^\vee \cong \mathrm{Sym}^{mq}((W)^{\oplus w})$$

of vector bundles on Y. Since W is nef, so is $\mathrm{Sym}^{mq}((W)^{\oplus w})$ (Proposition E.2.27) and therefore so is $(\pi_* \mathcal{O}_{\mathbb{P}}(mq))^\vee$ (Proposition 5.8.11).

Step 6: Conclude that $\det Q$ *is big.*

(Note that if (5.8.6) is surjective, the line bundle quotient $N := (\det Q)^{\otimes m} \otimes H^\vee$ is nef. Thus $(\det Q)^{\otimes m} \cong H \otimes N$ is written as the sum of an ample and nef divisor, which is necessarily big.)

Blowing up the image ideal sheaf of (5.8.6), we obtain a birational morphism $s \colon Y' \to Y$ and a quotient line bundle $s^* \pi_* \mathcal{O}_{\mathbb{P}}(mq)^\vee \twoheadrightarrow N \subset s^*(\det Q)^{\otimes m} \otimes s^* H^\vee$ which is nef. As N is nef and $s^* H$ is big and globally generated, the sub-line bundle $s^* H \otimes N \subset s^*(\det Q)^{\otimes m}$ is big. The difference of $s^*(\det Q)^{\otimes m}$ and $s^* H \otimes N$ is effective. Since the sum of a big and globally generated line bundle is big, we can conclude that $s^*(\det Q)^{\otimes m}$ is big, which in turn implies that $\det Q$ is big. $\square$

The proof above used the following property of nefness of vector bundles complementing the basic results from Section E.2.7.

Proposition 5.8.11. *Let X be a scheme of finite type over an algebraically closed field $\Bbbk$ of characteristic 0 and W be a vector bundle of rank w. Let G be a reductive group and suppose that W admits a reduction of the structure group $G \to \mathrm{GL}_w$. Let $V \subset W$ be a G-subbundle corresponding a G-invariant subspace $\Bbbk^v \subset \Bbbk^w$. If W is nef, then so is V.*

Proof. In characteristic 0, representations of reductive groups are completely reducible. Therefore $\Bbbk^v \subset \Bbbk^w$ has a G-invariant complement $\Bbbk^{w-v} \subset \Bbbk^w$. Since this expresses V as a quotient of W, we see that V is nef. $\square$

5.8.2 Application to $\overline{M}_g$

To apply Kollár's Criterion to $\overline{M}_g$, we will make use of multiplication maps between pluri-canonical bundles and their symmetric products. Given a morphism $S \to \overline{\mathcal{M}}_g$ corresponding to a family of stable curves $\pi \colon \mathcal{C} \to S$ and an integer $d \geq 0$, we will consider the *multiplication map*

$$\mathrm{Sym}^d \pi_*(\omega_{\mathcal{C}/S}^{\otimes k}) \to \pi_*(\omega_{\mathcal{C}/S}^{\otimes dk}). \tag{5.8.7}$$

For a stable curve C defined over a field $\Bbbk$, this multiplication map is

$$\mathrm{Sym}^d H^0(C, \omega_C^{\otimes k}) \to H^0(C, \omega_C^{\otimes dk})$$

and its kernel consists of degree d equations cutting out the image of $C \xrightarrow{|\omega_C^{\otimes k}|} \mathbb{P}^{r(k)-1}$. If $k \geq 3$, then $\omega_{\mathcal{C}/S}^{\otimes k}$ is relatively very ample and thus $\mathcal{C} \to S$ can be recovered from the kernel of the multiplication map.

Remark 5.8.12. We emphasize here that this construction depends on k and d, the same two integers which the GIT construction depends on (see Section 5.8.4).

Proposition 5.8.13. *Let $g \geq 2$. Assume that*

(a) $\overline{\mathcal{M}}_g$ is a proper Deligne–Mumford stack; and

(b) There exists a $k_0 > 0$ such that for every family of stable curves $\mathcal{C} \to T$ over a smooth projective curve T, $\pi_*(\omega_{\mathcal{C}/T}^{\otimes k})$ is nef for $k \geq k_0$.

Then for $k \gg 0$ and N sufficiently divisible, the line bundle $\lambda_k^{\otimes N}$ on $\overline{\mathcal{M}}_g$ descends to an ample line bundle on the coarse moduli space $\overline{M}_g$. In particular, $\overline{M}_g$ is projective.

Proof. Consider the universal curve $\mathcal{C} = \mathcal{U}_g$ over $S = \overline{\mathcal{M}}_g$. Choose integers k and d such that

- $\omega_{\mathcal{C}/S}^{\otimes k}$ is relatively very ample and $R^1\pi_*\omega_{\mathcal{C}/S}^{\otimes k} = 0$;
- Every stable curve $C \overset{|\omega_C^{\otimes k}|}{\hookrightarrow} \mathbb{P}^{r(k)-1}$ is cut out by equations of degree d; and
- $\pi_*(\omega_{\mathcal{C}/S}^{\otimes k})$ is nef.

The conditions imply that the multiplication map

$$W := \operatorname{Sym}^d \pi_*(\omega_{\mathcal{C}/S}^{\otimes k}) \twoheadrightarrow \pi_*(\omega_{\mathcal{C}/S}^{\otimes dk}) =: Q$$

is surjective. Let $w = \binom{r(k)+d-1}{d}$ and $q = r(dk)$ be the ranks of W and Q, respectively. Note that W has a reduction of the structure group to $G := \mathrm{SL}_{r(k)}$. The classifying map

$$\begin{aligned} \overline{\mathcal{M}}_g &\to [\operatorname{Gr}(q, \Bbbk^w)/G] \\ [C] &\mapsto \big[\underbrace{\operatorname{Sym}^d H^0(C, \omega_C^{\otimes k})}_{\Gamma(\mathbb{P}^{r(k)-1}, \mathcal{O}(d))} \twoheadrightarrow \underbrace{H^0(C, \omega_C^{\otimes dk})}_{\Gamma(C, \mathcal{O}(d))}\big] \end{aligned}$$

is injective as the conditions on d and k imply that the kernel of the multiplication map uniquely determines C.

Let $X \to \overline{\mathcal{M}}_g$ be a finite cover where X is a proper algebraic space (Theorem 4.5.1). By Kollár's Criterion (Theorem 5.8.5), the pullback of λ_k to X is ample for $k \gg 0$. By Proposition 4.3.28, for N sufficiently divisible, $\lambda_k^{\otimes N}$ descends to a line bundle L on $\overline{M}_g$. Since the pullback of L under the finite morphism $X \to \overline{\mathcal{M}}_g \to \overline{M}_g$ is ample, we conclude by Exercise 4.4.21 that L is ample. □

In the next section, we will establish condition (b), the nefness of the pluricanonical bundles. This will allow us to conclude:

Theorem 5.8.14. *If $2g - 2 + n > 0$, then $\overline{M}_{g,n}$ is projective.*

Proof. It suffices to handle the $n = 0$ case as $\overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ is the universal family (Proposition 5.6.4) and is a projective morphism (Proposition 5.3.9). The fact that λ_k descends to an ample line bundle on $\overline{M}_g$ follows from Proposition 5.8.13 as Condition (a) is a consequence of Stable Reduction (see Theorem 5.5.3) while (b) is Theorem 5.8.17. □

Remark 5.8.15. It also possible to show projectivity of $\overline{M}_{g,n}$ directly using Kollár's Criterion applied to the determinant of $\pi_*(L^k)$ where $L := \omega_{\mathcal{U}_{g,n}/\overline{\mathcal{M}}_{g,n}}(\sigma_1 + \cdots + \sigma_n)$ and $\mathcal{U}_{g,n} \to \overline{\mathcal{M}}_{g,n}$ is the universal family with sections $\sigma_1, \ldots, \sigma_n$.

Remark 5.8.16. The criteria of Proposition 5.8.13 for ampleness generalizes to every moduli of polarized varieties (see [Kol90, Thm. 2.6]); this was one of the original motivations of Kollár's paper. In recent years, Kollár's Criterion has been applied in more and more general settings to establish projectivity, e.g. Hassett's moduli space of weighted pointed curves [Has03], the moduli of stable varieties of any dimension [KP17], and the moduli of K-polystable Fano varieties [CP21, XZ20, LXZ21].

5.8.3 Nefness of pluri-canonical bundles

In this section, we establish that $\pi_*(\omega_{\mathcal{C}/T}^{\otimes k})$ is nef for every $k \geq 2$.

Theorem 5.8.17. *For family of stable curves $\mathcal{C} \to T$ over a smooth projective curve T, $\pi_*(\omega_{\mathcal{C}/T}^{\otimes k})$ is nef for $k \geq 2$.*

Proof sketch: Let $\Bbbk$ be the base field.

Step 1: Reduction to characteristic p. Assume that $\mathrm{char}(\Bbbk) = 0$. Since $\mathcal{C}$ and T are finite type over $\Bbbk$, their defining equations only involve finitely many coefficients of $\Bbbk$. Thus there exists a finitely generated $\mathbb{Z}$-subalgebra $A \subset \Bbbk$ and a cartesian diagram

$$\begin{tikzcd}
\mathcal{C} \arrow[r] \arrow[d] & \widetilde{\mathcal{C}} \arrow[d] \\
T \arrow[r] \arrow[d] & \widetilde{T} \arrow[d] \\
\operatorname{Spec} \Bbbk \arrow[r] & \operatorname{Spec} A
\end{tikzcd}$$

where $\widetilde{C}$ and $\widetilde{T}$ are schemes of finite type over A. By possibly enlarging A, we can arrange that $\widetilde{T} \to \operatorname{Spec} A$ is a smooth and projective family of curves and that $\widetilde{C} \to \widetilde{T}$ is a family of stable curves. Finally, by restricting along a morphism $\operatorname{Spec} R \to \operatorname{Spec} A$ from a DVR such that the images of the closed and generic points have characteristic p and 0, respectively, we may assume that A is a DVR. Since nefness is an open condition for such proper flat families (Proposition E.2.28), it suffices to prove the theorem when $\mathrm{char}(\Bbbk) = p > 0$.

Step 2: Second reductions. We reduce to the case where

(a) $\mathcal{C}$ is a smooth and minimal surface;

(b) $\mathcal{C} \to T$ is generically smooth; and

(c) the genus of T is at least 2

(details to be added). These conditions imply that $\mathcal{C}$ is of general type.

Step 3: Positive characteristic case. Let $p = \mathrm{char}(\Bbbk)$. If $\pi_*(\omega_{\mathcal{C}/T}^{\otimes k})$ is not nef, then there exists a quotient line bundle $\pi_*(\omega_{\mathcal{C}/T}^{\otimes k}) \twoheadrightarrow M^\vee$ where $d = \deg M > 0$. Consider the absolute Frobenius morphisms $F \colon \mathcal{C} \to \mathcal{C}$ and $F \colon T \to T$ which fit into a commutative diagram

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "F"] \arrow[d] & \mathcal{C} \arrow[d] \\
T \arrow[r, "F"] & T.
\end{tikzcd}$$

By properties of the dualizing sheaf, we have $F^*\pi_*(\omega_{\mathcal{C}/T}^{\otimes k}) = \pi_*(\omega_{\mathcal{C}/T}^{\otimes k})$. Since $\deg F^*M = pd$, we can apply the Frobenius repeatedly to arrange that d, the degree of M, is as large as we want. Specifically, we can arrange that $M \cong \omega_T^{\otimes k} \otimes L$ where L is a very ample line bundle on T. (This was the entire point of reducing to characteristic p: to repeatedly apply the Frobenius to jack-up the degree.)

The surjection $\pi_*(\omega_{\mathcal{C}/T}^{\otimes k}) \twoheadrightarrow M^\vee \cong (\omega_T^{\otimes k} \otimes L)^\vee$ yields a surjection

$$\pi_*(\omega_{\mathcal{C}/T}^{\otimes k}) \otimes \omega_T^{\otimes k} \otimes L \twoheadrightarrow \mathcal{O}_T$$

Since $\mathrm{h}^1(T, \mathcal{O}_T) \geq 2$, we have $\mathrm{h}^1(T, \pi_*(\omega_{\mathcal{C}/T}^{\otimes k}) \otimes \omega_T^{\otimes k} \otimes L) \geq 2$. Using the Leray spectral sequence to relate $H^1(\pi_*(\omega_{\mathcal{C}/T}^{\otimes k}) \otimes \omega_T^{\otimes k} \otimes L)$ to $H^1(\mathcal{C}, \omega_{\mathcal{C}}^{\otimes k} \otimes \pi^* L)$, one can show that $\mathrm{h}^1(\mathcal{C}, \omega_{\mathcal{C}}^{\otimes k} \otimes \pi^* L) \geq 2$ (details omitted). This however contradicts Bombieri–Ekedahl vanishing in the form of Lemma 5.8.19 with $D = \pi^* L$. $\square$

Remark 5.8.18. For families of *smooth* curves, $\pi_*(\omega_{\mathcal{C}/T})$ is nef; this fact is somewhat easier and was known earlier. If $\mathcal{C} \to S$ has no hyperelliptic fibers, then Max Noether's theorem on projective normality implies that $\mathrm{Sym}^d \pi_*(\omega_{\mathcal{C}/T}) \to \pi_*(\omega_{\mathcal{C}/T}^{\otimes d})$ is surjective. Therefore, the nefness of $\pi_*(\omega_{\mathcal{C}/T})$ implies the nefness of both $\mathrm{Sym}^d \pi_*(\omega_{\mathcal{C}/T})$ and the quotient $\pi_*(\omega_{\mathcal{C}/T}^{\otimes d})$ (Proposition E.2.27).

Lemma 5.8.19. *Let S be a smooth projective surface over an algebraically closed field $\Bbbk$ which is minimal and of general type. Let D be an effective divisor with $D^2 = 0$. If $\mathrm{char}(\Bbbk) \neq 2$, then $H^1(S, \omega_S^{\otimes n}(D)) = 0$ for all $n \geq 2$. If $\mathrm{char}(\Bbbk) = 2$, then $\mathrm{h}^1(S, \omega_S^{\otimes n}(D)) \leq 1$ for all $n \geq 2$.*

Proof. Bombieri–Ekedahl vanishing (Theorem E.3.1) implies that $H^1(S, K_S^{\otimes -n}) = 0$ for all $n \geq 1$. The Serre dual of this statement is that $H^1(S, K_S^{\otimes n}) = 0$ for all $n \geq 2$. The statement follows from using the short exact sequence

$$0 \to \omega_S^{\otimes n} \to \omega_S^{\otimes n}(D) \to \omega_S^{\otimes n}|_D \to 0$$

and adjunction (details omitted). $\square$

5.8.4 Projectivity via Geometric Invariant Theory

The Geometric Invariant Theory (GIT) construction depends on two integers:

- k, the multiple of the dualizing sheaf used to obtain an embedding $C \xrightarrow{|\omega_C^{\otimes k}|} \mathbb{P}^{r(k)-1}$. We need $k \geq 3$ for $\omega_C^{\otimes k}$ to be very ample for a stable curve C but we need $k \geq 5$ for the GIT construction to yield $\overline{M}_g$.
- d, the degree of the equations that we use to embed the Hilbert scheme of k-canonically embedded curves into a Grassmanian. We need $d \gg 0$ to obtain an embedding of the Hilbert scheme.

Assuming that $k \geq 3$, a stable curve C of genus g is pluricanonically embedded via

$$C \xrightarrow{|\omega_C^{\otimes k}|} \mathbb{P}^{r(k)-1}$$

where $r(k) = (2k-1)(g-1)$. Let $P(t) = \chi(C, \omega^{\otimes kt}) = (2kt-1)(g-1)$ be the Hilbert polynomial of C in $\mathbb{P}^{r(k)-1}$. Let $H' \subset \mathrm{Hilb}^P(\mathbb{P}^{r(k)-1})$ be the locally closed subscheme of the Hilbert scheme parameterizing stable curves $[C \hookrightarrow \mathbb{P}^{r(k)-1}]$ embedded via $\omega_C^{\otimes k}$. Note that $\mathrm{PGL}_{r(k)}$ acts naturally on $\mathrm{Hilb}^P(\mathbb{P}^{r(k)-1})$ and that the subscheme H' is $\mathrm{PGL}_{r(k)}$-invariant.

Exercise 5.8.20. Extend Theorem 3.1.15 by establishing that:

(a) $H' \subset \mathrm{Hilb}^P(\mathbb{P}^{r(k)-1}$ is a locally closed $\mathrm{PGL}_{r(k)}$-invariant subscheme, and

(b) $\overline{\mathcal{M}}_g \cong [H'/\mathrm{PGL}_{r(k)}]$.

Let $H = \overline{H'} \subset \mathrm{Hilb}^P(\mathbb{P}^{r(k)-1}$ be the closure of H'. For $d \gg 0$, we have an embedding into the Grassmanian of $P(d)$-dimensional quotients of $\Gamma(\mathbb{P}^{r(k)-1}, \mathcal{O}(d))$

$$\begin{aligned} H &\hookrightarrow \mathrm{Gr}(P(d), \Gamma(\mathbb{P}^{r(k)-1}, \mathcal{O}(d)) \\ [C \hookrightarrow \mathbb{P}^{r(k-1)}] &\mapsto \big[\Gamma(\mathbb{P}^{r(k)-1}, \mathcal{O}(d)) \twoheadrightarrow \Gamma(C, \mathcal{O}(d))\big] \end{aligned}$$

Note that there is a natural identification of this quotient with the multiplication map

$$\begin{tikzcd} \Gamma(\mathbb{P}^{r(k)-1}, \mathcal{O}(d)) \arrow[r, two heads] \arrow[d, equal] & \Gamma(C, \mathcal{O}(d)) \arrow[d, equal] \\ \mathrm{Sym}^d H^0(C, \omega_C^{\otimes k}) \arrow[r, two heads] & H^0(C, \omega_C^{\otimes dk}). \end{tikzcd}$$

Let $\mathcal{O}_{\mathrm{Gr}}(1)$ be the very ample line bundle on $\mathrm{Gr}(P(d), \Gamma(\mathbb{P}^{r(k)-1}, \mathcal{O}(d))$ obtained via the Plücker embedding

$$\begin{aligned} \mathrm{Gr}(P(d), \Gamma(\mathbb{P}^{r(k)-1}, \mathcal{O}(d)) &\hookrightarrow \mathbb{P}(\wedge^{P(d)}\Gamma(\mathbb{P}^{r(k)-1}, \mathcal{O}(d)) \\ [\Gamma(\mathbb{P}^{r(k)-1}, \mathcal{O}(d)) \twoheadrightarrow \Gamma(C, \mathcal{O}(d))] &\mapsto [\wedge^{P(d)}\Gamma(\mathbb{P}^{r(k)-1}, \mathcal{O}(d)) \twoheadrightarrow \wedge^{P(d)}\Gamma(C, \mathcal{O}(d))]; \end{aligned}$$

see Section 1.2. Finally, let $L_d = \mathcal{O}_{\mathrm{Gr}}(1)|_H$ be the very ample line bundle on H obtained by restricting $\mathcal{O}(1)$ under the composition

$$H \hookrightarrow \mathrm{Gr}(P(d), \Gamma(\mathbb{P}^{r(k)-1}, \mathcal{O}(d)) \hookrightarrow \mathbb{P}(\wedge^{P(d)}\Gamma(\mathbb{P}^{r(k)-1}, \mathcal{O}(d)) \tag{5.8.8}$$

As each morphism in (5.8.8) is $\mathrm{PGL}_{r(k)}$-equivariant, the line bundle L_d inherits a $\mathrm{PGL}_{r(k)}$-linearization.

Definition 5.8.21. A point $h \in H$ is said to be *GIT semistable with respect to* L_d if there exists an equivariant section $s \in \Gamma(H, L_d^{\otimes N})^{\mathrm{PGL}_{r(k)}}$ with $N > 0$ such that $s(h) \neq 0$. The *semistable locus* H^{ss} consisting of GIT semistable points is an open $\mathrm{PGL}_{r(k)}$-invariant subscheme.

Remark 5.8.22. Stack-theoretically, the $\mathrm{PGL}_{r(k)}$-linearization L_d defines a line bundle, which we will also denote by L_d, on the quotient stack $[H/\mathrm{PGL}_{r(k)}]$ and the open substack $[H^{\mathrm{ss}}/\mathrm{PGL}_{r(k)}]$ is the largest open substack such the restriction of L_d is semiample. In other words, $h \in H$ is GIT semistable if and only h does not lie in the stable base locus of L_d on $[H/\mathrm{PGL}_{r(k)}]$.

Remark 5.8.23. These definitions clearly extend to the action of every algebraic group G on a projective scheme X embedded G-equivariantly $X \hookrightarrow \mathbb{P}^N$ by a G-linearization L. One of the main results of GIT is that if G is reductive, then the graded ring $\bigoplus_{N\geq 0}\Gamma(H, L_d^{\otimes N})^{\mathrm{PGL}_{r(k)}}$ is finitely generated and that the morphism

$$X^{\mathrm{ss}} \to X^{\mathrm{ss}}/\!\!/G := \mathrm{Proj} \bigoplus_{N\geq 0} \Gamma(H, L_d^{\otimes N})^{\mathrm{PGL}_{r(k)}}$$

is a good quotient. Note that X^{ss} is precisely the maximal locus where the rational map $X \dashrightarrow X^{\mathrm{ss}}/\!\!/G$ is defined.

The GIT construction of $\overline{M}_g$ rests on the following difficult theorem:

Theorem 5.8.24. *Let $k \geq 5$ and $d \gg 0$. For $h = [C \hookrightarrow \mathbb{P}^{r(k)-1}] \in H$, the curve C is a stable if and only if $h \in H$ is GIT semistable with respect to L_d.*

Remark 5.8.25. This theorem can be established using the Hilbert–Mumford criteria. It is rather difficult to explicitly exhibit sections of $\Gamma(H, L_d^{\otimes N})^{\mathrm{PGL}_{r(k)}}$ and the Hilbert–Mumford criteria allows us to verify that a given point $h \in H$ is semistable by checking that for each one-parameter subgroup $\lambda \colon \mathbb{G}_m \to \mathrm{PGL}_{r(k)}$, the Hilbert–Mumford index $\mu(h, L_d)$, defined as the weight of $\mathbb{G}_m$ on the line in the affine cone $\mathbb{A}^{r(k)}$ over $\lim_{t\to 0} \lambda(t) \cdot h \in H \subset \mathbb{P}^{r(k)-1}$, is negative. The beauty of the Hilbert–Mumford criterion is that it magically guarantees the existence of sections for you! Nevertheless, verifying the Hilbert–Mumford criterion even for a smooth pluricanonical embedded curve is no easy task.

Given Theorem 5.8.24, we obtain $\overline{M}_g$ as the projective variety

$$\overline{M}_g = \operatorname{Proj} \Gamma(H, L_d^{\otimes N})^{\mathrm{PGL}_{r(k)}}.$$

Remark 5.8.26. As a spectacular corollary of Theorem 5.8.24, one obtains an alternative proof of Stable Reduction (Theorem 5.5.1) in arbitrary characteristic. This is perhaps surprising as the GIT argument uses rather little about the geometry of stable curves and their families.

Remark 5.8.27 (The ample cone)**.** For each $k \geq 5$ and $d \gg 0$, GIT constructs a line bundle on on $\overline{\mathcal{M}}_g$ which descends to an ample line bundle on $\overline{M}_g$. This class can be expressed as

$$r(k)\lambda_{dk} - r(dk)\lambda_k.$$

Grothendieck–Riemann–Roch can be used to express each of the line bundles λ_k as a linear combination of λ_1 and δ, the boundary divisor. The asymptotic limit of this class as d goes to infinity is proportional to

$$(12 - \frac{4}{k})\lambda_1 - \delta.$$

Taking $k = 5$, shows that $11.2\lambda - \delta$ is ample.

However, even more is true! By bootstrapping the positivity deduced from GIT, Cornalba and Harris showed that $a\lambda - \delta$ is ample if and only if $a > 11$, thus determining the ample cone of $\overline{M}_g$ in the $\lambda_1\delta$-plane of $\mathrm{NS}^1(\overline{\mathrm{M}}_\mathrm{g})$ [CH88].

Chapter 6

Geometry of algebraic stacks

6.1 Quasi-coherent sheaves and quotient stacks

We will define quasi-coherent sheaves on an algebraic stack in the same way that we did for Deligne–Mumford stacks in §4.1 but using the lisse-étale site on $\mathcal{X}$ instead of the small étale site. The entirety of §4.1 on sheaves, $\mathcal{O}_{\mathcal{X}}$-modules and quasi-coherent sheaves remains valid for algebraic stacks (with the same affine diagonal hypotheses).

6.1.1 Sheaves and $\mathcal{O}_{\mathcal{X}}$-modules

To develop abelian sheaf theory on an algebraic stack, we use the lisse-étale site.

Definition 6.1.1 (Lisse-étale site)**.** The *lisse-étale site* $\mathcal{X}_{\text{lis-ét}}$ on an algebraic stack $\mathcal{X}$ is the category of schemes smooth over $\mathcal{X}$ where morphisms are arbitrary maps of schemes smooth over $\mathcal{X}$. A covering $\{U_i \to U\}$ is a collection of morphisms such that $\coprod_i U_i \to U$ is surjective and étale.

This allows us to discuss sheaves of abelian groups on $\mathcal{X}_{\text{lis-ét}}$ and their morphisms. Extending §4.1.1, we can define sections $\Gamma(\mathcal{U}, F)$ or $F(\mathcal{U})$ of an abelian sheaf on an algebraic stack $\mathcal{U}$ smooth over $\mathcal{X}$. The structure sheaf $\mathcal{O}_{\mathcal{X}}$, defined as $\mathcal{O}_{\mathcal{X}}(U) = \Gamma(U, \mathcal{O}_U)$, is a ring object in the abelian category $\mathrm{Ab}(\mathcal{X}_{\text{lis-ét}})$. We can therefore define $\mathcal{O}_{\mathcal{X}}$-modules as in Definition 4.1.8 and the abelian category $\mathrm{Mod}(\mathcal{O}_{\mathcal{X}})$ of $\mathcal{O}_{\mathcal{X}}$-modules. Given a morphism $f\colon \mathcal{X} \to \mathcal{Y}$ of algebraic stacks, there are adjoint functors

$$\mathrm{Ab}(\mathcal{X}_{\text{lis-ét}}) \underset{f^{-1}}{\overset{f_*}{\rightleftarrows}} \mathrm{Ab}(\mathcal{Y}_{\text{lis-ét}}) \qquad \mathrm{Mod}(\mathcal{O}_{\mathcal{X}}) \underset{f^{*}}{\overset{f_*}{\rightleftarrows}} \mathrm{Mod}(\mathcal{O}_{\mathcal{Y}}).$$

Given two $\mathcal{O}_{\mathcal{X}}$-modules F and G, the *tensor product* $F \otimes G := F \otimes_{\mathcal{O}_{\mathcal{X}}} G$ is the sheafification of $U \mapsto F(U) \otimes_{\mathcal{O}_{\mathcal{X}}(U)} G(U)$, and the *Hom sheaf* $\mathscr{H}om_{\mathcal{O}_{\mathcal{X}}}(F, G)$ is the sheaf given by $U \mapsto \mathrm{Hom}_{\mathcal{O}_U}(F|_U, G|_U)$, where $F|_U$ denotes the restriction of F to $U_{\text{lis-ét}}$.

6.1.2 Quasi-coherent sheaves

Following §4.1.3, given an $\mathcal{O}_{\mathcal{X}}$-module F on an algebraic stack $\mathcal{X}$ and a smooth $\mathcal{X}$-scheme U, we let $F|_U$ be the restriction of F to the lisse-étale site of U and $F|_{U_{\mathrm{Zar}}}$ the further restriction to the small Zariski site.

Definition 6.1.2. Let $\mathcal{X}$ be an algebraic stack. An $\mathcal{O}_{\mathcal{X}}$-module F is *quasi-coherent* if

(1) for every smooth $\mathcal{X}$-scheme U, the restriction $F|_{U_{\mathrm{Zar}}}$ is a quasi-coherent $\mathcal{O}_{U_{\mathrm{Zar}}}$-module, and

(2) for every morphism $f\colon U \to V$ of smooth $\mathcal{X}$-schemes, the natural morphism $f^*(F|_{V_{\mathrm{Zar}}}) \to F|_{U_{\mathrm{Zar}}}$ is an isomorphism.

A quasi-coherent sheaf F on $\mathcal{X}$ is a *vector bundle* (resp. *vector bundle of rank* r, *line bundle*) if $F|_{U_{\mathrm{Zar}}}$ is for every smooth $\mathcal{X}$-scheme U.

If in addition $\mathcal{X}$ is locally noetherian, we say F is *coherent* if $F|_{U_{\mathrm{Zar}}}$ is coherent for every smooth $\mathcal{X}$-scheme U

We denote by $\mathrm{QCoh}(\mathcal{X})$ and $\mathrm{Coh}(\mathcal{X})$ (in the noetherian setting) the categories of quasi-coherent and coherent sheaves. We encourage the reader to check that the equivalent formulations of quasi-coherent given in Exercises 4.1.14 to 4.1.16 still hold, and that the above definition of quasi-coherence is consistent with the definition of quasi-coherence on a Deligne–Mumford stack (Definition 4.1.11) and with the usual definition on a scheme. For a quasi-compact and quasi-separated morphism $f\colon \mathcal{X} \to \mathcal{Y}$ of algebraic stacks, f_* and f^* preserve quasi-coherence (by the same argument as for Exercise 4.1.17).

Exercise 6.1.3. Let G be an affine algebraic group over a field $\Bbbk$. Recall that a G-representation is a $\Bbbk$-vector space with a dual action $\sigma\colon V \to \Gamma(G, \mathcal{O}_G) \otimes_{\Bbbk} V$ satisfying two natural compatibility conditions (see §C.1.3).

(a) Show that $\mathrm{QCoh}(\mathbf{B}G)$ is equivalent to the category $\mathrm{Rep}(G)$ of G-representations.

(b) If $\operatorname{Spec} A$ is an affine $\Bbbk$-scheme with a G-action, show that a quasi-coherent sheaf on $[\operatorname{Spec} A/G]$ is the data of an A-module M together with a coaction $\sigma\colon M \to \Gamma(G, \mathcal{O}_G) \otimes_{\Bbbk} M$ over $\Bbbk$ (i.e. a map of $\Bbbk$-vector spaces giving M the structure of a G-representation) such that multiplication $A \otimes_{\Bbbk} M \to M$ is a map of G-representations. This extends Example 4.1.13 where G is finite.

(c) Considering the diagram

$$\begin{tikzcd} \operatorname{Spec} A \arrow[r, "p"] & {[\operatorname{Spec} A/G]} \arrow[r, "\pi"] \arrow[d, "q"] & \operatorname{Spec} A^G \\ & \mathbf{B}G, & \end{tikzcd}$$

extend Exercise 4.1.18 by providing descriptions of the functors $p_*, p^*, \pi_*, \pi^*, q_*$ and q^* on quasi-coherent sheaves.

(d) If U is a $\Bbbk$-scheme with an action of G, then a *line bundle with a G-action* is a line bundle L on U together with an isomorphism $\alpha\colon \sigma^* L \xrightarrow{\sim} p_2^* L$ satisfying a cocycle condition $p_{23}^*\alpha \circ (\mathrm{id}_G \times \sigma)^*\alpha = (\mu \times \mathrm{id}_U)^*\alpha$; see C.3.2. Show that a line bundle with a G-action is the same a line bundle on the quotient stack $[U/G]$.

Example 6.1.4. If G and H are affine algebraic group over a field $\Bbbk$ such that $\mathbf{B}G \cong \mathbf{B}H$, then G and H have equivalent categories of representations. For example, if $O(q)$ and $O(q')$ are orthogonal groups with respect to non-degenerate quadratic forms q and q' on an n-dimensional $\Bbbk$-vector space V, then $\mathbf{B}O(q) \cong \mathbf{B}O(q')$ (see Example 3.1.11), and thus $O(q)$ and $O(Q')$ have equivalent categories of representations.

Recall that one of the first examples we gave of a quasi-coherent sheaf on a Deligne–Mumford stack was the Hodge line bundle on $\mathfrak{M}_g$ (Examples 4.1.12) which we later generalized the pluri-canonical line bundles λ_k on $\overline{\mathfrak{M}}_g$ (see §5.8). These line bundles played an important role in our argument for the projectivity of $\overline{M}_g$ and are equally essential in the study of its geometry. Determinantal line bundles play a similar role in the study of the moduli stack of vector bundles.

Example 6.1.5 (Determinantal line bundles)**.** Consider the stack $\mathcal{M} := \mathcal{B}\mathrm{un}_{C,r,d}$ of vector bundles on a smooth connected projective curve C over $\Bbbk$. Consider the diagram

$$\begin{tikzcd} & C \times \mathcal{M} \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ C & & \mathcal{M}. \end{tikzcd}$$

The projection $p_2 \colon C \times \mathcal{M} \to \mathcal{M}$ is representable, projective and smooth of relative dimension 1. For every vector bundle F on $C \times \mathcal{M}$, applying Proposition A.7.10 and smooth descent shows that the cohomology $\mathrm{R}^i p_{2,*}F$ (as defined below) is computed as a 2-term complex $[K^0 \to K^1]$ of vector bundles and that the line bundle

$$\det \mathrm{R}p_{2,*}F := \det(K_0) \otimes \det(K_1)^\vee$$

is well-defined on $\mathcal{M}$. Note that if $\operatorname{rk} K_0 = \operatorname{rk} K_1$, i.e. $\operatorname{rk} \mathrm{R}p_{2,*}F = 0$, then we have a map $\det K^0 \to \det K^1$ of line bundles and the corresponding map $\mathcal{O}_\mathcal{M} \to \det(K^0)^\vee \otimes \det(K_1)$ defines a section of the dual $(\det \mathrm{R}p_{2,*}F)^\vee$.

Let $\mathcal{E}_{\mathrm{univ}}$ be the universal vector bundle on $C \times \mathcal{M}$. For every vector bundle V on C, we define the *determinantal line bundle*

$$\mathcal{L}_V := \big(\det \mathrm{R}p_{2,*}(\mathcal{E}_{\mathrm{univ}} \otimes p_1^* V)\big)^\vee.$$

associated to V.

Example 6.1.6. If $\mathcal{X}$ is an algebraic stack of finite presentation over a scheme S, then the relative sheaf of differentials $\Omega_{\mathcal{X}/S}$ on $\mathcal{X}_{\text{lis-ét}}$, defined on a smooth $\mathcal{X}$-scheme U by $\Omega_{\mathcal{X}/S}(U) = \Omega_{U/S}$, is not quasi-coherent. This is because for a non-étale map $f \colon U \to V$ of smooth $\mathcal{X}$-schemes, $f^*\Omega_{V/S} \to \Omega_{U/S}$ is not an isomorphism. This differs from the Deligne–Mumford case where the sheaf $\Omega_{\mathcal{X}/S}$ on $\mathcal{X}_{\text{ét}}$ is quasi-coherent (Examples 4.1.12). When $\mathcal{X}$ is Deligne–Mumford, $\Omega_{\mathcal{X}/S}$ extends to a quasi-coherent sheaf on $\mathcal{X}_{\text{lis-ét}}$ by defining $\Omega_{\mathcal{X}/S}(U)$, for a smooth map $f \colon U \to \mathcal{X}$ from a scheme, to be the global sections of the sheaf $f^*\Omega_{\mathcal{X}/S}$ on $U_{\text{lis-ét}}$.

Exercises 4.1.19 and 4.1.20 generalize to algebraic stacks.

Proposition 6.1.7 (Flat Base Change)**.** *Consider a cartesian diagram*

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] \arrow[dr, phantom, "\square"] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

of algebraic stacks, and let F be a quasi-coherent sheaf on X. If $g\colon Y' \to Y$ is flat and $f\colon X \to Y$ is quasi-compact and quasi-separated, the natural adjunction map

$$g^* f_* F \to f'_* g'^* F$$

is an isomorphism. □

Proposition 6.1.8. *Let $\mathcal{X}$ be a noetherian algebraic stack. Every quasi-coherent sheaf on $\mathcal{X}$ is a directed colimit of its coherent subsheaves. If $\mathcal{U} \subset \mathcal{X}$ is an open substack, then every coherent sheaf on $\mathcal{U}$ extends to a coherent sheaf on $\mathcal{X}$.*

Exercise 6.1.9. Let $\mathcal{X} \to \mathcal{Y}$ be a smooth affine morphism of noetherian algebraic stacks with affine diagonal.

(1) Show that there is a vector bundle $\Omega_{\mathcal{X}/\mathcal{Y}}$ on $\mathcal{X}$ with the property that if $V \to \mathcal{Y}$ is a morphism from a scheme, the pullback of $\Omega_{\mathcal{X}/\mathcal{Y}}$ to $X_V := \mathcal{X} \times_{\mathcal{Y}} V$ is $\Omega_{\mathcal{X}_V/V}$.

(2) Given a commutative diagram

$$\begin{tikzcd} \operatorname{Spec} A_0 \arrow[r, "f_0"] \arrow[d, hook] & \mathcal{X} \arrow[d] \\ \operatorname{Spec} A \arrow[r] \arrow[ur, dashed] & \mathcal{Y} \end{tikzcd}$$

where $A \twoheadrightarrow A_0$ is a surjection of noetherian rings with square-zero kernel J, show that the set of liftings is a torsor under $\operatorname{Hom}_{A_0}(f_0^* \Omega_{\mathcal{X}/\mathcal{Y}}, J)$ and in particular is non-empty.

(3) Can you weaken the hypotheses?

6.1.3 Quasi-coherent constructions

Extending the constructions of §4.1.5 on a Deligne–Mumford stack to an algebraic stack $\mathcal{X}$, a *quasi-coherent $\mathcal{O}_{\mathcal{X}}$-algebra* is a quasi-coherent $\mathcal{O}_{\mathcal{X}}$-module $\mathcal{A}$ with a compatible structure as a ring object. The *relative spectrum* $\mathcal{S}\mathrm{pec}_{\mathcal{X}}\, \mathcal{A}$, defined as the stack of pairs (f, α) where $f\colon S \to \mathcal{X}$ is a morphism from a scheme and $\alpha\colon f^*\mathcal{A} \to \mathcal{O}_S$ is a map of $\mathcal{O}_S$-algebras, is an algebraic stack affine over $\mathcal{X}$. On a noetherian algebraic stack, every quasi-coherent $\mathcal{O}_{\mathcal{X}}$-algebra is a directed colimit of finite type subalgebras.

The *reduction of* $\mathcal{X}$ is $\mathcal{X}_{\mathrm{red}} := \mathcal{S}\mathrm{pec}_{\mathcal{X}}\, \mathcal{O}_{\mathcal{X}}^{\mathrm{red}}$ where $\mathcal{O}_{\mathcal{X}}^{\mathrm{red}}$ is the sheaf of $\mathcal{O}_{\mathcal{X}}$-algebras defined by $\mathcal{O}_{\mathcal{X}}^{\mathrm{red}}(U) = \Gamma(U, \mathcal{O}_U)_{\mathrm{red}}$ for a smooth $\mathcal{X}$-scheme U. If $\mathcal{X}$ is integral, the *normalization of* $\mathcal{X}$ is defined as $\widetilde{\mathcal{X}} := \mathcal{S}\mathrm{pec}_{\mathcal{X}}\, \mathcal{A}$, where $\mathcal{A}$ is the $\mathcal{O}_{\mathcal{X}}$-algebra whose ring of sections over a smooth $\mathcal{X}$-scheme U is the normalization of $\Gamma(U, \mathcal{O}_U)$; this is well-defined since normalization commutes with smooth base change (Proposition A.5.4). For a quasi-compact and quasi-separated morphism $f\colon \mathcal{X} \to \mathcal{Y}$ of algebraic stacks stacks, there is a factorization $f\colon \mathcal{X} \to \mathcal{S}\mathrm{pec}\, f_*\mathcal{O}_{\mathcal{X}} \to \mathcal{Y}$. The morphism f is affine if and only if $\mathcal{X} \to \mathcal{S}\mathrm{pec}\, f_*\mathcal{O}_{\mathcal{X}}$ is an isomorphism, and quasi-affine if and only if $\mathcal{X} \to \mathcal{S}\mathrm{pec}\, f_*\mathcal{O}_{\mathcal{X}}$ is an open immersion.

The proof of Zariski's Main Theorem (4.4.9) in the case of Deligne–Mumford stacks extends to algebraic stacks.

Theorem 6.1.10 (Zariski's Main Theorem)**.** *A separated, quasi-finite and representable morphism $f\colon \mathcal{X} \to \mathcal{Y}$ of noetherian algebraic stacks factors as the composition of a dense open immersion $\mathcal{X} \hookrightarrow \widetilde{\mathcal{Y}}$ and a finite morphism $\widetilde{\mathcal{Y}} \to \mathcal{X}$.* □

6.1.4 Picard groups

If $\mathcal{X}$ is an algebraic stack, we let $\mathrm{Pic}(\mathcal{X})$ denote the set of isomorphism classes of line bundles on $\mathcal{X}$. It is a abelian group under tensor product.

Example 6.1.11. If G is an affine algebraic group over a field $\Bbbk$, then $\mathrm{Pic}(\mathbf{B}G)$ is equivalent to the group of characters $G \to \mathbb{G}_m$. For example, $\mathrm{Pic}(\mathbf{B}\mathbb{G}_m) = \mathbb{Z}$, $\mathrm{Pic}(\mathbf{B}\,\mathrm{GL}_n) = \mathbb{Z}$, and $\mathrm{Pic}(\mathrm{PGL}_n) = \{0\}$.

Exercise 6.1.12. Let $\mathcal{X}$ be a smooth and irreducible algebraic stack over a field $\Bbbk$.

(a) If $\mathcal{D} \subset \mathcal{X}$ is a reduced substack with complement $\mathcal{U}$, show that there is a naturally defined line bundle $\mathcal{O}(\mathcal{D})$ (generalizing the usual construction for schemes) such that $\mathcal{O}(\mathcal{D})|_{\mathcal{U}} \cong \mathcal{O}_{\mathcal{U}}$.

(b) If V is a vector bundle on $\mathcal{X}$, show that

$$\mathrm{Pic}(\mathbb{A}(V)) = \mathrm{Pic}(\mathcal{X}) \quad \text{and} \quad \mathrm{Pic}(\mathbb{P}(V)) = \mathrm{Pic}(\mathcal{X}) \times \mathbb{Z}.$$

Exercise 6.1.13. Let $\mathbb{G}_m$ acts on $\mathbb{A}^n$ over a field $\Bbbk$ with weights $d_1, \ldots, d_n$. Let $\mathcal{O}(1)$ be the line bundle on $[\mathbb{A}^n/\mathbb{G}_m]$ corresponding to the projection $[\mathbb{A}^n/\mathbb{G}_m] \to \mathbf{B}\mathbb{G}_m$.

(a) Show that $\mathrm{Pic}([\mathbb{A}^n/\mathbb{G}_m]) \cong \mathbb{Z}$ generated by $\mathcal{O}(1)$.

(b) Show that the restriction $\mathrm{Pic}([\mathbb{A}^n/\mathbb{G}_m]) \to \mathrm{Pic}(\mathcal{P}(d_1, \ldots, d_n))$ is an isomorphism, where $\mathcal{P}(d_1, \ldots, d_n)$ is the weighted projective stack (see Example 3.9.6).

(c) If $f \in \Gamma(\mathbb{A}^n, \mathcal{O}_{\mathbb{A}^n})$ is a homogenous polynomial of degree d such that $V(f) \subset \mathbb{A}^n$ is reduced, show that $\mathcal{O}(V(f)) \cong \mathcal{O}(d)$.

Exercise 6.1.14. Let $\Bbbk$ be a field with $\mathrm{char}(\Bbbk) \neq 2, 3$.

(a) Show that $\mathrm{Pic}(\overline{\mathcal{M}}_{1,1}) = \mathbb{Z}$.

Hint: Use the description $\overline{\mathcal{M}}_{1,1} = [(\mathbb{A}^2 \smallsetminus 0)/\mathbb{G}_m]$ of Exercise 3.1.17(c) where $\mathbb{G}_m$ acts with weights 4 and 6. Show that the restriction $\mathrm{Pic}(\mathbb{A}^2/\mathbb{G}_m]) \to \mathrm{Pic}(\overline{\mathcal{M}}_{1,1})$ is an equivalence.

(b) Show that $\mathrm{Pic}(\mathcal{M}_{1,1}) = \mathbb{Z}/12$.

Hint: Show that the restriction $\mathrm{Pic}(\overline{\mathcal{M}}_{1,1}) \to \mathrm{Pic}(\mathcal{M}_{1,1})$ is surjective and that the image of $\mathcal{O}(\Delta) = \mathcal{O}(12)$ is trivial. Show that the images of $\mathcal{O}(4)$ and $\mathcal{O}(6)$ are non-trivial by considering their restrictions to the residual gerbes of the unique elliptic curves with $\mathbb{Z}/4$ and $\mathbb{Z}/6$ automorphism groups. See also [Mum65].

6.1.5 Global quotient stacks and the resolution property

Definition 6.1.15. An algebraic stack $\mathcal{X}$ is a *global quotient stack* if there exists an isomorphism $\mathcal{X} \cong [U/\operatorname{GL}_n]$ where U is an algebraic space.

In other words, $\mathcal{X}$ is a global quotient stack if and only if there is a principal GL_n-bundle $U \to \mathcal{X}$ from an algebraic space, or equivalently a representable morphism $\mathcal{X} \to \mathbf{B}\operatorname{GL}_n$.

Exercise 6.1.16. Show that a noetherian algebraic stack $\mathcal{X}$ is a global quotient stack if and only if there exists a vector bundle E on $\mathcal{X}$ such that for every geometric point $x\colon \operatorname{Spec} \Bbbk \to \mathcal{X}$ with closed image, the stabilizer G_x acts faithfully on the fiber $E \otimes \Bbbk$.

Hint: Use the correspondence between principal GL_n*-bundles and vector bundles from Exercise C.2.11.*

Exercise 6.1.17. Let $\mathcal{X} \to \mathcal{Y}$ be a surjective, flat, and projective morphism of noetherian algebraic stacks. If $\mathcal{X}$ is a quotient stack, show that $\mathcal{Y}$ is a quotient stack.

Being a quotient stack is also related to the following notion:

Definition 6.1.18. A noetherian algebraic stack has the *resolution property* if every coherent sheaf is the quotient of a vector bundle.

A smooth or quasi-projective scheme over a field has the resolution property. More generally, a scheme admitting an "ample family" of line bundles has the resolution property and this implies that every noetherian normal $\mathbb{Q}$-factorial scheme with affine diagonal has the resolution property [BS03].

Proposition 6.1.19. *Let G be an affine algebraic group over a field $\Bbbk$ acting on an quasi-projective $\Bbbk$-scheme U. Assume that there is an ample line bundle L with an action of G (e.g. U is quasi-affine and $L = \mathcal{O}_U$). Then $[\operatorname{Spec} A/G]$ has the resolution property.*

Remark 6.1.20. It is a general fact that every line bundle on a normal scheme over $\Bbbk$ has a positive power that has a G-action.

Proof. The line bundle L corresponds to a line bundle $\mathcal{L}$ on $[U/G]$ which is relatively ample with respect to the morphism $p\colon [U/G] \to \mathbf{B}G$. For a coherent sheaf F on $[U/G]$, the natural map

$$\mathcal{L}^{-N} \otimes p^* p_*(\mathcal{L}^N \otimes F) \twoheadrightarrow F$$

is surjective for $N \gg 0$. The pushforward $p_*(\mathcal{L}^N \otimes F)$ is a quasi-coherent sheaf on $\mathbf{B}G$, i.e. a G-representation, which we can write as a union of finite dimensional G-representations V_i (Algebraic Group Facts C.3.1(1)). We therefore obtain a surjection $\operatorname{colim}_i(\mathcal{L}^{-N} \otimes p^* V_i) \twoheadrightarrow F$. Since F is coherent, $\mathcal{L}^{-N} \otimes p^* V_i \twoheadrightarrow F$ is surjective for $i \gg 0$. □

An interesting converse was established by Totaro [Tot04] and generalized by Gross [Gro17].

Theorem 6.1.21. *Let $\mathcal{X}$ be a quasi-separated normal algebraic stack of finite type over a field $\Bbbk$. Assume that the stabilizer group at every closed point is smooth and affine. Then the following are equivalent:*

(1) $\mathcal{X}$ has the resolution property,

(2) $\mathcal{X} \cong [U/\operatorname{GL}_n]$ with U quasi-affine, and

(3) $\mathcal{X} \cong [\operatorname{Spec} A/G]$ with G an affine algebraic group.

In particular, $\mathcal{X}$ has affine diagonal.

Remark 6.1.22. While the normal hypothesis on $\mathcal{X}$ and smoothness hypothesis on the stabilizers are unnecessary, the affineness hypothesis on the stabilizers is necessary, e.g. the classifying stack $\mathbf{B}E$ of an elliptic curve has the resolution property.

Proof. The implications that (2) and (3) imply (1) were established in Proposition 6.1.19.

To see (3) $\Rightarrow$ (2), it suffices to find a faithful representation $G \hookrightarrow \operatorname{GL}_N$ such that GL_N/G is quasi-affine. Indeed, in this case, $[\operatorname{Spec} A/G] \cong [(\operatorname{Spec} A \times^G \operatorname{GL}_N)/\operatorname{GL}_N]$ (Exercise 3.4.14) and $\operatorname{Spec} A \times^G \operatorname{GL}_N$ is affine over GL_N/G. We begin by choosing a faithful representation $G \subset \operatorname{GL}_n$. By Algebraic Group Facts C.3.1(8), there is a GL_n-representation V and a $\Bbbk$-point $x \in \mathbb{P}(V)$ with stabilizer G. Under the action of $\operatorname{GL}_n \times \mathbb{G}_m$ on $\mathbb{A}(V)$ (where $\mathbb{G}_m$ acts via scaling), the stabilizer of a lift $\widetilde{x} \in \mathbb{A}(V)$ of x is G. The map $(\operatorname{GL}_n \times \mathbb{G}_m)/G \hookrightarrow \mathbb{A}(V)$, defined by $g \mapsto g\widetilde{x}$, is a locally closed immersion and thus $(\operatorname{GL}_n \times \mathbb{G}_m)/G$ is quasi-affine. Under the natural inclusion $\operatorname{GL}_n \times \mathbb{G}_m \hookrightarrow \operatorname{GL}_{n+1}$, the quotient $\operatorname{GL}_{n+1}/(\operatorname{GL}_n \times \mathbb{G}_m)$ is affine (and is sometimes called the "Steifel manifold"). The composition $\mathbf{B}G \to \mathbf{B}(\operatorname{GL}_n \times \mathbb{G}_m) \to \mathbf{B}\operatorname{GL}_{n+1}$ is quasi-affine and therefore so is $\operatorname{GL}_{n+1}/G$.

Conversely for (2) $\Rightarrow$ (3), we may choose a GL_n-equivariant open immersion $U \hookrightarrow \operatorname{Spec} A$ into an affine scheme of finite type over $\Bbbk$. Indeed, the morphism $p\colon [U/\operatorname{GL}_n] \to \mathbf{B}\operatorname{GL}_n$ is quasi-affine and $[U/\operatorname{GL}_n] \to \operatorname{Spec}_{\mathbf{B}\operatorname{GL}_n} p_*\mathcal{O}_{[U/\operatorname{GL}_n]}$ is an open immersion. By writing $p_*\mathcal{O}_{[U/\operatorname{GL}_n]} = \operatorname{colim}_\lambda \mathcal{A}_\lambda$ as a colimit of finite type $\mathcal{O}_{\mathbf{B}\operatorname{GL}_n}$-algebras, then limit methods imply that $[U/\operatorname{GL}_n] \to \operatorname{Spec}_{\mathbf{B}\operatorname{GL}_n} \mathcal{A}_\lambda$ is an open immersion for $\lambda \gg 0$. Let $Z \subset \operatorname{Spec} A$ be the reduced complement of U. By Lemma C.3.2(2), there is a GL_n-equivariant morphism $f\colon \operatorname{Spec} A \to \mathbb{A}^r$ such that $f^{-1}(0) = Z$. This induces an affine morphism $U \to \mathbb{A}^r \smallsetminus 0$. The complement $\mathbb{A}^r \setminus 0$ can be realized as the quotient GL_r/H where $H \subset \operatorname{GL}_r$ is the subgroup consisting of matrices whose last row is $(0, \ldots, 0, 1)$; H is identified with the semi-direct product $\mathbb{G}_a^{r-1} \rtimes \operatorname{GL}_{r-1}$. In the GL_n-equivariant cartesian diagram

$$\begin{tikzcd} P \arrow[r] \arrow[d] & \operatorname{GL}_r \arrow[d] \\ U \arrow[r] & \mathbb{A}^r \smallsetminus 0, \end{tikzcd}$$

P is affine over GL_r, thus affine. We conclude using the equivalent $[U/\operatorname{GL}_n] \cong [P/(\operatorname{GL}_n \times H)]$.

It remains to show (1) $\Rightarrow$ (2). We first show that $\mathcal{X} \cong [U/\operatorname{GL}_n]$ with U an algebraic space. Given a vector bundle E on $\mathcal{X}$ of rank n, the frame bundle $\operatorname{Frame}(E)$ is a principal GL_n-torsor and $\mathcal{X} \cong [\operatorname{Frame}(E)/\operatorname{GL}_n]$ (Exercise 6.1.16).

For every closed point $x \in \mathcal{X}$, let $i_x\colon \mathcal{G}_x \hookrightarrow \mathcal{X}$ be the inclusion of the residual gerbe (Proposition 3.5.16). Let $\kappa(x) \to \Bbbk$ be a finite field extension trivializing $\mathcal{G}_x$, i.e. there is a map $\widetilde{x}\colon \operatorname{Spec} \Bbbk \to \mathcal{X}$ representing x inducing a finite cover $p\colon \mathbf{B}G_{\widetilde{x}} \to \mathcal{G}_x$. Since $G_{\widetilde{x}}$ is affine, we can choose a faithful representation W. Using the resolution property, there is a vector bundle E and a surjection $E \twoheadrightarrow (i_x \circ p)_* W$. The associated frame bundle $\mathrm{Frame}(E) \to \mathcal{X}$ has trivial stabilizers over x. In other words, the kernel subgroup $S_E \subset I_{\mathcal{X}}$ of E (i.e. the subgroup stack of the inertia stack parameterizing elements acting trivially on E) is trivial over x. If F is another vector bundle, then $S_{E\oplus F} \subset S_E$ is a closed subgroup. Since $I_{\mathcal{X}}$ is noetherian, we can inductively enlarge the vector bundle E so that $U := \mathrm{Frame}(E)$ is an algebraic space and $\mathcal{X} \cong [U/\operatorname{GL}_n]$.

Since $\mathcal{X}$ is normal, U is also normal and we may apply Exercise 4.5.3 to conclude that U is the coarse moduli space of the action of a finite group H acting on a normal scheme U'. Let $p\colon U' \to U$ be the quotient morphism, and let $U'_1, \dots, U'_r$ be an affine covering of U' with reduced complements $Z'_1, \dots, Z'_r$. Then $F := p_*(\bigoplus_i I_{Z'_i})$ is a coherent sheaf on U. Moreover, since $q\colon U \to [U/\operatorname{GL}_n]$ is affine, $q^* q_* F \twoheadrightarrow F$ is surjective and by writing $q_* F$ as a colimit of coherent sheaves, we may find a coherent sheaf G on $\mathcal{X} \cong [U/\operatorname{GL}_n]$ and a surjection $q^* G \to F$. Since $\mathcal{X}$ has the resolution property, we see that there is even a vector bundle G and a surjection $q^* G \to F$. Since $p\colon U' \to U$ is affine, we have a surjection $p^* q^* G \twoheadrightarrow p^* F \twoheadrightarrow \bigoplus_i I_{Z'_i}$. Let $V = \mathrm{Frame}(G)$ and consider the cartesian diagram

$$\begin{tikzcd} U'_V \arrow[r] \arrow[d, "\beta"] & U_V \arrow[r] \arrow[d] & V \arrow[d] \\ U' \arrow[r, "p"] & U \arrow[r, "q"] & \mathcal{X} \end{tikzcd}$$

where the horizontal arrows are principal GL_n-bundles and the vertical arrows are GL_m-bundles where $m = \operatorname{rk}(G)$. Since the pullback of G to V is trivial, the pullback $\beta^*(\bigoplus_i I_{Z'_i})$ is globally generated. This implies that $\beta^{-1}(Z'_i)$ is defined by global functions on U'_V and that the complement $\beta^{-1}(U'_i)$ is covered by affine opens of the form $\{f \neq 0\}$ for $f \in \Gamma(U'_V, \mathcal{O}_{U'_V})$. This implies that $\mathcal{O}_{U'_V}$ is ample and that U'_V is a quasi-affine scheme. Since $\beta\colon U'_V \to U_V$ is the quotient by a finite group, U_V is also quasi-affine (Exercise 4.2.8). We have thus shown that $\mathcal{X} \cong [U_V/(\operatorname{GL}_n \times \operatorname{GL}_m)]$. Under the embedding $\operatorname{GL}_n \times \operatorname{GL}_m \hookrightarrow \operatorname{GL}_{n+m}$, the quotient $\operatorname{GL}_{n+m}/(\operatorname{GL}_n \times \operatorname{GL}_m)$ is quasi-affine. Setting $W = U_V \times^{(\operatorname{GL}_n \times \operatorname{GL}_m)} \operatorname{GL}_{n+m}$, we conclude that $\mathcal{X} \cong [W/\operatorname{GL}_{n+m}]$. $\square$

6.1.6 Sheaf cohomology

Abelian sheaf cohomology for algebraic stacks can be developed using essentially the same approach as we used in §4.1.6 for Deligne–Mumford stacks.

Lemma 6.1.23. *If $\mathcal{X}$ is an algebraic stack, the categories $\mathrm{Ab}(\mathcal{X}_{\text{lis-ét}})$ and $\mathrm{Mod}(\mathcal{O}_{\mathcal{X}})$ have enough injectives. If in addition $\mathcal{X}$ is quasi-separated, then $\mathrm{QCoh}(\mathcal{X})$ has enough injectives.*

Proof. The argument of Lemma 4.1.26 generalizes. $\square$

Definition 6.1.24 (Cohomology)**.** Let $\mathcal{X}$ be an algebraic stack and F a sheaf of abelian groups on $\mathcal{X}_{\text{lis-ét}}$. The *cohomology group* $\mathrm{H}^i(\mathcal{X}_{\text{lis-ét}}, F)$ is defined as the ith right derived functor of the global sections functor $\Gamma\colon \mathrm{Ab}(\mathcal{X}_{\text{lis-ét}}) \to \mathrm{Ab}$.

Given a morphism $f\colon \mathcal{X} \to \mathcal{Y}$ of algebraic stacks, the *higher direct image* $R^i f_* F$ is defined as the ith right derived functor of $f_*\colon \mathrm{Ab}(\mathcal{X}_{\text{lis-ét}}) \to \mathrm{Ab}(\mathcal{Y}_{\text{lis-ét}})$.

Definition 6.1.25 (Čech cohomology)**.** Given a smooth covering $\mathcal{U} = \{U_i \to \mathcal{X}\}_{i \in I}$ of algebraic stacks and an abelian sheaf F on $\mathcal{X}_{\text{lis-ét}}$, the *Čech complex of F with respect to $\mathcal{U}$* is $\check{C}^\bullet(\mathcal{U}, F)$ where

$$\check{C}^n(\mathcal{U}, F) = \prod_{(i_0, \ldots, i_n) \in I^{n+1}} F(U_{i_0} \times_{\mathcal{X}} \cdots \times_{\mathcal{X}} U_{i_n})$$

with differential

$$d^n \colon \check{C}^n(\mathcal{U}, F) \to \check{C}^{n+1}(\mathcal{U}, F), \qquad (s_{i_0, \ldots, i_n}) \mapsto \left(\sum_{k=0}^{n+1} (-1)^k p_{\widehat{k}}^* s_{i_0, \ldots, \widehat{i_k}, \ldots, i_n} \right)_{(i_0, \ldots, i_{n+1})}$$

where $p_{\widehat{k}} \colon U_{i_0} \times_{\mathcal{X}} \cdots \times_{\mathcal{X}} U_{i_n} \to U_{i_0} \times_{\mathcal{X}} \cdots \times_{\mathcal{X}} \widehat{U_{i_k}} \times_{\mathcal{X}} \cdots \times_{\mathcal{X}} U_{i_n}$ is the map forgetting the kth component (with indexing starting at 0). The *Čech cohomology of F with respect to $\mathcal{U}$* is

$$\check{\mathrm{H}}^i(\mathcal{U}, F) := \mathrm{H}^i(\check{C}^\bullet(\mathcal{U}, F)).$$

The arguments of Theorem 4.1.29 and Propositions 4.1.33, 4.1.35 and 4.1.36 as well as Exercise 4.1.38 extend.

Theorem 6.1.26. *For a quasi-coherent $\mathcal{O}_{X_{\text{lis-ét}}}$-module F on an affine scheme X, $\mathrm{H}^i(X_{\text{lis-ét}}, F) = 0$ for all $i > 0$.* □

Proposition 6.1.27. *Let $\mathcal{X}$ be an algebraic stack with affine diagonal and F be a quasi-coherent sheaf. If $\mathcal{U} = \{U_i \to \mathcal{X}\}$ is an étale covering with each U_i affine, then $\mathrm{H}^i(\mathcal{X}_{\text{lis-ét}}, F) = \check{\mathrm{H}}^i(\mathcal{U}, F)$.* □

Proposition 6.1.28. *If X is a scheme with affine diagonal and F be a quasi-coherent sheaf, then $\mathrm{H}^i(X, F) = \mathrm{H}^i(X_{\text{lis-ét}}, F_{\text{lis-ét}})$ for all i, where $F_{\text{lis-ét}}$ is the sheaf of $\mathcal{O}_{X_{\text{lis-ét}}}$-module defined by $F_{\text{ét}}(U) = \Gamma(U, f^*F)$ for a smooth map $f \colon U \to \mathcal{X}$ from a scheme.*

Similarly, if $\mathcal{X}$ is a Deligne–Mumford stack with affine diagonal and F is a quasi-coherent sheaf, then $\mathrm{H}^i(\mathcal{X}, F) = \mathrm{H}^i(\mathcal{X}_{\text{lis-ét}}, F_{\text{lis-ét}})$ for all i. □

Proposition 6.1.29. *Let $\mathcal{X}$ be an algebraic stack.*

(1) If F is an $\mathcal{O}_{\mathcal{X}}$-module, then the cohomology $\mathrm{H}^i(\mathcal{X}_{\text{lis-ét}}, F)$ of F as an abelian sheaf agrees with the ith right derived functor of $\Gamma \colon \mathrm{Mod}(\mathcal{O}_{\mathcal{X}}) \to \mathrm{Ab}$.

(2) If $\mathcal{X}$ has affine diagonal and F is a quasi-coherent sheaf on $\mathcal{X}$, then the cohomology $\mathrm{H}^i(\mathcal{X}_{\text{lis-ét}}, F)$ of F as an abelian sheaf agrees with the ith right derived functor of $\Gamma \colon \mathrm{QCoh}(\mathcal{X}) \to \mathrm{Ab}$.

For a morphism $f \colon \mathcal{X} \to \mathcal{Y}$ of algebraic stacks (resp. quasi-compact morphism of algebraic stacks with affine diagonals), then (1) (resp. (2)) holds also for the higher direct images $R^i f_* F$ of an $\mathcal{O}_{\mathcal{X}}$-module (resp. quasi-coherent sheaf): it can be computed as the ith right derived functor of $f_* \colon \mathrm{Mod}(\mathcal{O}_{\mathcal{X}}) \to \mathrm{Mod}(\mathcal{O}_{\mathcal{Y}})$ (resp. $f_* \colon \mathrm{QCoh}(\mathcal{X}) \to \mathrm{QCoh}(\mathcal{Y})$). □

Remark 6.1.30. If $\mathcal{X}$ does not have affine diagonal, then the sheaf cohomology $\mathrm{H}^i(\mathcal{X}_{\text{lis-ét}}, F)$ of a quasi-coherent sheaf may differ from the ith right derived functor of $\Gamma(\mathcal{X}, -) \colon \mathrm{QCoh}(\mathcal{X}) \to \mathrm{Ab}$.

Proposition 6.1.31. *If $\mathcal{X}$ is an algebraic stack and F_i is a directed system of abelian sheaves in $\mathcal{X}_{\text{lis-ét}}$, then $\mathrm{colim}_i\, \mathrm{H}^i(\mathcal{X}, F_i) \to \mathrm{H}^i(\mathcal{X}, \mathrm{colim}_i\, F_i)$ is an isomorphism.* □

6.1.7 Chow groups

Following [Tot99] and [EG98], we introduce the Chow groups of a quotient stack. Let G be a smooth affine algebraic group over an algebraically closed field $\Bbbk$ of dimension g, and let X be an n-dimensional scheme of finite type over $\Bbbk$. For each i, choose an r-dimensional G-representation V such that there is a nonempty open subscheme $U \subset \mathbb{A}(V)$ such that (a) G acts freely on U, (b) the quotient U/G is a scheme, and (c) $\operatorname{codim} \mathbb{A}(V) \setminus U > n - i - g$. Such representations exist. We define the $(i-g)$*th equivariant Chow group of* X or equivariantly the i*th Chow group of* $[X/G]$ as

$$\mathrm{CH}^G_{i-g}(X) = \mathrm{CH}_i([X/G]) := \mathrm{CH}_{i+r}(X \times^G U).$$

This definition is independent of the choice of representation. The definition is forced upon us if we desire invariance of Chow groups under vector bundles and open immersions of high codimension:

$$\begin{tikzcd} X \times^G U \arrow[r, hook, "\text{open}"] & {[(X \times \mathbb{A}(V))/G]} \arrow[d, "\text{vect bdl}"] \\ & {[X/G]} \end{tikzcd} \qquad \begin{tikzcd} \mathrm{CH}_{i+r}(X \times^G U) & \mathrm{CH}_{i+r}([(X \times \mathbb{A}(V))/G]) \arrow[l, "\sim"'] \\ & \mathrm{CH}_i([X/G]) \arrow[u, "\wr"'] \end{tikzcd}$$

If $[X/G]$ is smooth of pure dimension $d = n - g$, then we define

$$\mathrm{CH}^i_G(X) = \mathrm{CH}^i([X/G]) := \mathrm{CH}_{d-i}([X/G])$$
$$\mathrm{CH}^*_G(X) = \mathrm{CH}^*([X/G]) := \bigoplus_i \mathrm{CH}^i((X/G]).$$

The intersection product gives a ring structure, and we call $\mathrm{CH}^*_G(X)$ the *equivariant Chow ring of* X and $\mathrm{CH}^*([X/G])$ the *Chow ring of* $[X/G]$.

Example 6.1.32 ($\mathrm{CH}^*(\mathbf{B}\mathbb{G}_m)$)**.** Let V be the r-dimensional $\mathbb{G}_m$-representation with equal weights 1. Then $\mathbb{G}_m$ acts freely on $\mathbb{A}^n \setminus 0$, and for $-1 \geq i > -r-1$, we have that $\mathrm{CH}_i(\mathbf{B}\mathbb{G}_m) = \mathrm{CH}_{i+r}(\mathbb{P}^{r-1}) = \mathbb{Z}$. It follows that $\mathrm{CH}_i(\mathbf{B}\mathbb{G}_m) = \mathbb{Z}$ for $i \leq -1$ and is 0 otherwise. Therefore $\mathrm{CH}^j(\mathbf{B}\mathbb{G}_m) = \mathbb{Z}$ for $j \geq 0$, and $\mathrm{CH}^*(\mathbf{B}\mathbb{G}_m) = \mathbb{Z}[x]$. More generally, if $T \cong \mathbb{G}_m^r$ is a rank r torus, then $\mathrm{CH}^*(\mathbf{B}T)$ is isomorphic to the character ring $\mathbb{Z}[x_1, \ldots, x_r]$ of T.

We summarize some of the important properties of equivariant Chow groups.

Properties 6.1.33.

(1) (Independent of quotient presentation) If $[X/G] \cong [X'/G']$, *then* $\mathrm{CH}^G_i(X) \cong \mathrm{CH}^{G'}_i(X')$, *and in particular the definition of* $\mathrm{CH}_i([X/G])$ *is independent of the quotient presentation. The definition of Chow groups can be extended to finite type algebraic stacks over* $\Bbbk$*; see [Kre99].*

(2) (Vector bundle invariance) If $Y \to X$ *is* G*-equivariant and a Zariski-local affine fibration of relative dimension* r *(e.g. the total space of a rank* r *vector bundle), then* $\mathrm{CH}^G_*(X) \cong \mathrm{CH}^G_{*+r}(Y)$.

(3) (Excision sequence) If $\mathcal{Z} \subset \mathcal{X} = [X/G]$ *is a closed substack with complement* $\mathcal{U}$*, then there is a right exact sequence*

$$\mathrm{CH}_*(\mathcal{Z}) \to \mathrm{CH}_*(\mathcal{X}) \to \mathrm{CH}_*(\mathcal{U}) \to 0.$$

(4) *(Comparison with coarse moduli space) If $\mathcal{X} \cong [U/G]$ is a separated Deligne–Mumford stack with coarse moduli space X, then $\mathrm{CH}_*(\mathcal{X}) \otimes \mathbb{Q} \cong \mathrm{CH}_*(X) \otimes \mathbb{Q}$.*

(5) *(Functoriality and self-intersection) Flat morphisms induce pullback maps on Chow groups while proper morphisms induce pushforward maps. If $\mathcal{X} = [X/G]$ is smooth and $i\colon \mathcal{Z} \hookrightarrow \mathcal{X}$ is a smooth substack of pure codimension d, then there is pullback $i^*\colon \mathrm{CH}^*(\mathcal{X}) \to \mathrm{CH}^*(\mathcal{Z})$ given by intersection with $\mathcal{Z}$ such that $i^* i_* \alpha = c_d(\mathcal{N}_{\mathcal{Z}/\mathcal{X}}) \cap \alpha$ for $\alpha \in \mathrm{CH}^*(\mathcal{Z})$, where c_d is the top Chern class of the normal bundle.*

(6) *Let T be a torus acting on a smooth scheme X such that $T = T_1 \times T_2$ is a product of two tori with T_2 acting trivially. Then $\mathrm{CH}^*_T(X) \cong \mathrm{CH}^*_{T_1}(X) \otimes \mathrm{CH}^*(\mathbf{B}T_2)$.*

(7) *If G is a connected reductive group with maximal torus T, and X is a smooth scheme with a G-action, then the Weyl group $W = N_G(T)/T$ acts on $\mathrm{CH}^*_T(X)$ and $\mathrm{CH}^*_G(X)_{\mathbb{Q}} = \mathrm{CH}^*_T(X)^W_{\mathbb{Q}}$.*

Exercise 6.1.34.

(a) Let $\mathcal{P}(d_0, \ldots, d_n)$ be the weighted projective stack of Example 3.9.6. Show that $\mathrm{CH}^*(\mathcal{P}(d_0, \ldots, d_n)) \cong \mathbb{Z}[x]/(d_1 \cdots d_n x^{n+1})$.

(b) If $\mathrm{char}(\Bbbk) \neq 2, 3$, show that $\mathrm{CH}^*(\mathcal{M}_{1,1}) \cong \mathbb{Z}[x]/(12x)$ and $\mathrm{CH}^*(\overline{\mathcal{M}}_{1,1}) \cong \mathbb{Z}[x]/(24x^2)$. (Compare with Exercise 6.1.14).

(c) Let $\mathbb{G}_m$ act on $\mathbb{P}^n$ with weights $d_0, \ldots, d_n$. Show that $A^*([\mathbb{P}^n/\mathbb{G}_m]) = \mathbb{Z}[h,t]/p(h,t)$ where $p(h,t) = \sum_{i=0}^n h^i e_i(a_0 t, \ldots, a_n t)$ and e_i is the ith symmetric polynomial.

6.1.8 de Rham and singular cohomology

We quickly discuss the de Rham and singular cohomology of an algebraic stack following [Beh04].

Analyticification. If $\mathcal{X}$ is a smooth algebraic stack over $\mathbb{C}$ with affine diagonal, there is an analyticification $\mathcal{X}^{\mathrm{an}}$, analogous to the analyticification of a finite type $\mathbb{C}$-scheme, such that $\mathcal{X}^{\mathrm{an}}$ is a *differentiable stack*. If $U_0 \to \mathcal{X}$ is a smooth presentation by a scheme so that $\mathcal{X}$ is the quotient of the smooth groupoid $U_1 := U_0 \times_{\mathcal{X}} U_0 \rightrightarrows U_0$, then $U_0^{\mathrm{an}} \to \mathcal{X}^{\mathrm{an}}$ is a smooth presentation and $\mathcal{X}^{\mathrm{an}}$ is the quotient of the *Lie groupoid* $U_1^{\mathrm{an}} \rightrightarrows U_0^{\mathrm{an}}$.

De Rham cohomology of a differential stack. Given a differentiable stack $\mathcal{X}$ with a smooth presentation $U_0 \to \mathcal{X}$, we can define a *simplicial manifold* $U_\bullet$

$$\cdots U_3 \Rrightarrow U_2 \Rrightarrow U_1 \longrightarrow U_0, \quad \text{where } U_p := \underbrace{U_0 \times_{\mathcal{X}} \cdots \times_{\mathcal{X}} U_0}_{p \text{ times}} \tag{6.1.1}$$

with maps $\partial_i\colon U_p \to U_{p-1}$ forgetting the ith term along with degeneracy maps $s_i\colon U_{p-1} \to U_p$ inserting an identity morphism in the ith term. This defines a double complex $\Omega^q(U_p)$ with differentials given by exterior differentiation $d\colon \Omega^{q-1}(U_p) \to \Omega^q(U_p)$ and $\partial := \sum_{i=0}^p (-1)^i \partial_i^*\colon \Omega^q(U_{p-1}) \to \Omega^q(U_p)$. We define the *de Rham complex* $C^\bullet_{\mathrm{dR}}(\mathcal{X})$ as the total complex

$$C^k_{\mathrm{dR}}(\mathcal{X}) := \bigoplus_{p+q=k} \Omega^q(U_p),$$

with differential $\delta\colon C^k_{\mathrm{dR}}(\mathcal{X}) \to C^{k+1}_{\mathrm{dR}}(\mathcal{X})$ defined by $\delta(\omega) = \partial(\omega) + (-1)^p d(\omega)$ for $\omega \in \Omega^p(U_q)$. The *de Rham cohomology* is

$$\mathrm{H}^n_{\mathrm{dR}}(\mathcal{X}) := \mathrm{H}^n(C^\bullet_{\mathrm{dR}}(\mathcal{X})),$$

and is independent of the choice of presentation. As with the case of smooth manifolds, there is an identification of $\mathrm{H}^n_{\mathrm{dR}}(\mathcal{X})$ with the sheaf cohomology of the constant sheaf $\mathbb{R}$ on the big smooth site of smooth manifolds over $\mathcal{X}$.

Singular homology/cohomology of a topological stack. For a *topological stack* $\mathcal{X}$, one can replicate the constructions of singular homology and cohomology. Let $U_0 \to \mathcal{X}$ be a presentation and $U_\bullet$ be the simplicial topological space as in (6.1.1). For each p, we have the singular chain complex $C_\bullet(U_p)$ with differentials $d : C_q(U_p) \to C_{q-1}(U_p)$. This defines a double complex $C_q(U_p)$ using the differential $\partial = \sum_{i=0}^p (-1)^i \partial_j \colon C_q(U_p) \to C_q(U_{p-1})$ induced by the maps $\partial_i \colon U_p \to U_{p-1}$. We define the *singular chain complex* $C_\bullet(\mathcal{X})$ *of* $\mathcal{X}$ as the total complex

$$C_k(\mathcal{X}) := \bigoplus_{p+q=k} C_q(U_p)$$

with the differential $\delta\colon C_k(\mathcal{X}) \to C_{k-1}(\mathcal{X})$ given by $\delta(\gamma) = (-1)^{p+q}\partial(\gamma) + (-1)^q d(\gamma)$ for $\gamma \in C_q(U_p)$. For an abelian group A, we can therefore define the *singular homology groups of* $\mathcal{X}$ *with coefficients in* A as

$$\mathrm{H}_n(\mathcal{X}, A) := \mathrm{H}_n(C_\bullet(\mathcal{X}) \otimes_{\mathbb{Z}} A).$$

Dualizing, we define the *singular cochain complex* $C^\bullet(\mathcal{X})$ by $C^n(\mathcal{X}) := \mathrm{Hom}(C_n(\mathcal{X}), \mathbb{Z})$ and the *singular cohomology groups of* $\mathcal{X}$ *with coefficients in* A as

$$\mathrm{H}^n(\mathcal{X}, A) := \mathrm{H}^n(C^\bullet(\mathcal{X}) \otimes_{\mathbb{Z}} A).$$

Comparisons.

- There are pairings $\mathrm{H}_k(\mathcal{X}, \mathbb{Z}) \otimes \mathrm{H}^k(\mathcal{X}, \mathbb{Z}) \to \mathbb{Z}$ which after tensoring with $\mathbb{Q}$ gives identifications $\mathrm{H}^k(\mathcal{X}, \mathbb{Q}) \cong \mathrm{H}_k(\mathcal{X}, \mathbb{Q})^\vee$.
- If G is a topological group acting on a space U, then the *equivariant cohomology* is defined as $\mathrm{H}^*_G(U, A) := \mathrm{H}^*(EG \times^G U, A)$, where EG is a contractible space with a free action of G, and there is an identification $\mathrm{H}^*([U/G], A) = \mathrm{H}^*_G(U, A)$.
- For a differential stack $\mathcal{X}$, there is an identification $\mathrm{H}^*_{\mathrm{dR}}(\mathcal{X}) = \mathrm{H}^*(\mathcal{X}, \mathbb{R})$.
- If $\mathcal{X}$ is a *topological Deligne–Mumford stack* (e.g. the topological stack associated to a separated Deligne–Mumford stack of finite type over $\mathbb{C}$) with coarse moduli space $\mathcal{X} \to X$, then $\mathrm{H}^*(\mathcal{X}, \mathbb{Q}) = \mathrm{H}^*(X, \mathbb{Q})$.

6.2 The fppf topology and gerbes

This section is not essential for the proofs of the two main theorems of this book and is included for completeness. We prove that algebraic spaces/stacks are sheaves/stacks in the fppf topology and that quotients by fppf groupoids/equivalence relations are algebraic. One upshot is that $\mathbf{B}G$ is an algebraic stack for any (non-necessarily smooth) algebraic group, e.g. $\boldsymbol{\mu}_p$ in characteristic p.

We also introduce gerbes, a central topic in the theory of stacks. For us, we want to know that residual gerbes are gerbes (justifying the terminology) and later that the moduli stack $\mathcal{B}\mathrm{un}^{\mathrm{s}}(C)_{r,d}$ of stable vector bundles is a $\mathbb{G}_m$-gerbe over its coarse moduli space.

6.2.1 Fppf criterion for algebraicity

Theorem 6.2.1 (Fppf Criterion for Algebraicity)**.**

(1) If X is a sheaf on $\mathrm{Sch}_{\mathrm{fppf}}$ *such that there exists an fppf representable morphism $U \to X$ from a scheme, then X is an algebraic space.*

(2) If $\mathcal{X}$ is a stack over $\mathrm{Sch}_{\mathrm{fppf}}$ *such that there exists an fppf representable morphism $U \to \mathcal{X}$ from a scheme, then $\mathcal{X}$ is an algebraic stack.*

Proof. To add. ☐

Algebraic spaces are by definition sheaves in the big étale topology but it turns out they are also sheaves in the big fppf topology.

Proposition 6.2.2.

(1) An algebraic space X over a scheme S is a sheaf on $(\mathrm{Sch}/S)_{\mathrm{fppf}}$.

(2) An algebraic stack $\mathcal{X}$ over a scheme S is a stack over $(\mathrm{Sch}/S)_{\mathrm{fppf}}$.

Proof. To add.

☐

This allows us to finally prove that many properties of representable morphisms of algebraic stacks descend in the fppf topology. Smooth descent was established in Proposition 3.3.3

Proposition 6.2.3. *Let $\mathcal{P}$ be one of the following properties of morphisms of algebraic stacks: representable, isomorphism, open immersion, closed immersion, locally closed immersion, affine, or quasi-affine. Consider a cartesian diagram*

$$\begin{tikzcd} \mathcal{X}' \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{Y}' \arrow[d] \\ \mathcal{X} \arrow[r] & \mathcal{Y} \end{tikzcd}$$

of algebraic stacks where $\mathcal{Y}' \to \mathcal{Y}$ is fppf. Then $\mathcal{X} \to \mathcal{Y}$ has $\mathcal{P}$ if and only if $\mathcal{X}' \to \mathcal{Y}'$ has $\mathcal{P}$.

6.2.2 Fppf equivalence relations and groupoids

If $R \rightrightarrows U$ is an fppf equivalence relation of algebraic spaces, we define U/R as the sheafification in big fppf topology $\mathrm{Sch}_{\mathrm{fppf}}$ of the presheaf $T \mapsto U(T)/R(T)$. Likewise, if $s, t \colon R \rightrightarrows U$ is an fppf groupoid of algebraic spaces, we define $[U/R]$ as the stackification in $\mathrm{Sch}_{\mathrm{fppf}}$ of the prestack $[U/R]^{\mathrm{pre}}$, whose fiber category over a scheme T is the category of T-points of U where a morphism from $a \in U(T)$ to $b \in U(T)$ is an element $r \in R(R)$ such that $s(r) = a$ and $t(r) = b$.

The definitions of U/R and $[U/R]$ are consistent with the quotient of a smooth equivalence relation or groupoid as defined in Definition 3.4.7 using in the big étale topology $\mathrm{Sch}_{\text{ét}}$. This is because the sheafification U/R in $\mathrm{Sch}_{\text{ét}}$ is an an algebraic space by Corollary 4.4.11 and thus a sheaf in the fppf topology by Proposition 6.2.2. Similarly, the stackification $[U/R]$ over $\mathrm{Sch}_{\text{ét}}$ is an algebraic stack by Theorem 3.4.11 and thus a stack in the fppf topology by Proposition 6.2.2.

Corollary 6.2.4.

(1) If $R \rightrightarrows U$ is an fppf equivalence relation of algebraic spaces, then the quotient U/R is an algebraic space.

(2) If $R \rightrightarrows U$ is an fppf groupoid of algebraic spaces, then the quotient $[U/R]$ is an algebraic stack.

Proof. To add. □

We will now show that a quotient stacks arising from the action of an fppf group algebraic space is an algebraic stacks; this was shown for the action of a smooth affine group schemes in Corollary 3.1.10. We first need to generalize the definition of a principal G-bundle given in Definition C.2.1 for an action by an fppf group algebraic space.

Definition 6.2.5 (Principal G-bundles)**.** If $G \to S$ is an fppf group algebraic space, then a *principal G-bundle over an S-scheme T* is an algebraic space P with an action of G via $\sigma\colon G \times_S P \to P$ such that $P \to X$ is a G-invariant fppf morphism and $(\sigma, p_2)\colon G \times_S P \to P \times_T P$ is an isomorphism. *Morphisms of principal G-bundles* are G-equivariant morphisms of schemes. We say that a principal G-bundle $P \to T$ is trivial if there is a G-equivariant isomorphism $P \cong G \times T$.

When $G \to S$ is smooth, then every principal G-bundle $P \to T$ is trivialized by the smooth cover $P \to T$ and since smooth morphisms étale locally have sections, there is an étale cover $T' \to T$ such that $P_{T'}$ is trivial.

Remark 6.2.6. It is important to require that P is an algebraic space and not a scheme since we want principal G-bundles to satisfy descent and to be equivalent to the notion of a G-torsor (Definition 6.2.12). If $G \to S$ is affine, then P is automatically a scheme and the above definition thus agrees with Definition C.2.1. Indeed, P is a sheaf in the fppf topology (Proposition 6.2.2) and if $U \to P$ is an étale presentation, then $P \to T$ pulls back under the fppf composition $U \to P \to T$ to the affine morphism $G \times_S U \to U$. By Effective Descent (Proposition 2.2.11), $P \to T$ is affine and in particular that P is a scheme.

Raynaud provides an example of an abelian variety G and a principal G-bundle that is not scheme [Ray70, XIII 3.2].

Definition 6.2.7 (Quotient stacks)**.** Let $G \to S$ be an fppf group algebraic space acting on an algebraic space U over S. We define the *quotient stack* $[U/G]$ as the category over Sch/S whose objects over an S-scheme T are diagrams

$$\begin{tikzcd} P \arrow[r] \arrow[d] & U \\ T & \end{tikzcd} \tag{6.2.1}$$

where $P \to T$ is a principal G-bundle and $P \to U$ is a G-equivariant morphism of schemes. A morphism $(P' \to T', P' \to U) \to (P \to T, P \to U)$ consists of a morphism $T' \to T$ and a G-equivariant morphism $P' \to P$ of schemes such that the diagram

$$\begin{tikzcd} P' \arrow[r] \arrow[d] \arrow[rr, bend left] \arrow[dr, phantom, "\square"] & P \arrow[r] \arrow[d] & U \\ T' \arrow[r] & T & \end{tikzcd}$$

is commutative and the left square is cartesian.

Definition 6.2.8 (Classifying stacks)**.** Let $G \to S$ be an fppf group algebraic space. The *classifying stack* $\mathbf{B}G$ of G is defined as the quotient stack $[S/G]$. It classifies principal G-bundles $P \to T$.

Proposition 6.2.9. *If $G \to S$ is an fppf group algebraic space acting on an algebraic space U over S, then the quotient stack $[U/G]$ is an algebraic stack. In particular, the classifying stack $\mathbf{B}G$ is algebraic.*

Proof. Given a map $T \to [U/G]$ corresponding to an object (6.2.1), there is a cartesian diagram

$$\begin{tikzcd} P \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & U \arrow[d] \\ T \arrow[r] & {[U/G]} \end{tikzcd}$$

of stacks over $\mathrm{Sch}_{\mathrm{fppf}}$; this extends Exercise 2.3.28. As $P \to T$ is an fppf morphism of algebraic spaces, $U \to [U/G]$ is an fppf representable morphism. It follows from Theorem 6.2.1 that $[U/G]$ is an algebraic stack. $\square$

Exercise 6.2.10. Let $G \to S$ be an fppf group algebraic space acting on an algebraic space U over S.

(a) Generalize Exercise 2.4.16 by showing that the stackification of the prestack $[U/G]^{\mathrm{pre}}$ in the fppf topology is $[U/G]$.

(b) Provide an example where the stackification of $[U/G]^{\mathrm{pre}}$ in the étale topology is not isomorphic to $[U/G]$.

Recalling that $\boldsymbol{\mu}_{n,\mathbb{Z}}$ is the subgroup of $\mathbb{G}_{m,\mathbb{Z}}$ defined by $\operatorname{Spec} \mathbb{Z}[x]/(x^n - 1)$, we can now deduce that $\mathbf{B}\boldsymbol{\mu}_{n,\mathbb{Z}}$ is an algebraic stack. If $\Bbbk$ is a field of characteristic p, then $\boldsymbol{\mu}_n := \boldsymbol{\mu}_{n,\Bbbk}$ is smooth if and only if p doesn't divide n.

Exercise 6.2.11. Let $\Bbbk$ be a field.

(a) Exhibit an explicit smooth presentation of $\mathbf{B}\boldsymbol{\mu}_n$.

(b) Show that $\mathbf{B}\boldsymbol{\mu}_n$ is equivalent to the stack over $(\mathrm{Sch}/\Bbbk)_{\text{ét}}$ whose objects over a scheme T are pairs (L, α) consisting of a line bundle L on T and a trivialization $\alpha\colon \mathcal{O}_T \xrightarrow{\sim} L^{\otimes n}$.

(c) Show that $\mathbf{B}\boldsymbol{\mu}_n$ is a smooth and proper algebraic stack of dimension 0.

(d) Show that $\mathbf{B}\boldsymbol{\mu}_n$ is a Deligne–Mumford stack if and only if n is prime to the characteristic.

(e) If $x\colon \operatorname{Spec} \Bbbk \to \mathbf{B}\boldsymbol{\mu}_n$ denotes the canonical presentation, compute the tangent space $T_{\mathbf{B}\boldsymbol{\mu}_n, x}$.

6.2.3 Torsors

If G is a sheaf of groups, then a G-torsor is a sheaf of sets locally isomorphic to G.

Definition 6.2.12 (Torsors)**.** Let $\mathcal{S}$ be a site and G a sheaf of (not necessarily abelian) groups on $\mathcal{S}$. A *G-torsor on $\mathcal{S}$* is a sheaf P of sets on $\mathcal{S}$ with a left action $\sigma\colon G \times P \to P$ of G such that

(1) for every object $T \in \mathcal{S}$, there exists a covering $\{T_i \to T\}$ such that $P(T_i) \neq 0$ for each i, and

(2) the action map $(\sigma, p_2)\colon G \times P \to P \times P$ is an isomorphism.

If $T \in \mathcal{S}$ is an object and G is a sheaf of groups on the localized site $\mathcal{S}/T$, then a *G-torsor over T* is by definition a G-torsor on the $\mathcal{S}/T$.

Morphisms of G-torsors are G-equivariant morphisms of sheaves. We say that a G-torsor P is *trivial* if P is G-equivalently isomorphic to G.

Exercise 6.2.13. Show that Any morphism of G-torsors is an isomorphism.

Example 6.2.14. Let $\mathcal{X}$ be a stack over a site $\mathcal{S}$, and let $a, b \in \mathcal{X}$ be objects over $S \in \mathcal{S}$. The sheaf $\underline{\mathrm{Isom}}_S(a, b)$ of isomorphisms is a torsor for $\underline{\mathrm{Aut}}(a)$ under the action given by precomposition.

Given a morphism $f\colon T' \to T$ and a G-torsor P over T, the restriction $P|_{T'}$ is the sheaf on $\mathcal{S}/T'$ whose whose sections over a T'-scheme S are $P(S)$; the restriction $P|_{T'}$ is naturally a G-torsor over T'.

Exercise 6.2.15. Let $\mathcal{S}$ be a site with a final object S and G be a sheaf of groups on $\mathcal{S}$.

(a) Show that Axiom (1) is equivalent to $P \to S$ being an epimorphism of sheaves.

(b) If P is a G-torsor, show that S is isomorphic to the quotient sheaf P/G.

(c) Show that a G-torsor P is trivial if and only if there exists a section $s\colon S \to P$ of the structure morphism $P \to S$.

(d) Show that a sheaf P of sets on $\mathcal{S}$ with a left action by G is a G-torsor if and only if there exist a covering $\{S_i \to S\}$ and isomorphisms $P|_{S_i} \cong G|_{S_i}$ of $G|_{S_i}$-torsors.

Example 6.2.16 (Principal G-bundles)**.** If $G \to S$ is an fppf group scheme, then there is an equivalence of categories between G-torsors in the fppf topology and principal G-bundles (as defined in Definition 6.2.5). To see this, first suppose that $P \to T$ is a principal G-bundle over an S-scheme T, i.e. $P \to T$ is an fppf morphisms of algebraic spaces where G is equipped with a free and transitive action of $G \times_S T$. Since algebraic spaces are sheaves in the fppf topology (Proposition 6.2.2), we may view $G \times_S T$ as a sheaf of groups on $(\mathrm{Sch}/T)_{\mathrm{fppf}}$ and P as a sheaf of sets on $(\mathrm{Sch}/T)_{\text{ét}}$. Since every principal G-bundle is locally trivial in the fppf topology (Proposition C.2.4), P is a $G \times_S T$-torsor on $(\mathrm{Sch}/T)_{\mathrm{fppf}}$. Conversely, given a $G \times_S T$-torsor P on $(\mathrm{Sch}/T)_{\mathrm{fppf}}$, then by Exercise 6.2.15 there is an fppf cover $T' \to T$ such that $P \times_T T' \cong G \times_T T'$. Therefore, $P \times_T T' \to P$ is an fppf morphism from an algebraic space and Corollary 6.2.4 implies that P is an algebraic space. It follows that $P \to T$ is a principal G-bundle.

If in addition $G \to S$ is smooth, then there is an equivalence of categories between G-torsors in the étale topology and principal G-bundles. This holds because every principal G-bundle $P \to T$ is étale locally trivial and therefore P is a $G \times_S T$-torsor on $(\mathrm{Sch}/T)_{\text{ét}}$.

6.2.4 Gerbes

Gerbes are a 2-categorical generalization of torsors. While torsors are locally isomorphic to a sheaf of groups G, gerbes are locally isomorphic to classifying stacks $\mathbf{B}G$.

Definition 6.2.17 (Gerbes)**.** A stack $\mathcal{X}$ over a site $\mathcal{S}$ is called a *gerbe* if

(1) for every object $T \in \mathcal{S}$, there exists a covering $\{T_i \to T\}$ in $\mathcal{S}$ such that each fiber category $\mathcal{X}(T_i)$ is non-empty; and

(2) for objects $x, y \in \mathcal{X}$ over $T \in \mathcal{S}$, there exists a covering $\{T_i \to T\}$ and isomorphisms $x|_{T_i} \xrightarrow{\sim} y|_{T_i}$ for each i.

We say that a gerbe $\mathcal{X}$ is *trivial* if there is a section $\mathcal{S} \to \mathcal{X}$ of $\mathcal{X} \to \mathcal{S}$. When $\mathcal{S}$ has a final object S, then the triviality of a gerbe $\mathcal{X}$ is equivalent to the existence of an element of $\mathcal{X}(S)$.

Example 6.2.18. If G is a sheaf of groups on a site $\mathcal{S}$, then we extend Definition 6.2.8 by defining the *classifying prestack of* G as the category $\mathbf{B}G$ over $\mathcal{S}$ consisting of pairs (P, T) where $T \in \mathcal{S}$ and P is G-torsor over $\mathcal{S}/T$ (Definition 6.2.12). A morphism $(P', T') \to (P, T)$ is the data of a morphism $T' \to T$ in $\mathcal{S}$ and an isomorphism $P' \to P|_{T'}$ of G-torsors, where $P|_{T'}$ denotes the restriction of P along $T' \to T$.

The classifying stack $\mathbf{B}G$ is a gerbe over $\mathcal{S}$ because every G-torsor over T is locally isomorphic to the trivial G-torsor $G \times T$.

Exercise 6.2.19 (Gerbes are locally classifying stacks)**.** Let $\mathcal{S}$ be a site with a final object $S \in \mathcal{S}$, and let $\mathcal{X}$ be a stack over $\mathcal{S}$. Show that $\mathcal{X}$ is a gerbe if and only if there exist a covering $\{S_i \to S\}$ and sheaves of groups G_i on $\mathcal{S}/S_i$ such that there is an isomorphism $\mathcal{X} \times_{\mathcal{S}/S} \mathcal{S}/S_i \cong \mathbf{B}G_i$ over $\mathcal{S}/S_i$.

Exercise 6.2.20. Let S be a scheme and let $\mathcal{X}$ be a gerbe over $(\mathrm{Sch}/S)_{\mathrm{fppf}}$. If the diagonal $\mathcal{X} \to \mathcal{X} \times_S \mathcal{X}$ is representable, show that $\mathcal{X}$ is an algebraic stack.

An important type of gerbe $\mathcal{X}$ is one that is *banded* by a sheaf of groups. This means that $\mathcal{X}$ is equipped with the additional data of a natural isomorphism $G(T) \to \mathrm{Aut}_T(x)$ for every object $x \in \mathcal{X}(t)$.

Definition 6.2.21 (Banded G-gerbes)**.** Let G be an *abelian* sheaf on a site $\mathcal{S}$. A stack $\mathcal{X}$ over $\mathcal{S}$ is a *gerbe banded by* G (or a *banded* G*-gerbe* or simply a G*-gerbe*) is a gerbe together with the data of isomorphisms $\psi_x \colon G|_T \to \underline{\mathrm{Aut}}_T(x)$ of sheaves for each object $x \in \mathcal{X}(T)$. We require that for each isomorphism $\alpha \colon x \xrightarrow{\sim} y$ over T, the diagram

$$\begin{tikzcd} & G|_T \arrow[dl, "\psi_x"'] \arrow[dr, "\psi_y"] & \\ \underline{\mathrm{Aut}}_T(x) \arrow[rr, "\mathrm{Inn}_\alpha"] & & \underline{\mathrm{Aut}}_T(y). \end{tikzcd} \tag{6.2.2}$$

commutes, where $\mathrm{Inn}_\alpha(\tau) = \alpha\tau\alpha^{-1}$. The data of the isomorphisms ψ_x is called the *band* of $\mathcal{X}$.

A *morphism of banded G-gerbes* is a morphism of stacks compatible with the bands.

Remark 6.2.22. Here's another way to think about a band of a gerbe. Let $\mathcal{X}_{\mathcal{S}}$ be the *restricted site* whose underlying category is $\mathcal{X}$ and where a covering of $a \in \mathcal{X}(S)$ is a covering of S. Then the inertia stack $I_{\mathcal{X}} = \mathcal{X} \times_{\mathcal{X}\times\mathcal{X}} \mathcal{X}$ is a sheaf of groups on $\mathcal{X}_{\mathcal{S}}$: for $a \in \mathcal{X}(S)$, we have $I_{\mathcal{X}}(a) = \mathrm{Isom}_S(a)$. The compatibility condition (6.2.2) ensures that there is an isomorphism $\psi\colon G|_{\mathcal{X}} \to I_{\mathcal{X}}$ of sheaves on $\mathcal{X}_{\mathcal{S}}$.

Example 6.2.23 (The trivial banded gerbe)**.** If G is an abelian sheaf on a site $\mathcal{S}$, then the classifying stack $\mathbf{B}G$ of Example 6.2.18 is a banded G-gerbe, which we refer to as the *trivial banded G-gerbe*.

Exercise 6.2.24 (Band associated to a gerbe)**.** Let $\mathcal{S}$ be a site with a final object S. Let $\mathcal{X}$ be an *abelian gerbe* over $\mathcal{S}$, i.e. a gerbe $\mathcal{X}$ such that $\mathrm{Aut}_T(a)$ is abelian for every object $a \in \mathcal{X}(T)$. Show that there is a sheaf of groups G on S such that $\mathcal{X}$ is banded by G.

Hint: Use Axiom (1) of a gerbe to find a covering $\{X_i \to X\}$ and elements $a_i \in \mathcal{X}(X_i)$. Use Axiom (2) to glue the sheaves $G_i := \underline{\mathrm{Aut}}_{X_i}(a_i)$ to a sheaf G.

6.2.5 Algebraic gerbes

Attached to any algebraic stack $\mathcal{Y}$ is the big fppf site $(\mathrm{Sch}/\mathcal{Y})_{\mathrm{fppf}}$ of schemes over $\mathcal{Y}$: the underlying category of $(\mathrm{Sch}/\mathcal{Y})_{\mathrm{fppf}}$ is $\mathcal{Y}$ and a covering of an object $y \in \mathcal{Y}(T)$ is a covering of T. Moreover, if $\mathcal{X} \to \mathcal{Y}$ is a morphism of algebraic stacks, then $\mathcal{X}$ is a stack over $(\mathrm{Sch}/\mathcal{Y})_{\mathrm{fppf}}$ thanks to Proposition 6.2.2.

Definition 6.2.25 (Gerbes)**.** A morphism $\mathcal{X} \to \mathcal{Y}$ of algebraic stacks is a *gerbe* if $\mathcal{X}$ is a gerbe over the big fppf site $(\mathrm{Sch}/\mathcal{Y})_{\mathrm{fppf}}$.

We say that an algebraic stack $\mathcal{X}$ is a *gerbe* if there exists a morphism $\mathcal{X} \to X$ to an algebraic space which is a gerbe.

Proposition 6.2.26. *Let $\mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks.*

(1) The morphism $\mathcal{X} \to \mathcal{Y}$ is a gerbe if and only if there exist an fppf morphism $V \to \mathcal{Y}$ from a scheme and an fppf group algebraic space $G \to V$ such that $\mathcal{X} \times_{\mathcal{Y}} V \cong \mathbf{B}G$.

(2) If $\mathcal{X} \to \mathcal{Y}$ is a gerbe, then $\mathcal{X} \to \mathcal{Y}$ is a smooth morphism.

Proof. The second statement follows from the first since $\mathbf{B}G \to V$ is a smooth morphism; indeed smoothness is an fppf local property on the source (Proposition B.4.2). For ($\Rightarrow$), Exercise 6.2.19 implies that there is an fppf morphism $V \to \mathcal{Y}$ and a sheaf of groups G on V such that $\mathcal{X} \times_{\mathcal{Y}} V \cong \mathbf{B}G$. Since $\mathcal{X} \times_{\mathcal{Y}} V$ is an algebraic stack, it's diagonal is representable and thus G is an algebraic space. Conversely, suppose that $\mathcal{X} \times_{\mathcal{Y}} V \cong \mathbf{B}G$ for an fppf morphism $V \to \mathcal{Y}$ and an fppf group algebraic space $G \to V$. To see Axiom (1) of a gerbe, if $a \in (\mathrm{Sch}/\mathcal{Y})$ is an object over a scheme T, then $V_T := V \times_{\mathcal{Y}} T \to T$ is an fppf covering and since $\mathcal{X} \times_{\mathcal{Y}} V_T \cong \mathbf{B}G_{V_T}$, there is an object of $\mathcal{X}$ over V_T. Similarly for Axiom (2), if $x_1, x_2 \in \mathcal{X}$ are objects over $y \in \mathcal{Y}(T)$, then pull backs of x_1 and x_2 become isomorphic under the fppf covering $V_T \to T$. □

Exercise 6.2.27.

(a) Show that an algebraic stack $\mathcal{X}$ is a gerbe if and only if $I_{\mathcal{X}} \to \mathcal{X}$ is fppf.

(b) Show that a morphism $\mathcal{X} \to \mathcal{Y}$ of algebraic stacks is a gerbe if and only if $\mathcal{X} \to \mathcal{Y}$ and $\mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ are fppf.

Exercise 6.2.28. Let $G \to S$ be a *commutative*, fppf, and affine group scheme.

(1) Show that $\mathbf{B}G \to S$ is a banded G-gerbe.

(2) Show that a banded G-gerbe $\mathcal{X} \to \mathcal{Y}$ over S is trivial (i.e. admits a section) if and only if $\mathcal{X} \cong \mathbf{B}G \times_S \mathcal{Y}$ over $\mathcal{Y}$.

6.2.6 Cohomological characterization

The following exercises provide cohomological characterizations of torsors and gerbes for an abelian sheaf G on the small fppf site S_{fppf} of a scheme S. If G is represented by a smooth, commutative, and quasi-projective group scheme, then it turns out that $\mathrm{H}^i((\mathrm{Sch}/S)_{\mathrm{fppf}}, G) = \mathrm{H}^i((\mathrm{Sch}/S)_{\text{ét}}, G)$ (see Remark 4.1.39) and thus in this case we can use étale cohomology. For an extra challenge, try to prove these statements for abelian sheaves over any site. The reader may consult [Gir71] and [Ols16, §12] for detailed proofs.

Exercise 6.2.29 (Torsors)**.** Let S be a scheme.

(a) If G is an abelian sheaf on S_{fppf}, show that $\mathrm{H}^1(S_{\mathrm{fppf}}, G)$ is in bijective correspondence with isomorphism classes of G-torsors.

Hint: Imitate the proof using Čech cohomology that $\mathrm{H}^1(X, \mathcal{O}_X^*) = \mathrm{Pic}(X)$ *for a scheme* X*.*

(b) Let $0 \to G' \to G \to G'' \to 0$ be an exact sequence of abelian sheaves on S_{fppf}, and let

$$\begin{aligned} 0 \to \mathrm{H}^0(S_{\mathrm{fppf}}, G') \to \mathrm{H}^0(S_{\mathrm{fppf}}, G) &\to \mathrm{H}^0(S_{\mathrm{fppf}}, G'') \xrightarrow{\delta} \\ \to \mathrm{H}^1(S_{\mathrm{fppf}}, G') \xrightarrow{\alpha} \mathrm{H}^1(S_{\mathrm{fppf}}, G) &\xrightarrow{\beta} \mathrm{H}^1(S_{\mathrm{fppf}}, G'') \to \cdots \end{aligned}$$

be the corresponding long exact sequence. Show that under the bijection in (a), the boundary map δ assigns a section $S \to G''$ to the G'-torsor defined by the fiber product $G \times_{G''} S$. Show also that α assigns a G'-torsor P' to the quotient $P' \times^{G'} G := (P' \times G)/G'$ while β assigns a G-torsor P to $P \times^G G''$.

Exercise 6.2.30 (Gerbes)**.** Let S be a scheme.

(a) If G is an abelian sheaf on S_{fppf}, show that $\mathrm{H}^2(\mathcal{S}, G)$ is in bijective correspondence with isomorphism classes of G-banded gerbes.

Hint: Let $0 \to G \to I^0 \xrightarrow{d^0} I^1 \xrightarrow{d^1} I^2 \xrightarrow{d^2} \cdots$ *be an injective resolution. For a cohomology class* $\alpha \in \mathrm{H}^2(\mathcal{S}, G)$*, define a stack* $\mathcal{G}_\alpha$ *over* $\mathcal{S}$ *as follows. Choose* $\tau \in \Gamma(\mathcal{S}, I^2)$ *with* $\mathrm{d}^2(\tau) = 0$ *such that the image of* τ *in* $\mathrm{H}^2(\mathcal{S}, G)$ *is* α*. Define* $\mathcal{G}_\alpha$ *as the category of pairs* (S, σ) *consisting of an object* $S \in \mathcal{S}$ *and a section* $\sigma \in \Gamma(S, I^1)$ *with* $\mathrm{d}^1(\sigma) = \tau|_S$*. A morphism* $(S', \sigma') \to (S, \sigma)$ *is the data of a morphism* $f \colon S' \to S$ *and an element* $\rho \in \Gamma(S', I^0)$ *with boundary* $\mathrm{d}^0(\rho) = \sigma' - f^*(\sigma)$*. Show that* $\mathcal{G}_\alpha$ *is a* G*-banded gerbe and that the assignment* $\alpha \mapsto \mathcal{G}_\alpha$ *gives the stated bijection.*

Let $0 \to G' \to G \to G'' \to 0$ be an exact sequence of abelian sheaves on S_{fppf}, and let

$$\cdots \to \mathrm{H}^1(S_{\mathrm{fppf}}, G'') \xrightarrow{\delta} \mathrm{H}^2(S_{\mathrm{fppf}}, G') \xrightarrow{\alpha} \mathrm{H}^2(S_{\mathrm{fppf}}, G) \xrightarrow{\beta} \mathrm{H}^2(S_{\mathrm{fppf}}, G'') \to \cdots$$

be the corresponding long exact sequence.

(b) Show that under the bijection in (a), the boundary map δ assigns a G''-torsor $P'' \to S$ to the *gerbe of trivializations* $\mathcal{G}_{P''}$. The objects of the prestack $\mathcal{G}_{P''}$ over an S-scheme T is a pair (P, α) consisting of a G-torsor $P \to T$ and a trivialization $\alpha\colon P \times^G G'' \cong P'' \times_S T$ of G''-torsors. Morphisms in $\mathcal{G}_{P''}$ are morphisms of G-torsors compatible with the trivializations.

(c) Suppose that G', G and G'' are represented by commutative and affine algebraic groups over a field $\Bbbk$. Show that if $P'' \to S$ is a G''-torsor, then $\mathcal{G}_{P''}$ is identified with both the quotient stack $[P''/G]$ and the fiber product of $\mathbf{B}G \to \mathbf{B}G''$ and the map $S \to \mathbf{B}G''$ corresponding the G''-torsor P''.

Exercise 6.2.31 (Group structure)**.** If G is an abelian sheaf on the small fppf site S_{fppf} of a scheme S, show that the group laws of $\mathrm{H}^1(S_{\mathrm{fppf}}, G)$ and $\mathrm{H}^2(S_{\mathrm{fppf}}, G)$ can be described geometrically as follows:

(a) The product of two G-torsors P_1 and P_2 is the *contracted product* $P_1 \wedge^G P_2$ defined as the sheaf quotient $(P_1 \times P_2)/G$ where $h \cdot (p_1, p_2) = (h^{-1}p_1, hp_2)$ with the G-action specified by $g \cdot (p_1, p_2) = (gp_1, p_2) = (p_1, gp_2)$. The inverse of a G-torsor P is the sheaf P with the inverted G-action: $g \cdot p = g^{-1}p$.

(b) The product of two banded G-gerbes $(\mathcal{X}_1, \psi_{1,x})$ and $(\mathcal{X}_2, \psi_{2,x})$ is the *contracted product* $\mathcal{X}_1 \wedge^G P_2$, which is defined as the rigidification $(\mathcal{X}_1 \times \mathcal{X}_2) /\!\!/ G$ (see Proposition 6.2.42) of the product $\mathcal{X}_1 \times \mathcal{X}_2$ along the subgroup $(\psi_1, \psi_2)\colon G|_{\mathcal{X}_1 \times \mathcal{X}_2} \to I_{\mathcal{X}_1 \times \mathcal{X}_2}$ defined by the bands ψ_1 and ψ_2. The inverse $(\mathcal{X}, \psi_x)^{-1} = (\mathcal{X}, \psi_x^{-1})$ inverts the band.

6.2.7 Examples of gerbes

Exercise 6.2.32. Show that there is a non-trivial isomorphism

$$\alpha\colon \mathbf{B}(\mathbb{Z}/2) \times (\mathbb{A}^1 \smallsetminus 0) \to \mathbf{B}(\mathbb{Z}/2) \times (\mathbb{A}^1 \smallsetminus 0)$$

of trivial banded $\mathbb{Z}/2$-gerbes over $\mathbb{A}^1$ which glues to a non-trivial banded $\mathbb{Z}/2$-gerbe over $\mathbb{P}^1$.

Exercise 6.2.33. Let $1 \to K \to G \to Q \to 1$ be a short exact sequence of affine algebraic groups over $\Bbbk$ such that K is commutative. Show that $\mathbf{B}G \to \mathbf{B}Q$ is a banded K-gerbe which is trivial if and only if the sequence splits.

Exercise 6.2.34. Assume that $\mathrm{char}(\Bbbk) \neq 2, 3$. Recall from Exercise 3.1.17(c) that the moduli stack of stable elliptic curves has a quotient description $\overline{\mathcal{M}}_{1,1} = \mathcal{P}(4,6) := [(\mathbb{A}^2 \smallsetminus 0)/\mathbb{G}_m]$ where $\mathbb{G}_m$ acts with weights 4 and 6.

(a) Show that the j-line $\pi\colon \mathcal{M}_{1,1} \to \mathbb{A}^1$ is a trival banded $\mathbb{Z}/2$-gerbe over $\mathbb{A}^1 \setminus \{0, 1728\}$.

Hint: Construct a family of elliptic curves over $\mathbb{A}^1_{\Bbbk} \setminus \{0, 1728\}$ via the Weierstrass equation

$$y^2z + xyz = x^3 - \frac{36}{t - 1728}xz^2 - \frac{1}{t - 1728}z^3,$$

where t is the coordinate on $\mathbb{A}^1$, where the discriminant $\Delta = t^2/(t-1728)^3$. See [Sil09, Prop. III.1.4(c)].

(b) Consider the map $\overline{\mathcal{M}}_{1,1} = \mathcal{P}(4,6) \to \mathcal{P}(2,3)$ induced the homomorphism $\mathbb{G}_m \to \mathbb{G}_m, t \mapsto t^2$; note that $\mathcal{P}(2,3)$ is the banded $\mathbb{Z}/2$-gerbe obtained by rigidifying along the hyperelliptic involution (see Proposition 6.2.42), and the restriction along $\mathcal{P}(2,3) \setminus \{0, 1728, \infty\}$ is tje gerbe from (a). Show that $\overline{\mathcal{M}}_{1,1} \to \mathcal{Y}$ is non-trivial.

Hint: If it's trivial, show that there are torsion line bundles contradicting that $\operatorname{Pic}(\overline{\mathcal{M}}_{1,1}) = \mathbb{Z}$ *from Exercise 6.1.14(a).*

(c) Show that the rigidification $\mathcal{M}_{1,1} \to \mathcal{P}(2,3) \setminus \infty$ is also non-trivial.

Hint: If it's trivial, show that $\mathcal{M}_{1,1}$ has three 2-torsion line bundles contradicting that $\operatorname{Pic}(\mathcal{M}_{1,1}) = \mathbb{Z}/12$ *from Exercise 6.1.14(b).*

(d) For $g \geq 2$, let $\mathcal{H}_g \subset \mathcal{M}_g$ be the closed substack classifying hyperelliptic curves. Show that the rigidification $\mathcal{H}_g \to \mathcal{Y}$ along the hyperelliptic involution is a non-trivial banded $\mathbb{Z}/2$-gerbe.

Exercise 6.2.35. Show that $\underline{\mathcal{P}ic}(X)$ is a banded $\mathbb{G}_m$-gerbe over $\underline{\operatorname{Pic}}(X)$.

Exercise 6.2.36.

(a) Show that every gerbe $\mathcal{X}$ over an algebraic space that is étale locally isomorphic to $\mathbf{B}\mathbb{Z}/2$ is in fact banded by $\mathbb{Z}/2$.

(b) Give an example of a gerbe over an algebraic space that is étale locally isomorphic to $\mathbf{B}\mathbb{G}_m$ but that is not banded by $\mathbb{G}_m$.

Hint: Consider the classifying stack of a form of $\mathbb{G}_m$ (see Exercise 4.1.40).

Exercise 6.2.37 (Root gerbes and stacks revisited)**.** Recall that root gerbes and stacks were introduced in Examples 3.9.12 and 3.9.13.

(a) Since we now know how to construct quotient stacks by actions of $\boldsymbol{\mu}_r$ over any base scheme S, show that Exercise 3.9.14 still holds without the condition that r is invertible in $\Gamma(S, \mathcal{O}_S)$.

(b) Given a scheme X, a line bundle L, and a section $s \in \Gamma(X, L)$, show that $\sqrt[r]{L/X} \to X$ and the restriction of $\sqrt[r]{(L,s)/X} \to X$ along $V(s)$ are banded $\boldsymbol{\mu}_r$-gerbe.

(c) Show that $\sqrt[r]{L/X} \to X$ is trivial if and only if L has an rth root.

(d) Consider an exact sequence $1 \to \mu_r \to \mathbb{G}_m \to \mathbb{G}_m \to 1$ and a $\mathbb{G}_m$-torsor P'' corresponding to a line bundle L''. Show that $\sqrt[r]{L/X}$ is isomorphic to the gerbe of trivializations $\mathcal{G}_{P''}$ defined in Exercise 6.2.30(b).

Remark 6.2.38 (Banded μ_n-gerbes over $\mathbb{P}^1$)**.** Over an algebraically closed field $\Bbbk$, isomorphism classes of banded μ_n-gerbes over $\mathbb{P}^1$ are in bijection with $\mathbb{Z}/n\mathbb{Z}$. To see this, observe that the exact sequence $1 \to \mu_n \to \mathbb{G}_m \xrightarrow{n} \mathbb{G}_m \to 1$ induces an exact sequence on cohomology

$$\mathrm{H}^1(\mathbb{P}^1_{\text{ét}}, \mathbb{G}_m) \xrightarrow{n} \mathrm{H}^1(\mathbb{P}^1_{\text{ét}}, \mathbb{G}_m) \to \mathrm{H}^2(\mathbb{P}^1_{\text{ét}}, \mu_n) \to \mathrm{H}^2(\mathbb{P}^1_{\text{ét}}, \mathbb{G}_m).$$

Since $\mathrm{H}^1(\mathbb{P}^1_{\text{ét}}, \mathbb{G}_m) = \operatorname{Pic}(\mathbb{P}^1_{\text{ét}}) = \mathbb{Z}$, we can use the fact that $\mathrm{H}^2(\mathbb{P}^1_{\text{ét}}, \mathbb{G}_m) = 0$ to conclude that $\mathrm{H}^2(\mathbb{P}^1_{\text{ét}}, \mu_n) = \mathbb{Z}/n\mathbb{Z}$. The image of a line bundle $\mathcal{O}(d)$ is equivalent to the root stack $\sqrt[n]{\mathcal{O}(d)/\mathbb{P}^1}$, and this gerbe is trivial if and only if n divides d.

The gerbe $\sqrt[n]{\mathcal{O}(1)/\mathbb{P}^1}$ is isomorphic to the quotient stack $[(\mathbb{A}^2 \smallsetminus 0)/\mathbb{G}_m]$ where $t \cdot (x, y) = (t^n x, t^n y)$.

Exercise 6.2.39 (Azumaya algebras)**.** An *Azumaya algebra of rank* r^2 over a noetherian scheme X is a (possibly non-commutative) associative $\mathcal{O}_X$-algebra A which is coherent as an $\mathcal{O}_X$-module and such that there is an étale covering $X' \to X$ where $A \otimes_{\mathcal{O}_X} \mathcal{O}_{X'}$ is isomorphic to the matrix algebra $M_r(\mathcal{O}_X)$. We say that A is trivial if it is isomorphic to $M_r(\mathcal{O}_X)$. By Exercise C.2.15, Azumaya algebras are in bijection with principal PGL_n-bundles (which are also in bijection with Brauer–Severi schemes).

Let A be an Azumaya algebra over a noetherian scheme X of rank r^2.

(a) Define the *gerbe of trivializations of* A as the stack $\mathcal{G}_A$ over $(\mathrm{Sch}/X)_{\text{ét}}$ where an object over a X-scheme T is a pair (E, α) consisting of a vector bundle E on T of rank r and a trivialization $\alpha\colon \mathrm{End}_{\mathcal{O}_X}(E) \xrightarrow{\sim} A \otimes_{\mathcal{O}_X} \mathcal{O}_T$. Morphisms in $\mathcal{G}_A(T)$ are isomorphisms of vector bundles compatible with the trivializations. Show that $\mathcal{G}_A \to X$ is a banded $\mathbb{G}_m$-gerbe.

(b) Identify $\mathcal{G}_A$ with the gerbe of trivializations $\mathcal{G}_{P_A}$ defined in Exercise 6.2.30(b) with respect to the PGL_n-torsor P_A and the surjection $\mathrm{GL}_n \to \mathrm{PGL}_n$.

(c) The exact sequence $1 \to \mathbb{G}_{m,X} \to \mathrm{GL}_{r,X} \to \mathrm{PGL}_{r,X} \to 1$ of sheaves on $X_{\text{ét}}$ induces a boundary map $\mathrm{H}^1(X_{\text{ét}}, \mathrm{PGL}_r) \to \mathrm{H}^2(X_{\text{ét}}, \mathbb{G}_m)$. Show that the image of the PGL_r-torsor P_A corresponding to A under this boundary map is the class of $\mathcal{G}_A$.

(d) Show that A is trivial if and only if $\mathcal{G}_A$ is trivial.

(e) Use the quarternions to construct a non-trivial $\mathbb{G}_m$-torsor over $\operatorname{Spec} \mathbb{R}$.

Remark 6.2.40 (Brauer groups)**.** Two Azumaya algebras A and A' on a noetherian scheme X are *similar* if there exist vector bundles E and E' on X such that $A \otimes_{\mathcal{O}_X} \mathrm{End}_{\mathcal{O}_X}(E) \cong A \otimes_{\mathcal{O}_X} \mathrm{End}_{\mathcal{O}_X}(A')$. This defines an equivalence relation and the *Brauer group of* X is the set $\mathrm{Br}(X)$ of equivalence classes of Azumaya algebras. The set $\mathrm{Br}(X)$ becomes a group under the operators $[A] \cdot [A'] = [A \otimes A']$ and $[A]^{-1} = [A^{\mathrm{op}}]$ (where A^{op} is the opposite algebra with same elements and addition as A but with multiplication reversed: $a \cdot^{A^{\mathrm{op}}} b = b \cdot^{A} a$).

The exact sequence $1 \to \mathbb{G}_{m,X} \to \mathrm{GL}_{r,X} \to \mathrm{PGL}_{r,X} \to 1$ induces a boundary map $\mathrm{H}^1(X_{\text{ét}}, \mathrm{PGL}_r) \to \mathrm{H}^2(X_{\text{ét}}, \mathbb{G}_m)$. Viewing $\mathrm{H}^1(X_{\text{ét}}, \mathrm{PGL}_r)$ as the set of Azumaya algebras of rank r^2 and $\mathrm{H}^2(X_{\text{ét}}, \mathbb{G}_m)$ as the set of banded $\mathbb{G}_m$-gerbes, the boundary map assigns an Azumaya algebra A to the gerbe of trivializations $\mathcal{G}_A$. The element $[\mathcal{G}_A] \in \mathrm{H}^2(X_{\text{ét}}, \mathbb{G}_m)$ is torsion and annihilated by r. Two Azumaya algebras A and A' (of possibly different rank) are similar if and only if $\mathcal{G}_A \cong \mathcal{G}_{A'}$, and thus there is an injective map

$$\mathrm{Br}(X) \hookrightarrow \mathrm{Br}'(X) := \mathrm{H}^2(X_{\text{ét}}, \mathbb{G}_m)_{\mathrm{tors}}, \qquad A \mapsto \mathcal{G}_A,$$

into the *cohomological Brauer group* $\mathrm{Br}'(X)$. See [Mil80, §IV.2] and [Gro68] for additional background.

Grothendieck asked whether $\mathrm{Br}(X) \hookrightarrow \mathrm{Br}'(X)$ is surjective? This is known in some cases. The strongest result is due to Gabber: if X admits an ample line bundle [dJ03]. It is however open in general, even for smooth separated schemes over a field.

Exercise 6.2.41. Let X be a noetherian scheme and $\mathcal{X} \to X$ be a banded $\mathbb{G}_m$-gerbe corresponding to a cohomology class $[\mathcal{X}] \in \mathrm{H}^2(X_{\text{ét}}, \mathbb{G}_m)$.

(a) Show that the following are equivalent:

  (i) There exists an Azumaya algebra A on X such that $\mathcal{X} \cong \mathcal{G}_A$, i.e. $[\mathcal{X}]$ is in the image of $\mathrm{Br}'(X) \to \mathrm{Br}(X)$,

  (ii) $\mathcal{X}$ is a global quotient stack, and

  (iii) there exists a 1-twisted vector bundle E on $\mathcal{X}$, i.e. a vector bundle E such that for every field-valued point $\operatorname{Spec} \Bbbk \to \mathcal{X}$, the $\mathbb{G}_m$-representation corresponding to the pullback of E under $\mathbf{B}G_x \to \mathcal{X}$ decomposes as a direct sum of one-dimensional representations of *weight one*.

(b) Let X be a normal separated surface over $\mathbb{C}$ such that $\mathrm{H}^2(X, \mathbb{G}_m)$ contains a non-torsion element α; for an example, see [Gro68, II.1.11.b]. Conclude that the banded $\mathbb{G}_m$-gerbe corresponding to α is not a global quotient stack.

(c) Let $Y = \operatorname{Spec} \mathbb{C}[x, y, z]/(xy - z^2)$. Show that there is a non-trivial involution α of $(Y \smallsetminus 0) \times \mathbf{B}(\mathbb{Z}/2)$ such that the stack $\mathcal{X}$, obtained by gluing the trivial banded $\mathbb{Z}/2$-gerbes over Y along α, is a banded $\mathbb{Z}/2$-gerbe over the non-separated union $Y \bigcup_{Y \smallsetminus 0} Y$ which is not a global quotient stack.

See also [EHKV01].

6.2.8 Rigidification

Proposition 6.2.42. *Let $\mathcal{X}$ be an algebraic stack such that $I_{\mathcal{X}} \to \mathcal{X}$ is fppf. Let X be the sheaf on $\mathrm{Sch}_{\mathrm{fppf}}$ defined by the sheafification of the functor assigning a scheme S to the set of isomorphism classes $\mathcal{X}(S)/\sim$ of objects. Then X is an algebraic space and $\mathcal{X} \to X$ is a gerbe.*

Proof. To show that X is an algebraic space, it suffices to show that $\mathcal{X} \to X$ is a smooth representable morphism. In this case, a smooth presentation $U \to \mathcal{X}$ induces a smooth presentation $U \to X$ and it follows from Corollary 4.4.11 (or Theorem 6.2.1) that X is an algebraic space. As gerbes are smooth morphisms, it suffices to show that for every morphism $S \to X$ from a scheme, the fiber product $\mathcal{X} \times_X S \to S$ is a gerbe. By construction, there is an fppf cover $S' \to S$ and a morphism $a' \colon S' \to \mathcal{X}$ lifting the composition $S' \to S \to X$. Since the property of being a gerbe is fppf local, after replacing S with S', we may assume that $S \to X$ lifts to a map $a \colon S \to \mathcal{X}$. We claim that there is an isomorphism

$$\Psi \colon \mathcal{X} \times_X S \to \mathbf{B}\underline{\mathrm{Aut}}_S(a).$$

An object of the fiber product $\mathcal{X} \times_X S$ consists of a pair (f, a') where $f \colon T \to S$ is a map of schemes and $b \in \mathcal{X}(T)$ such that $T \to S \to X$ and $T \xrightarrow{b} \mathcal{X} \to X$ agree. Define $\Psi(f, b)$ as the principal $\underline{\mathrm{Aut}}_S(a)$-bundle $\underline{\mathrm{Isom}}_T(f^*a, b)$. Observe that $\Psi(f, f^*a)$ maps to the trivial bundle.

Since $\mathcal{X} \times_X S$ and $\mathbf{B}\underline{\mathrm{Aut}}_S(a)$ are both stacks in the fppf topology, we may verify that Ψ is essentially surjective fppf locally: if $P \to T$ is a principal $\underline{\mathrm{Aut}}_S(a)$-bundle, then there is a fppf cover $T' \to T$ such that $P \times_T T'$ is the trivial bundle, which we've seen is in the essential image. Similarly, we may verify that Ψ is fully faithful fppf locally. Let $(f, b), (f', b') \in (\mathcal{X} \times_X S)(T)$. Since the objects $f^*a, b, b' \in \mathcal{X}(T)$ map to the same T-valued point of X, by the construction of X there is an fppf cover $T' \to T$ such that there pullbacks become isomorphic. By replacing T with T', we may assume that $f^*a \simeq b \simeq b'$ are isomorphic. In this case, the full faithfulness claim is clear as both $\Psi(f,b)$ and $\Psi(f,b')$ are trivial bundles.

Alternatively, we may construct X directly. Let $U \to \mathcal{X}$ be a smooth presentation and $R \rightrightarrows U$ the corresponding smooth groupoid. The stabilizer groupoid scheme $S_U = R \times_{U \times U} U = I_{\mathcal{X}} \times_{\mathcal{X}} U$ is fppf over U. There is an fppf equipvalence relation $S_U \times_U R \rightrightarrows R$ where one arrow is given by composition and the other is projection. By Corollary 6.2.4, the fppf quotient $R' := R/(S_U \times_U R)$ is an algebraic space. There is an induced fppf equivalence relation $R' \rightrightarrows U$ and X is isomorphic to the fppf quotient U/R'.

See also [LMB, Cor. 10.8] and [SP, Tags 06QD and 06QJ]. □

We now consider a more general situation. If $\mathcal{X}$ is an algebraic stack, then the inertia stack $I_{\mathcal{X}}$ can be viewed as a group scheme over the big étale site $(\mathrm{Sch}/\mathcal{X})_{\text{ét}}$ of $\mathcal{X}$. As a group functor, $I_{\mathcal{X}}$ assigns an object $a \in \mathcal{X}(S)$ to the group $\mathrm{Aut}_S(a)$, and a morphism $\alpha\colon a' \to a$ over $S' \to S$ to the natural pullback map $\alpha^*\colon \mathrm{Aut}_S(a) \to \mathrm{Aut}_{S'}(a')$ (see (3.2.2)). Given $a\colon S \to \mathcal{X}$, there is a canonical isomorphism $I_{\mathcal{X}} \times_{\mathcal{X}} S \cong \underline{\mathrm{Aut}}_S(a)$ of group schemes over S.

Suppose that $\mathcal{H} \subset I_{\mathcal{X}}$ is a closed subgroup scheme over $\mathcal{X}$ such that $\mathcal{H} \to \mathcal{X}$ is fppf. This is equivalent to requiring that for every $a \in \mathcal{X}(S)$, there is a closed subgroup scheme $\mathcal{H}_a \subset \underline{\mathrm{Aut}}_S(a)$ which is fppf over S and such that if $a' \to a$ is a morphism over $S' \to S$, the canonical isomorphism $\underline{\mathrm{Aut}}_{S'}(a') \cong \underline{\mathrm{Aut}}_S(a) \times_S S'$ restricts to an isomorphism $\mathcal{H}_{a'} \cong \mathcal{H}_a \times_S S'$. If $\alpha\colon a \xrightarrow{\sim} a$ is an automorphism over the identity, then the canonical isomorphism $\alpha^*\colon \mathrm{Aut}_S(a) \to \mathrm{Aut}_S(a)$ is conjugation by α. In particular, $\mathcal{H}_a \subset \underline{\mathrm{Aut}}_S(a)$ is a *normal* group scheme.

Frequently in applications when $\mathcal{X}$ is defined over scheme S, the closed subgroup $\mathcal{H} \subset I_{\mathcal{X}}$ is obtained by the pullback of a fppf group scheme $H \to S$, i.e. $\mathcal{H} = H \times_S \mathcal{X}$.

Definition 6.2.43 (Rigidification)**.** Let $\mathcal{X}$ be an algebraic stack and $\mathcal{H} \subset I_{\mathcal{X}}$ be an fppf closed subgroup scheme over $\mathcal{X}$. The *rigidification* $\mathcal{X} /\!\!/ \mathcal{H}$ is defined as the stackification in $\mathrm{Sch}_{\mathrm{fppf}}$ of the prestack with the same objects as $\mathcal{X}$ and where the set of morphisms beteen $b \in \mathcal{X}(T)$ and $a \in \mathcal{X}(S)$ over $f\colon T \to S$ is defined as $\mathrm{Mor}(b,a) = \mathrm{Mor}_{\mathcal{X}(T)}(b, f^*a)/\mathcal{H}(T)$.

If $\mathcal{X}$ is defined over S and $\mathcal{H} = H \times_S \mathcal{X}$ is the base change of an fppf group scheme $H \to S$, then we write $\mathcal{X} /\!\!/ H := \mathcal{X} /\!\!/ \mathcal{H}$.

One can think of the subgroup $\mathcal{H}$ as giving an action of $\mathbf{B}\mathcal{H}$ on $\mathcal{X}$ and the rigidification $\mathcal{X} /\!\!/ \mathcal{H}$ as the quotient $\mathcal{X}/\mathbf{B}\mathcal{H}$.

Example 6.2.44. If $I_{\mathcal{X}} \to \mathcal{X}$ is fppf, then we can take $\mathcal{H} = I_{\mathcal{X}}$ and the rigidification $\mathcal{X} /\!\!/ I_{\mathcal{X}}$ is the algebraic space X constructed in Proposition 6.2.42.

Proposition 6.2.45. *Let $\mathcal{X}$ be an algebraic stack and $\mathcal{H} \subset I_{\mathcal{X}}$ be an fppf closed subgroup scheme over $\mathcal{X}$. The rigidification $\mathcal{X} /\!\!/ \mathcal{H}$ is an algebraic stack such that*

(1) the natural morphism $\pi\colon \mathcal{X} \to \mathcal{X} /\!\!/ \mathcal{H}$ is a gerbe;

(2) for every object $a \in \mathcal{X}(S)$, the natural map $\mathrm{Aut}_S(a) \to \mathrm{Aut}_S(\pi(a))$ is surjective with kernel $\mathcal{H}(S)$;

(3) a morphism $f\colon \mathcal{X} \to \mathcal{Y}$ factors uniquely through $\mathcal{X} /\!\!/ \mathcal{H}$ if and only if for every object $a \in \mathcal{X}(S)$, the composition $\mathcal{H}(S) \subset \ker(\mathrm{Aut}_{\mathcal{X}(S)}(a) \to \mathrm{Aut}_{\mathcal{Y}(S)}(f(a)))$; and

(4) if $\mathcal{H}$ is a commutative group scheme, then $\mathcal{H}$ descends to an fppf group scheme $H \to X$ such that $\mathcal{X} \to X$ is banded H-gerbe. If in addition $\mathcal{X}$ is defined over a scheme S and $\mathcal{H} = H \times_S \mathcal{X}$ is the pullback of a commutative fppf group scheme $H \to S$, then $\mathcal{X} \to X$ is a banded H-gerbe.

Proof. To show that $\mathcal{X}$ is algebraic, it suffices to show that $\pi \colon \mathcal{X} \to \mathcal{X} /\!\!/ \mathcal{H}$ is a smooth representable morphism: if $U \to \mathcal{X}$ is a smooth presentation, then so is the composition $U \to \mathcal{X} \to \mathcal{X} /\!\!/ \mathcal{H}$. If $g \colon S \to \mathcal{X} /\!\!/ \mathcal{H}$, then by the definition of $\mathcal{X} /\!\!/ \mathcal{H}$ as the stackification, there is an fppf cover $S' \to S$ such that $S' \to S \to \mathcal{X} /\!\!/ \mathcal{H}$ lifts to a map $a' \colon S' \to \mathcal{X}$. By replacing S with S', we may assume that $g \colon S \to \mathcal{X} /\!\!/ \mathcal{H}$ lifts to a morphsm $a \colon S \to \mathcal{X}$.

We claim that there is an isomorphism

$$\Psi \colon \mathcal{X} \times_{\mathcal{X} /\!\!/ \mathcal{H}} S \to \mathbf{B}\mathcal{H}_a.$$

Since $\mathcal{H}_a \to S$ is fppf, the classifying stack $\mathbf{B}\mathcal{H}_a$ is algebraic (Proposition 6.2.9) and smooth over S' (Proposition B.4.2), and the isomorphism Ψ would imply that $\mathcal{X} \to \mathcal{X} /\!\!/ \mathcal{H}$ is smooth and representable. An object of $\mathcal{X} \times_{\mathcal{X} /\!\!/ \mathcal{H}} S$ consists of a triple (f, b, α) where $f \colon T \to S$, $b \in \mathbb{X}(T)$, and $\alpha \colon g \circ f \xrightarrow{\sim} \pi \circ b$. Define $\Psi(f, b, \alpha)$ as the principal $\mathcal{H}_a$-bundle $T \times_{\underline{\mathrm{Isom}}_T(f^*a, b)/\mathcal{H}_a} \underline{\mathrm{Isom}}_T(f^a, b)$. Noting that $\Psi(f, f^*a, \mathrm{id})$ is the trivial bundle, the proof that Ψ is an isomorphism follows exactly as in Proposition 6.2.42. The remaining statements are left to the reader.

See also [ACV03, Thm. 5.1.5], [AGV08, §C], [Rom05, §5], and [AOV08, §A]. □

Exercise 6.2.46.

(a) If H is a commutative group scheme over S, show that $\mathbf{B}H /\!\!/ H \cong S$.

(b) Let $G \to S$ be an fppf group scheme acting on a S-scheme U. Suppose that $H \subset G$ is a central commutative fppf subgroup scheme acting trivially on U. Show that $[U/G] /\!\!/ H \cong [U/(G/H)]$.

Exercise 6.2.47. Let $\mathcal{X} \to S$ be an smooth, integral, and separated Deligne–Mumford stack over a scheme S. Let $\operatorname{Spec} K \to \mathcal{X}$ be a representative of the generic point. Show that the closure $\mathcal{H} \subset I_{\mathcal{X}}$ of generic fiber $I_{\mathcal{X}} \times_{\mathcal{X}} K$ of the interia is a closed étale subgroup scheme and that the rigidification $\mathcal{X} /\!\!/ \mathcal{H}$ is a smooth, integral, and separated Deligne–Mumford stack over S with generically trivial inertia.

Exercise 6.2.48. Let $\mathcal{X}$ be an algebraic stack over a scheme S, $H \to S$ be an fppf group scheme, and $H \times_S \mathcal{X} \subset I_{\mathcal{X}}$ a closed subgroup scheme. Show that the rigidification $\mathcal{X} /\!\!/ H$ can be given the moduli interpretation where an object over a scheme S is a pair $(\mathcal{G}, f)$ where $\mathcal{G} \to S$ is a banded H-gerbe and $f \colon \mathcal{G} \to \mathcal{X}$ is an H-equivariant morphism (i.e. for every object $a \in \mathcal{G}(T)$ over an S-scheme T, the composition $H(T) \xrightarrow{\sim} \mathrm{Aut}_T(a) \to \mathrm{Aut}_T(f(a))$ agrees with the inclusion $H(T) \hookrightarrow \mathrm{Aut}_T(f(a))$ given by the subgroup $H \times_S \mathcal{X} \subset I_{\mathcal{X}}$.

6.2.9 Residual gerbes revisited

Given an algebraic stack $\mathcal{X}$ and $x \in |\mathcal{X}|$, recall from Definition 3.5.12 that *the residual gerbe at x* (if it exists) is a reduced noetherian algebraic stack $\mathcal{G}_x$ with a monomorphism $\mathcal{G}_x \hookrightarrow \mathcal{X}$ such that $|\mathcal{G}_x|$ is a point mapping to x. We've already shown that the residual gerbe at a finite type point exists (Proposition 3.5.16).

We can now prove that residual gerbes are unique.

Lemma 6.2.49. *Let $\mathcal{X}$ be a noetherian algebraic stack and $x \in |\mathcal{X}|$ be a point. If the residual gerbe at x exists, it is unique.*

Proof. To add. $\square$

We now establish the existence of residual gerbes at all points and moreover show that they are in fact gerbes.

Proposition 6.2.50. *If $\mathcal{X}$ is a noetherian algebraic stack and $x \in |\mathcal{X}|$ is a point, then the residual gerbe $\mathcal{G}_x$ exists and is a gerbe over a field $\kappa(x)$, called the* residue field *of x.*

Proof. To add. $\square$

If $\mathcal{X}$ is a quasi-separated algebraic stack of finite type over a field $\Bbbk$ and $x \in \mathcal{X}(\Bbbk)$, then $\mathcal{G}_x = \mathbf{B}G_x$ (Proposition 3.5.16). More generally, we have:

Exercise 6.2.51. Let $\mathcal{X}$ be a noetherian algebraic stack and $x \in |\mathcal{X}|$ be a finite type point.

(1) For any representative $\overline{x}\colon \operatorname{Spec} \Bbbk \to \mathcal{X}$ of x, there is a cartesian diagram

$$\begin{tikzcd} B_{\Bbbk} G_{\overline{x}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{G}_x \arrow[r, hook] \arrow[d] & \mathcal{X} \\ \operatorname{Spec} \Bbbk \arrow[r] & \operatorname{Spec} \kappa(x). \end{tikzcd}$$

(2) If the stabilizer of x is smooth, show that there is a finite separable extension $\kappa(x) \to \Bbbk$ and a representative of x over $\Bbbk$.

Exercise 6.2.52. Let $C \subset \mathbb{P}^2_{\Bbbk}$ be a non-split quadric over a field $\Bbbk$, and let $\Bbbk \to \Bbbk'$ be a quadratic extension such that $C \times_{\Bbbk} \Bbbk' \cong \mathbb{P}^1_{\Bbbk'}$. Let $D \subset C$ be a divisor of degree 6 and let $X \to \mathbb{P}^1_{\Bbbk'}$ be the double cover ramified over $D \times_{\Bbbk} \Bbbk'$. Show that the residual gerbe of $[X] \in \mathcal{M}_2$ is non-trivial and has residue field $\Bbbk$.

To give some context for the above exercise, the rigidification $\mathcal{M}_2 \to \mathcal{Y}$ of the hyperelliptic involution is a non-trivial banded $\mathbb{Z}/2$-gerbe (see Exercise 6.2.34(d)). Restricting to the locus of curves whose only non-trivial automorphism is the hyperelliptic involution, we have a coarse moduli space $\mathcal{M}_2^\circ \to M_2^\circ$. The above exercise implies that even the fibers of $\mathcal{M}_2^\circ \to M_2^\circ$ over (non-algebraically closed) residue fields may be non-trivial banded $\mathbb{Z}/2$-gerbes.

6.3 Affine Geometric Invariant Theory and good moduli spaces

Good moduli spaces capture the stack-intrinsic properties of quotients that appear in Geometric Invariant Theory (GIT). In the affine case, GIT concerns the action of a linearly reductive group on an affine scheme. Recall that an affine algebraic group G over a field $\Bbbk$ is *linearly reductive* if the functor $\operatorname{Rep}(G) \to \operatorname{Vect}_{\Bbbk}$, taking a G-representation V to its G-invariants V^G, is exact. Examples include:

- finite discrete groups G whose order is not divisible by $\operatorname{char}(\Bbbk)$ (Maschke's Theorem (C.4.4));

- tori $\mathbb{G}_m^n$ and diagonalizable group schemes (Proposition C.1.13); and
- reductive algebraic groups (e.g. GL_n, SL_n and PGL_n) in $\mathrm{char}(\Bbbk) = 0$ (Theorem C.4.7).

See §C.4 for further equivalences, properties, and a discussion of linearly reductive groups.

Given an action of G on an affine $\Bbbk$-scheme $\operatorname{Spec} A$, the inclusion $A^G \hookrightarrow A$ induces a commutative diagram

$$\begin{tikzcd} \operatorname{Spec} A \arrow[d] \arrow[dr, "\widetilde{\pi}"] & \\ {[\operatorname{Spec} A/G]} \arrow[r, "\pi"] & \operatorname{Spec} A^G. \end{tikzcd}$$

Let's observe the following two properties of $\pi\colon [\operatorname{Spec} A/G] \to \operatorname{Spec} A^G$:

(1) $\Gamma([\operatorname{Spec} A/G], \mathcal{O}_{[\operatorname{Spec} A/G]}) = A^G$; this follows from the definition of global sections.

(2) The functor $\pi_*\colon \mathrm{QCoh}([\operatorname{Spec} A/G]) \to \mathrm{QCoh}(\operatorname{Spec} A^G)$ is exact. This holds because functor π_* takes a quasi-coherent $\mathcal{O}_{\mathcal{X}}$-module $\widetilde{M}$, corresponding to an A-module M with a G-action, to $\widetilde{M^G}$ (Exercise 6.1.3) and is therefore exact by the defining property of linear reductivity.

In this case, following terminology of Mumford and Seshadri, we say that $\operatorname{Spec} A \to \operatorname{Spec} A^G$ is a *good quotient* or *GIT quotient*, and $\operatorname{Spec} A^G$ is sometimes denoted as $(\operatorname{Spec} A)/\!\!/G$. See §6.6 for a more general discussion of good quotients and the projective case of GIT.

6.3.1 Good moduli spaces

The definition of a good moduli space is inspired by properties of GIT quotients and specifically properties of the morphisms $\pi\colon [\operatorname{Spec} A/G] \to \operatorname{Spec} A^G$ and $\pi\colon [X^{\mathrm{ss}}/G] \to X^{\mathrm{ss}}/\!\!/G := \operatorname{Proj} \bigoplus_{d\geq 0} \Gamma(X, \mathcal{O}_X(d))^G$, where G is linearly reductive and $X \subset \mathbb{P}(V)$ is a G-invariant closed subscheme of a projectivized G-representation.

Definition 6.3.1 (Good moduli spaces)**.** A quasi-compact and quasi-separated morphism $\pi\colon \mathcal{X} \to X$ from an algebraic stack $\mathcal{X}$ to an algebraic space X is a *good moduli space* if

(1) $\mathcal{O}_X \to \pi_*\mathcal{O}_{\mathcal{X}}$ is an isomorphism, and

(2) $\pi_*\colon \mathrm{QCoh}(\mathcal{X}) \to \mathrm{QCoh}(X)$ is exact.

Example 6.3.2 (Basic example: affine GIT)**.** If G is a linearly reductive group over a field $\Bbbk$ acting on an affine $\Bbbk$-scheme $\operatorname{Spec} A$, then $[\operatorname{Spec} A/G] \to \operatorname{Spec} A^G$ is a good moduli space.

Example 6.3.3 (Concrete examples)**.** If $\mathbb{G}_m$ acts on $\mathbb{A}^n$ over a field $\Bbbk$ via $t \cdot (x_1, \ldots, x_n) = (tx_1, \ldots, tx_n)$, then $[\mathbb{A}^n/\mathbb{G}_m] \to \operatorname{Spec} \Bbbk$ is a good moduli space. Observe that a nonzero $\Bbbk$-point $[\mathbb{A}^n/\mathbb{G}_m]$ is not closed and contains 0 in its closures, or in other words every $\mathbb{G}_m$-orbit contains 0 in its closure. Note that $[\mathbb{A}^n/\mathbb{G}_m] \setminus 0 = \mathbb{P}^{n-1}$.

If $\mathbb{G}_m$ acts on $\mathbb{A}^2$ via $t \cdot (x, y) = (tx, t^{-1}y)$, then $[\mathbb{A}^2/\mathbb{G}_m] \to \operatorname{Spec} \Bbbk[xy] = \mathbb{A}^1$ is a good moduli space. The fiber over $a \neq 0 \in \mathbb{A}^1$ under the good quotient $\mathbb{A}^2 \to \mathbb{A}^1$ is the hyperbola $xy = a$ in $\mathbb{A}^2$ and the fiber under the good moduli space $[\mathbb{A}^2/\mathbb{G}_m] \to \mathbb{A}^1$ is the point $\operatorname{Spec} \Bbbk \cong [V(xy - a)/\mathbb{G}_m)$. The fiber over the origin is the union of the three orbits $\{(x, 0) | x \neq 0\} \cup \{(0, y) | y \neq 0\} \cup \{0, 0\}$ in $\mathbb{A}^2$. Note that $[\mathbb{A}^2/\mathbb{G}_m] \setminus 0 = \mathbb{A}^1 \bigcup_{\mathbb{A}^1 \setminus 0} \mathbb{A}^1$ is the non-separated affine line

Example 6.3.4 (Tame coarse moduli spaces)**.** If $\mathcal{X}$ is a separated Deligne–Mumford stack of finite type over a noetherian scheme S, then the Keel–Mori Theorem (4.3.11) implies that there exists a coarse moduli space $\pi \colon \mathcal{X} \to X$. We say that the coarse moduli space $\mathcal{X} \to X$ is *tame* if every automorphism group has order prime to the characteristic, i.e. invertible in $\Gamma(S, \mathcal{O}_S)$. A tame coarse moduli space is a good moduli space. Indeed, this will follow from the fact that the property of being a good moduli space is local on the base in the étale topology (Lemma 6.3.20) and the Local Structure of Coarse Moduli Spaces (4.3.14). If $\mathcal{X}$ has quasi-finite stabilizers, then in fact every good moduli space $\pi \colon \mathcal{X} \to X$ is a coarse moduli space and π is separated; see Proposition 6.3.28.

The goal of this section is to establish the following theorem.

Theorem 6.3.5. *Let $\pi \colon \mathcal{X} \to X$ be a good moduli space where $\mathcal{X}$ is a quasi-separated algebraic stack defined over an algebraic space S. Then*

(1) π is surjective and universally closed;

(2) For closed substacks $\mathcal{Z}_1, \mathcal{Z}_2 \subset \mathcal{X}$, $\operatorname{im}(\mathcal{Z}_1 \cap \mathcal{Z}_2) = \operatorname{im}(\mathcal{Z}_1) \cap \operatorname{im}(\mathcal{Z}_2)$. For geometric points $x_1, x_2 \in \mathcal{X}(\Bbbk)$, $\pi(x_1) = \pi(x_2) \in X(\Bbbk)$ if and only if $\overline{\{x_1\}} \cap \overline{\{x_2\}} \neq \emptyset$ in $|\mathcal{X} \times_S \Bbbk|$. In particular, π induces a bijection between closed points in $\mathcal{X}$ and closed points in X;

(3) If $\mathcal{X}$ is noetherian, so is X. If $\mathcal{X}$ is of finite type over S and S is noetherian, then X is of finite type over S and π_* preserves coherence, i.e. for $F \in \operatorname{Coh}(\mathcal{X})$, $\pi_* F \in \operatorname{Coh}(X)$; and

(4) If $\mathcal{X}$ is noetherian, then π is universal for maps to algebraic spaces.

Remark 6.3.6. In (2), the images and intersections are taken scheme-theoretically. Note that since π is closed, the set-theoretic image of a closed substack $\mathcal{Z}$ is identified with the topological space of its scheme-theoretic image $\operatorname{im}(\mathcal{Z})$. If $I \subset \mathcal{O}_{\mathcal{X}}$ is the sheaf of ideals defining $\mathcal{Z}$, the image $\operatorname{im}(\mathcal{Z})$ is defined by $\pi_* I \subset \pi_* \mathcal{O}_{\mathcal{X}} = \mathcal{O}_X$.

In the case of affine GIT where we have a good moduli space $\pi \colon [\operatorname{Spec} A/G] \to \operatorname{Spec} A$ and a good quotient $\widetilde{\pi} \colon \operatorname{Spec} A \to \operatorname{Spec} A^G$, this theorem translates to:

Corollary 6.3.7 (Affine GIT)**.** *Let G be a linearly reductive algebraic group over an algebraically closed field $\Bbbk$. Then $\widetilde{\pi} \colon U = \operatorname{Spec} A \to U/\!/G := \operatorname{Spec} A^G$ satisfies: Then*

(1) $\widetilde{\pi}$ is surjective and for every G-invariant closed subscheme $Z \subset U$, $\operatorname{im}(Z) \subset U/\!/G$ is closed. The same holds for the base change $T \to U/\!/G$ by a morphism from a scheme;

(2) For closed G-invariant closed subschemes $Z_1, Z_2 \subset U$, $\operatorname{im}(Z_1 \cap Z_2) = \operatorname{im}(Z_1) \cap \operatorname{im}(Z_2)$. In particular, for $x_1, x_2 \in X(\Bbbk)$, $\widetilde{\pi}(x_1) = \widetilde{\pi}(x_2)$ if and only if $\overline{Gx_1} \cap \overline{Gx_2} \neq \emptyset$ and $\widetilde{\pi}$ induces a bijection between closed G-orbits of $\Bbbk$-points in U and $\Bbbk$-points of $U/\!/G$.

(3) *If A is noetherian, so is A^G. If A is finite generated over $\Bbbk$, then A^G is also finitely generated over $\Bbbk$ and for every finitely generated A-module M with a G-action, M^G is a finitely generated A^G-module; and*

(4) *If A is noetherian, then $\widetilde{\pi}$ is universal for G-invariant maps to algebraic spaces.* $\square$

Remark 6.3.8. If $Z \subset U = \operatorname{Spec} A$ is defined by a G-invariant ideal I, then (1) implies that $\pi(Z)$ is defined by $I^G \subset A^G$. If Z_1, Z_2 are defined by G-invariant ideals $I_1, I_2 \subset A$, then (2) implies that $(I_1 + I_2)^G = I_1^G + I_2^G$. In particular, if Z_1 and Z_2 are disjoint, then so are $\operatorname{im}(Z_1)$ and $\operatorname{im}(Z_2)$ and we can write $1 = f_1 + f_2$ with $f_1 \in I_1^G$ and $f_2 \in I_2^G$; the function f_1 restricts to 0 on Z_1 and 1 on Z_2. We see that G-invariant functions separate disjoint G-invariant closed subschemes.

Remark 6.3.9 (Hilbert's 14th problem)**.** Hilbert's 14th problem asks when the invariant ring A^G is finitely generated. While it is not true for every group G, Hilbert showed it is true when G is linearly reductive—this is what (3) above asserts. Hilbert's original argument in [Hil90] is so elegant and played such an important role in the development of modern algebra that we reproduce it here. Our proof of Theorem 6.3.5(3)—while similar in spirit—will not be as explicit.

Let $f_1, \dots, f_n$ be $\Bbbk$-algebra generators of A and let $V \subset A$ be a finite dimensional G-invariant subspace containing each f_i (Algebraic Group Facts C.3.1(1)). Then we have a surjection $\operatorname{Sym}^* V = \Bbbk[x_1, \dots, x_m] \twoheadrightarrow A$ of $\Bbbk$-algebras with G-actions and we set $I = \ker(\Bbbk[x_1, \dots, x_m] \to A)$. Since G is linearly reductive, $A^G = (\Bbbk[x_1, \dots, x_m]/I)^G = \Bbbk[x_1, \dots, x_m]^G / I^G$ and we can assume that $A = \Bbbk[x_1, \dots, x_m]$ is the polynomial ring so that A^G is a graded $\Bbbk$-algebra whose degree 0 component is $\Bbbk$. It therefore suffices to show that the ideal $J_+ := \sum_{d>0} A_d^G \subset A^G$ is finitely generated since its generators will then generate A^G as a $\Bbbk$-algebra.

Hilbert first showed that every ideal in $A = \Bbbk[x_1, \dots, x_n]$ is finitely generated—this is what is referred to today as Hilbert's Basis Theorem and was developed by Hilbert precisely to make this argument. It follows that $J_+ A \subset A$ is finitely generated by homogenous invariants $f_1, \dots, f_n \in A^G$. We will show that they also generate J_+ as an ideal in A^G. For $f \in A_d^G$, we can write

$$f = \sum_{i=1}^{n} f_i g_i \tag{6.3.1}$$

with $g_i \in A$ a homogeneous (not necessarily invariant) function of degree $d - \deg f_i$ (with $g_i = 0$ if $\deg f_i > d$). Since G is linearly reductive, there is a $\Bbbk$-linear map $R\colon A \to A^G$ called the *Reynolds operator* (see Remark C.4.6) which is the identity on A^G, respects the grading and satisfies $R(xy) = xR(y)$ for $x \in A^G$ and $y \in A$. Applying R to (6.3.1) shows that $f = R(f) = \sum_i f_i R(g_i)$ with $R(g_i) \in A^G$ and thus f lies in the ideal in A^G generated by the f_i.[1]

Hilbert gave a constructive proof of this theorem in [Hil93] which required the development of the Syzygy Theorem, the Nullstellensatz, a version of Noether normalization, and a version of the Hilbert–Mumford criterion. We strongly

[1]For an alternative argument that A^G is noetherian, linear reductivity can be used to show that $JA \cap A^G = J$ for every ideal $J \subset A^G$ (see Lemma 6.3.22(5)). If $J_1 \subset J_2 \subset \cdots \subset A^G$ is an ascending chain of ideals, then the ascending chain $J_1 A \subset J_2 A \subset \cdots \subset A$ terminates which implies that the original sequence $J_1 = J_1 A \cap A^G \subset J_2 = J_2 A \cap A^G \subset \cdots \subset A^G$ also terminates.

encourage you to read [Hil90] and [Hil93] (or Hilbert's translated lecture notes [Hil93]).

Remark 6.3.10 (Reductivity in positive characteristic)**.** In characteristic p, every smooth linear reductive group is an extension of a torus by a finite group prime to the characteristic. In particular, GL_n is not linearly reductive (see Example C.4.8). In characteristic p, there are the following variant notions for an affine algebraic group G over an algebraically closed field $\Bbbk$:

(1) *G is reductive* if G is smooth and every smooth, connected, unipotent and normal subgroup of G is trivial, and

(2) *G is geometrically reductive* if for every surjection $V \to W$ of G-representations and $w \in W^G$, there exists $n > 0$ such that w^{p^n} is in the image of $\mathrm{Sym}^{p^n} V \to \mathrm{Sym}^{p^n} W$.

It is a deep theorem due to Haboush [Hab75] that these notions are equivalent when G is smooth. See also §C.4.2-C.4.3 for further properties, equivalences and discussion.

Geometric reductivity (sometimes called semi-reductivity) was introduced by Mumford in [GIT, preface] in an effort to extend GIT—originally developed for linearly reductive groups—to reductive groups in positive characteristic. Indeed, it is precisely the geometric reductivity property that yield the same geometric properties that we saw for affine GIT quotients by linearly reductive groups: if G is geometrically reductive acting on an affine $\Bbbk$-scheme $\operatorname{Spec} A$, then $\widetilde{\pi}\colon \operatorname{Spec} A \to \operatorname{Spec} A^G$ satisfies Corollary 6.3.7(1)-(4) (with the exception that the noetherianness of A does not necessarily imply the noetherianess of A^G). The arguments are not substantially more complicated than the linearly reductive case. See [Nag64], [MFK94, App. 1.C], [New78, §3], [Dol03, §3.4], [Spr77, §2] and [DC71, §2].

Likewise, the notion of a good moduli space can be extended to characterize quotients by geometrically reductive groups: in [Alp14], a quasi-compact and quasi-separated morphism $\pi\colon \mathcal{X} \to X$, from an algebraic stack to an algebraic space, is called an *adequate moduli space* if (1) $\mathcal{O}_X \to \pi_*\mathcal{O}_{\mathcal{X}}$ is an isomorphism and (2) for every surjection $\mathcal{A} \to \mathcal{B}$ of quasi-coherent $\mathcal{O}_{\mathcal{X}}$-algebras, then every section s of $\pi_*(\mathcal{B})$ over a smooth morphism $\operatorname{Spec} A \to \mathcal{Y}$ has a positive power that lifts to a section of $\pi_*(\mathcal{A})$). An adequate moduli space satisfies Theorem 6.3.5(1)-(4) (except again for the noetherian implication). If G is geometrically reductive, then $\pi\colon [\operatorname{Spec} A/G] \to \operatorname{Spec} A^G$ is an adequate moduli space. In characteristic 0, an adequate moduli space is necessarily good.

In this book, we restrict to linearly reductive groups and good moduli spaces since the proofs of the basic properties are more elementary in this case and probably best seen first. In addition, there is currently no analogue of the Local Structure Theorem for Algebraic Stacks (6.5.1) around points with reductive stabilizers.

6.3.2 Cohomologically affine morphisms

The exactness condition on the pushforward π_* in the definition of a good moduli space (Definition 6.3.1(2)) is a non-representable analogue of affineness.

Definition 6.3.11 (Cohomologically affine)**.** A quasi-compact and quasi-separated morphism $f\colon \mathcal{X} \to \mathcal{Y}$ of algebraic stacks is *cohomologically affine* if

$$f_*\colon \mathrm{QCoh}(\mathcal{X}) \to \mathrm{QCoh}(\mathcal{Y})$$

is exact. A quasi-compact and quasi-separated algebraic stack $\mathcal{X}$ is *cohomologically affine* if $\mathcal{X} \to \operatorname{Spec} \mathbb{Z}$ is.

Example 6.3.12. An affine algebraic group G over a field $\Bbbk$ is linearly reductive (Definition C.4.1) if and only if $\mathbf{B}G$ is cohomologically affine.

Remark 6.3.13. By Serre's Criterion for Affineness (4.4.15), an algebraic space is cohomologically affine if and only if it is an affine scheme. An algebraic stack $\mathcal{X}$ with affine diagonal is cohomologically affine if and only if $\mathrm{H}^i(\mathcal{X}, F) = 0$ for all $i > 0$ and every quasi-coherent sheaf F; this follows because the cohomology $\mathrm{H}^i(\mathcal{X}, F)$ can be computed in $\mathrm{QCoh}(\mathcal{X})$ for such stacks $\mathcal{X}$ by Proposition 6.1.29(2). This is not true for algebraic stacks with non-affine diagonal, e.g. $\mathbf{B}E$ for an elliptic curve E.

Likewise, a morphism $f \colon \mathcal{X} \to \mathcal{Y}$ of algebraic stacks, with both $\mathcal{X}$ and $\mathcal{Y}$ having affine diagonal, is cohomologically affine if and only if $\mathrm{R}^i f_*(F) = 0$ for all $i > 0$ and every quasi-coherent sheaf F. If in addition f is representable, then f is cohomologically affine if and only if it is affine (see Corollary 6.3.16 below).

Remark 6.3.14 (Noetherian case)**.** If $\mathcal{X}$ is noetherian, then a quasi-compact, quasi-separated morphism $f \colon \mathcal{X} \to \mathcal{Y}$ is cohomologically affine if and only if $f_* \colon \mathrm{Coh}(\mathcal{X}) \to \mathrm{QCoh}(\mathcal{Y})$ is exact. This holds because every quasi-coherent sheaf is a colimit of coherent sheaves (Proposition 6.1.8) and f_* commutes with colimits. Since cohomology also commutes with colimits (Proposition 6.1.31), a morphism $f \colon \mathcal{X} \to \mathcal{Y}$ of noetherian algebraic stacks, both with affine diagonal, is cohomologically affine if and only if $\mathrm{R}^i f_*(F) = 0$ for all $i > 0$ and every *coherent* sheaf F.

Lemma 6.3.15. *Consider a cartesian diagram*

$$\begin{tikzcd} \mathcal{X}' \arrow[r, "g'"] \arrow[d, "\pi'"] & \mathcal{X} \arrow[d, "\pi"] \\ \mathcal{Y}' \arrow[r, "g"] & \mathcal{Y}. \end{tikzcd}$$

of algebraic stacks.

(1) *If g is faithfully flat and π' is cohomologically affine, then π is cohomologically affine.*

(2) *If $\mathcal{Y}$ has quasi-affine diagonal (e.g. a quasi-separated algebraic space) and π is cohomologically affine, then π' is cohomologically affine.*

Proof. For (1), by Flat Base Change (6.1.7) there is an equivalence $g^*\pi_* \simeq \pi'_* g'^*$ of functors defined on categories of quasi-coherent sheaves. Since π'_* and g'^* are exact and g^* is faithfully exact, π_* is exact.

For (2), we first show that if g is quasi-affine and π is cohomologically affine, then π' is also cohomologically affine. It suffices to handle the cases that g is an open immersion and g is affine. If g is an open immersion and $F' \twoheadrightarrow G'$ is a surjection in $\mathrm{QCoh}(\mathcal{X}')$, we define $G = \operatorname{im}(g'_* F' \to g'_* G')$. Note that $g'^* G \cong G'$. Since π_* is exact, $\pi_* g'_* F \twoheadrightarrow \pi_* G$. If we apply g^* and use the identifies $g^* \pi_* \simeq \pi'_* g'^*$ and $g'^* g'_* \simeq \mathrm{id}$, we obtain a surjection $\pi'_* F' \twoheadrightarrow \pi'_* g'^* G \cong \pi'_* G'$. On the other hand, if g is affine then g_* is faithfully exact. Since π_* and g'_* are are exact, the identity $g_* \pi'_* \simeq \pi_* g'_*$ implies that π'_* is also exact. To show (2), we may assume that $\mathcal{Y}$ and

$\mathcal{Y}'$ are quasi-compact and we can choose a smooth presentation $Y = \operatorname{Spec} A \to \mathcal{Y}$, which will be quasi-affine (since $\mathcal{Y}$ has quasi-affine diagonal). Then the base change $\mathcal{X}_Y \to Y$ of π along $Y \to \mathcal{Y}$ is cohomologically affine. To check that the base change $\mathcal{X}'_Y \to \mathcal{Y}'_Y$ is cohomologically affine, it suffices by (1) to check this after base changing by a smooth presentation $Y' = \operatorname{Spec} A' \to \mathcal{Y}' \times_{\mathcal{Y}} Y$ but this holds as $Y' \to Y$ is affine. Since $\mathcal{X}'_Y \to \mathcal{Y}'_Y$ is cohomologically affine so is $\pi' \colon \mathcal{X}' \to \mathcal{Y}'$ by invoking (1) again. □

Corollary 6.3.16. *Let $f \colon \mathcal{X} \to \mathcal{Y}$ be a representable and cohomologically affine morphism of algebraic stacks where $\mathcal{Y}$ has quasi-affine diagonal, then f is affine.*

Proof. Under the hypotheses, both affine and cohomologically affine morphisms descend under faithfully flat morphisms and we can reduce to the case where $\mathcal{X}$ is an algebraic space and $\mathcal{Y}$ is an affine scheme which is Serre's Criterion for Affineness (4.4.15). □

6.3.3 Properties of linearly reductive groups

Recall that an affine algebraic group G over a field $\Bbbk$ is *linearly reductive* if the functor $\operatorname{Rep}(G) \to \operatorname{Vect}_{\Bbbk}$, defined by $V \mapsto V^G$, is exact (Definition C.4.1). This is equivalent to the map $\mathbf{B}G \to \operatorname{Spec} \Bbbk$ being cohomologically affine.

Proposition 6.3.17. *Let $1 \to K \to G \to Q \to 1$ be an exact sequence of affine algebraic groups over a field $\Bbbk$. Then G is linearly reductive if and only if both K and Q are.*

Proof. We will use the cartesian diagram

$$\begin{tikzcd}
Q \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathbf{B}K \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \operatorname{Spec} \Bbbk \arrow[d] \\
\operatorname{Spec} \Bbbk \arrow[r] & \mathbf{B}G \arrow[r] & \mathbf{B}Q
\end{tikzcd}$$

of Exercise 2.3.32(c). To see ($\Rightarrow$), note that $\mathbf{B}K \to \mathbf{B}G$ is affine by descent since Q is affine. Therefore the composition $\mathbf{B}K \to \mathbf{B}G \to \operatorname{Spec} \Bbbk$ is cohomologically affine and K is linearly reductive. If V is a Q-representation, then its pullback under $q \colon \mathbf{B}G \to \mathbf{B}Q$ is the G-representation induced by the projection $G \to Q$ and in particular K acts trivially. On the other hand, the pushforward of a G-representation W under $q \colon \mathbf{B}G \to \mathbf{B}Q$ is the Q-representation W^K. Thus, the adjunction $V \to q_* q^* V$ is an isomorphism and $\Gamma(\mathbf{B}Q, -) = \Gamma(\mathbf{B}G, q^* -)$ is exact.

For the converse, descent (Lemma 6.3.15(2)) implies that $\mathbf{B}G \to \mathbf{B}Q$ is cohomologically affine and thus so is the composition $\mathbf{B}G \to \mathbf{B}Q \to \operatorname{Spec} \Bbbk$. □

Proposition 6.3.18. *Let H be a linearly reductive algebraic group over an algebraically closed field $\Bbbk$. If H acts freely on an affine scheme U over $\Bbbk$, then the algebraic space quotient U/H is affine.*

Proof. The algebraic space U/H and the good quotient $\operatorname{Spec} A^H$ are both universal for maps to algebraic spaces Theorem 6.3.5(4). Alternatively, the composition $U/H \to \mathbf{B}H \to \operatorname{Spec} \Bbbk$ is an affine morphism followed by a cohomologically affine morphism. It follows from Serre's Criterion for Affineness (4.4.15) that U/H is affine. □

In particular, if H is a linearly reductive subgroup of an affine algebraic group G, then the quotient G/H is affine. Matsushima's Theorem provides a converse.

Proposition 6.3.19 (Matsushima's Theorem)**.** *Let G be a linearly reductive group over an algebraically closed field $\Bbbk$.*

(1) A subgroup H of G is linearly reductive if and only if G/H is affine.

(2) Given an action of G on an algebraic space U of finite type over $\Bbbk$ and a $\Bbbk$-point $u \in U$ with stabilizer G_u, then G_u is linearly reductive if and only if the orbit Gu is affine.

Proof. Part (2) follows from (1) since $Gu = G/G_u$. For (1), the ($\Rightarrow$) implication follows from Proposition 6.3.18. For the converse, consider the cartesian diagram

$$\begin{tikzcd} G/H \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \operatorname{Spec} \Bbbk \arrow[d] \\ \mathbf{B}H \arrow[r] & \mathbf{B}G. \end{tikzcd}$$

If G/H is affine, then by smooth descent $\mathbf{B}H \to \mathbf{B}G$ is affine and therefore $\mathbf{B}H \to \mathbf{B}G \to \operatorname{Spec} k$ is cohomologically affine, i.e. H is linearly reductive. ☐

6.3.4 First properties of good moduli spaces

Lemma 6.3.20. *Consider a cartesian diagram*

$$\begin{tikzcd} \mathcal{X}' \arrow[r, "g'"] \arrow[d, "\pi'"] \arrow[dr, phantom, "\square"] & \mathcal{X} \arrow[d, "\pi"] \\ X' \arrow[r, "g"] & X \end{tikzcd}$$

of algebraic stacks where X and X' are quasi-separated algebraic spaces.

(1) If g is faithfully flat and π' is a good moduli space, then π is a good moduli space.

(2) If π is a good moduli space, so is π'.

(3) For $F \in \mathrm{QCoh}(\mathcal{X})$ and $G \in \mathrm{QCoh}(X)$, the adjunction map $\pi_* F \otimes G \to \pi_*(F \otimes \pi^* G)$ is an isomorphism. In particular, the adjunction map $G \xrightarrow{\sim} \pi_* \pi^* G$ is an isomorphism.

(4) For $F \in \mathrm{QCoh}(\mathcal{X})$, then the adjunction map $g^* \pi_* F \xrightarrow{\sim} \pi'_* g'^* F$ is an isomorphism.

(5) For a quasi-coherent sheaf of ideals $J \subset \mathcal{O}_X$, the natural map $J \to \pi_*(\pi^{-1} J \cdot \mathcal{O}_{\mathcal{X}})$ is an isomorphism.

Proof. If $g\colon X' \to X$ is flat, then the pullback of the natural map $\mathcal{O}_X \to \pi_* \mathcal{O}_{\mathcal{X}}$ under g is the map $\mathcal{O}_{X'} \to \pi'_* \mathcal{O}_{\mathcal{X}'}$. Thus (1) and the case of (2) when g is flat follows from Lemma 6.3.15 and descent. Note that since X' is quasi-separated, it has quasi-affine diagonal (Corollary 4.4.8).

Before proving the general case of (2), we first prove (3). Choose an étale presentation $U \to X$ with U the disjoint union of affine schemes. Since the base change $\pi_U \colon \mathcal{X}_U \to U$ is a good moduli space (by the flat case of (2)) and the adjunction map $\mathrm{id} \to \pi_*\pi^*$ pulls back to the adjunction map $\mathrm{id} \to \pi_{U,*}\pi_U^*$, we may assume that $X = \operatorname{Spec} A$ is affine. If $G_2 \to G_1 \to G \to 0$ is a free presentation, then the projection maps $\pi_*F \otimes G_i \to \pi_*(F \otimes \pi^*G_i)$ are isomorphisms. Since $\pi_*F \otimes -$ and $\pi_*(F \otimes \pi^*-)$ are right exact, we have a commutative diagram

$$\begin{tikzcd}
\pi_*F \otimes G_2 \arrow[r] \arrow[d] & \pi_*F \otimes G_1 \arrow[r] \arrow[d] & \pi_*F \otimes G \arrow[r] \arrow[d] & 0 \\
\pi_*(F \otimes \pi^*G_2) \arrow[r] & \pi_*(F \otimes \pi^*G_1) \arrow[r] & \pi_*(F \otimes \pi^*G) \arrow[r] & 0
\end{tikzcd}$$

Since the left two vertical maps are isomorphisms, so is the right one.

For (2), we must show that $\mathcal{O}_{X'} \to \pi'_*\mathcal{O}_{\mathcal{X}'}$ is an isomorphism as Lemma 6.3.15(2) already established that π'_* is exact. We can assume that X and X' are affine. In this case, g_* is faithfully exact so it suffices to show that

$$g_*\mathcal{O}_{X'} \to g_*\pi'_*\mathcal{O}_{\mathcal{X}'} \cong \pi_*g'_*\mathcal{O}_{\mathcal{X}'} \cong \pi_*\pi^*g_*\mathcal{O}_{X'} \tag{6.3.2}$$

is an isomorphism, where the last equivalence uses the identity $g'_*\pi'^*\mathcal{O}_{X'} \cong \pi^*g_*\mathcal{O}_{X'}$ following from the affineness of g. Thus the composition (6.3.2) is the adjunction isomorphism of (3) applied to $F = g_*\mathcal{O}_{X'}$.

For (4), we know by Flat Base Change (6.1.7) that (4) is fppf local on X and X' and that it holds when g is flat. We may therefore reduce to when $X' \to X$ is a morphism of affine schemes. By factoring $X' \to X$ as a closed immersion followed by a flat morphism, we can further reduce to the case that $X' \hookrightarrow X$ is a closed immersion defined by a quasi-coherent sheaf of ideals $J \subset \mathcal{O}_X$. We aim to show that $\pi_*F/J\pi_*F \cong \pi_*(F/(\pi^{-1}J \cdot \mathcal{O}_{\mathcal{X}})F)$. Using the exactness of π_*, this is equivalent to the inclusion $J\pi_*F \hookrightarrow \pi_*((\pi^{-1}J \cdot \mathcal{O}_{\mathcal{X}})F)$ being surjective. The sheaf $(\pi^{-1}J \cdot \mathcal{O}_{\mathcal{X}})F$ is the image of $\pi^*J \otimes F \to F$. By the exactness of π_*, the pushforward $\pi_*((\pi^{-1}J \cdot \mathcal{O}_{\mathcal{X}})F)$ is the image of $\pi_*(\pi^*J \otimes F) \to \pi_*F$, but by (3) this is identified with the image of $J \otimes \pi_*F \to \pi_*F$.

For (5), if $Z \subset X$ is the closed subspace defined by J, then the preimage ideal sheaf $\pi^{-1}J \cdot \mathcal{O}_{\mathcal{X}}$ defines the preimage $\pi^{-1}(Z)$. Exactness of π_* implies that there is a commutative diagram of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & J \arrow[r] \arrow[d] & \mathcal{O}_X \arrow[r] \arrow[d] & \mathcal{O}_Z \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \pi_*(\pi^{-1}J \cdot \mathcal{O}_{\mathcal{X}}) \arrow[r] & \pi_*\mathcal{O}_{\mathcal{X}} \arrow[r] & \pi_*\mathcal{O}_{\pi^{-1}(Z)} \arrow[r] & 0.
\end{tikzcd}$$

As $\mathcal{X} \to X$ and $\pi^{-1}(Z) \to Z$ are good moduli spaces, the right two vertical arrows are isomorphisms and therefore so is the left arrow. $\square$

Remark 6.3.21. The isomorphism $\pi_*F \otimes G \to \pi_*(F \otimes \pi^*G)$ in (3) is similar to the projection formula but holds even if G is not locally free. It holds as long as π is cohomologically affine.

Lemma 6.3.22. *Let $\pi\colon \mathcal{X} \to X$ be a good moduli space with X quasi-separated.*

(1) If $\mathcal{A}$ is a quasi-coherent sheaf of $\mathcal{O}_{\mathcal{X}}$-algebras, then $\mathcal{S}\mathrm{pec}_{\mathcal{X}}\, \mathcal{A} \to \mathcal{S}\mathrm{pec}_X\, \pi_*\mathcal{A}$ is a good moduli space.

(2) *If $\mathcal{Z} \subset \mathcal{X}$ is a closed substack defined by a sheaf of ideals I and* $\operatorname{im} \mathcal{Z} \subset X$ *is the scheme-theoretic image, i.e. the closed subspace defined by $\pi_* I \subset \mathcal{O}_X$, then $\mathcal{Z} \to \operatorname{im} \mathcal{Z}$ is a good moduli space.*

Proof. For (1), since $\mathcal{X} \times_X \mathcal{S}\mathrm{pec}_X\, \pi_*\mathcal{A} \to \mathcal{S}\mathrm{pec}_X\, \pi_*\mathcal{A}$ is cohomologically affine by Lemma 6.3.15 and $\mathcal{S}\mathrm{pec}_{\mathcal{X}}\, \mathcal{A} \to \mathcal{X} \times_X \mathcal{S}\mathrm{pec}_X\, \pi_*\mathcal{A}$ is affine, it follows that $\mathcal{S}\mathrm{pec}_{\mathcal{X}}\, \mathcal{A} \to \mathcal{S}\mathrm{pec}_X\, \pi_*\mathcal{A}$ is cohomologically affine and therefore a good moduli space as the push forward of $\mathcal{O}_{\mathcal{S}\mathrm{pec}_{\mathcal{X}} \mathcal{A}}$ is $\mathcal{O}_{\mathcal{S}\mathrm{pec}_X \pi_*\mathcal{A}}$ by construction. Applying (1) to $\mathcal{Z} = \mathcal{S}\mathrm{pec}_{\mathcal{X}}(\mathcal{O}_{\mathcal{X}}/I)$ recovers (2) using that $\pi_*(\mathcal{O}_{\mathcal{X}}/I) = \mathcal{O}_X/\pi_*I$. □

The above lemmas allow us to give quick proofs of the first two parts of Theorem 6.3.5.

Proof of Theorem 6.3.5(1). As $\mathcal{X}$ is quasi-separated, so is X. For every field-valued point $x \in X(\Bbbk)$, consider the base change $\mathcal{X} \times_X \operatorname{Spec} \Bbbk$. By Lemma 6.3.20(2), $\mathcal{X}_x \to \operatorname{Spec} \Bbbk$ is a good moduli space and in particular $\Gamma(\mathcal{X}_x, \mathcal{O}_{\mathcal{X}_x}) = \Bbbk$. It follows that $\mathcal{X}_x$ is non-empty and that $\pi \colon \mathcal{X} \to X$ is surjective. For a closed substack $\mathcal{Z} \subset \mathcal{X}$, Lemma 6.3.22(2) implies that $\mathcal{Z} \to \operatorname{im} \mathcal{Z}$ is a good moduli space and therefore also surjective. Thus, the set-theoretic image $\pi(\mathcal{Z})$ is identified with the scheme-theoretic image $\operatorname{im} \mathcal{Z}$ and is therefore closed. Since good moduli spaces are stable under base change, they are universally closed. □

Proof of Theorem 6.3.5(2). For two substacks $\mathcal{Z}_1, \mathcal{Z}_2 \subset X$ defined by ideal sheaves $I_1, I_2 \subset \mathcal{O}_X$, we apply the exact functor π_* to the short exact sequence $0 \to I_1 \to I_1 + I_2 \to I_2/I_1 \cap I_2 \to 0$ and surjection $I_2 \to I_2/I_1 \cap I_2$ to obtain a commutative diagram

$$\begin{tikzcd}
 & & \pi_* I_2 \arrow[d] \arrow[dr, two heads] & & \\
0 \arrow[r] & \pi_* I_1 \arrow[r] & \pi_*(I_1 + I_2) \arrow[r, two heads] & \pi_* I_2/\pi_*(I_1 \cap I_2) \arrow[r] & 0.
\end{tikzcd}$$

It follows that the natural inclusion $\pi_* I_1 + \pi_* I_2 \to \pi_*(I_1 + I_2)$ is surjective. □

6.3.5 Finite typeness of good moduli spaces

We will use the fact that a good moduli space $\mathcal{X} \to X$ is universally submersive and show that finite typeness descends under universally submersive morphisms. Recall from §A.4.3 that a morphism $f \colon X \to Y$ of schemes is *universally submersive* if f is surjective and Y has the quotient topology, and these properties are stable under base change. This notion extends to morphisms of algebraic stacks. Examples of universally submersive morphisms include fppf morphisms and universally closed morphisms.

Proposition 6.3.23. *Consider a commutative diagram*

$$\begin{tikzcd}
X \arrow[d] \arrow[dr] & \\
Y \arrow[r] & S
\end{tikzcd}$$

of noetherian schemes where $X \to Y$ is universally submersive. If $X \to S$ is of finite type, then so is $Y \to S$.

Proof. We can assume that $S = \operatorname{Spec} R$ and $Y = \operatorname{Spec} B$ are affine. Since a noetherian ring B is of finite type over R if and only if the reductions of the irreducible components of $\operatorname{Spec} B$ are of finite type over R, we can assume that B is an integral domain.

By Generic Flatness (A.2.11) and Raynaud-Gruson Flatification (A.2.16), there is a commutative diagram

$$\begin{tikzcd} \widetilde{X} \arrow[r] \arrow[d] & X \arrow[d, "f"] \arrow[dr] & \\ Y' \arrow[r, "g"] & Y \arrow[r] & S \end{tikzcd}$$

where $\widetilde{X} \to Y'$ is flat, $Y' = \operatorname{Bl}_I Y \to Y$ is the blow-up along an ideal $I \subset B$ and $\widetilde{X}$ is the strict transform of X, i.e. the closure of $(Y' \setminus g^{-1}(V(I))) \times_Y X$ in the base change $Y' \times_Y X$. We claim that $\widetilde{X} \to Y'$ is surjective. As $g\colon Y' \to Y$ is an isomorphism over $U = Y \setminus V(I)$ and $f\colon X \to Y$ is surjective, we know that $g^{-1}(U) \subset Y'$ is contained in the image. If $y' \in Y'$ is a point, we can choose a map $\operatorname{Spec} R \to Y'$ from a DVR whose generic point maps to $g^{-1}(U)$ and whose special point maps to y'. Since $X \to Y$ is universally submersive, there exists an extension of DVRs $R \to R'$ and a lift $\operatorname{Spec} R' \to X$ (see Exercise A.4.7). The induced map $\operatorname{Spec} R' \to X \times_Y Y'$ factors through $\widetilde{X}$ and we see that y' is thus in the image of $\widetilde{X}$.

Since X is of finite type over S, so is $\widetilde{X}$. Using faithfully flat descent, $Y' \to S$ is also of finite type. To show that $Y \to S$ is of finite type, we may choose generators $a_1, \dots, a_n \in I$ so that $Y' = \bigcup_i \operatorname{Spec} B_i$ where $B_i = B\langle f_j/f_i \rangle \subset K = \operatorname{Frac}(B)$ is the subalgebra generated by B and the elements f_j/f_i for $j \neq i$. Write $B = \bigcup_\lambda B_\lambda$ as a union of its finitely generated R-subalgebras. For $\lambda \gg 0$, each $f_i \in B_\lambda$ and we set $I_\lambda = (f_1, \dots, f_n) \subset B_\lambda$. Since Y' is finite type over B, each B_i is finitely generated over B and thus for $\lambda \gg 0$, we see that in the diagram

$$\begin{tikzcd} B_{\lambda,i} \arrow[r, equal] \arrow[d, hook] & B_\lambda\langle f_j/f_i \rangle \arrow[r, hook] & \operatorname{Frac}(B_\lambda) \arrow[d, hook] \\ B_i \arrow[r, equal] & B\langle f_j/f_i \rangle \arrow[r, hook] & \operatorname{Frac}(B) \end{tikzcd}$$

the inclusion $B_{\lambda,i} \hookrightarrow B_i$ is surjective. It follows that $Y' = \operatorname{Bl}_I \operatorname{Spec} B = \operatorname{Bl}_{I_\lambda} \operatorname{Spec} B_\lambda$ for $\lambda \gg 0$. Considering the composition

$$g_\lambda \colon Y' \xrightarrow{g} Y = \operatorname{Spec} B \xrightarrow{p_\lambda} \operatorname{Spec} B_\lambda,$$

the push forward of the injection $\mathcal{O}_Y \hookrightarrow g_* \mathcal{O}_{Y'}$ along p_λ yields an inclusion $p_{\lambda,*}\mathcal{O}_Y \hookrightarrow g_{\lambda,*}\mathcal{O}_{Y'}$. But $g_{\lambda,*}\mathcal{O}_{Y'}$ is a coherent module on $\operatorname{Spec} B_\lambda$ and thus so is $p_{\lambda,*}\mathcal{O}_Y$. This shows that B is a finite B_λ-module and thus finitely generated as an R-algebra. $\square$

We apply this proposition to show that good moduli spaces are of finite type.

Proof of Theorem 6.3.5(3). If $\mathcal{X}$ is noetherian, if $I_1 \subset I_2 \subset \cdots$ is an ascending chain of ideal sheaves of $\mathcal{O}_X$, then $\pi^{-1}I_1 \cdot \mathcal{O}_{\mathcal{X}} \subset \pi^{-1}I_2 \cdot \mathcal{O}_{\mathcal{X}} \subset$ is an ascending chain of ideal sheaves of $\mathcal{O}_{\mathcal{X}}$ which terminates. By Lemma 6.3.22(5), $I_n = \pi_*(\pi^{-1}I_n \cdot \mathcal{O}_{\mathcal{X}})$ and therefore the chain $I_1 \subset I_2 \subset \cdots$ terminates and X is noetherian.

Assume now that S is noetherian and $\mathcal{X}$ is of finite type over S. As $\mathcal{X} \to X$ is universally closed (Theorem 6.3.5(1)), it is also universally submersive. Choose a smooth presentation $U \to \mathcal{X}$ from a scheme. Since $U \to \mathcal{X}$ is universally submersive, so is the composition $U \to \mathcal{X} \to X$. Since $U \to S$ is of finite type and X is noetherian, Proposition 6.3.23 implies that $X \to S$ is also of finite type.

Given a coherent sheaf F on $\mathcal{X}$, to show that the pushforward $\pi_* F$ is coherent, we may assume that $X = \operatorname{Spec} A$ is affine and that $\mathcal{X}$ is irreducible. We first handle the case when $\mathcal{X}$ is reduced. By noetherian induction, we can assume that $\pi_* F$ is coherent if $\operatorname{Supp}(F) \subsetneq \mathcal{X}$. The maximal torsion subsheaf $F_{\mathrm{tors}} \subset F$ has support strictly contained in $\mathcal{X}$. Using the exact sequence $0 \to F_{\mathrm{tors}} \to F \to F/F_{\mathrm{tors}} \to 0$ and the exactness of π_*, we see the coherence of $\pi_*(F/F_{\mathrm{tors}})$ implies the coherence of $\pi_* F$. In other words, we can assume that F is torsion free. In this case, every section $s \colon \mathcal{O}_{\mathcal{X}} \to F$ is injective. We now argue by induction on the dimension of the vector space $\xi^* F$ where $\xi \colon \operatorname{Spec} K \to \mathcal{X}$ is a field-valued point whose image is the generic point. If F has no sections, then $\pi_* F = 0$ is coherent. Otherwise, a section induces a short exact sequence $0 \to \mathcal{O}_{\mathcal{X}} \to F \to F/\mathcal{O}_{\mathcal{X}} \to 0$ and $\xi^*(F/\mathcal{O}_{\mathcal{X}})$ has strictly smaller dimension. By again appealing to the exactness of π_*, we see that the coherence of $\pi_*(F/\mathcal{O}_{\mathcal{X}})$ implies the coherence of $\pi_* F$. Finally, to reduce to the reduced case, let $I \subset \mathcal{O}_{\mathcal{X}}$ be the ideal sheaf defining $\mathcal{X}_{\mathrm{red}} \hookrightarrow \mathcal{X}$. Then for some $N > 0$, we have that $I^N = 0$. By examining the exact sequences $0 \to \pi_*(I^{k+1}F) \to \pi_*(I^k F) \to \pi_*(I^k F/I^{k+1}F) \to 0$ and using that $\pi_*(I^k F/I^{k+1}F)$ is coherent (since $I^k F/I^{k+1}F$ is supported on $\mathcal{X}_{\mathrm{red}}$)), we conclude by induction that $\pi_* F$ is coherent. $\square$

6.3.6 Universality of good moduli spaces

We now complete the proof of Theorem 6.3.5 by showing that $\pi \colon \mathcal{X} \to X$ is universal for maps to algebraic spaces. Our argument follows the same logic for coarse moduli spaces in Theorem 4.3.6.

Proof of Theorem 6.3.5(4). We need to show that every diagram

$$\begin{tikzcd} \mathcal{X} \arrow[d, "\pi"'] \arrow[dr, "f"] & \\ X \arrow[r, dashed] & Y \end{tikzcd} \tag{6.3.3}$$

has a unique filling or in other words that the natural map $\operatorname{Mor}(X, Y) \to \operatorname{Mor}(\mathcal{X}, Y)$ is bijective.

The uniqueness follows as in the proof of Theorem 4.3.6 and uses only that $\pi \colon \mathcal{X} \to X$ is universally closed, schematically dominant and surjective: if $h_1, h_2 \colon X \to Y$ are two fillings of (6.3.3), then $\pi \colon \mathcal{X} \to X$ factors through the equalizer $E \to X$ of h_1 and h_2. Since $E \to X$ is universally closed, locally of finite type, surjective and a monomorphism, it is an isomorphism.

For existence, the case when Y is affine is easy:

$$\begin{aligned} \operatorname{Mor}(X, Y) = \operatorname{Hom}(\Gamma(Y, \mathcal{O}_Y), \Gamma(X, \mathcal{O}_X)) = \\ \operatorname{Hom}(\Gamma(Y, \mathcal{O}_Y), \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})) = \operatorname{Mor}(\mathcal{X}, Y). \end{aligned}$$

(Although unnecessary for the argument below, the case when Y is a scheme is also straightforward: if $\{Y_i\}$ is an affine cover of Y and we set $\mathcal{X}_i := f^{-1}(Y_i) \subset \mathcal{X}$

with complement $\mathcal{Z}_i$, then $\mathcal{X} \setminus \pi^{-1}(\pi(\mathcal{Z}_i)) \to X \setminus \pi(\mathcal{Z}_i)$ is a good moduli space and $\mathcal{X} \setminus \pi^{-1}(\pi(\mathcal{Z}_i)) \subset \mathcal{X}_i$. By the affine case, we have unique factorizations $X \setminus \pi(\mathcal{Z}_i) \to Y_i$ and since $\bigcap_i \pi(\mathcal{Z}_i) = \emptyset$, these maps glue to the desired map $X \to Y$; see also [GIT, §0.6].)

For the general case, since $\mathcal{X}$ is quasi-compact, the map $\mathcal{X} \to Y$ factors through a quasi-compact subspace so we can further assume that Y is quasi-compact. We can also use étale-descent and limit methods to reduce to the case that $X = \operatorname{Spec} A$ where A is a strictly henselian local ring. This reduction works just as in the case of coarse moduli spaces (Theorem 4.3.6). Since A is local, there is a unique closed point $x \in \mathcal{X}$; let $\mathcal{G}_x \hookrightarrow \mathcal{X}$ be the closed immersion of the residual gerbe (Proposition 3.5.16).

Let $(Y' = \operatorname{Spec} B, y') \to (Y, f(x))$ be an étale presentation. The base change $\mathcal{X}' := \mathcal{X} \times_Y Y' \to \mathcal{X}$ is an étale, separated, surjective and representable morphism. Let $x' \in \mathcal{X}'$ be a preimage of $x \in \mathcal{X}$ and $\mathcal{U}' \subset \mathcal{X}'$ be a quasi-compact open substack containing x'.

$$\begin{tikzcd}
\mathcal{U}' \arrow[r, hook] & \mathcal{X}' \arrow[r] \arrow[dd] & \mathcal{X} \arrow[d, "\pi"] \arrow[dd, bend left, "f"] \\
 & & X \\
 & Y' \arrow[r] & Y
\end{tikzcd}$$

Then $\mathcal{U}' \to \mathcal{X}$ is a quasi-finite, separated and representable morphism and Zariski's Main Theorem (6.1.10) implies that there is a factorization $\mathcal{U}' \to \widetilde{\mathcal{X}} \to \mathcal{X}$ with $\mathcal{U}' \hookrightarrow \widetilde{\mathcal{X}}$ an open immersion and $\widetilde{\mathcal{X}} \to \mathcal{X}$ a finite morphism. Writing $\widetilde{\mathcal{X}} = \mathcal{S}\mathrm{pec}_{\mathcal{X}}\, \mathcal{A}$ for a coherent sheaf of algebras $\mathcal{A}$, Lemma 6.3.22(1) implies that $\widetilde{\pi}\colon \widetilde{\mathcal{X}} \to \widetilde{X} := \operatorname{Spec}_X \pi_* \mathcal{A}$ is a good moduli space and we know from Theorem 6.3.5(3) that $\pi_*\mathcal{A}$ is coherent. As $\widetilde{X} \to X = \operatorname{Spec} A$ is finite with A henselian, we can write $\widetilde{X} = \coprod_i \operatorname{Spec} A_i$ with each A_i a henselian local ring (Proposition A.9.3). Replace $\widetilde{\mathcal{X}}$ with the copy of $\widetilde{\mathcal{X}}_i := \widetilde{\pi}^{-1}(\operatorname{Spec} A_i)$ containing x' and replace $\mathcal{U}'$ with $\widetilde{\mathcal{X}}_i \cap \mathcal{U}'$. Then $\widetilde{\mathcal{X}}$ has a unique closed point which is the point $x' \in \mathcal{U}'$ and thus the complement $\widetilde{\mathcal{X}} \setminus \mathcal{U}'$ is empty, i.e. $\mathcal{U}' = \widetilde{\mathcal{X}}$. We conclude that $\mathcal{U}' \to \mathcal{X}$ is a finite étale morphism and since it induces an isomorphism of residual gerbes at x', the map has degree one; it follows that $\mathcal{U}' \to \mathcal{X}$ is an isomorphism. Since Y' is an affine, the morphism $\mathcal{X} \cong \mathcal{U}' \to Y'$ factors through a map $X \to Y'$, and thus $f\colon \mathcal{X} \to Y$ factors through the composition $X \to Y' \to Y$. □

6.3.7 Luna's Fundamental Lemma

We will apply the following result in our construction of good moduli spaces (Theorem 6.7.1), to provide refinements of the Local Structure Theorem for Algebraic Stacks (6.5.1) and in the proof of Luna's Étale Slice Theorem (6.5.4) but it appears in many other arguments as well.

Theorem 6.3.24 (Luna's Fundamental Lemma)**.** *Consider a commutative diagram*

$$\begin{tikzcd} \mathcal{X}' \arrow[r, "f"] \arrow[d, "\pi'"] & \mathcal{X} \arrow[d, "\pi"] \\ X' \arrow[r, "g"] & X \end{tikzcd} \tag{6.3.4}$$

where $f\colon \mathcal{X}' \to \mathcal{X}$ is a separated and representable morphism of noetherian algebraic stacks, each with affine diagonal, and where π and π' are good moduli spaces. Let $x' \in \mathcal{X}'$ be a point such that

(a) f is étale at x',

(b) f induces an isomorphism of stabilizer groups at x', and

(c) $x' \in \mathcal{X}'$ and $x = f(x') \in \mathcal{X}$ are closed points.

Then there is an open neighborhood $U' \subset X'$ of $\pi'(x')$ such that $U' \to X$ is étale and such that $U' \times_X \mathcal{X} \cong \pi'^{-1}(U')$.

Remark 6.3.25. This result is really saying two things: (1) g is étale at $\pi'(x')$ and (2) after replacing X' with an open neighborhood of $\pi'(x')$ the diagram (6.3.4) is cartesian. In the case of quotients by finite groups, this was established in Proposition 4.3.7. Luna's original formulation [Lun73, p. 94] was the case when $\mathcal{X}' \cong [\operatorname{Spec} A'/G]$ and $\mathcal{X} \cong [\operatorname{Spec} A/G]$ with G linearly reductive and where $\mathcal{X}' \to \mathcal{X}$ is induced by a G-equivariant map $\operatorname{Spec} A' \to \operatorname{Spec} A$.

Proof. We will adapt the argument of Theorem 6.3.5(4). Since the question is étale local on X, limit methods (see the proof of Proposition 4.3.7) allow us to assume that $X = \operatorname{Spec} A$ with A a strictly henselian local ring. If $\mathcal{U}' \subset \mathcal{X}'$ is the étale locus of f, then $\mathcal{X}' \setminus \pi'^{-1}(\pi'(\mathcal{X}' \setminus \mathcal{U}'))$ contains x' since $\pi'(x')$ and $\pi'(\mathcal{X}' \setminus \mathcal{U}')$ are disjoint by Theorem 6.3.5(2). We can therefore replace $\mathcal{X}'$ with $\mathcal{X}' \setminus \pi'^{-1}(\pi'(\mathcal{X}' \setminus \mathcal{U}'))$ and assume that f is étale.

By Zariski's Main Theorem (6.1.10), we may choose a factorization $\mathcal{X}' \to \widetilde{\mathcal{X}} = \mathcal{S}\mathrm{pec}_{\mathcal{X}}\, \mathcal{A} \to \mathcal{X}$ with $\mathcal{X}' \hookrightarrow \widetilde{\mathcal{X}}$ an open immersion and $\widetilde{\mathcal{X}} \to \mathcal{X}$ a finite morphism. Then $\widetilde{\mathcal{X}} \to \widetilde{X} := \operatorname{Spec}_X \pi_* \mathcal{A}$ is a good moduli space and $\widetilde{X} \to X$ is finite. As A is henselian, we can write $\widetilde{X} = \coprod_i \operatorname{Spec} A_i$ with each A_i a henselian local ring. If $U' = \operatorname{Spec} A_i$ denotes the connected component containing the image of x', then $\widetilde{\pi}^{-1}(U') \subset \widetilde{\mathcal{X}}$ is an open substack containing a unique closed point, which is necessarily x'; it follows that $\mathcal{X}' = \pi^{-1}(U')$. Since $\mathcal{X}' \to \mathcal{X}$ is a finite étale morphism of degree one (as it preserves residual gerbes at x'), we see that $f\colon \mathcal{X}' \to \mathcal{X}$ is an isomorphism and thus so is $g\colon X' \to X$. ☐

Corollary 6.3.26. *With the same hypotheses as Theorem 6.3.24, suppose that f is étale and that for all closed points $x' \in \mathcal{X}'$*

(a) $f(x') \in \mathcal{X}$ is closed, and

(b) f induces an isomorphism of stabilizer groups at x'.

Then $g\colon X' \to X$ is étale and (6.3.4) is cartesian. ☐

6.3.8 Finite covers of good moduli spaces

Proposition 6.3.27. *Consider a commutative diagram*

$$\begin{tikzcd} \mathcal{X}' \arrow[r, "f"] \arrow[d, "\pi'"] & \mathcal{X} \arrow[d, "\pi"] \\ X' \arrow[r, "g"] & X \end{tikzcd}$$

where $\mathcal{X}$ and $\mathcal{X}'$ are noetherian algebraic stacks with affine diagonal, and π and π' are good moduli spaces. Assume that

(a) $f\colon \mathcal{X}' \to \mathcal{X}$ is quasi-finite, separated and representable,

(b) f maps closed points to closed points, and

(c) g is finite.

Then f is finite.

Proof. By Zariski's Main Theorem (6.1.10), there is a factorization $\mathcal{X}' \to \widetilde{\mathcal{X}} = \mathcal{S}\mathrm{pec}_{\mathcal{X}}\,\mathcal{A} \to \mathcal{X}$ with $\mathcal{X}' \hookrightarrow \widetilde{\mathcal{X}}$ an open immersion and $\widetilde{\mathcal{X}} \to \mathcal{X}$ a finite morphism. Then $\widetilde{X} = \mathrm{Spec}_X\, \pi_*\mathcal{A}$ is a finite over X and $\widetilde{\mathcal{X}} \to \widetilde{X}$ is a good moduli space. By replacing $\mathcal{X} \to X$ with $\widetilde{\mathcal{X}} \to \widetilde{X}$, we can assume that f is an open immersion. By replacing $\mathcal{X}$ with the fiber product $X' \times_X \mathcal{X}$, we can further reduce to the case that $X' = X$. For every closed point $x \in X$, let $x' \in \mathcal{X}'$ be the unique closed point over x. By (b), $f(x') \in \mathcal{X}$ is the unique closed point over x. Since $\mathcal{X}'$ contains all the closed points of $\mathcal{X}$, $f\colon \mathcal{X}' \to cX$ is an isomorphism. $\square$

Proposition 6.3.28. *Suppose $\mathcal{X}$ is a noetherian algebraic stack with affine diagonal and a good moduli space $\pi\colon \mathcal{X} \to X$. If the diagonal $\mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is quasi-finite, then it is finite (i.e. $\pi\colon \mathcal{X} \to X$ is separated).*

Proof. We claim that $\mathcal{X} \times_X \mathcal{X} \to X$ is a good moduli space. By Lemma 6.3.15, the projection $p_1\colon \mathcal{X} \times_X \mathcal{X} \to \mathcal{X}$ is cohomologically affine and therefore so is the composition $\mathcal{X} \times_X \mathcal{X} \xrightarrow{p_1} \mathcal{X} \xrightarrow{\pi} X$. On the other hand, if $U \to \mathcal{X}$ is a smooth presentation, then $p_1\colon U \times_X \mathcal{X} \to U$ is a good moduli space (Lemma 6.3.20) and in particular $\mathcal{O}_U \xrightarrow{\sim} p_{1,*}\mathcal{O}_{U \times_X \mathcal{X}}$. It follows from descent that $\mathcal{O}_{\mathcal{X}} \xrightarrow{\sim} p_{1,*}\mathcal{O}_{\mathcal{X} \times_X \mathcal{X}}$ and thus $\mathcal{O}_X \xrightarrow{\sim} (\pi \circ p_1)_*\mathcal{O}_{\mathcal{X} \times_X \mathcal{X}}$; the claim follows.

The diagonal $\mathcal{X} \to \mathcal{X} \times_X \mathcal{X}$ is a quasi-finite, separated and representable morphism that sends closed points to closed points and induces an isomorphism on good moduli spaces. Proposition 6.3.27 implies that $\mathcal{X} \to \mathcal{X} \times_X \mathcal{X}$ is finite. Note that since $\mathcal{X}$ has affine diagonal, the finiteness of the diagonal is equivalent to its properness. $\square$

6.3.9 Descending vector bundles

Proposition 6.3.29. *Let $\mathcal{X}$ be a noetherian algebraic stack and $\pi\colon \mathcal{X} \to X$ be a good moduli space. A vector bundle F on $\mathcal{X}$ descends to a vector bundle on X if and only if for every field-valued point $x\colon \mathrm{Spec}\,\Bbbk \to \mathcal{X}$ with closed image, the action of G_x on the fiber $F \otimes \Bbbk$ is trivial. In this case, π_*F is a vector bundle and the adjunction map $\pi^*\pi_*F \to F$ is an isomorphism.*

Proof. We follow the argument in the case of a tame coarse moduli space (Proposition 4.3.25). The condition is clearly necessary. To see that the condition is sufficient, consider the commutative diagram

$$\begin{tikzcd} \mathcal{G}_x \arrow[r, hook] \arrow[d, "p"'] & \mathcal{X} \arrow[d, "\pi"] \\ \operatorname{Spec} \kappa(x) \arrow[r, hook] & X. \end{tikzcd}$$

We first claim that $\pi^*\pi_*F \to F$ is surjective. For every closed point $x \in \mathcal{X}$, the hypotheses imply that $p^*p_*(F|_{\mathcal{G}_x}) \cong F|_{\mathcal{G}_x}$. Applying $\pi^*\pi_*(-)|_{\mathcal{G}_x}$ to the surjection $F \to F|_{\mathcal{G}_x}$ and using the exactness of π_*, we obtain that $(\pi^*\pi_*F)|_{\mathcal{G}_x} \to \pi^*(\pi_*(F|_{\mathcal{G}_x}))|_{\mathcal{G}_x} \cong p^*p_*(F|_{\mathcal{G}_x}) \cong F|_{\mathcal{G}_x}$ is surjective. The claim now follows from Lemma 6.3.30.

To show that π_*F is a vector bundle, we may assume that $X = \operatorname{Spec} A$ is affine and that the rank r of F is constant. The surjection $\bigoplus_{s \in \Gamma(X, \pi_*F)} A \to \pi_*F$ pulls back to a surjection $\bigoplus_{s \in \Gamma(\mathcal{X}, F)} \mathcal{O}_{\mathcal{X}} \to \pi^*\pi_*F$ and by the above claim, the composition $\bigoplus_{s \in \Gamma(\mathcal{X}, F)} \mathcal{O}_{\mathcal{X}} \to \pi^*\pi_*F \to F$ is surjective. As $F|_{\mathcal{G}_x} \cong \mathcal{O}^r_{\mathcal{G}_x}$ is trivial, for each closed point $x \in |\mathcal{X}|$, we can find r sections $\phi \colon \mathcal{O}^r_{\mathcal{X}} \to F$ such that $\phi|_{\mathcal{G}_x}$ is an isomorphism. By Lemma 6.3.30, there exists an open neighborhood $U \subset X$ of $\pi(x)$ such that $\phi|_{\pi^{-1}(U)}$ is an isomorphism. Thus $\pi_*\phi \colon \mathcal{O}^r_X \to \pi_*F$ is an isomorphism over U and we conclude that π_*F is a vector bundle of the same rank as F. Finally, since $\pi^*\pi_*F \to F$ is a surjection of vector bundles of the same rank, it is an isomorphism.

The case of a good quotient is due to Kempf. See also [KKV89, Prop. 4.2], [Alp13, Thm. 10.3] and [Ryd20, Thm. B]. ☐

Lemma 6.3.30. *Let $\mathcal{X}$ be a noetherian algebraic stack and $\pi \colon \mathcal{X} \to X$ be a good moduli space. Let $x \in |\mathcal{X}|$ be a closed point.*

(1) If F is a coherent sheaf on $\mathcal{X}$ such that $F|_{\mathcal{G}_x} = 0$, then there exists an open neighborhood $U \subset X$ of $\pi(x)$ such that $F|_{\pi^{-1}(U)} = 0$.

(2) If $\phi \colon F \to G$ is a morphism of coherent sheaves (resp. vector bundles of the same rank) on $\mathcal{X}$ such that $\phi|_{\mathcal{G}_x}$ is surjective, then there exists an open neighborhood $U \subset X$ of $\pi(x)$ such that $\phi|_{\pi^{-1}(U)}$ is surjective (resp. an isomorphism).

Proof. The argument of Lemma 4.3.21 applies. ☐

6.4 Coherent Tannaka duality and coherent completeness

We prove a version of Tannaka duality for noetherian algebraic stacks with affine diagonal (Theorem 6.4.1). We also introduce the notion of an algebraic stack $\mathcal{X}$ being coherently complete along a closed substack $\mathcal{X}_0$ (Definition 6.4.4) and show that certain quotient stacks with a unique closed point are coherently complete (Theorem 6.4.11). This includes the important examples of $[\mathbb{A}^1/\mathbb{G}_m]_R$ and Φ_R defined in §6.7.2 where R is a complete DVR.

The combined power of Tannaka duality and coherent completeness allows us to extend compatible maps $\mathcal{X}_n \to \mathcal{Y}$ from the nth nilpotent thickenings of $\mathcal{X}_0$ to a morphism $\mathcal{X} \to \mathcal{Y}$ (Corollary 6.4.8). This technique is used in an essential way in the proof of the Local Structure Theorem for Algebraic Stacks (6.5.1) and also appears in many other arguments—it becomes a powerful new addition in our algebraic stack toolkit.

6.4.1 Coherent Tannaka Duality

A classical theorem of Gabriel [Gab62] states that two noetherian schemes X and Y are isomorphic if and only if their abstract categories $\mathrm{Coh}(X)$ and $\mathrm{Coh}(Y)$ of coherent sheaves are equivalent, or in other words that a scheme X can be recovered from the category $\mathrm{Coh}(X)$. In representation theory, classical Tannaka duality by Saavedra Rivano [SR72] (see also Deligne and Milne's article [DMOS82, Ch. II]) asserts that an affine group scheme G over a field $\Bbbk$ can be recovered from the tensor category $\mathrm{Rep}^{\mathrm{fd}}(G)$ of finite dimensional representations and its forgetful functor $\mathrm{Rep}^{\mathrm{fd}}(G) \to \mathrm{Vect}_{\Bbbk}$.

Combining these two facts, one might hope than algebraic stack $\mathcal{X}$ is recovered by the tensor category $\mathrm{Coh}(\mathcal{X})$.[2] Following a brilliant observation of Lurie [Lur04], we will not only confirm this expectation, but we will show that in fact a tensor functor $\mathrm{Coh}(\mathcal{Y}) \to \mathrm{Coh}(\mathcal{X})$ is enough to recover a morphism $\mathcal{X} \to \mathcal{Y}$ of algebraic stacks.

Theorem 6.4.1 (Coherent Tannaka Duality)**.** *For noetherian algebraic stacks $\mathcal{X}$ and $\mathcal{Y}$ with affine diagonal, the functor*

$$\mathrm{MOR}(\mathcal{X}, \mathcal{Y}) \to \mathrm{MOR}^{\otimes}(\mathrm{Coh}(\mathcal{Y}), \mathrm{Coh}(\mathcal{X})), \qquad f \mapsto f^* \tag{6.4.1}$$

is an equivalence of categories, where $\mathrm{MOR}^{\otimes}(\mathrm{Coh}(\mathcal{Y}), \mathrm{Coh}(\mathcal{X}))$ denotes the category of right exact additive tensor functors $\mathrm{Coh}(\mathcal{Y}) \to \mathrm{Coh}(\mathcal{X})$ of symmetric monoidal abelian categories where morphisms are tensor natural transformations.

Remark 6.4.2. A *symmetric monoidal category* is a category $\mathcal{A}$ endowed with a bifunctor $\otimes : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ and a unit $1 \in \mathcal{A}$ together with associativity isomorphisms $\alpha_{A,B,C} : A \otimes (B \otimes C) \xrightarrow{\sim} (A \otimes B) \otimes C$, left and right unit isomorphisms $l_A : 1 \otimes A \xrightarrow{\sim} A \xrightarrow{\sim} A$ and $r_A : A \otimes 1 \xrightarrow{\sim} A$, and commutativity isomorphisms $s_{A,B} : A \otimes B \cong B \otimes A$ (with $s_{A,B} \circ s_{B,A} = \mathrm{id}$) satisfying certain coherence conditions [Mac71, §XI.1]. A *tensor functor* $F : \mathcal{A} \to \mathcal{B}$ between symmetric monoidal abelian categories is a functor equipped with isomorphisms $\Phi_{A,B} : F(A) \otimes F(B) \xrightarrow{\sim} F(A \otimes B)$ and $\varphi : 1_{\mathcal{B}} \xrightarrow{\sim} F(1_{\mathcal{A}})$ compatible with the isomorphisms $\alpha_{A,B,C}$, l_A, r_A and $s_{A,B}$ [Mac71, §XI.2]. A *tensor natural transformation* between tensor functors is a natural transformation of functors compatible with the isomorphisms $\Phi_{A,B}$ and φ [Mac71, §XI.2].

A *symmetric monoidal abelian category* (resp. *symmetric monoidal R-linear abelian category* for a ring R) is a symmetric monoidal (resp. R-linear) abelian category $\mathcal{A}$ such that $\otimes : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ is additive (resp. R-linear) in each variable. A tensor functor is *additive* or *R-linear* if the underlying functor is. When $\mathcal{X}$ and $\mathcal{Y}$ are defined over a noetherian ring R, then Theorem 6.4.1 induces an equivalence

$$\mathrm{MOR}_R(\mathcal{X}, \mathcal{Y}) \xrightarrow{\sim} \mathrm{MOR}_R^{\otimes}(\mathrm{Coh}(\mathcal{Y}), \mathrm{Coh}(\mathcal{X}))$$

between morphisms over R and right exact R-linear tensor functors.

[2] The structure as an abelian category is not enough, e.g. $\mathrm{Coh}(B\mathbb{Z}/2) \cong \mathrm{Coh}(\operatorname{Spec} \Bbbk \coprod \operatorname{Spec} \Bbbk)$ in $\mathrm{char}(\Bbbk) \neq 2$.

Proof. Since every quasi-coherent sheaf on a noetherian algebraic stack is a colimit of its coherent subsheaves (Proposition 6.1.8), every right exact tensor functor $F\colon \mathrm{Coh}(\mathcal{Y}) \to \mathrm{Coh}(\mathcal{X})$ extends to a tensor functor $F\colon \mathrm{QCoh}(\mathcal{Y}) \to \mathrm{QCoh}(\mathcal{X})$ preserving colimits. Likewise every tensor natural transformation between functors of coherent sheaves extends uniquely to one defined on quasi-coherent sheaves.

Fully faithfulness: Let $f, g\colon \mathcal{X} \to \mathcal{Y}$. Choose a smooth presentation $p\colon U \to \mathcal{Y}$ where U is an affine scheme. Since the question is smooth-local on $\mathcal{X}$, after replacing $\mathcal{X}$ with $\mathcal{X} \times_{f,\mathcal{Y},p} U$, we may assume there is a factorization $f\colon \mathcal{X} \xrightarrow{\widetilde{f}} U \xrightarrow{p} \mathcal{Y}$. Likewise, we may assume there is a factorization $g\colon \mathcal{X} \xrightarrow{\widetilde{g}} V \xrightarrow{q} \mathcal{Y}$ where V is an affine scheme. Since $\mathcal{Y}$ has affine diagonal, $p\colon U \to \mathcal{Y}$ is affine and we have identifications

$$\mathrm{Mor}_{\mathcal{Y}}(\mathcal{X}, U) \cong \mathrm{Hom}_{\mathcal{O}_{\mathcal{Y}}-\mathrm{alg}}(p_*\mathcal{O}_U, f_*\mathcal{O}_{\mathcal{X}}) \cong \mathrm{Hom}_{\mathcal{O}_{\mathcal{X}}-\mathrm{alg}}(f^*p_*\mathcal{O}_U, \mathcal{O}_{\mathcal{X}})$$

Therefore $\widetilde{f}$ and $\widetilde{g}$ correspond to sections $s_{\widetilde{f}}\colon f^*p_*\mathcal{O}_U \to \mathcal{O}_{\mathcal{X}}$ and $s_{\widetilde{g}}\colon g^*q_*\mathcal{O}_V \to \mathcal{O}_{\mathcal{X}}$. A 2-isomorphism $\alpha\colon f \to g$ (i.e. a morphism in $\mathrm{MOR}(\mathcal{X}, \mathcal{Y})$) is identified with a factorization

$$\begin{tikzcd}
\mathcal{X} \arrow[r, "(\widetilde{f}, \widetilde{g}, \alpha)"] \arrow[dr, "\mathrm{id}"'] & U \times_{\mathcal{X}} V \arrow[d, "\pi"] \\
& \mathcal{X}
\end{tikzcd}$$

which is the same data as a section s_α of $\mathcal{O}_{\mathcal{X}} \to f^*\pi_*\mathcal{O}_{U\times_{\mathcal{X}}V}$. Letting $\alpha^*\colon f^* \to g^*$ be the image of α under (6.4.1), i.e. the pullback tensor natural transformation, the section s_α can be written as

$$f^*\pi_*\mathcal{O}_{U\times_{\mathcal{X}}V} \cong f^*(p_*\mathcal{O}_U)\otimes f^*(q_*\mathcal{O}_V) \xrightarrow{\mathrm{id}\otimes\alpha^*_{q_*\mathcal{O}_V}} f^*(p_*\mathcal{O}_U)\otimes g^*(q_*\mathcal{O}_V) \xrightarrow{s_{\widetilde{f}}\otimes s_{\widetilde{g}}} \mathcal{O}_S.$$

To see the faithfulness of (6.4.1), if $\alpha, \alpha'\colon f \to g$ are 2-isomorphisms with $\alpha^* = \alpha'^*$, then $\alpha^*_{q_*\mathcal{O}_V} = \alpha'^*_{q_*\mathcal{O}_V}$ and therefore the two sections s_α and $s_{\alpha'}$ are equal and $\alpha = \alpha'$. For the fullness of (6.4.1), let $\beta\colon f^* \to g^*$ be a tensor natural transformation. Then $\mathrm{id}\otimes\beta_{q_*\mathcal{O}_V}$ defines a section $f^*\pi_*\mathcal{O}_{U\times_{\mathcal{X}}V} \to \mathcal{O}_S$ and thus a 2-isomorphism $\alpha\colon f \to g$ such that $\beta_{q_*\mathcal{O}_V} = \alpha^*_{q_*\mathcal{O}_V}$. To see that $\beta_E = \alpha^*_E$ for every $E \in \mathrm{QCoh}(\mathcal{Y})$, note that the factorization $g = q \circ \widetilde{g}$ yields a splitting of $g^*E \to g^*(q_*q^*E)$. Since f^* and g^* commute with direct sums, it suffices to assume that $E = q_*G$ for $G \in \mathrm{QCoh}(V)$. Writing $G = \mathrm{colim}(\mathcal{O}_V^{\oplus I} \to \mathcal{O}_V^{\oplus J})$ as a colimit of free $\mathcal{O}_V$-modules, we can conclude that $\beta_{q_*G} = \alpha^*_{q_*G}$ since f^* and g^* commute with colimits and q_* is exact.

Essential surjectivity: Let $F\colon \mathrm{QCoh}(\mathcal{Y}) \to \mathrm{QCoh}(\mathcal{X})$ be a tensor functor preserving colimits.

The affine case: If $\mathcal{X} = \operatorname{Spec} A$ and $\mathcal{Y} = \operatorname{Spec} B$ are noetherian affine schemes, then we have a map

$$\phi\colon B \cong \mathrm{End}(\mathcal{O}_{\mathcal{Y}}) \xrightarrow{F} \mathrm{End}(\mathcal{O}_{\mathcal{X}}) = A.$$

We claim that ϕ is a ring homomorphism and that there is a functorial isomorphism

$F(N) = N \otimes_B A$ for $N \in \mathrm{Mod}_B$. For $b, b' \in B$, consider the commutative diagrams

$$\begin{tikzcd} \mathcal{O}_{\mathcal{Y}} \otimes \mathcal{O}_{\mathcal{Y}} \arrow[r] \arrow[d, "b \otimes b'"] & \mathcal{O}_{\mathcal{Y}} \arrow[d, "bb'"] \\ \mathcal{O}_{\mathcal{Y}} \otimes \mathcal{O}_{\mathcal{Y}} \arrow[r] & \mathcal{O}_{\mathcal{Y}} \end{tikzcd} \quad \overset{F}{\longmapsto} \quad \begin{tikzcd} A \otimes A \arrow[r] \arrow[d, "\phi(b) \otimes \phi(b')"] & A \arrow[d, "\phi(bb')"] \\ A \otimes A \arrow[r] & A \end{tikzcd}$$

where the horizontal maps correspond to multiplication. The commutativity of the right square is implied by the fact that F preserves tensor products. This shows that $\phi(b)\phi(b') = \phi(bb')$. For a B-module N, choose a free presentation $B^{\oplus J} \to B^{\oplus I} \to N \to 0$. Since both F and $- \otimes_B A$ are right exact and preserves direct sums, applying them to the free presentation yields an identification $F(N) \cong N \otimes_B A$ as both are cokernels of $A^{\oplus J} \to A^{\oplus I}$. One checks similarly that this identification is functorial.

Reduction to the case that $\mathcal{X}$ is affine: Choose a smooth presentation $g \colon U \to \mathcal{X}$ from an affine scheme and consider the diagram

$$\begin{tikzcd} U \times_X U \arrow[d, shift right, "p_2"'] \arrow[d, shift left, "p_1"] & \\ U \arrow[d, "g"] \arrow[dr, dotted, "g^* \circ F"] & \\ \mathcal{X} \arrow[r, dotted, "F"] & \mathcal{Y} \end{tikzcd}$$

where the dashed arrow $\mathcal{X} \dashrightarrow \mathcal{Y}$ is denoting that we have a tensor functor $\mathrm{QCoh}(\mathcal{Y}) \to \mathrm{QCoh}(\mathcal{X})$ in the other direction. Assuming that the result holds when U is affine, there is a morphism $h \colon U \to \mathcal{Y}$ and an isomorphism $h \xrightarrow{\sim} g^* \circ F$ of functors. By full faithfulness, there is an isomorphism $p_1 \circ h \xrightarrow{\sim} p_2 \circ h$ satisfying the cocycle condition and thus smooth descent implies that there is a unique morphism $f \colon \mathcal{X} \to \mathcal{Y}$ with $F \simeq f^*$.

Reduction to the case that $\mathcal{Y}$ is affine: Let $\mathcal{X} = \operatorname{Spec} A$ and choose a smooth presentation $q \colon V = \operatorname{Spec} C \to \mathcal{Y}$. Since $\mathcal{Y}$ has affine diagonal, q is an affine morphism. Define $B := F(q_* \mathcal{O}_V)$ which is an A-algebra since $q_* \mathcal{O}_V$ is an $\mathcal{O}_{\mathcal{Y}}$-algebra. Consider the diagram

$$\begin{tikzcd} \operatorname{Spec} B \arrow[r, dotted, "F'"] \arrow[d] & V = \operatorname{Spec} C \arrow[d, "q"] \\ \operatorname{Spec} A = \mathcal{X} \arrow[r, dotted, "F"] & \mathcal{Y} \end{tikzcd}$$

where $F' \colon \mathrm{Mod}_C \to \mathrm{Mod}_B$ is the right exact tensor functor sending M to $F(q_* \widetilde{M})$ (which is a module over $B = F(q_* \mathcal{O}_V)$ because $q_* \widetilde{M}$ is a $q_* \mathcal{O}_V$-module). By the affine case, F' is induced by a morphism $f' \colon \operatorname{Spec} B \to \operatorname{Spec} C$. We can extend the above diagram into

$$\begin{tikzcd}
\operatorname{Spec} B \otimes_A B \arrow[r, "f''"] \arrow[d, shift left] \arrow[d, shift right] & V \times_{\mathcal{Y}} V \arrow[d, shift left] \arrow[d, shift right] \\
\operatorname{Spec} B \arrow[r, "f'"] \arrow[d] & V = \operatorname{Spec} C \arrow[d, "q"] \\
\operatorname{Spec} A = \mathcal{X} \arrow[r, dotted, "F"] & \mathcal{Y}.
\end{tikzcd}$$

Since q is affine, $V \times_{\mathcal{Y}} V$ is affine and the top square (under either set of projections) is cartesian.

If we could show that $A \to B$ is faithfully flat, we would be done as the full faithfulness in the affine case would imply that f' descends to our desired morphism $f: \mathcal{X} \to \mathcal{Y}$. This seems hard to directly check but we do know already that the maps $B \rightrightarrows B \otimes_A B$ are faithfully flat as they correspond to base changes of the smooth maps $V \times_{\mathcal{Y}} V \rightrightarrows V$. We will show instead that $A \to B$ is universally injective. Since faithful flatness descends under universal injectivity maps (Proposition A.2.21(4)), the faithful flatness of $A \to B$ follows from the universal injectivity.

Universal injectivity of $A \to B$: Recall from Definition A.2.20 that an injective map of A-modules is called universally injective if it remains injective after tensoring by every A-module. By Proposition A.2.21(3) this notion is local under faithfully flat morphisms and thus extends to morphisms $F \to G$ of quasi-coherent sheaves on an algebraic stack.

Since $q: V \to \mathcal{Y}$ is faithfully flat, $\mathcal{O}_{\mathcal{Y}} \to q_* \mathcal{O}_V$ is universally injective (Proposition A.2.21(1)). We write $q_* \mathcal{O}_V = \operatorname{colim} Q_i$ as a colimit of coherent subsheaves (Proposition 6.1.8) and we may assume that each Q_i contains the image of $\mathcal{O}_{\mathcal{Y}} \to q_* \mathcal{O}_V$. Then $\mathcal{O}_{\mathcal{Y}} \to Q_i$ is also universally injective and since Q_i is coherent, $\mathcal{O}_{\mathcal{Y}} \to Q_i$ is a split injection smooth-locally on $\mathcal{Y}$ (Proposition A.2.21(2)). Applying F to $\mathcal{O}_{\mathcal{Y}} \to q_* \mathcal{O}_V = \operatorname{colim} Q_i$ and using that it preserves colimits, we have a factorization

$$\begin{tikzcd}
 & F(Q_i) \arrow[d] \\
A = F(\mathcal{O}_{\mathcal{Y}}) \arrow[ur] \arrow[r] & B = F(q_* \mathcal{O}_V) = \operatorname{colim} F(Q_i).
\end{tikzcd}$$

It suffices to show that $A \to F(Q_i)$ is universally injective. We will show in fact that it is a split injection. As $\mathcal{O}_{\mathcal{Y}} \to Q_i$ is smooth-locally split, the map on duals $Q_i^\vee \to \mathcal{O}_{\mathcal{Y}}^\vee = \mathcal{O}_{\mathcal{Y}}$ is surjective. Applying F, we have a surjection $F(Q_i^\vee) \to F(\mathcal{O}_{\mathcal{Y}}) = A$ (using right exactness) and we can choose an element $\lambda \in F(Q_i^\vee)$ mapping to 1. Under the natural map $F(Q_i^\vee) \to F(Q_i)^\vee$, the element λ is sent to a map $F(Q_i) \to A$ which one checks to be a section of the given map $A \to F(Q_i)$.

See also [Lur04], [HR19b], [BHL17] and [SP, Tag 0GRR]. $\square$

Remark 6.4.3 (Relation to classical Tannaka duality)**.** If G is an affine group scheme over a field $\Bbbk$, then the category $\mathcal{C} = \operatorname{Rep}^{\mathrm{fd}}(G)$ of finite dimensional representations is a symmetric monoidal $\Bbbk$-linear category and there is a tensor functor $\omega\colon \mathrm{Rep}^{\mathrm{fd}}(G) \to \mathrm{Vect}_{\Bbbk}$. For $\Bbbk$-algebra R, let ω_R denote the composition $\mathrm{Rep}^{\mathrm{fd}}(G) \to \mathrm{Vect}_{\Bbbk} \xrightarrow{-\otimes_{\Bbbk} R} \mathrm{Mod}_R$ and let $\mathrm{Aut}^{\otimes}(\omega_R)$ denote the group of tensor natural isomorphisms of ω_R. Then G is recovered as the functor $\underline{\mathrm{Aut}}^{\otimes}(\omega)$ on affine $\Bbbk$-schemes assigning R to $\mathrm{Aut}^{\otimes}(\omega_R)$ [DMOS82, II.2.8].

On the other hand, Coherent Tannaka Duality for Algebraic Stacks (Theorem 6.4.1) implies that for every noetherian $\Bbbk$-algebra R, there is an equivalence of categories

$$\mathrm{MOR}_{\Bbbk}(\mathrm{Spec}\, R, \mathbf{B}G) \xrightarrow{\sim} \mathrm{MOR}^{\otimes}(\mathrm{Rep}(G)^{\mathrm{fd}}, \mathrm{Mod}_R).$$

In this way, we see that $\mathrm{Rep}(G)^{\mathrm{fd}}$ determines $\mathbf{B}G$. To recover G, the fiber functor $\omega\colon \mathrm{Rep}^{\mathrm{fd}}(G) \to \mathrm{Vect}_{\Bbbk}$ corresponds to a morphism $p\colon \mathrm{Spec}\, \Bbbk \to \mathbf{B}G$ and $G = \mathrm{Aut}_{\Bbbk}(p)$. For example, if $O(q)$ and $O(q')$ are orthogonal groups with respect to non-degenerate quadratic forms q and q' of the same dimension, then $\mathrm{Rep}(O(q)) \cong \mathrm{Rep}(O(q'))$ even though $O(q)$ and $O(q')$ may not be isomorphic; in this case the two maps $\mathrm{Spec}\, \Bbbk \to \mathbf{B}O(q)$ and $\mathrm{Spec}\, \Bbbk \to \mathbf{B}O(q')$ define two different fiber functors on the same category.

The classical version also provides conditions when the data of $(\mathcal{C}, \omega)$ is isomorphic to the category of representations of a group scheme. Namely, we say $\mathcal{C}$ is *rigid* if for every object of $X \in \mathcal{C}$, there is a 'dual' $X^{\vee} \in \mathcal{C}$, i.e. an object $X^{\vee}$ such that $X^{\vee} \otimes - \colon \mathcal{C} \to \mathcal{C}$ is right adjoint to $X \otimes - \colon \mathcal{C} \to \mathcal{C}$. If $\mathcal{C}$ is a rigid symmetric monoidal $\Bbbk$-linear abelian category with $\mathrm{End}(1) = \Bbbk$ and $\omega\colon \mathcal{C} \to \mathrm{Vect}_{\Bbbk}$ is an exact faithful $\Bbbk$-linear tensor functor, then $\underline{\mathrm{Aut}}^{\otimes}(\omega)$ is represented by an affine group scheme G over $\Bbbk$ and there is a tensor equivalence $\mathcal{C} \cong \mathrm{Rep}^{\mathrm{fd}}(G)$ under which ω corresponds to the forgetful functor [DMOS82, II.2.11]. Moreover, G is of finite type over $\Bbbk$ if and only if $\mathcal{C}$ has a tensor generator.

6.4.2 Coherent completeness

Coherent Tannaka Duality becomes especially powerful when combined with coherent completeness.

Definition 6.4.4. A noetherian algebraic stack $\mathcal{X}$ is *coherently complete along a closed substack* $\mathcal{X}_0$ if the natural functor

$$\mathrm{Coh}(\mathcal{X}) \to \varprojlim \mathrm{Coh}(\mathcal{X}_n), \quad F \mapsto (F_n)$$

is an equivalence of categories, where $\mathcal{X}_n$ denotes the nth nilpotent thickening of $\mathcal{X}_0$ and F_n is the pullback of F to $\mathcal{X}_n$.

Remark 6.4.5. If $\mathcal{I} \subset \mathcal{O}_{\mathcal{X}}$ is the coherent sheaf of ideals defining $\mathcal{X}_0$, then $\mathcal{X}_n$ is defined by $\mathcal{I}^{n+1}$. Letting $i_n\colon \mathcal{X}_n \hookrightarrow \mathcal{X}_{n+1}$ denote the natural inclusion, an object in $\varprojlim \mathrm{Coh}(\mathcal{X}_n)$ corresponds to a sequence $F_n \in \mathrm{Coh}(\mathcal{X}_n)$ of coherent sheaves together with maps $\alpha_n\colon i_{n,*}F_n \to F_{n+1}$ inducing isomorphism $F_n \to i_n^* F_{n+1}$. A morphism $(F_n, \alpha_n) \to (F'_n, \alpha'_n)$ is a sequence of maps $\phi_n\colon F_n \to F'_n$ such that $\phi_{n+1} \circ \alpha_n = \alpha_{n+1} \circ i_{n,*}\phi_n$.

Example 6.4.6. If $(R, \mathfrak{m})$ is a complete noetherian local ring, then the Artin–Rees Lemma [AM69, Prop. 10.9] implies that $\mathrm{Spec}\, R$ is coherent complete along $\mathrm{Spec}\, R/\mathfrak{m}$. The same is true if $R = \lim R/I^n$ is a noetherian I-adically complete ring.

Example 6.4.7. Grothendieck's Existence Theorem (D.4.4) asserts that if X is a proper scheme over a complete local ring $(R, \mathfrak{m})$ and $X_0 = X \times_R R/\mathfrak{m}$, then X is coherently complete along X_0. If $\mathcal{X}$ is a proper Deligne–Mumford stack or more generally a proper algebraic stack over $\operatorname{Spec} R$, the same is true. The result holds also if R is an I-adically complete noetherian ring. See [Ols05, Thm. 1.4] or [Con05a, Thm. 4.1].

Corollary 6.4.8 (Coherent Tannaka Duality)**.** *Let $\mathcal{X}$ and $\mathcal{Y}$ be noetherian algebraic stacks with affine diagonal. Suppose that $\mathcal{X}$ is coherently complete along $\mathcal{X}_0$. Then there is an equivalence of categories*

$$\mathrm{MOR}(\mathcal{X}, \mathcal{Y}) \to \varprojlim \mathrm{MOR}(\mathcal{X}_n, \mathcal{Y}), \quad f \mapsto (f_n),$$

where $f_n \colon \mathcal{X}_n \to Y$ denotes the restriction of f to the nth nilpotent thickening $\mathcal{X}_n$ of $\mathcal{X}_0$.

Proof. This follows from the equivalences

$$\begin{aligned}
\mathrm{MOR}(\mathcal{X}, \mathcal{Y}) &\simeq \mathrm{MOR}^{\otimes}\big(\mathrm{Coh}(\mathcal{Y}), \mathrm{Coh}(\mathcal{X})\big) && \text{(Coherent Tannaka Duality)} \\
&\simeq \mathrm{MOR}^{\otimes}\big(\mathrm{Coh}(\mathcal{Y}), \varprojlim \mathrm{Coh}(\mathcal{X}_n)\big) && \text{(coherent completeness)} \\
&\simeq \varprojlim \mathrm{MOR}^{\otimes}\big(\mathrm{Coh}(\mathcal{Y}), \mathrm{Coh}(\mathcal{X}_n)\big) \\
&\simeq \varprojlim \mathrm{MOR}(\mathcal{X}_n, \mathcal{Y}) && \text{(Coherent Tannaka Duality).}
\end{aligned}$$

□

Remark 6.4.9. If $\mathcal{X}$ and $\mathcal{Y}$ are defined over a noetherian ring R, then there is an equivalence $\mathrm{MOR}_R(\mathcal{X}, \mathcal{Y}) \to \varprojlim \mathrm{MOR}_R(\mathcal{X}_n \mathcal{Y})$. This follows in the same way using the Tannaka duality equivalence between the category of morphisms $\mathcal{X} \to \mathcal{Y}$ over R and the category of right exact R-linear tensor functors (Remark 6.4.2).

For example, in order to show that there is map $\operatorname{Spec} A \to \mathcal{Y}$ from the spectrum of a noetherian I-adically complete ring A, it suffices to construct compatible maps $\operatorname{Spec} A/I^n \to \mathcal{Y}$. This is only easy to see directly if A is local.

Exercise 6.4.10. Let G be an affine algebraic group acting a noetherian separated algebraic space W over $\Bbbk$. Let $W_0 \subset W$ be a G-invariant closed subspace and let W_n be its nth nilpotent thickenings. Suppose that $[W/G]$ is coherent complete along a closed substack $[W_0/G]$. For every noetherian algebraic space X over $\Bbbk$ with affine diagonal equipped with an action of G, the natural map on equivariant maps

$$\mathrm{Mor}^G(W, X) \to \varprojlim_n \mathrm{Mor}^G\big(W_n, X\big)$$

is bijective.

Hint: reduce to Corollary 6.4.8 by using that a G-equivariant map $W \to X$ corresponds to a morphism $[W/G] \to [X/G]$ over $\mathbf{B}G$, i.e $\mathrm{Mor}^G(W, X) = \{*\} \times_{\mathrm{Mor}([W/G], \mathbf{B}G)} \mathrm{Mor}([W/G], [X/G])$.

6.4.3 Coherent completeness of quotient stacks

The coherent completeness result that we will exploit through the rest of the book and in particular in the proof of the Local Structure Theorem for Algebraic Stacks (6.5.1) is the following:

Theorem 6.4.11. *Let $\Bbbk$ be an algebraically closed field and R be a complete noetherian local $\Bbbk$-algebra with residue field $\Bbbk$. Let G be a linearly reductive group over $\Bbbk$ acting on an affine scheme $\operatorname{Spec} A$ of finite type over R. Suppose that $A^G = R$ and that there is a G-fixed $\Bbbk$-point $x \in \operatorname{Spec} A$. Then $[\operatorname{Spec} A/G]$ is coherent complete along the closed substack $\mathbf{B}G$ defined by x.*

Example 6.4.12. If $\mathbb{G}_m$ acts diagonally on $\mathbb{A}^r$, then $[\mathbb{A}^r/\mathbb{G}_m]$ is coherent completely along the origin $B\mathbb{G}_m$. In other words a $\mathbb{G}_m$-equivariant coherent sheaf on $\mathbb{A}^r$ is equivalent to a compatible family of $\mathbb{G}_m$-equivariant modules over $\Bbbk[x_1, \dots, x_r]/(x_1, \dots, x_r)^{n+1}$.

Remark 6.4.13. We have a commutative diagram

$$\begin{tikzcd}
\mathbf{B}G \arrow[r, hook] & {[\operatorname{Spec} A/G] \times_{A^G} \Bbbk} \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & {[\operatorname{Spec} A/G]} \arrow[d] \\
& \operatorname{Spec} \Bbbk \arrow[r, hook] & \operatorname{Spec} A^G.
\end{tikzcd}$$

A formal consequence of the above theorem is that $[\operatorname{Spec} A/G]$ is also coherently complete with respect to the fiber $[\operatorname{Spec} A/G] \times_{A^G} \Bbbk$. This version is analogous to Grothendieck's Existence Theorem (6.4.7) but the coherent completeness along $\mathbf{B}G$ is a substantially stronger statement, e.g. for $[\mathbb{A}^n/\mathbb{G}_m]$ where the fiber of $[\mathbb{A}^n/\mathbb{G}_m] \to \operatorname{Spec} \Bbbk$ is everything.

Proof of Theorem 6.4.11. We need to show that $\operatorname{Coh}(\mathcal{X}) \to \varprojlim \operatorname{Coh}(\mathcal{X}_n)$ is an equivalence of categories, where $\mathcal{X} = [\operatorname{Spec} A/G]$ and $\mathcal{X}_n$ is the nth nilpotent thickening of $\mathbf{B}G \hookrightarrow \mathcal{X}$ of the inclusion of the residual gerbe at x.

Full faithfulness: Suppose that F and F' are coherent $\mathcal{O}_{\mathcal{X}}$-modules, and let F_n and F'_n denote the restrictions to $\mathcal{X}_n$, respectively. We need to show that

$$\operatorname{Hom}_{\mathcal{O}_{\mathcal{X}}}(F, F') \to \varprojlim \operatorname{Hom}_{\mathcal{O}_{\mathcal{X}}}(F_n, F'_n)$$

is bijective. Since $\mathcal{X}$ has the resolution property (Proposition 6.1.19), we can find a resolution $F_2 \to F_1 \to F \to 0$. by vector bundles. This induces a diagram

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Hom}_{\mathcal{O}_{\mathcal{X}}}(F, F') \arrow[r] \arrow[d] & \operatorname{Hom}_{\mathcal{O}_{\mathcal{X}}}(F_1, F') \arrow[r] \arrow[d] & \operatorname{Hom}_{\mathcal{O}_{\mathcal{X}}}(F_2, F') \arrow[d] \\
0 \arrow[r] & \varprojlim \operatorname{Hom}_{\mathcal{O}_{\mathcal{X}}}(F_n, F'_n) \arrow[r] & \varprojlim \operatorname{Hom}_{\mathcal{O}_{\mathcal{X}}}(F_{1,n}, F'_n) \arrow[r] & \varprojlim \operatorname{Hom}_{\mathcal{O}_{\mathcal{X}}}(F_{2,n}, F'_n)
\end{tikzcd}$$

with exact rows. We may therefore assume that F is a vector bundle. In this case,

$$\operatorname{Hom}_{\mathcal{O}_{\mathcal{X}}}(F, F') = \operatorname{Hom}_{\mathcal{O}_{\mathcal{X}}}(\mathcal{O}_{\mathcal{X}}, F^\vee \otimes F')$$
$$\operatorname{Hom}_{\mathcal{O}_{\mathcal{X}}}(F_n, F'_n) = \operatorname{Hom}_{\mathcal{O}_{\mathcal{X}}}\big(\mathcal{O}_{\mathcal{X}_n}, (F_n^\vee \otimes F'_n)\big).$$

Therefore, we can also assume that $F = \mathcal{O}_{\mathcal{X}}$ and we are reduced to showing that

$$\Gamma(\mathcal{X}, F') \to \varprojlim \Gamma\big(\mathcal{X}_n, F'_n\big) \tag{6.4.2}$$

is an isomorphism. Writing $F' = \widetilde{M}$ where M is a finitely generated A-module with an action of G and letting $\mathfrak{m} \subset A$ be the maximal ideal for x, then $\Gamma(\mathcal{X}_n, F'_n) = M^G/(\mathfrak{m}^n M)^G$ since G is linearly reductive. We must therefore verify that

$$M^G \to \varprojlim M^G/(\mathfrak{m}^n M)^G \tag{6.4.3}$$

is an isomorphism. To this end, we first show that

$$\bigcap_{n\geq 0} \big(\mathfrak{m}^n M\big)^G = 0. \tag{6.4.4}$$

or in other words that (6.4.3) is injective. Let $N := \bigcap_{n\geq 0} \mathfrak{m}^n M$. The Artin–Rees lemma [AM69, Prop. 10.9] applied to $N \subset M$ implies that there exists an integer c such that $\mathfrak{m}^n M \cap N = \mathfrak{m}^{n-c}(\mathfrak{m}^c M \cap N)$ for all $n \geq c$. Taking $n = c+1$, we see that $N = \mathfrak{m}N$ so $N \otimes_A A/\mathfrak{m} = 0$. Since the support of N is a closed G-invariant subscheme of $\operatorname{Spec} A$ which does not contain x, it follows that $N = 0$.

Note also that since G is linearly reductive, M^G is a finitely generated A^G-module (Corollary 6.3.7(3)). We next establish that (6.4.3) is an isomorphism if A^G is artinian. In this case, $\{(\mathfrak{m}^n M)^G\}$ automatically satisfies the Mittag–Leffler condition (it is a sequence of artinian A^G-modules). Therefore, taking the inverse limit of the exact sequences $0 \to (\mathfrak{m}^n M)^G \to M^G \to M^G/(\mathfrak{m}^n M)^G \to 0$ and applying (6.4.4), yields an exact sequence

$$0 \to 0 \to M^G \to \varprojlim M^G/(\mathfrak{m}^n M)^G \to 0$$

and shows that (6.4.3) is an isomorphism. To establish (6.4.3) in the general case, let $J = (\mathfrak{m}^G)A \subseteq A$ and observe that

$$M^G \cong \varprojlim M^G/\big(\mathfrak{m}^G\big)^n M^G \cong \varprojlim \big(M/J^n M\big)^G, \tag{6.4.5}$$

since G is linearly reductive. For each n, we know that

$$\big(M/J^n M\big)^G \cong \varprojlim_l M^G/\big((J^n + \mathfrak{m}^l)M\big)^G \tag{6.4.6}$$

using the artinian case proved above. Finally, combining (6.4.5) and (6.4.6) together with the observation that $J^n \subseteq \mathfrak{m}^l$ for $n \geq l$, we conclude that

$$\begin{aligned} M^G &\cong \varprojlim_n \big(M/J^n M\big)^G \\ &\cong \varprojlim_n \varprojlim_l M^G/\big((J^n + \mathfrak{m}^l)M\big)^G \\ &\cong \varprojlim_l M^G/\big(\mathfrak{m}^l M\big)^G. \end{aligned}$$

Essential surjectivity: The linear reductivity of G implies that every coherent sheaf $F = \widetilde{M}$ on $[\operatorname{Spec} A/G]$ decomposes as a direct sum

$$M = \bigoplus_{\rho \in \Gamma} M^{(\rho)}, \tag{6.4.7}$$

where Γ denotes the set of isomorphism classes of irreducible representations of G and $M^{(\rho)}$ is the isotypic component corresponding to ρ; explicitly if W_ρ denotes the irreducible representation corresponding to ρ, then $M^{(\rho)} = \operatorname{Hom}^G_{\Bbbk}(W_\rho, M) \otimes W_\rho$. Moreover, the decomposition (6.4.7) is compatible with the A-module structure of M and the decomposition $A = \bigoplus_{\rho\in\Gamma} A^{(\rho)}$.

Let us also note that if $F = \widetilde{M} \in \mathrm{Coh}(\mathcal{X})$ with restrictions $M_n = M/\mathfrak{m}^{n+1}M$, then applying (6.4.3) to $M \otimes W_\rho^\vee$ shows that $M^{(\rho)} = \varprojlim M_n^{(\rho)}$. By Theorem 6.3.5(3), we also know that $M^{(\rho)}$ is a finitely generated A^G-module. In particular, $A^{(\rho)} = \varprojlim (A/\mathfrak{m}^{n+1})^{(\rho)}$ is a finitely generated A^G-module.

This suggests that if $F_n = \widetilde{M}_n$ is a compatible system of coherent $\mathcal{O}_{\mathcal{X}_n}$-modules with $M_n = \bigoplus_\rho M_n^{(\rho)}$, we define

$$M^{(\rho)} := \varprojlim M_n^{(\rho)} \qquad \text{and} \qquad M := \bigoplus_{\rho \in \Gamma} M^{(\rho)}. \tag{6.4.8}$$

To see that M is an A-module with a G-action, let $\rho, \gamma \in \Gamma$ be irreducible representations and let $\Lambda \subset \Gamma$ denote the finite set of non-zero irreducible representations appearing $W_\rho \otimes W_\gamma$. Taking limits of the maps $A_n^{(\rho)} \otimes_{(A/\mathfrak{m}^{n+1})^G} M_n^{(\gamma)} \to \bigoplus_{\lambda\in\Lambda} M_n^{(\lambda)}$, defines multiplication

$$A^{(\rho)} \otimes_A M^{(\gamma)} \to \varprojlim \big(A_n^{(\rho)} \otimes_{(A/\mathfrak{m}^{n+1})^G} M_n^{(\gamma)}\big) \to \varprojlim \big(\bigoplus_{\lambda\in\Lambda} M_n^{(\lambda)}\big) \cong \bigoplus_{\lambda\in\Lambda} M^{(\lambda)}.$$

Note that we also have $M/\mathfrak{m}^{n+1}M \cong M_n$ by construction.

We need to show that the A-module M of (6.4.8) is finitely generated. The coherent sheaf $F_0 = \widetilde{M}_0$ on $\mathcal{X}_0 = \mathbf{B}G$ is a finite dimensional G-representation and we can consider the coherent $\mathcal{O}_{\mathcal{X}}$-module $F_0 \otimes_{\mathcal{O}_{\mathcal{X}_0}} \mathcal{O}_{\mathcal{X}}$ or equivalently the A-module $M_0 \otimes_{\Bbbk} A$ with its natural G-action. Since $\mathcal{X}$ is cohomologically affine, the functor

$$\mathrm{Hom}_{\mathcal{O}_{\mathcal{X}}}(F_0 \otimes_{\mathcal{O}_{\mathcal{X}_0}} \mathcal{O}_{\mathcal{X}}, -) = \Gamma(\mathcal{X}, (F_0^\vee \otimes_{\mathcal{O}_{\mathcal{X}_0}} \mathcal{O}_{\mathcal{X}}) \otimes_{\mathcal{O}_{\mathcal{X}}} -)$$

is exact. Apply the functor to the surjection $M \twoheadrightarrow M_0$ induces a map

$$M_0 \otimes_{\Bbbk} A \to M \tag{6.4.9}$$

which we would like to show is surjective. We do know that the restrictions $M_0 \otimes_{\Bbbk} (A/\mathfrak{m}^{n+1}) \to M_n$ are surjective as its cokernel is a coherent module on $\mathcal{X}_n$ not supported at the unique closed point.

As above, we first handle the case that A^G is artinian. Since $A^{(\rho)} \xrightarrow{\sim} \varprojlim (A/\mathfrak{m}^n)^{(\rho)}$ is a finitely generated A^G-module, it follows that $(A/\mathfrak{m}^n)^{(\rho)}$ stabilizes to $A^{(\rho)}$ for $n \gg 0$. Since (6.4.9) induces surjections $M_0 \otimes_{\Bbbk} (A/\mathfrak{m}^{n+1}) \to M_n$, it follows that the modules $M_n^{(\rho)}$ stabilize to $M_\infty^{(\rho)}$ for $n \gg 0$ and that $M = \bigoplus_\rho M_\infty^{(\rho)}$ is finitely generated. In the general case, let $X_m = \operatorname{Spec} A^G/(\mathfrak{m} \cap A^G)^{m+1}$ and consider the cartesian diagram

$$\begin{tikzcd}
\mathcal{X} \times_X X_m \arrow[r, hook, "i_m"] \arrow[d, "\pi_m"] & \mathcal{X} \times_X X_{m+1} \arrow[r, hook] \arrow[d, "\pi_{m+1}"] & \cdots \arrow[r, hook] & \mathcal{X} \arrow[d, "\pi"] \\
X_m \arrow[r, hook, "j_m"] & X_{m+1} \arrow[r, hook] & \cdots \arrow[r, hook] & X.
\end{tikzcd}$$

For each m, we may consider the nth nilpotent thickenings $\mathcal{Z}_{m,n}$ of $\mathcal{X}_0 \hookrightarrow \mathcal{X} \times_X X_m$ which are closed substacks $\mathcal{X}_n$. Since X_m is the spectrum of artinian ring, the restrictions $F_n|_{\mathcal{Z}_{m,n}}$ extend to a coherent sheaf $H_m = \widetilde{N}_m$ on $\mathcal{X} \times_X X_m$. Moreover, there is a canonical isomorphism between H_m and the restriction of H_{m+1} to

$\mathcal{X} \times_X X_m$. By Lemma 6.3.20(4), the adjunction morphism $j_m^* \pi_{m+1,*} \xrightarrow{\sim} \pi_{m,*} i_m^*$ is an isomorphism on quasi-coherent sheaves. This implies that $N_{m+1}^{(\rho)} = \Gamma(\mathcal{X} \times_X X_{m+1}, H_{m+1} \otimes W_\rho^\vee)$ restricts to $N_m^{(\rho)}$ and that $M^{(\rho)} = \varprojlim N_m^{(\rho)}$ is a finitely generated A^G-module. By Nakayama's lemma, the map (6.4.9) is surjective on each ρ-isotypical component. Thus (6.4.9) is surjective and M is finitely generated.

For an alternative (but similar) argument for essential surjectivity, we first choose a surjection $E \twoheadrightarrow F_0$ from a vector bundle E on $\mathcal{X}$. For this we can either apply the resolution property of $\mathcal{X}$ (Proposition 6.1.19) or take $E = F_0 \otimes_{\mathcal{O}_{\mathcal{X}_0}} \mathcal{O}_{\mathcal{X}}$ as above. Since each $F_{n+1} \to F_n$ is surjective and $\mathrm{Hom}_{\mathcal{O}_{\mathcal{X}}}(E, -) = \Gamma(\mathcal{X}, E^\vee \otimes_{\mathcal{O}_{\mathcal{X}}} -)$ is exact, we can lift $E \twoheadrightarrow F_0$ to compatible maps $E \to F_n$, each which is surjective (Nakayama's lemma). The sequence $(\ker(E_n \to F_n))$ not necessarily an adic sytem of coherent sheaves on $\mathcal{X}_n$ as the restriction $\ker(E_{n+1} \to F_{n+1})$ to $\mathcal{X}_n$ may not be $\ker(E_n \to F_n)$. But we can modify it as follows: For each $l \geq m \geq n$, the images of $\ker(E_l \to F_l)$ in E_m stabilize to K'_m for $l \gg m$ and $K'_m/\mathfrak{m}^{n+1}K'_m$ stabilize to K_n for $m \gg n$ (see also [SP, Tag 087X]). Then $(K_n) \in \varprojlim \mathrm{Coh}(\mathcal{X}_n)$ is an adic sequence. Repeating the construction, we can find a vector bundle E' on $\mathcal{X}$ and compatible surjections $E' \to K_n$. By full faithfulness, there is a morphism $E' \to E$ extending the maps $E'_n \to E_n$. Then $\mathrm{coker}(E' \to E)$ is a coherent $\mathcal{O}_{\mathcal{X}}$ extending (F_n).

See also [AHR20, Thm. 1.3] and [AHR19, Thm. 1.6]. □

Exercise 6.4.14. If S is a noetherian affine scheme, show that $[\mathbb{A}^1/\mathbb{G}_m]_S$ is coherently complete along $\mathbf{B}G_{m,S}$.

6.5 Local structure of algebraic stacks

We establish a local structure theorem for algebraic stack around points with linearly reductive stabilizer. The main theorem (Theorem 6.5.1) implies that quotient stacks of the form $[\mathrm{Spec}\, A/G]$, where G is a linearly reductive, are the building blocks of algebraic stacks near points with linearly reductive stabilizers in the similar way to how affine schemes are the building blocks of schemes and algebraic spaces. When $\mathcal{X}$ is Deligne–Mumford, we've already seen an analogous Local Structure Theorem for Deligne–Mumford Stacks (4.2.11). The local structure theorem will be applied to construct good moduli spaces in a similar way to how the result for Deligne–Mumford stacks was used to prove the Keel–Mori Theorem (4.3.11) on the existence of coarse moduli spaces.

Theorem 6.5.1 (Local Structure Theorem for Algebraic Stacks)**.** *Let $\mathcal{X}$ be an algebraic stack of finite type over an algebraically closed field $\Bbbk$ with affine diagonal. For every point $x \in \mathcal{X}(\Bbbk)$ with linearly reductive stabilizer G_x, there exist an affine étale morphism*

$$f\colon ([\mathrm{Spec}\, A/G_x], w) \to (\mathcal{X}, x)$$

which induces an isomorphism of stabilizer groups at w.

Remark 6.5.2. In the case that $x \in \mathcal{X}$ is a smooth point, then one can say more: there is also an étale morphism

$$[\mathrm{Spec}\, A/G_x], w) \to ([T_{\mathcal{X},x}/G_x], 0)$$

where $T_{\mathcal{X},x}$ is the Zariski tangent space equipped as a G_x-representation. This addendum follows from the proof but also follows from applying Luna's Étale

Slice Theorem (6.5.4) to $[\operatorname{Spec} A/G_x]$. The upshot is that we can reduce étale local properties of $\mathcal{X}$ to G_x-equivariant properties of $T_{\mathcal{X},x}$; for moduli problems this translates into studying the first order deformation space as a representation under the automorphism group.

By combining this theorem with Luna's Fundamental Lemma (6.3.24), we obtain the following result.

Corollary 6.5.3 (Local Structure for Good Moduli Spaces)**.** *Let $\mathcal{X}$ be an algebraic stack of finite type over an algebraically closed field $\Bbbk$ with affine diagonal. Suppose that there exists a good moduli space $\pi\colon \mathcal{X} \to X$. Then for every closed point $x \in \mathcal{X}$, there exists an étale neighborhood $W \to X$ of $\pi(x)$ and a cartesian diagram*

$$\begin{tikzcd}
{[\operatorname{Spec} A/G_x]} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{X} \arrow[d, "\pi"] \\
W = \operatorname{Spec} A^{G_x} \arrow[r] & X.
\end{tikzcd}$$

□

Section outline: We first discuss Luna's Étale Slice Theorem (6.5.4), a beautiful argument providing an explicit construction of an étale neighborhood in the case that $\mathcal{X}$ is already known to have the form $[\operatorname{Spec} B/G]$ with G reductive. The proof of the Local Structure Theorem (6.5.1) is far less explicit requiring: (1) deformation theory, (2) coherent completeness, (3) Coherent Tannaka Duality and (4) Artin Approximation or Equivariant Artin Algebraization.

Letting $\mathcal{T} = [T_{\mathcal{X},x}/G_x]$, deformation theory produces an embedding $\mathcal{X}_n \hookrightarrow \mathcal{T}_n$ of the nth nilpotent thickenings of x and 0. The key step in the proof is to show that the system of closed morphisms $\{\mathcal{X}_n \to \mathcal{X}\}$ algebraizes. The first step is effectivization: the fiber product $\widehat{\mathcal{T}} := \mathcal{T} \times_T \operatorname{Spec} \widehat{\mathcal{O}}_{T,\pi(0)}$, where $\pi\colon \mathcal{T} \to T := T_{\mathcal{X},x}/\!\!/G_x$, is coherently complete (Theorem 6.4.11). We can thus construct a closed substack $\widehat{\mathcal{X}} \hookrightarrow \widehat{\mathcal{T}}$ extending $\mathcal{X}_n \hookrightarrow \mathcal{T}$ and then apply Coherent Tannaka Duality (6.4.8) to construct a morphism $\widehat{\mathcal{X}} \to \mathcal{X}$ extending $\mathcal{X}_n \to \mathcal{X}$.

If $x \in \mathcal{X}$ is smooth, Artin Approximation over the GIT quotient $T_{\mathcal{X},x}/\!\!/G_x$ produces an étale neighborhood $U \to T_{\mathcal{X},x}/\!\!/G_x$ such that $\pi^{-1}(U) \to \mathcal{X}$ algebraizes $\widehat{\mathcal{T}} \to \mathcal{X}$. In the general case, Artin Approximation cannot handle this final step and we need to establish an equivariant version of Artin Algebraization (Theorem 6.5.14).

6.5.1 Luna's Étale Slice Theorem

The local structure theorem was inspired by Luna's étale slice theorem in equivariant geometry.

Theorem 6.5.4 (Luna's Étale Slice Theorem)**.** *Let G be a linearly reductive group over an algebraically closed field $\Bbbk$ and let X be an affine scheme of finite type over $\Bbbk$ with an action of G. If $x \in X(\Bbbk)$ has linearly reductive stabilizer, then there exists a G_x-invariant, locally closed, and affine subscheme $W \subset X$ such that the induced map*

$$[W/G_x] \to [X/G] \tag{6.5.1}$$

is affine étale. If in addition the orbit $Gx \subset X$ is closed, then there is a cartesian diagram

$$\begin{tikzcd} {[W/G_x]} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & {[X/G]} \arrow[d] \\ W/\!\!/G_x \arrow[r] & X/\!\!/G \end{tikzcd}$$

where $W/\!\!/G_x \to X/\!\!/G$ is also étale.

Moreover, if $x \in X$ is a smooth point and if we denote by $N_x = T_{X,x}/T_{Gx,x}$ the normal space to the orbit, then it can be arranged that there is an G_x-invariant étale morphism $W \to N_x$ which is the pullback of an étale map $W/\!\!/G_x \to N_x/\!\!/G_x$ of GIT quotients.

Remark 6.5.5. One can also formulate the statement G-equivariantly: G acts naturally on the quotient $G \times^{G_x} W := (G \times W)/G_x$ and there is an identification $[W/G_x] \cong [(G \times^{G_x} W)/G]$ and likewise $W/\!\!/G_x \cong (G \times^{G_x} W)/\!\!/G$ (see Exercise 3.4.14). The morphism (6.5.1) corresponds to an étale G-equivariant morphism $G \times^{G_x} W \to X$.

We also point out that if the orbit Gx is closed, then Matsushima's Theorem (6.3.19) implies that the stabilizer G_x is linearly reductive.

The proof will rely on the existence of a G_x-invariant morphism $X \to T_{X,x}$ which we refer to as the *Luna map*.

Lemma 6.5.6 (Luna map). *Let G be a linearly reductive group over an algebraically closed field $\Bbbk$ and let X be an affine scheme of finite type over $\Bbbk$ with an action of G. If $x \in X(\Bbbk)$ has linearly reductive stabilizer, there exists a G_x-equivariant morphism*

$$f\colon X \to T_{X,x} \tag{6.5.2}$$

sending x to the origin. If X is smooth at x, then f is étale at x.

Proof. Letting $X = \operatorname{Spec} A$ and $\mathfrak{m} \subset A$ be the maximal ideal of x, then $\mathfrak{m}$ and $\mathfrak{m}/\mathfrak{m}^2$ are G_x-representations and we see that G_x acts naturally on the tangent space $T_{X,x} := \operatorname{Spec} \operatorname{Sym}^* \mathfrak{m}/\mathfrak{m}^2$. Since G_x is linearly reductive, the surjection $\mathfrak{m} \to \mathfrak{m}/\mathfrak{m}^2$ of G_x-representations has a section $\mathfrak{m}/\mathfrak{m}^2 \hookrightarrow \mathfrak{m}$. This induces a G_x-equivariant ring map $\operatorname{Sym}^* \mathfrak{m}/\mathfrak{m}^2 \to A$ and thus a G_x-equivariant morphism $f\colon \operatorname{Spec} A \to T_{X,x}$ sending x to the origin. If $x \in X$ is smooth, then since f induces an isomorphism of tangent spaces at x, we conclude that f is étale at x (Étale Equivalences A.3.2). $\square$

Proof of Theorem 6.5.4. Since X is affine and of finite type, we can choose a finite dimensional G-representation V and a G-equivariant closed immersion $X \hookrightarrow \mathbb{A}(V)$ (Lemma C.3.2). If $W \subset \mathbb{A}(V)$ is an affine G_x-invariant locally closed subscheme such that $[W/G_x] \to [\mathbb{A}(V)/G]$ is étale, then the same is true for $W' := W \cap X \subset X$ and $[W'/G_x] \to [X/G]$. We can therefore immediately reduce to the case that $x \in X$ is smooth. In this case, there is a Luna map (6.5.6)$f\colon X \to T_{X,x}$ which is G_x-invariant, étale at x, and with $f(x) = 0$. The subspace $T_{Gx,x} \subset T_{X,x}$ is G_x-invariant and again since G_x is linearly reductive, the surjection $T_{X,x} \to N_x = T_{X,x}/T_{Gx,x}$ has a section $N_x \hookrightarrow T_{X,x}$. We define W

as the preimage of N_x under f:

$$\begin{tikzcd} W \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\square"] & N_x \arrow[d, hook] \\ X \arrow[r, "f"] & T_{X,x}. \end{tikzcd}$$

Since the maps $f\colon [W/G_x] \to [X/G]$ and $g\colon [W/G_x] \to [N_x/G_x]$ induce an isomorphism of tangent spaces and stabilizer groups at w, they are both étale at $x \in W$ (or equivalently the G-equivariant maps $G \times^{G_x} W \to X$ and $G \times^{G_x} W \to G \times^{G_x} N_x$ are étale at (id, x)). We have a commutative diagram

$$\begin{tikzcd} {[N_x/G_x]} \arrow[d] & {[W/G_x]} \arrow[l, "g"'] \arrow[r, "f"] \arrow[d] & {[X/G]} \arrow[d] \\ N_x/\!\!/G_x & W/\!\!/G_x \arrow[l] \arrow[r] & X/\!\!/G_x \end{tikzcd}$$

where both f and g are étale at x, preserve stabilizer groups at x and map x to closed points. We can therefore apply Luna's Fundamental Lemma (6.3.24) to replace W with a G_x-equivariant, open, and affine neighborhood of x so that the above squares are cartesian. $\square$

When $\mathcal{X}$ is already known to be a quotient stack of a normal quasi-projective scheme, the Local Structure Theorem follows from a direct argument. This case is sufficient to handle many moduli problems, e.g. $\mathcal{B}\mathrm{un}^{\mathrm{ss}}_{r,d}(C)$ in characteristic 0.

Exercise 6.5.7. If G is a connected affine algebraic group over an algebraically closed field $\Bbbk$ acting on a normal finite type $\Bbbk$-scheme X, and $x \in X(\Bbbk)$ has linearly reductive stabilizer, show that there is a G_x-invariant, locally closed, and affine subscheme $W \hookrightarrow X$ such that $[W/G_x] \to [X/G]$ is étale.

Hint: Sumihiro's Theorem on Linearizations (C.3.3) to reduce to the case that $X = \mathbb{P}(V)$. Choose a homogenous polynomial f not vanishing at x such that $\mathbb{P}(V)_f$ is G_x-invariant and then argue as in the proof of Luna's Étale Slice Theorem by considering the G_x-equivariant étale map $\mathbb{P}(V)_f \to T_x\mathbb{P}(V)$.

6.5.2 Deformation theory

In our proof of the Local Structure Theorem (6.5.1), we will need some deformation theory of algebraic stacks in the form of the following two propositions.

Proposition 6.5.8. *Consider a commutative diagram*

$$\begin{tikzcd} \mathcal{W} \arrow[r, "f"] \arrow[d, hook] & \mathcal{X} \arrow[d] \\ \mathcal{W}' \arrow[r] \arrow[ur, dashed] & \mathcal{Y} \end{tikzcd}$$

of noetherian algebraic stacks with affine diagonal where $\mathcal{X} \to \mathcal{Y}$ is smooth and affine and $\mathcal{W}' \hookrightarrow \mathcal{W}'$ is a closed immersion defined by a square-zero sheaf of ideals J. If $\mathcal{W}$ is cohomologically affine, there exists a lift in the above diagram.

Proof. When $\mathcal{W}$ is affine, the statement follows from the Infinitesimal Lifting Criterion (A.3.1). To reduce to this case, let $U' \to \mathcal{W}'$ be a smooth presentation with U' an affine scheme and set $U = U' \times_{\mathcal{W}'} \mathcal{W}$. Since $\mathcal{W}$ has affine diagonal, each n-fold fiber product $(U/\mathcal{W})^n := U \times_{\mathcal{W}} \cdots \times_{\mathcal{W}} U$ is affine. We have a commutative diagram

$$\begin{tikzcd}
(U/\mathcal{W})^2 \arrow[r, shift left] \arrow[r, shift right] \arrow[d, hook] & U \arrow[r, "q_1"] \arrow[d, hook] & \mathcal{W} \arrow[r, "f"] \arrow[d, hook] & \mathcal{X} \arrow[d] \\
(U'/\mathcal{W}')^2 \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & U' \arrow[r] \arrow[urr, "f'_U"] & \mathcal{W}' \arrow[r] \arrow[ur, dashed, "f'"'] & \mathcal{Y}.
\end{tikzcd}$$

where we have chosen a lift $f'_U \colon U' \to \mathcal{X}$. Defining the coherent sheaf $F = f^*(\Omega^\vee_{\mathcal{X}/\mathcal{Y}}) \otimes J$ on $\mathcal{W}$, we know by Exercise 6.1.9 that the set of lifts $U' \to \mathcal{X}$ is a torsor under $\Gamma(U, q_1^* F)$ so that any other lift differs from f'_U by an element of $\Gamma(U, q_1^* F)$. Because $\mathcal{X} \to \mathcal{Y}$ is representable, to check that f'_U descends to a morphism $f' \colon \mathcal{W}' \to \mathcal{X}$, we need to arrange that $f'_U \circ p_1 = f'_U \circ p_2$. Let $q_n \colon (U/\mathcal{W})^n \to \mathcal{W}$. The difference $f'_U \circ p_1 - f'_U \circ p_2$ can be viewed an element of $\Gamma((U/\mathcal{W})^2, q_2^* F)$.

Since $q_1 \colon U \to \mathcal{W}$ is a surjective, smooth and affine morphism, there is an exact sequence of quasi-coherent sheaves

$$0 \to F \to q_{1,*} q_1^* F \to q_{2,*} q_2^* F \to q_{3,*} q_3^* F \to \cdots ;$$

see Exercise B.1.2. Since $\mathcal{W}$ is cohomologically affine, taking global sections yields an exact sequence

$$\begin{array}{ccccc}
\Gamma(U, q_1^* F) & \xrightarrow{d_0} & \Gamma((U/\mathcal{W})^2, q_2^* F) & \xrightarrow{d_1} & \Gamma((U/\mathcal{W})^3, q_3^* F) \\
s & \longmapsto & p_1^* s - p_2^* s & & \\
 & & s & \longmapsto & p_{12}^* s - p_{13}^* s + p_{23}^* s.
\end{array}$$

One checks that $d_1(f'_U \circ p_1 - f'_U \circ p_2) = 0$ so there exists an element $s \in \Gamma(U, q_1^* F)$ with $d_0(s) = f'_U \circ p_1 - f'_U \circ p_2$. After modifying the lift f'_U by s, we see that $f'_U \circ p_1 - f'_U \circ p_2 = 0$ so that f'_U descends to $f' \colon \mathcal{W}' \to \mathcal{X}$. $\square$

Remark 6.5.9. Alternatively, one can show that the obstruction to this deformation problem lies in $\mathrm{Ext}^1_{\mathcal{O}_{\mathcal{W}}}(f^*\Omega_{\mathcal{X}/\mathcal{Y}}, J) = \mathrm{H}^1(\mathcal{W}, f^*(\Omega^\vee_{\mathcal{X}/\mathcal{Y}}) \otimes J)$, which vanishes since $\mathcal{W}$ is cohomologically affine. The above result holds more generally [Ols06, Thm. 1.5].

Proposition 6.5.10. *Let $\mathcal{W} \hookrightarrow \mathcal{W}'$ be a closed immersion of algebraic stacks of finite type over $\Bbbk$ with affine diagonal defined by a square-zero sheaf of ideals J. Let G be an affine algebraic group over $\Bbbk$. If $\mathcal{W}$ is cohomologically affine, then every principal G-bundle $\mathcal{P} \to \mathcal{W}$ extends to a principal G-bundle $\mathcal{P}' \to \mathcal{W}'$.*

Proof. Our proof will use smooth descent and the deformation theory of principal G-bundles over schemes (Exercise D.2.9). Let $U' \to \mathcal{W}'$ be a smooth presentation from an affine scheme and let $U := \mathcal{W} \times_{\mathcal{W}'} U'$. Since $\mathcal{W}$ has affine diagonal, each n-fold fiber products $(U/\mathcal{W})^n = U \times_{\mathcal{W}} \cdots \times_{\mathcal{W}} U$ is affine and we denote the projection by $q_n \colon (U/\mathcal{W})^n \to \mathcal{W}$. By descent theory, the principal G-bundle

$\mathcal{P} \to \mathcal{W}$ corresponds to a principal G-bundle $P \to U$ together with an isomorphism $\alpha\colon p_1^* P \xrightarrow{\sim} p_2^* P$ on $(U/\mathcal{W})^2$ satisfying the cocycle condition $p_{23}^*\alpha \circ p_{12}^*\alpha = p_{13}^*\alpha$ on $(U/\mathcal{W})^3$. Letting $F = \mathfrak{g} \otimes J$ be the coherent sheaf on $\mathcal{W}$ where $\mathfrak{g}$ denotes the Lie algebra of G, we know by Exercise D.2.9 that the deformation theory of $q_n^*\mathcal{P} \to (U/\mathcal{W})^n$ with respect to the closed immersion $(U/\mathcal{W})^n \hookrightarrow (U'/\mathcal{W}')^n$ is controlled by $q_n^* F$.

Since U is affine, we can choose a deformation $P' \to U'$ of $P \to U$. We can also choose an isomorphism $\alpha'\colon p_1^*P' \xrightarrow{\sim} p_2^*P'$ on $(U'/\mathcal{W}')^2$ lifting α where any other choice of an isomorphism differs by an element of $\Gamma((U/\mathcal{W})^2, q_2^*F)$. The isomorphism $(p_{13}^*\alpha')^{-1} \circ p_{23}^*\alpha' \circ p_{12}^*\alpha'$ restricts to the identity on $(U/\mathcal{W})^3$ and thus corresponds to an element $\Psi \in \Gamma((U/\mathcal{W})^3, q_3^*F)$. If $\Psi = 0$, then descent theory implies that $P' \to U'$ descends to the desired principal G-bundle $\mathcal{P}' \to \mathcal{W}'$. Since $0 \to F \to q_{1,*}q_1^*F \to q_{2,*}q_2^*F \to \cdots$ is an exact sequence and $\mathcal{W}$ is cohomologically affine, taking global sections gives an exact sequence

$$\begin{aligned}
\Gamma((U/\mathcal{W})^2, q_2^*F) \xrightarrow{d_2} \Gamma((U/\mathcal{W})^3, q_3^*F) &\xrightarrow{\quad d_3 \quad} \Gamma((U/\mathcal{W})^4, q_4^*F) \\
s \longmapsto p_{12}^*s - p_{13}^*s + p_{23}^*s & \\
& s \longmapsto p_{123}^*s - p_{134}^*s + p_{124}^*s - p_{234}^*s.
\end{aligned} \tag{6.5.3}$$

While Ψ may be nonzero, one can check that $d_3(\Psi) = 0$ and thus there exists an element $s \in \Gamma((U/\mathcal{W})^2, q_2^*F)$ such that $d_2(s) = \Psi$. Thus modifying the isomorphism α' by s, we see that we can arrange the cocycle condition to hold. $\square$

Remark 6.5.11. The deformation question is equivalent to deforming the morphism $f\colon \mathcal{W} \to \mathbf{B}G$ classified by $P \to \mathcal{W}$ to a morphism $\mathcal{W}' \to \mathbf{B}G$, which is analogous to Proposition 6.5.8 except that $\mathcal{X} = \mathbf{B}G \to \mathcal{Y} = \operatorname{Spec} \Bbbk$ is not affine. The obstruction to deforming a principal G-bundle lies in the group $\mathrm{H}^2(\mathcal{W}, \mathfrak{g} \otimes J)$. When $\mathcal{W} \to \mathbf{B}G$ is representable, one can see this as a consequence of [Ols06, Thm. 1.5] (see Remarks D.5.12 and D.7.5): the obstruction lies in $\mathrm{Ext}^1_{\mathcal{O}_\mathcal{W}}(Lf^*L_{\mathbf{B}G/\Bbbk}, J)$. Under the composition $\operatorname{Spec}\Bbbk \xrightarrow{p} \mathbf{B}G \to \operatorname{Spec}\Bbbk$, we have an exact triangle $p^*L_{\mathbf{B}G/\Bbbk} \to L_{\Bbbk/\Bbbk} \to L_{\Bbbk/\mathbf{B}G}$. Since $L_{\Bbbk/\Bbbk} = 0$, we obtain that $p^*L_{\mathbf{B}G/\Bbbk} = L_{\Bbbk/\mathbf{B}G}[-1] \cong \mathfrak{g}^\vee[-1]$ and $L_{\mathbf{B}G/\Bbbk} \cong \mathfrak{g}^\vee[-1]$, where the Lie algebra $\mathfrak{g}$ is equipped with the adjoint representation. Thus $\mathrm{Ext}^1_{\mathcal{O}_\mathcal{W}}(Lf^*L_{\mathbf{B}G/\Bbbk}, J) = \mathrm{H}^1(\mathcal{W}, f^*\mathfrak{g}[1] \otimes J) = \mathrm{H}^2(\mathcal{W}, \mathfrak{g} \otimes J)$. Since $\mathcal{W}$ is cohomologically affine with affine diagonal, this cohomology group is 0 and the obstruction vanishes.

Here's a third approach in the case that $\mathcal{W} = [\operatorname{Spec} A/G]$ where G is linearly reductive and A^G is an artinian $\Bbbk$-algebra. Since $\mathcal{W}$ is global quotient stack, there exists a vector bundle E on $\mathcal{W}$ such that the stabilizer groups act faithfully on the fibers (Exercise 6.1.16). Generalizing the deformation theory of vector bundles on schemes (Proposition D.2.15), the obstruction to deforming E to a vector bundle E' lies in $\mathrm{H}^2(\mathcal{W}, \mathscr{E}nd_{\mathcal{O}_\mathcal{W}}(E) \otimes J)$ which vanishes as $\mathcal{W}$ is cohomologically affine. Since the stabilizer groups also act faithfully on the fibers of E', we have that $\mathcal{W}' \cong [V'/\operatorname{GL}_n]$ where V' is an algebraic space. Then $\mathcal{W} \cong [V/\operatorname{GL}_n]$ with $V_{\mathrm{red}} = V'_{\mathrm{red}}$. Since $\mathcal{W}$ is cohomologically affine and $V \to \mathcal{W}$ is affine, V is cohomologically affine and thus affine by Serre's Criterion for Affineness (4.4.15). It follows that V' is also affine (Proposition 4.4.18). Since $\Gamma(\mathcal{W}', \mathcal{O}_{\mathcal{W}'})$ is an artinian $\Bbbk$-algebra and has no non-trivial affine étale covers, Luna's Étale Slice Theorem (6.5.4) implies that we can arrange that $\mathcal{W}' \cong [\operatorname{Spec} A'/G]$.

We will also need the following criteria for morphisms to be closed immersions or isomorphisms.

Lemma 6.5.12. *Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a representable morphism of algebraic stacks of finite type over an algebraically closed field $\Bbbk$ with affine diagonal. Assume that $|\mathcal{X}| = \{x\}$ and $|\mathcal{Y}| = \{y\}$ consist of a single point and that f induces an isomorphism $\mathcal{X}_0 := \mathbf{B}G_x$ with $\mathcal{Y}_0 := \mathbf{B}G_y$. Let $\mathfrak{m}_x \subset \mathcal{O}_{\mathcal{X}}$ and $\mathfrak{m}_y \subset \mathcal{O}_{\mathcal{Y}}$ be the ideal sheaves defining $\mathcal{X}_0$ and $\mathcal{Y}_0$, and let $f_1\colon \mathcal{X}_1 \to \mathcal{Y}_1$ be the induced morphism between the first nilpotent thickenings of $\mathcal{X}_0$ and $\mathcal{Y}_0$.*

(1) If f_1 is a closed immersion, then so is f.

(2) If f_1 is a closed immersion and there is an isomorphism $\bigoplus_{n\geq 0} \mathfrak{m}_y^n/\mathfrak{m}_y^{n+1} \cong \bigoplus_{n\geq 0} \mathfrak{m}_x^n/\mathfrak{m}_x^{n+1}$ of graded $\mathcal{O}_{\mathcal{X}_0}$-modules, then f is an isomorphism.

Proof. Choose a smooth presentation $V = \operatorname{Spec} B \to \mathcal{Y}$ from an affine scheme such that $V \times_{\mathcal{Y}} \mathcal{Y}_0 \cong \operatorname{Spec} \Bbbk$ (Theorem 3.6.1). Then B is a local artinian $\Bbbk$-algebra as $\mathcal{Y}$ consists of only one point. The base change $U = V \times_{\mathcal{Y}} \mathcal{X}$ is an algebraic space and since $U_{\mathrm{red}} = V_{\mathrm{red}}$ is a point, it follows from Proposition 4.4.18 that $U = \operatorname{Spec} A$ with A a local artinian $\Bbbk$-algebra. We can therefore assume that $f\colon \operatorname{Spec} A \to \operatorname{Spec} B$ is a morphism of local artinian schemes.

For (1), we need to show that if $B/\mathfrak{m}_B^2 \to A/\mathfrak{m}_A^2$ is surjective, so is $B \to A$. We first claim that the inclusion $\mathfrak{m}_B A \hookrightarrow \mathfrak{m}_A$ is surjective. By Nakayama's Lemma, it suffices to show that $\mathfrak{m}_B A/\mathfrak{m}_A \mathfrak{m}_B A \to \mathfrak{m}_A/\mathfrak{m}_A^2$ is surjective but this follows from the hypothesis that the composition $\mathfrak{m}_A/\mathfrak{m}_A^2 \to \mathfrak{m}_B A/\mathfrak{m}_A\mathfrak{m}_B A \to \mathfrak{m}_A/\mathfrak{m}_A^2$ is surjective. Since $B/\mathfrak{m}_B \to A/\mathfrak{m}_B A = A/\mathfrak{m}_A$ is surjective, another application of Nakayama's Lemma shows that $B \to A$ is surjective. See also [Har77, Lem. II.7.4] for a related criterion.

For (2), since $\dim_\Bbbk \mathfrak{m}_B^n/\mathfrak{m}_B^{n+1} = \dim_\Bbbk \mathfrak{m}_A^n/\mathfrak{m}_A^{n+1}$, the surjections $\mathfrak{m}_B^n/\mathfrak{m}_B^{n+1} \to \mathfrak{m}_A^n/\mathfrak{m}_A^{n+1}$ are isomorphisms and it follows that f is an isomorphism. ☐

6.5.3 Proof of the Local Structure Theorem—smooth case

Proof of Theorem 6.5.1—smooth case. Since the $\Bbbk$-point $x \in \mathcal{X}$ is locally closed (Proposition 3.5.16), by replacing $\mathcal{X}$ by an open substack we may assume that $x \in \mathcal{X}$ is a closed point. Let $\mathcal{I}$ be the coherent sheaf of ideals defining $\mathcal{X}_0 := \mathbf{B}G_x \hookrightarrow \mathcal{X}$ and set $\mathcal{X}_n$ to be the nth nilpotent thickening defined by $\mathcal{I}^{n+1}$. The Zariski tangent space $T_{\mathcal{X},x}$ can be identified with the normal space $(\mathcal{I}/\mathcal{I}^2)^\vee$ to the orbit, viewed as a G_x-representation. (Note that when $\mathcal{X} = [X/G]$ with G a smooth affine algebraic group, then $T_{\mathcal{X},x}$ is identified with the normal space to the orbit $T_{X,\widetilde{x}}/T_{Gx,\widetilde{x}}$ for a point $\widetilde{x} \in X(\Bbbk)$ over x.)

Define the quotient stack $\mathcal{T} = [T_{\mathcal{X},x}/G_x]$ and let $\mathcal{T}_0 = \mathbf{B}G_x$ be the closed substack supported at the origin and $\mathcal{T}_n$ its nth nilpotent thickenings. We claim that there are compatible isomorphisms $\mathcal{X}_n \cong \mathcal{T}_n$. Since G_x is linearly reductive, $\mathcal{X}_0 = \mathbf{B}G_x$ is cohomologically affine. By the deformation theory of principal G_x-bundles (Proposition 6.5.10), we can inductively extending the principal G_x-bundle $\operatorname{Spec} \Bbbk \to \mathcal{X}_0$ to principal G_x-bundles $\operatorname{Spec} A_n \to \mathcal{X}_n$. This yields isomorphisms $\mathcal{X}_n \cong [\operatorname{Spec} A_n/G_x]$ and affine morphisms $\mathcal{X}_n \to \mathbf{B}G_x$. We have a closed immersion

$\mathcal{X}_0 \hookrightarrow \mathcal{T}$ and we can inductively find lifts

$$\begin{tikzcd} \mathcal{X}_n \arrow[r] \arrow[d, hook] & \mathcal{T} \arrow[d] \\ \mathcal{X}_{n+1} \arrow[r] \arrow[ur, dashed] & \mathbf{B}G_x \end{tikzcd}$$

since $\mathcal{T} \to \mathbf{B}G_x$ is smooth and affine (Proposition 6.5.8). The induced morphism $\mathcal{X}_1 \to \mathcal{T}_1$ is an isomorphism since it is a morphism between deformations $\mathbf{B}G_x \hookrightarrow \mathcal{X}_1$ and $\mathbf{B}G_x \hookrightarrow \mathcal{T}_1$ of the coherent sheaf $\mathcal{I}/\mathcal{I}^2$ and any such morphism is an isomorphism (by reducing to Lemma D.1.7 by smooth descent). (In fact, both $\mathcal{X}_1$ and $\mathcal{T}_1$ are trivial deformations as they admit retractions to $\mathbf{B}G_x$.) Lemma 6.5.12(2) now implies that the maps $\mathcal{X}_n \to \mathcal{T}_n$ are isomorphisms.

Let $\pi\colon \mathcal{T} \to T = T_{\mathcal{X},x}/\!\!/G_x$ be the morphism to the GIT quotient. The fiber product $\widehat{\mathcal{T}} := \operatorname{Spec} \widehat{\mathcal{O}}_{T,\pi(0)} \times_T \mathcal{T}$ is a quotient stack of the form $[\operatorname{Spec} B/G]$ where B is of finite type over the noetherian complete local $\Bbbk$-algebra $B^G = \widehat{\mathcal{O}}_{T,\pi(0)}$. Therefore $\widehat{\mathcal{T}}$ is coherently complete along $\mathcal{T}_0$ (Theorem 6.4.11) and $\mathrm{MOR}(\mathcal{T},\mathcal{X}) \xrightarrow{\sim} \varprojlim \mathrm{MOR}(\mathcal{T}_n,\mathcal{X})$ is an equivalence by Coherent Tannaka Duality (6.4.8). It follows that the morphisms $\mathcal{X}_n \cong \mathcal{T}_n \hookrightarrow \mathcal{X}$ extend to a morphism $\widehat{\mathcal{T}} \to \mathcal{X}$ filling in the diagram

$$\begin{tikzcd} \mathcal{X}_n \cong \mathcal{T}_n \arrow[r] \arrow[rrr, bend left] & \widehat{\mathcal{T}} \arrow[r] \arrow[d] \arrow[rr, dashed, bend left] \arrow[dr, phantom, "\square"] & \mathcal{T} \arrow[d] & \mathcal{X} \\ & \operatorname{Spec} \widehat{\mathcal{O}}_{T,\pi(0)} \arrow[r] & T. & \end{tikzcd}$$

The functor parameterizing isomorphism classes of morphisms

$$F\colon \mathrm{Sch}/T \to \mathrm{Sets}, \qquad (T' \to T) \mapsto \{T' \times_T \mathcal{T} \to \mathcal{X}\}/\sim$$

is limit preserving as $\mathcal{X}$ is of finite type over $\Bbbk$ (see Exercise 3.3.31). The morphism $\widehat{\mathcal{T}} \to \mathcal{X}$ yields an element of F over $\operatorname{Spec} \widehat{\mathcal{O}}_{T,\pi(0)}$. By Artin Approximation (A.10.9), there exist an étale morphism $(U,u) \to (T,0)$ where U is an affine scheme with a $\Bbbk$-point $u \in U$ and a morphism $(U \times_T \mathcal{T},(u,0)) \to (\mathcal{X},x)$ agreeing with $(\widehat{\mathcal{T}},0) \to (\mathcal{X},x)$ to first order. Since $U \times_T \mathcal{T}$ is smooth at $(u,0)$ and $\mathcal{X}$ is smooth at x, and since $U \times_T \mathcal{T} \to \mathcal{X}$ induces an isomorphism of tangent spaces and stabilizer groups at $(u,0)$, the morphism $U \times_T \mathcal{T} \to \mathcal{X}$ is étale at $(u,0)$. Observe that $U \times_T \mathcal{T}$ is of the form $[\operatorname{Spec} A/G_x]$ for a finitely generated $\Bbbk$-algebra A such that $U = \operatorname{Spec} A^{G_x}$. We can arrange that $[\operatorname{Spec} A/G_x] \to \mathcal{X}$ is étale everywhere after replacing U with an open affine subscheme and $\operatorname{Spec} A$ with its preimage. That $[\operatorname{Spec} A/G_x] \to \mathcal{X}$ can be arranged to be affine follows from Proposition 6.5.13. $\square$

Proposition 6.5.13. *Let $\mathcal{X}$ be an algebraic stack of finite type over an algebraically closed field with affine diagonal. Let $f\colon [\operatorname{Spec} A/G] \to \mathcal{X}$ be a finite type morphism with G is a linearly reductive group. If $w \in \operatorname{Spec} A$ has closed G-orbit and f induces an isomorphism of stabilizer groups at w, then there exists a G-invariant, affine, and open subscheme $U \subset \operatorname{Spec} A$ containing w such that $f|_{[U/G]}$ is affine.*

Proof. Set $\mathcal{W} = [\operatorname{Spec} A/G]$ with $\pi\colon \mathcal{W} \to \operatorname{Spec} A^G$. Since $f\colon \mathcal{W} \to \mathcal{X}$ is quasi-finite on an open subset $\mathcal{U}$, then $\{\pi(w)\}$ and $\pi(\mathcal{W} \setminus \mathcal{U})$ are disjoint closed subspaces and choosing an affine open $V \subset \operatorname{Spec} A^G \setminus \pi(\mathcal{W} \setminus \mathcal{U})$ containing $\pi(w)$, we may replace $\mathcal{W}$ with $\pi^{-1}(V)$ and we can assume that $f : \mathcal{W} \to \mathcal{X}$ is quasi-finite.

Choose a smooth presentation $V = \operatorname{Spec} B \to \mathcal{X}$ and consider the fiber product

$$\begin{tikzcd} \mathcal{W}_V \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & V = \operatorname{Spec} B \arrow[d] \\ \mathcal{W} = [\operatorname{Spec} A/G] \arrow[r] & \mathcal{X}. \end{tikzcd}$$

Since $\mathcal{X}$ has affine diagonal, $\operatorname{Spec} B \to \mathcal{X}$ is affine and therefore $\mathcal{W}_V$ is cohomologically affine. As $\mathcal{W}_V$ has quasi-finite diagonal, Proposition 6.3.28 implies that $\mathcal{W}_V \to V$ is separated and it follows from descent that $\mathcal{W} \to \mathcal{X}$ is also separated and that the relative inertia $I_{\mathcal{W}/\mathcal{X}} \to \mathcal{W}$ is finite. Since the fiber over $w \in \mathcal{W}$ is trivial, there is an open neighborhood $\mathcal{U}$ over which the relative inertia is trivial. As in the first paragraph, we may replace $\mathcal{U}$ with an open substack of the form $[\operatorname{Spec} C/G]$ containing w. Since $f|_{\mathcal{U}} : \mathcal{U} \to \mathcal{X}$ is a representable and cohomologically affine morphism, Serre's Criterion for Affineness (6.3.16) implies that $f|_{\mathcal{U}}$ is affine. □

6.5.4 Equivariant Artin Algebraization

The smoothness hypothesis of $x \in \mathcal{X}$ was used above to establish that $\mathcal{T}_n \cong \mathcal{X}_n$ and that $U \times_T \mathcal{T} \to \mathcal{X}$ is étale. More critically, it implied that $\varprojlim \Gamma(\mathcal{X}_n, \mathcal{O}_{\mathcal{X}_n})$, which is identified with the G_x-invariants of a miniversal deformation space, is the completion of a finitely generated $\Bbbk$-algebra, namely $\widehat{\mathcal{O}}_{T,0}$. If $x \in \mathcal{X}$ is not smooth, it seems difficult to directly establish that $\varprojlim \Gamma(\mathcal{X}_n, \mathcal{O}_{\mathcal{X}_n})$ is the completion of a finitely generated $\Bbbk$-algebra. Recall that we ran into a similar issue when discussing Artin Algebraization (D.6.6). When the complete local ring R is known to be the completion of a finitely generated algebra, then Artin Algebraization is an easy consequence of Artin Approximation (see Remark D.6.8). To circumvent this issue in our general proof of Artin Algebraization, we wrote $R = \widehat{\mathcal{O}}_{V,v}/I$ where V is a finite type $\Bbbk$-scheme and used Artin Approximation to simultaneously approximate both the given object over R and the equations defining I. We follow a similar strategy but this time we proceed G-equivariantly.

We will use the following extension of the notion of formal versality introduced in Definition D.3.5: for an algebraic stack $\widehat{\mathcal{T}}$ with a unique closed point t, a morphism $\widehat{\xi} : \widehat{\mathcal{T}} \to \mathcal{X}$ of prestacks over Sch is *formally versal at* t if every commutative diagram

$$\begin{tikzcd} \mathcal{Z} \arrow[r] \arrow[d, hook] & \widehat{\mathcal{T}} \arrow[d, "\widehat{\xi}"] \\ \mathcal{Z}' \arrow[r] \arrow[ur, dashed] & \mathcal{X} \end{tikzcd}$$

has a lift, where $\mathcal{Z} \hookrightarrow \mathcal{Z}'$ is a closed immersion of noetherian algebraic stacks with affine diagonal, $|\mathcal{Z}| = |\mathcal{Z}'|$ consists of a single point and the image of $\mathcal{Z} \to \widehat{\mathcal{T}}$ is t.

Theorem 6.5.14 (Equivariant Artin Algebraization)**.** *Let $\Bbbk$ be an algebraically closed field and R be a complete noetherian local $\Bbbk$-algebra with residue field $\Bbbk$. Let $\widehat{\mathcal{T}} = [\operatorname{Spec} B/G]$ be an algebraic stack of finite type over $R = B^G$, where G is linearly reductive. Assume that the unique closed point $t \in \widehat{\mathcal{T}}$ has stabilizer equal* *to G. If $\mathcal{X}$ is a limit preserving prestack over* $\mathrm{Sch}/\Bbbk$ *and* $\eta\colon \widehat{\mathcal{T}} \to \mathcal{X}$ *is a morphism of prestacks formally versal at* t*, then there exists*

(1) *an algebraic stack* $\mathcal{W} = [\operatorname{Spec} A/G]$ *of finite type over* $\Bbbk$ *and a closed point* $w \in \mathcal{W}$*;*

(2) *morphisms* $f\colon \mathcal{W} \to \mathcal{X}$ *and* $\varphi\colon \widehat{\mathcal{T}} \to \mathcal{W}$ *such that in the diagram*

$$\begin{tikzcd} \widehat{\mathcal{T}} \arrow[d, "\varphi"'] \arrow[dr, "\eta"] & \\ \mathcal{W} \arrow[r, "f"] & \mathcal{X} \end{tikzcd} \tag{6.5.4}$$

the induced morphisms $\varphi_n\colon \widehat{\mathcal{T}}_n \to \mathcal{W}_n$ *between the nth nilpotent thickenings of* t *and* w *are isomorphisms, and there exist compatible 2-isomorphisms* $\eta_n \xrightarrow{\sim} f_n \circ \varphi_n$*.*

Moreover, if $\mathcal{X}$ is an algebraic stack of finite type over $\Bbbk$ with affine diagonal, then it can be arranged that (6.5.4) is commutative and that φ induces an isomorphism $\widehat{\mathcal{T}} \to \widehat{\mathcal{W}} := \mathcal{W} \times_W \operatorname{Spec} \widehat{\mathcal{O}}_{W,\pi(w)}$*, where* $\pi\colon \mathcal{W} \to W = \operatorname{Spec} A^G$*.*

Remark 6.5.15. If one takes G to be the trivial group, one recovers the classical version of Artin Algebraization (D.6.6).

As in the proof of Theorem D.6.6, we will apply Artin Approximation to a well chosen integer N to construct $\mathcal{W}$ such that there are isomorphisms $\mathcal{W}_n \cong \widehat{\mathcal{T}}_n$ for $n \leq N$ and such that the Artin–Rees Lemma implies that there are also isomorphisms $\mathcal{W}_n \cong \widehat{\mathcal{T}}_n$ for $n > N$. To get control over the constant in the Artin–Rees Lemma, we need to generalize Definition D.6.3: for a noetherian algebraic stack $\mathcal{X}$ with a closed point x defined by a a sheaf of ideals $\mathfrak{m}_x$ and an integer $c \geq 0$, we say that $(\mathsf{AR})_c$ *holds at* x for a map $\varphi\colon E \to F$ of coherent sheaves on $\mathcal{X}$ if

$$\varphi(E) \cap \mathfrak{m}_x^n F \subseteq \varphi(\mathfrak{m}_x^{n-c} E), \quad \forall n \geq c.$$

When $\mathcal{X}$ is a scheme, $(\mathsf{AR})_c$ holds for all sufficiently large c by the Artin–Rees lemma and in fact it holds replacing $\{x\}$ with a closed subscheme. By smooth descent, $(\mathsf{AR})_c$ also holds for algebraic stacks for $c \gg 0$.

Proof. The morphism $\eta\colon \widehat{\mathcal{T}} \to \mathcal{X}$ and the structure morphism $\widehat{\mathcal{T}} \to \mathbf{B}G$ induce a morphism $\widehat{\mathcal{T}} \to \mathcal{X} \times \mathbf{B}G$. We let $\widehat{T} = \operatorname{Spec} R$ be the GIT quotient of $\widehat{\mathcal{T}} = [\operatorname{Spec} B/G]$. Since R is the colimit of its finitely generated $\Bbbk$-subalgebras and $\mathcal{X} \times \mathbf{B}G$ is limit preserving, limit methods (§A.6) imply that there is a commutative diagram

$$\begin{tikzcd} \widehat{\mathcal{T}} \arrow[r] \arrow[rr, bend left] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{S} \arrow[r] \arrow[d] & \mathcal{X} \times \mathbf{B}G \arrow[d] \\ \widehat{T} \arrow[r] & S \arrow[r] & \operatorname{Spec} \Bbbk \end{tikzcd}$$

where $S = \operatorname{Spec} R'$ is an affine scheme of finite type over $\Bbbk$, $\mathcal{S}$ is an algebraic stack of finite type over S with affine diagonal such that $\widehat{\mathcal{T}} = \widehat{T} \times_S \mathcal{S}$, and $\widehat{\mathcal{T}} \to \mathcal{X} \times \mathbf{B}G$ factors as $\widehat{\mathcal{T}} \to \mathcal{S} \to \mathcal{X} \times \mathbf{B}G$. Moreover, we can arrange that $\mathcal{S} \to \mathbf{B}G$ is affine. Let $\widetilde{s} \in \mathcal{S}$ and $s \in S$ be the images of t. By possibly adding generators to R' so that $R' \to R \to R/\mathfrak{m}_R^2$ is surjective, we can arrange that $\widehat{\mathcal{O}}_{S,s} \to R$ is surjective by Lemma A.10.15, or in other words that $\widehat{T} \to \widehat{S} := \operatorname{Spec} \widehat{\mathcal{O}}_{S,s}$ is a closed immersion.

Note that $\widehat{\mathcal{T}}$ is a closed substack of $\mathcal{S} \times_S \widehat{S}$. By choosing a resolution $\mathcal{O}_{\widehat{S}}^{\oplus r} \to \mathcal{O}_{\widehat{S}} \twoheadrightarrow R$ and pulling it back $\mathcal{S} \times_S \widehat{S}$, we obtain a resolution

$$\ker(\beta) \xhookrightarrow{\alpha} \mathcal{O}^{\oplus r}_{\mathcal{S}\times_S \widehat{S}} \xrightarrow{\beta} \mathcal{O}_{\mathcal{S}\times_S \widehat{S}} \twoheadrightarrow \mathcal{O}_{\widehat{\mathcal{T}}}. \tag{6.5.5}$$

Consider the functor $F\colon \mathrm{Sch}/S \to \mathrm{Sets}$ assigning $(U \to S)$ to the set of isomorphism classes of complexes

$$L \xrightarrow{\alpha} \mathcal{O}^{\oplus r}_{\mathcal{S}\times_S U} \xrightarrow{\beta} \mathcal{O}_{\mathcal{S}\times_S U}$$

of finitely presented quasi-coherent $\mathcal{O}_{\mathcal{S}\times_S U}$-modules. By standard limit arguments, F is limit preserving. The complex (6.5.5) defines an element $(\alpha, \beta) \in F(\widehat{S})$ such that $\operatorname{coker}(\beta) = \mathcal{O}_{\widehat{\mathcal{T}}}$. Let N be an integer such that $(\mathsf{AR})_N$ holds for α and β at $(\widetilde{s}, s)$.

Artin Approximation (A.10.9) gives an étale neighborhood $(S', s') \to (S, s)$ and an element $(\alpha', \beta') \in F(S')$ such that $(\alpha, \beta) = (\alpha', \beta')$ in $F(\mathcal{O}_{S,s}/\mathfrak{m}_s^{N+1})$. We let $\mathcal{W} \hookrightarrow \mathcal{S} \times_S S'$ be the closed substack defined by $\operatorname{coker}(\beta')$ and set $w = (\widetilde{s}, s') \in \mathcal{W}$. Letting S_n, S'_n and $\widehat{T}_n$ be the nth nilpotent thickenings of S, S' and $\widehat{T}$ at the images of $t \in \mathcal{T}$, we have that $\widehat{\mathcal{T}} \times_{\widehat{T}} \widehat{T}_N$ and $\mathcal{W} \times_{S'} S'_N$ are equal as closed substacks of $\mathcal{S} \times_S S_N$. This gives (1)-(2) for $n \leq N$. In particular, we have an isomorphism $\varphi_N\colon \widehat{\mathcal{T}}_N \to \mathcal{W}_N$ and we let $\psi_N\colon \mathcal{W}_N \to \widehat{\mathcal{T}}_N$ be its inverse.

Using that $\eta\colon \widehat{\mathcal{T}} \to \mathcal{X}$ is formally versal, we can inductively find compatible lifts for $n \geq N$

$$\begin{tikzcd}
\mathcal{W}_n \arrow[r, "\psi_n"] \arrow[d, hook] & \widehat{\mathcal{T}} \arrow[d, "\eta"] \\
\mathcal{W}_{n+1} \arrow[r] \arrow[ur, dashed, "\psi_{n+1}"] & \mathcal{X}.
\end{tikzcd}$$

On the other hand, applying Lemma D.6.4 (generalized to stacks by smooth descent) on $\mathcal{S} \times_S \widehat{S}$ with $c = N$ to the complex (6.5.5) and the restriction of the complex defined by (α', β'), we obtain an isomorphism $\operatorname{Gr}_{\mathfrak{m}_t} \mathcal{O}_{\widehat{\mathcal{T}}} \cong \operatorname{Gr}_{\mathfrak{m}_w} \mathcal{O}_{\mathcal{W}}$ of graded $\mathcal{O}_{\mathbf{B}G}$-modules. By Lemma 6.5.12, the induced morphisms $\psi_n\colon \mathcal{W}_n \to \widehat{\mathcal{T}}_n$ are isomorphisms for all n. As $\widehat{\mathcal{T}}$ is coherently complete (Theorem 6.4.11), Coherent Tannaka Duality (6.4.8) implies that the inverses $\varphi_n = \psi_n^{-1}\colon \widehat{\mathcal{T}}_n \to \mathcal{W}_n$ effectivize to a morphism $\varphi\colon \widehat{\mathcal{T}} \to \mathcal{W}$. This completes (1)-(2).

For the final statement when $\mathcal{X}$ is algebraic, we again apply Coherent Tannaka Duality this time using the coherent completeness of both $\widehat{\mathcal{T}}$ and $\mathcal{W}$. By applying Corollary 6.4.8 to the inverses $\psi_n = \varphi_n^{-1}$, we can construct an inverse $\psi\colon \widehat{\mathcal{W}} \to \widehat{\mathcal{T}}$ of φ. Thus $\varphi\colon \widehat{\mathcal{T}} \to \widehat{\mathcal{W}}$ is an isomorphism. Using the fully faithfulness of Corollary 6.4.8, there is a 2-isomorphism $\eta \to f \circ \varphi$ extending the given 2-isomorphisms $\eta_n \xrightarrow{\sim} f_n \circ \varphi_n$ and thus $\widehat{\mathcal{T}} \to \widehat{\mathcal{W}}$ is a morphism over $\mathcal{X}$. □

6.5.5 Proof of the Local Structure Theorem—general case

Proof of Theorem 6.5.1. We may assume that $x \in \mathcal{X}$ is a closed point. Let $\mathcal{T} := [T_{\mathcal{X},x}/G_x]$, let $\pi\colon \mathcal{T} \to T = T_{\mathcal{X},x}/\!\!/G_x$ be the morphism to the GIT quotient, and let $\widehat{\mathcal{T}} := \operatorname{Spec} \widehat{\mathcal{O}}_{T,\pi(0)} \times_T \mathcal{T}$. Let $\mathcal{T}_0 = \mathbf{B}G_x$ be the closed substack supported at the origin and $\mathcal{T}_n$ its nth nilpotent thickenings.

We will construct compatible closed immersions $\mathcal{X}_n \hookrightarrow \mathcal{T}_n$. Since G_x is linearly reductive, $\mathcal{X}_0 = \mathbf{B}G_x$ is cohomologically affine. By deforming the principal G_x-bundle $\operatorname{Spec} \Bbbk \to \mathcal{X}_0$ using Proposition 6.5.10, we can inductively construct isomorphisms $\mathcal{X}_n \cong [\operatorname{Spec} A_n/G_x]$. By the deformation theory of the smooth and affine morphism $\widehat{\mathcal{T}} \to \mathbf{B}G_x$ (Proposition 6.5.8), we can inductively find lifts

$$\begin{tikzcd}
\mathcal{X}_n \arrow[r] \arrow[d, hook] & \mathcal{T} \arrow[d] \\
\mathcal{X}_{n+1} \arrow[r] \arrow[ur, dashed] & \mathbf{B}G_x.
\end{tikzcd}$$

As in the smooth case, $\mathcal{X}_1 \to \mathcal{T}_1$ is an isomorphism. By Lemma 6.5.12(1), each morphism $\mathcal{X}_n \to \mathcal{T}_n$ is a closed immersion.

If I_n denotes the ideal sheaf defining $\mathcal{X}_n \hookrightarrow \mathcal{T}_n$, then $\mathcal{O}_{\mathcal{T}_n}/I_n$ is a system of coherent $\mathcal{O}_{\mathcal{T}_n}$-modules. Since $\widehat{\mathcal{T}}$ is coherently complete (Theorem 6.4.11), there exists a coherent sheaf of ideals $I \subset \mathcal{O}_{\widehat{\mathcal{T}}}$ such that the surjection $\mathcal{O}_{\widehat{\mathcal{T}}} \to \mathcal{O}_{\widehat{\mathcal{T}}}/I$ extends the surjections $\mathcal{O}_{\mathcal{T}_n} \to \mathcal{O}_{\mathcal{X}_n}$. The closed immersion $\widehat{\mathcal{X}} \hookrightarrow \widehat{\mathcal{T}}$ defined by I extends the given closed immersions $\mathcal{X}_n \hookrightarrow \mathcal{T}_n$ yielding a commutative diagram

$$\begin{tikzcd}
\mathcal{X}_n \arrow[r, hook] \arrow[d, hook] \arrow[rr, bend left] & \widehat{\mathcal{X}} \arrow[r, dashed, "\eta"] \arrow[d, hook] & \mathcal{X} \\
\mathcal{T}_n \arrow[r, hook] & \widehat{\mathcal{T}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{T} \arrow[d] \\
& \operatorname{Spec} \widehat{\mathcal{O}}_{T,0} \arrow[r] & T.
\end{tikzcd}$$

of solid arrows. Since $\widehat{\mathcal{X}}$ is also coherently complete, Coherent Tannaka Duality (6.4.8) gives a morphism $\eta \colon \widehat{\mathcal{X}} \to \mathcal{X}$ extending the above diagram. Since $\widehat{\mathcal{X}}$ has the same nilpotent thickenings of $\widehat{\mathcal{X}}$, the morphism $\eta \colon \widehat{\mathcal{X}} \to \mathcal{X}$ is formally versal at 0. By Equivariant Artin Algebraization (6.5.14) with $G = G_x$, we obtain a morphism $f \colon \mathcal{W} = [\operatorname{Spec} B/G_x] \to \mathcal{X}$ from an algebraic stack $\mathcal{W}$ of finite type over $\Bbbk$ with a closed point $w \in \mathcal{W}$ and a morphism $\varphi \colon \widehat{\mathcal{X}} \to \mathcal{W}$ over $\mathcal{X}$ inducing an isomorphism $\widehat{\mathcal{X}} \to \mathcal{W} \times_W \operatorname{Spec} \widehat{\mathcal{O}}_{W,\pi(w)}$ where $\pi \colon \mathcal{W} \to \operatorname{Spec} B^{G_x}$. Since $f \colon \mathcal{W} \to \mathcal{X}$ induces isomorphisms $\mathcal{W}_n \to \mathcal{X}_n$, f is étale at w. After replacing $\mathcal{W}$ with an open substack, we can arrange that f is étale everywhere. By Proposition 6.5.13, we can also arrange that f is affine.

See also [AHR20, AHR19, AHLHR22]. □

6.5.6 The coherent completion at a point

We say that $(\mathcal{X}, x)$ is a *complete local stack* if $\mathcal{X}$ is a noetherian algebraic stack with affine stabilizers and with a unique closed point x such that $\mathcal{X}$ is coherently complete along the residual gerbe $\mathcal{G}_x$. An important example is $([\operatorname{Spec} A/G], x)$ where G is linearly reductive over an algebraically closed field $\Bbbk$, A^G is a complete noetherian local $\Bbbk$-algebra with residue field $\Bbbk$, A is of finite type over R, and the unique closed point x is fixed by G (Theorem 6.4.11). For instance, $([\mathbb{A}^n/\mathbb{G}_m], 0)$ is complete local.

The *coherent completion* of a noetherian algebraic stack $\mathcal{X}$ at a point x is a complete local stack $(\widehat{\mathcal{X}}_x, \widehat{x})$ together with a morphism $\eta\colon (\widehat{\mathcal{X}}_x, \widehat{x}) \to (\mathcal{X}, x)$ inducing isomorphisms of nth infinitesimal neighborhoods of $\widehat{x}$ and x. If $\mathcal{X}$ has affine stabilizers, then the pair $(\widehat{\mathcal{X}}_x, \eta)$ is unique up to unique 2-isomorphism by Coherent Tannaka Duality (6.4.8).

Theorem 6.5.16. *Let $\mathcal{X}$ be an algebraic stack of finite type over an algebraically closed field $\Bbbk$ with affine diagonal. For every point $x \in \mathcal{X}(k)$ with linearly reductive stabilizer G_x, the coherent completion $\widehat{\mathcal{X}}_x$ exists. Moreover,*

(1) The coherent completion is a quotient stack $\widehat{\mathcal{X}}_x = [\operatorname{Spec} B/G_x]$ such that the invariant ring B^{G_x} is the completion of a finite type $\Bbbk$-algebra and $B^{G_x} \to B$ is of finite type.

(2) Let $f\colon (\mathcal{W}, w) \to (\mathcal{X}, x)$ be an étale morphism where $\mathcal{W} = [\operatorname{Spec} A/G_x]$, the point $w \in |\mathcal{W}|$ is closed, and f induces an isomorphism of stabilizer groups at w. Then $\widehat{\mathcal{X}}_x = \mathcal{W} \times_W \operatorname{Spec} \widehat{\mathcal{O}}_{W,\pi(w)}$, where $\pi\colon \mathcal{W} \to W = \operatorname{Spec} A^{G_x}$ is the morphism to the GIT quotient.

(3) If $\pi\colon \mathcal{X} \to X$ is a good moduli space, then $\widehat{\mathcal{X}}_x = \mathcal{X} \times_X \operatorname{Spec} \widehat{\mathcal{O}}_{X,\pi(x)}$.

Proof. The Local Structure Theorem (6.5.1) gives an étale morphism $f\colon (\mathcal{W}, w) \to (\mathcal{X}, x)$, where $\mathcal{W} = [\operatorname{Spec} A/G_x]$ and f induces an isomorphism of stabilizer groups at the closed point w. The main statement and Parts (1) and (2) follow by taking $\widehat{\mathcal{X}}_x = \mathcal{W} \times_W \operatorname{Spec} \widehat{\mathcal{O}}_{W,\pi(w)}$ and $B = A \otimes_{A^{G_x}} \widehat{A^{G_x}}$. Indeed, $\widehat{\mathcal{X}}_x = [\operatorname{Spec} B/G_x]$ is coherently complete by Theorem 6.4.11. Part (3) follows from (2) using Corollary 6.5.3. $\square$

We have the following stacky generalization of the fact that completions determine the étale local structure of finite type schemes (Corollary A.10.13).

Theorem 6.5.17. *Let $\mathcal{X}$ and $\mathcal{Y}$ be algebraic stack of finite type over an algebraically closed field $\Bbbk$ with affine diagonal. Suppose $x \in \mathcal{X}$ and $y \in \mathcal{Y}$ are $\Bbbk$-points with linearly reductive stabilizer group schemes G_x and G_y, respectively. Then the following are equivalent:*

(1) There exist compatible isomorphisms $\mathcal{X}_n \to \mathcal{Y}_n$.

(2) There exists an isomorphism $\widehat{\mathcal{X}}_x \to \widehat{\mathcal{Y}}_y$.

(3) There exist an affine scheme $\operatorname{Spec} A$ with an action of G_x, a point $w \in \operatorname{Spec} A$ fixed by G_x, and a diagram of étale morphisms

$$\begin{tikzcd} & {[\operatorname{Spec} A/G_x]} \arrow[dl, "f"'] \arrow[dr, "g"] & \\ \mathcal{X} & & \mathcal{Y} \end{tikzcd}$$

such that $f(w) = x$ and $g(w) = y$, and both f and g induce isomorphisms of stabilizer groups at w.

If, in addition, the points $x \in \mathcal{X}$ and $y \in \mathcal{Y}$ are smooth, then the conditions above are equivalent to the existence of an isomorphism $G_x \to G_y$ of group schemes and an isomorphism $T_{\mathcal{X},x} \to T_{\mathcal{Y},y}$ of tangent spaces which is equivariant under $G_x \to G_y$.

Proof. The implications (3) $\implies$ (2) $\implies$ (1) are immediate. We also have (1) $\implies$ (2) by Coherent Tannaka Duality (6.4.8) To show that (2) $\implies$ (3), let $(\mathcal{W} = [\operatorname{Spec} A/G_x], w) \to (\mathcal{X}, x)$ be an étale neighborhood as given by the Local Structure Theorem (6.5.1). Let $\pi\colon \mathcal{W} \to W = \operatorname{Spec} A^{G_x}$ denote the good moduli space. Then $\widehat{\mathcal{X}}_x = \mathcal{W} \times_W \operatorname{Spec} \widehat{\mathcal{O}}_{W,\pi(w)}$. The functor

$$F\colon \operatorname{Sch}/W \to \operatorname{Sets}, \quad (T \to W) \mapsto \operatorname{Hom}(\mathcal{W} \times_W T, \mathcal{Y})$$

is locally of finite presentation. Artin Approximation (A.10.9) applied to F and $\alpha \in F(\operatorname{Spec} \widehat{\mathcal{O}}_{W,\pi(w)})$ provides an étale morphism $(W', w') \to (W, w)$ and a morphism $\varphi\colon \mathcal{W}' := \mathcal{W} \times_W W' \to \mathcal{Y}$ such that $\varphi|_{\mathcal{W}'_1}\colon \mathcal{W}'_1 \to \mathcal{Y}_1$ is an isomorphism. Since $\widehat{\mathcal{W}'}_{w'} \cong \widehat{\mathcal{X}}_x \cong \widehat{\mathcal{Y}}_y$, it follows that φ induces an isomorphism $\widehat{\mathcal{W}'} \to \widehat{\mathcal{Y}}$ by Lemma 6.5.12. After replacing W' with an open neighborhood we thus obtain an étale morphism $(\mathcal{W}', w') \to (\mathcal{Y}, y)$. The final statement is clear from Luna's Etale Slice Theorem (6.5.4). □

6.5.7 Applications to equivariant geometry

Sumihiro's Theorem on Torus Action (C.3.4) asserts that for a normal scheme of finite type over $\Bbbk$ with the action of a torus T, every $\Bbbk$-point has a T-invariant affine open neighborhood. If X is not normal, there are not necessarily T-invariant affine open neighborhoods, e.g. consider nodal cubic X equipped with a $\mathbb{G}_m$-action near its node $x \in X$. However, there is always a T-equivariant affine *étale* neighborhood.

Theorem 6.5.18. *Let X be an algebraic space locally of finite type over an algebraically closed field $\Bbbk$ with affine diagonal. Suppose that X has an action of an affine algebraic group G. If $x \in X(\Bbbk)$ has linearly reductive stabilizer, then there exist a G-equivariant étale neighborhood $(\operatorname{Spec} A, u) \to (X, x)$ inducing an isomorphism of stabilizer groups at u.*

If G is a torus, then every point has a G-invariant étale neighborhood $(\operatorname{Spec} A, u) \to (X, x)$ inducing an isomorphism of stabilizer groups at u.

Proof. By the Local Structure Theorem (6.5.1), there is an étale neighborhood $([\operatorname{Spec} A/G_x], u) \to ([X/G], x)$ such that w is a closed point and f induces an isomorphism of stabilizer groups at w. By Proposition 6.5.13, after replacing $\operatorname{Spec} A$ with a G_x-invariant open affine neighborhood of w, we can arrange that the composition $[\operatorname{Spec} A/G_x] \to [X/G] \to \mathbf{B}G$ is affine. Therefore, $W := [\operatorname{Spec} A/G_x] \times_{[X/G]} X$ is an affine scheme and $W \to X$ is a G-equivariant étale neighborhood of x.

When G is a torus, then any subgroup of G and in particular each stabilizer group is linearly reductive. □

6.6 Geometric Invariant Theory (GIT)

Geometric Invariant Theory (GIT) was developed by Mumford in [GIT] as a means to construct quotients and moduli spaces in algebraic geometry. For other expository accounts, we recommend [New78], [Kra84], [Dol03], [Muk03] and [Stu08].

6.6.1 Good quotients

Let G be an affine algebraic group over an algebraically closed field $\Bbbk$ acting on an algebraic space U of finite type over $\Bbbk$. In the following cases, we've already established the existence of a *geometric quotient* U/G (Definition 4.3.1), i.e. a G-invariant map $U \to U/G$ inducing a bijection $U(\Bbbk)/G(\Bbbk) \to (U/G)(\Bbbk)$ and universal for G-invariant maps to algebraic spaces; in other words $[U/G] \to U/G$ is a coarse moduli space.

- If G is a (reduced) finite group and the action is free (i.e. the action map $G \times U \to U \times U$ is a monomorphism), then $U/G := [U/G]$ exists as an algebraic space of finite type over $\Bbbk$ (Corollary 3.1.12). This also holds in the non-finite case: if G is an algebraic group and the action is free, then $[U/G]$ is an algebraic stack (Proposition 6.2.9) such that $[U/G] \to [U/G] \times [U/G]$ is a monomorphism and therefore $U/G := [U/G]$ is an algebraic space (Theorem 4.4.10).
- If G is finite and $U = \operatorname{Spec} A$ is affine, then $U/G := \operatorname{Spec} A^G$ is a geometric quotient (Theorem 4.3.6).
- If G is finite and U is projective (resp. quasi-projective, quasi-affine), then the quotient U/G exists as a projective (resp. quasi-projective, quasi-affine) $\Bbbk$-scheme (Exercise 4.2.8).
- If G is finite and U is separated, then U/G exists as a separated algebraic space as a consequence of the Keel–Mori Theorem (4.3.11). This also holds in the non-finite case: if G is an affine algebraic group, the stabilizers of the action are finite and reduced, and the action map $G \times U \to U \times U$ is proper, then $[U/G]$ is a separated Deligne–Mumford stack (Theorem 3.6.4) and the existence of a geometric quotient follows from the Keel–Mori Theorem.

GIT studies the case where G is linearly reductive[3] but not necessarily finite. GIT allows for the possibility of points $u \in U$ where the stabilizer G_u may not be finite and the orbit Gu may not be closed, e.g. $\mathbb{G}_m$ acting on $\mathbb{A}^1$.

In Corollary 6.3.7, we've already considered the affine case of GIT where G is a linearly reductive algebraic group over an algebraically closed field $\Bbbk$ acting on an affine $\Bbbk$-scheme $\operatorname{Spec} A$. In this case, we have a commutative diagram

$$\begin{tikzcd}
\operatorname{Spec} A \arrow[d] \arrow[dr, "\widetilde{\pi}"] & \\
{[\operatorname{Spec} A/G]} \arrow[r, "\pi"] & (\operatorname{Spec} A)/\!\!/G := \operatorname{Spec} A^G
\end{tikzcd}$$

where $\pi\colon [\operatorname{Spec} A/G] \to \operatorname{Spec} A^G$ is a good moduli space and $\widetilde{\pi}\colon \operatorname{Spec} A \to \operatorname{Spec} A^G$ is a *good quotient*.

Definition 6.6.1 (Good quotients)**.** Given an action of a linearly reductive algebraic group G over an algebraically closed field $\Bbbk$ on an algebraic space U over $\Bbbk$, a G-invariant map $\widetilde{\pi}\colon U \to X$ is a *good quotient* if

(1) $\mathcal{O}_X \to (\pi_* \mathcal{O}_U)^G$ is an isomorphism (where $(\pi_*\mathcal{O}_U)^G(V) = \Gamma(U_V, \mathcal{O}_{U_V})^G$ for an étale X-scheme V) and

[3] GIT can be developed in the more general setting of actions by *reductive* algebraic groups; see Remark 6.3.10.

(2) $\widetilde{\pi}$ is affine.[4]

The good quotient of U by G is often denoted as $U/\!\!/G = X$.

Remark 6.6.2. The map $\widetilde{\pi}\colon U \to X$ is a good quotient if and only if $\pi\colon [U/G] \to X$ is a good moduli space. To see the equivalence, we may assume that $X = \operatorname{Spec} B$ is affine since both properties are étale local (Lemma 6.3.20(1)). For ($\Rightarrow$), $U = \operatorname{Spec} A$ is also affine and $B = A^G$, and thus $[\operatorname{Spec} A/G] \to \operatorname{Spec} A^G$ is a good moduli space. To see ($\Leftarrow$), observe that since $U \to [U/G]$ is affine and π_* is exact on quasi-coherent sheaves, the pushforward $\widetilde{\pi}_*$ is exact on quasi-coherent sheaves and thus $\widetilde{\pi}$ is affine by Serre's Criterion for Affineness (4.4.15).

Proposition 6.6.3. *Let G be a linearly reductive algebraic group over an algebraically closed field $\Bbbk$ acting on an algebraic space U over $\Bbbk$. If $\widetilde{\pi}\colon U \to X$ is a good quotient, then*

(1) $\widetilde{\pi}$ is surjective and the image of a closed G-invariant subscheme is closed. The same holds for the base change $T \to X$ by a morphism from a scheme;

(2) for closed G-invariant closed subschemes $Z_1, Z_2 \subset U$, $\operatorname{im}(Z_1 \cap Z_2) = \operatorname{im}(Z_1) \cap \operatorname{im}(Z_2)$. In particular, for $x_1, x_2 \in X(\Bbbk)$, $\widetilde{\pi}(x_1) = \widetilde{\pi}(x_2)$ if and only if $\overline{Gx_1} \cap \overline{Gx_2} \neq \emptyset$, and $\widetilde{\pi}$ induces a bijection between closed G-orbits in U and $\Bbbk$-points of X;

(3) if U is noetherian, so is X. If U is finite type over $\Bbbk$, then so is X, and for every coherent $\mathcal{O}_U$-module F with a G-action, $(\pi_* F)^G$ is coherent; and

(4) $\widetilde{\pi}$ is universal for G-invariant maps to algebraic spaces.

Proof. This follows from Theorem 6.3.5 as $[U/G] \to X$ is a good moduli space. $\square$

Remark 6.6.4 (Semistable reduction in GIT)**.** Since $[U/G] \to X$ is universally closed (Theorem 6.3.5(1)), it satisfies the valuative criterion for universally closedness (Theorem 3.8.5). This translates into the following: for every DVR R over $\Bbbk$ with fraction field K and every map $\operatorname{Spec} R \to X$ with a lift $\eta\colon \operatorname{Spec} K \to U$, there exist an extension $R \to R'$ of DVRs, an element $g' \in G(K')$ over the fraction field of R', and a lift in the commutative diagram

$$\begin{tikzcd}
\operatorname{Spec} K' \arrow[d] \arrow[rr, bend left, "g'\cdot\eta|_{K'}"] & \operatorname{Spec} K \arrow[r, "\eta"] \arrow[d] & U \arrow[d, "\widetilde{\pi}"] \\
\operatorname{Spec} R' \arrow[r] \arrow[urr, dashed] & \operatorname{Spec} R \arrow[r] & X.
\end{tikzcd}$$

In fact, if $R = \Bbbk[\![x]\!]$, it can be arranged that $R \to R'$ is finite; see [Mum77, Lem. 5.3] and [AHLH18, Thm. A.8].

6.6.2 Projective GIT

Let U be a projective scheme over an algebraically closed field $\Bbbk$ with an action of a linearly reductive algebraic group G. Suppose that there is a G-equivariant embedding $U \hookrightarrow \mathbb{P}(V)$, where V is a finite dimension G-representation; this is

[4] A *good quotient* is sometimes defined as an affine G-invariant morphism $\widetilde{\pi}\colon U \to X$ such that $\mathcal{O}_X \xrightarrow{\sim} (\pi_* \mathcal{O}_U)^G$ and properties Proposition 6.6.3(1)-(2) holds, c.f. [Ses72, Def. 1.5].

equivalent to giving a very ample line bundle $\mathcal{O}_U(1)$ with a G-action, i.e. a very ample G-linearization (see §C.3.2).

Definition 6.6.5. We define the *semistable* and *stable* locus as

$$U^{\mathrm{ss}} := \{u \in U \mid \text{there exists } f \in \Gamma(U, \mathcal{O}_U(d))^G \text{ with } d > 0 \text{ such that } f(u) \neq 0\},$$

$$U^{\mathrm{s}} := \left\{ u \in U \;\middle|\; \begin{array}{l} \text{there exists } f \in \Gamma(U, \mathcal{O}_U(d))^G \text{ with } d > 0 \text{ such that} \\ \quad -f(u) \neq 0, \\ \quad -\text{the orbit } Gu \subset U_f \text{ is closed, and} \\ \quad -\text{the function } U \to \mathbb{Z},\ x \mapsto \dim G_x \text{ is constant in an open} \\ \qquad \text{neighborhood of } u^{5} \end{array} \right\}.$$

A point $u \in U$ is called *semistable* (resp. *stable*) if $u \in U^{\mathrm{ss}}$ (resp. $u \in U^{\mathrm{s}}$).[6] The *nullcone* $\widehat{N} \subset \mathbb{A}(V)$ is by definition the affine cone over $U \setminus U^{\mathrm{ss}}$: it is set of points u in the affine cone $\widehat{U} \subset \mathbb{A}(V)$ such that $f(u) = 0$ for every non-constant G-invariant polynomial on $\mathbb{A}(V)$.

We stress that the stable and semistable loci depend on the choice of G-equivariant embedding $U \hookrightarrow \mathbb{P}(V)$. When U is a normal projective variety, then every line bundle L has a positive tensor power $L^{\otimes n}$ that has a G-linearization by Sumihiro's Theorem on Linearizations (C.3.3). For example, $\mathcal{O}(1)$ on $\mathbb{P}^n$ does not have a PGL_{n+1}-linearization, but $\mathcal{O}(n+1)$ does.

Let $R = \bigoplus_{d \geq 0} \Gamma(U, \mathcal{O}_U(d))$ be the projective coordinate ring. We consider the map

$$\widetilde{\pi} \colon U^{\mathrm{ss}} \to U^{\mathrm{ss}} /\!\!/ G := \operatorname{Proj} R^G. \tag{6.6.1}$$

Note that U^{ss} may be empty in which case $\operatorname{Proj} R^G$ is the empty scheme. If U^{ss} is non-empty, it is precisely the locus where the rational map $\operatorname{Proj} R \dashrightarrow \operatorname{Proj} R^G$ is defined.

Theorem 6.6.6. *Let G be a linearly reductive algebraic group over an algebraically closed field $\Bbbk$. Let $U \subset \mathbb{P}(V)$ be a G-equivariant closed subscheme where V is a finite dimension G-representation. Then there is a cartesian diagram*

$$\begin{tikzcd} U^{\mathrm{s}} \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & U^{\mathrm{ss}} \arrow[r, hook] \arrow[d, "\widetilde{\pi}"] & U \\ U^{\mathrm{s}}/G \arrow[r, hook] & U^{\mathrm{ss}} /\!\!/ G & \end{tikzcd}$$

where $U^{\mathrm{s}}/G \subset U^{\mathrm{ss}} /\!\!/ G$ is an open subscheme, the map $\widetilde{\pi}$ of (6.6.1) is a good quotient, and the restriction $\widetilde{\pi}|_{U^{\mathrm{s}}} \colon U^{\mathrm{s}} \to U^{\mathrm{s}}/G$ is a geometric quotient. Moreover, $U^{\mathrm{ss}} /\!\!/ G$ is projective with an ample line bundle L such that $\widetilde{\pi}^* L \cong \mathcal{O}_U(N)$ for some N.

If in addition the action of G on U has generically finite stabilizers, then the action of G on U^{s} is proper (i.e. the action map $G \times U^{\mathrm{s}} \to U^{\mathrm{s}} \times U^{\mathrm{s}}$ is proper) or in other words $[U^{\mathrm{s}}/G]$ is separated.

[5] Since the function $x \mapsto \dim G_x$ is upper semi-continuous, this condition is automatic if $\dim G_u = 0$.

[6] In the literature, a point $u \in U$ is sometimes called 'unstable' if it is not semistable; we avoid this potentially misleading terminology.

Proof. Since U is projective, $R = \bigoplus_{d \geq 0} \Gamma(U, \mathcal{O}_U(d))$ is finitely generated over $\Bbbk$. Thus by Corollary 6.3.7(3), R^G is also finitely generated over $\Bbbk$ and $U /\!\!/ G = \operatorname{Proj} R^G$ is projective. As localization commutes with taking invariants, $(R^G)_{(f)} = (R_{(f)})^G$ for every homogeneous element $f \in R^G$ of positive degree. We thus have a cartesian diagram

$$\begin{tikzcd} U_f = \operatorname{Spec} R_{(f)} \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & U^{\mathrm{ss}} \arrow[r, hook] \arrow[d, "\widetilde{\pi}"] & U \\ U_f /\!\!/ G = (U /\!\!/ G)_f \arrow[r, hook] & U /\!\!/ G. \end{tikzcd}$$

Since the property of being a good quotient is Zariski local and since the loci $(U/\!\!/G)_f$ cover $U/\!\!/G$, we conclude that $\widetilde{\pi}\colon U^{\mathrm{ss}} \to U/\!\!/G$ is a good quotient. By construction, $U^{\mathrm{ss}}/\!\!/G$ is projective and there is an integer N such that $L := \mathcal{O}_{U/\!\!/G}(N)$ is an ample line bundle which pulls back to $\mathcal{O}_U(N)|_{U^{\mathrm{ss}}}$.

To show that $U^{\mathrm{s}} \to U^{\mathrm{s}}/G$ is a geometric quotient, it suffices to show that every G-orbit in U^{s} is closed. Since the dimension of the stabilizer increases under orbit degeneration, it in fact suffices to show that the dimension of the stabilizers in U^{s} is locally constant. Every point $u \in U^{\mathrm{s}}$ has by definition an open neighborhood $V \subset U$ such that $\dim G_v = \dim G_u$ for all $v \in V$. Since $\dim G = \dim G_v + \dim Gv$, we see that the dimension of the orbit is constant on V. Finally, if there is a dense open subset of U which has dimension 0 stabilizers, then it follows from the definition of stability that every $u \in U^{\mathrm{s}}$ has a finite (possibly non-reduced) stabilizer. Since $[U^{\mathrm{s}}/G] \to U^s/G$ is also a good moduli space and $[U^{\mathrm{s}}/G]$ has quasi-finite diagonal, it follows from Proposition 6.3.28 that $[U^{\mathrm{s}}/G]$ is separated. ∎

Example 6.6.7. Given $\mathbb{G}_m$ acting on $\mathbb{P}^2$ via $t \cdot [x : y : z] = [tx : t^{-1}y : z]$, the semistable locus is the complement of $V(xy, z) = \{[0 : 1 : 0], [1 : 0 : 0]\}$ and the good quotient is $(\mathbb{P}^2)^{\mathrm{ss}} \to \operatorname{Proj} \Bbbk[xy, z] = \mathbb{P}^1$. The fiber over $xy = 0$ is the union of three orbits and its complement is the stable locus. Observe that the restriction to $z \neq 0$ is the good quotient $\mathbb{A}^2 \to \mathbb{A}^1$ taking $(x, y) \mapsto xy$ while the fiber over $z = 0$ is the line at infinity with $[0 : 1 : 0]$ and $[1 : 0 : 0]$ removed.

Example 6.6.8. Consider the diagonal action of SL_2 on $X = (\mathbb{P}^1)^4$ and the SL_2-equivariant Segre embedding

$$(\mathbb{P}^1)^4 \to \mathbb{P}^{15}, \quad ([x_1 : y_1], \dots, [x_4 : y_4]) \mapsto [x_1x_2x_3x_4, \dots, y_1y_2y_3y_4].$$

This corresponds to the SL_2-linearization of $L := \mathcal{O}(1) \boxtimes \cdots \boxtimes \mathcal{O}(1)$. The invariant ring $\bigoplus_{d \geq 0} \Gamma(X, L^{\otimes d})$ is generated in degree 1 by the *generalized cross ratios*

$$\begin{aligned} I_1 &= (x_1y_2 - x_2y_1)(x_3y_4 - x_4y_3) \\ I_2 &= (x_1y_3 - x_3y_1)(x_2y_4 - x_4y_2) \\ I_3 &= (x_1y_4 - x_4y_1)(x_2y_3 - x_3y_2) \end{aligned}$$

with the linear relation $I_1 - I_2 + I_3 = 0$. The invariant ring is $\Bbbk[I_1, I_2]$ and the quotient $X^{\mathrm{ss}} /\!\!/ \mathrm{SL}_2 = \mathbb{P}^1$. The semistable locus X^{ss} consists of tuples where at most 2 points are equal while the stable locus consists of tuples of distinct points.

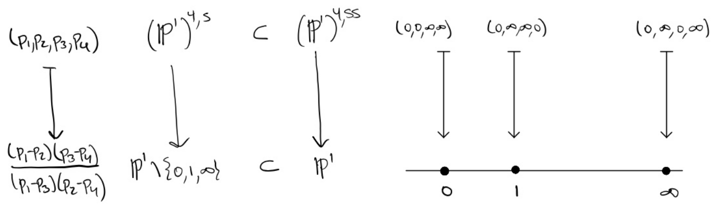


Figure 6.1: 4 unordered points up to projective equivalence

An ordered tuple $(p_1, \ldots, p_4)$ of distinct points is mapped to the *cross ratio*

$$\frac{(p_1 - p_2)(p_3 - p_4)}{(p_1 - p_3)(p_2 - p_4)}.$$

In particular, two stable tuples are projectively equivalent (i.e. in the same SL_2 orbit) if and only if they have the same cross ratio. The complement $X^{\mathrm{ss}} \setminus X^{\mathrm{s}}$ contains 3 closed orbits: the SL_2-orbits of $(0, 0, \infty, \infty)$, $(0, \infty, 0, \infty)$, and $(0, \infty, \infty, 0)$. Tuples such as $(0, 0, 1, \infty)$ or $(1, \infty, 0, 0)$ have non-closed SL_2-orbits in X^{ss} with $\mathrm{SL}_2 \cdot (0, 0, \infty, \infty)$ in the orbit closure. See Example 6.6.27 to see the computations of the stable and semistable locus for the more general case of n ordered points in $\mathbb{P}^1$.

Remark 6.6.9 (Symplectic reduction)**.** There is an interesting connection between GIT and symplectic geometry. Let G is reductive algebraic group over $\mathbb{C}$ acting on a smooth projective variety $U \subset \mathbb{P}(V)$ where V is a $n+1$ dimensional G representation. Let ω be a symplectic form on U, and let $K \subset G$ be a maximal compact subgroup K and $\mathfrak{k}$ its Lie algebra. There is a *moment map*

$$\mu \colon U \to \mathfrak{k}^\vee$$

which is K-equivariant with respect to the coadjoint action on $\mathfrak{k}^\vee$ and satisfies $d\mu(x)(\xi) \cdot a = \omega_x(\xi, v_x)$ for $u \in U$, $\xi \in T_x U$, and $a \in \mathfrak{k}$, where v_x is the vector field on U obtained by the infinitesimal action of K on U. Then

$$u \in U \text{ is semistable} \iff \overline{Gu} \cap \mu^{-1}(0) \neq \emptyset$$

and the inclusion $\mu^{-1}(0) \hookrightarrow U$ induces a homeomorphism $\mu^{-1}(0)/K \to U/\!\!/G$. See [MFK94, §8].

Exercise 6.6.10 (Affine GIT with respect to a character)**.** Let $U = \operatorname{Spec} A$ be a finite type scheme over an algebraically closed field $\Bbbk$ with an action of an affine algebraic group G specified by a coaction $\sigma \colon A \to \Gamma(G, \mathcal{O}_G) \otimes A$. Let $\chi \colon G \to \mathbb{G}_m = \operatorname{Spec} \Bbbk[t]_t$ be a character. Define the *semistable* and *stable* locus as

$$U^{\mathrm{ss}} := \left\{ u \in U \;\middle|\; \begin{array}{l} \text{there exists } f \in A \text{ such that } f(u) \neq 0 \\ \text{and } \sigma(f) = \chi^*(t)^d \otimes f \text{ for } d > 0 \end{array} \right\}$$

$$U^{\mathrm{s}} := \left\{ u \in U \;\middle|\; \begin{array}{l} \text{there exists } f \in A \text{ such that } f(u) \neq 0,\ \sigma(f) = \chi^*(t)^d \otimes f \\ \text{for } d > 0, \text{ the orbit } Gu \subset U_f \text{ is closed, and the function} \\ x \mapsto \dim G_x \text{ is constant in an open neighborhood of } u \end{array} \right\}.$$

Defining $U^{\mathrm{ss}}/\!\!/G := \operatorname{Proj} \bigoplus_{d \geq 0} A_d$ where $A_d = \{f \in A \mid \sigma(f) = \chi^*(t)^d \otimes f\}$, show that the conclusion of Theorem 6.6.6 holds except that $U^{\mathrm{ss}}/\!\!/G$ is projective over $A^G = A_0$ (rather than $\Bbbk$).

For example, under the scaling $\mathbb{G}_m$-action on $U = \mathbb{A}^n$ and with respect to the the identity character $\chi = \mathrm{id}$, then $U^{\mathrm{ss}} = U^{\mathrm{s}} = \mathbb{A}^n$ and the quotient is $\mathbb{P}^{n-1}$.

Exercise 6.6.11 (Projective GIT over an affine)**.** Let U be a projective scheme over a finitely generated $\Bbbk$-algebra B, where $\Bbbk$ is an algebraically closed field, and let G be an affine algebraic group acting on U. Suppose that there is a G-equivariant embedding $U \hookrightarrow \mathbb{P}_R(\mathcal{E})$, where $\mathcal{E}$ is a vector bundle with a G-action. Defining the semistable locus U^{ss} and stable locus U^{s} exactly as in Definition 6.6.5, show that the conclusion of Theorem 6.6.6 holds except that $U^{\mathrm{ss}}/\!\!/G$ is projective over B^G (rather than $\Bbbk$).

6.6.3 One-parameter subgroups and limits

If G is an algebraic group over a field $\Bbbk$, a *one-parameter subgroup* is a homomorphism $\lambda\colon \mathbb{G}_m \to G$ of algebraic groups (that is not required to be injective). If U is a separated algebraic space over $\Bbbk$ with an action of G and $u \in U(\Bbbk)$, then we say that the limit $\lim_{t\to 0} \lambda(t)\cdot u$ exists if there exists an extension of the diagram

$$\begin{tikzcd} \mathbb{G}_m \arrow[r, "t\mapsto\lambda(t)\cdot u"] \arrow[d, hook] & U \\ \mathbb{A}^1 \arrow[ur, dashed] & \end{tikzcd}$$

Since U is separated, the limit is unique if it exists. If U is proper, then a limit necessarily exists. For example, if $U = \mathbb{P}(V)$ where V is a finite dimensional representation and λ is a one-parameter subgroup, then we can choose a basis of V such that $\lambda(t)\cdot(v_1,\dots,v_n) = (t^{d_1}v_1,\dots,t^{d_n}v_n)$ with $d_1 \le \dots \le d_n$. If $d = \min\{d_i \mid v_i \neq 0\}$, then $\lim_{t\to 0}\lambda(t)\cdot[v_0 : \ldots : v_n] = [v_0' : \ldots : v_n']$ where $v_i' = v_i$ for all i such that $d_i = d$ and is 0 otherwise.

Given a point $u \in U$ such that $\lim_{t\to 0}\lambda(t)\cdot u \in U$ exists, the $\mathbb{G}_m$-equivariant extension $\mathbb{A}^1 \to U$ induces a morphism $[\mathbb{A}^1/\mathbb{G}_m] \to [U/G]$ of algebraic stacks. The next proposition asserts that the converse is also true, i.e. such limits characterize morphism $[\mathbb{A}^1/\mathbb{G}_m] \to [U/G]$. The most important case below is when R is the algebraically closed field $\Bbbk$ and the reader is encouraged to keep this case in mind; it is stated more generally for future use.

Proposition 6.6.12. *Let G be a smooth affine algebraic group over an algebraically closed field $\Bbbk$, and let U be a separated algebraic space of finite type over $\Bbbk$. For every complete noetherian local $\Bbbk$-algebra R with algebraically closed residue field κ, $\mathrm{MOR}_{\Bbbk}(\Theta_R, [U/G])$ is equivalent to the groupoid of pairs (u,λ) consisting of a point $u \in U(R)$ and a one-parameter subgroup $\lambda\colon \mathbb{G}_m \to G$ such that $\lim_{t\to 0}\lambda(t)\cdot u \in U(R)$ exists. A morphism $(u,\lambda) \to (u',\lambda')$ is an element $g \in G(R)$ such that $u' = g\cdot u$, $\lambda' = g\lambda g^{-1}$ and $\lim_{t\to 0}\lambda(t)g\lambda(t)^{-1} \in G(R)$ exists.*

Under this correspondence, the morphism $\Theta_R \to [U/G]$ sends 1 to u and 0 to $\lim_{t\to 0}\lambda(t)\cdot u$.

Remark 6.6.13. Above the existence of $\lim_{t\to 0}\lambda(t)\cdot u \in U(R)$ means that the map $\mathbb{G}_{m,R} \to U$, defined by $t \mapsto \lambda(t)\cdot u$, extends to a map $\mathbb{A}^1_R \to U$. The extension is unique as U is separated.

Proof. Given (u,λ), the $\mathbb{G}_m$-equivariant map $m_{u,\lambda}\colon \mathbb{G}_{m,R} \to U$ defined by $t \mapsto \lambda(t)\cdot u$ extends to a $\mathbb{G}_{m,R}$-equivariant map $\widetilde{m}_{u,\lambda}\colon \mathbb{A}^1_R \to U$; this induces a morphism of quotient prestacks $f^{\mathrm{pre}}_{u,\lambda}\colon [\mathbb{A}^1_R/\mathbb{G}_m] \to [U/G]$ and a morphism of quotient stacks $f_{u,\lambda}\colon \Theta_R \to [U/G]$. A 2-isomorphism $\alpha\colon f^{\mathrm{pre}}_{u,\lambda} \to f^{\mathrm{pre}}_{u',\lambda'}$ corresponds to an element $\Gamma \in G(\mathbb{A}^1_R)$ such that $\widetilde{m}_{u',\lambda'} = \Gamma \cdot \widetilde{m}_{u,\lambda} \in U(\mathbb{A}^1_R)$ and such that for every $t \in \mathbb{G}_m(\mathbb{A}^1_R)$, with $m_t\colon \mathbb{A}^1_R \to \mathbb{A}^1_R$ denoting multiplication by t, $\lambda'(t)\Gamma = (\Gamma \circ m_t)\lambda(t) \in G(\mathbb{A}^1_R)$. On points, the second condition asserts that for $p \in \mathbb{A}^1_R$, $\Gamma(t(p)p) = \lambda'(t(p))\Gamma(p)\lambda(t(p))^{-1}$. We therefore see that Γ is determined by the element $g := \Gamma(1) \in G(R)$, evaluating at $1\colon \operatorname{Spec} R \to \mathbb{A}^1$, such that the map $\mathbb{G}_{m,R} \to G$, taking t to $\lambda'(t)g\lambda(t)^{-1}$, extends to $\mathbb{A}^1_R \to G$.

We thus obtain a fully faithful functor

$$\{(u,\lambda) \mid \lim_{t\to 0} \lambda(t)\cdot u \text{ exists}\} \to \mathrm{MOR}_{\Bbbk}(\Theta_R, [U/G])$$
$$(u,\lambda) \mapsto f_{u,\lambda}.$$

To see essential surjectivity, let $f\colon \Theta_R \to [U/G]$ be a morphism. In the fiber diagram

$$\begin{tikzcd} \mathcal{P} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & U \arrow[d] \\ \Theta_R \arrow[r, "f"] & {[U/G]} \end{tikzcd}$$

$\mathcal{P} \to \Theta_R$ is a principal G-bundle. The restriction $\mathcal{P}|_{\mathbf{B}\mathbb{G}_{m,\kappa}}$ along the unique closed point $0\colon \mathbf{B}\mathbb{G}_{m,\kappa} \to \Theta_R$, corresponds to a $\mathbb{G}_m$-equivariant principal G-bundle P on $\operatorname{Spec}\kappa$. After choosing an isomorphism $P \cong G$, we see that $\mathcal{P}$ corresponds to a one-parameter subgroup $\lambda'\colon \mathbb{G}_{m,\kappa} \to G_\kappa$. The one-parameter subgroup λ' is contained in some maximal torus T' of G_κ and all maximal tori are conjugate (C.3.1(8)-(9)) as κ is algebraically closed. It follows that λ' is conjugate to the base change of a one-parameter subgroup $\lambda\colon \mathbb{G}_m \to G$. On the other hand, every such λ induces a principal G-bundle $\mathcal{P}_\lambda := [(\mathbb{A}^1_R \times G)/\mathbb{G}_m]$ over Θ_R. We claim that there is an isomorphism $\alpha\colon \mathcal{P} \to \mathcal{P}_\lambda$ of principal G-bundles. By construction, we have an isomorphism $\alpha_0\colon \mathcal{P}|_{\mathbf{B}\mathbb{G}_{m,\kappa}} \to \mathcal{P}_\lambda|_{\mathbf{B}\mathbb{G}_{m,\kappa}}$. Since $\underline{\mathrm{Isom}}_{\Theta_R}(\mathcal{P},\mathcal{P}_\lambda) \to \Theta_R$ is smooth (as it's a principal G-bundle and G is smooth), we may use deformation theory (Proposition 6.5.8) to construct compatible isomorphisms $\alpha_n\colon \mathcal{P}|_{\mathcal{X}_n} \to \mathcal{P}_\lambda|_{\mathcal{X}_n}$ over the nilpotent thickenings $\mathcal{X}_n$ of $0\colon \mathbf{B}\mathbb{G}_{m,\kappa} \hookrightarrow \Theta_R$. Coherent Tannaka Duality (6.4.8) coupled with the coherent completeness of Θ_R along $\mathbf{B}\mathbb{G}_{m,\kappa}$ (Theorem 6.4.11) implies that the isomorphisms α_n extend to an isomorphism $\alpha\colon \mathcal{P} \to \mathcal{P}_\lambda$.

Restricting the composition $\mathbb{A}^1_R \times G \to \mathcal{P}_\lambda \xrightarrow{\alpha^{-1}} \mathcal{P} \to U$ to the identity in G yields a $\mathbb{G}_m$-equivariant morphism $\mathbb{A}^1_R \to U$. One checks that this corresponds to the given map $f\colon \Theta_R \to [U/G]$ on quotient stacks. Letting $u \in U(R)$ be the image of 1, we see that $f_{u,\lambda}$ is 2-isomorphic to f. $\square$

Remark 6.6.14. Proposition 6.6.12 can be upgraded to a description of the stack of morphisms from $[\mathbb{A}^1/\mathbb{G}_m]$ to $[U/G]$. Namely, there is a decomposition

$$\underline{\mathrm{Mor}}([\mathbb{A}^1/\mathbb{G}_m], [U/\mathbb{G}_m]) \cong \coprod_\lambda [U^+_\lambda/P_\lambda]$$

where λ varies over conjugacy classes of one-parameter subgroups. See [HL14, Thm. 1.37].

6.6.4 Cartan Decompositions

It is frequently intractable to show that a point $u \in U$ is semistable by explicitly exhibiting an invariant section $s \in \Gamma(U, \mathcal{O}_U(d))^G$ not vanishing at u. Fortunately, there is an alternative—the Hilbert–Mumford Criterion (6.6.23)—which often reduces the question of whether a point is semistable into a combinatorial question.

The key algebraic input in the proof of the Hilbert–Mumford Criterion is the Cartan Decomposition, sometimes known as the Iwahori decomposition or the Cartan–Iwahori–Matsumoto Decomposition. Given a one-parameter subgroup $\lambda \colon \mathbb{G}_m \to G$, and a DVR R with fraction field K, we denote by $\lambda|_K$ the image of the composition $\operatorname{Spec} K \to \mathbb{G}_m \xrightarrow{\lambda} G$ where the first map is defined by the $\Bbbk$-algebra map $\Bbbk[t]_t \to K$ taking t to a uniformizer in R.

Theorem 6.6.15 (Cartan Decomposition)**.** *Let G be a reductive*[7] *Let R be a complete DVR over $\Bbbk$ with residue field $\Bbbk$ and fraction field K. Then for every element $g \in G(K)$, there exists $h_1, h_2 \in G(R)$ and a one-parameter subgroup $\lambda \colon \mathbb{G}_m \to G$ such that*

$$g = h_1 \lambda|_K h_2.$$

Proof. As our proof will utilize that $\mathbf{B}G$ is S-complete (Definition 6.7.9), a concept developed in the next section, we postpone the proof until Proposition 6.7.48. In fact, we show not only that the theorem holds for reductive groups but that it characterizes reductivity. See also [IM65, Cor. 2.17], [Ses72, Thm. 2.1] and [BT72, §4]. □

Remark 6.6.16 (Equivalent formulation)**.** Let $T \subset G$ be a maximal torus. The above theorem is equivalent to the identity

$$G(K) = G(R)T(K)G(R).$$

For ($\Rightarrow$), choose $h \in G(R)$ such that $h\lambda|_K h^{-1} \in T(K)$. Then

$$g = h_1 \lambda|_K h_2 = \underbrace{(h_1 h^{-1})}_{\in G(R)} \underbrace{(h\lambda|_K h^{-1})}_{\in T(K)} \underbrace{(hh_2)}_{\in G(R)}.$$

Conversely, suppose $g = h_1 t h_2$ for $h_1, h_2 \in G(R)$ and $t \in T(K)$. If we write $T \cong \mathbb{G}_m^r$ and $\pi \in R$ as the uniformizing parameter, then $t = (u_1 \pi^{d_1}, \ldots, u_r \pi^{d_r})$ for units $u_i \in R^\times$ and integers $d_i \in \mathbb{Z}$. After replacing h_1 with $h_1 \cdot (u_1, \ldots, u_r)$, we can write $g = h_1 \lambda|_K h_2$ where $\lambda \colon \mathbb{G}_m \to T \subset G$ is the one-parameter subgroup given by $t \mapsto (t^{d_1}, \ldots, t^{d_r})$.

Remark 6.6.17 (Case of GL_n)**.** The Cartan Decomposition for GL_n can be established by an elementary linear algebra argument. Let $g = (g_{ij}) \in \mathrm{GL}_n(K)$. After performing row and column operations, we can assume that $g_{1,1} = \pi^d$ has minimal valuation among the g_{ij}, where $\pi \in R$ is a uniformizer. For each $k \geq 2$, we write $g_{1,k} = u\pi^e$. Now perform the row operations where the nth row r_n is exchanged for $r_n - u\pi^{e-d} r_1$. In this way, we can arrange that $g_{1,k} = 0$ for $k \geq 2$ and by performing analogous column operations, we can also arrange that $g_{k,1} = 0$ for $k \geq 2$. The statement is thus established by induction.

Exercise 6.6.18. Let $\Bbbk$ be a field.

[7] While we are mainly focused on developing GIT for linearly reductive groups, it turns out that it is not much more difficult to prove this theorem for reductive groups.

(a) Let $U \subset \mathbb{P}(V)$ be a $\mathbb{G}_m$-equivariant locally closed subscheme where V is a finite dimensional $\mathbb{G}_m$-representation. Show that $[U/\mathbb{G}_m]$ is separated if and only if U has no $\mathbb{G}_m$-fixed points, or in other words that the diagonal $[U/\mathbb{G}_m] \to [U/\mathbb{G}_m] \times [U/\mathbb{G}_m]$ is finite if and only if it is quasi-finite.

(b) Let G be a reductive algebraic group acting on an algebraic space U over $\Bbbk$. Show that $[U/G]$ is separated if and only if for every one-parameter subgroup $\lambda\colon \mathbb{G}_m \to G$, the corresponding quotient stack $[U/\mathbb{G}_m]$ is separated.

Hint: Verify the valuative criterion by applying the Cartan Decomposition.

Remark 6.6.19. Unlike the case of $\mathbb{G}_m$ in (a), it is not true $[U/G]$ is separated for an action of an affine algebraic group G acting linearly on a quasi-projective scheme U with finite stabilizers. See Exercise 3.9.3(d) for such an example by a free action of SL_2 on a quasi-affine variety.

6.6.5 The Destabilization Theorem

Theorem 6.6.20 (Destabilization Theorem). *Let G be a reductive algebraic group over an algebraically closed field $\Bbbk$ acting on an affine scheme U of finite type over $\Bbbk$. Given $u \in U(\Bbbk)$, there exists a one-parameter subgroup $\lambda\colon \mathbb{G}_m \to G$ such that $u_0 := \lim_{t\to 0} \lambda(t) \cdot u$ exists and has closed G-orbit.*

Proof. Let $R = \Bbbk[\![t]\!]$ with fraction field $K = \Bbbk(\!(t)\!)$. We can choose an element $g \in G(K)$ and a commutative diagram

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d, hook] & Gx \arrow[d, hook] \\ \operatorname{Spec} R \arrow[r, "\widetilde{g}"] & X \end{tikzcd}$$

where the top map is given by the composition $\operatorname{Spec} K \xrightarrow{g} G \to Gx$ and such that $y := \widetilde{g}(0) \in Gu_0$. By the Cartan decomposition, there exists $h_1, h_2 \in G(R)$ and a one-parameter subgroup $\lambda\colon \mathbb{G}_m \to G$ such that $h_1 g = \lambda|_K h_2$. By applying the general fact that for $a \in G(R)$ and $b \in X(R)$, $(a \cdot b)(0) = a(0) \cdot b(0)$ to $h_1 \in G(R)$ and $\widetilde{g} \in X(R)$, we obtain that

$$\lim_{t\to 0} \lambda(t) h_2(t) \cdot u = \lim_{t\to 0} h_1(t) g(t) \cdot u = h_1(0) \cdot \widetilde{g}(0) = h_1(0) \cdot y \in Gu_0. \tag{6.6.2}$$

We claim that the related but *possibly different* limit $\lim_{t\to 0} \lambda(t) h_2(0) \cdot u$ exists and is also contained in the closed orbit Gu_0. Once this is established, the theorem would be established by using the one-parameter subgroup $h_2(0)^{-1} \lambda h_2(0)$:

$$\lim_{t\to 0} (h_2(0)^{-1} \lambda h_2(0))(t) \cdot u = h_2^{-1}(0) \cdot \lim_{t\to 0} \lambda(t) h_2(0) \cdot u \in Gu_0.$$

First, to see that $\lim_{t\to 0} \lambda(t) h_2(0) \cdot u$ exists, we may apply Lemma C.3.2(1) below to reduce to the case that $X = \mathbb{A}(V)$ is a G-representation. We may choose a basis of $V \cong \Bbbk^n$ such that action of λ-action has weights $\lambda_1, \ldots, \lambda_n$. We may also write $h_2 \cdot u = (a_1, \ldots, a_n) \in X(R)$ with each $a_i \in \Bbbk[\![t]\!]$ and further decompose $a_i = a_i(0) + a_i'$ with $a_i' \in (t)$. Since

$$\lim_{t\to 0} \lambda(t) h_2(t) \cdot u = \lim_{t\to 0} (t^{\lambda_1}(a_1(0) + a_1'), \ldots, t^{\lambda_n}(a_n(0) + a_n')) \tag{6.6.3}$$

exists, we see that for each i with $\lambda_i < 0$, we must have that $a_i(0) = 0$, which in turn implies that $\lim_{t\to 0} \lambda(t)h_2(0) \cdot u$ exists.

Finally, to see that this limit lies in Gu_0, we may apply Lemma C.3.2(2) to obtain a G-equivariant map $f\colon X \to \mathbb{A}(W)$ such that $f^{-1}(0) = Gu_0$. We are thus reduced to showing that $\lim_{t\to 0} \lambda(t)h_2(0) \cdot f(x) = 0$. By computing the limit $\lim_{t\to 0} \lambda(t)h_2(t) \cdot f(x)$ as in (6.6.3), the same argument shows that since $\lim_{t\to 0} \lambda(t)h_2(t) \cdot f(x) = 0$, we must also have that $\lim_{t\to 0} \lambda(t)h_2(0) \cdot f(x) = 0$. See also [GIT, p. 53] and [Kem78, Thm. 1.4]. □

Given a one-parameter subgroup $\lambda\colon \mathbb{G}_m \to G$ and a point $u \in U$ such that $\lim_{t\to 0} \lambda(t) \cdot u$ exists, the $\mathbb{G}_m$-equivariant map $\mathbb{A}^1 \to U$ extending $t \mapsto \lambda(t) \cdot u$ defines a morphism of algebraic stacks $f_{u,\lambda}\colon [\mathbb{A}^1/\mathbb{G}_m] \to [U/G]$ such that such that the image of the specialization $1 \rightsquigarrow 0$ is $u \rightsquigarrow u_0$. Combining this observation with the above theorem and the Local Structure Theorem (6.5.1) yields:

Corollary 6.6.21. *Let $\mathcal{X}$ be an algebraic stack of finite type over an algebraically closed field $\Bbbk$ with affine diagonal. Let $x \rightsquigarrow x_0$ be a specialization of $\Bbbk$-points such that the stabilizer G_{x_0} is linearly reductive. Then there exists a morphism $[\mathbb{A}^1/\mathbb{G}_m] \to \mathcal{X}$ representing the specialization $x \rightsquigarrow x_0$.* □

6.6.6 Hilbert–Mumford Criterion

The stable and semistable locus can often be effectively computed using the Hilbert–Mumford Criterion. To setup the formulation, let $U \subset \mathbb{P}(V)$ be a G-equivariant closed subscheme where V is a finite dimensional G-representation, and let $u \in U$ be a $\Bbbk$-point with a lift $\widetilde{u} \in \mathbb{A}(V)$. Given a one-parameter subgroup $\lambda\colon \mathbb{G}_m \to G$, we can choose a basis $V \cong \Bbbk^n$ such that $\lambda(t) \cdot (v_1, \dots, v_n) = (t^{d_1}v_1, \dots, t^{d_n}v_n)$. Define the *Hilbert–Mumford index* as

$$\mu(u, \lambda) := \max_{i, \widetilde{u}_i \neq 0} -d_i. \tag{6.6.4}$$

Equivalently, if $u_0 = \lim_{t\to 0} \lambda(t) \cdot u \in \mathbb{P}(V)$ (which exists since $\mathbb{P}(V)$ is proper), then $\mathbb{G}_m$ fixes u_0 and $\mu(u, \lambda)$ is the opposite of the weight of the induced $\mathbb{G}_m$-action on the line $L_{u_0} \subset V$ classified by u_0.

Remark 6.6.22. From the definition of the Hilbert–Mumford index, we see that

(a) $\lim_{t\to 0} \lambda(t) \cdot \widetilde{u}$ exists if and only if $\mu(u, \lambda) \le 0$,

(b) $\lim_{t\to 0} t \cdot \widetilde{u} = 0$ if and only if $\mu(u, \lambda) < 0$, and

(c) $\mu(gx, g\lambda g^{-1}) = \mu(x, \lambda)$.

Theorem 6.6.23 (Hilbert–Mumford Criterion)**.** *Let G be a linearly reductive algebraic group over an algebraically closed field $\Bbbk$ acting on a G-equivariant closed subscheme $U \subset \mathbb{P}(V)$, where V is a finite dimension G-representation. Let $u \in \mathbb{P}(V)$ be a $\Bbbk$-point with a lift $\widetilde{u} \in \mathbb{A}(V)$. Then*

$$\begin{aligned} u \in U^{\mathrm{ss}} &\iff 0 \notin \overline{G\widetilde{u}} \\ &\iff \lim_{t\to 0} \lambda(t) \cdot \widetilde{u} \neq 0 \textit{ for all } \lambda\colon \mathbb{G}_m \to G \\ &\iff \mu(u, \lambda) \ge 0 \textit{ for all } \lambda\colon \mathbb{G}_m \to G. \end{aligned}$$

If in addition the action of G on U has generically finite stabilizers, then

$$\begin{aligned} u \in U^{\mathrm{s}} &\iff G\widetilde{u} \subset \mathbb{A}(V) \textit{ is closed} \\ &\iff \mu(u,\lambda) > 0 \textit{ for all non-trivial } \lambda\colon \mathbb{G}_m \to G. \end{aligned}$$

Remark 6.6.24. The criterion that is now referred to as the "Hilbert–Mumford Criterion" was first developed by Hilbert in [Hil93, § 15-16] and then adapted by Mumford in [GIT, p. 53]. It holds more generally when G is reductive.

Proof. For semistability, the $(\Rightarrow)$ implication is clear: if $0 \in \overline{G\widetilde{u}}$, then for every non-constant invariant function,we have that $f(\widetilde{u}) = f(0) = 0$; hence $u \notin U^{\mathrm{ss}}$. For the converse, if $0 \notin \overline{G\widetilde{u}}$, then 0 and $\overline{G\widetilde{u}}$ are disjoint closed G-invariant subschemes of $\mathbb{A}(V)$. Therefore their images in $\mathbb{A}(V)/\!\!/G = \operatorname{Spec}(\operatorname{Sym}^* V^\vee)^G$ are disjoint (Corollary 6.3.7(2)). We may thus find an invariant function $f \in (\operatorname{Sym}^* V^\vee)^G$ with $f(0) = 0$ and $f(\widetilde{u}) \neq 0$ which we may assume to be homogeneous of positive degree, i.e $f \in \operatorname{Sym}^d V^\vee = \Gamma(\mathbb{P}(V), \mathcal{O}(d))$ for $d > 0$. In the second equivalence, $(\Rightarrow)$ is again clear: if there is a λ such that $\lim_{t\to 0} \lambda(t) \cdot \widetilde{u} = 0$, then $0 \in \overline{G\widetilde{u}}$. Conversely, if $0 \in \overline{G\widetilde{u}}$, Theorem 6.6.20 provides a one-parameter subgroup λ such that the limit of u under λ is 0. The third equivalence follows from the definition of the Hilbert–Mumford index (see Remark 6.6.22).

For stability, we may assume that $u \in U^{\mathrm{ss}}$; otherwise 0 is in the closure of $G\widetilde{u}$ and thus $G\widetilde{u}$ is not closed. By definition, there is an invariant section $f \in \Gamma(U, \mathcal{O}(d))^G$ of positive degree not vanishing at u. After possibly increasing d, we can arrange that f extends to an invariant section $f \in \Gamma(\mathbb{P}(V), \mathcal{O}(d))^G$: this follows from the exact sequence $0 \to I_U \to \mathcal{O}_{\mathbb{P}(V)} \to \mathcal{O}_U \to 0$ and the vanishing of $\mathrm{H}^1(\mathbb{P}(V), I_U(N))$ for $N \gg 0$. We may thus view f as homogeneous polynomial of degree d on $\mathbb{A}(V)$. Letting $\alpha = f(\widetilde{u})$, we have a commutative diagram

$$\begin{tikzcd} G \arrow[r, "\Psi_{\widetilde{u}}"] \arrow[dr, "\Psi_u"'] & V(f-\alpha) \arrow[r, hook] \arrow[d] & \mathbb{A}(V) \\ & \mathbb{P}(V)_f & \end{tikzcd}$$

where $\Psi_u(g) = g \cdot u$ and $\Psi_{\widetilde{u}}(g) = g \cdot \widetilde{u}$. By assumption, we have that $\dim G_u = \dim G_{\widetilde{u}} = 0$ so both stabilizers are finite, thus proper. By Exercise 3.3.15(b), $Gu \subset \mathbb{P}(V)_f$ is closed if and only if Ψ_u is proper, and $G\widetilde{u} \subset \mathbb{A}(V)$ is closed if and only if $\Psi_{\widetilde{u}}$ is proper. On the other hand, $V(f-\alpha) \to \mathbb{P}(V)_f$ is proper, and thus Ψ_u is proper if and only if $\Psi_{\widetilde{u}}$ is. Thus $Gu \subset U_f$ is closed if and only if $G\widetilde{u} \subset \mathbb{A}(V)$ is closed giving the first equivalence. For the second equivalence, if $G\widetilde{u}$ is not closed, then there exists a one-parameter subgroup $\lambda\colon \mathbb{G}_m \to G$ such that $\lim_{t\to 0} \lambda(t) \cdot \widetilde{u}$ exists and is not contained in $G\widetilde{u}$. This gives a non-trivial λ with $\mu(u,\lambda) \leq 0$. Conversely, if $G\widetilde{u}$ is closed, then $\Psi_{\widetilde{u}}$ is proper and therefore for every non-trivial λ, the map $\mathbb{G}_m \to \mathbb{A}(V)$, defined by $t \mapsto \lambda(t) \cdot \widetilde{u}$, is also proper. This implies that $\lim_{t\to 0} \lambda(t)\widetilde{u}$ does not exist as otherwise the limit would define an extension $\mathbb{A}^1 \to \mathbb{A}(V)$ of $\mathbb{G}_m \to \mathbb{A}(V)$ and applying the valuative criterion

$$\begin{tikzcd} \mathbb{G}_m \arrow[r, equal] \arrow[d, hook] & \mathbb{G}_m \arrow[d] \\ \mathbb{A}^1 \arrow[r] \arrow[ur, dashed] & \mathbb{A}(V) \end{tikzcd}$$

would yield a contradiction. Since the limit doesn't exist, $\mu(u,\lambda) > 0$. $\square$

We also provide a stack-theoretic formulation. The data of a G-equivariant embedding $U \subset \mathbb{P}(V)$ is classified by a line bundle L on $[U/G]$ such that the pullback of L under $U \to [U/G]$ is very ample. Since the stable and semistable locus are G-invariant, they define open substacks of $[U/G]$. The data of point $u \in U(\Bbbk)$ and a one-parameter subgroup $\lambda\colon \mathbb{G}_m \to G$ up to conjugation is classified by a map $f_{u,\lambda}\colon [\mathbb{A}^1/\mathbb{G}_m] \to [U/G]$ such that the induced map $\mathbf{B}\mathbb{G}_m \overset{0}{\hookrightarrow} [\mathbb{A}^1/\mathbb{G}_m] \xrightarrow{f_{u,\lambda}} [U/G] \to \mathbf{B}G$ is determined by λ. The Hilbert–Mumford index is $\mu(u,\lambda) = -\operatorname{wt}(f_{u,\lambda}^* L)|_{B\mathbb{G}_m}$.

Corollary 6.6.25 (Hilbert–Mumford Criterion)**.** *Let G be a linearly reductive algebraic group over an algebraically closed field $\Bbbk$ acting on a projective $\Bbbk$-scheme U. Let L be a line bundle on $[U/G]$ corresponding to a very ample G-linearization. Then $u \in [U/G]$ is semistable if and only if $\operatorname{wt}((f^*L)|_{B\mathbb{G}_m}) \geq 0$ for all maps*

$$f\colon [\mathbb{A}^1/\mathbb{G}_m] \to [U/G], \quad \textit{with } f(1) \simeq u.$$

If in addition the action of G on U has generically finite stabilizers, then u is stable if and only if $\operatorname{wt}((f^*L)|_{B\mathbb{G}_m}) > 0$ for all maps $f\colon [\mathbb{A}^1/\mathbb{G}_m] \to [U/G]$ such that $f(1) \simeq u$ and the induced map $\mathbb{G}_m \to G_{f(0)}$ on stabilizers is non-trivial. $\square$

Exercise 6.6.26 (Affine Hilbert-Mumford Criterion)**.** Let G be a linearly reductive group over an algebraically closed field $\Bbbk$ acting on an affine scheme $U = \operatorname{Spec} A$ of finite type. Let $\chi\colon G \to \mathbb{G}_m$ be a character, and let U^{ss} and U^{s} be the semistable and stable locus with respect to χ as defined in Exercise 6.6.10. For $u \in U(\Bbbk)$, show that

$$\begin{aligned} u \in U^{\mathrm{ss}} \iff & \text{for all one-parameter subgroups } \lambda\colon \mathbb{G}_m \to G \text{ such that} \\ & \lim_{t\to 0} \lambda(t)\cdot u \text{ exists, } \langle \chi, \lambda\rangle \geq 0 \end{aligned}$$

If in addition the action of G on U has generically finite stabilizers, show that $u \in U^{\mathrm{s}}$ if and only if the same condition holds with strict inequality $\langle \chi, \lambda\rangle > 0$.

Hint: Consider the action of G on $U \times \mathbb{A}^1$ induced by χ defined by $g\cdot(u,z) = (g\cdot u, \chi(g)^{-1}\cdot z)$, and show that $u \notin U^{\mathrm{ss}}$ if and only if $\overline{G\cdot(u,1)} \cap (U \times \{0\}) \neq \emptyset$. Use the Destabilization Theorem (6.6.20) to show that this is equivalent to the existence of a one-parameter subgroup λ such that

$$\lim_{t\to 0} \lambda(t)\cdot(u,1) = \lim_{t\to 0}(\lambda(t)\cdot u, t^{-\langle\chi,\lambda\rangle}) \in U \times \{0\}.$$

6.6.7 Examples

Example 6.6.27. Consider the diagonal action of SL_2 on $X = (\mathbb{P}^1)^n$, and consider the SL_2-equivariant Segre embedding $(\mathbb{P}^1)^n \to \mathbb{P}^{2^n-1}$ (or equivalently the SL_2-linearization $\mathcal{O}(1) \boxtimes \cdots \boxtimes \mathcal{O}(1)$). We claim that

$$\begin{aligned} X^{\mathrm{s}} &= \{(p_1,\ldots,p_n) \mid \text{for all } q \in \mathbb{P}^1,\ \#\{i \mid p_i = q\} < n/2\} \\ X^{\mathrm{ss}} &= \{(p_1,\ldots,p_n) \mid \text{for all } q \in \mathbb{P}^1,\ \#\{i \mid p_i = q\} \leq n/2\}. \end{aligned}$$

To see this, let $(p_1,\ldots,p_n) \in X(\Bbbk)$ and $\lambda\colon \mathbb{G}_m \to \mathrm{SL}_2$ be a one-parameter subgroup. There exists $g \in \mathrm{SL}_2(\Bbbk)$ such that $g\lambda g^{-1} = \lambda_0^d$ for some $d \in \mathbb{Z}$ where

$\lambda_0(t) = \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix}$. We can assume $d \geq 0$ as the case $d < 0$ is handled similarly. Since $\mu(x, \lambda) = \mu(gx, \lambda_0^d) = d\mu(gx, \lambda_0)$, it suffices to compute $\mu(gx, \lambda_0)$. Since $\mu(-, \lambda_0)$ is symmetric with respect to the S_n-action, we can assume that $gx = (0, \ldots, 0, p_k, \ldots, p_n)$ with $p_k, \ldots, p_n \neq 0$. A coordinate of the Segre embedding is of the form $(\prod_{i \in \Sigma} x_i)(\prod_{i \notin \Sigma} y_i)$ for a subset $\Sigma \subset \{1, \ldots, n\}$, and its weight is $n - 2(\#\Sigma)$. The coordinate where gx is nonzero with the largest weight is $y_1 \cdots y_k x_{k+1} \cdots x_n$ with weight $2k - n$. Thus $\mu(gx, \lambda_0) = n - 2k$. Therefore, if no more than (resp. less than) $n/2$ of the points p_i are the same, then x is semistable (resp. stable) if and only if $n \geq 2k$ (resp. $n > 2k$). Conversely, if more than (resp. at least) $n/2$ of the same, then after translating by an element of SL_2 and using the symmetry of the S_n-action, we can write $u = (0, \ldots, 0, p_k, \ldots, p_n)$ with $k > n/2$ (resp. $k \geq n/2$) and $\lambda_0 = \mathrm{diag}(t^{-1}, t)$ destabilizes u.

If n is odd, then $X^{\mathrm{ss}} = X^{\mathrm{s}}$ and $X^{\mathrm{ss}} \to X^{\mathrm{ss}} /\!\!/ \mathrm{SL}_2$ is a geometric quotient. If n is even, the map $X^{\mathrm{ss}} \to X^{\mathrm{ss}} /\!\!/ \mathrm{SL}_2$ identifies $(p_1, \ldots, p_n)$ and $(q_1, \ldots, q_n)$ if there is a subset $\Sigma \subset \{1, \ldots, n\}$ of size $n/2$ such that $p_i = p_j$ and $q_i = q_j$ for all $i, j \in \Sigma$; in this case, the unique closed orbit in fiber is the orbit of the n-tuple with 0's in positions in Σ and ∞'s elsewhere. The complement $X^{\mathrm{ss}} \setminus X^{\mathrm{s}}$ has precisely $\frac{1}{2}\binom{n}{n/2}$ closed orbits.

A modification of the argument yields the same stable and semistable locus for the action of PGL_2 on $(\mathbb{P}^1)^n$ under the PGL_2-linearization $\mathcal{O}(2) \boxtimes \cdots \boxtimes \mathcal{O}(2)$. Since $\mathrm{Aut}(\mathbb{P}^1) = \mathrm{PGL}_2$, the quotient $X^{\mathrm{ss}} /\!\!/ \mathrm{SL}_2 = X^{\mathrm{ss}} /\!\!/ \mathrm{PGL}_2$ can be viewed as a compactification of the moduli of n ordered points in $\mathbb{P}^1$ up to projective equivalence.

Exercise 6.6.28.

(a) Under the action of SL_2 on the projectivization $\mathbb{P}(\Gamma(\mathbb{P}^1, \mathcal{O}(n))) \cong \mathbb{P}^n$ of binary forms of degree n, show that the semistable (resp. stable) locus consists of binary forms $f(x, y)$ such that every linear factor has multiplicity less than or equal to (resp. less than) $n/2$.

(b) Under the SL_2-linearization $\mathcal{O}(a_1) \boxtimes \cdots \boxtimes \mathcal{O}(a_n)$ on $(\mathbb{P}^1)^n$ with each $a_i > 0$, show that the semistable (resp. stable) locus consists of tuples $(p_1, \ldots, p_n)$ such that for all $q \in \mathbb{P}^1(\Bbbk)$,

$$\sum_{p_i = q} a_i \leq (\sum_{i=1}^n a_i)/2$$

(resp. strict inequality holds).

(c) Under the SL_{r+1} action on $(\mathbb{P}^r)^n$ and the SL_{r+1}-linearization $\mathcal{O}(a_1) \boxtimes \cdots \boxtimes \mathcal{O}(a_n)$ with each $a_i > 0$, show that the semistable (resp. stable) locus consists of tuples $(p_1, \ldots, p_n)$ such that for every linear subspace $W \subsetneq \mathbb{P}^r$

$$\sum_{p_i \in W} a_i \leq \frac{\dim W + 1}{r + 1} (\sum_{i=1}^n a_i)$$

(resp. strict inequality holds).

Exercise 6.6.29 (Cubic curves)**.** Consider the action of SL_3 on the projective space $\mathbb{P}(\mathrm{H}^0(\mathbb{P}^2, \mathcal{O}(3)))$ of cubic curves in $\mathbb{P}^2$. Show that the semistable locus consists of curves with at worst nodal singularities and that the stable locus consists of smooth curves.

Remark 6.6.30 (Quartic curves)**.** A more involved calculation shows that under the SL_3 action on $\mathbb{P}(\mathrm{H}^0(\mathbb{P}^2, \mathcal{O}(4)))$, a quartic curve is semistable if and only if it doesn't contain a triple point and is not the union of a cubic curve and an inflection tangent line, and is stable if and only if it has at worst nodal and cuspidal singularities. See also [Mum77, §1.13].

Remark 6.6.31 (Cubic surfaces)**.** Under the action of SL_4 on $\mathbb{P}(\mathrm{H}^0(\mathbb{P}^3, \mathcal{O}(3)))$, a cubic surface is stable (resp. semistable) if and only if it has finitely many singular points and the singularities are ordinary double points (resp. ordinary double points or rank 2 double points whose axes are not contained in the surface). See [Muk03, Thm. 7.14] and [Hil93].

Exercise 6.6.32 (Quiver GIT)**.** A *quiver* $Q = (Q_0, Q_1)$ is a directed graph where Q_0 is a finite set of vertices and Q_1 is a finite set of arrows; there are source and target maps $s, t\colon Q_1 \to Q_0$. A $\Bbbk$-representation of Q consists of a vector space V_i for every $i \in Q_0$ together with linear maps $L_\alpha\colon V_i \to V_j$ for every arrow $\alpha\colon i \to j$. If each V_i is finite dimensional with $d_i = \dim V_i$, we say that $d = (d_i)$ is the dimension vector of V.

Fix $d = (d_i)$ and consider the space

$$R(Q, d) = \prod_{\alpha \in Q_1} \mathrm{Hom}(\Bbbk^{s(\alpha)}, \Bbbk^{t(\alpha)})$$

of representations with dimension vector d. This inherits an action of $\prod_i \mathrm{GL}_{d_i}$ via $(g_i) \cdot (L_\alpha) = (g_{t(\alpha)} L_\alpha g_{s(\alpha)}^{-1})$. The diagonal subgroup $\mathbb{G}_m \subset \prod_i \mathrm{GL}_{d_i}$ consisting of tuples $(t\,\mathrm{id}_{\Bbbk^{d_i}})$ of scalar matrices for $t \in \mathbb{G}_m$ is normal and acts trivially. Therefore the quotient $G := (\prod_i \mathrm{GL}_{d_i})/\mathbb{G}_m$ also acts on $R(Q, d)$.

For any tuple $a = (a_i)_{i \in Q_0}$ of integers such that $\sum_i a_i d_i = 0$, consider the character

$$\chi_a\colon \prod_i G \to \mathbb{G}_m, \quad (g_i) \mapsto \prod_i \det(g_i)^{a_i}.$$

Use the Affine Hilbert–Mumford Criterion (6.6.26) to show that a representation $V \in R(Q, d)$ is semistable (resp. stable) with respect to χ if and only if for every subrepresentation $W \subset V$ (i.e. subspaces $W_i \subset V_i$ such that $L_\alpha(W_{s(\alpha)}) \subset W_{t(\alpha)}$),

$$\sum_i a_i \dim W_i \geq 0$$

(resp. strict inequality holds). See also [Kin94, Prop. 3.1].

Remark 6.6.33 (Cox construction of toric varieties)**.** Let $X = X(\Sigma)$ be a proper toric variety with fan $\Sigma \subset N_\mathbb{R}$ and torus T_N, where N is a lattice with dual M. Letting $\Sigma(1)$ denote the rays of the fan, the divisors D_ρ associated to $\rho \in \Sigma(1)$ generate the class group. There is a short exact sequence

$$0 \to M \to \mathbb{Z}^{\Sigma(1)} \to \mathrm{Cl}(X) \to 0.$$

The algebraic group $G := \mathrm{Hom}(\mathrm{Cl}(X), \mathbb{G}_m)$ is diagonalizable (hence linearly reductive) and sits in a short exact sequence

$$1 \to G \to \mathbb{G}_m^{\Sigma(1)} \to T_N \to 1$$

obtained by applying $\mathrm{Hom}(-, \mathbb{G}_m)$ to the above sequence. The group G acts naturally on $\mathbb{A}^{\Sigma(1)}$.

For a cone $\sigma \in \Sigma$, let $x^\sigma := \prod_{\rho \in \sigma(1)} x_\rho$. Define the closed subset $Z \subset \mathbb{A}^{\Sigma(1)}$ by the vanishing of the ideal generated by the monomials x^σ as σ varies over maximal dimensional cones; this set can also be described as the union $\bigcup_C V(x_\rho \mid \rho \in C)$ where the union runs over primitive collections $C \subset \Sigma(1)$, i.e. subsets C such that C is not contained in $\sigma(1)$ for any $\sigma \in \Sigma$ and such that for any $C' \subsetneq C$, there exists $\sigma \in \Sigma$ with $C' \subset \sigma(1)$. This locus Z is G-invariant.

The main theorem here is that X is isomorphic to the good quotient $(\mathbb{A}^{\Sigma(1)} \setminus Z)/\!\!/G$. This is the so-called 'Cox construction of X', and it gives X homogeneous coordinates in a similar fashion to how $\mathbb{A}^{n+1}$ gives homogeneous coordinates for $\mathbb{P}^n = (\mathbb{A}^{n+1} \setminus 0)/\mathbb{G}_m$. When Σ is a simplicial fan, X is a geometric quotient $(\mathbb{A}^{\Sigma(1)} \setminus Z)/G$. Moreover, the class group $\mathrm{Cl}(X)$ is identified with group of character $\mathbb{X}^*(G)$, and if L is an ample line bundle on X corresponding to a character χ, then $\mathbb{A}^{\Sigma(1)} \setminus Z$ is the semistable locus for the action of G on $\mathbb{A}^{\Sigma(1)}$ with respect to the character χ. See [Cox95] and [CLS11, §5].

Example 6.6.34 (Variation of GIT for $\mathbb{G}_m$-actions)**.** Consider a $\mathbb{G}_m$-action on an affine scheme $X = \operatorname{Spec} A$ of finite type over $\Bbbk$. In this example, we will consider how the GIT quotients (with respect to a character in the sense Exercise 6.6.10) vary as we vary the character of $\mathbb{G}_m$. There is a bijection $\mathrm{Hom}(\mathbb{G}_m, \mathbb{G}_m) \cong \mathbb{Z}$ and we write $\chi_d(t) = t^d$ as the character corresponding to $d \in \mathbb{Z}$.

Let $A = \bigoplus_{n \in \mathbb{Z}} A_n$ be the induced grading. There are three cases for the semistable locus $X^{\mathrm{ss}}_{\chi_d}$ with respect to the character χ_d:

(1) $d = 0$: $X^{\mathrm{ss}}(0) := X^{\mathrm{ss}}_{\chi_0} = X$ and $X^{\mathrm{ss}}_{\chi_0}/\!\!/\mathbb{G}_m = \operatorname{Spec} A_0$.

(2) $d > 0$: $X^{\mathrm{ss}}(+) := X^{\mathrm{ss}}_{\chi_d} = X \setminus V(\sum_{n<0} A_n)$ and $X^{\mathrm{ss}}_{\chi_d} = \operatorname{Proj} \bigoplus_{d \geq 0} A_{nd}$ is independent of d; moreover $X^{\mathrm{ss}}(0)$ is identified with $X^+_{\chi_d}$ with respect to the one-parameter subgroup χ_d (Exercise 6.6.53).

(3) $d < 0$: $X^{\mathrm{ss}}(-) := X^{\mathrm{ss}}_{\chi_d} = X \setminus V(\sum_{n>0} A_n) = X^+_{\chi_d}$ and $X^{\mathrm{ss}}_{\chi_d} = \operatorname{Proj} \bigoplus_{d \geq 0} A_{-nd}$ is independent of d.

There is a commutative diagram

$$\begin{tikzcd}
X^{\mathrm{ss}}(+) \arrow[r, hook] \arrow[d] & X \arrow[d] & X^{\mathrm{ss}}(-) \arrow[l, hook'] \arrow[d] \\
X^{\mathrm{ss}}(+)/\!\!/\mathbb{G}_m \arrow[r] & X/\!\!/\mathbb{G}_m & X^{\mathrm{ss}}(-)/\!\!/\mathbb{G}_m \arrow[l]
\end{tikzcd}$$

where the vertical maps are good quotients, the top maps are open immersions, and the bottom maps are projective. The Affine Hilbert–Mumford Criterion (6.6.26) implies that there are identifications of the stable loci with respect to χ_0, χ_1, and χ_{-1}: $X^{\mathrm{s}}(0) = X \setminus (X^{\mathrm{ss}}(+) \cap X^{\mathrm{ss}}(-))$, $X^{\mathrm{s}}(+) = X^{\mathrm{ss}}(+) = X \setminus X^{\mathrm{ss}}(-)$, and $X^{\mathrm{s}}(-) = X^{\mathrm{ss}}(-) = X \setminus X^{\mathrm{ss}}(+)$. Therefore, we see that if both $X^{\mathrm{ss}}(+)$ and $X^{\mathrm{ss}}(-)$ are nonempty, then $X^{\mathrm{ss}}(+)/\mathbb{G}_m \to X/\!\!/\mathbb{G}_m$ and $X^{\mathrm{ss}}(-)/\mathbb{G}_m \to X/\!\!/\mathbb{G}_m$ are isomorphisms over $X^{\mathrm{s}}(0)/\mathbb{G}_m$, and in particular birational. We also see that if the complements of $X^{\mathrm{ss}}(+)$ and $X^{\mathrm{ss}}(-)$ in X each have codimension at least two, then the birational map $X^{\mathrm{ss}}(+)/\!\!/\mathbb{G}_m \dashrightarrow X^{\mathrm{ss}}(-)/\!\!/\mathbb{G}_m$ is an isomorphism in codimension 2 such that the divisor $\mathcal{O}(1)$ (which is relatively ample over $X/\!\!/\mathbb{G}_m$) pushes forward to a divisor on $X^{\mathrm{ss}}(-)/\!\!/\mathbb{G}_m$ whose dual is relatively ample, i.e. $X^{\mathrm{ss}}(+)/\!\!/\mathbb{G}_m \dashrightarrow X^{\mathrm{ss}}(-)/\!\!/\mathbb{G}_m$ is a flip with respect to $\mathcal{O}(1)$.

Remark 6.6.35 (Variation of GIT)**.** Extending the previous example, consider a projective variety X over $\Bbbk$ with an action of a linearly reductive group G. Two line bundles (resp. G-linearizations) L_1 and L_2 on X are algebraically equivalent (resp. G-algebraically equivalent) if there is a connected variety T, points $t_1, t_2 \in T(\Bbbk)$, and a line bundle (resp. G-linearization) $\mathcal{L}$ on $X \times T$ such that $L_i = \mathcal{L}|_{X \times \{t_i\}}$. The Neron–Severi group $\mathrm{NS}(X)$ (resp. G-equivariant Neron–Severi group $\mathrm{NS}^G(X)$) of line bundles (resp. G-linearizations) on X up to (G-)algebraic equivalence is finitely generated. The kernel of $\mathrm{NS}^G(X)_{\mathbb{R}} \to \mathrm{NS}(X)$ is identified with the rational character group $\mathbb{X}^*(G)_{\mathbb{R}}$. We let $\mathrm{Eff}^G(X) \subset \mathrm{NS}^G(X)_{\mathbb{R}}$ be the cone of G-effective linearizations, i.e. G-linearizations L such that there is a nonzero section of $L^{\otimes d}$) for some $d > 0$ or in other words such that $X^{\mathrm{ss}}_L \neq \emptyset$. We also let $\mathrm{Amp}^G(X) \subset \mathrm{NS}^G(X)_{\mathbb{R}}$ be the cone of ample G-linearizations.

The main results of variation of GIT can be formulated as follows. The semistable locus X^{ss}_L only depends on the G-algebraic equivalence class of L. There is a polyhedral decomposition of the cone $\mathrm{Amp}^G(X) \cap \mathrm{Eff}^G(X)$ defined by codimension 1 walls such that the semistable locus is constant in any open chamber. If L_0 is on a wall while L_+ and L_- are on opposite adjacent chambers, then there is a commutative diagram

$$\begin{tikzcd}
X^{\mathrm{ss}}_{L_+} \arrow[r, hook] \arrow[d] & X^{\mathrm{ss}}_{L_0} \arrow[d] & X^{\mathrm{ss}}_{L_-} \arrow[l, hook'] \arrow[d] \\
X^{\mathrm{ss}}_{L_+}/\!\!/G \arrow[r] & X^{\mathrm{ss}}_{L_0}/\!\!/G & X^{\mathrm{ss}}_{L_-}/\!\!/G \arrow[l]
\end{tikzcd}$$

where the vertical maps are good quotients, the top maps are open immersions, and the bottom maps are projective. If $X^{\mathrm{ss}}_{L_+}$ and $X^{\mathrm{ss}}_{L_-}$ are non-empty, the bottom maps are birational; when the bottom maps are isomorphisms in codimension 2, then $X^{\mathrm{ss}}_{L_+}/\!\!/G \dashrightarrow X^{\mathrm{ss}}_{L_-}/\!\!/G$ is a flip with respect to the line bundle $\mathcal{O}(1)$ on $X^{\mathrm{ss}}_{L_+}/\!\!/G$, which is relatively ample over $X^{\mathrm{ss}}_{L_0}/\!\!/G$ and which pulls back to $L_+|_{X^{\mathrm{ss}}_{L_+}}$.
See [Tha96] and [DH98].

Remark 6.6.36 (Mori Dream Spaces)**.** There is an interesting connection between the Mori program and variation of GIT. A normal $\mathbb{Q}$-factorial projective variety X is a *Mori dream space* if (1) $\mathrm{Pic}(X)_{\mathbb{Q}} = \mathrm{NS}(X)_{\mathbb{Q}}$, (2) the cone $\mathrm{Nef}(X)$ of nef line bundles is the affine hull of finitely many semiample line bundles, and (3) there are finitely many birational maps $f_i \colon X \dashrightarrow X_i$, which are isomorphisms in codimension 1, to a $\mathbb{Q}$-factorial normal projective variety X_i such that the movable cone $\mathrm{Mov}(X)$ is the union of $f_i^{-1}(\mathrm{Amp}(X_i)_{\mathbb{Q}})$; a line bundle is *movable* if its stable base locus has codimension at least 2. In other words, X is a Mori dream space if $\mathrm{Mov}(X)$ has a finite wall and chamber decomposition such the projective variety determined by line bundle is constant within an open chamber.

Equivalently, X is a Mori dream space if $\mathrm{Pic}(X)_{\mathbb{Q}} = \mathrm{NS}(X)_{\mathbb{Q}}$ and the Cox ring

$$\mathrm{Cox}(X) := \bigoplus_{(d_1,\ldots,d_n)\in\mathbb{N}^n} \Gamma(X, L_1^{d_1} \otimes \cdots \otimes L_n^{d_n})$$

is finitely generated, where $L_1, \ldots, L_n$ is a basis for $\mathrm{Pic}(X)_{\mathbb{Q}}$ such that their affine hull contains $\mathrm{Eff}(X)_{\mathbb{Q}}$. If X is a Mori dream space, then X along with each birational model X_i is a GIT quotient of the semistable locus of $\mathrm{Spec}(\mathrm{Cox}(X))$ by the torus $\mathbb{G}_m^n$ with respect to a character. Moreover there is an identification of the Mori chambers of $\mathrm{Mov}(X)$ with the variation of GIT chambers for the action of $\mathbb{G}_m^n$ on $\mathrm{Spec}(\mathrm{Cox}(X))$. See also [HK00].

Example 6.6.37 (Partial desingularization)**.** If U is a smooth variety and $U \to X$ is a geometric quotient by a linearly reductive group, then X necessarily has finite quotient singularities; this is a consequence of the Local Structure Theorem (4.3.14). On the other hand, if $U \to X$ is a good quotient, then X can have worse singularities. Nevertheless, there is a canonical procedure to partially resolve the singularities of X so that they have finite quotient singularities.

Suppose that there is an open subset $X' \subset X$ such that $\pi_0(X') \to X'$ is a geometric quotient; this happens for example if $U = V^{\mathrm{ss}}$ is the semistable locus with respect to the action of G on a projective variety $V \subset \mathbb{P}^N$ and the stable locus V^{s} is nonempty. Then there is a commutative diagram

$$\begin{tikzcd} U_n \arrow[r] \arrow[d, "\pi_n"] & U_{n-1} \arrow[r] \arrow[d, "\pi_{n-1}"] & \cdots \arrow[r] & U = U_0 \arrow[d, "\pi_0"] \\ X_n \arrow[r] & X_{n-1} \arrow[r] & \cdots \arrow[r] & X = X_0 \end{tikzcd}$$

such that:

- Each U_{i+1} is a G-invariant open subscheme of the blow-up $\mathrm{Bl}_Z U_i$, where Z is a G-invariant smooth closed subscheme whose stabilizers are of maximal dimension, and $U_{i+1} \subset \mathrm{Bl}_Z U_i$ is the complement of the strict transform of $\pi_i^{-1}(\pi_i(Z))$. If $U = V^{\mathrm{ss}}$ is the semistable locus of a projective variety with respect to a G-linearization L, then U_{i+1} is the semistable locus with respect to $(q^*L)^{\otimes n} \otimes \mathcal{O}(-E)$ for $n \gg 0$, where $q\colon \mathrm{Bl}_Z U_i \to U_i$ and E denotes the exceptional divisor.

- The maps $X_{i+1} \to X_i$ are projective birational.

- The maps $\pi_i\colon U_i \to X_i$ are good quotients by G, and the map $\pi_n\colon U_n \to X_n$ is a geometric quotient. In particular, X_n has finite quotient singularities.

For a simple example of this procedure in action, consider the $\mathbb{G}_m$-action on $\mathbb{A}^2$ with weights 1 and -1. In this case, the quotient $\mathbb{A}^2/\!\!/\mathbb{G}_m \cong \mathbb{A}^1$ is smooth but it is not a geometric quotient. The procedure tells us to take the blow-up $\mathrm{Bl}_0 \mathbb{A}^2$ at the origin and the complement U_1 of the strict transform of $V(xy)$. Then $\mathbb{G}_m$-acts with finite stabilizers on U_1 and $U_1 \to \mathbb{A}^2$ is $\mathbb{G}_m$-invariant birational (but not proper) map inducing an isomorphism $U_1/\mathbb{G}_m \to \mathbb{A}^2/\!\!/\mathbb{G}_m$ on quotients.

See [Kir85], [Rei89], and [ER21].

6.6.8 Kempf's Optimal Destabilization Theorem

Given an algebraic group G over an algebraically closed field $\Bbbk$, we define $\mathbb{X}_*(G)$ as the set of one-parameter subgroups $\mathbb{G}_m \to G$.

Definition 6.6.38. A *length* $\|-\|$ *on* $\mathbb{X}_*(G)$ is a non-negative real-valued function on $\mathbb{X}_*(G)$ which is conjugation invariant, i.e. $\|g\lambda g^{-1}\| = \|\lambda\|$ for $\lambda \in \mathbb{X}_*(G)$ and $g \in G(\Bbbk)$, and such that for every maximal torus $T \subset G$, there is a positive definite integral-valued bilinear form $(-,-)$ on $\mathbb{X}_*(T)$ with $(\lambda,\lambda) = \|\lambda\|^2$ for $\lambda \in \mathbb{X}_*(T)$.

Example 6.6.39. If $G = \mathrm{GL}_n$, then any one-parameter subgroup λ is conjugate to a one-parameter subgroup of the form $t \mapsto \mathrm{diag}(t^{d_1}, \dots, t^{d_n})$ and we can define $\|\lambda\| = \sqrt{d_1^2 + \cdots + d_n^2}$.

Example 6.6.40. For every reductive algebraic group G, there is a length $\|-\|$ on $\mathbb{X}_*(G)$. To see this, let $T \subset G$ be a maximal torus and choose a positive definite integral-valued bilinear form $(-,-)$ on $\mathbb{X}_*(T)$ which is invariant under the conjugation action of the Weyl group $W := N(T)/T$. There is a bijection $\mathbb{X}_*(G)/G \cong \mathbb{X}_*(T)/W$ between conjugacy classes of $\mathbb{X}_*(G)$ under G and conjugacy classes of $\mathbb{X}_*(T)$ under W. In other words, for every $\lambda \in \mathbb{X}_*(G)$ there exists $g \in G(\Bbbk)$ such that $g\lambda g^{-1} \in \mathbb{X}_*(T)$, and moreover for any other element $g' \in G(\Bbbk)$ such that $g'\lambda g'^{-1} \in \mathbb{X}_*(T)$, then $g\lambda g^{-1}$ and $g'\lambda g'^{-1}$ are conjugate under W. It follows that $\|\lambda\|^2 := (g\lambda g^{-1}, g\lambda g^{-1})$ is well-defined.

Let $X = \operatorname{Spec} A$ be an affine $\Bbbk$-scheme with the action of G and let $x_0 \in X(\Bbbk)$ be a point with closed orbit. For every point $x \in X(\Bbbk)$ with $Gx_0 \subset \overline{Gx}$ and a one-parameter subgroup $\lambda \colon \mathbb{G}_m \to G$ such that $\lim_{t \to 0} \lambda(t) \cdot x$ exists, we define the *Hilbert–Mumford index of x with respect to λ* as

$$\mu(x, \lambda) = -\deg f_{x,\lambda}^{-1}(Gx_0). \tag{6.6.5}$$

where $f_{x,\lambda} \colon \mathbb{A}^1 \to X$ is the map extending $\mathbb{G}_m \to X$, $t \mapsto \lambda(t) \cdot x$. Note that if $\lim_{t\to 0} \lambda(t) \cdot x \notin Gx_0$, then $\mu(x, \lambda) = 0$.

Since $\mu(x, \lambda^n) = n \cdot \mu(x, \lambda)$, it natural to consider the *normalized Hilbert–Mumford index*

$$\frac{\mu(x, \lambda)}{\|\lambda\|}$$

as a measure of how quickly $\lambda(t) \cdot x$ approaches the closed orbit Gx_0. The more negative the normalized Hilbert–Mumford index is, the faster $\lambda(t) \cdot x$ approaches Gx_0. Kempf proved that there is a one-parameter subgroup minimizing this index and that it is unique up to conjugation.

Theorem 6.6.41 (Kempf's Optimal Destabilization Theorem—affine version)**.** *Let G be a reductive algebraic group over an algebraically closed field $\Bbbk$ with a length $\|-\|$ on $\mathbb{X}_*(G)$. Let $X = \operatorname{Spec} A$ be an affine scheme of finite type over $\Bbbk$ with an action of G. Let $x_0 \in X(\Bbbk)$ be a point with closed orbit. For every point $x \in X(\Bbbk)$ with $Gx_0 \subset \overline{Gx}$, there exists a one-parameter subgroup $\lambda_0 \colon \mathbb{G}_m \to G$ such that $\mu(x, \lambda_0)/\|\lambda_0\|$ achieves a minimal value $M(x)$ over all $\lambda \in \mathbb{X}_*(G)$ such that $\lim_{t\to 0} \lambda(t) \cdot x \in Gx_0$.*

If λ_0' is another such one-parameter subgroup, then $P(\lambda_0) = P(\lambda_0')$ and $\lambda_0' = u\lambda_0 u^{-1}$ for a unique element $x \in X(\lambda_0)$. Every maximal torus $T \subset P_{\lambda_0}$ contains a unique indivisible element achieving this minimum value.

Remark 6.6.42. The subgroup $P_{\lambda_0} = \{g \in G \mid \lim_{t\to 0} \lambda_0(t) g \lambda_0(t)^{-1} \text{ exists}\}$ is the parabolic associated to λ_0 and $U_{\lambda_0} = \{g \in G \mid \lim_{t \to 0} \lambda_0(t) g \lambda_0(t)^{-1} = 1\}$ is the unipotent radical of P_{λ_0}; see §C.3.3.

In the projective case where there is a G-equivariant embedding $X \hookrightarrow \mathbb{P}(V)$, we have already defined the Hilbert–Mumford index $\mu(x, \lambda)$ in (6.6.4) as follows: choosing a basis of V such that $\mathbb{G}_m$ acts on $\mathbb{A}(V) = \mathbb{A}^n$ with weights $d_1, \dots, d_n$ and a lift $\widehat{x} = (u_1, \dots, u_n) \in \mathbb{A}(V)$ of x, then $-\mu(x, \lambda)$ is defined as the smallest d_i with $u_i \neq 0$. If $\lim_{t\to 0} \lambda(t) \cdot \widehat{x}$ exists, then this agrees with the definition in (6.6.5).

To see this, observe that the extension $f_{\widehat{x},\lambda}\colon \mathbb{A}^1 \to \mathbb{A}^n$ of the map $t \mapsto \lambda(t)\cdot\widehat{x}$ is the map $t \mapsto (t^{d_i}u_i)$ and $f^{-1}_{\widehat{x},\lambda}(0) = \operatorname{Spec}\Bbbk[t]/(t^d)$ where d is the smallest d_i with $u_i \neq 0$.

The projective version below follows from applying the affine version (Theorem 6.6.41) to a lift $\widehat{x} \in \mathbb{A}(V)$ of a non-semistable point $x \in \mathbb{P}(V)$. In this case, the closed orbit in $\overline{G\widehat{x}}$ is the fixed point 0. The following theorem also holds for reductive groups but we restrict to linearly reductive groups as we've only discussed semistability in that context.

Theorem 6.6.43 (Kempf's Optimal Destabilization Theorem—projective version)**.** *Let G be a linearly reductive algebraic group over an algebraically closed field $\Bbbk$ with a length $\|-\|$ on $\mathbb{X}_*(G)$. Let $X \subset \mathbb{P}(V)$ be a G-equivariant closed subscheme where V is a finite dimensional G-representation. For every non-semistable point $x \in X(\Bbbk)$, there exists a one-parameter subgroup $\lambda_0\colon \mathbb{G}_m \to G$ such that $\mu(x,\lambda_0)/\|\lambda_0\|$ achieves a minimal value $M(x)$ over all $\lambda \in \mathbb{X}_*(G)$.*

If λ_0' is another such one-parameter subgroup, then $P_{\lambda_0} = P_{\lambda_0'}$ and $\lambda_0' = g\lambda_0 g^{-1}$ for a unique element $g \in X(\lambda_0)$. Every maximal torus $T \subset P_{\lambda_0}$ contains a unique indivisible element achieving this minimum value. □

Definition 6.6.44. We call any λ_0 satisfying Theorem 6.6.41 or Theorem 6.6.43 a *Kempf optimal destabilizing one-parameter subgroup for x*, and we call $M(x)$ the *optimal normalized Hilbert–Mumford index* for x.

Proof of Theorem 6.6.41. The proof is simpler when $x_0 \in \overline{Gx}$ is a fixed point such as in the projective version when the closed orbit is $0 \in \mathbb{A}(V)$; the reader is encouraged to keep this case in mind. By Lemma C.3.2, we may choose finite dimensional G-representations V and W along with G-equivariant maps

$$\begin{tikzcd} & \mathbb{A}(V) \\ X \arrow[ur, hook, "i"] \arrow[dr, "f"'] & \\ & \mathbb{A}(W), \end{tikzcd} \tag{6.6.6}$$

where $i\colon X \hookrightarrow \mathbb{A}(V)$ is a closed immersion with $i(x_0) = 0$ and $f\colon X \to \mathbb{A}(W)$ is a morphism with $f^{-1}(0) = Gx_0$. When x_0 is a fixed point, we can take $f = i$ in (6.6.6).

A one-parameter subgroup $\lambda\colon \mathbb{G}_m \to G$ induces $\mathbb{G}_m$-actions on V and W, and thus gradings $V = \bigoplus_{d\in\mathbb{Z}} V_d$ and $W = \bigoplus_{d\in\mathbb{Z}} W_d$. We define

$$\begin{aligned} m(i(x),\lambda) &= \min\{d \,|\, \text{the projection of } i(x) \text{ to } V_d \text{ is non-zero}\}, \\ m(f(x),\lambda) &= \min\{d \,|\, \text{the projection of } f(x) \text{ to } W_d \text{ is non-zero}\}. \end{aligned}$$

For any $g \in G$, we have the identities $m(i(v),\lambda) = m(i(g\cdot v), g\lambda g^{-1})$ and $m(f(v),\lambda) = m(f(g\cdot v), g\lambda g^{-1})$.

It is easy to see that if $\lim_{t\to 0}\lambda(t)\cdot x$ exists, then $\mu(x,\lambda) = -m(f(x),\lambda)$, and that

$$\begin{aligned} \lim_{t\to 0}\lambda(t)\cdot x \text{ exists} &\iff m(i(x),\lambda) \geq 0, \\ \lim_{t\to 0}\lambda(t)\cdot x \in Gx_0 &\iff m(i(x),\lambda) \geq 0 \text{ and } m(f(x),\lambda) > 0. \end{aligned}$$

By the Destabilization Theorem (6.6.20), there exists $\lambda_x \in \mathbb{X}_*(G)$ such that $m(i(x),\lambda_x) \geq 0$ and $m(f(x),\lambda_x) > 0$.

Case of a torus: Let $T \subset G$ be a maximal torus containing λ_x. We can decompose $V = \bigoplus_{\chi \in \mathbb{X}^*(T)} V_\chi$ as a T-representation where $\mathbb{X}^*(T)$ denotes the set of characters of T. We define *the state of* $i(x) \in V$ *with respect to* T to be the set

$$\mathrm{State}_T(i(x)) = \{\chi \in \mathbb{X}^*(T) \,|\, \text{the projection of } i(x) \text{ to } V_\chi \text{ is nonzero}\}.$$

Likewise, we have the state $\mathrm{State}_T(f(x)) \subset \mathbb{X}^*(T)$ of $f(x) \in W$ with respect to T.

Let $\langle -, - \rangle$ be the natural pairing $\mathbb{X}^*(T) \times \mathbb{X}_*(T) \to \mathbb{Z}$. For a one-parameter subgroup $\lambda \in \mathbb{X}_*(T)$, we have identifications

$$m(i(x), \lambda) = \min_{\chi \in \mathrm{State}_T(i(x))} \langle \chi, \lambda \rangle \quad \text{and} \quad m(f(x), \lambda) = \min_{\chi \in \mathrm{State}_T(f(x))} \langle \chi, \lambda \rangle.$$

We claim that the function $\lambda \mapsto m(f(x), \lambda) / \|\lambda\|$ achieves a maximum value on the set $\{\lambda \neq 0 \in \mathbb{X}_*(T) \,|\, m(i(x), \lambda_T) \geq 0\}$ at a one-parameter subgroup λ_T, and that any other one-parameter subgroup achieving this minimum is a positive multiple of λ_T. This is precisely the conclusion of Lemma 6.6.45 below applied to the lattice $L = \mathbb{X}_*(T) \cong \mathbb{Z}^r$ and the subsets of $\mathbb{X}^*(T) \cong \mathrm{Hom}_{\mathbb{Z}}(L, \mathbb{Z})$ given by $F := \mathrm{State}_T(i(x))$ and $G := \mathrm{State}_T(f(x))$.

General case: If $T \subset G$ is a maximal torus and $g \in G(\Bbbk)$, then there is an identification $\mathbb{X}^*(T) \cong \mathbb{X}^*(gTg^{-1})$ given by identifying $\chi \in \mathbb{X}^*(T)$ with the character $gTg^{-1} \to \mathbb{G}_m$ defined by $gtg^{-1} \mapsto \chi(t)$. Under this identification, $\mathrm{State}_T(i(x)) = \mathrm{State}_{gTg^{-1}}(i(gx))$. Given a one-parameter subgroup $\lambda \in \mathbb{X}_*(G)$, we've seen that $m(f(x), \lambda) = m(f(gx), g\lambda g^{-1})$ for $g \in G(\Bbbk)$. We claim that in fact $m(f(x), \lambda) = m(f(x), p\lambda p^{-1})$ for $p \in P_\lambda$. By symmetry, it suffices to show that $m(f(x), \lambda) \leq m(f(x), p\lambda p^{-1})$. Interpreting $-m(f(x), \lambda)$ as the smallest integer d such that $\lim_{t \to 0} t^d \lambda(t) \cdot f(x) \in \mathbb{A}(W)$ exists, we need to show that $\lim_{t \to 0} t^d p\lambda(t)p^{-1} \cdot f(x) \in \mathbb{A}(W)$ exists. This follows from the computation

$$\begin{aligned}
\lim_{t \to 0} \left(t^d p\lambda(t)p^{-1} \cdot f(x)\right) &= \lim_{t \to 0} \left(p \cdot \left(\lambda(t)p^{-1}\lambda(t)^{-1}\right) \cdot \left(t^d \lambda(t) f(x)\right)\right) \\
&= p \cdot \left(\lim_{t \to 0} \lambda(t)p^{-1}\lambda(t)^{-1}\right) \cdot \left(\lim_{t \to 0} t^d \lambda(t) f(x)\right).
\end{aligned}$$

We now show that the function $\lambda \mapsto m(f(x), \lambda) / \|\lambda\|$ achieves a minimum value on

$$\Sigma := \{\lambda \in \mathbb{X}_*(G) \,|\, m(i(x), \lambda) \geq 0\}.$$

If T is a maximal torus, by the torus case we know that for every $g \in G(\Bbbk)$ there is a minimum value on each *non-empty* set $\mathbb{X}_*(gTg^{-1}) \cap \Sigma$, and that the minimum is determined by the subsets of $\mathbb{X}_*(T)$ given by $\mathrm{State}_{gTg^{-1}}(i(x)) \cong \mathrm{State}_T(i(g^{-1}u))$ and $\mathrm{State}_{gTg^{-1}}(f(x)) \cong \mathrm{State}_T(f(g^{-1}u))$. Since these subsets are contained in the finite set of characters χ with $V_\chi \neq 0$ (resp. $W_\chi \neq 0$), there are only finitely many minimum values as g ranges over $G(\Bbbk)$. Since the image of any $\lambda \in \mathbb{X}_*(G)$ is contained in gTg^{-1} for some $g \in G(\Bbbk)$, it follows that there is a global minimum value achieved by a one-parameter subgroup $\lambda_0 \in \Sigma$. We may assume that λ_0 is indivisible, i.e. λ_0 cannot be written as a positive multiple of another one-parameter subgroup.

To establish the uniqueness, we choose a maximal torus $T \subset G$ containing λ_0. By the torus case, $\lambda_0 \in \mathbb{X}_*(T) \cap \Sigma$ is the unique indivisible one-parameter subgroup achieving the minimal value. For $p \in P_{\lambda_0}$, the conjugate one-parameter subgroup $p\lambda_0 p^{-1}$ also achieves this minimal value. Since any other maximal torus

$T' \subset P_{\lambda_0}$ is pTp^{-1} for some $p \in P_{\lambda_0}$, we see that $\mathbb{X}_*(T') \cap \Sigma$ also contains a unique indivisible element achieving the minimum value. Finally, let $\lambda_1 \in \mathbb{X}_*(G)$ be another indivisible element achieving the minimum value. The intersection $P_{\lambda_0} \cap P_{\lambda_1}$ contains a maximal torus T of G (Proposition C.3.7(a)), and we can write $\lambda_T = p_0\lambda_0 p_0^{-1} = p_1\lambda_1 p_1^{-1}$ for $p_0, p_1 \in P_{\lambda_T}$. It follows that $P_{\lambda_0} = P_{\lambda_T} = P_{\lambda_1}$, and that λ_0 and λ_1 are conjugate by a unique element element of U_{λ_T} (Proposition C.3.7(b)).

See also [Kem78, Thm. 3.4]. □

The argument above used the following lemma in convex geometry.

Lemma 6.6.45. *Let Λ be a finite dimensional lattice, and let F and G be non-empty finite subsets of $\Lambda^\vee = \mathrm{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{Z})$. Assume that $\Lambda_{\mathbb{R}} = \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$ has a positive definite inner product which is integral valued on Λ. Define*

$$f_{\min} \colon \Lambda_{\mathbb{R}} \to \mathbb{R},\ \lambda \mapsto \min_{f \in F} f(\lambda) \quad \textit{and} \quad g_{\min} \colon \Lambda_{\mathbb{R}} \to \mathbb{R},\ \lambda \mapsto \min_{g \in G} g(\lambda).$$

Suppose that there exists $\lambda \in \Lambda_{\mathbb{R}}$ such that $f_{\min}(\lambda) \geq 0$ and $g_{\min}(\lambda) > 0$. Then the function

$$\begin{aligned} C_F := \{\lambda \neq 0 \in \Lambda_{\mathbb{R}} \mid f_{\min}(\lambda) \geq 0\} &\to \mathbb{R}, \\ \lambda &\mapsto g_{\min}(\lambda)/\left\|\lambda\right\| \end{aligned}$$

obtains a maximum value M, and there exists a unique element $\lambda_0 \in C_F \cap \Lambda$ such that $M = g_{\min}(\lambda_0)/\left\|\lambda_0\right\|$ and such that any other element $\lambda \in C_F \cap \Lambda$ with $M = g_{\min}(\lambda)/\left\|\lambda\right\|$ is an integral multiple of λ_0.

Proof. The set $\{\lambda \in C_F \mid g_{\min}(\lambda) \geq 1\}$ is closed and convex, and therefore contains a unique point λ' closest to the origin. Since $g_{\min}(\alpha\lambda') = \alpha g_{\min}(\lambda')$ for $\alpha \in \mathbb{R}$, we must have that $g_{\min}(\lambda') = 1$ and that $\lambda' \in C_F$ is the unique point with $g_{\min}(\lambda') = 1$ and $g_{\min}(\lambda')/\left\|\lambda'\right\| = M$.

We now argue that the ray spanned by λ' contains an integral point. If λ' is in the interior of $\{\lambda \in C_F \mid g_{\min}(\lambda) \geq 1\}$, i.e. $f(\lambda') > 0$ for all $f \in F$ and there is a unique $g \in G$ with $g(\lambda') = 1$, then λ' is the closed point to the origin on the affine plane defined by $g = 1$. We claim that $\lambda' = g^*/\langle g^*, g^*\rangle$ where $g^* \in \Lambda_{\mathbb{R}}$ is the unique point such that $\langle g^*, \lambda\rangle = g(\lambda)$ for all $\lambda \in \Lambda_{\mathbb{R}}$. Indeed, the point λ' is contained in the plane $g = 1$, and for any other point λ on this plane, we have that $\langle \lambda', \lambda\rangle = 1/\langle g^*, g^*\rangle = \langle \lambda', \lambda'\rangle$ and the Cauchy–Schwarz inequality implies that $\langle \lambda', \lambda'\rangle^2 = \langle \lambda', \lambda\rangle^2 \leq \langle \lambda', \lambda'\rangle\langle \lambda, \lambda\rangle$ so that $\langle \lambda', \lambda'\rangle \leq \langle \lambda, \lambda\rangle$. Since the inner product and g take integral values, $g^* \in \Lambda$. We then take λ_0 to be the unique indivisible element in the ray spanned by g^*.

To reduce to this case, let $f_1, \ldots, f_t \in F$ be the functions satisfying $f_i(\lambda') = 0$, and let $g_1, \ldots, g_s \in G$ be the functions satisfying $g_i(\lambda') = g_{\min}(\lambda')$. Since each f_i and g_j take integral values, we may restrict to the subspace

$$W := \left\{ \lambda \in \Lambda_{\mathbb{R}} \;\middle|\; \begin{array}{l} f_1(\lambda) = \cdots = f_t(\lambda) = 0 \\ g_1(\lambda) = \cdots = g_s(\lambda) \end{array} \right\},$$

and the lattice $W \cap \Lambda$. Then λ' is in the interior of $\{\lambda \in C_F \cap W \mid g_{\min}(\lambda) \geq 1\}$ and thus is the closest point to the origin contained in the affine plane define by $g_1 = 1$. □

Corollary 6.6.46. *In the setting of Theorem 6.6.41 or Theorem 6.6.43, there is a unique morphism $f\colon [\mathbb{A}^1/\mathbb{G}_m] \to [X/G]$ with $f(1) \simeq x$ and $f(0) \simeq x_0$.*

Proof. By Proposition 6.6.12, a morphism $[\mathbb{A}^1/\mathbb{G}_m] \to [X/G]$ is determined by a one-parameter subgroup λ such that $\lim_{t\to 0} \lambda(t)x \in Gx_0$, and that λ is unique up to conjugation by P_λ. Since any two Kempf's worst one-parameter subgroups are conjugate under U_λ (and thus P_λ), the statement follows. $\square$

Example 6.6.47. We revisit the SL_2 action on $(\mathbb{P}^1)^n$ with the linearization given by the Segre embedding $(\mathbb{P}^1)^n \hookrightarrow \mathbb{P}^{2^n-1}$ (Example 6.6.27). The non-semistable consists of tuples $x = (p_1, \ldots, p_n)$ where more than $n/2$ points are equal. Suppose that precisely $k > n/2$ points are equal. Since the Hilbert–Mumford index is symmetric, we can assume that the first k are equal. If $\lambda\colon \mathbb{G}_m \to \mathrm{SL}_2$ is a one-parameter subgroup, we can choose $g \in \mathrm{SL}_2(\Bbbk)$ with $g\lambda g^{-1} = \lambda_0^d$ where $d \in \mathbb{Z}$ and $\lambda_0(t) = \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix}$. After rescaling the norm, we can assume that $\|\lambda_0\| = 1$. We also assume that $d \geq 0$ as the $d < 0$ case can be handled similarly. Then

$$\frac{\mu(x,\lambda)}{\|\lambda\|} = \frac{\mu(gx, g\lambda g^{-1})}{\|g\lambda g^{-1}\|} = \mu(gx, \lambda_0)$$

This index is negative if and only if $gx = \{0, \ldots, 0, p_{k+1}, \ldots, p_n\}$ in which case $\mu(gx, \lambda_0) = n - 2k$. It follows that λ_0 (resp. $g^{-1}\lambda_0 g$) is a Kempf optimal destabilizing one parameter subgroup for gx (resp x). Observe that the parabolic $P_{\lambda_0} \subset \mathrm{SL}_2$ of lower triangular matrices is also the stabilizer of $0 \in \mathbb{P}^1$, and thus $G_{gx} \subset P_{\lambda_0}$. For any $h \in P_{\lambda_0}$, $h^{-1}\lambda_0 h$ (resp. $(hg)^{-1}\lambda_0 hg$) is also a Kempf optimal destabilizing subgroup for gx (resp. x).

Exercise 6.6.48. Let G be a reductive algebraic group over an algebraically closed field $\Bbbk$ with a length $\|-\|$ on $\mathbb{X}_*(G)$. Let $X = \operatorname{Spec} A$ be an affine scheme of finite type over $\Bbbk$ with an action of G. Let $x_0 \in X(\Bbbk)$ have closed G-orbit. Let $x \in X(\Bbbk)$ be a point such that $Gx_0 \subset \overline{Gx}$, and let P_x be the parabolic determined by Kempf's Optimal Destabilization Theorem (6.6.41).

(a) Show that for all $g \in G(\Bbbk)$ that $gP_xg^{-1} = P_{gx}$.

Hint: Show that if $P_x = P_\lambda$ for a one-parameter subgroup λ, then $P_{gx} = P_{g\lambda g^{-1}}$.

(b) Show that $G_x \subset P_x$.

Hint: Use that for a parabolic P, $N_G(P) = P$ (Proposition C.3.7).

The following criterion can sometimes be used to check stability/semistability by computing Hilbert–Mumford indices only for one-parameter subgroups in a fixed maximal torus.

Exercise 6.6.49 (Kempf–Morrison Criterion)**.** Let $G = \mathrm{GL}(W)$ or $\mathrm{SL}(W)$, where W is finite dimensional vector space over an algebraically closed field $\Bbbk$ of characteristic 0. Let $X \subset \mathbb{P}(V)$ be a G-invariant closed subscheme, where V is a finite dimension G-representation. Let $x \in X(\Bbbk)$. Assume that there is a linearly reductive subgroup $H \subset G_x$ such that W decomposes as a direct sum of distinct H-representations, and let $T \subset G$ be a maximal torus compatible with this decomposition. Show that

$$\begin{aligned} x \in X^{\mathrm{ss}} &\iff \mu(x,\lambda) \leq 0 \text{ for all } \lambda\colon \mathbb{G}_m \to T, \\ x \in X^{\mathrm{s}} &\iff \mu(x,\lambda) < 0 \text{ for all } \lambda\colon \mathbb{G}_m \to T. \end{aligned}$$

Hint: If $u \notin X^{\mathrm{ss}}$, let $\lambda_0 \colon \mathbb{G}_m \to G$ be a Kempf optimal destabilization one-parameter subgroup and $0 = V_0 \subset V_1 \subset \cdots \subset V_k = V$ be the filtration induced by the parabolic P_{λ_0}. Use Exercise 6.6.48 to conclude that each V_i is H-invariant, and use the hypothesis on the H-representation V to show that each V_i is T-invariant; thus $T \subset P_{\lambda_0}$. Apply Kempf's Optimal Destabilization Theorem again to find λ in T with $\mu(x,\lambda) < 0$. If $u \notin X^{\mathrm{s}}$, letting $\widehat{x} \in \mathbb{A}(V)$ be a lift of x and $\widehat{x}_0 \in \overline{G\widehat{x}}$ be a point with closed orbit, repeat the above argument using the affine version of Kempf's Optimal Destabilization Theorem.

Exercise 6.6.50 (Existence of destabilizing one-parameter subgroups over a perfect field)**.** Let X be an affine scheme of finite type over a perfect field $\Bbbk$, and let G be a reductive algebraic group over $\Bbbk$ acting on X. This exercise will show that for every point $x \in X(\Bbbk)$, there exists a one-parameter subgroup $\lambda \colon \mathbb{G}_m \to G$ *defined over* $\Bbbk$ such that $\lim_{t\to 0} \lambda(t) \cdot x$ has closed G-orbit. See also [Kem78, §4].

1. Show that if $\mathrm{Gal} := \mathrm{Gal}(\overline{\Bbbk}/\Bbbk)$ is the geometric Galois group, then Gal acts on the set $\mathbb{X}_*(G_{\Bbbk})$ of one-parameters subgroups such that $\mathbb{X}_*(G) = \mathbb{X}_*(G_{\overline{\Bbbk}})^{\mathrm{Gal}}$.
2. Show that there exists a length $\|{-}\|$ on $\mathbb{X}_*(G_{\overline{\Bbbk}})$ which is invariant under the action of Gal.
3. Show that the subsets $\{\lambda \in \mathbb{X}_*(G_{\overline{\Bbbk}}) \mid \lim_{t\to 0} \lambda(t) \cdot x \in X(\overline{\Bbbk}) \text{ exists}\}$ and $\{\lambda \in \mathbb{X}_*(G_{\overline{\Bbbk}}) \mid \lim_{t\to 0} \lambda(t) \cdot x \in G_{\overline{\Bbbk}} x_0\}$ are Gal-invariant where $G_{\overline{\Bbbk}} x_0$ is the unique closed orbit in $\overline{G_{\overline{\Bbbk}} u}$. Moreover, show that if V and W are G-representations as in (6.6.6), then the functions $m(i(x),\lambda)$ and $m(f(x),\lambda)$ are Gal-invariant.
4. Generalize Theorem 6.6.41 and Theorem 6.6.43 to the case when $\Bbbk$ is a perfect field and $x \in X(\Bbbk)$.

Note in particular that if G has no non-trivial one-parameter subgroups defined over $\Bbbk$, then the G-orbit of any $\Bbbk$-point is closed.

Finally we record the following consequence of *the proof* of Kempf's Optimal Destabilization Theorem (6.6.41). This will play a key role in the proof of the HKKN Stratification (6.6.70).

Proposition 6.6.51. *Let G be a reductive algebraic group over an algebraically closed field $\Bbbk$ with a length $\|{-}\|$ on $\mathbb{X}_*(G)$. Let $X = \operatorname{Spec} A$ be an affine scheme of finite type over $\Bbbk$ with an action of G with a unique closed orbit Gx_0. Fix a maximal torus $T \subset G$. There are finitely many one-parameter subgroups $\lambda_1, \ldots, \lambda_n \in \mathbb{X}_*(T)$ and numbers $M_1, \ldots, M_n \in \mathbb{R}_{<0}$ such that for every point $x \in X(\Bbbk)$, there exist a unique $i = 1, \ldots, n$ such that λ_i is an optimal Kempf one-parameter subgroup for gx for some $g \in G$, and such that $M_i = \mu(x,\lambda_i)/\|\lambda_i\|$.*

Proof. We will use the notation of the proof of Theorem 6.6.41. For $x \in X(\Bbbk)$, the unique parabolic subgroup of a Kempf optimal destabilization one-parameter subgroup is determined by the subsets $\mathrm{State}_{gTg^{-1}}(i(x)) \cong \mathrm{State}_T(i(gx)) \subset \mathbb{X}_*(T)$ and $\mathrm{State}_{gTg^{-1}}(f(x)) \cong \mathrm{State}_T(f(gx)) \subset \mathbb{X}_*(T)$ as g ranges over $G(\Bbbk)$. These subsets are contained in the finite subset of characters $\chi \in \mathbb{X}_*(T)$ with $V_\chi \neq 0$ or $W_\chi \neq 0$. Thus there are only finitely many possibilities for an optimal destabilizing subgroup of T, and the statement follows. $\square$

6.6.9 Fixed loci, $\mathbb{G}_m$-actions, and attractor loci

If X is an algebraic space over a field $\Bbbk$ equipped with an action of an affine algebraic group G, we define the *fixed locus* as the functor

$$X^G := \underline{\mathrm{Mor}}^G(\operatorname{Spec} \Bbbk, X) \colon \mathrm{Sch}/\Bbbk \to \mathrm{Sets}$$

assigning a $\Bbbk$-scheme S to the set of G-equivariant maps from S to X, where S is endowed with the trivial G action.

Theorem 6.6.52. *Let X be an algebraic space of finite type over an algebraically closed field $\Bbbk$ with affine diagonal equipped with an action of a linearly reductive algebraic group G. Then*

(1) The fixed locus X^G is represented by a subscheme of X;

(2) If G is a torus then X^G is a closed subscheme.

(3) If X is smooth, so is X^G.

Proof. If G is connected and $U \to X$ is a G-invariant étale morphism, we claim that

$$\begin{tikzcd} U^G \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & U \arrow[d] \\ X^G \arrow[r, hook] & X \end{tikzcd} \tag{6.6.7}$$

is cartesian. Indeed, suppose $S \to U$ is map such that $S \to U \to X$ is G-invariant. Let $U_S \to S$ be the base change of $U \to X$ by $S \to X$. Since $U_S \to S$ is G-invariant, it suffices to show that the section $j \colon S \to U_S$ is G-invariant. As $U \to X$ is étale, $j \colon S \to U_S$ is an open immersion. Because G is connected, for each point $s \in S$, the G-orbit $Gj(s) \subset U_S$ is connected and thus contained in S.

For (1), given a fixed point $x \in X^G(\Bbbk)$, Theorem 6.5.18 produces a G-invariant étale neighborhood $(U,u) \to (X,x)$ with U affine and $u \in U^G(\Bbbk)$. If G is connected, then $U^G \to X^G$ is étale and representable by (6.6.7). Thus it suffices to show that U^G is representable. Since U is affine, we can choose a G-equivariant embedding $U \hookrightarrow \mathbb{A}(V)$ into a finite dimensional G-representation. In this case, $\mathbb{A}(V)^G = \mathbb{A}(V^G)$ and thus $U^G = U \cap \mathbb{A}(V)^G$ is representable. In general, let $G^0 \subset G$ be the connected component of the identity, and let $g_1, \ldots, g_n \in G(\Bbbk)$ be representatives of the finitely many cosets $G(\Bbbk)/G^0(\Bbbk)$. Then G/G_0 acts on X^{G_0} and $X^G = \bigcap_i (X^{G_0})^{g_i}$, where $(X^{G_0})^{g_i}$ is identified with the fiber product of the diagonal $X^G \to X^G \times X^G$ and the map $X^G \to X^G \times X^G$ given by $x \mapsto (x, gx)$.

For (2), every subgroup of G is linearly reductive and Theorem 6.5.18 therefore produces a G-invariant étale surjective morphism $U \to X$ from an affine scheme. As G is connected, the argument above shows that $U^G \hookrightarrow U$ is a closed immersion and thus by étale descent so is $X^G \hookrightarrow X$.

For (3), if $x \in X^G(\Bbbk)$, there is a G-invariant étale morphism $(U,u) \to (X,x)$ from an affine scheme and a G-invariant étale morphism $U \to T_{U,u}$ as in the proof of Luna's Étale Slice Theorem (see (6.5.2)). Since $T_{U,u}^G$ is a linear subspace, it is smooth. Since $U^G \to X^G$ and $U^G \to T_{U,u}^G$ are étale at u, the statement follows from étale descent. See also [Ive72, Prop. 1.3] and [Mil17, Thm. 13.1]. $\square$

Let X be a separated algebraic space of finite type over $\Bbbk$ equipped with an action of $\mathbb{G}_m$. Define the *attractor locus* as the functor

$$X^+ := \underline{\mathrm{Mor}}^{\mathbb{G}_m}(\mathbb{A}^1, X) \colon \mathrm{Sch}/\Bbbk \to \mathrm{Sets}$$

assigning a $\Bbbk$-scheme S to the set of $\mathbb{G}_m$-equivariant maps from $S \times \mathbb{A}^1$ to X, where $\mathbb{G}_m$ acts trivially on S and with the usual scaling action on $\mathbb{A}^1$. Evaluation at 0 defines a morphism of functors

$$\mathrm{ev}_0 \colon X^+ \to X^{\mathbb{G}_m}.$$

On $\Bbbk$-points, $X^+(\Bbbk)$ is the set of points $x \in X(\Bbbk)$ such that $\lim_{t \to 0} t \cdot x$ exists, and $\mathrm{ev}_0(x)$ is this limit. Since X is separated, the limit is unique if it exists. If X is proper, the limit always exists and $X^+(\Bbbk) = X(\Bbbk)$. The functorial definition of X^+ endows it with an interesting scheme-structure, e.g. when $\mathbb{G}_m$ acts on $X = \mathbb{P}^1$ via $t \cdot [x:y] = [tx:y]$, then $X^+ = \mathbb{A}^1 \coprod \{\infty\}$.

Exercise 6.6.53. If $X = \operatorname{Spec} A$ is affine, then the $\mathbb{G}_m$-action induces a grading $A = \bigoplus_{d \in \mathbb{A}} A_d$. Show that the functors $X^{\mathbb{G}_m}$ and X^+ are representable by the closed subschemes of X defined by the ideals $\sum_{d \neq 0} A_d$ and $\sum_{d<0} A_d$.

Example 6.6.54 (Centralizers and parabolics)**.** Let G be an affine algebraic group over an algebraically closed field $\Bbbk$. A one-parameter subgroup $\lambda \colon \mathbb{G}_m \to G$ induces a $\mathbb{G}_m$-action on G via conjugation $t \cdot g := \lambda(t) g \lambda(t)^{-1}$. Under this action, the fixed locus $G^{\mathbb{G}_m} = C_\lambda$ is identified with the centralizer of λ and the attractor locus $G^+_\lambda = P_\lambda$ is identified with the subgroup consisting of elements $g \in G$ such that $\lim_{t \to 0} \lambda(t) g \lambda(t)^{-1}$ exists. The unipotent subgroup U_λ is identified with kernel of $\mathrm{ev}_0 \colon P_\lambda \to C_\lambda$.

When G is reductive, $P_\lambda \subset G$ is a parabolic subgroup or in other words G/P_λ is projective. See §C.3.3 for more properties of these subgroups.

We say that a map $X \to Y$ is an *affine fibration* (resp. *Zariski-local affine fibration*) if there exists an étale (resp. Zariski) cover $\{Y_i \to Y\}$ such that $X \times_Y Y_i \cong \mathbb{A}^n_{Y_i}$ over Y_i. Since the transition functions are not required to be linear, this notion is more general than a vector bundle.

Theorem 6.6.55. *Let X be a separated algebraic space of finite type over an algebraically closed field $\Bbbk$ equipped with an action of $\mathbb{G}_m$. The functor X^+ is representable by an algebraic space of finite type over $\Bbbk$ and $\mathrm{ev}_0 \colon X^+ \to X^{\mathbb{G}_m}$ is an affine morphism.*

Assume in addition that X is smooth (resp. smooth scheme). Then $X^{\mathbb{G}_m}$ is also smooth and $\mathrm{ev}_0 \colon X^+ \to X^{\mathbb{G}_m}$ is an affine fibration (resp. Zariski-local affine fibration). If $x \in X^{\mathbb{G}_m}$ and $T_{X,x} = T_{>0} \oplus T_0 \oplus T_{<0}$ is the $\mathbb{G}_m$-equivariant decomposition into nonnegative, zero, and positive weights, then $T_{X_i,x} = T_0 \oplus T_{>0}$, $T_{F_i,x} = T_0$, and $X_i \to F_i$ has relative dimension $\dim T_{>0}$.

Proof. If $X = \operatorname{Spec} A$ is affine, then $X^{\mathbb{G}_m}$ and X^+ are closed subschemes of X (Exercise 6.6.53). In the special case that $X = \mathbb{A}(V)$ where V is a finite dimensional G-representation, then $X^{\mathbb{G}_m} = \mathbb{A}(V^G)$ and $X^+ = \mathbb{A}(V_{\geq 0})$ where $V_{\geq 0}$ is the direct sum of the non-negative isotypic components, and moreover $\mathrm{ev}_0 \colon X^+ \to X^{\mathbb{G}_m}$ is a relative affine space.

We claim that if $U \to X$ is a $\mathbb{G}_m$-invariant étale morphism, then the diagram

$$\begin{tikzcd} U^+ \arrow[r, "\mathrm{ev}_0"] \arrow[d] \arrow[dr, phantom, "\square"] & U^{\mathbb{G}_m} \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & U \arrow[d] \\ X^+ \arrow[r, "\mathrm{ev}_0"] & X^{\mathbb{G}_m} \arrow[r, hook] & X \end{tikzcd} \tag{6.6.8}$$

is cartesian. The right square was verified in the proof of Theorem 6.6.52. For the left square, we need to show that there exists a unique $\mathbb{G}_m$-equivariant morphism filling in a $\mathbb{G}_m$-equivariant diagram

$$\begin{tikzcd} \operatorname{Spec} \Bbbk \times S \arrow[r] \arrow[d] & U \arrow[d] \\ \mathbb{A}^1 \times S \arrow[r] \arrow[ur, dashed] & X \end{tikzcd} \tag{6.6.9}$$

where S is an affine scheme of finite type over k, and the vertical left arrow is the inclusion of the origin. For each $n \geq 1$, the formal lifting property of étaleness yields a unique $\mathbb{G}_m$-equivariant map $\operatorname{Spec} k[x]/x^n \times S \to U$ such that

$$\begin{tikzcd} \operatorname{Spec} \Bbbk \times S \arrow[r] \arrow[d] & U \arrow[d] \\ \operatorname{Spec}(\Bbbk[x]/x^n) \times S \arrow[r] \arrow[ur, dashed] & X \end{tikzcd}$$

commutes. As $[\mathbb{A}^1/\mathbb{G}_m] \times S$ is coherently complete along $\mathbf{B}\mathbb{G}_{m,S}$ (Exercise 6.4.14), Coherent Tannaka Duality in the form of Exercise 6.4.10 yields a unique $\mathbb{G}_m$-equivariant morphism $\mathbb{A}^1 \times S \to U$ such that (6.6.9) commutes.

Choose a $\mathbb{G}_m$-invariant étale surjective morphism $U \to X$ from an affine scheme (Theorem 6.5.18). Then (6.6.8) implies that $U^+ \to X^+$ is etale and representable, and since U^+ is an affine scheme of finite type, it follows that X^+ is an algebraic space of finite type. Since $U^+ \to U^{\mathbb{G}_m}$ is affine, étale descent implies that $X^+ \to X^{\mathbb{G}_m}$ is also affine.

If X is smooth, then $X^{\mathbb{G}_m}$ is smooth by Theorem 6.6.52. As U is also smooth, for each $u \in U^{\mathbb{G}_m}(\Bbbk)$, there is a $\mathbb{G}_m$-equivariant morphism $U \to T_{U,u}$ étale at u with $f(u) = 0$ (Lemma 6.5.6). Then $U^+ \to T^+_{U,u}$ is also étale at u. Let $V \subset U$ be the open locus where $U^+ \to T^+_{U,u}$ is étale. Since V is $\mathbb{G}_m$-equivariant, if $v \in V^{\mathbb{G}_m}$, then $\mathrm{ev}_0^{-1}(v) \subset V$. Choosing an affine subscheme $V' \subset V^{\mathbb{G}_m}$ containing u and replacing U^+ with $\mathrm{ev}_0^{-1}(V')$, we may assume that $U^+ \to T^+_{U,u}$ is everywhere étale. By (6.6.8), we have a cartesian diagram

$$\begin{tikzcd} T^+_{U,u} \arrow[d] \arrow[dr, phantom, "\square"] & U^+ \arrow[l] \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X^+ \arrow[d] \\ T^{\mathbb{G}_m}_{U,u} & U^{\mathbb{G}_m} \arrow[l] \arrow[r] & X^{\mathbb{G}_m} \end{tikzcd} \tag{6.6.10}$$

where the horizontal arrows are étale. With $T_{X,x} = T_{>0} \oplus T_0 \oplus T_{<0}$, there are identifications $T^{\mathbb{G}_m}_{X,x} = T_0$ and $T^+_{X,x} = T_{>0} \oplus T_0$. Since $T^+_{U,u} \to T^{\mathbb{G}_m}_{U,u}$ a surjection of vector spaces, $U \to U^{\mathbb{G}_m}$ is a Zariski-local affine fibration. By étale descent, $X \to X^{\mathbb{G}_m}$ is an affine fibration of relative dimension $\dim T_{>0}$.

If X is a smooth scheme, then by Sumihiro's Theorem on Torus Actions (C.3.4) we may choose $U = \coprod_i U_i \to X$ such that $\{U_i\}$ is a $\mathbb{G}_m$-invariant affine open covering. Then (6.6.10) implies that $X^+ \to X^{\mathbb{G}_m}$ is a Zariski-local affine fibration.

See also [Dri13, Prop. 1.2.2, Thm. 1.4.2] and [AHR20, Thm. 5.16]. $\square$

Remark 6.6.56. Another approach to establish the algebraicity of X^+ in Theorem 6.6.55 is to show that the stack $\underline{\mathrm{Mor}}([\mathbb{A}^1/\mathbb{G}_m], \mathcal{X})$, whose objects over a $\Bbbk$-scheme S are morphisms $[\mathbb{A}^1/\mathbb{G}_m]_S \to \mathcal{X}$, is algebraic when $\mathcal{X}$ has affine diagonal. This can be shown by verifying Artin's Axioms (D.7.4) where the crucial step is to verify the effectivity condition (AA$_5$): this follows from the coherent completeness of $[\mathbb{A}^1/\mathbb{G}_m]_R$, where R is a noetherian local $\Bbbk$-algebra, along the unique closed point (Theorem 6.4.11) together with Coherent Tannaka Duality (6.4.8).

When $\mathcal{X} = [X/\mathbb{G}_m]$, then a $\mathbb{G}_m$-equivariant morphism $\mathbb{A}^1 \to X$ corresponds to a morphism $[\mathbb{A}^1/\mathbb{G}_m] \to [X/\mathbb{G}_m]$ over $\mathbf{B}\mathbb{G}_m$ (Exercise 3.1.14), and there is a cartesian diagram

$$\begin{tikzcd}
\underline{\mathrm{Mor}}^{\mathbb{G}_m}(\mathbb{A}^1, X) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \underline{\mathrm{Mor}}([\mathbb{A}^1/\mathbb{G}_m], [X/\mathbb{G}_m]) \arrow[d] \\
\operatorname{Spec} \Bbbk \arrow[r] & \underline{\mathrm{Mor}}([\mathbb{A}^1/\mathbb{G}_m], \mathbf{B}\mathbb{G}_m).
\end{tikzcd}$$

The algebraicity of the stacks of morphisms implies that $\underline{\mathrm{Mor}}^{\mathbb{G}_m}(\mathbb{A}^1, X)$ is an algebraic space.

6.6.10 The Białynicki-Birula Stratification

Theorem 6.6.57 (Białynicki-Birula Stratification[8])**.** *Let X be a separated algebraic space of finite type over an algebraically closed field $\Bbbk$ with an action of $\mathbb{G}_m$. Let $X^{\mathbb{G}_m} = \coprod_{i=1}^n F_i$ be the fixed locus with connected components F_i. There exists an affine morphism $X_i \to F_i$ for each i and a monomorphism $\coprod_i X_i \to X$. Moreover,*

(1) If X is proper, then $\coprod_i X_i \to X$ is surjective.

(2) If X is smooth (resp. smooth scheme), then F_i is smooth and $X_i \to F_i$ is a (resp. Zariski-local) affine fibration. If $x \in F_i$ and $T_{X,x} = T_{x,>0} \oplus T_{x,0} \oplus T_{x,<0}$ is the $\mathbb{G}_m$-equivariant decomposition into nonnegative, zero, and positive weights, then $T_{X_i,x} = T_{x,>0} \oplus T_{x,0}$, $T_{F_i,x} = T_{x,0}$, and $X_i \to F_i$ has relative dimension $\dim T_{x,>0}$.

(3) The map $X_i \hookrightarrow X$ is a locally closed immersion under any of the following conditions:

(a) X is affine,

(b) X is a smooth scheme, or

(c) there exists a $\mathbb{G}_m$-equivariant locally closed immersion $X \hookrightarrow \mathbb{P}(V)$ where V is a $\mathbb{G}_m$-representation (e.g., X is a normal quasi-projective variety).

(4) If X is smooth, irreducible, and quasi-projective, then the stratification $X^+ = \coprod_i X_i$ is filterable, *i.e. there is an ordering of the indices such that $X_{\geq i} := \bigcup_{j \geq i} X_j$ is closed for each i. If in addition there are finitely many fixed points $\{x_1, \ldots, x_n\}$, then $T_{x_i,0} = 0$ and $X_i = \mathbb{A}(T_{x_i,>0})$ is an affine*

[8] This is frequently referred to as the 'Białynicki-Birula Decomposition' as some authors prefer to reserve the term 'stratifications' to decomposisions where each strata has a neighborhood which is a topologically locally trivial.

space; in particular,

$$X^+ = X_{\geq 1} \supset X_{\geq 2} \supset \cdots \supset X_{\geq n} \supset \emptyset$$

is a cell decomposition, *i.e. each* $X_{\geq i} \setminus X_{\geq i-1} = X_i$ *is an affine space.*

Proof. By Theorem 6.6.55, X^+ is representable and there is affine morphism $\mathrm{ev}_0 \colon X^+ \to X^{\mathbb{G}_m}$ of finite type. We define X_i as the preimage $\mathrm{ev}_0^{-1}(F_i)$. Since X is separated, the inclusion $X^+ \hookrightarrow X$ is a monomorphism. This gives the main statement. If X is proper, then $X^+ \to X$ is surjective (i.e. (1) holds) as $\lim_{t\to 0} t \cdot x$ exists for every $x \in X(\Bbbk)$. Statement (2) follows directly from Theorem 6.6.55.

For (3), if $X = \operatorname{Spec} A$ and $A = \bigoplus_d A_d$ is the grading induced by the $\mathbb{G}_m$-action, then X^+ is the closed subscheme defined by the ideal $\sum_{d<0} A_d$ (Exercise 6.6.53) and in particular affine. If X is a smooth scheme, then there exists a $\mathbb{G}_m$-invariant affine open cover (Theorem C.3.3). For any point $x \in X^+$, let x_0 be the image of x under $\mathrm{ev}_0 \colon X^+ \to X^0$, and choose a $\mathbb{G}_m$-invariant affine open neighborhood $U \subset X$ of x_0. This induces a diagram

$$\begin{tikzcd} U^+ \arrow[r, hook] \arrow[dr, hook] & \mathrm{ev}_1^{-1}(U) \arrow[r] \arrow[d, hook] & U \arrow[d, hook] \\ & X^+ \arrow[r, "\mathrm{ev}_1"] & X. \end{tikzcd} \tag{6.6.11}$$

Since $U^+ \to U$ is a closed immersion (as U is affine) and $X^+ \to X$ is separated (it is a monomorphism), $U^+ \to \mathrm{ev}_1^{-1}(U)$ is a closed immersion. Since $U^+ = X^+ \times_{X^0} U^0$ (see (6.6.8)), $x \in U^+$ and $U^+ \to X^+$ is an open immersion. In particular, $U^+ \subset \mathrm{ev}_1^{-1}(U)$ is an open and closed subscheme containing x. On the other hand, X_i is smooth and connected (as $X_i \to F_i$ is an affine fibration), and thus irreducible. It follows that $X_i \cap U^+ = X_i \cap \mathrm{ev}_1^{-1}(U)$ and that $X_i \cap \mathrm{ev}_1^{-1}(U) \to U$ is a closed immersion which in turn implies that $X_i \to X$ is a locally closed immersion. The final case (3)(c) easy reduces to the case of $X = \mathbb{P}(V)$ in which a direct calculation shows that each X_i is of the form $\mathbb{P}(W) \setminus \mathbb{P}(W')$ for linear subspaces $W' \subset W \subset V$. See also [BB73, Thm. 4.1], [Hes81, Thm. 4.5,p. 69], [Dri13, Thm. B.0.3], [AHR20, Thm. 5.27], and [JS21, Thm. 1.5].

For (4), by Sumihiro's Theorem on Linearizations (C.3.3), we can choose a G-equivariant locally closed immersion $X \hookrightarrow \mathbb{P}^n$, where $\mathbb{G}_m$ acts on $\mathbb{P}^n$ via $t \cdot [x_0 : \ldots : x_n] = [t^{d_0} x_0 : \ldots : t^{d_n} x_n]$ with $d_0 \leq \cdots \leq d_n$. Let $D_1, \ldots, D_s$ be the distinct weights and set $J_i = \{j \mid d_j = D_i\}$ so that $J_1 \cup \cdots \cup J_s$ is a partition of $\{0, 1, \ldots, n\}$. Then $(\mathbb{P}^n)^{\mathbb{G}_m} = \coprod_{i=1}^s F_i$ where $F_i = V(x_j \mid j \in J_i)$. The preimage of F_i under the morphism $\mathrm{ev}_0 \colon \mathbb{P}^n \to (\mathbb{P}^n)^{\mathbb{G}_m}$, given by $p \mapsto \lim_{t \to 0} t \cdot p$, is

$$P_i := \mathrm{ev}_0^{-1}(F_i) = \left\{ [x_0 : \ldots : x_n] \;\middle|\; \begin{array}{l} x_j = 0 \text{ for all } j \in J_1 \cup \cdots \cup J_{i-1} \\ x_k \neq 0 \text{ for some } k \in J_i \end{array} \right\},$$

Moreover, the union

$$P_{\geq i} := \bigcup_{j \geq i} P_j = V(x_k \mid k \in J_1 \cup \cdots \cup J_{i-1}) \subset \mathbb{P}^n$$

is closed. The fixed locus for X is $X^{\mathbb{G}_m} = (\mathbb{P}^n)^{\mathbb{G}_m} \cap X = \coprod_i F_i \cap X$. For each i, we write $F_i \cap X = \coprod_{j=1}^{l_i} F_{ij}$ and $P_i \cap X = \coprod_{j=1}^{l_i} X_{ij}$ as the irreducible decompositions.

Then $\mathrm{ev}_0: \mathbb{P}^n \to (\mathbb{P}^n)^{\mathbb{G}_m}$ restricts to morphisms $\mathrm{ev}_0: X_{ij} \to F_{ij}$. For $j \neq k$, the strata X_{ij} and X_{ik} are disjoint, and thus $\overline{X}_{ij} \cap \overline{X}_{ik} \subset P_{\geq i+1} \cap X$. It follows that

$$(P_{\geq i+1} \cap X) \cup X_{i1} \cup \cdots \cup X_{ij} \subset X$$

is closed for each $j = 1, \ldots, s$. Ordering the strata as $X_{11}, \ldots, X_{1l_1}, \ldots, X_{s1}, \ldots, X_{sl}$ establishes the claim. See also [Bir76, Thm. 3]. $\square$

Remark 6.6.58. It is not true in general that $X_i \hookrightarrow X$ is a locally closed immersion. Based on Hironaka's example of a proper, non-projective, smooth 3-fold, Sommese constructed a smooth algebraic space X such that $X_i \hookrightarrow X$ is not a locally closed immersion [Som82]. On the other hand, Konarski provided an example of a normal proper toric variety X such that $X_i \hookrightarrow X$ is not a locally closed immersion [Kon82].

Remark 6.6.59 (Morse stratifications)**.** The Białynicki-Birula stratification of X can be obtained as the Morse stratification corresponding to the non-degenerate Morse function $\mu: X \to \mathrm{Lie}(S^1)^\vee = \mathbb{R}$: a point $x \in X$ lies in X_i if only if the limit of its forward trajectory under the gradient flow of μ lies in F_i. See [CS79].

Example 6.6.60. Suppose $\mathbb{G}_m$ acts on $X = \mathbb{P}^2$ via $t \cdot [x : y : z] = [x : ty : t^2]$. Then $X^{\mathbb{G}_m} = F_1 \coprod F_2 \coprod F_3$ where $F_1 = \{[1 : 0 : 0]\}$, $F_2 = \{[0 : 1 : 0]\}$, and $F_3 = \{[0 : 0 : 1]\}$, and $X_1 = \{x \neq 0\} = \mathbb{A}^2$, $X_2 = \{[0 : y : z] \mid y \neq 0\} = \mathbb{A}^1$ and $X_3 = F_3$.

Let $\widetilde{X}$ be the blowup $\mathrm{Bl}_p X$ at the fixed point $p = [0 : 1 : 0]$. Then $\mathbb{G}_m$ acts on the exceptional divisor $E \cong \mathbb{P}^1$ via $t \cdot [u : v] = [u : t^2 v]$ with fixed points $q_1 = [1 : 0]$ and $q_2 = [0 : 1]$. The fixed locus $\widetilde{X}^{\mathbb{G}_m}$ contains four points $\widetilde{F}_1 = \{[1 : 0 : 0]\}$, $\widetilde{F}_2 = \{q_1\}$, $\widetilde{F}_3 = \{q_2\}$, and $\widetilde{F}_4 = \{[0 : 0 : 1]\}$. We have that $\widetilde{X}_1 = X_1 \cong \mathbb{A}^2$, $\widetilde{X}_2 = X_2 \cong \mathbb{A}^1$, $X_3 = E \setminus \{q_2\} \cong \mathbb{A}^1$, and $\widetilde{X}_4 = X_4 = \widetilde{F}_4$ as illustrated in Figure 6.2. Observe that $\overline{X}_3 \setminus X_3 = \{q_2\}$ is not the union of other strata.

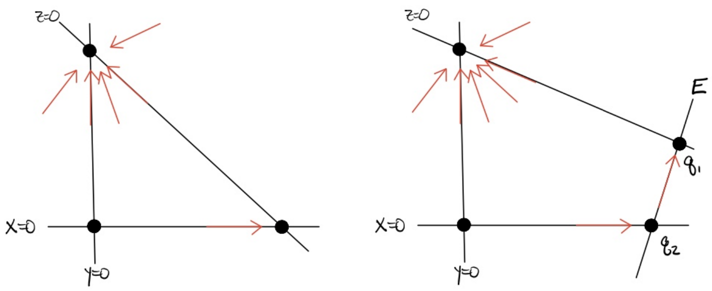


Figure 6.2: Białynicki-Birula stratifications for $\mathbb{P}^2$ (left) and $\mathrm{Bl}_p \mathbb{P}^2$ (right).

Corollary 6.6.61. *Let X be a smooth, irreducible, and quasi-projective scheme over an algebraically closed field $\Bbbk$ with an action of $\mathbb{G}_m$ such that there are only finitely many fixed points. Then $A_i(X)$ is a free $\mathbb{Z}$-module generated by the closures of the i-dimensional cells. If in addition $\Bbbk = \mathbb{C}$, then the cycle map $\mathrm{CH}_i(X) \to \mathrm{H}^{\mathrm{BM}}_{2i}(X, \mathbb{Z})$ to Borel–Moore homology is an isomorphism and $\mathrm{H}^{\mathrm{BM}}_{2i+1}(X, \mathbb{Z}) = 0$.*

Remark 6.6.62. When X is compact (e.g. projective), then $H_{2i}^{\mathrm{BM}}(X,\mathbb{Z})$ is ordinary integral singular homology.

Proof. Theorem 6.6.57(4) implies that X has a cell decomposition and the statement follows from [Ful98, Ex. 19.1.11]. See also [Bri97, §3.2]. □

Example 6.6.63 (Chow groups of $\mathrm{Hilb}_n(\mathbb{A}^2)$)**.** Let $X = \mathrm{Hilb}_n(\mathbb{A}^2)$ be the Hilbert scheme of n points; this is a smooth irreducible scheme (see 1.5.4). The natural action of $T = \mathbb{G}_m^2$ induces a T-action on X. Under the $\mathbb{G}_m$-action induced by a one-parameter subgroup $\mathbb{G}_m \to T$ given by positive weights, the evaluation map $\mathrm{ev}_0\colon X^+ \to X$ is surjective, and the $\mathbb{G}_m$-fixed points correspond to subschemes $Z = V(I) \subset \mathbb{A}^2$ supported at the origin where I is a monomial ideal. We see that there are only finitely many $\mathbb{G}_m$-fixed points. We may therefore use Corollary 6.6.61 to compute $\mathrm{CH}^*(X)$.

For a monomial ideal $I \subset R := \Bbbk[x,y]$, for each integer i, define

$$a_i := \min\{j \mid x^i y^j \in I\}$$

and let r be the largest integer such that $a_r > 0$. Then $a_0 \geq \ldots \geq a_r$ is a partition of n and $I = (y^{a_0}, xy^{a_1}, \ldots, x^{r+1})$. We need to compute the dimension of the positive weight space $T_{I,>0}$ of the $\mathbb{G}_m$-action on the tangent space

$$T_I = \mathrm{Hom}_R(I, R/I)$$

of X at the monomial ideal I; see Exercise 1.5.1 for the identification of the tangent space. To accomplish this, we first argue that

$$T_I = \sum_{0\leq i\leq j\leq r} \sum_{s=a_{j+1}}^{a_j-1} (\chi_1^{i-j-1}\chi_2^{a_i-s-1} + \chi_1^{j-i}\chi_2^{s-a_i}), \tag{6.6.12}$$

as $T = \mathbb{G}_m^2$ representations, where $\chi_i\colon T \to \mathbb{G}_m$ denotes the one-dimensional representation giving by $(t_1,t_2) \mapsto t_i^{-1}$. There are

$$\sum_{0\leq i\leq j\leq r} 2(a_j - a_{j+1}) = 2\sum_{0\leq i\leq r} a_i = 2n = \dim T_I$$

one-dimensional representations appearing on the right-hand side, and they are linearly independent. It thus suffices to show that each of them occurs in T_I. An R-module map $\phi\colon I \to R/I$ is given by the values $\phi(x^i y^{a_i})$ subject to the relations

$$\phi(x^{i+1}y^{a_i}) = x\phi(x^i a_i) \quad \text{and} \quad \phi(x^i y^{a_{i-1}}) = y^{a_{i-1}-a_i}\phi(x^i a_i).$$

Let $0 \leq i \leq j \leq r$ and $a_{j+1} \leq s < a_j$. Defining

$$\begin{aligned}
\phi_{i,j,s}\colon I \to R/I, \quad x^l y^{a_l} &\mapsto \begin{cases} x^{l+j-i}y^{a_l+s-a_i} & \text{if } l \leq i \\ 0 & \text{otherwise} \end{cases} \\
\psi_{i,j,s}\colon I \to R/I, \quad x^l y^{a_l} &\mapsto \begin{cases} x^{l+i-j-1}y^{a_l+s-a_i} & \text{if } l \geq j+1 \\ 0 & \text{otherwise,} \end{cases}
\end{aligned}$$

one checks that $\phi_{i,j,s}$ and $\psi_{i,j,s}$ are R-module maps that are eigenvectors for $\chi_1^{j-i}\chi_2^{s-a_i}$ and $\chi_1^{i-j-1}\chi_2^{a_i-s-1}$. Thus (6.6.12) holds.

Choose $\lambda = (\lambda_1, \lambda_2) \colon \mathbb{G}_m \to T$ with $\lambda_1 \gg \lambda_2$. Under our sign conventions, a character $\chi_1^a \chi_2^b$ appearing in (6.6.12) has positive weight with respect to λ if $a < 0$, or if $a = 0$ and $b < 0$. Thus

$$T_{I,>0} = \sum_{0 \le i \le j \le r} \sum_{s=a_{j+1}}^{a_j - 1} \chi_1^{i-j-1} \chi_2^{a_i - s - 1} + \sum_{j=0}^{r} \sum_{s=a_{j+1}}^{a_j - 1} \chi_1^{j-i} \chi_2^{s-a_i}$$

and

$$\begin{aligned}\dim T_{I,>0} &= \left(\sum_{i=0}^{r} \sum_{j=i}^{r} (a_j - a_{j+1}) \right) + \left(\sum_{j=0}^{r} (a_j - a_{j+1}) \right) \\ &= \left(\sum_{i=0}^{r} a_i \right) + a_0 = n + a_0\end{aligned}$$

Since there is a bijection between monomial ideals $I \subset R = \Bbbk[x,y]$ with $\dim_{\Bbbk} R/I = n$ and partitions $a_0 \ge \cdots \ge a_r$ of n, for every $d \ge 0$, the number of monomial ideals I such that $\dim T_{I,>0} = d$ is equal to

$$P(2n-d, d-n) := \#\text{ partitions } a_1 \ge \cdots \ge a_r \text{ of } 2n-d \text{ with each } a_i \le d-n. \tag{6.6.13}$$

It follows from Corollary 6.6.61 that

$$\dim \mathrm{CH}_d(\mathrm{Hilb}_n(\mathbb{A}^2))_{\mathbb{Q}} = P(2n-d, d-n).$$

See also [ESm87, Thm. 1.1] and [Göt94, §2.2].

Exercise 6.6.64 (Chow groups of $\mathrm{Hilb}_n(\mathbb{P}^2)$)**.** Follow the above strategy to show that the dth Betti number b_d of $\mathrm{Hilb}_n(\mathbb{P}^2)$ (or equivalently $\dim \mathrm{CH}_d(\mathrm{Hilb}_n(\mathbb{P}^2))$) is equal to

$$b_d = \sum_{n_0+n_1+n_2=n} \sum_{p+r=d-n_1} P(p, n_0 - p) P(n_1) P(2n_2 - r, r - n_2),$$

where $P(a)$ is the number of partitions of a and $P(a,b)$ is defined by (6.6.13).

Remark 6.6.65. Göttsche used the Weil conjectures in [Göt90, Thm. 0.1] (see also [Göt94, Thm. 2.3.10]) to show that for any smooth projective surface S over $\mathbb{C}$ or $\overline{\mathbb{F}}_q$ that the Poincaré polynomial $p(S^{[n]}, z) = \sum_i b_i(S^{[n]}) z^n$ of $S^{[n]} := \mathrm{Hilb}_n(S)$ satisfies

$$\sum_{n=0}^{\infty} p(S^{[n]}, z) t^n = \prod_{m=1}^{\infty} \frac{(1 + z^{2m-1} t^m)^{b_1(S)} (1 + z^{2m+1} t^m)^{b_3(S)}}{(1 - z^{2m-2} t^m)^{b_0(S)} (1 - z^{2m} t^m)^{b_2(S)} (1 - z^{2m+2} t^m)^{b_4(S)}}$$

In particular, the betti numbers of $S^{[n]}$ only depend on the Betti numbers of S. While each term $p(S^{[n]}, z)$ does not admit a particularly nice expression, the generating function involving *all* n does.

On the other hand, Nakajima constructed in [Nak97] (see also [Nak99b]) an action of the Heisenberg algebra on $\mathrm{H}_*(S^{[n]})$ which can be used to recover the Betti number formula above as well as additional properties of the cohomology ring.

We can also use the Białynicki-Birula Stratification to compute equivariant Chow rings $\mathrm{CH}^*_G(X)$ (or equivalently the Chow ring $\mathrm{CH}^*([X/G])$ of the quotient stack) as introduced in §6.1.7. The following statements can also be made in de Rham or singular cohomology (§6.1.8) where instead of the excision sequence above, one uses the Thom–Gysin long exact sequence.

We will use the following two lemmas, which we state in a generality that we can also apply to the HKKN stratification of §6.6.11.

Lemma 6.6.66. *Let X be a smooth irreducible scheme over an algebraically closed $\Bbbk$ with an action of a smooth affine algebraic group G. Let $S_1, \ldots, S_r \subset X$ be nonempty, disjoint, smooth, irreducible, and locally closed G-invariant subschemes such that $X = \coprod_i S_i$ and such that $S_{\geq i} := \bigcup_{j \geq i} S_j$ is closed for each i. Let d_i be the codimension of S_i in X. If the top Chern class $c^G_{d_i}(N_{S_i/X}) \in \mathrm{CH}^*_G(S_i)_{\mathbb{Q}}$ is a nonzero divisor for each i, then*

$$\dim \mathrm{CH}^k_G(X)_{\mathbb{Q}} = \sum_{i=1}^{r} \dim \mathrm{CH}^{k-d_i}_G(S_i)_{\mathbb{Q}}$$

for each k.

Proof. By assumption, $S_{\leq i} = \bigcup_{j \leq i} S_j$ is open for each i, and $S_i \subset S_{\leq i}$ is a closed subscheme with open complement $S_{<i}$. We have a commutative diagram

$$\begin{tikzcd}
\mathrm{CH}^{k-d_i}_G(S_i) \arrow[r] \arrow[dr] & \mathrm{CH}^k_G(S_{\leq i}) \arrow[r] \arrow[d] & \mathrm{CH}^k_G(S_{<i}) \arrow[r] & 0 \\
 & \mathrm{CH}^k_G(S_i) & &
\end{tikzcd}$$

where the top row is the right exact excision sequence (6.1.33(3)) and the vertical downward arrow is given by intersecting with S_i. By the self-intersection formula (6.1.33(5)), the composition $\mathrm{CH}^{k-d_i}_G(S_i) \to \mathrm{CH}^k_G(S_i)$ is multiplication by $c^G_{d_i}(N_{S_i/X})$. By hypothesis, this map is injective after tensoring with $\mathbb{Q}$. It follows that the top row is an exact sequence after tensoring with $\mathbb{Q}$, and that

$$\dim \mathrm{CH}^k_G(S_{\leq i})_{\mathbb{Q}} = \dim \mathrm{CH}^{k-d_i}_G(S_i)_{\mathbb{Q}} + \dim \mathrm{CH}^k_G(S_{<i})_{\mathbb{Q}}.$$

The formula follows from induction. See [AB83, Prop. 1.9]. □

Remark 6.6.67. If $[S_i/G]$ is Deligne–Mumford, then $\mathrm{CH}^k_G(S_i)$ vanishes for $k \gg 0$ and $c^G_{d_i}(N_{S_i/X})$ is a zero divisor.

The following gives a condition for the top Chern class to be a nonzero divisor.

Lemma 6.6.68. *Let X be a smooth irreducible scheme over an algebraically closed $\Bbbk$ with an action of a smooth affine algebraic group G, and let N be a G-equivariant vector bundle of rank d on X. Suppose that there is a subgroup $\mathbb{G}_m \subset G$ acting trivially on X and a pont $x \in X(\Bbbk)$ such that $N \otimes \kappa(x)$ contains no $\mathbb{G}_m$-invariant vectors. Then $c^G_d(N) \in \mathrm{CH}^*_G(X)_{\mathbb{Q}}$ is a non-zero divisor.*

Proof. Choose a maximal torus T containing $\mathbb{G}_m$ and a character $T \to \mathbb{G}_m$ such that the composition $\mathbb{G}_m \hookrightarrow T \to \mathbb{G}_m$ is given by $t \mapsto t^d$ for $d > 0$. By (6.1.33(7)), $\mathrm{CH}^*_G(X)_{\mathbb{Q}} = \mathrm{CH}^*_T(X)^W_{\mathbb{Q}}$ where W is the Weyl group. Since $\mathrm{CH}^*_G(X)_{\mathbb{Q}}$ is a subring of $\mathrm{CH}^*_T(X)_{\mathbb{Q}}$, we are reduced to show that $c_d^T(N) \in \mathrm{CH}^*_T(X)_{\mathbb{Q}}$ is a non-zero divisor. Writing T as the product of the giving $\mathbb{G}_m$ and a subtorus T', then since $\mathbb{G}_m$ acts trivially by (6.1.33(6))

$$\mathrm{CH}^*_T(X) \cong \mathrm{CH}^*_{T'}(X) \otimes \mathrm{CH}^*(\mathbf{B}\mathbb{G}_m) \cong \mathrm{CH}^*_{T'}(X)[T]$$

For $x \in X(\Bbbk)$, we can write

$$\begin{aligned} c_d^T(N) &= \sum c_i^{T'}(N) \otimes c_{d-i}^{\mathbb{G}_m}(N \otimes \kappa(x)) \\ &= 1 \otimes c_d^{\mathbb{G}_m}(N \otimes \kappa(x)) + \text{higher degree terms.} \end{aligned}$$

If $a_1, \ldots, a_{d_i}$ denote the $\mathbb{G}_m$-weights of $N \otimes \kappa(x)$, then by hypothesis each $a_i \neq 0$ and

$$c_d^{\mathbb{G}_m}(N \otimes \kappa(x)) = \big(\prod_i a_i\big) T^{d_i} \in \mathrm{CH}^*(\mathbf{B}\mathbb{G}_m)_{\mathbb{Q}} \cong \mathbb{Q}[T]$$

is a non-zero divisor,and therefore $c_d^T(N)$ is also a non-zero divisor.

See also [AB83, Prop. 13.4] and [Bri97, §3.2]. □

We define the *G-equivariant Chow–Poincaré polynomial* of a G-equivariant scheme X as

$$p_G(X,t) = \sum_{d=0}^{\infty} \big(\dim \mathrm{CH}^d_G(X)_{\mathbb{Q}}\big) t^d.$$

We also denote $p(X,t) = \sum_{d=0}^{\infty} \big(\dim \mathrm{CH}^d(X)_{\mathbb{Q}}\big) t^d$ as the (non-equivariant) *Chow–Poincaré polynomial.*

Proposition 6.6.69. *Let X be a smooth, irreducible, and quasi-projective scheme over an algebraically closed field $\Bbbk$ with an action of $\mathbb{G}_m$ such that $X^+ \to X$ is surjective (i.e. X is projective). Let $X = \coprod_{i=1}^r X_i$ and $X^{\mathbb{G}_m} = \coprod_{i=1}^r F_i$ be the Białynicki-Birula Stratification (6.6.57), and let d_i be the codimension of X_i in X. Then*

$$p_{\mathbb{G}_m}(X,t) = \sum_{i=1}^{r} p(F_i) \cdot t^{d_i}(1-t)^{-1}.$$

Proof. Since each F_i is smooth and $X_i \to F_i$ is a Zariski-local affine fibration (Theorem 6.6.55), the pullback map $\mathrm{CH}^*_{\mathbb{G}_m}(F_i) \xrightarrow{\sim} \mathrm{CH}^*_{\mathbb{G}_m}(X_i)$ is an isomorphism (6.1.33(2)). Under this isomorphism, $N_{X_i/X}$ is the image of its restriction $(N_{X_i/X})|_{F_i}$. For $x \in F_i$, $N_{X_i/X} \otimes \kappa(x) = T_{x,<0}$ has no $\mathbb{G}_m$-invariant vectors and thus Lemma 6.6.68 implies that $c_{d_i}^{\mathbb{G}_m}((N_{X_i/X})|_{F_i})$ is a non-zero divisor. Lemma 6.6.66 therefore implies that $p_{\mathbb{G}_m}(X,t) = \sum_i p_{\mathbb{G}_m}(X_i,t)$. Since

$$\mathrm{CH}^*_{\mathbb{G}_m}(X_i) \cong \mathrm{CH}^*_{\mathbb{G}_m}(F_i) \cong \mathrm{CH}^*(F_i) \otimes \mathrm{CH}^*(\mathbf{B}\mathbb{G}_m) \cong \mathrm{CH}^*(F_i)[T],$$

where the second equality uses (6.1.33(6)), we have the identity $p_{\mathbb{G}_m}(F_i,t) = p(F_i)(1-t)^{-1}$ and the statement follows. □

6.6.11 The Hesselink–Kempf–Kirwan–Ness Stratification

For an action of a reductive group G on a projective variety $X \subset \mathbb{P}^n$, we show that the non-semistable locus admits a stratification into locally closed subschemes according to the normalized Hilbert–Mumford index

$$M(x) := \mu(x,\lambda)/\left\|\lambda\right\| \in \mathbb{R}_{<0}$$

of a Kempf optimal destabilizing one-parameter subgroup λ of a point $x \in X \setminus X^{\mathrm{ss}}$. The more negative the index $M(x)$ is, the more non-semistable (or 'unstable') the pont x is. The strata will be indexed by pairs (λ, M) where $\lambda \in \mathbb{X}_*(G)$ and $M \in \mathbb{R}_{<0}$.

Recall from the Białynicki-Birula decomposition that for a one-parameter subgroup $\lambda\colon \mathbb{G}_m \to G$, the attractor locus $X_\lambda^+ = \mathrm{Mor}^{\mathbb{G}_m}(\mathbb{A}^1, X)$ for the induced $\mathbb{G}_m$-action is a disjoint union of locally closed subschemes.

Theorem 6.6.70 (The HKKN Statification)**.** *Let G be a linearly reductive algebraic group over an algebraically closed field $\Bbbk$ with a maximal torus T and a length $\|-\|$ on $\mathbb{X}_*(G)$. Let $X \subset \mathbb{P}(V)$ be a G-equivariant closed subscheme where V is a finite dimension G-representation. There is a finite subset $\Sigma \subset \mathbb{X}_*(T) \times \mathbb{R}_{<0}$ and a stratification of the non-semistable locus into G-invariant locally closed subschemes*

$$X \setminus X^{\mathrm{ss}} = \coprod_{(\lambda, M) \in \Sigma} S_{\lambda, M}$$

such that for each $(\lambda, M) \in \Sigma$,

(1) $X^+_{\lambda,M} := \{x \in X^+_\lambda \mid M(x) = M\}$ is a P_λ-invariant locally closed subscheme of X consisting of points x such that λ is a Kempf optimal destabilizing one-parameter subgroup for x, and $S_{\lambda,M} = G \cdot X^+_{\lambda,M}$;

(2) a point $x \in X^+_{\lambda,M}$ if and only if $\mathrm{ev}_0(x) = \lim_{t\to 0} \lambda(t) \cdot x \in X^+_{\lambda,M} \cap X^\lambda$; thus $Z_{\lambda,M} := \{x \in X^\lambda \mid M(x) = M\}$ is a C_λ-invariant closed subscheme of $X^+_{\lambda,M}$ such that $X^+_{\lambda,M} = \mathrm{ev}_0^{-1}(Z_{\lambda,M})$.

(3) the natural map $G \times^{P_\lambda} X^+_{\lambda,M} \to S_{\lambda,M}$ is finite, surjective, and universally injective; if $\mathrm{char}(\Bbbk) = 0$, then $G \times^{P_\lambda} X^+_{\lambda,M} \to S_{\lambda,M}$ is an isomorphism.

(4) the locus

$$\bigcup_{(\lambda', M') \in \Sigma, M' \leq M} S_{\lambda', M'}$$

is closed and in particular contains $\overline{S}_{\lambda,M}$;

(5) if X is smooth, then so is each $X^+_{\lambda,M}$; in $\mathrm{char}(\Bbbk) = 0$, the strata $S_{\lambda,M}$ is also smooth.

Remark 6.6.71. The locus $S_{\lambda,M}$ is called a *stratum* while $X^+_{\lambda,M}$ and $Z_{\lambda,M}$ are sometimes called a *blade* and *center* of the stratum. In characteristic 0, we have stack-theoretic equivalences $[X^+_{\lambda,M}/P_\lambda] \cong [S_{\lambda,M}/G]$ and a stratification

$$[(X \setminus X^{\mathrm{ss}})/G] = \coprod_{(\lambda,M) \in \Sigma} [S_{\lambda,M}/G].$$

For each (λ, M), there is a diagram

$$\begin{tikzcd} {[Z_{\lambda,M}/C_\lambda]} \arrow[r, "i"'] & {[X^+_{\lambda,M}/P_\lambda]} \arrow[l, bend right, "\mathrm{ev}_0"'] \cong [S_{\lambda,M}/G] \arrow[r, hook] & {[X/G]} \end{tikzcd} \tag{6.6.14}$$

such that $\mathrm{ev}_0 \circ i = \mathrm{id}$.

Proof. Let $\widehat{X} \subset \mathbb{A}(V)$ be the affine cone of X, and let $\widehat{N} \subset \widehat{X}$ be the nullcone, i.e. the affine cone of $X \setminus X^{\mathrm{ss}}$. Then $0 \in \widehat{N}$ is the unique closed G-orbit.

Applying Proposition 6.6.51 to the nullcone $\widehat{N} \subset \mathbb{A}(V)$, there is a finite subset $\Sigma \subset \mathbb{X}_*(T) \times \mathbb{R}_{<0}$ such that for every point $\widehat{x} \in \widehat{N} \setminus 0$, there is a unique $(\lambda, M) \in \Sigma$ such that λ is a Kempf optimal destabilizing one-parameter subgroup for $\widehat{x}$ with $M = \mu(\widehat{x}, \lambda)/\|\lambda\|$.

Since $\widehat{N}$ is affine, the locus $\widehat{N}^+_\lambda \subset \widehat{N}$ is a closed subscheme for each $(\lambda, M) \in \Sigma$ (Exercise 6.6.53). Since G is reductive, $P_\lambda \subset G$ is parabolic and

$$[\widehat{N}^+_\lambda / P_\lambda] \cong [G \times^{P_\lambda} \widehat{N}^+_\lambda / G] \to [\widehat{N}/G]$$

is projective. The image of this morphism is a closed substack corresponding to a closed G-invariant subscheme $\widehat{S}_\lambda$ such that $\widehat{S}_\lambda = G \cdot \widehat{N}^+_\lambda$. The loci $\widehat{N}^+_\lambda$ and $\widehat{S}_\lambda$ are invariant under scaling and are thus the affine cones over closed subschemes N_λ and S_λ of $X \setminus X^{\mathrm{ss}}$ such that $S_\lambda = G \cdot N_\lambda$.

The locus $X^+_{\lambda,M} := \{x \in X^+_\lambda \mid M(x) = M\}$ is identified with the points $x \in N_\lambda$ with $M(x) = M$. Moreover, $S_{\lambda,M} := \{x \in S_\lambda \mid M(x) = M\}$ is identified with $G \cdot X^+_{\lambda,M}$. There are identifications

$$X^+_{\lambda,M} = X^+_\lambda \setminus \bigcup_{(\lambda',M'),M'<M} X^+_{\lambda',M'} \quad \text{and} \quad S_\lambda = S_{\lambda,M} \setminus \bigcup_{(\lambda',M'),M'<M} S_{\lambda',M'}.$$

Thus $X^+_{\lambda,M}$ and $S_{\lambda,M}$ are open in X^+_λ and S_λ, and each are locally closed in $U \setminus U^{\mathrm{ss}}$. From the conclusion of Proposition 6.6.51, the loci $S_{\lambda,M}$ are disjoint and cover $U \setminus U^{\mathrm{ss}}$. This gives (1).

For (2), if $x \in X \subset \mathbb{P}(V)$, then the limit $x_0 = \lim_{t\to 0} \lambda(t) \cdot x$ is the projection onto the subspace $W = \oplus V_\chi$ ranging over characters $\chi \in \mathbb{X}^*(T)$ such that the projection $\mathrm{proj}_\chi(x)$ of x to V_χ is non-zero and $\langle \chi, \lambda \rangle = -\mu(x, \lambda)$. By Lemma 6.6.45, λ lies on the ray spanned by the unique point closest to the origin in the closed convex set of $C_x = \{\lambda \in \mathbb{X}_*(T)_\mathbb{R} \mid \langle \chi, \lambda \rangle \geq 1, \mathrm{proj}_\chi(x) \neq 0\}$. It follows that λ is also the closest point to the origin in the analogously defined set C_{x_0}. Alternatively, one can check that if $\lambda_0 \in \mathbb{X}_*(T)$ is a optimal destabilizing one-parameter subgroup for x_0, then $\mu(x_0, \lambda_0)/\|\lambda_0\| \leq \mu(x, \lambda)/\|\lambda\|$ (giving the implication $x_0 \in X^+_{\lambda,M} \Rightarrow x \in X^+_{\lambda,M}$) and $\mu(x, \lambda^N \lambda_0)/\|\lambda^N \lambda_0\| \leq \mu(x, \lambda)/\|\lambda\|$ for $N \gg 0$ (giving the implication $x \in X^+_{\lambda,M} \Rightarrow x_0 \in X^+_{\lambda,M}$).

For (3), for $x \in X^+_{\lambda,M}$ we claim that

$$P_\lambda = \{g \in G(\Bbbk) \mid gx \in X^+_{\lambda,M}\}. \tag{6.6.15}$$

Since $X^+_{\lambda,M}$ is P_λ-invariant, we have the inclusion '$\subset$'. Conversely, if $gx \in X^+_{\lambda,M}$, then both λ and $g\lambda g^{-1}$ are optimal destabilization one-parameter subgroups for x. By Kempf's Optimal Destabilization Theorem (6.6.41), the parabolics P_λ and $P_{g\lambda g^{-1}} = gP_\lambda g^{-1}$ are equal. Since $N_G(P_\lambda) = P_\lambda$ (Proposition C.3.7), we conclude that $g \in P_\lambda$. Since $[X^+_\lambda/P_\lambda] \to [X/G]$ is proper, so is $[X^+_{\lambda,m}/P_\lambda] \to [S_{\lambda,M}/G]$. The map $[X^+_{\lambda,m}/P_\lambda] \to [S_{\lambda,M}/G]$ is surjective by construction, and injective on $\Bbbk$-points by (6.6.15); it is thus finite, surjective, and universally injective, and moreover an isomorphism if $\mathrm{char}(\Bbbk) = 0$.

For (4), given $M < 0$, assume by induction that $\bigcup_{(\lambda',M')\in\Sigma,M'<M} S_{\lambda',M'}$ is closed. Then for each (λ, M), we have that

$$S_{\lambda,M} = S_\lambda \setminus \bigcup_{(\lambda',M')\in\Sigma,M'<M} S_{\lambda',M'}$$

and it follows that $\bigcup_{(\lambda',M')\in\Sigma,M'\leq M} S_{\lambda',M'}$ is closed.

For (5), if X is smooth, then each X_λ^+ is smooth (Theorem 6.6.55). Since $X_{\lambda,M}^+\subset X_\lambda^+$ is open, $X_{\lambda,M}^+$ is also smooth. In $\mathrm{char}(\Bbbk)=0$, $S_{\lambda,M}=G\times^{P_\lambda}X_{\lambda,M}^+$ by Part (3) and thus also smooth.

See also [Hes78, §4-6] and [Kir84, §12-13]. □

Remark 6.6.72. When X is a smooth projective variety over $\mathbb{C}$, the HKKN stratification coincides with the Morse stratification of the square-norm of the moment map $\|-\|^2: X\to\mathbb{R}$. Given $x\in X$, the optimal destabilizing one-parameter subgroup corresponds to the path of steepest descent starting from x. The centers $Z_{\lambda,M}$ correspond to the set of critical values of $\|-\|^2$ while the strata $S_{\lambda,M}$ are the locally closed submanifolds consisting of points which flow to $Z_{\lambda,M}$. See [Kir84, §6] and [Nes84].

Example 6.6.73. Let $\mathbb{G}_m$ act linearly on $X=\mathbb{P}^2$ with weights $-1,2,3$. Letting $\lambda=\mathrm{id}$ be the identity one-parameter subgroup, the non-semistable locus is $V(x^2y,x^3z)$ has the stratification $S_{\lambda^{-1},-1}\cup S_{\lambda,-2}\cup S_{\lambda,-3}$ where $S_{\lambda^{-1},-1}=\{[1:0:0]\}$, $S_{\lambda,-2}=\{[0:y:z]\mid y\neq 0\}$, and $S_{\lambda,-3}=\{[0:0:1]\}$.

Example 6.6.74. Revisiting the action of SL_2 on $X=(\mathbb{P}^1)^n$ with the Segre linearization (Example 6.6.47), let $\lambda_0\colon\mathbb{G}_m\to\mathrm{SL}_2$ be the one-parameter subgroup defined by $\lambda_0(t)=\mathrm{diag}(t^{-1},t)$. The strata are indexed by $(\lambda_0,-1),(\lambda_0,-3),\ldots,(\lambda_0,-n)$ if n is odd and by $(\lambda_0,-2),(\lambda_0,-4),\ldots,(\lambda_0,-n)$ if n is even. The strata $S_{\lambda_0,n-2k}$ consists of tuples with precisely $k>n/2$ points in common and has codimension $k-1$. The blade $X^+_{\lambda_0,n-2k}$ consists of tuples where precisely k points are 0 while the center $Z_{\lambda_0,n-2k}$ is the set of $\mathbb{G}_m$-fixed points where k points are 0 and $n-k$ points are ∞.

Remark 6.6.75 (Θ-stratifications)**.** As indicated in Remark 6.6.14, there is an identification

$$\underline{\mathrm{Mor}}([\mathbb{A}^1/\mathbb{G}_m],[X/G])=\coprod_{\lambda\in\mathbb{X}_*(G)/\sim}[X_\lambda^+/P_\lambda],$$

where $\mathbb{X}_*(G)/\sim$ represents the set of one-parameter subgroups up to conjugation.

A Θ-*stratification* of an algebraic stack $\mathcal{X}$ locally of finite type over $\Bbbk$ is the data of a totally ordered set Σ with a minimal element $0\in\Sigma$ and a stratification into locally closed substacks

$$\mathcal{X}=\coprod_{\lambda\in\Sigma}\mathcal{S}_\lambda$$

such that

(1) for each $\lambda\in\Sigma$, there is a union of connected components

$$\mathcal{S}'_\lambda\subset\underline{\mathrm{Mor}}([\mathbb{A}^1/\mathbb{G}_m],\mathcal{X})$$

such that $\mathrm{ev}_0\colon\mathcal{S}'_\lambda\to\mathcal{X}$ is a locally closed immersion mapping isomorphically onto $\mathcal{S}_\lambda$.

(2) for each $\lambda\in\Sigma$, $\bigcup_{\lambda'\leq\lambda}\mathcal{S}_{\lambda'}$ is an open substack containing $\mathcal{S}_\lambda$ as a closed substack;

see [HL14]. Let $\mathcal{Z}'_\lambda$ be the preimage of $\mathcal{S}'_\lambda$ under the map $i\colon\underline{\mathrm{Mor}}(\mathbf{B}\mathbb{G}_m,[X/G])\to\underline{\mathrm{Mor}}([\mathbb{A}^1/\mathbb{G}_m],\mathcal{X})$. The map $\mathrm{ev}_0\colon\underline{\mathrm{Mor}}([\mathbb{A}^1/\mathbb{G}_m],[X/G])\to\underline{\mathrm{Mor}}(B\mathbb{G}_m,\mathcal{X})$ obtained by restricting to 0 is a section of i, and there is a diagram analogous to

(6.6.14)

$$\begin{tikzcd} \mathcal{Z}'_\lambda \arrow[r, "i"] & \mathcal{S}'_\lambda \arrow[l, bend right, "\mathrm{ev}_0"'] \arrow[r, hook] & \mathcal{X}. \end{tikzcd}$$

In characteristic 0, the HKKN stratification is an example of a Θ-stratification, where one orders the indices (λ, M) first by $-M$ and then arbitrary by λ. In the next chapter, we will see that the moduli stack $\mathcal{B}\mathrm{un}_{r,d}(C)$ has a Θ-stratification called the Harder–Narasimhan–Shatz stratification.

Recall that the *Chow–Poincare polynomial* of a G-equivariant scheme X is $p_G(X,t) = \sum_{d=0}^{\infty} \big(\dim \mathrm{CH}^d_G(U)_{\mathbb{Q}}\big)t^d$.

Proposition 6.6.76 (Kirwan Surjectivity)**.** *Under the hypotheses of Theorem 6.6.70, assume further assume that X is smooth and irreducible, and that $\mathrm{char}(\Bbbk) = 0$. Suppose that for all (λ, M), the stratum $S_{\lambda,M}$ is equidimensional of codimension $d_{\lambda,M}$. Then*

$$\dim \mathrm{CH}^k_G(X)_{\mathbb{Q}} = \dim \mathrm{CH}^k_G(X^{\mathrm{ss}})_{\mathbb{Q}} + \sum_{(\lambda,M)} \dim \mathrm{CH}^{k-d_{\lambda,M}}_{C_\lambda}(Z_{\lambda,M})_{\mathbb{Q}}$$

and

$$p_G(X,t) = p_G(X^{\mathrm{ss}},t) + \sum_{(\lambda,M)} p_{C_\lambda}(Z_{\lambda,M},t)t^{d_{\lambda,M}}.$$

Proof. From Theorem 6.6.70, we know that $[S_{\lambda,M}/G] \cong [X^+_{\lambda,M}/P_\lambda]$. From Theorem 6.6.55, we know that $\mathrm{ev}_0 \colon X^+_{\lambda,M} \to Z_{\lambda,M}$ sending a point to its limit is a Zariski-local affine fibration and equivariant with respect to $P_\lambda \to C_\lambda$. We claim that $[X^+_{\lambda,M}/P_\lambda] \to [Z_{\lambda,M}/C_\lambda]$ induces an isomorphism

$$\mathrm{CH}^*_{C_\lambda}(Z_{\lambda,M}) \to \mathrm{CH}^*_{P_\lambda}(X^+_{\lambda,M}). \tag{6.6.16}$$

By the definition of the equivariant Chow groups, $\mathrm{CH}^i_{P_\lambda}(X^+_{\lambda,M})$ is identified by $\mathrm{CH}^i(X^+_{\lambda,M} \times^{P_\lambda} V)$ where V is an open subspace $\mathbb{A}(W)$ of a P_λ-representation such that P_λ acts freely on V and $\mathbb{A}(W) \setminus V$ has sufficiently high codimension. On the other hand, $\mathrm{CH}^i_{C_\lambda}(Z_{\lambda,M})$ is identified with $\mathrm{CH}^i(Z_{\lambda,M} \times^{C_\lambda} V)$ and the map (6.6.16) corresponds to the pullback map on Chow induced from the composition

$$X^+_{\lambda,M} \times^{P_\lambda} V \to Z_{\lambda,M} \times^{P_\lambda} V \to Z_{\lambda,M} \times^{C_\lambda} V.$$

The first map is a Zariski local affine fibration and the second map is a principal bundle under $U_\lambda = \ker(P_\lambda \to C_\lambda)$. Since U_λ is unipotent, U_λ is isomorphic to affine space and principal U_λ-bundles are locally trivial in the Zariski topology. We conclude that (6.6.16) is an isomorphism.

We also claim that $c_{d_{\lambda,M}}(N_{S_{\lambda,M}/X}) \in \mathrm{CH}^*_G(S_{\lambda,M})$ is a nonzero divisor. Since $N_{S_{\lambda,M}/X}|_{Z_{\lambda,M}}$ is identified with $N_{S_{\lambda,M}/X}$ under $\mathrm{CH}^*_{C_\lambda}(Z_{\lambda,M}) \cong \mathrm{CH}^*_G(S_{\lambda,M})$, it suffices to show that $c_{d_{\lambda,M}}((N_{S_{\lambda,M}/X})|_{Z_{\lambda,M}}) \in \mathrm{CH}^*_{C_\lambda}(Z_{\lambda,M})_{\mathbb{Q}}$ is a nonzero divisor where $d = d_{\lambda,M}$. By Theorem 6.6.55, λ acts on a fiber of the normal bundle with non-zero weights. Thus Lemma 6.6.68 implies that $c_{d_{\lambda,M}}((N_{S_{\lambda,M}/X})|_{Z_{\lambda,M}})$ is a non-zero divisor.

We therefore can apply Lemma 6.6.66 with the strata $S_{\lambda,M}$ ordered first by $-M$ and then with any ordering of the λ's; the semistable locus U^{ss} is viewed as a strata with the smallest index. This yields

$$\begin{aligned}\dim \mathrm{CH}^k_G(X)_{\mathbb{Q}} &= \dim \mathrm{CH}^k_G(X^{\mathrm{ss}})_{\mathbb{Q}} + \sum_{(\lambda,M)} \dim \mathrm{CH}^{k-d_{\lambda,M}}_G(S_{\lambda,M})_{\mathbb{Q}} \\ &= \dim \mathrm{CH}^k_G(X^{\mathrm{ss}})_{\mathbb{Q}} + \sum_{(\lambda,M)} \dim \mathrm{CH}^{k-d_{\lambda,M}}_{C_{\lambda,M}}(Z_{\lambda,M})_{\mathbb{Q}}.\end{aligned}$$

□

Remark 6.6.77. This formula was established for de Rham cohomology in [Kir84, Thm. 5.4]. Instead of the excision sequence

$$\mathrm{CH}^{k-d_{\lambda,M}}_G(S_{\lambda,M}) \to \mathrm{CH}^k_G(S_{\leq(\lambda,M)}) \to \mathrm{CH}^k_G(S_{<(\lambda,M)}) \to 0,$$

one uses the Thom–Gysin long exact sequence

$$\cdots \to \mathrm{H}^{k-d_{\lambda,M}}_G(S_{\lambda,M}) \to \mathrm{H}^k_G(S_{\leq(\lambda,M)}) \to \mathrm{H}^k_G(S_{<(\lambda,M)}) \to \cdots .$$

In this case, the surjectivity of the right map for all (λ, M) is equivalent to the injectivity of the left map for all (λ, M), and the latter condition is verified as above by the showing the top Chern class of the normal bundle is a non-zero divisor.

Example 6.6.78. As an application, we can compute the dimension of the rational Chow groups of $[(\mathbb{P}^1)^{n,\mathrm{ss}}/\mathrm{SL}_2]$ using the computation of the stratification in Example 6.6.74. When n is odd, this also gives the dimension of the rational Chow groups of the GIT quotient $(\mathbb{P}^1)^{n,\mathrm{ss}}/\mathrm{SL}_2$ by Properties 6.1.33(4).

Since $[(\mathbb{P}^1)^n/\mathrm{SL}_2] \to \mathbf{B}\,\mathrm{SL}_2$ is an iterated $\mathbb{P}^1$-bundle and $\mathrm{CH}^*(\mathrm{SL}_2) \cong \mathbb{Z}[T]$ generated in degree 2,

$$\begin{aligned}\mathrm{CH}^*([(\mathbb{P}^1)^n/\mathrm{SL}_2]) &\cong \mathrm{CH}^*((\mathbb{P}^1)^n) \otimes \mathrm{CH}^*(\mathbf{B}\,\mathrm{SL}_2) \\ &\cong \mathbb{Z}[H_1,\ldots,H_n]/(H_1,\ldots,H_n)^2 \otimes \mathbb{Z}[T]\end{aligned}$$

and the Chow–Poincare polynomial is $p_{\mathrm{SL}_2}((\mathbb{P}^1)^n, t) = (1+t)^n(1-t^2)^{-1}$. On the other hand, the strata $S_{\lambda,n-2k}$ where precisely k points are the same has codimension $k-1$ and its center $Z_{\lambda,n-2k}$ consists of $\binom{n}{k}$ $\mathbb{G}_m$-fixed points. Thus $p_{\mathbb{G}_m}(Z_{\lambda,n-2k}, t) = \binom{n}{k}(1-t)^{-1}$ and

$$\begin{aligned}p_G((\mathbb{P}^1)^{\mathrm{ss}}, t) &= (1+t)^n(1-t)^{-1} - \sum_{k>n/2} \binom{n}{k} t^{k-1}(1-t)^{-1} \\ &= 1 + nt + \cdots + \left(1 + (n-1) + \binom{n-1}{2} + \cdots + \binom{n-1}{\min(d, n-3-d)}\right)t^d \\ &\quad + \cdots + nt^{n-4} + t^{n-3}.\end{aligned}$$

See also [Kir84, §16.1].

6.7 Existence of good moduli spaces

In this section, we provide necessary and sufficient conditions for the existence of a separated good moduli space in characteristic 0.

Theorem 6.7.1 (Existence Theorem of Good Moduli Spaces). *Let $\mathcal{X}$ be an algebraic stack, of finite type over an algebraic closed field $\Bbbk$ of characteristic 0, with affine diagonal. There exists a good moduli space $\pi\colon \mathcal{X} \to X$ with X a separated algebraic space if and only if $\mathcal{X}$ is Θ-complete (Definition 6.7.7) and S-complete (Definition 6.7.9).*

Moreover, X is proper if and only $\mathcal{X}$ satisfies the existence part of the valuative criterion for properness.

The conditions of Θ-completeness and S-completeness are defined and discussed in detail in §6.7.2.

6.7.1 Strategy for constructing good moduli spaces

We first explain how the Local Structure Theorem for Algebraic Stacks (6.5.1) gives us a natural strategy to construct the good moduli space X. Namely, for each closed point $x \in \mathcal{X}$, we have an étale quotient presentation

$$\begin{tikzcd} \mathcal{W} = [\operatorname{Spec} A/G_x] \arrow[r, "f"] \arrow[d] & \mathcal{X} \\ W = \operatorname{Spec} A^{G_x} \end{tikzcd}$$

where f is affine étale, and there is a preimage $w \in \mathcal{W}$ of x such that f induces an isomorphism of stabilizer groups at w. We would like to show that the GIT quotients $W = \operatorname{Spec} A^{G_x}$ as x ranges over closed points provides étale models that can be glued to a good moduli space of $\mathcal{X}$. To this end, we need to construct an étale equivalence relation on W. Since f is affine, the fiber product $\mathcal{R} := \mathcal{W} \times_{\mathcal{X}} \mathcal{W}$ is isomorphic to a quotient stack $[\operatorname{Spec} B/G_x]$ and we have a diagram

$$\begin{tikzcd} \mathcal{R} \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] \arrow[d] & \mathcal{W} \arrow[r, "f"] \arrow[d] & \mathcal{X} \\ R \arrow[r, shift left, "q_1"] \arrow[r, shift right, "q_2"'] & W \end{tikzcd}$$

where $R = \operatorname{Spec} B^{G_x}$. If $q_1, q_2 \colon R \rightrightarrows W$ defines an étale equivalence relation, the algebraic space quotient W/R gives a candidate for a good moduli space of $f(\mathcal{W}) \subset \mathcal{X}$.

Luna's Fundamental Lemma (6.3.26) provides condition on when $q_1, q_2 \colon R \rightrightarrows W$ are étale: we need that for all closed points $r \in \mathcal{R}$ that

(a) $p_1(r), p_2(r) \in \mathcal{W}$ are closed points; and

(b) p_1 and p_2 induce isomorphisms of stabilizer groups at r.

On the other hand, we know that $f(w) \in \mathcal{X}$ is closed and f induces an isomorphism of stabilizer groups at the given preimage w of x. We would like to show that there is an open neighborhood $\mathcal{U}$ of w such that the restriction $f|_{\mathcal{U}}$ satisfies: (a) $f|_{\mathcal{U}}$ sends closed points map to closed points and (b) $f|_{\mathcal{U}}$ induces isomorphism of stabilizer groups at closed points, and moreover that these conditions are stable under base change. While property (a) is stable under base change, property (b) is not, and we will introduce a stronger condition below—called Θ-surjectivity (Definition 6.7.30)—which is stable under base change and implies (b).

The role of Θ-completeness and S-completeness in the construction of the good moduli space is the following: the Θ-completeness of $\mathcal{X}$ implies that Θ-surjectivity holds (and thus condition (a) and its base changes hold) in an open neighborhood of w (Proposition 6.7.34) while S-completeness implies that condition (b) holds in an open neighborhood of w (Proposition 6.7.40).

Counterexamples

The following examples do not admit good moduli spaces. We will explain why the approach outlined above fails and then later explain how they violate the conditions of Θ-completeness and S-completeness. We work over an algebraically closed field $\Bbbk$.

Example 6.7.2. Consider the action of $\mathbb{G}_m$ on $\mathbb{P}^1$ given by $t \cdot [x : y] = [tx : y]$. The quotient stack $\mathcal{X} = [\mathbb{P}^1/\mathbb{G}_m]$ does not admit a good moduli space. Note that Theorem 6.3.5(2) implies that every $\Bbbk$-point has a unique closed point in its closure. Here we see that $[1 : 1]$ specializes to two closed points $[1 : 0]$ and $[0 : 1]$. Alternatively, if there were a good moduli space, it would have to be $\mathcal{X} \to \operatorname{Spec} \Bbbk$ (which is universal for maps to algebraic spaces) but then the composition $\mathbb{P}^1 \to \mathcal{X} \to \operatorname{Spec} \Bbbk$ would be affine by Serre's Criterion for Affineness (4.4.15) , a contradiction.

There are two open substacks $\mathcal{U}_1, \mathcal{U}_2 \subset [\mathbb{P}^1/\mathbb{G}_m]$ isomorphic to $[\mathbb{A}^1/\mathbb{G}_m]$ each which admits a good moduli space $\pi_i : \mathcal{U}_i \to \operatorname{Spec} \Bbbk$ but they do not glue to a good moduli space of $\mathcal{X}$: the intersection $\mathcal{U}_1 \cap \mathcal{U}_2$ is the open point in both $\mathcal{U}_1$ and $\mathcal{U}_2$ and not the preimage of an open subscheme under π_i. To see how the approach above fails, observe that the étale presentation $f : \mathcal{W} := \mathcal{U}_1 \coprod \mathcal{U}_2 \to \mathcal{X}$ satisfies (a) and (b) but the base changes $p_1, p_2 : \mathcal{W} \times_{\mathcal{X}} \mathcal{W} = \mathcal{U}_1 \coprod \mathcal{U}_2 \coprod \mathcal{U}_1 \cap \mathcal{U}_2 \to \mathcal{W}$ fails (b), i.e. the closed point in $\mathcal{U}_1 \cap \mathcal{U}_2$ is mapped to a non-closed point under either projection.

Example 6.7.3. For a related example, let C be the projective nodal cubic with its $\mathbb{G}_m$-action. The quotient $\mathcal{X} = [C/\mathbb{G}_m]$ has two points—one open and one closed—but while there is no topological obstruction as above, $\mathcal{X}$ again does not admit a good moduli space because C is projective, not affine. Viewing the nodal cubic as the quotient of nodal union X' of two $\mathbb{P}^1$'s along 0 and ∞ modulo the rotation action of $\mathbb{Z}/2$, we have a finite étale cover $[X'/\mathbb{G}_m] \to [X/\mathbb{G}_m]$. Removing one of the origins, we have an affine étale cover $\mathcal{W} = [\operatorname{Spec}(k[x,y]/xy)/\mathbb{G}_m] \to \mathcal{X}$ where $\mathbb{G}_m$ acts via $t \cdot (x, y) = (tx, t^{-1}y)$. Again, this map sends closed points to closed points, but the projections $\mathcal{W} \times_{\mathcal{X}} \mathcal{W} \rightrightarrows \mathcal{W}$ do not.

Example 6.7.4. Let $\mathbb{G}_m$ act on $\mathbb{A}^2$ via $t \cdot (x, y) = (tx, y)$ and set $\mathcal{X} = [\mathbb{A}^2/\mathbb{G}_m] \smallsetminus 0$. The point $p = (1, 0) \in \mathcal{X}$ is closed with trivial stabilizer, and the open immersion $f : \mathbb{A}^1 \hookrightarrow \mathcal{X}$, sending z to $(z, 1)$, is an étale quotient presentation. Note that while $f(0)$ is closed, the image $f(z)$ is not closed for $z \neq 0$. The map $\mathcal{X} \to \mathbb{A}^1$ defined by $(x, y) \mapsto y$ is not a good moduli space as $\mathbb{A}^2 \smallsetminus 0$ is not affine.

We will see in the next section that the previous examples violate Θ-completeness. Similar phenomenon can naturally occur in moduli, e.g. by removing a single polystable but not stable vector bundle from $\mathcal{B}\mathrm{un}_{r,d}(C)^{\mathrm{ss}}$. The next examples violate S-completeness.

Example 6.7.5. Suppose $\operatorname{char}(\Bbbk) \neq 2$ and let $G = \mathbb{Z}/2$ act on the non-separated union $U = \mathbb{A}^1 \bigcup_{x\neq 0} \mathbb{A}^1$ by exchanging the copies of $\mathbb{A}^1$. The quotient stack $[U/G]$ has a $\mathbb{Z}/2$ stabilizer everywhere except at the origin. This is a Deligne–Mumford stack with quasi-finite but not finite inertia; in fact we've seen this before in Exercise 4.3.19 to illustrate the necessity of the separatedness condition in the Keel–Mori Theorem (4.3.11). By precomposing by the inclusion of one of the $\mathbb{A}^1$'s, we have an affine étale morphism $\mathbb{A}^1 \to [U/G]$ which is stabilizer preserving at 0 but not in any open neighborhood of 0.

For a related example, the Deligne–Mumford locus $\mathcal{X}^{\mathrm{DM}}$ in the moduli stack $\mathcal{X} = [\operatorname{Sym}^4 \mathbb{P}^1 / \operatorname{PGL}_2]$ of four unordered points in $\mathbb{P}^1$ is not separated (see Example 4.3.20). Note however that the stable locus $\mathcal{X}^{\mathrm{s}}$ consisting of four distinct points is separated and the semistable locus $\mathcal{X}^{\mathrm{ss}} = \mathcal{X}^{\mathrm{DM}} \cup \{[0:0:\infty:\infty]\}$ has a projective good moduli space.

Example 6.7.6. Consider the action of $G = \mathbb{G}_m \rtimes \mathbb{Z}/2$ on $X = \mathbb{A}^2 \smallsetminus 0$ via $t \cdot (a,b) = (ta, t^{-1}b)$ and $-1 \cdot (a,b) = (b,a)$. Note that every point $(a,b) \in X$ with $ab \neq 0$ is fixed by the order 2 element $(a/b, -1) \in G$. The quotient stack $[X/G]$ is a non-separated Deligne–Mumford stack which does not admit a good moduli space; note however that $[\mathbb{A}^2/G] \to \operatorname{Spec} \Bbbk[xy]$ is a good moduli space.

6.7.2 The valuative criteria: Θ- and S-completeness

We define the stack 'Theta' as

$$\Theta := [\mathbb{A}^1/\mathbb{G}_m]$$

over $\operatorname{Spec} \mathbb{Z}$.[9] If R is a DVR with fraction field K and residue field κ, we define $\Theta_R := \Theta \times \operatorname{Spec} R$ and set $0 \in \Theta_R$ to be the unique closed point. Observe that Θ_R is a local model of the quotient stack $[\mathbb{A}^2/\mathbb{G}_m]$ with weights $0, 1$ as it is identified with the base change of the good moduli space $[\mathbb{A}^2/\mathbb{G}_m] \to \operatorname{Spec} \Bbbk[x]$ along the map $\operatorname{Spec} R \to \operatorname{Spec} \Bbbk[x]$ where x maps to a uniformizer π in R.

The following cartesian diagram gives a schematic picture of Θ_R (where x is the coordinate on $\mathbb{A}^1$ and $\pi \in R$ is the uniformizer).

$$\begin{tikzcd}
 & \operatorname{Spec} R \arrow[dr, hook, "x\neq 0"] & & \mathbf{B}\mathbb{G}_{m,R} \arrow[dl, hook', "x=0"'] & \\
\operatorname{Spec} K \arrow[ur, hook] \arrow[dr, hook] & & \Theta_R & & \mathbf{B}\mathbb{G}_{m,\kappa} \arrow[ul, hook'] \arrow[dl, hook'] \\
 & \Theta_K \arrow[ur, hook, "\pi\neq 0"'] & & \Theta_\kappa \arrow[ul, hook', "\pi=0"] &
\end{tikzcd} \tag{6.7.1}$$

where the maps to the left are open immersions and to the right are closed immersions.

In particular, a morphism $\Theta_R \smallsetminus 0 = \operatorname{Spec} R \bigcup_{\operatorname{Spec} K} \Theta_K \to \mathcal{X}$ to an algebraic stack is the data of morphisms $\operatorname{Spec} R \to \mathcal{X}$ and $\Theta_K \to \mathcal{X}$ together with an isomorphism of their restrictions to $\operatorname{Spec} K$.

[9]The symbol Θ is used as it resembles the picture of the two orbits of $\mathbb{G}_m$ on the complex plane.

Definition 6.7.7. A noetherian algebraic stack $\mathcal{X}$ is Θ-*complete*[10] if for every DVR R, every commutative diagram

$$\begin{tikzcd} \Theta_R \smallsetminus 0 \arrow[r] \arrow[d] & \mathcal{X} \\ \Theta_R \arrow[ur, dashed] & \end{tikzcd} \tag{6.7.2}$$

of solid arrows can be uniquely filled in.

Remark 6.7.8. We can state an equivalent formulation using the stack $\underline{\mathrm{Mor}}(\Theta, \mathcal{X})$ classifying morphisms $\Theta \to \mathcal{X}$. Evaluation at 1 gives a morphism

$$\mathrm{ev}_1 \colon \underline{\mathrm{Mor}}(\Theta, \mathcal{X}) \to \mathcal{X}, \quad f \mapsto f(1)$$

of stacks, and the Θ-completeness of $\mathcal{X}$ is equivalent to the morphism ev_1 satisfying the valuative criterion for properness. If $\mathcal{X}$ is of finite type over an algebraically closed field $\Bbbk$, then the stack $\underline{\mathrm{Mor}}(\Theta, \mathcal{X})$ is an algebraic stack *locally* of finite type over $\Bbbk$; see Remark 6.6.14 where an explicit description is given when $\mathcal{X}$ is a quotient stack. The stack $\underline{\mathrm{Mor}}(\Theta, \mathcal{X})$ is however rarely quasi-compact, e.g. for $\mathcal{X} = \mathbf{B}\mathbb{G}_m$, and ev_1 is thus rarely proper.

For a DVR R with fraction field K, residue field κ, and uniformizer π, we define

$$\phi_R := [\mathrm{Spec}\,\big(R[s,t]/(st-\pi)\big)/\mathbb{G}_m], \tag{6.7.3}$$

where s and t have $\mathbb{G}_m$-weights 1 and -1 respectively.[11] The quotient ϕ_R is a local model of the quotient stack $[\mathbb{A}^2/\mathbb{G}_m]$ with weights $1, -1$ as it is identified with the base change of the good moduli space $[\mathbb{A}^2/\mathbb{G}_m] \to \mathrm{Spec}\,\Bbbk[xy]$ along the map $\mathrm{Spec}\,R \to \mathrm{Spec}\,\Bbbk[xy]$ given by $xy \mapsto \pi$.

The locus where $s \neq 0$ in ϕ_R is isomorphic to $[\mathrm{Spec}\,\big(R[s,t]_s/(t-\pi/s)\big)/\mathbb{G}_m] \cong [\mathrm{Spec}(R[s]_s)/\mathbb{G}_m] \cong \mathrm{Spec}\,R$ and the locus where $t \neq 0$ has a similar description. We thus have cartesian diagrams analogous to (6.7.1)

$$\begin{tikzcd} & \mathrm{Spec}\,R \arrow[dr, hook, "s\neq 0"] & & \Theta_\kappa \arrow[dl, hook', "s=0"'] & \\ \mathrm{Spec}\,K \arrow[ur, hook] \arrow[dr, hook] & & \phi_R & & \mathbf{B}\mathbb{G}_{m,\kappa} \arrow[ul, hook'] \arrow[dl, hook] \\ & \mathrm{Spec}\,R \arrow[ur, hook, "t\neq 0"'] & & \Theta_\kappa \arrow[ul, hook', "t=0"] & \end{tikzcd} \tag{6.7.4}$$

where the maps to the left are open immersions and to the right are closed immersions.

In particular, a morphism $\phi_R \smallsetminus 0 = \mathrm{Spec}\,R \bigcup_{\mathrm{Spec}\,K} \mathrm{Spec}\,R \to \mathcal{X}$ to an algebraic stack is the data of two morphisms $\xi, \xi' \colon \mathrm{Spec}\,R \to \mathcal{X}$ together with an isomorphism $\xi_K \simeq \xi'_K$ over $\mathrm{Spec}\,K$.

[10] In the literature, the term 'Θ-reductive' is often used.

[11] The symbol ϕ is used because it looks like the non-separated affine line with an additional origin. In the literature, $\overline{\mathrm{ST}}_R$ is used as it is a compactification of $\mathrm{ST}_R = \overline{\mathrm{ST}}_R \smallsetminus 0 = \mathrm{Spec}\,R \bigcup_{\mathrm{Spec}\,K} \mathrm{Spec}\,R$, which is the 'standard test' scheme for separatedness.

Definition 6.7.9. A noetherian algebraic stack $\mathcal{X}$ is *S-complete* if for every DVR R, every commutative diagram

$$\begin{tikzcd} \phi_R \setminus 0 \arrow[r] \arrow[d] & \mathcal{X} \\ \phi_R \arrow[ur, dashed] & \end{tikzcd} \tag{6.7.5}$$

of solid arrows can be uniquely filled in.[12]

Remark 6.7.10. There are obvious extension of the definition of Θ-completeness and S-completeness to morphisms $f\colon \mathcal{X} \to \mathcal{Y}$ but we will not need such notions.

Lemma 6.7.11. *A noetherian algebraic stack with affine diagonal is Θ-complete (resp. S-complete), if and only if every diagram (6.7.2) (resp. (6.7.5)), there exists a lift after an extension of DVRs $R \subset R'$. In particular, Θ-completeness and S-completeness can be verified on complete DVRs with algebraically closed residue field.*

Proof. We begin with the observation that if $\mathcal{X} \to \mathcal{Y}$ has affine diagonal and $j\colon \mathcal{U} \to \mathcal{T}$ is an open immersion of algebraic stacks over $\mathcal{Y}$ with $j_*\mathcal{O}_{\mathcal{U}} = \mathcal{O}_{\mathcal{T}}$, then two extensions $f_1, f_2\colon \mathcal{T} \to \mathcal{X}$ of a $\mathcal{Y}$-morphism $\mathcal{U} \to \mathcal{X}$ are canonically 2-isomorphic. Indeed, since $\mathrm{Isom}_{\mathcal{T}}(f_1, f_2) \to \mathcal{T}$ is affine, the section over $\mathcal{U}$ induced by the 2-isomorphism $f_1|_{\mathcal{U}} \xrightarrow{\sim} f_2|_{\mathcal{U}}$ extends uniquely to a section of $\mathcal{T}$.

Consider a diagram (6.7.2), an extension of DVRs $R \subset R'$, and a lifting $\Theta_{R'} \to \mathcal{X}$. The open immersion $j\colon \Theta_R \setminus 0 \to \Theta_R$ satisfies $j_*\mathcal{O}_{\Theta_R \setminus 0} = \mathcal{O}_{\Theta_R}$ and by flat base change, the same property holds for the morphisms obtained by base changing j along $\Theta_{R'} \to \Theta_R$, $\Theta_{R'} \times_{\Theta_R} \Theta_{R'} \to \Theta_R$, and $\Theta_{R'} \times_{\Theta_R} \Theta_{R'} \times_{\Theta_R} \Theta_{R'} \to \Theta_R$. By the above observation, there exists a canonical 2-isomorphism between the two extensions $\Theta_{R'} \times_{\Theta_R} \Theta_{R'} \rightrightarrows \Theta_{R'} \to \mathcal{X}$ which necessarily satisfies the cocycle condition. By fpqc descent, the lifting $\Theta_{R'} \to \mathcal{X}$ descends to a lifting $\Theta_R \to \mathcal{X}$. The same argument works for S-completeness. $\square$

Remark 6.7.12. It is even true that when $\mathcal{X}$ is of finite type over $\Bbbk$, these criteria can be verified on DVRs essentially of finite type over $\Bbbk$; see [AHLH18, §4]. We will not use this fact.

Lemma 6.7.13. *Let $f\colon \mathcal{X} \to \mathcal{Y}$ be an affine morphism of noetherian algebraic stacks. If $\mathcal{Y}$ is Θ-complete (resp. S-complete), so is $\mathcal{X}$.*

Proof. Since Θ_R is regular and $0 \in \Theta_R$ is codimension 2, the pushforward of the structure sheaf along $\Theta_R \setminus 0 \to \Theta_R$ is the structure sheaf. We therefore have canonical equivalences

$$\begin{aligned} \mathrm{MOR}_{\mathcal{Y}}(\Theta_R \setminus 0, \mathcal{X}) &\cong \mathrm{MOR}_{\mathcal{O}_{\mathcal{Y}}\text{-alg}}(f_*\mathcal{O}_{\mathcal{X}}, (\Theta_R \setminus 0 \to \mathcal{Y})_*\mathcal{O}_{\Theta_R \setminus 0}) \\ &\cong \mathrm{MOR}_{\mathcal{O}_{\mathcal{Y}}\text{-alg}}(f_*\mathcal{O}_{\mathcal{X}}, (\Theta_R \to \mathcal{Y})_*\mathcal{O}_{\Theta_R}) \\ &\cong \mathrm{MOR}_{\mathcal{Y}}(\Theta_R, \mathcal{X}). \end{aligned}$$

The case of S-completeness is identical. $\square$

[12] The 'S' stands for 'Seshadri' as S-completeness is a geometric property reminiscent of how the S-equivalence relation on sheaves implies separatedness of the moduli space.

Proposition 6.7.14. *If G is a reductive group over an algebraically closed field* $\Bbbk$*, then every quotient stack* $[\operatorname{Spec} A/G]$ *is* Θ*-complete and* S*-complete.*

Proof. We first show that $\mathbf{B}\operatorname{GL}_n$ is Θ-complete. A morphism $\Theta_R \setminus 0 \to \mathcal{X}$ corresponds to a vector bundle E on $\Theta_R \setminus 0$. The algebraic stack Θ_R is regular and $0 \in \Theta_R$ is a codimension 2 point. If $\widetilde{E}$ is a coherent sheaf on Θ_R extending E, then the double dual $\widetilde{E}^{\vee\vee}$ is a vector bundle extending E. (In fact the pushforward of E along $\Theta_R \setminus 0 \hookrightarrow \Theta_R$ is a vector bundle.) This provides the desired extension $\Theta_R \to \mathcal{X}$. As G is affine, we can choose a faithful representation $G \subset \operatorname{GL}_n$. As G is reductive, the quotient GL_n/G is affine by Matushima's Theorem (C.4.9). Using the cartesian diagram

$$\begin{tikzcd} \operatorname{GL}_n/G \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \operatorname{Spec}\Bbbk \arrow[d] \\ \mathbf{B}G \arrow[r] & \mathbf{B}\operatorname{GL}_n \end{tikzcd}$$

and smooth descent, we see that $\mathbf{B}G \to \mathbf{B}\operatorname{GL}_n$ is affine. We conclude that $\mathbf{B}G$ and $[\operatorname{Spec} A/G]$ are Θ-complete by Lemma 6.7.13. $\square$

As a result, we see that the hypotheses of Θ-completeness and S-completeness in Theorem 6.7.1 are necessary.

Corollary 6.7.15. *Let $\mathcal{X}$ be an algebraic stack of finite type over an algebraically closed field $\Bbbk$ with affine diagonal. If $\pi\colon \mathcal{X} \to X$ be a good moduli space, then $\mathcal{X}$ is Θ-complete. Moreover, $\mathcal{X}$ is S-complete if and only if X is separated.*

Proof. For a $\Bbbk$-algebra A, the map $\Theta_A \to \operatorname{Spec} A$ is a good moduli space and thus every map $\Theta_A \to X$ factors through $\operatorname{Spec} A$ by the universality of good moduli spaces (Theorem 6.3.5(4)). If R is a DVR with fraction field K, then every map $\Theta_R \to X$ (resp. $\Theta_K \to X$) factors through $\operatorname{Spec} R$ (resp. $\operatorname{Spec} K$). To see that $\mathcal{X}$ is Θ-complete, it therefore suffices to a find a lift of every commutative diagram

$$\begin{tikzcd} \Theta_R \setminus 0 \arrow[r] \arrow[d] & \mathcal{X} \arrow[d, "\pi"] \\ \operatorname{Spec} R \arrow[r] \arrow[ur, dashed] & X \end{tikzcd}$$

of solid arrows. By the Local Structure for Good Moduli Spaces (6.5.3), there exists an étale morphism $\operatorname{Spec} B \to X$ containing the image of $\operatorname{Spec} R$ such that $\mathcal{X} \times_X \operatorname{Spec} B \cong [\operatorname{Spec} A/G]$ with G linearly reductive and $B = A^G$. Since $\operatorname{Spec} R \to X$ lifts to $\operatorname{Spec} B$ after an extension of DVRs and since Θ-completeness can be checked after an extension (Lemma 6.7.11), we are reduced to the case of $[\operatorname{Spec} A/G]$. This is Proposition 6.7.14.

If X is separated, then X is S-complete as $\phi_R \setminus 0 = \operatorname{Spec} R \cup_{\operatorname{Spec} K} \operatorname{Spec} R \to X$ factors through $\operatorname{Spec} R$ by the valuative criterion for separatedness. The above argument can be repeated to show that $\mathcal{X}$ is S-complete. Conversely, suppose $f, g\colon \operatorname{Spec} R \to X$ are two maps such that $f|_K = g|_K$. After possibly an extension of R, we may choose a lift $\operatorname{Spec} K \to \mathcal{X}$ of $f|_K = g|_K$. Since $\mathcal{X} \to X$ is universally closed (Theorem 6.3.5(1)), after possibly further extensions of R, we may choose lifts $\widetilde{f}, \widetilde{g}\colon \operatorname{Spec} R \to \mathcal{X}$ of f, g such that $\widetilde{f}|_K \cong \widetilde{g}|_K$ by the Valuative Criterion for Universal Closedness (3.8.5). Since $\mathcal{X}$ is S-complete, we can extend $\widetilde{f}$ and $\widetilde{g}$ to a morphism $\phi_R \to \mathcal{X}$. As $\phi_R \to \operatorname{Spec} R$ is a good moduli space and hence universal for maps to algebraic spaces, the morphism $\phi_R \to \mathcal{X}$ descends to a unique morphism $\operatorname{Spec} R \to X$ which necessarily must be equal to both f and g. We conclude that X is separated by the Valuative Criterion for Separatedness. $\square$

Lemma 6.7.16. *Let $\mathcal{X}$ be a noetherian algebraic stack with affine and quasi-finite diagonal. If R is a complete DVR, every map $\Theta_R \to \mathcal{X}$ (resp. $\phi_R \to \mathcal{X}$) factors through $\Theta_R \to \operatorname{Spec} R$ (resp. $\phi_R \to \operatorname{Spec} R$).*

Proof. Since good moduli spaces are universal for maps to algebraic spaces, we already know the claim when $\mathcal{X}$ is an algebraic space. In fact, we will reduce to the case when $\mathcal{X}$ is affine in which case the factorizations follow easily from the fact that $\Gamma(\Theta_R, \mathcal{O}_{\Theta_R}) = \Gamma(\phi_R, \mathcal{O}_{\phi_R}) = R$.

Let $x \in \mathcal{X}(\kappa)$ be the image of $0 \in \Theta_R$. Since $\mathbb{G}_m$ has no nontrivial finite quotients, the induced map $\mathbb{G}_m \to G_x$ on stabilizers is trivial. By Proposition 4.2.15, we may find a smooth presentation $U \to \mathcal{X}$ from an affine scheme together with a lift $u \in U(\kappa)$ of x. The map $\mathbf{B}\mathbb{G}_{m,\kappa} \to \mathcal{X}$ factors through $u\colon \operatorname{Spec} \kappa \to U$ and thus lifts to a map $\mathbf{B}\mathbb{G}_{m,\kappa} \to \operatorname{Spec} \kappa \xrightarrow{u} U$. Letting $\mathcal{T}_n$ be the nth nilpotent thickening of $\mathbf{B}\mathbb{G}_{m,\kappa} \hookrightarrow \Theta_R$, deformation theory (Proposition 6.5.8) implies that we may find compatible lifts $\mathcal{T}_n \to U$ of $\mathcal{T}_n \hookrightarrow \Theta_R \to \mathcal{X}$. By Coherent Tannaka Duality (6.4.8), there is an extension $\Theta_R \to U$. Since $\Theta_R \to U$ factors through $\operatorname{Spec} R$, so does $\Theta_R \to \mathcal{X}$. $\square$

Proposition 6.7.17. *Every noetherian algebraic stack $\mathcal{X}$ with affine and quasi-finite diagonal (e.g. a Deligne–Mumford stack with affine diagonal) is Θ-complete. Moreover, $\mathcal{X}$ is S-complete if and only if it is separated.*

Proof. By Lemma 6.7.11, Θ-completeness and S-completeness can be tested on a complete DVR R. Lemma 6.7.16 implies that that $\mathcal{X}$ is Θ-complete and also implies that $\mathcal{X}$ is S-complete if only if every diagram

$$\begin{tikzcd}
\operatorname{Spec} R \bigcup_{\operatorname{Spec} K} \operatorname{Spec} R \arrow[r] \arrow[d] & \mathcal{X} \\
\operatorname{Spec} R \arrow[ur, dashed] &
\end{tikzcd}$$

has a lift, which is the usual valuative criterion for separatedness. $\square$

Example 6.7.18. The examples in Examples 6.7.5 and 6.7.6 of non-separated Deligne–Mumford stacks are not S-complete.

6.7.3 Examples of Θ- and S-completeness

We discuss the valuative criteria of Θ-completeness and S-completeness for quotient stacks, stacks of coherent sheaves,f and the stack of all curves. In each case, it is useful to provide geometric descriptions of morphisms from $\Theta = [\mathbb{A}^1/\mathbb{G}_m]$ to the stack.

Quotient stacks

Proposition 6.6.12 implies that giving a map $\Theta \to [U/G]$ is equivalent to giving a point $u \in U$ and a one-parameter subgroup $\lambda\colon \mathbb{G}_m \to G$ such that $\lim_{t\to 0} \lambda(t)\cdot u$ exists. We now use this interpretation to provide a geometric characterization of Θ-completeness for quotient stacks. Recall that the attractor locus U_λ^+ represents the functor $\mathrm{Mor}_{\Bbbk}^{\mathbb{G}_m}(\mathbb{A}^1, U)$ (Theorem 6.6.55). The evaluation map $\mathrm{ev}_1\colon U_\lambda^+ \to U$ is defined by sending $f\colon \mathbb{A}^1 \to U$ to $f(1)$.

Proposition 6.7.19. *Let G be a smooth linearly reductive group over an algebraically closed field $\Bbbk$, and U be a separated algebraic space of finite type over $\Bbbk$ with an action of G. Then*

$[U/G]$ is Θ-complete	$\iff$	*for every map $u\colon \operatorname{Spec} R \to U$ from a complete DVR over $\Bbbk$ with algebraically closed residue field and one-parameter subgroup $\lambda\colon \mathbb{G}_m \to G$ such that $\lim_{t\to 0}\lambda(t)\cdot u_K \in U(K)$ exists, then $\lim_{t\to 0}\lambda(t)\cdot u \in U(R)$ also exists;*
	$\iff$	*for every one-parameter subgroup $\lambda\colon \mathbb{G}_m \to G$, the morphism $\mathrm{ev}_1\colon U_\lambda^+ \to U$ is a closed immersion.*

Proof. Since G is linearly reductive, $\mathbf{B}G$ is Θ-complete (Proposition 6.7.14). Therefore Θ-completeness of $[U/G]$ is equivalent to the existence of a lift in every diagram

$$\begin{tikzcd} \Theta_R \smallsetminus 0 \arrow[r] \arrow[d, hook] & {[U/G]} \arrow[d] \\ \Theta_R \arrow[r] \arrow[ur, dashed] & \mathbf{B}G \end{tikzcd} \tag{6.7.6}$$

where R is a complete DVR with algebraically closed residue field (Lemma 6.7.11). By Proposition 6.6.12, the map $\Theta_R \to \mathbf{B}G$ corresponds to a one-parameter subgroup $\lambda\colon \mathbb{G}_m \to G$ while $\Theta_R \smallsetminus 0 \to [U/G]$ corresponds to a map $u\colon \operatorname{Spec} R \to U$ such that $\lim_{t\to 0}\lambda(t)\cdot u_K \in U(K)$ exists. In other words, we have a commutative diagram

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & U_\lambda^+ \arrow[d, "\mathrm{ev}_1"] \\ \operatorname{Spec} R \arrow[r, "u"] \arrow[ur, dashed] & U \end{tikzcd} \tag{6.7.7}$$

of solid arrows. A lift of (6.7.6) corresponds to the existence of $\lim_{t\to 0}\lambda(t)\cdot u \in U(R)$ or equivalently to a lift of (6.7.7). Since $\mathrm{ev}_1\colon U_\lambda^+ \to U$ is a monomorphism of finite type, it is closed immersion if and only if it is proper or equivalently satisfies the existence part of the valuative criterion. $\square$

Example 6.7.20. When $U = \operatorname{Spec} A$ is affine, a one-parameter subgroup $\lambda\colon \mathbb{G}_m \to G$ induces a grading $A = \bigoplus_{d\in\mathbb{Z}} A_d$, and U_λ^+ is represented by $V(\sum_{d<0} A_d)$ (Exercise 6.6.53). We see thus that $\mathrm{ev}_1\colon U_\lambda^+ \hookrightarrow U$ is a closed immersion; this recovers the fact that $[U/G]$ is Θ-complete (Proposition 6.7.14).

Example 6.7.21. We can use this criteria to see that Examples 6.7.2 to 6.7.4 are not Θ-complete. For $[\mathbb{P}^1/\mathbb{G}_m]$ with action $t \cdot [x : y] = [tx : y]$, taking $\lambda = \mathrm{id}$ we have that $(\mathbb{P}^1)^+_\lambda = \mathbb{A}^1 \coprod \{\infty\}$. For the quotient $[C/\mathbb{G}_m]$ of the nodal cubic C with normalization $\mathbb{P}^1 \to C$ identifying 0 and ∞, then $C^+_\infty = \mathbb{P}^1 \setminus \infty$ for $\lambda = \mathrm{id}$. Finally, for $[X/\mathbb{G}_m]$ with $X = \mathbb{A}^2 \setminus 0$ with action $t \cdot (x, y) = (tx, y)$, then $X^+_\lambda = \{y \neq 0\}$ for $\lambda = \mathrm{id}$.

Example 6.7.22. We can also provide a interpretation using the algebraic stack $\underline{\mathrm{Mor}}(\Theta, [X/G])$ of morphisms which decomposes as a disjoint union $\coprod_\lambda [X^+_\lambda/P_\lambda]$ where λ varies over conjugation classes of one-parameter subgroups $\lambda\colon \mathbb{G}_m \to G$ (Remark 6.6.14). The evaluation morphism $\mathrm{ev}_0\colon [X^+_\lambda/P_\lambda] \to [X/G]$ is induced by the inclusion $X^+_\lambda \to X$. The Θ-completeness of $[X/G]$ corresponds to the properness of the maps $[X^+_\lambda/P_\lambda] \to [X/G]$.

One can also give a criteria for when $[U/G]$ is S-complete in terms of one-parameter subgroups $\lambda\colon \mathbb{G}_m \to G$ and properties of the morphism $\mathrm{Mor}^{\mathbb{G}_m}_{\mathbb{A}^1}(\mathbb{A}^2, U \times \mathbb{A}^1) \to U \times U \times \mathbb{A}^1$, where the maps to U are obtained by restricting along the two maps $\mathbb{A}^1 \to \mathbb{A}^2$ given by $x \mapsto (x, 1)$ and $x \mapsto (1, x)$.

Stacks of coherent sheaves

Given a projective scheme X, let $\underline{\mathrm{Coh}}(X)$ denote the algebraic stack of coherent sheaves on X (see Exercise 3.1.21). Maps $\Theta \to \underline{\mathrm{Coh}}(X)$ correspond to filtrations.

Proposition 6.7.23. *Let X be a projective scheme over an algebraically closed field $\Bbbk$. For a noetherian $\Bbbk$-algebra R, $\mathrm{MOR}_\Bbbk(\Theta_R, \underline{\mathrm{Coh}}(X))$ is equivalent to the groupoid of pairs $(E, E_\bullet)$ where E is a coherent sheaf on X_R flat over R and*

$$E_\bullet : 0 \subset \cdots \subset E_{i-1} \subset E_i \subset E_{i+1} \subset \cdots \subset E$$

is a $\mathbb{Z}$-graded filtration such that $E_i = 0$ for $i \ll 0$, $E_i = E$ for $i \gg 0$, and each factor E_i/E_{i-1} is flat over R. A morphism $(E, E_\bullet) \to (E', E'_\bullet)$ is an isomorphism $E \to E'$ of coherent sheaves compatible with the filtration.

Under this correspondence, the morphism $\Theta_R \to \underline{\mathrm{Coh}}(X)$ sends 1 to E and 0 to the associated graded $\mathrm{gr}\, E_\bullet := \bigoplus_i E_i/E_{i-1}$, and factors through $\mathcal{B}\mathrm{un}(X) \subset \underline{\mathrm{Coh}}(X)$ if and only if E and each factor E_i/E_{i-1} is a vector bundle.

Proof. A morphism $\Theta_R \to \underline{\mathrm{Coh}}(X)$ corresponds to a coherent sheaf $\mathcal{F}$ on $C \times \Theta_R$ flat over Θ_R. By smooth descent this corresponds to a coherent sheaf on $C \times \mathbb{A}^1_R$ flat over $\mathbb{A}^1_R$ together with a $\mathbb{G}_m$-action. Pushing forward $\mathcal{F}$ along the affine morphism $C \times \Theta_R \to C \times \mathbf{B}\mathbb{G}_{m,R}$, we see that $\mathcal{F}$ also corresponds to a graded $\mathcal{O}_{C_R}[x]$-module flat over $R[x]$. Writing $\mathcal{F} = \bigoplus_i E_i$ with each E_i a coherent sheaf on C_R, then multiplication by x induces maps $x\colon E_i \to E_{i+1}$ which are necessarily injective as $\mathcal{F}$ is flat over $R[x]$, hence torsion free. Since $\mathcal{F}$ is finitely generated as a graded $R[x]$-module, there exist finitely many homogeneous generators with bounded degree. Thus $E_i = E$ for $i \gg 0$. On the other hand, considering the $\mathcal{O}_{C_R}[x]$-submodule $E_{\geq d} := \bigoplus_{i \geq n} E_i \subset \mathcal{F}$, the ascending chain $\cdots \subset E_{\geq d} \subset E_{\geq d-1} \subset \cdots \subset \mathcal{F}$ must terminate as $\mathcal{F}$ is noetherian. It follows that $E_i = 0$ for $i \ll 0$. Since $\mathcal{F}$ is flat as an $R[x]$-module, the quotient $\mathcal{F}/x\mathcal{F} = \bigoplus_i E_i/E_{i-1}$ is flat as an R-module and thus each factor E_i/E_{i-1} is flat over R.

Conversely, given E and a filtration $E_\bullet$ satisfying the above conditions, consider the graded $\mathcal{O}_{C_R}[x]$-module $\mathcal{F} := \bigoplus_i E_i$. We will show by induction that $E_{\geq d} :=$

$\bigoplus_{i \geq d} E_i$ is flat and finitely generated over $R[x]$; this implies that $\mathcal{F}$ is flat and finitely generated over $R[x]$ since $E_i = 0$ for $i \ll 0$. For $d \gg 0$, $E_{\geq d}$ is isomorphic to the graded $R[x]$-module $(E \otimes_R R[x])\langle d \rangle$, where $\langle d \rangle$ denotes the grading shift, and is thus flat and finitely generated. For every d, we have an exact sequence

$$0 \to (E_d \otimes_R R[x])\langle d \rangle \to E_{\geq d} \to ((E_{d+1}/E_d) \otimes_R R[x])\langle d+1 \rangle \to 0.$$

The flatness of E and the quotients E_{d+1}/E_d implies the flatness of each E_d. Thus the left and right term above are flat and finitely generated as $R[x]$-modules, and thus so is the middle term. $\square$

Proposition 6.7.24. *For every projective scheme X over an algebraically closed field $\Bbbk$, the algebraic stack $\underline{\mathrm{Coh}}(X)$ is Θ-complete and S-complete.*

Proof. Given a DVR R, Proposition 6.7.23 implies that a map $\Theta_R \smallsetminus 0 \to \underline{\mathrm{Coh}}(X)$ corresponds to a coherent sheaf E on X_R flat over R and a $\mathbb{Z}$-graded filtration $F_\bullet \colon \cdots F_{i-1} \subset F_i \subset \cdots \subset E_K$ such that $F_i = E_K$ for $i \gg 0$, $F_i = 0$ for $i \ll 0$, and F_i/F_{i-1} is flat over R. Viewing E is a subsheaf of E_K, we define $E_i := F_i \cap E$ as the intersection in E_K. Since E_i/E_{i-1} is a subsheaf of F_i/F_{i-1}, it is torsion free, hence flat as an R-module. The filtration $E_\bullet$ defines an extension $\Theta_R \to \underline{\mathrm{Coh}}(X)$. (Aside: this is exactly the argument for the valuative criterion of properness of the Quot scheme (Proposition 1.4.2). Note also that if we let $j \colon \Theta_R \smallsetminus 0 \hookrightarrow \Theta_R$, let $j_x \colon \operatorname{Spec} R \hookrightarrow \Theta_R$ and let $j_\pi \colon \Theta_K \to \Theta_R$ denote the open immersions and we let $\mathcal{E}$ be the coherent sheaf on $C \times (\Theta_R \setminus 0)$ denoting the union of E and $F_\bullet$, then the extension is given by $(\mathrm{id} \times j)_* \mathcal{E} = j_{x,*} E \cap j_{\pi,*} F_\bullet = E[x^{\pm 1}] \cap F_\bullet = E_\bullet$, where $E[x^{\pm 1}]$ is the $\mathbb{Z}$-graded filtration given by placing E in every degree.)

For S-completeness, suppose we are given a map $\phi_R \smallsetminus 0 \to \underline{\mathrm{Coh}}(X)$ corresponding to coherent sheaves E and F flat over R and an isomorphism $\alpha \colon E_K \to F_K$. Recalling the quotient presentation $\phi_R = [\operatorname{Spec}(R[s,t]/(st - \pi))/\mathbb{G}_m]$, we have several natural open immersions: $j \colon \phi_R \smallsetminus 0 \hookrightarrow \phi_R$, $j_s, j_t \colon \operatorname{Spec} R \hookrightarrow \phi_R$ (with $s \neq 0$ and $t \neq 0$), and $j_{st} \colon \operatorname{Spec} K \to \phi_R$ (with $st \neq 0$). We compute the pushforward as the equalizer

$$\begin{array}{ccccccc} 0 \longrightarrow (\mathrm{id} \times j)_* \mathcal{E} & \longrightarrow & (\mathrm{id} \times j_s)_* E \oplus (\mathrm{id} \times j_t)_* F & \longrightarrow & (\mathrm{id} \times j_{st})_* F_K \\ & & (a, b) & \longmapsto & a - \alpha(b). \end{array}$$

The pushforwards can be computed as graded modules over $R[s,t]/(st - \pi)$:

$$\begin{aligned} (\mathrm{id} \times j_{st})_* F_K &= F_K \otimes_R R[t^{\pm 1}] = \bigoplus_{n \in \mathbb{Z}} F_K t^n, \\ (\mathrm{id} \times j_s)_* E &= E \otimes_R R[t^{\pm 1}] = \bigoplus_{n \in \mathbb{Z}} E t^n, \\ (\mathrm{id} \times j_t)_* F &= F \otimes_R R[s^{\pm 1}] \cong \bigoplus_{n \in \mathbb{Z}} (\pi^{-n} \cdot F) t^n \subset (\mathrm{id} \times j_{st})_* F_K \end{aligned}$$

where we've used that $s = t^{-1}\pi$. Thus

$$j_* \mathcal{E} \cong \bigoplus_{n \in \mathbb{Z}} \big(E \cap (\pi^{-n} \cdot F)\big) t^n \subset (\mathrm{id} \times j_{st})_* F_K.$$

Each R-module $E \cap (\pi^{-n} \cdot F) \subset E$ is finitely generated since E is. Moreover, the ascending chain $\cdots \subset E \cap (\pi^{-n} \cdot F) \subset E \cap (\pi^{-n-1} \cdot F) \subset \cdots$ terminates to E and it follows that $j_*\mathcal{E}$ is coherent. To show that $j_*\mathcal{E}$ is flat over ϕ_R, we only need to check that it is flat over 0. By the Local Criterion for Flatness (Theorem A.2.5), we need to show that $\mathrm{Tor}_1^A(A/\mathfrak{m}, j_*\mathcal{E}) = 0$ where $A = R[s,t]/(st-\pi)$ and $\mathfrak{m} = (s,t)$. The Koszul complex gives a resolution of the residue field $\kappa = A/\mathfrak{m} = R/\pi$:

$$0 \to A \xrightarrow{(t,-s)} A \oplus A \xrightarrow{(s,t)} A \to \kappa \to 0.$$

Tensoring with $j_*\mathcal{E}$ yields a complex

$$0 \to j_*\mathcal{E} \xrightarrow{(t,-s)} j_*\mathcal{E} \oplus j_*\mathcal{E} \xrightarrow{(s,t)} j_*\mathcal{E}. \tag{6.7.8}$$

The pushforward of the exact sequence

$$0 \to \mathcal{O}_{\phi_R \smallsetminus 0} \xrightarrow{(t,-s)} \mathcal{O}_{\phi_R \smallsetminus 0} \oplus \mathcal{O}_{\phi_R \smallsetminus 0} \xrightarrow{(s,t)} \mathcal{O}_{\phi_R \smallsetminus 0} \to 0$$

along $\mathrm{id} \times j\colon C \times \phi_R \smallsetminus 0 \hookrightarrow C \times \phi_R$ is a left exact sequence of vector bundles and tensoring with $j_*\mathcal{E}$ yields a left exact sequence which identified with (6.7.8). Thus $\mathrm{Tor}_1^A(A/\mathfrak{m}, j_*\mathcal{E}) = 0$. $\square$

The description in Proposition 6.7.23 interpreting maps from Θ as filtrations allows us to prove a simple criteria for an open substack $\mathcal{U} \subset \underline{\mathrm{Coh}}(X)$ to be Θ-complete or S-complete. We call two $\mathbb{Z}$-graded filtrations

$$E_\bullet : 0 \subset \cdots \subset E_{i-1} \subset E_i \subset E_{i+1} \subset \cdots \subset E$$

and

$$F^\bullet : F \supset \cdots \supset F^{i-1} \supset F^i \supset F^{i+1} \supset \cdots \supset 0$$

are *opposite* if $E_i/E_{i-1} \cong F^i/F^{i+1}$ for all i. Observe that $F_\bullet$ defined by $F_i = F^{-i}$ is a $\mathbb{Z}$-graded filtration with the same indexing as $E_\bullet$ and being opposite means that $\mathrm{gr}\, E_\bullet$ is isomorphic as a $\mathbb{Z}$-graded sheaf to $\mathrm{gr}\, F_\bullet$ with the opposite grading. A map $[(\operatorname{Spec} \Bbbk[x,y]/xy)/\mathbb{G}_m] \to \underline{\mathrm{Coh}}(X)$, where $t \cdot (x,y) = (tx, t^{-1}y)$, is the same data as two opposite filtration $E_\bullet$ and $F^\bullet$ such that $E_i = 0$ and $F^i = F$ for $i \ll 0$, and $E_i = E$ and $F^i = 0$ for $i \gg 0$; in this case, under this map $(1,0) \mapsto E$, $(0,1) \mapsto F$, and $(0,0) \mapsto \mathrm{gr}\, E_\bullet$.

Proposition 6.7.25. *Let C be a smooth, connected, and projective scheme over an algebraically closed field $\Bbbk$, and let $\mathcal{U} \subset \underline{\mathrm{Coh}}(C)$ be an open substack.*

(1) *The substack $\mathcal{U}$ is Θ-complete if and only if for every DVR R (with fraction field K and residue field κ), coherent sheaf E on C_R flat over R, and $\mathbb{Z}$-graded filtration $E_\bullet$ with $E_i = 0$ for $i \ll 0$, $E_i = E$ for $i \gg 0$ and with each E_i/E_{i-1} flat over R, then if E and $\mathrm{gr}(E_\bullet|_K)$ are in $\mathcal{U}$, so is $\mathrm{gr}(E_\bullet|_\kappa)$.*

(2) *The substack is S-complete if and only if for every pair of opposite filtrations $E_\bullet$ and $F^\bullet$ of $E, F \in \mathcal{U}(\Bbbk)$, the associated graded $\mathrm{gr}\, E_\bullet$ is in $\mathcal{U}$.*

Remark 6.7.26. For a projective scheme of arbitrary dimension, Part (1) and the ($\Leftarrow$) implication in (2) hold with the same proof.

Proof. Since we already know that $\underline{\mathrm{Coh}}(C)$ is S-complete and Θ-complete, the valuative criteria for $\mathcal{U}$ are equivalent to the existence of lifts for all commutative diagrams

$$\begin{tikzcd} \Theta_R \smallsetminus 0 \arrow[r] \arrow[d, hook] & \mathcal{U} \arrow[d, hook] \\ \Theta_R \arrow[r] \arrow[ur, dashed] & \underline{\mathrm{Coh}}(C) \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \phi_R \smallsetminus 0 \arrow[r] \arrow[d, hook] & \mathcal{U} \arrow[d, hook] \\ \phi_R \arrow[r] \arrow[ur, dashed] & \underline{\mathrm{Coh}}(C) \end{tikzcd}$$

where R is a DVR. In other words, we need to show that the images of 0 under the unique fillings $\Theta_R \to \underline{\mathrm{Coh}}(C)$ and $\phi_R \to \underline{\mathrm{Coh}}(C)$ are contained in $\mathcal{U}$. Therefore (1) holds as the image of 0 under $\Theta_R \to \underline{\mathrm{Coh}}(C)$ is $\mathrm{gr}(E_\bullet|_\kappa)$.

For the ($\Leftarrow$) implication in (2), the restriction of $\phi_R \to \underline{\mathrm{Coh}}(C)$ along $\pi = 0$ yields a map $[\mathrm{Spec}(\Bbbk[x,y]/xy)/\mathbb{G}_m] \to \underline{\mathrm{Coh}}(C)$ corresponding to opposite filtrations $E_\bullet$ and $F^\bullet$. If we $\mathrm{gr}\, E_\bullet \in \mathcal{U}(\Bbbk)$, then the image of $\phi_R \to \underline{\mathrm{Coh}}(C)$ is contained in $\mathcal{U}$. Conversely, let $[\mathrm{Spec}(\Bbbk[x,y]/xy)/\mathbb{G}_m] \to \underline{\mathrm{Coh}}(X)$ be a map such that the images of $(1,0)$ and $(0,1)$ are in $\mathcal{U}$ but the image of $(0,0)$ is not in $\mathcal{U}$. Let $\mathcal{X}_n$ be the nth nilpotent thickening of the closed immersion $[\mathrm{Spec}(\Bbbk[x,y]/xy)/\mathbb{G}_m] \hookrightarrow \phi_R$. Since the obstruction to lifting a coherent sheaf E lies in the second coherent cohomology of $\mathcal{X}_0$ and since $\mathcal{X}_0$ is cohomologically affine, deformation theory and Coherent Tannaka Duality (6.4.8) yield an extension $\phi_R \to \underline{\mathrm{Coh}}(X)$ with the image of $\phi_R \setminus 0$ contained in $\mathcal{U}$. □

Remark 6.7.27. If the genus of C is at least 2, then the stack of vector bundles $\mathcal{B}\mathrm{un}(C)$ is not Θ-complete nor S-complete. Let $p \in C$ be a point defined by the vanishing of a section $s \in \Gamma(C, \mathcal{O}(p))$, and let $I \subset \mathcal{O}_{C_R}$ be the ideal sheaf of $(p,0) \in C \times \mathrm{Spec}\, R$. The injection $(s, -\pi)\colon \mathcal{O}_{C_R}(-p) \hookrightarrow \mathcal{O}_{C_R} \oplus \mathcal{O}_{C_R}(p)$ has quotient I, which is torsion free, hence flat over R, but is not a vector bundle. By Proposition 6.7.25, we see that $\mathcal{B}\mathrm{un}(C)$ is not Θ-complete.

Let L and M be line bundles on C, and let $p \in C$ be a point such that $\mathrm{Ext}^1_{\mathcal{O}_C}(M, L(p))$ and $\mathrm{Ext}^1_{\mathcal{O}_C}(L, M(p))$ are nonzero; if L and M have the same degree, then a Riemann–Roch calculation shows that both Ext^1 groups are nonzero. Let Q (resp. Q') be a nontrivial extension of M by $L(p)$ (resp. L by $M(p)$). Then

$$E_\bullet : 0 \subset L \subset L(p) \subset Q \quad \text{and} \quad F^\bullet : Q' \supset M(p) \supset M \supset 0$$

define opposite filtrations where $E_0 = L$ and $F^0 = Q'$. The associated graded $\mathrm{gr}\, E_\bullet = L \oplus \kappa(p) \oplus M$ is not a vector bundle and thus $\mathcal{B}\mathrm{un}(C)$ is not S-complete by Proposition 6.7.25.

We will apply the above criteria later to verify that the stack $\mathcal{B}\mathrm{un}^{\mathrm{ss}}_{r,d}(C)$ of semistable vector bundles on a smooth, connected, and projective curve is both Θ-complete and S-complete.

Stack of all curves

Proposition 6.7.28. *Let $\mathscr{M}_g^{\mathrm{all}}$ be the algebraic stack of all proper curves (Theorem 5.4.7) over an algebraically closed field $\Bbbk$. For every $\Bbbk$-algebra R, $\mathrm{MOR}_{\Bbbk}(\Theta_R, \mathscr{M}_g^{\mathrm{all}})$ is the groupoid whose objects are $\mathbb{G}_m$-equivariant families of proper curves $\mathcal{C} \to \mathbb{A}^1_R$, where $\mathbb{G}_m$ acts on $\mathbb{A}^1_R$ with the usual scaling action. Morphisms are $\mathbb{G}_m$-equivariant morphisms.*

Proof. The statement follows from smooth descent applied to $\mathbb{A}^1_R \to \Theta_R$. □

Remark 6.7.29. A similar description holds for other moduli stacks of varieties. Such $\mathbb{G}_m$-equivariant maps are often called 'test configurations' in the literature.

The stacks $\mathscr{M}_g$ and $\overline{\mathscr{M}}_g$ of smooth and stable curves are both Θ-complete and S-complete as they are separated Deligne–Mumford stacks. There is unfortunately no known simple criteria—similar to the above criteria for quotient stacks and stacks of coherent sheaves—to verify whether a given substack of the stack $\mathscr{M}_g^{\mathrm{all}}$ of all curves is Θ-complete or S-complete.

6.7.4 Θ-completeness and Θ-surjectivity

The property that a morphism $\mathcal{X} \to \mathcal{Y}$ sends closed points to closed points is not stable under base change (see Examples 6.7.2 and 6.7.3). We introduce a stronger and better behaved property called Θ-surjectivity. The main result of this section is that an étale quotient presentation $([\operatorname{Spec} A/G_x], w) \to (\mathcal{X}, x)$ is Θ-surjective in an open neighborhood of w as long as $\mathcal{X}$ is Θ-complete (prop:theta-surjective-in-open-neighborhood). As motivated in §6.7.1, this result will be crucial in proving the main existence theorem (Theorem 6.7.1) of this section.

Definition 6.7.30. Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks and $x \in \mathcal{X}(\Bbbk)$ be a geometric point. We say that f is Θ-*surjective at* x if every diagram

$$\begin{tikzcd} \operatorname{Spec} \Bbbk \arrow[r, "x"] \arrow[d, hook, "1"] & \mathcal{X} \arrow[d, "f"] \\ \Theta_{\Bbbk} \arrow[r] \arrow[ur, dashed] & \mathcal{Y} \end{tikzcd} \tag{6.7.9}$$

has a lift. We say that f is Θ-*surjective* if it is Θ-surjective at every geometric point.

This notion is clearly stable under base change. Every morphism $f\colon \mathcal{X} \to \mathcal{Y}$ of noetherian algebraic stacks where $\mathcal{Y}$ has affine and quasi-finite diagonal is Θ-surjective since in this case every map $\Theta_{\Bbbk} \to \mathcal{Y}$ factors through $\operatorname{Spec} \Bbbk$ (Lemma 6.7.16). The next lemma gives condition for when the lift is unique and when the definition is independent of the choice of geometric point.

Lemma 6.7.31. *Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a separated, representable, and finite type morphism of noetherian algebraic stacks.*

(1) Every lift of (6.7.9) is unique.

(2) If f is Θ-surjective at a geometric point $x \in \mathcal{X}(\Bbbk)$, then f is Θ-surjective at every other geometric point $x' \in \mathcal{X}(\Bbbk')$ representing the same point in $|\mathcal{X}|$ as x.

Proof. Part (1) follows from descent and the valuative criterion for separatedness. To show (2), it suffices to show that given an extension $\Bbbk \to \Bbbk'$ of algebraically closed fields, a lift $\Theta_{\Bbbk'} \to \mathcal{X}$ implies the existence of a lift $\Theta_{\Bbbk} \to \mathcal{X}$. We write $\Bbbk' = \bigcup_\lambda A_\lambda$ as a union of finitely generated $\Bbbk$-subalgebras. By Limit Methods (§A.6), there exists a lift $\Theta_{A_\lambda} \to \mathcal{X}$ of $\operatorname{Spec} A_\lambda \to \mathcal{X}$. Restricting along a closed point of $\operatorname{Spec} A_\lambda$ provides a lift over $\Bbbk$. $\square$

Proposition 6.7.32. *Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks, each of finite type over an algebraically closed field $\Bbbk$ with affine diagonal. Suppose that the closed points of $\mathcal{Y}$ have linearly reductive stabilizer. If f is Θ-surjective, then f sends closed points to closed points.*

Proof. Let $x \in \mathcal{X}$ be a closed point. Let $f(x) \rightsquigarrow y_0$ be a specialization to a closed point. By Corollary 6.6.21, this specialization can be realized by a map $\Theta \to \mathcal{Y}$. Since f is Θ-surjective, this can be lifted to a map $g\colon \Theta \to \mathcal{X}$ with $g(1) = x$. But $x \in \mathcal{X}$ is a closed point so this lift must correspond to the trivial specialization $x \rightsquigarrow x$. It follows that $f(x) = y_0$ is a closed point. $\square$

Remark 6.7.33. The converse is not true. In Example 6.7.3, where C is the nodal cubic with $\mathbb{G}_m$-action, the étale morphism $[\operatorname{Spec}(\Bbbk[x,y]/(xy))/\mathbb{G}_m] \to [C/\mathbb{G}_m]$ sends closed points to closed points but is not Θ-surjective.

Proposition 6.7.34. *Let $\mathcal{X}$ be an algebraic stack of finite type over an algebraically closed field $\Bbbk$ with affine diagonal such that the closed points of $\mathcal{X}$ have linearly reductive stabilizers. Let $x \in \mathcal{X}$ be a closed point, let $f\colon ([\operatorname{Spec} A/G_x], w) \to (\mathcal{X}, x)$ be an affine étale morphism inducing an isomorphism of stabilizer groups at w, and let $\pi\colon [\operatorname{Spec} A/G_x] \to \operatorname{Spec} A^{G_x}$. If $\mathcal{X}$ is Θ-complete, there exists an open affine neighborhood $U \subset \operatorname{Spec} A^{G_x}$ of $\pi(w)$ such that $f|_{\pi^{-1}(U)}\colon \pi^{-1}(U) \to \mathcal{X}$ is Θ-surjective.*

Proof. Let $\mathcal{W} = [\operatorname{Spec} A/G_x]$ and define $\Sigma_f \subset |\mathcal{W}|$ as the set of points $y \in \mathcal{W}$ such that f is Θ-surjective at y. We first show that $\Sigma_f \subset \mathcal{W}$ is open if $\mathcal{X} \cong [\operatorname{Spec} B/G]$ with G linearly reductive. Zariski's Main Theorem (6.1.10) provides a factorization

$$f\colon \mathcal{W} \overset{j}{\hookrightarrow} \widetilde{\mathcal{X}} \overset{\nu}{\to} \mathcal{X}$$

where j is an open immersion and ν is a finite morphism. By Lemma 6.7.13, $[\operatorname{Spec} B/G]$ is Θ-complete, and by Proposition 6.7.14, $\widetilde{\mathcal{X}}$ is also Θ-complete. As ν is finite, $\Sigma_j = \Sigma_f$ and we may assume that f is an open immersion. Let $\mathcal{Z} \subset \mathcal{X}$ be the reduced complement of $\mathcal{W}$ and let $\pi\colon \mathcal{X} \to \operatorname{Spec} B^G$ denote the good moduli space. We claim that $|\mathcal{W}| \smallsetminus \Sigma_f = \pi^{-1}(\pi(|\mathcal{Z}|)) \cap |\mathcal{W}|$. The inclusion "$\subset$" is clear: the morphism $\mathcal{X} \smallsetminus \pi^{-1}(\pi(|\mathcal{Z}|)) \hookrightarrow \mathcal{X}$ is the base change of the Θ-surjective morphism $X \smallsetminus \pi(|\mathcal{Z}|) \hookrightarrow X$ of algebraic spaces. Conversely, let $y \in \pi^{-1}(\pi(|\mathcal{Z}|)) \cap |\mathcal{W}|$ represented by a geometric point $\operatorname{Spec} K \to \mathcal{X}$. Let $z \in |\mathcal{Z}_K|$ be the unique closed point in the closure of $y \in |\mathcal{X}_K|$ and let $\Theta_K \to \mathcal{X}_K$ be a morphism representing the specialization $y \rightsquigarrow z$ (Corollary 6.6.21). Since $\Theta_K \to \mathcal{X}$ does not lift to $\mathcal{W}$, $y \notin \Sigma_f$.

We now claim that $\Sigma_f \subset \mathcal{W}$ is constructible. Use the Local Structure Theorem (6.5.1) to choose an affine, étale, and surjective morphism $g\colon \mathcal{X}' = [\operatorname{Spec} B/G] \to \mathcal{X}$ with G linearly reductive. Let $\mathcal{W}' = \mathcal{W} \times_{\mathcal{X}} \mathcal{X}'$ with projections $g'\colon \mathcal{W}' \to \mathcal{W}$ and $f'\colon \mathcal{W}' \to \mathcal{X}'$. Since we already know that $\Sigma_{f'}$ is open, the claim follows from Chevalley's Theorem (3.3.29) once we show that $\mathcal{W} \setminus \Sigma_f = g'(\mathcal{W}' \setminus \Sigma_{f'})$. To see this, it suffices to show that for an algebraically closed field K, every map $h\colon \Theta_K \to \mathcal{X}$ lifts to a map $h'\colon \Theta_K \to \mathcal{X}'$. Let $x' \in \mathcal{X}'(K)$ be a preimage of $h(0) \in \mathcal{X}(K)$. Since g is representable and étale, the induced map $G_{x'} \to G_{h(0)}$ on stabilizers is injective with finite cokernel. Thus the map $\mathbb{G}_{m,K} \to G_{h(0)}$ on stabilizers induced by $h\colon \Theta_K \to \mathcal{X}$ factors through $G_{x'}$. We may therefore lift the map $h|_{\mathbf{B}\mathbb{G}_{m,K}}$ to a map $\mathbf{B}\mathbb{G}_{m,K} \to \mathcal{X}'$. Letting $\mathcal{X}_n$ be the nth nilpotent thickening of $\mathbf{B}\mathbb{G}_{m,K} \hookrightarrow \Theta_K$, there are compatible lifts $\mathcal{X}_n \to \mathcal{X}'$ of $\mathcal{X}_n \to \mathcal{X}$ by deformation theory (Proposition 6.5.8) which extends to a lift $\Theta_K \to \mathcal{X}'$ by Coherent Tannaka Duality (6.4.8).

Since $\Sigma_f \subset \mathcal{W}$ is constructible and $w \in \Sigma_f$, to show that Σ_f is open, it suffices to show that for every generization $\xi \rightsquigarrow w$ of w is contained in Σ_f. Let $h\colon \operatorname{Spec} R \to \mathcal{W}$ be a morphism from a complete DVR representing the specialization $\xi \rightsquigarrow w$. Letting K and κ be the fraction and residue field of R, we claim that there exists a lift (necessarily unique as f is separated)

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r, "h_K"] \arrow[d, hook] & \mathcal{W} \arrow[d, "f"] \\ \Theta_K \arrow[r, "g"] \arrow[ur, dashed, "\widetilde{g}"] & \mathcal{X}. \end{tikzcd} \tag{6.7.10}$$

This claim implies that f is Θ-surjective at ξ, i.e. $\xi \in \Sigma_f$. To show the claim, we first apply the Θ-completeness of $\mathcal{X}$ to construct a lift

$$\begin{tikzcd} \Theta_R \setminus 0 \arrow[r, "(f \circ h) \cup g"] \arrow[d, hook] & \mathcal{X} \\ \Theta_R. \arrow[ur, dashed, "q"'] & \end{tikzcd}$$

Since $\mathcal{W} \to \mathcal{X}$ is stabilizer preserving at w, we have a lift $\mathbf{B}\mathbb{G}_{m,\kappa} \to \mathcal{W}$ of $q|_{\mathbf{B}\mathbb{G}_{m,\kappa}}$. Since Θ_R is coherently complete along $\mathbf{B}\mathbb{G}_{m,\kappa}$ (6.4.11), we may apply deformation theory (Proposition 6.5.8) and Coherent Tannaka Duality (6.4.8) to construct a lift

$$\begin{tikzcd} \mathbf{B}\mathbb{G}_{m,\kappa} \arrow[r] \arrow[d, hook] & \mathcal{W} \arrow[d, "f"] \\ \Theta_R \arrow[r, "q"] \arrow[ur, dashed, "\widetilde{q}"] & \mathcal{X} \end{tikzcd}$$

The restriction $\widetilde{q}|_{\operatorname{Spec} R}$ is 2-isomorphic to h since it agrees at the closed point and f is étale. It follows that $\widetilde{g} := \widetilde{q}|_{\Theta_K}$ is a lift of (6.7.10). $\square$

The topology of $\Bbbk$-points of Θ-complete stacks is analogous to the topology of quotient stacks arising from GIT.

Proposition 6.7.35. *Let $\mathcal{X}$ be an algebraic stack of finite type over an algebraically closed field $\Bbbk$ with affine diagonal. Assume that $\mathcal{X}$ is Θ-complete and that the closed points of $\mathcal{X}$ have linearly reductive stabilizer. Then the closure of every $\Bbbk$-point contains a unique closed point.*

Proof. Assume that x and x' are two closed points in the closure of $p \in \mathcal{X}(\Bbbk)$. By Corollary 6.6.21, there are maps $f, f' \colon \Theta \to \mathcal{X}$ realizing the specializations $p \rightsquigarrow x$ and $p \rightsquigarrow x'$. Under the action of $\mathbb{G}_m^2$ on $\mathbb{A}^2$ given by $(t_1, t_2) \cdot (y_1, y_2) = (t_1 y_1, t_2 y_2)$, the maps f and f' glue to define a map $[\mathbb{A}^2/\mathbb{G}_m^2] \setminus 0 \to \mathcal{X}$. By considering only the diagonal $\mathbb{G}_m$-action, the map $[\mathbb{A}^2/\mathbb{G}_m] \setminus 0 \to \mathcal{X}$ extends to $\Psi \colon [\mathbb{A}^2/\mathbb{G}_m] \to \mathcal{X}$ by the Θ-completeness of $\mathcal{X}$. Then $\Psi(0,0)$ is a common specialization of $x = \Psi(1,0)$ and $x' = \Psi(0,1)$. Since x and x' are closed points, we have that $x = \Psi(0,0) = x'$. $\square$

Exercise 6.7.36. With the hypotheses of Proposition 6.7.35, show that if in addition $\mathcal{X}$ has a unique closed point, then $\mathcal{X} \cong [\operatorname{Spec}(A)/G_x]$ such that A^{G_x} is an artinian local $\Bbbk$-algebra with residue field $\Bbbk$.

6.7.5 Unpunctured inertia

We prove that an S-complete stack $\mathcal{X}$ has 'unpunctured inertia' (Theorem 6.7.43) and the consequence that an étale quotient presentation $f \colon ([\operatorname{Spec} A/G_x], w) \to (\mathcal{X}, x)$ is stabilizer preserving in an open neighborhood of w (Proposition 6.7.40).

Definition 6.7.37. We say that a noetherian algebraic stack has *unpunctured inertia* if for every closed point $x \in |\mathcal{X}|$ and every formally versal morphism $p\colon (T,t) \to (\mathcal{X}, x)$ where T is the spectrum of a local ring with closed point t, every connected component of the inertia group scheme $\underline{\mathrm{Aut}}_{\mathcal{X}}(p) \to T$ has non-empty intersection with the fiber over t.

Remark 6.7.38. Here $(T,t) \to (\mathcal{X}, x)$ is formally versal if the map $\widehat{T} \to \mathcal{X}$ from the completion is is formally versal at t as in Definition D.3.5.

Remark 6.7.39. Unpuncturedness is related to the purity of the morphism $\underline{\mathrm{Aut}}_{\mathcal{X}}(p) \to T$ as defined in [RG71, §3.3] (see also [SP, Tag 0CV5]). If T is the spectrum of a strictly henselian local ring, then purity requires that if $s \in T$ is an *arbitrary* point and γ is an associated point in the fiber $\underline{\mathrm{Aut}}_{\mathcal{X}}(p)_s$, then the closure of γ in $\underline{\mathrm{Aut}}_{\mathcal{X}}(p)$ has non-empty intersection with the fiber over the closed point t of T.

Proposition 6.7.40. *Let $\mathcal{X}$ be an algebraic stack of finite type over an algebraically closed field $\Bbbk$ with affine diagonal. Let $x \in \mathcal{X}$ be a closed point with linearly reductive stabilizer. Let $f\colon ([\operatorname{Spec} A/G_x], w) \to (\mathcal{X}, x)$ be an affine étale morphism inducing an isomorphism of stabilizer groups at w, and let $\pi\colon [\operatorname{Spec} A/G_x] \to \operatorname{Spec} A^{G_x}$. If $\mathcal{X}$ has unpunctured inertia, there exists an open affine neighborhood $U \subset \operatorname{Spec} A^{G_x}$ of $\pi(w)$ such that $f|_{\pi^{-1}(U)}\colon \pi^{-1}(U) \to \mathcal{X}$ induces isomorphisms of stabilizer groups at all points.*

Proof. Set $\mathcal{W} = [\operatorname{Spec} A/G_x]$. It suffices to find an open neighborhood $\mathcal{U} \subset \mathcal{W}$ of w such that $f|_{\mathcal{U}}\colon \mathcal{U} \to \mathcal{X}$ induces an isomorphism $I_{\mathcal{U}} \to \mathcal{U} \times_{\mathcal{X}} I_{\mathcal{X}}$. Consider the cartesian diagram

$$\begin{tikzcd} I_{\mathcal{W}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{W} \times_{\mathcal{X}} I_{\mathcal{X}} \arrow[d] \\ \mathcal{W} \arrow[r] & \mathcal{W} \times_{\mathcal{X}} \mathcal{W}; \end{tikzcd}$$

see Exercise 3.2.14. Since f is separated and étale, the morphism $I_{\mathcal{W}} \to \mathcal{W} \times_{\mathcal{X}} I_{\mathcal{X}}$ is finite and étale. We set $\mathcal{Z} \subset \mathcal{W} \times_{\mathcal{X}} I_{\mathcal{X}}$ to be the open and closed substack over which $I_{\mathcal{W}} \to \mathcal{W} \times_{\mathcal{X}} I_{\mathcal{X}}$ is *not* an isomorphism. Since f is stabilizer preserving at w, the point w is not contained in the image of $\mathcal{Z}$ under $p_1\colon \mathcal{W} \times_{\mathcal{X}} I_{\mathcal{X}} \to \mathcal{W}$.

Consider a formally smooth morphism $(T,t) \to (\mathcal{X}, x)$ from the spectrum of a local ring with closed point t. Since $\mathcal{X}$ has unpunctured inertia, the preimage of $\mathcal{Z}$ in $\mathcal{W} \times_{\mathcal{X}} I_{\mathcal{X}} \times_{\mathcal{X}} T$ is empty; indeed, if there were a non-empty connected component of this preimage, it must intersect the fiber over t non-trivially contradicting that $w \notin p_1(\mathcal{Z})$. This in turn implies that $w \notin \overline{p_1(\mathcal{Z})}$. Therefore, if we set $\mathcal{U} = \mathcal{W} \smallsetminus \overline{p_1(\mathcal{Z})}$, the induced morphism $I_{\mathcal{U}} \to \mathcal{U} \times_{\mathcal{X}} I_{\mathcal{X}}$ is an isomorphism. $\square$

Proposition 6.7.41. *Let $\mathcal{X}$ be a noetherian algebraic stack.*

(1) If $\mathcal{X}$ has quasi-finite inertia, then $\mathcal{X}$ has unpunctured inertia if and only if $\mathcal{X}$ has finite inertia.

(2) If $\mathcal{X}$ has connected stabilizer groups, then $\mathcal{X}$ has unpunctured inertia.

Proof. If $\mathcal{X}$ has finite inertia, then $\underline{\mathrm{Aut}}_{\mathcal{X}}(p) \to T$ is finite so clearly the image of each connected component contains the unique closed point $t \in T$. For the converse, we may assume that T is the spectrum of a Henselian local ring in which case $\underline{\mathrm{Aut}}_{\mathcal{X}}(p) = G \coprod H$ where $G \to T$ finite and the fiber of $H \to T$ over t is empty (Proposition A.9.3). If T is nonempty (i.e. $\underline{\mathrm{Aut}}_{\mathcal{X}}(p) \to T$ is not finite), then any connected component of T doesn't meet the central fiber and thus $\mathcal{X}$ does not have unpunctured inertia.

For (2), by definition all fibers of $\underline{\mathrm{Aut}}_{\mathcal{X}}(p) \to U$ are connected, so every connected component of $\underline{\mathrm{Aut}}_{\mathcal{X}}(p)$ intersects the component containing the identity section. $\square$

Remark 6.7.42. For algebraic stacks with connected stabilizer groups (e.g. the moduli stack $\mathcal{B}\mathrm{un}^{\mathrm{ss}}_{r,d}(C)$ of semistable vector bundles on a curve), Proposition 6.7.41(2) implies unpunctured inertia. The deeper result below (Theorem 6.7.43) is therefore unneeded in the proof of existence of a good moduli space of $\mathcal{B}\mathrm{un}^{\mathrm{ss}}_{r,d}(C)$.

The rest of this section is dedicated to proving the following theorem.

Theorem 6.7.43. *Let $\mathcal{X}$ be an algebraic stack of finite type over an algebraically closed field $\Bbbk$ with affine diagonal. Assume that the closed points have linearly reductive stabilizers. If $\mathcal{X}$ is S-complete, then $\mathcal{X}$ has unpunctured inertia.*

Proof. Let $x \in |\mathcal{X}|$ be a closed point, let $p\colon (U,u) \to (\mathcal{X},x)$ be a formally smooth morphism from the spectrum of a local ring, and let $H \subset \underline{\mathrm{Aut}}_{\mathcal{X}}(p)$ be a connected component. The image of the projection $H \to U$ is a constructible set whose closure contains u. It follows that we can find a DVR R with residue field $\Bbbk$ and a map $\operatorname{Spec} R \to U$ whose special point maps to u and whose generic point lies in the image of $H \to U$. Let $\xi\colon \operatorname{Spec} R \to U \xrightarrow{p} \mathcal{X}$ denote the composition. After a residually-trivial extension of DVRs, we may assume that the generic point $\operatorname{Spec} K \to U$ lifts to H. This gives a commutative diagram

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & \operatorname{Spec} R \arrow[d] \arrow[dr, "\xi"] & \\ H \arrow[r] & U \arrow[r, "p"] & \mathcal{X}. \end{tikzcd}$$

Let H_K be the base change of $H \to U$ along $\operatorname{Spec} K \to U$. We claim we can choose a finite type point $g \in H_K$ of finite order. If $g \in H_K$ is a finite type point, then after replacing K with a finite field extension, we can decompose $g = g_s g_u$ under the Jordan decomposition, where g_s is semisimple and g_u is unipotent. Now consider the reduced Zariski closed K-subgroup $H' \subset \underline{\mathrm{Aut}}_{\mathcal{X}}(p)_K$ generated by g_s. Because g_s is semisimple, H' is a diagonalizable group scheme over K, and we may replace g_s with a finite order element in H' which still commutes with g_u. If $\mathrm{char}(K) > 0$, then g_u has finite order and we are finished. If $\mathrm{char}(K) = 0$, then g_u lies in the identity component of G, so g lies on the same component as the finite order element g_s. This gives the desired element.

We claim that after replacing R with a residually-trivial extension, there is a map $\xi'\colon \operatorname{Spec} R \to \mathcal{X}$ such that $\xi'_K \simeq \xi_K$ and $g \in H_K$ extends to an automorphism of ξ'. This would finish the proof: since the closure of g meets the fiber of $\underline{\mathrm{Aut}}_{\mathcal{X}}(p) \to U$ over u, the component H must also meet the central fiber.

If $\mathcal{X} \cong [\operatorname{Spec} A/\operatorname{GL}_n]$, then this claim is precisely the content of Proposition 6.7.44 below. We will use the Local Structure Theorem (6.5.1) to reduce to this case: let $f\colon (\operatorname{Spec} A/G_x], w) \to (\mathcal{X}, x)$ be an étale quotient presentation. After replacing R with a residually-trivial extension, we may lift ξ to a map $\widetilde{\xi}\colon \operatorname{Spec} R \to [\operatorname{Spec} A/G_x]$ such that $\widetilde{\xi}(0) = w$. To show that g lifts to an element

$\widetilde{g} \in \operatorname{Aut}(\widetilde{\xi}_K)$, we will use S-completeness. We may glue ξ to itself along g to define a morphism

$$\operatorname{Spec} R \bigcup_{\operatorname{Spec} K} \operatorname{Spec} R = \phi_R \smallsetminus 0 \to \mathcal{X}.$$

Since $\mathcal{X}$ is S-complete, this map extends to a morphism $h\colon \phi_R \to \mathcal{X}$. Since $\xi(0) = x$ and x is a closed point, the image $h(0)$ of $0 \in \phi_R$ is also x. Since f is stabilizer preserving at w, we may lift $h|_{\mathbf{B}\mathbb{G}_m}$ to a map $\widetilde{h}_0\colon \mathbf{B}\mathbb{G}_m \to [\operatorname{Spec} A/G_x]$ with image w. By Deformation Theory (6.5.8), we may find compatible lifts to $[\operatorname{Spec} A/G_x]$ of the restrictions of h to the nilpotent thickenings of ϕ_R along 0, and by Coherent Tannaka Duality (6.4.8), we may find construct a lift $\widetilde{h}$ below

$$\begin{tikzcd}
\mathbf{B}\mathbb{G}_m \arrow[r, "\widetilde{h}_0"] \arrow[d, hook] & {[\operatorname{Spec} A/G_x]} \arrow[d, "f"] \\
\phi_R \arrow[r, "h"] \arrow[ur, dashed, "\widetilde{h}"] & \mathcal{X}.
\end{tikzcd}$$

Since f is affine and étale, both restrictions $\widetilde{h}|_{s\neq 0}$ and $\widetilde{h}|_{t\neq 0}$ to $\operatorname{Spec} R$ are isomorphic to $\widetilde{\xi}$ and thus $\widetilde{h}|_{\phi_R \smallsetminus 0}$ gives a lift $\widetilde{g} \in \operatorname{Aut}(\widetilde{\xi}_K)$ of g. Finally, we apply Proposition 6.7.44 to construct a map $\widetilde{\xi}'\colon \operatorname{Spec} R \to [\operatorname{Spec} A/G_x]$ with $\widetilde{\xi}'(0) = w$ such that $\widetilde{\xi}_K \simeq \widetilde{\xi}'_K$ and $\widetilde{g}$ extends to an automorphism of $\widetilde{\xi}'$. The composition $\xi'' := f \circ \widetilde{\xi}''\colon \operatorname{Spec} R \to \mathcal{X}$ then satisfies the claim.

See also [AHLH18, Thm. 5.2]. □

Our proof used the following valuative criterion for a quotient stack.

Proposition 6.7.44. *Let $\mathcal{X} = [\operatorname{Spec} A/G]$ where $\operatorname{Spec} A$ is an affine scheme of finite type over an algebraically closed field $\Bbbk$ equipped with an action by a linearly reductive group G. Let $x \in \mathcal{X}$ be a closed point. Then $\mathcal{X}$ satisfies the following property:*

($\star$) *For every DVR R with residue field $\Bbbk$ and fraction field K, for every morphism $\xi\colon \operatorname{Spec} R \to \mathcal{X}$ with $\xi(0) \simeq x$, and for every K-pint $g \in \operatorname{Aut}_{\mathcal{X}}(\xi_K)$ of finite order, there is an extension $R \to R'$ of DVRs (with $K' = \operatorname{Frac}(R')$) and a morphism $\xi'\colon \operatorname{Spec} R' \to \mathcal{X}$ such that $\xi'(0) \simeq x$, $\xi'_{K'} \simeq \xi_{K'}$ and $g|_{K'}$ extends to an automorphism of ξ'.*

Remark 6.7.45. In other words, for every map $\xi\colon \operatorname{Spec} R \to \operatorname{Spec} A$ and element $g \in G_{\xi_K} \subset G(K)$ of finite order, there exists after an extension $R \subset R'$ of DVRs and an element $h \in G(K')$ such that $h \cdot \xi_{K'}$ extends to a map $\xi'\colon \operatorname{Spec} R' \to \operatorname{Spec} A$ with $\xi'(0) \in Gx$ and such that $h^{-1} g|_{K'} h$ extends to an R'-point of G.

To illustrate this criterion, consider the the action of $G = \mathbb{G}_m \rtimes \mathbb{Z}/2$ on $\mathbb{A}^2$ via $t \cdot (a,b) = (ta, t^{-1}b)$ and $-1 \cdot (a,b) = (b,a)$. Note that every point $(a,b) \in \mathbb{A}^2$ with $ab \neq 0$ is fixed by the order 2 element $(a/b, -1) \in G$. Consider $\xi\colon \operatorname{Spec} R = \Bbbk[\![z]\!] \to \mathbb{A}^2$ via $z \mapsto (z^2, z)$. The element $g = (z^{-1}, -1) \in G(\Bbbk(\!(z)\!))$ stabilizes ξ_K but does not extend to $G(\Bbbk[\![z]\!])$. However, we may take the degree 2 ramified extension $\Bbbk[\![z]\!] \to \Bbbk[\![\sqrt{z}]\!]$ and define $\xi'\colon \operatorname{Spec} \Bbbk[\![\sqrt{z}]\!] \to \mathbb{A}^2$ by $\sqrt{z} \mapsto ((\sqrt{z})^3, (\sqrt{z})^3)$. Over the generic point, there is an isomorphism $\xi'_{\Bbbk(\!(\sqrt{z})\!)} \simeq \xi_{\Bbbk(\!(\sqrt{z})\!)}$ given by $h = (\sqrt{z}, -1) \in G(\Bbbk(\!(\sqrt{z})\!))$ and the element $g|_{\Bbbk(\!(\sqrt{z})\!)} = (\sqrt{z}, -1)^{-1} \cdot g|_{K'} \cdot (\sqrt{z}, -1) = (1, -1) \in G(\Bbbk(\!(\sqrt{z})\!))$ extends to an element of $G(\Bbbk[\![\sqrt{z}]\!])$-point.

Proof. After choosing an embedding $G \hookrightarrow \mathrm{GL}_n$ and replacing $[\operatorname{Spec} A/G]$ with $[(\operatorname{Spec} A \times^{G_x} \mathrm{GL}_n)/\mathrm{GL}_n]$, we may assume that $G = \mathrm{GL}_n$.

We first verify $(\star)$ for quotient stacks $[\operatorname{Spec} A/G] = \operatorname{Spec} A \times \mathbf{B}G$ with a trivial action. As R is local and $G = \mathrm{GL}_n$, the composition $\operatorname{Spec} R \to [\operatorname{Spec} A/G] \to \mathbf{B}G$ corresponds to the trivial G-bundle. We need to prove that every finite order element $g \in G(K)$ is conjugate to an element of $G(R)$ after passing to an extension of the DVR R. We can conjugate g to its Jordan canonical form (after an extension of R). Since g has finite order, the diagonal entries of the resulting matrix are rth roots of unity for some r. Because the group μ_r of r^{th} roots of unity is a finite group scheme over $\operatorname{Spec} R$, the entries of the Jordan canonical form must lie in R.

If $\mathcal{X} = [X/G]$ with X proper over $\Bbbk$, we show that $(\star)$ holds except that $\xi'(0)$ may not be isomorphic to x. Since $p\colon \mathcal{X} \to \mathbf{B}G$ is proper and representable, for every morphism $\xi\colon \operatorname{Spec} R \to \mathcal{X}$ from a DVR, we have a closed immersion $\underline{\mathrm{Aut}}_{\mathcal{X}}(\xi) \hookrightarrow \underline{\mathrm{Aut}}_{\mathbf{B}G}(p \circ \xi)$ of group schemes over $\operatorname{Spec} R$. Moreover, any lift of the generic point of a morphism $\operatorname{Spec} R \to \mathbf{B}G$ to $[X/G]$ extends to a unique morphism $\operatorname{Spec} R \to \mathbf{B}G$. Therefore, given an element $g \in \mathrm{Aut}_{\mathcal{X}}(\xi_K)$, we use that $(\star)$ holds for $\mathbf{B}G$ to find (after replacing R with an extension) a morphism $\eta\colon \operatorname{Spec} R \to \mathbf{B}G$ such that $\eta_K \simeq (p \circ \xi)_K$ and $g|_K$ extends to a R-point of $\mathrm{Aut}_{\mathbf{B}G}(\eta)$. If we lift η to a morphism $\xi'\colon \operatorname{Spec} R \to [X/G]$ such that $\xi'_K \simeq \xi_K$, then the element $g|_K$ extends to an automorphism of ξ'.

In verifying $(\star)$ for $[\operatorname{Spec} A/G]$, we may assume that A is reduced. Viewing $[\operatorname{Spec} A/G]$ as an algebraic stack which is affine and of finite type over $\operatorname{Spec} A^G \times \mathbf{B}G$, we can choose a vector bundle $\mathcal{E}$ on $\operatorname{Spec} A^G \times \mathbf{B}G$ and a G-equivariant embedding $\operatorname{Spec} A \hookrightarrow \mathbb{A}_{A^G}(\mathcal{E})$ over A^G. Viewing $\mathbb{A}_{A^G}(\mathcal{E})$ as an open subscheme of $\mathbb{P}_{A^G}(\mathcal{E} \oplus \mathcal{O})$, we let X be the closure of $\operatorname{Spec} A$ in $\mathbb{P}_{A^G}(\mathcal{E} \oplus \mathcal{O})$. This gives a G-equivariant diagram

$$
\begin{tikzcd}
\operatorname{Spec} A \arrow[r, hook] \arrow[dr] & X \arrow[d] \\
& \operatorname{Spec} A^G
\end{tikzcd}
\tag{6.7.11}
$$

where X is a reduced projective scheme and the complement $X \smallsetminus \operatorname{Spec} A$ is the support of an ample G-invariant Cartier divisor E. We also claim that $\operatorname{Spec} A$ is precisely the semistable locus of X with respect to $\mathcal{O}_X(E)$ in the sense of Exercise 6.6.11. Indeed the tautological invariant section $s\colon \mathcal{O}_X \to \mathcal{O}_X(E)$ restricts to an isomorphism over $\operatorname{Spec} A$ and thus $\operatorname{Spec} A \subset X^{\mathrm{ss}}$. Conversely s^n defines an isomorphism

$$A^G \xrightarrow{\sim} \Gamma(X, \mathcal{O}_X(nE))^G$$

for all $n \geq 0$. Under this isomorphism, for every invariant global section $f \in \Gamma(X, \mathcal{O}_X(nE))^G$, the restriction $f|_{\operatorname{Spec} A}$ agrees with a section of the form gs^n, where g is the pullback of a function under the map $X \to \operatorname{Spec} A^G$. It follows that $f = g \cdot s^n$ because X is reduced. This shows that $X^{\mathrm{ss}} \subset \operatorname{Spec} A$.

We now verify that $(\star)$ holds for $[\operatorname{Spec} A/G]$. Let $\xi\colon \operatorname{Spec} R \to [\operatorname{Spec} A/G]$ be a map with $\xi(0) \simeq x$, and let $g \in \mathrm{Aut}_{\mathcal{X}}(\xi_K)$ be a finite order K-point. By applying the above result to $[X/G]$, there exists (after an extension of R) a map $\xi'\colon \operatorname{Spec} R \to [X/G]$ such that $\xi'_K \simeq \xi_K$ and g extends to an element of $\mathrm{Aut}_{\mathcal{X}}(\xi')$ but where $\xi'(0)$ may not be isomorphic to x. The stabilizer group scheme $\mathrm{Stab}_G(X) \subset X \times G$ is a closed subscheme equivariant with respect to the product action of G on $X \times G$ where G acts on itself via conjugation. The pair (ξ', g)

defines a morphism

$$\eta\colon \operatorname{Spec} R \to [\operatorname{Stab}_G(X)/G].$$

We will show that after an extension of R, there is a map $\eta'\colon \operatorname{Spec} R \to [\operatorname{Stab}_G(\operatorname{Spec} A)/G]$ with $\eta'_K \simeq \eta_K$. Similar to (6.7.11), we have a G-equivariant diagram

$$\begin{tikzcd} \operatorname{Stab}_G(\operatorname{Spec} A) \arrow[r, hook] \arrow[dr] & \operatorname{Stab}_G(X) \arrow[d] \\ & \operatorname{Spec} A^G \times G \end{tikzcd}$$

with $\operatorname{Stab}_G(X)$ projective over $\operatorname{Spec} A^G \times G$. We claim that the semistable locus of $\operatorname{Stab}_G(X)$ for the action of G with respect to the pullback of $\mathcal{O}_X(E)$ is precisely $\operatorname{Stab}_G(\operatorname{Spec} A)$ in the sense Exercise 6.6.11. The invariant section $s \in \Gamma(X, \mathcal{O}_X(E))^G$ pulls back to an invariant section on $\operatorname{Stab}_G(X)$ and thus $\operatorname{Stab}_G(\operatorname{Spec} A) \subset \operatorname{Stab}_G(X)^{\mathrm{ss}}$. To see the converse, suppose that $(y, h) \in \operatorname{Stab}_G(X)$ with $y \notin X^{\mathrm{ss}} = \operatorname{Spec} A$. Applying Kempf's Optimal Destabilizing Theorem (6.6.43) to a lift $\widehat{y}$ of y to the affine cone $\widehat{X} \to \operatorname{Spec} A^G$ of X yields a one-parameter subgroup $\lambda\colon \mathbb{G}_m \to G$ such that $\lim_{t\to 0} \lambda(t)\cdot \widehat{y} \in \widehat{X}$ exists and is contained in the zero section $\operatorname{Spec} A^G$. Moreover, since $G_y \subset P_\lambda$ (Exercise 6.6.48), $\lim_{t\to 0} \lambda(t)\cdot(\widehat{y}, h)$ also exists and is contained in the zero section of the affine cone over of $\operatorname{Stab}_G(X)$; thus (y, h) is not semistable.

The induced morphism $\operatorname{Stab}_G(\operatorname{Spec} A)/\!\!/G \to (\operatorname{Spec} A^G \times G)/\!\!/G$ of GIT quotients is proper, and the good moduli space $[\operatorname{Stab}_G(\operatorname{Spec} A)/G] \to \operatorname{Stab}_G(\operatorname{Spec} A)/\!\!/G$ is universally closed. By the valuative criterion, the composition

$$\operatorname{Spec} R \xrightarrow{\eta} \operatorname{Stab}_G(X) \to \operatorname{Spec} A^G \times G \to (\operatorname{Spec} A^G \times G)/\!\!/G$$

lifts a morphism $\chi\colon \operatorname{Spec} R \to [\operatorname{Stab}_G(\operatorname{Spec} A)/G]$ such that $\chi_K \simeq \xi_K$ after an extension of R. The composition $\xi'\colon \operatorname{Spec} R \xrightarrow{\chi} [\operatorname{Stab}_G(\operatorname{Spec} A)/G] \to [\operatorname{Spec} A/G]$ has the property that $\xi'_K \simeq \xi_K$ and that g extends to an element of $\operatorname{Aut}_{\mathcal{X}}(\xi')$. To arrange that $\xi'(0) \simeq x$, we apply Lemma 6.7.46 below. □

Lemma 6.7.46. *Let $\mathcal{X} = [\operatorname{Spec} A/G]$ where $\operatorname{Spec} A$ is an affine scheme of finite type over an algebraically closed field $\Bbbk$ equipped with an action of a reductive group G. Let $\xi, \xi'\colon \operatorname{Spec} R \to \mathcal{X}$ be morphisms from a DVR with residue field $\Bbbk$ such that $\xi_K \simeq \xi'_K$ and $\xi(0) \in \mathcal{X}$ is a closed point. For every element $g \in \operatorname{Aut}_{\mathcal{X}}(\xi')$, there exists (after replacing R with an extension) a morphism $\xi''\colon \operatorname{Spec} R \to \mathcal{X}$ such that $\xi''_K \simeq \xi'_K$, $g|_K$ extends to an automorphism of ξ'', and $\xi''(0) \simeq \xi(0)$.*

Proof. Since $\xi(0)$ and $\xi'(0)$ lie in the same fiber of $\mathcal{X} \to \operatorname{Spec} A^G$, the closure of $\xi'(0)$ in $|\mathcal{X}|$ must contain $\xi(0)$. Kempf's Criterion (6.6.41) yields a canonical map $f\colon \Theta \to [\operatorname{Spec} A/G]$ with $f(1) \simeq \xi'(0)$ and $f(0) \simeq \xi(0)$. Since f is canonical, every automorphism of $f(1)$ extends to an automorphism of the map f. In particular the restriction of $g \in \operatorname{Aut}_{\mathcal{X}}(\xi')$ to $f(1) = \xi'(0)$ extends uniquely to an automorphism g_f of f.

We now apply the Strange Gluing Lemma (6.7.47), which after replacing R with $R[\pi^{1/N}]$ and precomposing f with the map $\Theta \to \Theta$ defined by $x \mapsto x^N$ for $N \gg 0$, yields a *unique* map $\gamma\colon \phi_R \to \mathcal{X}$, such that $\gamma|_{s=0} \simeq f$ and $\gamma|_{t\neq 0} \simeq \xi'$. The uniqueness γ guarantees that the automorphism g of ξ' and g_f of f extends uniquely to an automorphism g_γ of γ. Finally we construct the desired map ξ'' as the composition

$$\xi''\colon \operatorname{Spec}(R[\sqrt{\pi}]) \xrightarrow{q} \phi_R \xrightarrow{\gamma} \mathcal{X},$$

where in (s,t,π) coordinates the first map q is defined by $(\sqrt{\pi},\sqrt{\pi},\pi)$. Under q, the special point of $\operatorname{Spec}(R[\sqrt{\pi}])$ maps to the point $0 \in \phi_R$. By construction, $\xi''(0) \simeq \xi(0)$ and the automorphism g of γ restricts to an automorphism of ξ'' extending $g|_{K(\sqrt{\pi})}$. □

Lemma 6.7.47 (Strange Gluing Lemma)**.** *Let $\mathcal{X}$ be an algebraic stack of finite type over an algebraically closed field $\Bbbk$ with affine diagonal. Let R be a DVR with residue field $\Bbbk$. Let $f\colon \Theta \to \mathcal{X}$ and $\xi\colon \operatorname{Spec} R \to \mathcal{X}$ be morphisms with an isomorphism $f(1) \simeq \xi(0)$. For $N \gg 0$, after replacing R with $R[\pi^{1/N}]$ and f with the composition $\Theta \xrightarrow{N} \Theta \xrightarrow{f} \mathcal{X}$, there is a unique morphism $\gamma\colon \phi_R \to \mathcal{X}$ such that $\gamma|_{s=0} \simeq f$ and $\gamma|_{t\neq 0} \simeq \xi$.*

Proof. For $n > 0$, define

$$\phi_R^{n,1} = [\operatorname{Spec}(R[s,t]/(st^n - \pi))/\mathbb{G}_m]$$

where the $\mathbb{G}_m$-acts with weight n on s and -1 on t. We have a closed immersion $\Theta \hookrightarrow \phi_R^{n,1}$ defined by $s = 0$ and an open immersion $\operatorname{Spec} R \hookrightarrow \phi_R^{n,1}$ defined by $t \neq 0$. Note that any morphism $\phi_R^{n,1} \to \mathcal{X}$ restricts to morphisms $f\colon \Theta \to \mathcal{X}$ and $\xi\colon \operatorname{Spec} R \to \mathcal{X}$ along with an isomorphism $\xi(0) \simeq f(1)$. We will show conversely that for $n \gg 0$, any $f\colon \Theta \to \mathcal{X}$ and $\xi\colon \operatorname{Spec} R \to \mathcal{X}$ with $\xi(0) \simeq f(1)$ extends canonically to a map $\phi_R^{n,1} \to \mathcal{X}$.

Letting $C = R[t, \pi/t, \pi/t^2, \ldots] \subset R[t]_t$, the diagram

$$\begin{tikzcd}
\operatorname{Spec} \Bbbk[t]_t \arrow[r] \arrow[d] & \operatorname{Spec} \Bbbk[t] \arrow[d] \\
\operatorname{Spec} R[t]_t \arrow[r] & \operatorname{Spec} C
\end{tikzcd}$$

is a pushout in the category of schemes (Theorem A.8.1). This diagram is $\mathbb{G}_m$-equivariant and the diagram obtained by taking the fiber product with $\mathbb{G}_m$ is also a pushout. It follows from Corollary A.8.6 that taking quotients by $\mathbb{G}_m$ yields a diagram

$$\begin{tikzcd}
\operatorname{Spec} \Bbbk \arrow[r] \arrow[d] & \Theta \arrow[d] \arrow[ddr, bend left, "f"] & \\
\operatorname{Spec} R \arrow[r] \arrow[drr, bend right, "\xi"'] & {[\operatorname{Spec}(C)/\mathbb{G}_m]} \arrow[dr, dashed, "\Psi"] & \\
 & & \mathcal{X}
\end{tikzcd} \tag{6.7.12}$$

where the square is a pushout in the category of algebraic stacks with affine diagonal; this induces the dotted arrow Ψ. We can write C as a union $C = \bigcup C_n$ where $C_n := R[t, \pi/t^n] \subset R[t]_t$. Note that $C_n \cong R[s,t]/(st^n - \pi)$ so in particular $[\operatorname{Spec}(C_n)/\mathbb{G}_m] \cong \phi_R^{n,1}$. As $\mathcal{X} \to S$ is locally of finite presentation, for $n \gg 0$ the morphism Ψ factors uniquely as $[\operatorname{Spec}(C)/\mathbb{G}_m] \to \phi_R^{n,1} \to \mathcal{X}$ (Exercise 3.3.31).

To finish the proof, compose the uniquely defined map $\Phi_R^{n,1} \to \mathcal{X}$ with the canonical map $\Phi_{R[\pi^{1/n}]} \to \mathcal{X}$ induced by the map of graded algebras $R[s,t]/(st^n - \pi) \to R[\pi^{1/n}][s^{1/n}, t]/(s^{1/n}t - \pi)$, where $s^{1/n}$ has weight 1. $\square$

6.7.6 S-completeness and reductivity

We've already seen that S-completeness characterizes separatedness (Proposition 6.7.17 and Corollary 6.7.15). We've also seen that it implies unpunctured inertia (Theorem 6.7.43) and therefore implies the existence of stabilizer preserving local quotient presentations (Proposition 6.7.40). We now prove a third remarkable property of S-completeness: it characterizes reductivity. More precisely, a smooth affine algebraic group G is reducitive if and only if $\mathbf{B}G$ is S-complete if and only if G has Cartan Decompositions (Proposition 6.7.48). This also completes the proof of Theorem 6.6.15

Proposition 6.7.48. *Let G be a smooth affine algebraic group over an algebraically closed field $\Bbbk$. The following are equivalent:*

(1) G is reductive,

(2) $\mathbf{B}G$ is S-complete, and

(3) G satisfies the Cartan Decomposition: for every complete DVR R over $\Bbbk$ with residue field $\Bbbk$ and fraction field K and for every element $g \in G(K)$, there exists elements $h_1, h_2 \in G(R)$ and a one-parameter subgroup $\lambda\colon \mathbb{G}_m \to G$ such that

$$g = h_1 \lambda|_K h_2.$$

In particular, if $\mathcal{X}$ is an S-complete algebraic stack and $x \in \mathcal{X}$ is closed point with smooth affine stabilizer G_x, then G_x is reductive.

Proof. For (2) $\Rightarrow$ (3), observe that since $\Phi_R \smallsetminus 0 = \operatorname{Spec} R \bigcup_{\operatorname{Spec} K} \operatorname{Spec} R$, an element $g \in G(K)$ determines a morphism

$$\rho_g\colon \Phi_R \smallsetminus 0 \to \mathbf{B}G$$

by gluing two trivial G-torsors over $\operatorname{Spec} R$ via the isomorphism induced by g of their restrictions to $\operatorname{Spec} K$. Since $\mathbf{B}G$ is S-complete, we have a lift

$$\begin{tikzcd} \Phi_R \smallsetminus 0 \arrow[r, "\rho_g"] \arrow[d] & \mathbf{B}G \\ \Phi_R. \arrow[ur, dashed, "h"] & \end{tikzcd} \tag{6.7.13}$$

Restricting h to the origin gives a map $\mathbf{B}\mathbb{G}_m \hookrightarrow \Phi_R \xrightarrow{h} \mathbf{B}G$ which corresponds to a map $\lambda\colon \mathbb{G}_m \to G$ (up to conjugation); this provides us with our candidate one-parameter subgroup. We make two observations:

- If $g, g' \in G(K)$ are elements, the morphisms $\rho_g, \rho_{g'}\colon \Phi_R \smallsetminus 0 \to \mathbf{B}G$ are isomorphic if and only if there are elements $h, h' \in G(R)$ such that $hg = g'h'$.
- If $\lambda\colon \mathbb{G}_m \to G$ is a one-parameter subgroup and $\lambda|_{\Phi_R \smallsetminus 0}$ denotes the composition $\Phi_R \smallsetminus 0 \hookrightarrow \Phi_R \to \mathbf{B}\mathbb{G}_m \xrightarrow{\lambda} \mathbf{B}G$, then $\lambda|_{\Phi_R \smallsetminus 0}$ and $\rho_{g'}$, where $g' = \lambda|_K$, are isomorphic.

It therefore suffices to show that the extension h in (6.7.13) is isomorphic to $\lambda|_{\phi_R}\colon \phi_R \to \mathbf{B}\mathbb{G}_m \xrightarrow{\lambda} \mathbf{B}G$. To see this, let $\mathcal{P}$ and $\mathcal{P}'$ denote the principal G-bundles over ϕ_R classifying h and $\lambda|_{\phi_R}$. Since G is smooth and affine, $\underline{\mathrm{Isom}}_{\phi_R}(\mathcal{P}, \mathcal{P}') \to \phi_R$ is smooth and affine. We have a section over the inclusion $\mathcal{X}_0 := \mathbf{B}\mathbb{G}_m \hookrightarrow \phi_R$ of 0. Letting $\mathcal{X}_n$ denote the nth nilpotent thickening, deformation theory (Proposition 6.5.8) and the cohomological affineness of $\mathcal{X}_n$ implies that we may find compatible sections over $\mathcal{X}_n$. Coherent Tannaka Duality (6.4.8) and the coherent completeness of ϕ_R along $\mathbf{B}\mathbb{G}_m$ (Theorem 6.4.11) implies that the map

$$\mathrm{MOR}_{\phi_R}(\phi_R, \underline{\mathrm{Isom}}_{\phi_R}(\mathcal{P}, \mathcal{P}')) \to \varprojlim \mathrm{MOR}_{\phi_R}(\mathcal{X}_n, \underline{\mathrm{Isom}}_{\phi_R}(\mathcal{P}, \mathcal{P}'))$$

is an equivalence. We thus obtain a section of $\underline{\mathrm{Isom}}_{\phi_R}(\mathcal{P}, \mathcal{P}') \to \phi_R$, i.e. an isomorphism between $\mathcal{P}$ and $\mathcal{P}'$.

To see that (3) $\Rightarrow$ (2), it suffices to show that every map $\phi_R \smallsetminus 0 \to \mathbf{B}G$ extends to a map $\phi_R \to \mathbf{B}G$ where R is a complete DVR over $\Bbbk$ with residue field $\Bbbk$ (Lemma 6.7.11). Since every principal G-bundle over $\operatorname{Spec} R$ is trivial, the map $\phi_R \smallsetminus 0 \to \mathbf{B}G$ is isomorphic to ρ_g for some element $g \in G(K)$. Writing $g = h_1 \lambda|_K h_2$, the two observations above show that $\phi_R \to \mathbf{B}\mathbb{G}_m \to \mathbf{B}G$ is an extension.

We've already seen that (1) $\Rightarrow$ (2) in Proposition 6.7.14. Conversely, if G is not reductive, there is a normal subgroup $\mathbb{G}_a \triangleleft R_u(G)$ of the unipotent radical. As $G/R_u(G)$ and $R_u(G)/\mathbb{G}_a$ are both affine, the composition $\mathbf{B}\mathbb{G}_a \to \mathbf{B}R_u G \to \mathbf{B}G$ is affine. By Lemma 6.7.13, this would imply that $\mathbf{B}\mathbb{G}_a$ is S-complete but this is a contradiction: taking $R = \Bbbk[\![x]\!]$ and $K = \Bbbk(\!(x)\!)$ the element $x \in \mathbb{G}_a(K)$ cannot be written as $h_1 \lambda|_K h_2$.

See also [AHHL21, Thm. A]. □

6.7.7 Proof of the Existence Theorem of Good Moduli Spaces

The necessity of Θ-completeness and S-completeness for the existence of a good moduli space was established in Corollary 6.7.15. We now establish the sufficiency following the strategy outlined in §6.7.1.

Proof of Theorem 6.7.1. Since $\mathcal{X}$ is S-complete and $\mathrm{char}(\Bbbk) = 0$, the stabilizer G_x of every closed point $x \in \mathcal{X}$ is linearly reductive (Proposition 6.7.48). By the Local Structure Theorem (6.5.1), there exists an affine étale morphism $f\colon ([\operatorname{Spec} A/G_x], w) \to (\mathcal{X}, x)$ inducing an isomorphism of stabilizer groups at x. Since $\mathcal{X}$ is Θ-complete and S-complete, we may assume that f is Θ-surjective and stabilizer preserving at all points after replacing $[\operatorname{Spec} A/G_x]$ with an open neighborhood of x (Propositions 6.7.34 and 6.7.40). Since $\mathcal{X}$ is quasi-compact, there exists finitely many closed points $x_i \in \mathcal{X}$ and morphisms $f_i\colon [\operatorname{Spec} A_i/G_{x_i}] \to \mathcal{X}$ as above whose images cover $\mathcal{X}$. Choosing embeddings $G_{x_i} \hookrightarrow \mathrm{GL}_n$ for some n, there are equivalence $[\operatorname{Spec} A_i/G_{x_i}] \cong [(\operatorname{Spec} A_i \times^{G_{x_i}} \mathrm{GL}_n)/\mathrm{GL}_n]$. Setting $A = \prod_i (A_i \times^{G_{x_i}} \mathrm{GL}_n)$, there is an surjective, affine, and étale morphism

$$f\colon \mathcal{X}_1 := [\operatorname{Spec} A/\mathrm{GL}_n] \to \mathcal{X}$$

which is Θ-surjective and stabilizer preserving at all points. Since $\mathrm{char}(\Bbbk) = 0$, there is a good moduli space $\mathcal{X}_1 \to X_1 := \operatorname{Spec} A^{\mathrm{GL}_N}$.

Set $\mathcal{X}_2 = \mathcal{X}_1 \times_{\mathcal{X}} \mathcal{X}_1$. The projections $p_1, p_2 \colon \mathcal{X}_2 \to \mathcal{X}_1$ are also affine, étale, Θ-surjective, and stabilizer preserving. Since f is affine, $\mathcal{X}_2 \cong [\operatorname{Spec} B/\operatorname{GL}_n]$ and there is a good moduli space $\mathcal{X}_2 \to X_2 := \operatorname{Spec} B^{\operatorname{GL}_n}$. This provides a commutative diagram

$$
\begin{tikzcd}
\mathcal{X}_2 \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] \arrow[d] & \mathcal{X}_1 \arrow[r, "f"] \arrow[d] & \mathcal{X} \arrow[d, dashed] \\
X_2 \arrow[r, shift left, "q_1"] \arrow[r, shift right, "q_2"'] & X_1 \arrow[r, dashed] & X
\end{tikzcd}
\tag{6.7.14}
$$

which each square on left is cartesian by Luna's Fundamental Lemma (6.3.26). Moreover, by the universality of good moduli spaces (Theorem 6.3.5(4)), the étale groupoid structure on $\mathcal{X}_2 \rightrightarrows \mathcal{X}_1$ induces a étale groupoid structure on $X_2 \rightrightarrows X_1$.

We claim that $X_2 \rightrightarrows X_1$ is an étale equivalence relation, i.e. that the quotient stack $[X_1/X_2]$ is an algebraic space. By the Characterization of Algebraic Spaces (3.6.5), it suffices to show that if $x_1 \in X_1$ is a $\Bbbk$-point, then (x_1, x_1) has a unique preimage under $(q_1, q_2) \colon X_2 \to X_1 \times X_1$. Let $x_2, x_2' \in X_2$ be two points mapping to $(x_1, x_1) \in X_1 \times X_1$, and let $\widetilde{x}_2, \widetilde{x}_2' \in \mathcal{X}_2$ be the unique closed points in their preimages. Since f is Θ-surjective, the images $p_1(\widetilde{x}_2)$, $p_2(\widetilde{x}_2)$, $p_1(\widetilde{x}_2')$, and $p_2(\widetilde{x}_2')$ are all closed points of $\mathcal{X}_1$ over x_1, and therefore they are all identified with the unique closed point $\widetilde{x}_1$ over x_1. On the other hand, since f is stabilizer preserving, the stabilizer groups of $\widetilde{x}_2$ and $\widetilde{x}_2'$ are the same as the stabilizer groups of $\widetilde{x}_1$ and of its image in $\mathcal{X}$. Let's denote this stabilizer group by G. It follows that the fiber product of $(p_1, p_2) \colon \mathcal{X}_2 \to \mathcal{X}_1 \times \mathcal{X}_1$ along the inclusion of the residual gerbe $\mathcal{G}_{(\widetilde{x}_1, \widetilde{x}_1)} = \mathbf{B}G \times \mathbf{B}G \to \mathcal{X}_1 \times \mathcal{X}_1$ is isomorphic to $\mathbf{B}G$ and thus identified with the residual gerbe of a *unique* closed point. Therefore $x_2 = x_2'$.

Since $X_2 \rightrightarrows X_1$ is an étale equivalence relation, the quotient $X = X_1/X_2$ is an algebraic space. From étale descent, there is a morphism $\mathcal{X} \to X$ which pulls back under $X_1 \to X$ to the good moduli space $\mathcal{X}_1 \to X_1$. By descent of good moduli spaces (Lemma 6.3.20(2)), $\mathcal{X} \to X$ is a good moduli space. Finally, we use that $\mathcal{X}$ is S-complete to conclude that X is separated (Corollary 6.7.15). $\square$

Appendix A

Morphisms of schemes

In this appendix, we recall definitions and summarize certain properties of morphisms of schemes: locally of finite presentation, flat, smooth, étale and unramified.

We pay particular attention to properties that can be described *functorially*, i.e. properties of schemes and their morphisms that can be characterized in terms of their functors. For instance, the properties of being separated, universally closed, proper, locally of finite presentation, smooth, étale and unramified can be characterized functorially. Such descriptions are particularly advantageous for us since we systematically study moduli problems via functors and stacks. For example, the valuative criterion for properness for $\overline{\mathcal{M}}_g$ amounts to checking that every family of curves over a punctured curve (i.e. over the generic point of a DVR) can be extended uniquely (after possibly a finite extension) to the entire curve (i.e. DVR). Similarly, the smoothness of $\overline{\mathcal{M}}_g$ can be shown by using the functorial formal lifting criterion.

A.1 Morphisms locally of finite presentation

A morphism of schemes $f\colon X \to Y$ is *locally of finite type* (resp. *locally of finite presentation*) if for all affine open subschemes $\operatorname{Spec} B \subset Y$ and $\operatorname{Spec} A \subset f^{-1}(\operatorname{Spec} B)$, there is surjection $A[x_1, \ldots, x_n] \to B$ of A-algebras (resp. a surjection $A[x_1, \ldots, x_n] \to B$ with finitely generated kernel). If in addition f is quasi-compact (resp. quasi-compact and quasi-separated), we say that f is *of finite type* (resp. *of finite presentation*).

Remark A.1.1. When Y is locally noetherian, these two notions coincide. However, in the non-noetherian setting even closed immersions may not be locally of finite presentation; e.g. $\operatorname{Spec} \mathbb{C} \hookrightarrow \operatorname{Spec} \mathbb{C}[x_1, x_2, \ldots]$. Since functors and stacks are defined in these notes on the entire category of schemes, it is often necessary to work with non-noetherian schemes. In particular, when defining a moduli functor or stack, we need to specify what families of objects are over possibly non-noetherian schemes. Morphisms of finite presentation are better behaved than morphisms of finite type and so we often use the former condition. For example, when defining a family of smooth curves $\pi\colon \mathcal{C} \to S$, we require not only that π is proper and smooth, but also of finite presentation.

Before stating the functorial characterization of locally of finite presentation morphism, we recall the notion of systems.

Definition A.1.2. A *directed system* (resp. *inverse system*) in a category $\mathcal{C}$ is a partially ordered set $(I, \geq)$ which is directed (i.e. for every $i, j \in I$ there exists $k \in I$ such that $k \geq i$ and $k \geq j$) together with a covariant (resp. contravariant) $I \to \mathcal{C}$.

Proposition A.1.3. *A morphism $f \colon X \to Y$ of schemes is locally of finite presentation if and only if for every inverse system $\{\operatorname{Spec} A_\lambda\}_{\lambda \in I}$ of affine schemes over Y, the natural map*

$$\operatorname{colim}_\lambda \operatorname{Mor}_Y(\operatorname{Spec} A_\lambda, X) \to \operatorname{Mor}_Y(\operatorname{Spec}(\operatorname{colim}_\lambda A_\lambda), X) \tag{A.1.1}$$

is bijective.

There's a conceptual reason for this: every ring A (e.g. $\mathbb{C}[x_1, x_2, \ldots]$) is the union (or colimit) of its finitely generated subalgebras A_λ. The condition that every map $\operatorname{Spec} A \to X$ factors through some $\operatorname{Spec} A_\lambda \to X$ can be viewed as the condition that specifying $\operatorname{Spec} A \to X$ over Y depends on only finite data (i.e. local generators and relations for the ring maps defining $X \to Y$) and therefore translates to a finiteness condition on X over Y. We encourage the reader to verify the proposition, especially in the case of a morphism of affine schemes. See [EGA, IV.8.14.2] or [SP, Tag 01ZC].

A.2 Flatness

You can't get very far in moduli theory without flatness. While its definition is seemingly abstract and algebraic, it is a magical geometric property of a morphism $X \to Y$ that ensures that fibers X_y 'vary nicely' as $y \in Y$ varies. This principle is nicely evidenced by Flatness via the Hilbert Polynomial (A.2.4). It is the reason that we define objects of our moduli stacks as flat families.

A.2.1 Flatness criteria

A module M over a ring A is *flat* if the functor

$$- \otimes_A M \colon \operatorname{Mod}(A) \to \operatorname{Mod}(A)$$

is exact. We recall the following criteria:

(1) (Stalk Criterion) M is flat over A if and only if $M_{\mathfrak{p}}$ is flat over $A_{\mathfrak{p}}$ for every prime (equivalently maximal) ideal $\mathfrak{p}$. More generally, if $A \to B$ is a ring map, a B-module N is flat if and only if for every prime $\mathfrak{q} \subset B$ with preimage $\mathfrak{p} \subset A$, $N_{\mathfrak{q}}$ is flat over $A_{\mathfrak{p}}$.

(2) (Ideal Criterion) M is flat if and only if for every finitely generated ideal $I \subset A$, the map $I \otimes_A M \to M$ is injective [Eis95, Prop. 6.1]. (When A is a PID, this implies that M is flat if and only if M is torsion free.)

(3) (Tor Criterion) M is flat if and only if $\operatorname{Tor}_1^A(A/I, M) = 0$ for all finitely generated ideals $I \subset A$ [Eis95, Prop. 6.1].

(4) (Finitely Presented Criterion) M is finitely presented and flat over A if and only if M is finite and projective if and only if M is finite and locally free (i.e. $M_{\mathfrak{p}}$ is finite and free for all prime—or equivalently maximal—ideals $\mathfrak{p}$); see [SP, Tag 00NX]. (Without the finitely presented hypothesis, Lazard's

Theorem states that M is flat over A if and only if M can be written as a directed colimit $\operatorname{colim}_{i \in I} M_i$ of free finite A-modules M_i; see [Eis95, A6.6] or [SP, Tag 058G].)

(5) (Equational Criterion) M is flat if and only if for every relation $\sum_{i=1}^n a_i m_i = 0$ with $a_i \in A$ and $m_i \in M$, there exists $m'_j \in M$ for $j = 1, \ldots, r$ and $a'_{ij} \in A$ such that $\sum_{j=1}^r a'_{ij} m'_j = m_i$ for all i and $\sum_{i=1}^n a'_{ij} a_i = 0$ for all j [Eis95, Cor. 6.5].

If $f\colon X \to Y$ is a morphism of schemes, then a quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ is *flat* if for all affine opens $\operatorname{Spec} B \subset Y$ and $\operatorname{Spec} A \subset f^{-1}(\operatorname{Spec} B)$, the B-module $\Gamma(\operatorname{Spec} A, \mathcal{F})$ is a flat.

Proposition A.2.1 (Flat Equivalences)**.** *Let $f\colon X \to Y$ be a morphism of schemes and $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. The following are equivalent:*

(1) $\mathcal{F}$ is flat over Y;

(2) There exists a Zariski-cover $\{\operatorname{Spec} B_i\}$ of Y and $\{\operatorname{Spec} A_{ij}\}$ of $f^{-1}(\operatorname{Spec} B_i)$ such that $\Gamma(\operatorname{Spec} A_{ij}, \mathcal{F})$ is flat as an B_i-module under the ring map $B_i \to A_{ij}$;

(3) For all $x \in X$, the $\mathcal{O}_{X,x}$-module $\mathcal{F}_x$ is flat as an $\mathcal{O}_{Y,y}$-module.

(4) The functor

$$\operatorname{QCoh}(Y) \to \operatorname{QCoh}(X), \qquad \mathcal{G} \mapsto f^*\mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{F}$$

is exact.

Proof. See [Har77, §III.9] or [SP, Tag 01U2]. □

We say that a morphism $f\colon X \to Y$ of schemes is *flat at $x \in X$* (resp. a quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ is *flat at $x \in X$*) if there exists a Zariski open neighborhood $U \subset X$ containing x such that $f|_U$ (resp. $\mathcal{F}|_U$) is flat over Y. This is equivalent to the flatness of $\mathcal{O}_{X,x}$ (resp. $\mathcal{F}_x$) as an $\mathcal{O}_{Y,y}$-module.

Proposition A.2.2 (Flatness Criterion over Smooth Curves)**.** *Let C be an integral and regular scheme of dimension 1 (e.g. the spectrum of a DVR or a smooth connected curve over a field), and let $X \to C$ be a morphism of schemes. A quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ is flat over C if and only if every associated point of $\mathcal{F}$ maps to the generic point of C.*

Proof. A short argument shows that this follows from the fact that a module over a DVR is flat if and only if it is torsion free; see [Har77, III.9.7]. □

Over higher dimensional bases, it is sometimes possible to check flatness by reducing to the above criterion over a smooth curve. This is called the valuative criterion for flatness: if $f\colon X \to S$ is a finite type morphism of noetherian schemes over a reduced scheme S and $\mathcal{F}$ is a coherent $\mathcal{O}_X$-module, then $\mathcal{F}$ is flat at $x \in X$ if and only if for every map $(\operatorname{Spec} R, 0) \to (S, f(x))$ from a DVR, the restriction $\mathcal{F}|_{X_R}$ is flat over R at all points in $X_R := X \times_S \operatorname{Spec} R$ over 0 and x [EGA, IV.11.8.1]. Despite providing a conceptual geometric criterion for flatness, surprisingly it is rarely used in moduli theory.

Proposition A.2.3 (Flatness Criterion over Artinian Rings)**.** *A module over an artinian ring is flat if and only if it is free if and only if it is projective.*

Proof. See [SP, Tag 051E]. □

Recall that if $X \subset \mathbb{P}^n_K$ is a subscheme and $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_X$-module, the *Hilbert polynomial* of $\mathcal{F}$ is $P_{\mathcal{F}}(z) = \chi(X, \mathcal{F}(z)) \in \mathbb{Q}[z]$.

Proposition A.2.4 (Flatness via the Hilbert Polynomial)**.** *Let S be a connected, reduced and noetherian scheme and let $X \subset \mathbb{P}^n_S$ be a closed subscheme. A coherent $\mathcal{O}_X$-module $\mathcal{F}$ is flat over S if and only if the function*

$$S \to \mathbb{Q}[z], \qquad s \mapsto P_{\mathcal{F}|_{X_s}}$$

assigning a point $s \in S$ to the Hilbert polynomial of the restriction $\mathcal{F}|_{X_s}$ to the fiber $X_s \subset \mathbb{P}^n_{\kappa(s)}$ is constant.

Proof. See [Har77, Thm. 9.9]. □

Theorem A.2.5 (Local and Infinitesimal Criteria for Flatness)**.** *Let $A \to B$ be a local homomorphism of noetherian local rings, and let M be a finite B-module. The following are equivalent:*

(1) M is flat over A,

(2) (Local Criterion) $\mathrm{Tor}_1^A(A/\mathfrak{m}_A, M) = 0$, and

(3) (Infinitesimal Criterion) $M/\mathfrak{m}_A^n M$ is flat over $A/\mathfrak{m}_A^n$ for every $n \geq 1$.

Proof. See [Eis95, Thm. 6.8, Exer. 6.5] and [SP, Tag 00MK]. □

The following consequence of the Local Criterion for Flatness is particularly useful in deformation theory.

Corollary A.2.6. *Let $A \twoheadrightarrow A_0$ be a surjective homomorphism of noetherian rings with kernel I such that $I^2 = 0$. An A-module M is flat over A if and only if*

(1) $M_0 := M \otimes_A A_0$ is flat over A_0, and

(2) the map $M_0 \otimes_{A_0} I \to M$ is injective.

Proof. For ($\Rightarrow$), condition (1) holds by base change and condition (2) holds by tensoring the exact sequence $0 \to I \to A \to A_0 \to 0$ with M and using the identification $M \otimes_A I \cong M_0 \otimes_{A_0} I$. For ($\Leftarrow$), by the Local Criterion for Flatness (A.2.5) it suffices to show that $\mathrm{Tor}_1^A(A/\mathfrak{p}, M) = 0$ for all prime ideals $\mathfrak{p} \subset A$. Let $\mathfrak{p}_0 := \mathfrak{p}/I \subset A$. Consider the following diagram which is obtained by tensoring the exact sequences $0 \to I \to \mathfrak{p} \to \mathfrak{p}_0 \to 0$ and $0 \to I \to A \to A_0 \to 0$ with M:

$$\begin{tikzcd}
 & & \mathrm{Tor}_1^A(M, A/\mathfrak{p}) \arrow[r] \arrow[d] & \mathrm{Tor}_1^{A_0}(M_0, A_0/\mathfrak{p}_0) \arrow[d] & \\
0 \arrow[r] & M \otimes_A I \arrow[r] \arrow[d, equal] & M \otimes_A \mathfrak{p} \arrow[r] \arrow[d] & M_0 \otimes_{A_0} \mathfrak{p}_0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M \otimes_A I \arrow[r] & M \arrow[r] \arrow[d, two heads] & M_0 \arrow[r] \arrow[d, two heads] & 0 \\
 & & M \otimes_A A/\mathfrak{p} \arrow[r] & M_0 \otimes_{A_0} A_0/\mathfrak{p}_0 &
\end{tikzcd}$$

Condition (2) implies that the second row is exact, and it follows that the first row is also exact, where we've used the identification $M \otimes_A \mathfrak{p}_0 \cong M_0 \otimes_{A_0} \mathfrak{p}_0$. Condition (1) implies that $\mathrm{Tor}_1^{A_0}(M_0, A_0/\mathfrak{p}_0) = 0$ and it follows from the snake lemma that $\mathrm{Tor}_1^A(M, A/\mathfrak{p}) = 0$. See also [Har10, Prop. 2.2]. $\square$

Remark A.2.7. Applying this with $A = \Bbbk[\epsilon]/(\epsilon^2)$ being the dual numbers and $A' = \Bbbk$, we recover the fact that an A-module M is flat if and only if $M \otimes_{\Bbbk[\epsilon]/(\epsilon^2)} \Bbbk \xrightarrow{\epsilon} M$ is injective. This also follows from the fact that a module N over a ring B is flat if and only if for every ideal $I \subset B$, the map $I \otimes_B M \to M$ is injective, and using that the only ideal in $\Bbbk[\epsilon]/(\epsilon^2)$ is (ϵ).

The Local Criterion for Flatness also provides the following useful criterion for when slicing preserves flatness.

Corollary A.2.8. *Let $f\colon X \to S$ be a morphism locally of finite presentation, and let $x \in X$ be a point with image $s \in S$. If f is flat at x and the image of $h \in \mathfrak{m}_x \subset \mathcal{O}_{X,x}$ in the local ring $\mathcal{O}_{X_s,x}$ of the fiber is a nonzerodivisor, then there exists an open neighborhood $U \subset X$ of x such that h extends to a global function on U and the composition $V(h) \hookrightarrow U \to S$ is locally of finite presentation and flat at x.*

Proof. The noetherian case is a consequence of the following algebraic statement: if $A \to B$ is a flat local ring homomorphism of noetherian local rings and $f \in \mathfrak{m}_B$ is a nonzerodivisor in $B \otimes_A A/\mathfrak{m}_A$, then $A \to B/(f)$ is flat. It suffices to show that f is also a nonzerodivisor in B. Indeed, in this case $0 \to B \xrightarrow{f} B \to B/(f) \to 0$ is an exact sequence which implies that $\mathrm{Tor}_1^A(A/\mathfrak{m}_A, B/(f)) = 0$ and thus $B/(f)$ is flat over A by the Local Criterion for Flatness (A.2.5). Let $b \in B$ and suppose that $fb = 0$. We already know that $b \in \mathfrak{m}_A B$ and we claim that $b \in \mathfrak{m}_A^n B$ for $n > 0$ implies that $b \in \mathfrak{m}_A^{n+1} B$. Given this claim, then $b \in \bigcap_{n>0} \mathfrak{m}_A^n B$ and thus $b = 0$ by Krull's Intersection Theorem. Let $a_1, \ldots, a_r$ be minimal generators of $\mathfrak{m}_A^n$ as an A-module. Write $b = \sum_i a_i b_i$ for $b_i \in B$. Then $0 = fb = \sum_i a_i(fb_i)$. Since B is A-flat, the Equational Criterion implies that there exists $m'_j \in B$ and $a'_{ij} \in A$ such that $\sum_j a'_{ij} a'_j = fb_i$ for all i and $\sum_i a'_{ij} a_i = 0$ for all j. By Nakayama's Lemma and our choice of the a_i's, each a'_{ij} is in $\mathfrak{m}_A$ (as otherwise the images $\overline{a}_i \in \mathfrak{m}_A^n/\mathfrak{m}_A^{n+1}$ would be linearly dependent over $A/\mathfrak{m}_A$). This implies that $fb_i \in \mathfrak{m}_A B$. As f is a nonzerodivisor in $B \otimes_A A/\mathfrak{m}_A$, we see that $b_i \in \mathfrak{m}_A B$ for each i and thus $b \in \mathfrak{m}_A^{n+1}$.

The general case can be reduced to the noetherian case using limit methods (A.6). See also [Mat89, Thm. 22.5] or [SP, Tag 056X]. $\square$

Theorem A.2.9 (Fibral Flatness Criterion)**.** *Consider a commutative diagram*

$$\begin{tikzcd} X \arrow[rr] \arrow[dr] & & Y \arrow[dl] \\ & S & \end{tikzcd}$$

of schemes, and let F be a quasi-coherent $\mathcal{O}_X$-module of finite presentation. Assume that $X \to S$ is locally of finite presentation and $Y \to S$ is locally of finite type. Let $x \in X$ with images $y \in Y$ and $s \in S$. If the stalk $\mathcal{F}_x$ is nonzero, then the following are equivalent:

(1) F is flat over S at x, and $\mathcal{F}_s := \mathcal{F}|_{X_s}$ is flat over Y_s at x, and

(2) Y is flat over S at y and $\mathcal{F}$ is flat over Y at x.

Proof. See [SP, Tag 039A] □

If $A \to B$ is local ring map of noetherian local rings, then $\dim B = \dim A + \dim B/\mathfrak{m}_A B$. The following is a partial converse.

Theorem A.2.10 (Miracle Flatness). *Let $A \to B$ be a local homomorphism of noetherian local rings. Assume that*

1. *A is regular,*
2. *B is Cohen–Macaulay, and*
3. *$\dim B = \dim A + \dim B/\mathfrak{m}_A B$.*

Then $A \to B$ is flat.

Proof. See [Nag62, Thm. 25.16] or [SP, Tag 00R4]. □

A.2.2 Properties of flatness

Theorem A.2.11 (Generic Flatness). *Let $f: X \to S$ be a finite type morphism of schemes and $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_X$-module. If S is reduced, there exists a dense open subscheme $U \subset S$ such that $X_U \to U$ is flat and of presentation and such that $\mathcal{F}|_{X_U}$ is flat over U and of finite presentation as on $\mathcal{O}_{X_U}$-module.*

Proof. See [SP, Tag 052B]. □

Proposition A.2.12 (Fppf Morphisms are Open). *Let $f: X \to Y$ be a morphism of schemes. If f is flat and locally of finite presentation, then for every open subset $U \subset X$, the image $f(U) \subset Y$ is open.*

Proof. See [SP, Tag 01UA]. □

Proposition A.2.13. *A flat monomorphism locally of finite presentation (e.g. an étale monomorphism) is an open immersion.*

Theorem A.2.14 (Existence of Flattening Stratifications). *Let $X \to S$ be a projective morphism of noetherian schemes, $\mathcal{O}_X(1)$ be a relatively ample line bundle and $\mathcal{F}$ be a coherent sheaf on X. For each polynomial $P \in \mathbb{Q}[z]$, there exists a locally closed subscheme $S_P \hookrightarrow S$ such that a morphism $T \to S$ factors through S_P if and only if the pullback $\mathcal{F}_T$ of $\mathcal{F}$ to X_T is flat over T and for every $t \in T$, the pullback $\mathcal{F}_{\kappa(t)}$ to $X_{\kappa(t)}$ has Hilbert polynomial P.*

Moreover, there exists a finite indexing set I of polynomials such that $S = \coprod_{P \in I} S_P$ set-theoretically. The closure of S_P in S is contained set-theoretically in the union $\bigcup_{P \leq Q} S_Q$, where $P \leq Q$ if and only if $P(z) \leq Q(z)$ for $z \gg 0$.

Proof. See [FGA_{IV}] or [Mum66, §8]. □

Remark A.2.15. When $X \to S$ is only proper, there is a *universal flattening*, i.e. an algebraic space S' and a morphism $S' \to S$ such that a map $T \to S$ factors through $S' \to S$ if and only if the pullback $\mathcal{F}|_{X_T}$ to $X_T := X \times_S T$ is flat over T [SP, Tag 05UG]. In general, S' may not be a disjoint union of locally closed subschemes of S; see [Kre13].

Theorem A.2.16 (Raynaud-Gruson Flatification). *Let Y be a quasi-compact and quasi-separated scheme and $X \to Y$ be a finitely presented morphism which is flat over a quasi-compact open subscheme $U \subset Y$. Then there is a commutative diagram*

$$\begin{tikzcd} \widetilde{X} \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ Y' \arrow[r, "p"] & Y \end{tikzcd}$$

where $p\colon Y' \to Y$ is a blow-up of a finitely presented closed subscheme $Z \subset Y$ disjoint from U and the strict transform $\widetilde{X}$ of X is flat over Y'.

The *strict transform* $\widetilde{X}$ above is by definition the closure of $(Y' \setminus p^{-1}(Z)) \times_Y X$ in the base change $Y' \times_Y X$.

Proof. See [RG71, Thm. I.5.2.2] or [SP, Tag 0815]. □

A.2.3 Faithful flatness

A module M over a ring A is *faithfully flat* if the functor $- \otimes_A M\colon \operatorname{Mod}(A) \to \operatorname{Mod}(A)$ is faithfully exact, i.e. a sequence $N' \to N \to N''$ of A-modules is exact if and only if $N' \otimes_A M \to N \otimes_A M \to N'' \otimes_A M$ is exact

for every non-zero map $\phi\colon N \to N'$ of A-modules, the induced map $\phi \otimes_A M\colon N \otimes_A M \to N' \otimes_A M$ is also non-zero.

Proposition A.2.17 (Faithfully Flat Equivalences). *Let A be a ring and M be a* flat *A-module. The following are equivalent:*

(1) M is faithfully flat;

(2) for every non-zero map $\phi\colon N \to N'$ of A-modules, the induced map $\phi \otimes_A M\colon N \otimes_A M \to N' \otimes_A M$ is also non-zero;

(3) for every non-zero A-module N, the tensor product $N \otimes_A M$ is non-zero;

(4) for every prime ideal $\mathfrak{p} \subset A$, the tensor product $M \otimes_A \kappa(\mathfrak{p})$ is non-zero; and

(5) for every maximal ideal $\mathfrak{m} \subset A$, the tensor product $M \otimes_A \kappa(\mathfrak{m}) \cong M/\mathfrak{m}M$ is non-zero.

Proof. See [SP, Tag 00H9]. □

When $M = B$ is an A-algebra, then by (4) a flat ring map $A \to B$ is faithfully flat if $\operatorname{Spec} B \to \operatorname{Spec} A$ is surjective , or equivalently by (5) every maximal ideal of A is in the image of $\operatorname{Spec} B \to \operatorname{Spec} A$. The latter equivalence implies that any flat local ring map is faithfully flat.

A morphism $f\colon X \to Y$ of schemes is *faithfully flat* if f is flat and surjective. This is equivalent to the condition that $f^*\colon \operatorname{QCoh}(Y) \to \operatorname{QCoh}(X)$ is faithfully exact. It is also equivalent to the condition that a quasi-coherent $\mathcal{O}_Y$-module (resp. a morphism of quasi-coherent $\mathcal{O}_Y$-modules) is zero if and only if its pullback is. Faithfully flat morphisms play an important role in descent theory; see §B.

A.2.4 Fppf and fpqc morphisms

Fppf and fpqc morphisms are acronyms for 'fidèlement plate de présentation finie' and 'fidèlement plat et quasi-compact,' respectively. Despite this terminology, an fpqc morphism is more general than a faithfully flat and quasi-compact map.

Definition A.2.18. We define a morphism $f\colon X \to Y$ of schemes to be:

(1) *fppf* if f is faithfully flat and locally of finite presentation, and

(2) *fpqc* is f is faithfully flat and every quasi-compact open subset of Y is the image of a quasi-compact open subset of X.

Remark A.2.19. A quasi-compact and faithfully flat morphism is fpqc. In addition, an open and faithfully flat moprhism is fpqc: for a quasi-compact open subset $V \subset Y$, we can write $f^{-1}(V) = \bigcup_i U_i$ as a union of affines, and since each $f(U_i) \subset V$ is open and V is quasi-compact, we see that V is the image of finitely many of the U_i's. In particular, since every fppf morphism is open (Proposition A.2.12), an fppf morphism is also fpqc.

An fppf (resp. fpqc) cover $\{X_i \to X\}$ is a collection of morphisms such that $\coprod_i X_i \to X$ is fppf (resp. fpqc).

A.2.5 Universally injective homomorphisms

The defining characteristic of a flat module is that it preserves every injection under tensoring. The dual notion of an injection of modules that is preserved under tensoring is also a very useful property.

Definition A.2.20. A homomorphism $M \to N$ of A-modules if *universally injective* if for every A-module P, the map $M \otimes_A P \to N \otimes_A P$ is injective. A ring map $A \to B$ is *universally injective* if it is as a map of A-modules.

We will use this notion in a fundamental way in our proof of Coherent Tannaka Duality (Theorem 6.4.1). To this end, the following properties will be used:

Proposition A.2.21.

(1) A faithfully flat ring map $A \to B$ is universally injective.

(2) A split injective $M \to N$ of A-modules is universally injective. The converse is true if N/M is finitely presented.

(3) If $A \to A'$ is faithfully flat, then a map $M \to N$ of A-modules is universally injective if and only if $M \otimes_A A' \to N \otimes_A A'$ is.

(4) If $A \to B$ is universally injective and $B \to B \otimes_A B, b \mapsto b \otimes 1$ is faithfully flat, then $A \to B$ is faithfully flat.

Proof. For (1), (2) and (1), see [SP, Tags 08WP, 058L and 08XD]. Part (3) follows directly from the faithful exactness of $- \otimes_A A'$. See also [Laz69] and [Lam99, §4J]. □

Remark A.2.22. It is also true that an A-module map $M \to N$ is universally injective if and only if $\mathrm{Hom}_A(P, N) \to \mathrm{Hom}_A(P, N/M)$ is surjective for all finitely presented A-modules P [SP, Tag 058F]. If M/N is flat, then $M \to N$ is universally injective; if in addition N is flat, then the converse is true (in which case M is also flat [SP, Tag 058P].

Remarkably universally injective ring maps are precisely those maps that satisfy effective descent for modules; see Remark B.1.5.

A.3 Étale, smooth, unramified, and syntomic morphisms

A.3.1 Smooth morphisms

A morphism $f\colon X \to Y$ of schemes is *smooth* if f is locally of finite presentation, flat and for every $y \in Y$ the geometric fiber $X_{\overline{\kappa(y)}} = X \times_Y \operatorname{Spec} \overline{\kappa(y)}$ is regular.

Smooth Equivalences A.3.1. Let $f\colon X \to Y$ be morphism of (resp. noetherian) schemes locally of finite presentation. The following are equivalent:

(1) f is smooth;

(2) f satisfies the *Infinitesimal Lifting Criterion for Smoothness* (sometimes referred to as *formal smoothness*): for every surjection $A \to A_0$ of rings with nilpotent kernel (resp. surjection $A \to A_0$ of local artinian rings whose kernel is isomorphic to the residue field $A/\mathfrak{m}_A$) and every commutative diagram

$$\begin{tikzcd} \operatorname{Spec} A_0 \arrow[r] \arrow[d, hook] & X \arrow[d, "f"] \\ \operatorname{Spec} A \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

of solid arrows, there exists a dotted arrow filling in the diagram;

(3) f satisfies the *Jacobi Criterion for Smoothness*: for every point $x \in X$, there exist affine open neighborhoods $\operatorname{Spec} B$ of $f(x)$ and $\operatorname{Spec} A \subset f^{-1}(\operatorname{Spec} B)$ of x and an A-algebra isomorphism

$$B \cong \big(A[x_1, \ldots, x_n]/(f_1, \ldots, f_r)\big)_g$$

for some $f_1, \ldots, f_r, g \in A[x_1, \ldots, x_n]$ with $r \leq n$ such that the determinant $\det(\frac{\delta f_j}{\delta x_i})_{1 \leq i,j \leq r} \in B$ of the Jacobi matrix, defined by the partial derivatives with respect to the *first* r x_i's, is a unit.

If in addition X and Y are locally of finite type over an algebraically closed field $\Bbbk$, then the above are equivalent to:

(4) for all $x \in X(\Bbbk)$, there is an isomorphism $\widehat{\mathcal{O}}_{X,x} \cong \widehat{\mathcal{O}}_{Y,y}[\![x_1, \ldots, x_r]\!]$ of $\widehat{\mathcal{O}}_{Y,y}$-algebras.

We say that a morphism $f\colon X \to Y$ of schemes is *smooth at $x \in X$* if there exists an open neighborhood $U \subset X$ of x such that $f|_U\colon U \to Y$ is smooth.

If $f\colon X \to Y$ is a smooth morphism of schemes, then $\Omega_{X/Y}$ is a locally free $\mathcal{O}_X$-module of finite rank. If Y is connected, the rank of $\Omega_{X/Y}$ is the dimension of any fiber.

A.3.2 Étale morphisms

A morphism $f\colon X \to Y$ of schemes is *étale* if f is smooth of relative dimension 0 (i.e. f is smooth and $\dim X_y = 0$ for all $y \in Y$).

Étale Equivalences A.3.2. Let $f\colon X \to Y$ be morphism of (resp. noetherian) schemes locally of finite presentation. The following are equivalent:

(1) f is étale;

(2) f is smooth and $\Omega_{X/Y} = 0$;

(3) f is flat and for all $y \in Y$, the fiber X_y is isomorphic to a disjoint union $\coprod_i \operatorname{Spec} K_i$ where each K_i is separable field extension of $\kappa(y)$; (This is exactly the condition that f is flat and unramified; see §A.3.3.)

(4) f satisfies the *Infinitesimal Lifting Criterion for Étaleness*: for every surjection $A \to A_0$ of rings with nilpotent kernel (resp. surjection $A \to A_0$ of local artinian rings whose kernel is isomorphic to the residue field $A/\mathfrak{m}_A$) and every commutative diagram

$$\begin{tikzcd} \operatorname{Spec} A_0 \arrow[r] \arrow[d, hook] & X \arrow[d, "f"] \\ \operatorname{Spec} A \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

of solid arrows, there exists a unique dotted arrow filling in the diagram;

(5) f satisfies the *Jacobi Criterion for Étaleness*: for every point $x \in X$, there exist affine open neighborhoods $\operatorname{Spec} B$ of $f(x)$ and $\operatorname{Spec} A \subset f^{-1}(\operatorname{Spec} B)$ of x and an A-algebra isomorphism

$$B \cong \big(A[x_1, \dots, x_n]/(f_1, \dots, f_n)\big)_g$$

for some $f_1, \dots, f_n, g \in A[x_1, \dots, x_n]$ such that the determinant $\det(\frac{\delta f_j}{\delta x_i})_{1 \le i,j \le n} \in B$ is a unit.

If in addition X and Y are locally of finite type over an algebraically closed field $\Bbbk$, then the above are equivalent to:

(6) for all $x \in X(\Bbbk)$, the induced map $\widehat{\mathcal{O}}_{Y,y} \to \widehat{\mathcal{O}}_{X,x}$ on completions is an isomorphism

If in addition X and Y are smooth over $\Bbbk$, then the above are equivalent to:

(7) for all $x \in X(\Bbbk)$, the induced map $T_{X,x} \to T_{Y,y}$ on tangent spaces is an isomorphism.

We say that a morphism $f\colon X \to Y$ of schemes is *étale at* $x \in X$ if there exists an open neighborhood $U \subset X$ of x such that $f|_U\colon U \to Y$ is étale.

A.3.3 Unramified morphisms

A morphism $f\colon X \to Y$ of schemes is *unramified* if f is locally of finite type and every geometric fiber is discrete and reduced. Note that this second condition is equivalent to requiring that for all $y \in Y$, the fiber X_y is isomorphic to a disjoint union $\coprod_i \operatorname{Spec} K_i$ where each K_i is separable field extension of $\kappa(y)$.

Caution A.3.3. We are following the conventions of [RG71] and [SP] rather than [EGA] as we only require that f is locally of finite type rather than locally of finite presentation.

Unramified Equivalences A.3.4. Let $f\colon X \to Y$ be morphism of schemes locally of finite type. The following are equivalent:

(1) f is unramified;

(2) $\Omega_{X/Y} = 0$;

(3) f satisfies the *Infinitesimal Lifting Criterion for Unramifiedness*: for every surjection $A \to A_0$ of rings with nilpotent kernel (resp. surjection $A \to A_0$ of local artinian rings whose kernel is isomorphic to the residue field $A/\mathfrak{m}_A$) and every commutative diagram

$$\begin{tikzcd} \operatorname{Spec} A_0 \arrow[r] \arrow[d, hook] & X \arrow[d, "f"] \\ \operatorname{Spec} A \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

of solid arrows, there exists at most one dotted arrow filling in the diagram.

If in addition X and Y are locally of finite type over an algebraically closed field $\Bbbk$, then the above are equivalent to:

(4) for all $x \in X(\Bbbk)$, the induced map $\widehat{\mathcal{O}}_{Y,y} \to \widehat{\mathcal{O}}_{X,x}$ on completions is surjective.

We say that a morphism $f: X \to Y$ of schemes is *unramified at $x \in X$* if there exists an open neighborhood $U \subset X$ of x such that $f|_U: U \to Y$ is unramified.

A.3.4 Étale-local structure of smooth, étale, and unramified morphisms

Every smooth morphism looks étale-locally like relative affine space $\mathbb{A}^n_B \to \operatorname{Spec} B$.

Proposition A.3.5. *Let $X \to Y$ be a morphism of schemes smooth at $x \in X$. There exists affine open subschemes $\operatorname{Spec} A \subset X$ and $\operatorname{Spec} B \subset Y$ with $x \in \operatorname{Spec} A$, and a commutative diagram*

$$\begin{tikzcd} X \arrow[d] & \operatorname{Spec} A \arrow[l, hook', "\mathrm{op}"'] \arrow[r, "\text{ét}"] \arrow[d] & \mathbb{A}^n_B \arrow[dl] \\ Y & \operatorname{Spec} B \arrow[l, hook', "\mathrm{op}"'] & \end{tikzcd}$$

where $\operatorname{Spec} A \to \mathbb{A}^n_B$ is étale.

Proof. See [SP, Tag 039P]. □

Corollary A.3.6. *Let $f: X \to Y$ be a morphism of schemes smooth at $x \in X$. Then there exists an étale neighborhood $Y' \to Y$ of $f(x)$ such that $X \times_Y Y' \to Y'$ has a section.*

Proof. We apply the proposition. The morphism $\mathbb{A}^n_B \to \operatorname{Spec} B$ admits the zero section $\operatorname{Spec} B \to \mathbb{A}^n_B$ and we let $Y' := \operatorname{Spec} B \times_{\mathbb{A}^n_B} \operatorname{Spec} A$. Then $Y' \to \operatorname{Spec} B$ is étale and $Y' \to \operatorname{Spec} A \hookrightarrow X$ defines a section $Y' \to X \times_Y Y'$ of $X \times_Y Y' \to Y'$. □

Every étale (resp. unramified) morphism is étale-locally an isomorphism (resp. closed immersion).

Proposition A.3.7. *Let $f: X \to S$ be a separated morphism of schemes étale at $x \in X$. Then there exists an étale neighborhood $(U, u) \to (S, f(x))$ and a finite disjoint union decomposition*

$$X_U = W \amalg \coprod_i V_i$$

such that each $V_i \to U$ is an isomorphism and the the fiber W_u contains no point over x.

Proof. See [SP, Tags 04HM and 04HG]. □

A.3.5 Further properties

Proposition A.3.8 (Fiberwise Criteria for Étaleness/Smoothness/Unramifiedness)**.** *Consider a diagram*

$$\begin{tikzcd} X \arrow[rr] \arrow[dr] & & Y \arrow[dl] \\ & S & \end{tikzcd}$$

of schemes where $X \to S$ and $Y \to S$ are locally of finite presentation.

1. *$X \to Y$ is unramified if and only if $X_s \to Y_s$ is for all $s \in S$.*
2. *If $X \to S$ is flat, then $X \to Y$ is étale (resp. smooth) if and only if $X_s \to Y_s$ is for all $s \in S$.*

Remark A.3.9. With the same hypotheses, let $x \in X$ be a point with image $s \in S$. Then $X \to Y$ is étale (resp. smooth, unramified) at $x \in X$ if and only if $X_s \to Y_s$ is at x.

Corollary A.3.10. *If $f\colon X \to Y$ is a proper flat morphism of finite presentation, then the set Σ_f, consisting of points $y \in Y$ where $X_y \to \operatorname{Spec} \kappa(y)$ is smooth, is open.*

Proof. By Remark A.3.9, if $y \in Y$ is a point such that $X_y \to \operatorname{Spec} \kappa(y)$ is smooth, then $f\colon X \to Y$ is smooth in an open neighborhood of X_y. If $Z \subset X$ is the closed locus where $f\colon X \to Y$ is not smooth, then $f(Z) \subset Y$ is precisely the locus where the fibers of f are not smooth. Since f is proper, $f(Z)$ is closed. □

Proposition A.3.11. *Let $X \to Y$ be a smooth morphism of noetherian schemes. For every point $x \in X$ with image $y \in Y$,*

$$\dim_x(X) = \dim_y(Y) + \dim_x(X_y).$$

Proposition A.3.12. *If $X \to Y$ is a finite étale morphism, there exists a finite étale cover $Y' \to Y$ such that $X \times_Y Y' \to Y'$ is a trivial covering, i.e. $X \times_Y Y'$ is isomorphism to $\coprod_i Y'$ over Y'.*

Proof. We may assume that the degree d of $X \to Y$ is constant. The scheme $(X/Y)^d = \underbrace{X \times_Y \cdots \times_Y X}_{d}$ represents the functor on Sch/Y assigning a Y-scheme T to the set of d sections of $X \times_Y T \to T$. Each pairwise diagonal $(X/Y)^{d-1} \to (X/Y)^d$ is an open and closed immersion and we set $(X/Y)^d_0 \subset (X/Y)^d$ to be the complement of all pairwise diagonals. The projection morphism $(X/Y)^d_0 \to Y$ is finite étale and the functorial description gives d disjoint sections of $X \times_Y (X/Y)^d_0 \to (X/Y)^d_0$. □

Proposition A.3.13. *A dominant unramified morphism $X \to Y$ of schemes with Y normal and X connected is étale.*

Proof. See [SGA1, Cor. I.9.11]. □

A.3.6 Fitting ideals and the singular locus

For background references on fitting ideals, we recommend [SP, Tag 07Z6] and [Eis95, §20]. If R is a noetherian ring and M is a finitely generated R-module, the *kth fitting ideal* $\mathrm{Fit}_k(M)$ *of* M is the ideal generated by the $(n-k)\times(n-k)$ minors of a matrix A defining a presentation

$$\bigoplus_{i\in I} R \xrightarrow{A} R^n \to M \to 0.$$

(Of course, when M is finitely presented (e.g. R is noetherian), then left term can be assume to a finite free module R^m, in which case A is an $m \times n$ matrix.) The fitting ideal is independent of the choice of presentation. This defines an increase sequence of ideals

$$0 = \mathrm{Fit}_{-1}(M) \subset \mathrm{Fit}_0(M) \subset \mathrm{Fit}_1(M) \subset \cdots$$

such that $\mathrm{Fit}_k(M) = R$ if M can be generated by k elements. The R-module M is locally free of rank r if and only if $\mathrm{Fit}_{r-1}(M) = 0$ and $\mathrm{Fit}_r(M) = R$, and in this case $\mathrm{Fit}_k(M) = 0$ for all $k < r$. For $\mathfrak{p} \in \operatorname{Spec} R$, $\mathrm{Fit}_k(M)_{\mathfrak{p}} = \mathrm{Fit}_k(M_{\mathfrak{p}})$ and is equal to $R_{\mathfrak{p}}$ if and only if $M_{\mathfrak{p}}$ is generated by k sections. Moreover, fitting ideals commute with taking completion, i.e. $\mathrm{Fit}_k(M_{\mathfrak{p}}) \otimes_{R_{\mathfrak{p}}} \widehat{R}_{\mathfrak{p}} = \mathrm{Fit}_k(\widehat{M_{\mathfrak{p}}})$.

If X is a noetherian scheme and F is a coherent sheaf on X, the *kth fitting ideal sheaf* $\mathrm{Fit}_k(F)$ *of* F is the coherent sheaf of ideals defined by the property that for an affine open $U \subset X$, $\Gamma(U, \mathrm{Fit}_k(F)) = \mathrm{Fit}_k(\Gamma(F, U))$.

Fitting ideals allow us to define a scheme structure on the singular locus.

Definition A.3.14. If X is a noetherian scheme of pure dimension d over a field $\Bbbk$, we define the *singular locus of* X as the subscheme $\mathrm{Sing}(X) := V(\mathrm{Fit}_d(\Omega_{X/\Bbbk}))$ defined by the dth fitting ideal of of $\Omega_{X/\Bbbk}$.

More generally, if $X \to S$ is an fppf morphism such that every fiber has pure dimension d, we define the *relative singular locus* as the subscheme $\mathrm{Sing}(X/S) := V(\mathrm{Fit}_d(\Omega_{X/S}))$.

For example, if $X = \operatorname{Spec} R$ with $R = \Bbbk[x_1, \dots, x_n]/I$ and $I = (f_1, \dots, f_m)$, then using the exact sequence $I/I^2 \to \Omega_{\mathbb{A}^n/\Bbbk}|_X \to \Omega_{X/\Bbbk} \to 0$, we see that there is a resolution

$$\mathcal{O}_X^m \xrightarrow{J} \mathcal{O}_X^n \to \Omega_{X/\Bbbk} \to 0 \quad \text{with } J = \left(\frac{\partial f_j}{\partial x_i}\right),$$

and $\mathrm{Sing}(X)$ is defined by all $(n-d)\times(n-d)$ minors of the $n \times m$ Jacobian matrix J.

A.3.7 Local complete intersections

A scheme X locally of finite type over a field $\Bbbk$ is a *local complete intersection at* $p \in X$ if there exists an affine open neighborhood $p \in \operatorname{Spec} A \subset A$ such that A is a global complete intersection over $\Bbbk$, i.e. $A \cong \Bbbk[x_1, \dots, x_n]/(f_1, \dots, f_c)$ with $\dim A = n - c$. The scheme X is a *local complete intersection* if it is at every point.

Proposition A.3.15. *For a scheme X locally of finite type over a field $\Bbbk$ and a point $p \in X$, the following are equivalent:*

(1) *X is a local complete intersection at p,*
(2) *the local ring $\mathcal{O}_{X,x} \cong R/(f_1, \dots, f_c)$ where R is a regular local ring and $f_1, \dots, f_c \in R$ is a regular sequence, and*
(3) *the completion $\widehat{\mathcal{O}}_{X,x} \cong R/(f_1, \dots, f_c)$ where R is a regular local ring and $f_1, \dots, f_c \in R$ is a regular sequence.*

Proof. See [SP, Tags 00S8 and 09PY]. □

For a scheme locally of finite type over a field $\Bbbk$, we have the following implications:

$$\text{smooth} \implies \text{local complete intersection} \implies \text{Cohen–Macaulay}.$$

A.3.8 Syntomic morphisms

There is also a well-behaved relative notion of local complete intersections: a morphism of schemes $f\colon X \to S$ is *syntomic* (or a *flat local complete intersection morphism*) if f is fppf and every fiber is a local complete intersection. We say that $f\colon X \to S$ is *syntomic at $x \in X$* if there is an open neighborhood U of x such that $f|_U$ is syntomic; this is equivalent to f being fppf in an open neighborhood of x and the fiber X_s being a local complete intersection at x, where $s = f(x)$. Moreover, syntomic morphisms have a local structure analogous to local complete intersections: a morphism $f\colon X \to S$ is syntomic at $x \in X$ if and only if there are affine open neighborhood $x \in \operatorname{Spec} A \subset X$ and $\operatorname{Spec} B \subset Y$ with $f(\operatorname{Spec} A) \subset \operatorname{Spec} B$ such that $A \cong B[x_1, \dots, x_n]/(f_1, \dots, f_c)$ and every nonempty fiber of $\operatorname{Spec} A \to \operatorname{Spec} B$ has dimension $n - c$. See [EGA, IV §19.3], [SGA6, VIII §1] and [SP, Tag 01UB].

A.3.9 Lifting étale, smooth, and syntomic morphisms along closed immersions

The following fact is sometimes convenient.

Proposition A.3.16. *Consider a diagram*

$$\begin{tikzcd}
\operatorname{Spec} A_0 \arrow[r, hook, dashed] \arrow[d] \arrow[dr, phantom, "\square"] & \operatorname{Spec} A \arrow[d, dashed] \\
\operatorname{Spec} B_0 \arrow[r, hook] & \operatorname{Spec} B
\end{tikzcd}$$

of solid arrows where $\operatorname{Spec} B \hookrightarrow \operatorname{Spec} B_0$ is a closed immersion. If $\operatorname{Spec} A_0 \to \operatorname{Spec} B_0$ is étale (resp. smooth, syntomic), then there exists an étale (resp. smooth, syntomic) morphism $\operatorname{Spec} A \to \operatorname{Spec} B$ making the above diagram cartesian.

Proof. See [SP, Tags 04D1 and 07M8]. □

A.4 Properness and the valuative criterion

One of the most important applications of the valuative criterion is in moduli theory where it can be applied for instance to show that $\overline{\mathcal{M}}_g$ is proper and $\mathcal{B}\mathrm{un}_C^{\mathrm{ss}}$ is universally closed. As we generalize the criterion to algebraic stacks, we provide a quick recap for how it's established for schemes.

A.4.1 Preliminaries

The starting point is the following lifting criterion for quasi-compact morphisms to be closed.

Lemma A.4.1. *A quasi-compact morphism $f\colon X \to Y$ of schemes is closed if and only if for every point $x \in X$, every specialization $f(x) \rightsquigarrow y_0$ lifts to a specialization $x \rightsquigarrow x_0$:*

$$\begin{tikzcd} X \arrow[d, "f"] & x \arrow[r, dashed, squiggly] \arrow[d, mapsto] & x_0 \arrow[d, mapsto, dashed] \\ Y & f(x) \arrow[r, squiggly] & y_0. \end{tikzcd}$$

Proof. The implication ($\Rightarrow$) is clear as $f(\overline{\{x\}}) \subset Y$ is closed. For the converse, after replacing X with a closed subscheme it suffices to show that $f(X)$ is closed. We can assume that $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$ are affine (since f is quasi-compact) and reduced (since the question is topological). The scheme-theoretic image of $\operatorname{Spec} A \to \operatorname{Spec} B$ is defined by $I := \ker(B \to A)$. By replacing B with B/I, we can assume that $B \to A$ is injective. For every minimal prime $\mathfrak{p} \in \operatorname{Spec} B$, the localization $B_\mathfrak{p} \to A_\mathfrak{p}$ is injective and thus $A_\mathfrak{p} \neq 0$ and the fiber $f^{-1}(\mathfrak{p}) = \operatorname{Spec} A_\mathfrak{p}$ is non-empty. Since $f(X)$ contains all the minimal primes and is closed under specialization, $f(X) = Y$ is closed. $\square$

The noetherian valuative criterion depends on the following algebraic fact:

Proposition A.4.2. *Let $(A, \mathfrak{m}_A)$ be a noetherian local domain with fraction field K such that A is not a field. If $K \to L$ is a finitely generated field extension, then there exists a DVR R with fraction field L dominating A (i.e. $A \subset R$ and $\mathfrak{m}_A \cap K = \mathfrak{m}_R$ is the maximal ideal of R).*

Proof. We first reduce to the case that $K \to L$ is a finite field extension. To this end, choose a transcendence basis $x_1, \ldots, x_n \in L$ over K and replace A with $A[x_1, \ldots, x_n]_\mathfrak{n}$ where $\mathfrak{n} = \mathfrak{m}_A A[x_1, \ldots, x_n] + (x_1, \ldots, x_n)$.

Let $X = \operatorname{Spec} A$ with closed point $x = \mathfrak{m}_A$. Let $B = \operatorname{Bl}_x X$ be the blow-up of X at x with exceptional divisor E. If $\xi \in E$ is a generic point, then $\mathcal{O}_{B,\xi}$ is a noetherian domain of dimension 1 (by Krull's Hauptidealsatz) with fraction field K. We now let $R \subset L$ be the integral closure of $\mathcal{O}_{B,\xi}$ in L. By Krull–Akizuki (Proposition A.4.3), R is noetherian. Since R is also normal of dimension 1, it is a DVR. $\square$

Proposition A.4.3 (Krull–Akizuki)**.** *Let R be a noetherian domain of dimension 1 with fraction field K. If $K \to L$ is a finite extension of fields, then every ring A with $R \subset A \subset L$ is noetherian.*

Proof. See [Nag62, p. 115] or [SP, Tag 00PG]. $\square$

Krull–Akizuki has the following geometric implication:

Proposition A.4.4. *If $f\colon X \to Y$ is a finite type morphism of noetherian schemes, $x \in X$ and $f(x) \rightsquigarrow y_0$ is a specialization, then there exists a commutative diagram*

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & X \arrow[d, "f"] & x \arrow[d, mapsto] \\ \operatorname{Spec} R \arrow[r] & Y & f(x) \arrow[r, squiggly] & y_0. \end{tikzcd}$$

where R is a DVR with fraction field K, the image of $\operatorname{Spec} K \to X$ *is x and* $\operatorname{Spec} R \to Y$ *realizes the specialization $f(x) \rightsquigarrow y_0$. In particular, every specialization $x \rightsquigarrow x_0$ in a noetherian scheme is realized by a map* $\operatorname{Spec} R \to X$ *from a DVR.*

Proof. After replacing X with $\overline{\{f(x)\}}$ and Y with $\overline{\{x\}}$, we may assume that X and Y are integral with generic points x and $f(x)$. Then $\mathcal{O}_{Y,y_0}$ is a noetherian local domain with fraction field $\kappa(f(x))$. By applying Proposition A.4.2 to the field extension $\kappa(f(x)) \to \kappa(x)$, we obtain a DVR R with fraction field $\kappa(x)$ dominating $\mathcal{O}_{Y,y_0}$ which yields the desired diagram. $\square$

A.4.2 The Valuative Criteria

Theorem A.4.5 (Valuative Criteria for Proper/Separated/Universally Closed Morphisms)**.** *Let $f\colon X \to Y$ be a finite type morphism of noetherian schemes. Consider a commutative diagram*

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \operatorname{Spec} R \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd} \tag{A.4.1}$$

of solid arrows where R is a DVR with fraction field K. Then

(1) *f is proper if and only if for every diagram (A.4.1), there exists a unique lift.*

(2) *f is separated if and only if for every diagram (A.4.1), any two lifts are equal.*

(3) *f is universally closed if and only if for every diagram (A.4.1), there exists a lift.*

Proof. We first claim that it suffices to handle the universally closed case. Indeed, a morphism $X \to Y$ is separated if and only if the diagonal $X \to X \times_Y X$ is universally closed, and the equality of two lifts in the valuative criterion for $X \to Y$ corresponds to the existence of a lift in the valuative criterion for $X \to X \times_Y X$.

Suppose that $X \to Y$ satisfies the valuative criterion for universally closedness. To show that $X \to Y$ is universally closed, we claim that it suffices to check that for every *finite type* morphism $T \to Y$, the base change $X_T \to T$ is closed. Indeed, suppose that for an arbitrary morphism $T \to Y$ of schemes, the base change $f_T\colon X_T \to T$ is not closed. By Lemma A.4.1, there exists $x \in X_T$ and a specialization $\underline{f_T(x)} \rightsquigarrow t_0$ which doesn't lift to a specialization $x \rightsquigarrow x_0$. This implies that $Z = \overline{\{x\}} \subset X_T$ has trivial intersection with the fiber $(X_T)_{t_0}$. Applying Lemma A.4.6 (with its notation) yields, after replacing T with an open neighborhood of t_0, a commutative diagram

$$\begin{tikzcd} & x \arrow[d, mapsto] & X_T \arrow[r] \arrow[d, "f_T"] & X_{T'} \arrow[r] \arrow[d, "f_{T'}"] & X \arrow[d, "f"] \\ f_T(x) \arrow[r, squiggly] & t_0 & T \arrow[r, "g"] & T' \arrow[r] & Y \end{tikzcd}$$

where $T' \to Y$ is finite type and a closed subscheme $Z' \subset X_{T'}$ such that $f_{T'}(Z')$ contains $g(f_T(x))$ but not $g(t_0)$. This shows that $f_{T'}\colon X_{T'} \to T'$ is not closed.

Since the valuative criterion holds for $X \to Y$, it also holds for the morphism $X_T \to T$ of *noetherian* schemes. It therefore suffices to show that $X \to Y$ is closed. By Lemma A.4.1, it suffices to show that given $x \in X$, every specialization $f(x) \rightsquigarrow y_0$ lifts to a specialization $x \rightsquigarrow x_0$. By Proposition A.4.4, there exists a diagram (A.4.1) such that $\operatorname{Spec} R \to Y$ realizes $f(x) \rightsquigarrow y_0$ with a lift $\operatorname{Spec} K \to X$ whose image is x. The valuative criterion implies the existence of a lift $\operatorname{Spec} R \to X$ which in turn yields a specialization $x \rightsquigarrow x_0$ lifting $f(x) \rightsquigarrow y_0$.

Conversely, assume that $f\colon X \to Y$ is universally closed and that we are given a diagram (A.4.1). By replacing Y with $\operatorname{Spec} R$ and X with $X \times_Y \operatorname{Spec} R$, we may assume that $Y = \operatorname{Spec} R$ and that we have a diagram

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r, "x"] \arrow[d] & X \arrow[dl] \\ \operatorname{Spec} R & \end{tikzcd}$$

By replacing X with $\overline{\{x\}}$, we may assume that X is integral with generic point x. Since $X \to \operatorname{Spec} R$ is closed, there exists a specialization $x \rightsquigarrow x_0$ mapping to the specialization of the generic point to the closed point in $\operatorname{Spec} R$. This gives an inclusion of local rings $R \hookrightarrow \mathcal{O}_{X,x_0}$ in K. Since R is a valuation ring with fraction field K (i.e. is maximal among local rings properly contained in K), we see that $R = \mathcal{O}_{X,x_0}$ and the inclusion $\operatorname{Spec} \mathcal{O}_{X,x_0} \to X$ gives the desired lift. $\square$

Lemma A.4.6. *Let $f\colon X \to Y$ be a quasi-compact morphism of schemes. Let $T \to Y$ be a morphism of schemes, $t_0 \in T$ be a point and $Z \subset X_T$ a closed subscheme with trivial intersection with the fiber $(X_T)_{t_0}$. Then after replacing T with an open neighborhood of t_0, there exist a finite type morphism $T' \to Y$ of schemes with a factorization $T \xrightarrow{g} T' \to Y$ and a closed subscheme $Z' \subset X_{T'}$ with trivial intersection with the fiber $(X_{T'})_{g(t_0)}$ such that $\operatorname{im}(Z \hookrightarrow X_T \to X_{T'}) \subset Z'$.*

Proof. See [SP, Tag 05BD]. $\square$

A.4.3 Universally submersive morphisms

A morphism $f\colon X \to Y$ of schemes is *submersive* if f is surjective and Y has the quotient topology, i.e. a subset $U \subset Y$ is open if and only if $f^{-1}(U)$ is open, and $f\colon X \to Y$ is *universally submersive* if for every map $Y' \to Y$, the base change $X \times_Y Y' \to Y'$ is submersive.

Exercise A.4.7. (1) Show that a morphism $f\colon X \to Y$ of noetherian schemes is universally submersive if and only if for every map $\operatorname{Spec} R \to Y$ from a DVR, there is a commutative diagram

$$\begin{tikzcd} \operatorname{Spec} R' \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \operatorname{Spec} R \arrow[r] & Y \end{tikzcd}$$

where $R \to R'$ is local homomorphism of DVRs.

(2) Show that every fppf morphism or universally closed morphism of noetherian schemes is universally submersive.

A.5 Quasi-finite morphisms and Zariski's Main Theorem

We say that a locally of finite type morphism $x\colon X \to Y$ of schemes is *quasi-finite at* $x \in X$ if x is isolated in the fiber $X_{f(x)} = X \times_Y \operatorname{Spec} \kappa(k(y))$. When $f\colon X \to Y$ is quasi-compact, then this is equivalent to $f^{-1}(f(x))$ being a finite set. We say that $f\colon X \to Y$ is *locally quasi-finite* if f is is locally of finite type and quasi-finite at every point, and *quasi-finite* if f is of finite type and quasi-finite.

Theorem A.5.1 (Étale Localization of Quasi-finite Morphisms)**.** *Let $f\colon X \to S$ be a separated and finite type morphisms of schemes which is quasi-finite at $x \in X$. There exists an étale neighborhood $(S', s') \to (S, f(x))$ with $\kappa(s') = \kappa(f(x))$ and a decomposition $X \times_S S' = F \sqcup W$ into open and closed subschemes such that $F \to S'$ is finite and the fiber $W_{s'}$ is empty.*

Proof. See [EGA, IV.8.12.3] or [SP, Tag 02LP]. $\square$

Proposition A.5.2. *A separated and quasi-finite morphism $f\colon X \to Y$ of schemes factors as*

$$f\colon X \to \mathcal{S}\mathrm{pec}_Y\, f_*\mathcal{O}_X \to Y$$

where $X \hookrightarrow \mathcal{S}\mathrm{pec}_Y\, f_*\mathcal{O}_X$ is an open immersion and $\mathcal{S}\mathrm{pec}_Y\, f_*\mathcal{O}_X \to Y$ is affine.

Proof. As $f_*\mathcal{O}_X$ commutes with étale (even flat) base change on Y, so does the above factorization of $f\colon X \to Y$. Therefore, it suffices to show that every point $y \in Y$ has an étale neighborhood where the proposition holds. By Theorem A.5.1 we may assume that $X = X_1 \sqcup X_2$ with X_1 finite over Y and $(X_2)_y = \emptyset$. After replacing Y with $\mathcal{S}\mathrm{pec}_Y\, f_*\mathcal{O}_X$, we may also assume that $f_*\mathcal{O}_X = \mathcal{O}_Y$. As $\mathcal{O}_X = \mathcal{A}_1 \times \mathcal{A}_2$ is the product of quasi-coherent $\mathcal{O}_X$-algebras, $\mathcal{O}_Y = f_*\mathcal{O}_X = f_*\mathcal{A}_1 \times f_*\mathcal{A}_2$ and thus Y decomposes as $Y_1 \sqcup Y_2$ such that $y \in Y_1$ and $f(X_i) \subset Y_i$ for $i = 1, 2$. After replacing Y with Y_1, we see that $X \to Y$ is finite. Thus X is affine and $X = Y = \mathcal{S}\mathrm{pec}_Y\, f_*\mathcal{O}_X$. $\square$

In the above factorization $f_*\mathcal{O}_Y$ may not be a finite type $\mathcal{O}_Y$-algebra (even if Y is a noetherian affine scheme, then $\Gamma(X, \mathcal{O}_X)$ may not be a noetherian ring; see [Ols16, Ex. 7.2.15]). However, we may modify the factorization to arrange that $X \to Y$ factors as an open immersion followed by a *finite* morphism.

Theorem A.5.3 (Zariski's Main Theorem)**.** *A separated and quasi-finite morphism $f\colon X \to Y$ of schemes factors as the composition of a dense open immersion $X \hookrightarrow \widetilde{Y}$ and a finite morphism $\widetilde{X} \to X$. In particular, f is quasi-affine.*

Proof. If $\mathcal{A} \subset f_*\mathcal{O}_X$ denotes the integral closure of $\mathcal{O}_Y \to f_*\mathcal{O}_{\mathcal{X}}$, there is a factorization $X \xrightarrow{j} \mathcal{S}\mathrm{pec}_Y\, \mathcal{A} \to Y$. We claim that $j\colon X \to \mathcal{S}\mathrm{pec}_Y\, \mathcal{A}$ is an open immersion. To show this claim, it suffices to show that for every point $x \in X$, there is an open neighborhood $V \subset \mathcal{S}\mathrm{pec}_Y\, \mathcal{A}$ of $j(x)$ such that $j^{-1}(V) \to V$ is an isomorphism. Since normalization commutes with étale base change (Proposition A.5.4) and being an open immersion at a point is an étale local property, we are free to replace Y by an étale neighborhood of $f(x)$. By Theorem A.5.1, we can assume that $X = F \sqcup W$ with F finite over Y and $x \in F$. In this case, the normalization $\mathcal{S}\mathrm{pec}_Y\, \mathcal{A}$ of Y in X is $F \sqcup \widetilde{W}$ where $\widetilde{W}$ is the normalization of Y in W, and the claim follows.

By construction $\mathcal{S}\mathrm{pec}_Y\, \mathcal{A} \to Y$ is integral. We can write $\mathcal{A} = \operatorname{colim} \mathcal{A}_\lambda$ as the colimit of finite type $\mathcal{O}_Y$-algebras and since open immersions descent under limits (Proposition A.6.7), we see that $X \to \mathcal{S}\mathrm{pec}_Y\, \mathcal{A}_\lambda$ is an open immersion for $\lambda \gg 0$. Since $\mathcal{S}\mathrm{pec}_Y\, \mathcal{A}_\lambda \to Y$ is integral and finite type, it is finite.

See also [EGA, IV.8.12.6] or [SP, Tag 05K0]. □

The following algebra result was used above and will be useful to generalize Zariski's Main Theorem to algebraic spaces (Theorem 4.4.9) and stacks (Theorem 4.4.9).

Proposition A.5.4. *Let Y be a scheme, $\mathcal{B}$ be a quasi-coherent $\mathcal{O}_Y$-algebra and $\widetilde{\mathcal{B}}$ be the integral closure of $\mathcal{O}_Y$ in $\mathcal{B}$. If $f\colon X \to Y$ is a smooth morphism, then $f^*\widetilde{\mathcal{B}}$ is identified with the integral closure of $\mathcal{O}_X$ in $f^*\mathcal{B}$.*

Proof. See [SP, Tag 03GG] or [LMB, Prop. 16.2]. □

Zariski's Main Theorem has some useful corollaries.

Corollary A.5.5. *A quasi-finite and proper morphism (resp. proper monomorphism) of schemes is finite (resp. a closed immersion).* □

Proof. If $f\colon X \to Y$ is a quasi-finite and proper, Zariski's Main Theorem (A.5.3) gives a factorization $f\colon X \hookrightarrow \widetilde{X} \to Y$ and the dense open immersion $X \hookrightarrow \widetilde{X}$ is also closed, thus an isomorphism. On the other hand, if $f\colon X \to Y$ is a proper monomorphism, then it is also quasi-finite, thus finite. The statement reduces to the algebra fact that a finite epimorphism of rings is surjective (c.f. [SP, Tag 04VT]). □

Remark A.5.6. Every universally closed morphism is necessarily quasi-compact [SP, Tag 04XU]. It follows that every morphism which is universally closed, locally of finite type, and a monomorphism is a closed immersion; see also [SP, Tag 04XV].

A.6 Limits of schemes

In moduli theory, we often need to deal with non-noetherian rings for the simple reason that moduli functors and stacks are defined over the category Sch of all schemes. Working instead with the category of locally noetherian schemes has the limitation that it is not closed under fiber products while working instead with the category of schemes finite type over a field or $\mathbb{Z}$ doesn't contain local rings of schemes or their completions.

In any case, using the limit methods presented in this section, it is usually straightforward to reduce properties of schemes and their morphisms to the noetherian case.

A.6.1 Limits of schemes

The first result states that a limit exists for an inverse system $(S_\lambda, f_{\lambda\mu})_{\lambda\in\Lambda}$ of schemes over a directed set Λ (see Definition A.1.2) where the transition map $f_{\lambda\mu}\colon S_\lambda \to S_\mu$ for every $\lambda \geq \mu$ is affine.

Proposition A.6.1 (Existence of Limits)**.** *If $(S_\lambda, f_{\lambda\mu})_{\lambda\in\Lambda}$ is an inverse system of schemes with affine transition maps, then the limit $S = \lim_\lambda S_\lambda$ exists in the category of schemes such that each morphism $f_\lambda \colon S \to S_\lambda$ is affine.*

Proof. If each $S_\lambda = \operatorname{Spec} A_\lambda$ is affine, one takes $S = \operatorname{Spec}(\operatorname{colim}_\lambda A_\lambda)$. In general, choose an element $0 \in I$ and set $S = \mathcal{S}\mathrm{pec}_{S_0}(\operatorname{colim}_{\lambda\geq 0} f_{\lambda 0,*}\mathcal{O}_{S_\lambda})$. Details can be found in [EGA, IV.8.2] and [SP, Tag 01YX]. □

A morphism $f\colon X \to Y$ of schemes is locally of finite presentation if and only if for every inverse system $(S_\lambda, f_{\lambda\mu})$ of affine schemes over Y, the map $\operatorname{colim}_\lambda \operatorname{Mor}_Y(S_\lambda, X) \to \operatorname{Mor}_Y(\lim_\lambda S_\lambda, X)$ is bijective (Proposition A.1.3) The same holds for inverse systems of quasi-compact and quasi-separated schemes over S with affine transition maps; see [EGA, IV.8.14.1] and [SP, Tag 01ZC].

A.6.2 Noetherian approximation

Every affine scheme $\operatorname{Spec} A$ is the limit of affine schemes $\operatorname{Spec} A_\lambda$ of finite type over $\mathbb{Z}$. This follows from the fact that the ring A is the union of its finitely generated $\mathbb{Z}$-subalgebras. More generally, we have:

Proposition A.6.2 (Relative Noetherian Approximation)**.** *Let $X \to S$ be a morphism of schemes with X quasi-compact and quasi-separated and with S quasi-separated. Then $X = \lim_\lambda X_\lambda$ is a limit of an inverse system $(X_\lambda, f_{\lambda\mu})$ of schemes of finite presentation over S with affine transition maps over S.*

Proof. See [SP, Tag 09MV]. □

When $S = \operatorname{Spec}\mathbb{Z}$, this is often referred to as *Absolute Noetherian Approximation* and was first established in [TT90, Thm. C.9].

A.6.3 Descending properties under limits

Proposition A.6.3 (Descending Properties of Schemes under Limits)**.** *Let $S = \lim_\lambda S_\lambda$ be a limit of an inverse system of quasi-compact and quasi-separated schemes with affine transition maps. If S is affine (resp. quasi-affine, separated), then so is S_λ for $\lambda \gg 0$.*

Proof. See [SP, Tags 01Z6, 086Q and 01Z5] and [TT90, Props C.6-7]. □

Proposition A.6.4 (Descending Morphisms under Limits)**.** *Let $S = \lim_\lambda S_{\lambda\in\Lambda}$ be a limit of an inverse system of quasi-compact and quasi-separated schemes with affine transition maps.*

(1) For a finitely presented morphism $X \to S$ of schemes, there exists an index $0 \in \Lambda$ and a finitely presented morphism $X_0 \to S_0$ of schemes such that $X \cong X_0 \times_{S_0} S$. Moreover, if we define $X_\lambda := X_0 \times_{S_0} S_\lambda$ for $\lambda > 0$, then $X = \lim_{\lambda\geq 0} X_\lambda$ is the limit of the inverse system $(X_\lambda, f_{\lambda\mu})$ where the (affine) transition map $f_{\lambda\mu}\colon X_\lambda \to X_\mu$ is the base change of $S_\lambda \to S_\mu$ for $\lambda \geq \mu$.

(2) Let X_0 and Y_0 be finitely presented schemes over S_0 for some index $0 \in \Lambda$. For $\lambda > 0$, set $X_\lambda = X_0 \times_{S_0} S_\lambda$ and $Y_\lambda = Y_0 \times_{S_0} S_\lambda$, and let $X = \lim_\lambda X_\lambda$ and $Y = \lim_\lambda Y_\lambda$ be the limits (Proposition A.6.1). Then the natural map

$$\operatorname{colim}_{\lambda\geq 0} \operatorname{Mor}_{S_\lambda}(X_\lambda, Y_\lambda) \to \operatorname{Mor}_S(X, Y)$$

is bijective.

Remark A.6.5. In other words, the category of schemes finitely presented over S is the colimit of the categories of schemes finitely presented over S_λ.

Proof. See [EGA, IV.8.8] and [SP, Tag 01ZM]. □

Definition A.6.6. We say that a property $\mathcal{P}$ of morphisms of schemes *descends under limits* if the following holds for every limit $S = \lim_{\lambda \in \Lambda} S_\lambda$ of an inverse system of quasi-compact and quasi-separated schemes with affine transition maps: for every index $0 \in \Lambda$, and for every morphism $g_0 \colon X_0 \to Y_0$ of quasi-compact and quasi-separated schemes with base changes $g_\lambda \colon X_\lambda \to Y_\lambda$ over S_λ and $g \colon X \to Y$ over S, we require that if g has $\mathcal{P}$, then g_λ has $\mathcal{P}$ for $\lambda \gg 0$.

Proposition A.6.7 (Descending Properties of Morphisms under Limits)**.** *The following properties of morphisms of schemes descends under limits: isomorphism, closed immersion, open immersion, affine, quasi-affine, finite, quasi-finite, proper, projective, quasi-projective, separated, monomorphism, surjective, flat, locally of finite presentation, unramified, étale, smooth, syntomic, and for any integer d the property that every fiber is connected and has pure dimension d.*

Proof. See [EGA, IV 8.10.5] and [SP, Tags 081C and 05M5]. □

Suppose $S = \operatorname{colim} S_\lambda$ is the colimit of an inverse system $(S_\lambda, f_{\lambda\mu})$ of quasi-compact and quasi-separated schemes with affine transition maps. If F_0 is a quasi-coherent sheaf on S_0 for an index $0 \in \Lambda$, and $F = f_\lambda^* F$ and $F_\lambda = f_{\lambda 0}^* F_0$ are the pullbacks to S and S_λ for $\lambda \geq 0$, then $\Gamma(S, F) = \operatorname{colim}_{\lambda \geq 0} \Gamma(S_\lambda, F_\lambda)$ [SP, Tag 01Z0]. Moreover, a quasi-coherent sheaf on S and its properties often descend to some S_λ.

Proposition A.6.8 (Descending Sheaves under Limits)**.** *Let $(S_\lambda, f_{\lambda\mu})$ be an inverse system of quasi-compact and quasi-separated schemes with affine transition maps and limit $S = \lim_{\lambda \in \Lambda} S_\lambda$. Denote the projection maps by $f_\lambda \colon S \to S_\lambda$.*

(1) If F is an $\mathcal{O}_S$-module of finite presentation (resp. vector bundle, line bundle), then there exists an index $i \in \Lambda$ and an $\mathcal{O}_{S_\lambda}$ module F_λ of finite presentation (resp. vector bundle, line bundle) such that $F \cong f_\lambda^* F_\lambda$.

(2) For an index $0 \in \Lambda$, let F_0 and G_0 be $\mathcal{O}_{S_0}$-modules of finite presentation. The natural map

$$\operatorname{colim}_{\lambda \geq 0} \operatorname{Hom}_{\mathcal{O}_{S_\lambda}}(f_{\lambda 0}^* F_0, f_{\lambda 0}^* G_0) \to \operatorname{Hom}_{\mathcal{O}_S}(f_0^* F_0, f_0^* G_0)$$

is bijective.

(3) For an index $0 \in \Lambda$, let $f_0 \colon X_0 \to Y_0$ be a finitely presented morphism of schemes over S_0 and let F_0 be a quasi-coherent sheaf on X_0 of finite presentation. If the pullback of F_0 under $X_0 \times_{S_0} S \to X_0$ is flat over $Y_0 \times_{S_0} S$, then the pullback of F_0 under $X_0 \times_{S_0} S_\lambda \to X_0$ is flat over $Y_0 \times_{S_0} S_\lambda$ for $\lambda \gg 0$.

Remark A.6.9. In other words, the category of finitely presented modules over S is the colimit of the categories of finitely presented modules over S_λ.

Proof. See [EGA, IV.8.5.2] and [SP, Tags 01ZR, 0B8W and 05LY]. □

A.6.4 Application

For a typical application of noetherian approximation in moduli theory, we illustrate here how properties of an arbitrary family of curves can be reduced to a family over a noetherian base.

Proposition A.6.10. *Let S be a quasi-compact and quasi-separated scheme (e.g. an affine scheme), and let $\mathcal{C} \to S$ be a flat, proper and finitely presented morphism of schemes such that every geometric fiber has dimension at most 1. Then there exist a cartesian diagram*

$$\begin{tikzcd} \mathcal{C} \arrow[r] \arrow[d] & \mathcal{C}' \arrow[d] \\ S \arrow[r] & S' \end{tikzcd}$$

where S' is a scheme of finite type over $\mathbb{Z}$ and $\mathcal{C}' \to S'$ is a flat, proper morphism of schemes such that every geometric fiber has dimension at most 1. Moreover, if $\mathcal{C} \to S$ is smooth, then $\mathcal{C}' \to S'$ can also be arranged to be smooth.

Remark A.6.11. The upshot is that we can now establish properties of the morphism $\mathcal{C}_\lambda \to S_\lambda$ of *noetherian* schemes and then deduce properties of $\mathcal{C} \to S$ by base change. In Lemma 5.2.17 we show if $\mathcal{C} \to S$ is a nodal family of curves, then $\mathcal{C}' \to S'$ can be arranged to be nodal.

Proof. Write $S = \lim_{\lambda \in \Lambda} S_\lambda$ as a limit of an inverse system of schemes of finite type over $\mathbb{Z}$ (Proposition A.6.1). Note that each S_λ is quasi-compact and quasi-separated. Since $\mathcal{C} \to S$ is finitely presented, there exists an index $0 \in \Lambda$ and a finitely presented morphism $\mathcal{C}_0 \to S_0$ such that $\mathcal{C} \cong \mathcal{C}_0 \times_{S_0} S$ (Proposition A.6.4). For each $\lambda > 0$, we can define $\mathcal{C}_\lambda = \mathcal{C}_0 \times_{S_0} S_\lambda$ and we have a cartesian diagram

$$\begin{tikzcd} \mathcal{C} \arrow[r] \arrow[d] & \mathcal{C}_\lambda \arrow[d] \\ S \arrow[r] & S_\lambda. \end{tikzcd}$$

Since $\mathcal{C} \to S$ is flat and proper with fiber of dimension at most 1 (resp. smooth), then there exists $\lambda_0 \in I$ such that the same is true for $\mathcal{C}_\lambda \to S_\lambda$ for all $\lambda \geq \lambda_0$ (Proposition A.6.7). We now take $S' = S_\lambda$ and $\mathcal{C}' = \mathcal{C}_\lambda$ for every $\lambda \geq \lambda_0$. $\square$

A.7 Cohomology and base change

Given a proper morphism $f \colon X \to Y$ of noetherian schemes and a coherent sheaf F on X, we would like to know:

(a) When is $\mathrm{R}^i f_* F$ a vector bundle on Y?

(b) For a morphism of schemes $Y' \to Y$ inducing a cartesian diagram

$$\begin{tikzcd} X_{Y'} \arrow[r, "g'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y, \end{tikzcd}$$

when is the comparison map

$$\phi^i_{Y'}\colon g^*\mathrm{R}^i f_*F \to \mathrm{R}^i f'_*g'^*F \tag{A.7.1}$$

an isomorphism?

When $f\colon X \to Y$ is flat, Flat Base Change tells us that (A.7.1) is always an isomorphism. Cohomology and Base Change provides an answer when F is flat over Y.

Cohomology and Base Change is an essential tool in moduli theory. It can be applied to verify properties of families of objects and construct vector bundles on moduli spaces. For instance, for a family $\pi\colon \mathcal{C} \to S$ of smooth curves, we can verify that $\pi_*\Omega^{\otimes k}_{\mathcal{C}/S}$ is a vector bundle for $k > 0$ whose construction commutes with base change on S and that $\mathcal{C}$ embeds canonically into $\mathbb{P}_S(\pi_*\Omega^{\otimes k}_{\mathcal{C}/S})$ for $k \geq 3$ (Proposition 5.1.9). These properties are used for instance to verify the algebraicity of $\mathscr{M}_g$ (Theorem 3.1.15). On the other hand, applying this result to the universal family $\pi\colon \mathcal{U}_g \to \mathscr{M}_g$ yields vector bundles $\pi_*\Omega^{\otimes k}_{\mathcal{U}_g/\mathscr{M}_g}$ on $\mathscr{M}_g$; when $k = 1$, this is a vector bundle of rank g called the *Hodge bundle*.

A.7.1 Algebraic input

The key algebraic input to Cohomology and Base Change is the following:

Theorem A.7.1. *Let $X \to \operatorname{Spec} A$ be a proper morphism of noetherian schemes and F be a coherent sheaf on X which is flat over A. There is a complex*

$$K^\bullet\colon 0 \to K^0 \to K^1 \to \cdots \to K^n \to 0$$

of finitely generated, projective A-modules such that $\mathrm{H}^i(X, F) = \mathrm{H}^i(K^\bullet)$ for all i.

Moreover, for every A-module M, $\mathrm{H}^i(X, F \otimes_A M) = H^i(K^\bullet \otimes_A M)$. In particular, for a morphism $\operatorname{Spec} B \to \operatorname{Spec} A$ of schemes, $\mathrm{H}^i(X_B, F_B) = \mathrm{H}^i(K^\bullet \otimes_A B)$ where $X_B := X \times_{\operatorname{Spec} A} \operatorname{Spec} B$ and F_B is the pullback of F to X_B.

Proof. This is established by choosing a finite affine cover $\{U_i\}$ of X and considering the corresponding alternating Čech complex $C^\bullet$ on $\{U_i\}$ with coefficients in F. Then $C^\bullet$ is a finite complex of free (but not finitely generated) A-modules and $\mathrm{H}^i(X, F) = \mathrm{H}^i(C^\bullet)$. The result is then obtained by inductively refining $C^\bullet$ to build a finite complex $K^\bullet$ of finitely generated, projective A-modules which is quasi-isomorphic to $C^\bullet$.

See [Mum70, p.46] where the last statement is established for A-algebras B but the argument goes through for every A-module M. See also [SP, Tag 07VK] or [Vak17, 28.2.1]. □

Remark A.7.2 (Perfect complexes)**.** A bounded complex $K^\bullet$ of coherent sheaves on a noetherian scheme X is *perfect* if there is an affine cover $X = \bigcup_i U_i$ such that each $K^\bullet|_{U_i}$ is quasi-isomorphic to a bounded complex of vector bundles on U_i. (By a vector bundle, we mean a locally free sheaf of finite rank—this is equivalent to the corresponding module on $\Gamma(U_i, \mathcal{O}_{U_i})$ to be finitely generated and projective.) If X is affine (or more generally has the resolution property, i.e. every coherent sheaf is the quotient of a vector bundle), then $K^\bullet$ is perfect if and only if it is quasi-isomorphic to a bounded complex of vector bundles on X [SP, Tags 066Y and 0F8F]). Moreover, the compact objects in $D_{\mathrm{QCoh}}(X)$ are precisely the perfect complexes [SP, Tag 09M8].

With this terminology in place, Theorem A.7.1 has the following translation: $\mathrm{R}f_*F \in D^b_{\mathrm{Coh}}(\operatorname{Spec} A)$ is perfect [SP, Tag 07VK]. More generally, if $F^\bullet$ is a perfect complex on X, then $\mathrm{R}f_*F^\bullet$ is also perfect [SP, Tag 0A1H].

A.7.2 Theorems of Semicontinuity, Grauert and Cohomology and Base Change

Theorem A.7.1 tells us for a proper morphism $X \to \operatorname{Spec} A$ and coherent sheaf F on X flat over A, the cohomology $\mathrm{H}^i(X, F)$ can be computed using a perfect complex $K^\bullet$. Since Zariski-locally on the base, the complex $K^\bullet$ is a finite complex of *free* objects, this reduces cohomological questions to linear algebra.

Theorem A.7.3 (Semicontinuity Theorem)**.** *Let $f: X \to Y$ be a proper morphism of noetherian schemes and F be a coherent sheaf on X which is flat over Y.*

(1) For each $i \geq 0$, the function

$$Y \to \mathbb{Z}, \quad y \mapsto \mathrm{h}^i(X_y, F_y)$$

is upper semicontinuous.

(2) The function

$$Y \to \mathbb{Z}, \quad y \mapsto \chi(X_y, F_y) = \sum_{i=0}^{\infty} (-1)^i \mathrm{h}^i(X_y, F_y)$$

is locally constant.

Proof. This is a direct consequence of Theorem A.7.1; see also [Mum70, p. 47], [Har77, Thm. 12.8] or [Vak17, 28.2.4]. □

When the base scheme is reduced, Grauert's Theorem provides a criterion for when the higher pushforward sheaves $\mathrm{R}^i f_* F$ are vector bundles.

Theorem A.7.4 (Grauert's Theorem)**.** *Let $f: X \to Y$ be a proper morphism of noetherian schemes and let F be a coherent sheaf on X which is flat over Y. Assume that Y is reduced and connected. For each integer i, the following are equivalent:*

(1) the function $y \mapsto \mathrm{h}^i(X_y, F_y)$ is constant; and

(2) $\mathrm{R}^i f_* F$ is a vector bundle and the comparison map

$$\phi^i_y : \mathrm{R}^i f_* F \otimes \kappa(y) \to \mathrm{H}^i(X_y, F_y)$$

is an isomorphism for all $y \in Y$.

If these conditions hold, then we have the following additional properties:

(a) for all maps $g: Y' \to Y$ of schemes, the comparison map $\phi^i_{Y'} : g^* \mathrm{R}^p f_* F \to \mathrm{R}^i f'_* g'^* F$ is an isomorphism; and

(b) The comparison map $\phi^{i-1}_y : \mathrm{R}^{i-1} f_* F \otimes \kappa(y) \to \mathrm{H}^{i-1}(X_y, F_y)$ is an isomorphism.

Proof. See [Mum70, p.51-2], [Har77, Cor. 12.9] and [Vak17, 28.1.5]. □

Grauert's Theorem is proved by using that $\mathrm{R}f_*F$ is a perfect complex and a linear algebra argument to show that $\mathrm{R}^i f_*F \otimes \kappa(y)$ has constant dimension. Since Y is reduced, this implies that $\mathrm{R}^i f_*F$ is a vector bundle. When Y is not reduced, the local criterion for flatness can be leveraged to provide the following useful criteria.

Theorem A.7.5 (Cohomology and Base Change)**.** *Let $f: X \to Y$ be a proper and finitely presented morphism of schemes, and let F be a finitely presented sheaf on X which is flat over Y. Suppose that for a point $y \in Y$ and integer i, the comparison map $\phi_y^i: \mathrm{R}^i f_*F \otimes \kappa(y) \to \mathrm{H}^i(X_y, F_y)$ is surjective. Then the following hold*

(a) There is an open neighborhood $V \subset Y$ of y such that for every morphism $Y' \to V$ of schemes, the comparison map $\phi_{Y'}^i: g^*\mathrm{R}^p f_*F \to \mathrm{R}^i f'_* g'^* F$ is an isomorphism. In particular, ϕ_y^i is an isomorphism.

(b) ϕ_y^{i-1} is surjective if and only if $\mathrm{R}^i f_*F$ is a vector bundle in an open neighborhood of y.

Proof. See [EGA, III.7.7.5, III.7.7.10, III.7.8.4], [Har77, Thm. 12.11] and [Vak17, 28.1.6]. □

Remark A.7.6. For moduli-theoretic applications, it is sometimes convenient to apply Cohomology and Base Change in the non-noetherian setting. Using the methods of Noetherian Approximation from §A.6, it is not hard to see how the general statement follows from the noetherian version. Since the statement is local on Y, we can assume Y is affine and we can write $Y = \lim_{\lambda \in \Lambda} Y_\lambda$ as a limit of affine schemes of finite type over $\mathbb{Z}$. Since $X \to Y$ is finitely presented, there exists an index $0 \in \Lambda$ and a finitely presented morphism $X_0 \to Y_0$ such that $X \cong X_0 \times_{Y_0} Y$ (Proposition A.6.4). For each $\lambda > 0$, we can define $X_\lambda = X_0 \times_{Y_0} Y_\lambda$ and we have $X \cong X_\lambda \times_{Y_\lambda} Y$. By Proposition A.6.7, $X_\lambda \to Y_\lambda$ is proper for $\lambda \gg 0$. By Proposition A.6.8(1), there exist an index $\mu \in \Lambda$ and a coherent sheaf F_μ on X_μ that pulls back to F under $X \to X_\mu$. For $\lambda > \mu$, set F_λ to be the pullback of F_μ under $X_\lambda \to X_\mu$. By Proposition A.6.8(3), F_λ is flat over Y_λ for $\lambda \gg 0$. We may now apply noetherian Cohomology and Base Change to the data of $X_\lambda \to Y_\lambda$ and F_λ for $\lambda \gg 0$, and we may deduce the same properties for $X \to Y$ and F under the base change $Y \to Y_\lambda$.

Corollary A.7.7. *Let $f: X \to Y$ be a proper morphism of noetherian schemes and let F be a coherent sheaf on X which is flat over Y. The following are equivalent:*

(1) $\mathrm{H}^i(X_y, F_y) = 0$ for all $y \in Y$ and $i > 0$; and

(2) $\mathrm{R}^i f_*F = 0$ for all $i > 0$, and f_*F is a vector bundle whose construction commutes with base change on Y (i.e. for all morphisms $g: Y' \to Y$ of schemes, the comparison map $\phi_{Y'}^0: g^* f_*F \to f'_* g'^* F$ is an isomorphism).

Proof. The implication (2) ⇒ (1) follows from choosing $N > \dim X_y$ so that $\mathrm{H}^N(X_y, F_y) = 0$ and ϕ_y^N is surjective, and then inductively applying Cohomology and Base Change (A.7.5(b)) to conclude ϕ_y^i is surjective for all $i \leq N$.

For the converse, since $\phi_y^i: \mathrm{R}^i f_*F \otimes \kappa(y) \to \mathrm{H}^i(X_y, F_y) = 0$ is surjective for all $y \in Y$ and $i > 0$, A.7.5(a) implies that each ϕ_y^i is an isomorphism and therefore $\mathrm{R}^i f_*F = 0$ for $i > 0$. We now apply Cohomology and Base Change three more times: A.7.5(b) with $i = 1$ implies that ϕ_y^0 is surjective for all $y \in Y$, A.7.5(b) with $i = 0$ (as ϕ_y^{-1} is necessarily surjective) implies that f_*F is a vector bundle, and A.7.5(a) with $i = 0$ implies that the construction of f_*F commutes with base change on Y. $\square$

A.7.3 Applications to moduli theory

Here is a typical application of Cohomology and Base Change to moduli theory. The following proposition is used to establish properties of smooth families of curves (Proposition 5.1.9) and its argument applies in the same way to families of stable curves (Proposition 5.3.9).

Proposition A.7.8. *Let $\pi\colon \mathcal{C} \to S$ be a family of smooth curves of genus $g \geq 2$ (i.e. $\mathcal{C} \to S$ is a smooth, proper morphism of schemes such that every geometric fiber is a connected curve of genus g). Then*

(1) $\pi_*\mathcal{O}_\mathcal{C} = \mathcal{O}_S$;

(2) For $k > 1$, the pushforward $\pi_*(\Omega_{\mathcal{C}/S}^{\otimes k})$ is a vector bundle of rank $(2k-1)(g-1)$ whose construction commutes with base change on S and $\mathrm{R}^i\pi_*(\Omega_{\mathcal{C}/S}^{\otimes k}) = 0$ for $i > 0$.

(3) The pushforward $\pi_*(\Omega_{\mathcal{C}/S})$ is a vector bundle of rank g whose construction commutes with base change on S and $\mathrm{R}^1\pi_*(\Omega_{\mathcal{C}/S}) \cong \mathcal{O}_S$ while $\mathrm{R}^i\pi_*(\Omega_{\mathcal{C}/S}) = 0$ for $i \geq 2$.

Proof. To see (1), observe that $\mathrm{H}^0(\mathcal{C}_s, \mathcal{O}_{\mathcal{C}_s}) = \kappa(s)$ for all $s \in S$ since $\mathcal{C}_s$ is proper and geometrically connected. It follows that $\phi_s^0\colon \pi_*\mathcal{O}_\mathcal{C} \otimes \kappa(s) \to \mathrm{H}^0(\mathcal{C}_s, \mathcal{O}_{\mathcal{C}_s})$ is surjective. Cohomology and Base Change (A.7.5(a)–(b) with $i = 0$) implies that ϕ_s^0 is an isomorphism and that $\pi_*\mathcal{O}_\mathcal{C}$ is a line bundle. On a fiber over $s \in S$, the natural map $\mathcal{O}_S \to \pi_*\mathcal{O}_\mathcal{C}$ induces a surjective map $\kappa(s) \to \pi_*\mathcal{O}_\mathcal{C} \otimes \kappa(s)$ (as post-composing with $\phi_s^0\colon \pi_*\mathcal{O}_\mathcal{C} \otimes \kappa(s) \to \mathrm{H}^0(\mathcal{C}_s, \mathcal{O}_{\mathcal{C}_s}) = \kappa(s)$ is the identity). Thus $\mathcal{O}_S \to \pi_*\mathcal{O}_\mathcal{C}$ is a surjective morphism of line bundles, hence an isomorphism.

For (2) with $k > 1$, $\mathrm{H}^1(\mathcal{C}_s, \Omega_{\mathcal{C}_s/\kappa(s)}^{\otimes k}) = \mathrm{H}^0(\mathcal{C}_s, \Omega_{\mathcal{C}_s/\kappa(s)}^{\otimes(1-k)})$ for all $s \in S$ by Serre Duality (5.1.2) and this vanishes as $\deg(\Omega_{\mathcal{C}_s/\kappa(s)}^{\otimes(1-k)}) < 0$. Note that we also have the vanishing of $\mathrm{H}^i(\mathcal{C}_s, \Omega_{\mathcal{C}_s/\kappa(s)}^{\otimes k})$ for $i \geq 2$ since $\dim \mathcal{C}_s = 1$. Cohomology and Base Change (A.7.5(a)) gives the vanishing of the higher pushforward $\mathrm{R}^i\pi_*(\Omega_{\mathcal{C}/S}^{\otimes k}) = 0$ for $i > 0$. On the other hand, $\mathrm{h}^0(\mathcal{C}_s, \Omega_{\mathcal{C}_s/\kappa(s)}^{\otimes k}) = \deg(\Omega_{\mathcal{C}_s/\kappa(s)}^{\otimes k}) + 1 - g = (2k-1)(g-1)$ by Riemann–Roch (5.1.1). Corollary A.7.7 implies that $\pi_*(\Omega_{\mathcal{C}/S}^{\otimes k})$ is a vector bundle of rank $(2k-1)(g-1)$.

For (3), observe that since $\Omega_{\mathcal{C}/S}$ is a relative dualizing sheaf, Grothendieck–Serre Duality implies that $\mathrm{R}^1\pi_*\Omega_{\mathcal{C}/S} \cong \pi_*\mathcal{O}_\mathcal{C}$ and this is identified with $\mathcal{O}_S$ by (1). For $i \geq 2$, $\mathrm{H}^i(\mathcal{C}_s, \Omega_{\mathcal{C}/S} \otimes \kappa(s)) = 0$ and A.7.5(a) implies that $\mathrm{R}^i\pi_*\Omega_{\mathcal{C}/S} = 0$. Applying A.7.5(b) with $i = 2$ yields that $\phi_s^1\colon \mathrm{R}^1\pi_*\Omega_{\mathcal{C}/S} \otimes \kappa(s) \to \mathrm{H}^1(\mathcal{C}_s, \Omega_{\mathcal{C}_s/\kappa(s)})$ is surjective for every $s \in S$ and thus an isomorphism (A.7.5(a) with $i = 1$). Since $\mathrm{R}^1\pi_*\Omega_{\mathcal{C}/S} \cong \pi_*\mathcal{O}_\mathcal{C} \cong \mathcal{O}_S$ is a line bundle, applying A.7.5(b) with $i = 1$ implies that $\phi_s^0\colon \pi_*\Omega_{\mathcal{C}/S} \otimes \kappa(s) \to \mathrm{H}^0(\mathcal{C}_s, \Omega_{\mathcal{C}_s/\kappa(s)})$ is surjective, and applying A.7.5(a)–(b) with $i = 0$ implies that $\pi_*\Omega_{\mathcal{C}/S}$ is a vector bundle of rank $\mathrm{h}^0(\mathcal{C}_s, \Omega_{\mathcal{C}_s/\kappa(s)}) = g$ whose construction commutes with base change. $\square$

The following proposition is useful to verify the algebraicity of stacks of coherent sheaves and vector bundles (Theorem 3.1.19).

Proposition A.7.9. *Let $p\colon X \to S$ be a proper morphism of schemes and F be a finitely presented sheaf on X of finite presentation and flat over S. Suppose that $\dim X_s \le d$ for all $s \in S$. The subset S' of points $s \in S$ such that $\mathrm{H}^j(X_s, F_s) = 0$ for all $j > 0$ is open. Denoting $X' = p^{-1}(S')$, $p' := p|_{X'}\colon X' \to S$ and $F' = F|_{X'}$, we have that $\mathrm{R}^j p'_* F' = 0$ for all $j > 0$ and that $p'_* F'$ is a vector bundle whose construction commutes with base change.*

Proof. For each $j = 1, \ldots, d$, A.7.5(a) implies that the locus of points $s \in S$ such that $\mathrm{H}^j(X_s, F_s) = 0$ is open and the comparison map $\phi^j_s\colon \mathrm{R}^j p_* F \otimes \kappa(s) \to \mathrm{H}^j(X_s, F_s)$ is an isomorphism. It follows that $\mathrm{R}^j p'_* F = 0$ which allows us to apply A.7.5(b) (with $i = 1$) to conclude that $\phi^0_s\colon p'_* F' \otimes \kappa(s) \to \mathrm{H}^0(X_s, F_s)$ is surjective. Apply A.7.5(a)-(b) (with $i = 0$) now gives the final statement. □

For the following proposition (specialized to $n = 1$) is convenient to define determinantal line bundles on $\mathcal{B}\mathrm{un}_{C,r,d}$.

Proposition A.7.10. *Let $f\colon X \to S$ be a smooth projective morphism of relative dimension n between noetherian schemes. If F is a coherent sheaf on X flat over S, there is a locally free resolution*

$$0 \to F_n \to F_{n-1} \to \cdots \to F_0 \to F$$

such that

- *$\mathrm{R}^i f_* F_d = 0$ for $i \neq n$ and $d = 0, \ldots, n$,*
- *$\mathrm{R}^n f_* F_d$ is locally free for $d = 0, \ldots, n$,*
- *$\mathrm{R}^i f_* F$ is the $(n-i)$th homology $H_{n-i}(\mathrm{R}^n f_* F_\bullet)$ of the complex $\mathrm{R}^n f_* F_\bullet$.*
- *the determinant*
$$\det \mathrm{R} f_* F := \bigotimes_i (\det \mathrm{R}^i f_* F)^{(-1)^i}$$
is identified with $\bigotimes_d (\det \mathrm{R}^n f_* F_d)^{(-1)^{n-d}}$.

Moreover, the construction is compatible with base change on S.

Proof. See [HL10, Prop. 2.1.10]. □

A.7.4 Applications to line bundles

Given a flat, proper morphism $f\colon X \to Y$, when is a line bundle L on X the pullback of a line bundle on Y? More generally, is there a largest subscheme $Z \subset Y$ where L_Z on $X_Z = X \times_Y Z$ is the pullback of a line bundle on Z? In this section, we provide three answers in increasing complexity.

As we will need to impose conditions on the fibers X_y, we first discuss relationships between various conditions.

Lemma A.7.11. *Let $f\colon X \to Y$ be a flat, proper morphism of noetherian schemes. Consider the following conditions:*

(1) the geometric fibers of $f\colon X \to Y$ are non-empty, connected and reduced;

(2) $\mathrm{h}^0(X_y, \mathcal{O}_{X_y}) = 1$ for all $y \in Y$; and

(3) $\mathcal{O}_Y = f_* \mathcal{O}_X$ and this holds after arbitrary base change (i.e. $\mathcal{O}_T = f_{T,*} \mathcal{O}_{X_T}$ for a morphism $T \to Y$ of schemes).

Then (1) ⇒ (2) ⟺ (3).

Proof. If (1) holds, then $\mathrm{H}^0(X_y, \mathcal{O}_{X_y}) \otimes_{\kappa(y)} \overline{\kappa(y)} = \mathrm{H}^0(X \times_Y \overline{\kappa(y)}, \mathcal{O}_{X \times_Y \overline{\kappa(y)}}) = \overline{\kappa(y)}$ by Flat Base Change and the fact that a connected, reduced and proper scheme over an algebraically closed field has only constant functions. This gives (2).

If (2) holds, then the comparison map $\phi_y^0 \colon f_* \mathcal{O}_X \otimes \kappa(y) \to \mathrm{H}^0(X_y, \mathcal{O}_{X_y}) = \kappa(y)$ is necessarily surjective as we have the global section $1 \in \mathrm{H}^0(Y, f_* \mathcal{O}_X)$. Theorem A.7.5 (with $i = 0$) implies that $f_* \mathcal{O}_X$ is a line bundle and that $\mathcal{O}_Y \to f_* \mathcal{O}_X$ is a surjection of line bundles, hence an isomorphism. Since the same argument applies to the base change $X_T \to T$, this gives (3). The converse (3) ⇒ (2) follows by consider the base change $T = \operatorname{Spec} \kappa(y) \to Y$. □

When Y is reduced, Grauert's Theorem provides a complete answer to when a line bundle is a pullback.

Proposition A.7.12 (Version 1)**.** *Let $f \colon X \to Y$ be a flat, proper morphism of noetherian schemes such that $\mathrm{h}^0(X_y, \mathcal{O}_{X_y}) = 1$ for all $y \in Y$. Let L be a line bundle on X. If Y is reduced, then $L = f^* M$ for a line bundle M on Y if and only if L_y is trivial for all $y \in Y$. Moreover, if these conditions hold, then $M = f_* L$ and the adjunction morphism $f^* f_* L \to L$ is an isomorphism.*

Proof. The condition on geometric fibers implies that $\mathrm{h}^0(X_y, L_y) = 1$ and Grauert's Theorem (A.7.4) implies that $f_* L$ is a line bundle and that $f_* L \otimes \kappa(y) \xrightarrow{\sim} \mathrm{H}^0(X_y, L_y)$ is an isomorphism. We claim that $f^* f_* L \to L$ is surjective. It suffices to show that $(f^* f_* L)|_{X_y} \to L|_{X_y}$ is surjective. Denoting $f_y \colon X_y \to \operatorname{Spec} \kappa(y)$, we have identifications $(f^* f_* L)|_{X_y} = f_y^*(f_* L \otimes \kappa(y)) = f_y^*(\mathrm{H}^0(X_y, L_y)) = \mathcal{O}_{X_y}$ and the claim follows. Since $f^* f_* L \to L$ is a surjection of line bundles, it is an isomorphism. □

Exercise A.7.13. Show that if Y is a connected and reduced noetherian scheme and E is a vector bundle, then $\operatorname{Pic}(\mathbb{P}_Y(E)) = \operatorname{Pic}(Y) \times \mathbb{Z}$. See also [Har77, Exer. III.12.5].

Proposition A.7.14 (Version 2)**.** *Let $f \colon X \to Y$ be a flat, proper morphism of noetherian schemes with integral geometric fibers. For a line bundle L on X, the locus*

$$\{y \in Y \mid L_y \text{ is trivial}\} \subset Y$$

is closed.

Proof. The important observation here is that for a geometrically integral and proper scheme Z over field $\Bbbk$, a line bundle M is trivial if and only if $\mathrm{h}^0(Z, M) > 0$ and $\mathrm{h}^0(Z, M^\vee) > 0$. To see that the latter condition is sufficient, observe that we have non-zero homomorphisms $\mathcal{O}_Z \to M$ and $\mathcal{O}_Z \to M^\vee$, the latter of which dualizes to a non-zero map $M \to \mathcal{O}_Z$. Since Z is integral, the composition $\mathcal{O}_Z \to M \to \mathcal{O}_Z$ is also non-zero and is defined by a constant in $\mathrm{H}^0(Z, \mathcal{O}_Z) = \Bbbk$. It follows that $M \to \mathcal{O}_Z$ is a surjective map of line bundles, hence an isomorphism. By the Semicontinuity Theorem (A.7.3) the condition that $\mathrm{h}^0(X_y, L_y) > 0$ and $\mathrm{h}^0(X_y, L_y^\vee) > 0$ is closed, and the statement follows. See also [Mum70, p. 51]. □

Remark A.7.15. If the geometric fibers are only connected and reduced, the locus may fail to be closed. For example consider a family of smooth curves $f\colon X \to Y$ where Y is a curve and X is a smooth surface. For a closed point $x \in X$, consider the blow-up $\mathrm{Bl}_x X \to X$ and let E be the exceptional divisor. Then $\mathrm{Bl}_x X \to Y$ is a flat, proper morphism and the fiber over $f(x) \in Y$ is connected and reduced, but reducible. The line bundle $L = \mathcal{O}_{\mathrm{Bl}_x X}(E)$ has the property that the fiber L_y is trivial if and only if $y \neq f(x)$.

The two versions above can be combined to the following powerful statement for a flat, proper morphism $X \to Y$. For moduli-theoretic applications, it is essential that we allow the possibly that Y is non-reduced and that the fibers X_y be reducible. The proposition will be applied in the proof of Theorem 3.1.15 to show that the locus of curves C in a Hilbert scheme $\mathrm{Hilb}^P(\mathbb{P}^{5g-6}_{\mathbb{Z}}/\mathbb{Z})$ which are tri-canonically is a closed condition.

Proposition A.7.16 (Version 3)**.** *Let $f\colon X \to Y$ be a flat, proper morphism of noetherian schemes such that $\mathrm{h}^0(X_y, \mathcal{O}_{X_y}) = 1$ for all $y \in Y$ (resp. the geometric fibers are integral). For a line bundle L on X, there is a unique locally closed (resp. closed) subscheme $Z \subset Y$ such that*

(1) L_Z on $X_Z = X \times_Y Z$ is the pullback of a line bundle on Z; and

(2) if $T \to Y$ is a morphism of schemes such that L_T on X_T is the pullback of a line bundle on T, then $T \to Y$ factors through Z.

Remark A.7.17. In other words, the functor

$$\mathrm{Sch}/Y \to \mathrm{Sets},$$
$$(T \to Y) \mapsto \begin{cases} \{*\} & \text{if } L_T \text{ is the pullback of a line bundle on } T \\ \emptyset & \text{otherwise} \end{cases}$$

is representable by a locally closed (resp. closed) subscheme of Y.

Proof. By the Semicontinuity Theorem (A.7.3), the locus $V = \{y \in Y \mid \mathrm{h}^0(X_y, L_y) \leq 1\}$ is open. Since for points $y \notin V$, L_y is not trivial, we may replace Y with V and assume that $\mathrm{h}^0(X_y, L_y) \leq 1$ for all $y \in Y$.

Observe that if $L = f^*M$ for a line bundle M on Y, then by using the projection formula and the fact that $\mathcal{O}_Y = f_*\mathcal{O}_X$ (Lemma A.7.11), we see that $f_*L \cong f_*f^*M \cong f_*f^*\mathcal{O}_Y \otimes M \cong f_*\mathcal{O}_X \otimes M \cong M$ is a line bundle and that the adjunction map $f^*f_*L \to L$ is an isomorphism. The same holds for the base change $X_T \to T$, and we conclude that L_T is a pullback of a line bundle on T if and only if $f_{T,*}L$ is a line bundle and $f_T^*f_{T,*}L \to L$ is an isomorphism. We see therefore that the question is Zariski-local on Y and T. We will show that every point $y \in Y$ has an open neighborhood where the proposition holds.

By applying Theorem A.7.1 and after replacing Y with an open affine neighborhood of y, we may assume that there is a homomorphism $d\colon A^{r_0} \xrightarrow{d} A^{r_1}$ of finitely generated and free A-modules such that for every ring map $A \to B$, $\mathrm{H}^0(X_B, L_B) = \ker(d \otimes B)$. Consider the dual $d^\vee$ of d, we define M as the cokernel in the sequence

$$A^{r_1} \xrightarrow{d^\vee} A^{r_0} \to M \to 0.$$

Tensoring over $A \to B$ yields a right exact sequence

$$B^{r_1} \xrightarrow{d^\vee \otimes B} B^{r_0} \to M \otimes_A B \to 0$$

which after applying the contravariant left-exact functor $\mathrm{Hom}_B(-, B)$ becomes

$$0 \to \mathrm{Hom}_B(M \otimes_A B, B) \to B^{r_0} \xrightarrow{d \otimes_A B} B^{r_1}.$$

We conclude that

$$\mathrm{H}^0(X_B, L_B) = \mathrm{Hom}_B(M \otimes_A B, B) = \mathrm{Hom}_A(M, B). \tag{A.7.2}$$

Applying this to $A \to \kappa(y)$ for every point $y \in \operatorname{Spec} A$, we have $\mathrm{H}^0(X_y, L_y) = \mathrm{Hom}_A(M, \kappa(y)) = M \otimes_A \kappa(y)$.

If $\mathrm{h}^0(X_y, L_y) = 0$, then L_y is not trivial and there is an open neighborhood U of y such that $\widetilde{M}|_U = 0$. The proposition holds over U since there are no morphisms $T \to U$ from a non-empty scheme such that L_T is a pullback.

If $\mathrm{h}^0(X_y, L_y) = 1$, then $M \otimes_A \kappa(y) = \kappa(y)$ and by Nakayama's lemma, after replacing Y with an open affine neighborhood of y, there is a surjection $A \to M$. Write $M = A/I$ for an ideal I and define the closed subscheme $Z = V(I) \subset Y$. Observe that $\mathrm{H}^0(Z, L_Z) = \mathrm{Hom}_A(A/I, A/I) = A/I$ so that $f_{Z,*}L_Z$ is the trivial line bundle. For an A/I-algebra B, we have that $\mathrm{H}^0(X_B, L_B) = \mathrm{Hom}_A(A/I, B) = B$. It follows that the comparison map $\mathrm{H}^0(X_Z, L_Z) \otimes_{A/I} B \to \mathrm{H}^0(X_B, L_B)$ is an isomorphism, or in other words the construction of $f_{Z,*}L_Z$ commutes with base change. We claim that $T \to Y$ factors through Z if and only if $f_{T,*}L_T$ is a line bundle. This question is Zariski-local on T so we may assume $T = \operatorname{Spec} B$ is affine. If $f_{T,*}L_T$ is a line bundle, we may assume $f_{T,*}L_T = \mathcal{O}_T$ is trivial since the question is local on T. Then $B = \mathrm{Hom}_A(A/I, B)$ implies that $I \subset \ker(A \to B)$ or in other words that $A \to B$ factors as $A \to A/I \to B$. Finally, considering the adjunction morphism $\lambda\colon f_Z^* f_{Z,*}L_Z \to L_Z$ on X_Z, we claim that for $y \in Z$, L_y is trivial if and only if $\lambda|_{X_y}$ is surjective. If $\lambda|_{X_y}$ is surjective, then using that $f_{Z,*}L_Z = \mathcal{O}_Z$, we have a surjection $\mathcal{O}_{X_y} \to L_y$ of line bundles, hence an isomorphism. For converse, observe that since $f_{Z,*}L_Z$ commutes with base change, the comparison map $f_{Z,*}L_Z \otimes \kappa(y) = \mathrm{H}^0(X_y, L_y)$ is an isomorphism. Denoting $f_y\colon X_y \to \operatorname{Spec} \kappa(y)$, we have identifications $(f_Z^* f_{Z,*}L_Z)|_{X_y} = f_y^*(f_{Z,*}L_Z \otimes \kappa(y)) = f_y^* f_{y,*}L_y$ under which $\lambda|_{X_y}$ corresponds to the adjunction map $f_y^* f_{y,*}L_y \to L_y$ which is an isomorphism. Replacing Z with $Z \setminus \operatorname{Supp}(\operatorname{coker}(\lambda))$ establishes the proposition in the case that $\mathrm{h}^0(X_y, \mathcal{O}_{X_y}) = 1$ for all $y \in Y$. If the fibers are geometrically integral, then Proposition A.7.14 implies that Z is closed.

See also [Mum70, p. 90], [Vie95, Lem. 1.19] and [SP, Tags 0BEZ and 0BF0]. □

Remark A.7.18. Note that to prove the strongest version, we needed the strongest version of our various cohomology and base change results, namely Theorem A.7.1.

Remark A.7.19. For a flat, proper morphism $X \to S$, define the *Picard functor* as

$$\mathrm{Pic}_{X/S}\colon \mathrm{Sch}/S \to \mathrm{Sets}, \quad T \mapsto \mathrm{Pic}(X_T)/f_T^* \mathrm{Pic}(T).$$

If $f\colon X \to S$ has geometrically integral fibers, then the existence of a closed subscheme $Z \subset Y$ characterized by Proposition A.7.16 is equivalent to the diagonal morphism $\mathrm{Pic}_{X/S} \to \mathrm{Pic}_{X/S} \times_S \mathrm{Pic}_{X/S}$ of presheaves over Sch/S being representable by closed immersions, i.e. $\mathrm{Pic}_{X/S}$ is separated over S.

A.8 Pushouts

A.8.1 Existence of pushouts

Pushouts are the dual notion of fiber product. Unlike fiber products, pushouts may not always exist. However, Ferrand showed that they often exist when one of the maps is a closed immersion and the other is an affine morphism.

Theorem A.8.1 (Ferrand's Theorem on the Existence of Pushouts)**.** *Consider a diagram*

$$\begin{tikzcd} X_0 \arrow[r, hook, "i"] \arrow[d, "f_0"'] & X \arrow[d, dashed, "f"] \\ Y_0 \arrow[r, hook, dashed, "j"] & Y \end{tikzcd} \tag{A.8.1}$$

of schemes where $i\colon X_0 \hookrightarrow X$ *is a closed immersion and* $f_0\colon X_0 \to Y_0$ *is affine. If*

($\star$) *for every point* $y_0 \in Y_0$*, the subspace* $f_0^{-1}(\operatorname{Spec} \mathcal{O}_{Y_0,y_0}) \subset X_0$ *has a basis of open affine neighborhoods of* X*,*

then there exists a closed immersion $j\colon Y_0 \hookrightarrow Y$ *and an affine morphism* $f\colon X \to Y$ *of schemes such that* (A.8.1) *is cocartesian (i.e. a pushout). Moreover, we have the following properties:*

(a) the square (A.8.1) *is cartesian,* $X \to Y$ *restricts to an isomorphism* $X \setminus X_0 \to Y \setminus Y_0$ *and the induced map* $X \amalg Y_0 \to Y$ *is universally submersive;*

(b) the induced map

$$\mathcal{O}_Y \to j_*\mathcal{O}_{Y_0} \times_{(j\circ f_0)_*\mathcal{O}_{X_0}} f_*\mathcal{O}_X$$

is an isomorphism of sheaves; and

(c) if f_0 *is finite (resp. integral), then so is* f*. In this case, Condition* ($\star$) *can be replaced with the condition that every finite set of points in* X_0 *and* Y *is contained in an open affine (resp. for every* $y_0 \in Y_0$*,* $f_0^{-1}(y_0)$ *is contained in an open affine). Finally if* X_0*,* X *and* Y_0 *are of finite type over a noetherian scheme, then so is* Y*.*

Proof. See [Fer03, Thm. 5.4 and 7.1] and [SP, Tag 0ECH]. □

Example A.8.2 (Affine case)**.** In the affine case where $X = \operatorname{Spec} A$, $X_0 = \operatorname{Spec} A_0$, $Y_0 = \operatorname{Spec} B_0$, then $\operatorname{Spec}(A \times_{A_0} B_0)$ is the pushout $X \amalg_{X_0} Y_0$.

Example A.8.3 (Gluing and pinching)**.** If $X_0 \hookrightarrow X$ and $X_0 \hookrightarrow Y_0$ are closed immersions, the pushout $X \amalg_{X_0} Y_0$ can be viewed as the gluing of X and Y_0 along X_0. For example, the nodal curve $\operatorname{Spec} k[x,y]/xy$ is the union of $\mathbb{A}^1$ and $\mathbb{A}^1$ along their origins. If $X_0 = Z \sqcup Z$ is the union of two isomorphic disjoint subschemes of X and $X_0 \to Z$ is the projection, then the pushout $X \amalg_{Z \sqcup Z} Z$ can viewed as the pinching of the two copies of Z in X. For example, the nodal cubic curve is the pinching of 0 and ∞ in $\mathbb{P}^1$.

Example A.8.4 (Non-noetherianness)**.** When $f_0\colon X_0 \to Y_0$ is affine but not finite, then the pushout $X \amalg_{X_0} Y_0$ is often not noetherian. For example, if $X_0 = V(x) \subset X = \mathbb{A}^2_{\Bbbk}$ and $f_0\colon X_0 \to \operatorname{Spec} \Bbbk$, the pushout is the non-noetherian affine scheme defined by

$$k[x,y] \times_{k[x]} k = k[x, xy, xy^2, xy^3, \ldots] \subset k[x,y].$$

On the other hand, we wouldn't expect a finite type pushout as one cannot contract the y-axis in $\mathbb{A}^2_{\Bbbk}$.

A.8.2 Properties of pushouts

Given a fiber product diagram of rings

$$\begin{tikzcd} B \arrow[r] \arrow[d, two heads] & A \arrow[d, two heads] \\ B_0 \arrow[r] & A_0 \end{tikzcd}$$

with $A \twoheadrightarrow A_0$ surjective and $B := B_0 \times_{A_0} A$, the fiber product $\mathrm{Mod}(B_0) \times_{\mathrm{Mod}(A_0)} \mathrm{Mod}(A)$ is the category of triples (N_0, M, α) consisting of a B_0-module N_0, an A-module M and isomorphism $\alpha\colon N_0 \otimes_{B_0} A_0 \xrightarrow{\sim} M \otimes_A A_0$. Equivalently, an object is a diagram

$$\begin{tikzcd} & & M \arrow[dl] \arrow[dd, dash] \\ N_0 \arrow[r] \arrow[dd, dash] & M_0 \arrow[dd, dash] & \\ & & A \arrow[dl] \\ B_0 \arrow[r] & A_0 & \end{tikzcd} \tag{A.8.2}$$

where N_0, M_0 and M are modules over B_0, A_0 and A, and the maps $N_0 \to M_0$ and $M \to M_0$ are morphisms of B_0 and A-modules inducing isomorphisms $N_0 \otimes_{B_0} A_0 \to M_0$ and $M \otimes_A A_0 \to M_0$.

We define functors

$$\mathrm{Mod}(B) \underset{L}{\overset{R}{\leftrightarrows}} \mathrm{Mod}(B_0) \times_{\mathrm{Mod}(A_0)} \mathrm{Mod}(A) \tag{A.8.3}$$

where for a B-module N, $L(N) := (N \otimes_B B_0, N \otimes_B A, \alpha)$ with α being the canonical isomorphism $(N \otimes_B B_0) \otimes_{B_0} A_0 \xrightarrow{\sim} (N \otimes_B A) \otimes_A A_0$. For an object (N_0, M, α) corresponding to a diagram (A.8.2), we define $R(N_0, M, \alpha) := N_0 \times_{M_0} M$, which we can view as:

$$\begin{tikzcd} & N_0 \times_{M_0} M \arrow[rr] \arrow[dl] \arrow[dd, dash] & & M \arrow[dl] \arrow[dd, dash] \\ N_0 \arrow[rr, crossing over] \arrow[dd, dash] & & M_0 & \\ & B_0 \times_{A_0} A \arrow[rr] \arrow[dl] & & A \arrow[dl] \\ B_0 \arrow[rr] & & A_0. \arrow[from=uu, dash, crossing over] & \end{tikzcd}$$

Proposition A.8.5. *The functors L and R restrict to an equivalence on the full subcategories of flat (resp. finite) modules.*

Proof. See [Fer03, Thm. 2.2], [Sch68, Lem. 3.4] and [SP, Tag 0D2G] where it is established more generally that R is the right adjoint of L, the adjunction morphism $L \circ R \to \mathrm{id}$ is an isomorphism and for a B-module N, the adjunction map $N \to R \circ L(N)$ is surjective whose kernel is annihilated by $I = \ker(B \to B_0)$ and contained in IM. $\square$

Corollary A.8.6. *Consider a commutative cube of schemes*

$$\begin{tikzcd}
& X_0' \arrow[rr, hook] \arrow[dl] \arrow[dd] & & X' \arrow[dl] \arrow[dd] \\
X_0 \arrow[rr, hook, crossing over] \arrow[dd] & & X & \\
& Y_0' \arrow[rr, hook] \arrow[dl] & & Y' \arrow[dl] \\
Y_0 \arrow[rr, hook] & & Y \arrow[from=uu, crossing over] &
\end{tikzcd}$$

of schemes where $X_0 \hookrightarrow X$ is a closed immersion and $X_0 \to Y_0$ is affine.

(1) *Assume that $Y' \to Y$ is a flat morphism of schemes and X_0', Y_0' and X' are the base changes under $Y' \to Y$ (i.e. the bottom, left, top and right faces are cartesian).*

  (a) *If the front face is a pushout, then so is the back face and the natural functor*

$$\mathrm{QCoh}(Y') \to \mathrm{QCoh}(Y_0') \times_{\mathrm{QCoh}(X_0')} \mathrm{QCoh}(X'),$$

  restricts to an equivalence on the full subcategories of $\mathrm{QCoh}(Y')$, $\mathrm{QCoh}(Y_0')$ *and* $\mathrm{QCoh}(X')$ *containing finitely presented $\mathcal{O}$-modules flat over Y', Y_0' and X'.*

  (b) *If in addition $Y' \to Y$ is faithfully flat and locally of finite presentation, then back face being a pushout implies that the front face is as well.*

(2) *If the top and left faces are cartesian, and the front and back faces are pushouts, then all faces are cartesian. Moreover, if $Y_0' \to Y_0$ and $X' \to X$ are étale (resp. smooth, flat), then so is $Y' \to Y$.*

A.9 Henselizations

A.9.1 Henselian and strictly henselian local rings

Let $(R, \mathfrak{m})$ be a local ring with residue field κ. We will denote the image of $a \in R$ (resp. $f \in R[x]$) as $\overline{a} \in \kappa$ (resp. $\overline{f} \in \kappa[x]$). If $f \in R[t]$, we denote its derivative by $f' \in R[t]$. Note that $\overline{f'} = \overline{f}'$.

Definition A.9.1. Let $(R, \mathfrak{m})$ be a local ring with residue field κ.

(1) We say that R is *henselian* if for a monic polynomial $f \in R[t]$, every root $\alpha_0 \in \kappa$ of $\overline{f}$ with $\overline{f'}(\alpha_0) \neq 0$ lifts to a root $\alpha \in R$ of f.

(2) We say that R is *strictly henselian* if R is henselian and κ is separably closed.

Remark A.9.2. Hensel's lemma states that a complete DVR R (e.g. $\mathbb{Z}_p$) is henselian.

Proposition A.9.3 (Henselian Equivalences)**.** *The following are equivalent for a local ring $(R, \mathfrak{m})$ with residue field κ:*

(1) *R is henselian;*

(2) *for a polynomial $f \in R[t]$, every factorization $\overline{f} = g_0 h_0$ with $\gcd(g_0, h_0) = 1$ lifts to a factorization $f = gh$ with $\overline{g} = g_0$ and $\overline{h} = h_0$;*

(3) *every finite R-algebra is a finite product of local rings finite over R;*

(4) *every quasi-finite R-algebra A is isomorphic to a product $A \cong B \times C$ where B is a finite over R and $C \otimes_R \kappa = 0$;*

(5) *every étale ring homomorphism $\phi\colon R \to A$ and a prime $\mathfrak{p} \subset A$ with $\phi^{-1}(\mathfrak{p}) = \mathfrak{m}$ and $\kappa = \kappa(\mathfrak{p})$ has a unique section $s\colon A \to R$ with $s^{-1}(\mathfrak{m}) = \mathfrak{p}$.*

Moreover R is strictly henselian if and only if for every étale ring homomorphism $\phi\colon R \to A$ and prime $\mathfrak{p} \subset A$ with $\phi^{-1}(\mathfrak{p}) = \mathfrak{m}$, there is a unique section $s\colon A \to R$ with $s^{-1}(\mathfrak{m}) = \mathfrak{p}$.

Proof. See [EGA, IV.18.5.11], [Mil80, Thm. I.4.2] and [SP, Tag 04GG]. □

Remark A.9.4. The following stronger version of (4) holds for a henselian ring R: for every quasi-finite and separated morphism $X \to \operatorname{Spec} R$ of schemes, $X \cong F \sqcup W$ with F finite over R and $W \times_R \kappa = \emptyset$. This is a reformulation of Étale Localization of Quasi-finite Morphisms (Theorem A.5.1); see Exercise A.9.10.

Remark A.9.5. Both property (5) and the analogous property of strictly henselian rings generalize to étale morphisms $X \to \operatorname{Spec} A$ of schemes: every section over $\operatorname{Spec} A/\mathfrak{m}$

$$\begin{tikzcd} & X \arrow[d] \\ \operatorname{Spec} A/\mathfrak{m} \arrow[ur] \arrow[r, hook] & \operatorname{Spec} A \arrow[u, dashed, bend right] \end{tikzcd}$$

extends to a global section.

Proposition A.9.6. *Let $(R, \mathfrak{m})$ be a henselian (resp. strictly henselian) local ring with residue field κ.*

(1) *Every finite R-algebra is a product of finite henselian local (resp. strictly henselian) R-algebras.*

(2) *Every complete local ring is henselian.*

(3) *The functor $A \mapsto A \otimes_R \kappa$ gives an equivalence of categories between finite étale A-algebras and finite étale κ-algebras.*

Proof. See [EGA, IV.18.5.10, IV.18.5.14-15], [Mil80, 4.3-4.5] and [SP, Tag 04GE]. □

Remark A.9.7. The more general notion of henselian pairs is sometimes useful (although won't be used in these notes). A pair (X, X_0) consisting of a scheme X and a closed subscheme $X_0 \subset X$ is *henselian* if every finite morphism $f\colon U \to X$ induces a bijection $\operatorname{ClOpen}(U) \to \operatorname{ClOpen}(f^{-1}(X_0))$ between open and closed subschemes of U and those of $f^{-1}(X_0)$. If $(R, \mathfrak{m})$ is a henselian local ring, then $(\operatorname{Spec} R, \operatorname{Spec}(R/\mathfrak{m}))$ is a henselian pair by Proposition A.9.3(3). See [EGA, IV.18.5.5] or [SP, Tag 09XD] for further discussion and equivalences.

A.9.2 Henselizations and strict henselizations

Definition A.9.8. Let $(R, \mathfrak{m})$ be a local ring with residue field κ. The *henselization of R* is a local homomorphism $R \to R^{\mathrm{h}}$ into a henselian local ring R^{h} such that every other local homomorphism $R \to A$ into a henselian local ring factors uniquely through $R \to R^{\mathrm{h}}$.

Given a separable closure $\kappa \to \kappa^s$, the *strict henselization of R with respect to $\kappa \to \kappa^s$* is a local homomorphism $R \to R^{\mathrm{sh}}$ into a strictly henselian local ring $(R^{\mathrm{sh}}, \mathfrak{m}^{\mathrm{sh}})$ inducing $\kappa \to \kappa^s$ on residue fields such that every other local homomorphism $R \to A$ into a strictly henselian local ring $(A, \mathfrak{m}_A)$ factors through $R \to R^{\mathrm{sh}}$ and the factorization is uniquely determined by the inclusion $R^{\mathrm{sh}}/\mathfrak{m}^{\mathrm{sh}} \to A/\mathfrak{m}_A$ of residue fields.

Proposition A.9.9. *Let $(R, \mathfrak{m})$ be a local ring with residue field κ and let $\kappa \to \kappa^s$ be a separable closure. The henselization $R \to R^{\mathrm{h}}$ and strict henselization $R \to R^{\mathrm{sh}}$ exist and can be constructed as*

$$R^{\mathrm{h}} = \operatorname{colim}_{(R,\mathfrak{m}) \xrightarrow{\text{ét}} (A, \mathfrak{m}_A),\ \kappa = A/\mathfrak{m}_A} A$$

$$R^{\mathrm{sh}} = \operatorname{colim}_{\substack{(R,\mathfrak{m}) \xrightarrow{\text{ét}} (A, \mathfrak{m}_A) \\ \downarrow \quad \swarrow \quad \\ \kappa^s \qquad\qquad}} A$$

where the colimits are taken over the directed system of étale R-algebras A and maximal ideals $\mathfrak{m}_A$ over $\mathfrak{m}$; in the henselian case, we require that $\kappa = R/\mathfrak{m}_A$ while in the strictly henselian case, the data includes a homomorphism $A \to \kappa^s$ over R. Moreover,

(a) the residue fields of R^{h} and R^{sh} are κ and κ^s, respectively;

(b) the maps $R \to R^{\mathrm{h}}$ and $R \to R^{\mathrm{sh}}$ are faithfully flat; and

(c) if R is noetherian, then so is R^{h} and R^{sh}.

Proof. See [EGA, IV.18.5-8], [Mil80, I.4] and [SP, Tags 0BSK and 07QL]. □

For a scheme X and a point $x \in X$ with a choice of separable closure $\kappa(x) \to \kappa^s$, the henselization $\mathcal{O}^{\mathrm{h}}_{X,x}$ and strict henselization $\mathcal{O}^{\mathrm{h}}_{X,x}$ are the colimits of $\Gamma(U, \mathcal{O}_U)$ taken over diagrams

$$\begin{tikzcd} \operatorname{Spec} \kappa \arrow[r, "x"] \arrow[dr] & U \arrow[d, "\text{ét}"] \\ & X \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \operatorname{Spec} \kappa^s \arrow[r] \arrow[dr] & U \arrow[d, "\text{ét}"] \\ & X, \end{tikzcd}$$

respectively, where $U \to X$ is an étale morphism of schemes and $\operatorname{Spec} \kappa^s \to U$ is a lift of $\operatorname{Spec} \kappa^s \to \operatorname{Spec} \kappa(x) \xrightarrow{x} X$. Both $\mathcal{O}^{\mathrm{h}}_{X,x}$ and $\mathcal{O}^{\mathrm{sh}}_{X,x}$ can be thought of as local rings in étale topology.

Exercise A.9.10. Show that Étale Localization of Quasi-finite Morphisms (Theorem A.5.1) follows from the case when S is the spectrum of a henselian ring (see Remark A.9.4).

Hint: Use limit methods (Propositions A.6.4 and A.6.7) to extend a decomposition $X \times_S \operatorname{Spec} \mathcal{O}^{\mathrm{h}}_{S,s} \cong F^{\mathrm{h}} \sqcup W^{\mathrm{h}}$ to an étale neighborhood of s.

A.10 Artin Approximation

In this section, we discuss the deep result of Artin Approximation (Theorem A.10.9) which can be vaguely expressed as the following principle:

algebraic properties that hold for the completion $\widehat{\mathcal{O}}_{S,s}$ of the local ring of a scheme S at a point s also hold in an étale neighborhood $(S', s') \to (S, s)$.

Artin Approximation is related to another equally deep and powerful result known as Néron–Popescu Desingularization (Theorem A.10.4). Both Artin Approximation and Néron–Popescu are difficult theorems which we will not attempt to prove here. However, we will show at least how Artin Approximation easily follows from Néron–Popescu Desingularization.

A.10.1 Néron–Popescu Desingularization

Definition A.10.1. A ring homomorphism $A \to B$ of noetherian rings is called *geometrically regular* if $A \to B$ is flat and for every prime ideal $\mathfrak{p} \subset A$ and every finite field extension $\kappa(\mathfrak{p}) \to \kappa'$ (where $\kappa(\mathfrak{p}) = A_{\mathfrak{p}}/\mathfrak{p}$), the fiber $B \otimes_A \kappa'$ is regular.

Remark A.10.2. It is important to note that $A \to B$ is *not* assumed to be of finite type. If it is, then $A \to B$ is geometrically regular if and only if $A \to B$ is smooth.

Remark A.10.3. It can be shown that it is equivalent to require that each geometric fiber $B \otimes_A \overline{\kappa(\mathfrak{p})}$ is regular, or equivalently that $B \otimes_A \kappa'$ is regular for every *inseparable* finite field extensions $\kappa(\mathfrak{p}) \to \kappa'$. In particular, in characteristic 0, $A \to B$ is geometrically regular if it is flat and and every fiber $B \otimes_A \kappa(\mathfrak{p})$ is regular.

Theorem A.10.4 (Néron–Popescu Desingularization)**.** *A homomorphism $A \to B$ of noetherian rings is geometrically regular if and only if there is a directed system B_λ of smooth A-algebras over a directed set Λ such that $B = \operatorname{colim} B_{\lambda \in \Lambda}$.*

Proof. This was result was proved by Néron in [Nér64] in the case that A and B are DVRs and in general by Popescu in [Pop85], [Pop86] and [Pop90]. We recommend [Swa98] and [SP, Tag 07GC] for an exposition of this result. □

Example A.10.5. A field extension $k \to l$ is geometrically regular if and only if it is separable. When $k \to l$ is algebraic, then l is the colimit of separable finite (i.e. étale) extensions.

Definition A.10.6. A noetherian local ring A is called a *G-ring* if the homomorphism $A \to \widehat{A}$ is geometrically regular.

Remark A.10.7. One of the defining features of excellent schemes is that their local rings are G-rings.

In order to apply Néron–Popescu Desingularization, we will need that local rings of schemes are frequently G-rings.

Theorem A.10.8. *If A is the localization of a finitely generated algebra over a field or $\mathbb{Z}$, then A is a G-ring.*

Proof. While substantially easier than Néron–Popescu Desingularization, this result nevertheless requires some effort. See [EGA, IV.7.4.4] or [SP, Tag 07PX]. □

A.10.2 Artin Approximation

Let S be a scheme and consider a contravariant functor

$$F \colon \mathrm{Sch}/S \to \mathrm{Sets}$$

where Sch/S denotes the category of schemes over S. We say that F is *limit preserving* (or *locally of finite presentation*) if for system of $\mathcal{O}_S$-algebras A_λ (i.e. each $\operatorname{Spec} A_\lambda$ is an S-scheme), the natural map

$$\operatorname{colim} F(B_\lambda) \to F(\operatorname{colim} B_\lambda)$$

is bijective. When F is a functor $\mathrm{Mor}_S(-, X)$ representable by a scheme X over S, then this equivalent to $X \to S$ being locally of finite presentation (Proposition A.1.3).

Theorem A.10.9 (Artin Approximation)**.** *Let S be a scheme and $s \in S$ a point such that $\mathcal{O}_{S,s}$ is a G-ring (Definition A.10.6), e.g. a scheme of finite type over a field or $\mathbb{Z}$. Let*

$$F \colon \mathrm{Sch}/S \to \mathrm{Sets}$$

be a limit preserving contravariant functor and $\widehat{\xi} \in F(\operatorname{Spec} \widehat{\mathcal{O}}_{S,s})$. For every integer $N \geq 0$, there exist an étale morphism

$$(S', s') \to (S, s) \qquad \textit{and} \qquad \xi' \in F(S')$$

with $\kappa(s) = \kappa(s')$ such that the restrictions of $\widehat{\xi}$ and ξ' to $\operatorname{Spec}(\mathcal{O}_{S,s}/\mathfrak{m}_s^{N+1})$ are equal.

Remark A.10.10. To make sense of the restriction ξ' to $\operatorname{Spec}(\mathcal{O}_{S,s}/\mathfrak{m}_s^{N+1})$, note that since $(S', s') \to (S, s)$ is a residually-trivial étale morphism, there are compatible identifications $\mathcal{O}_{S,s}/\mathfrak{m}_s^{N+1} \cong \mathcal{O}_{S',s'}/\mathfrak{m}_{s'}^{N+1}$.

Remark A.10.11. It is not possible in general to find $\xi' \in F(S')$ restricting to $\widehat{\xi}$ or even such that the restrictions of ξ' and $\widehat{\xi}$ to $\operatorname{Spec} \mathcal{O}_{S,s}/\mathfrak{m}_s^{n+1}$ agree for all $n \geq 0$. For instance, F could be the functor $\mathrm{Mor}(-, \mathbb{A}^1)$ representing the affine line $\mathbb{A}^1$ and $\widehat{\xi} \in \widehat{\mathcal{O}}_{S,s}$ could be a non-algebraic power series.

Proof. The theorem was originally proven in [Art69a, Cor. 2.2] in the case that S is of finite type over a field or an excellent dedekind domain. We also recommend [BLR90, §3.6] for an accessible account of the case of excellent and henselian DVRs. We will show how Artin Approximation follows from Néron–Popescu Desingularization (Theorem A.10.4).

Néron–Popescu Desingularization implies that $\widehat{\mathcal{O}}_{S,s} = \operatorname{colim}_{\lambda \in \Lambda} B_\lambda$ is a directed colimit of smooth $\mathcal{O}_{S,s}$-algebras. Since F is limit preserving, there exist $\lambda \in \Lambda$, a factorization $\mathcal{O}_{S,s} \to B_\lambda \to \widehat{\mathcal{O}}_{S,s}$ and an element $\xi_\lambda \in F(\operatorname{Spec} B_\lambda)$ whose restriction to $F(\operatorname{Spec} \widehat{\mathcal{O}}_{S,s})$ is $\widehat{\xi}$. Letting $B = B_\lambda$ and $\xi = \xi_\lambda$, we have a commutative diagram

$$\begin{tikzcd}
\operatorname{Spec} \widehat{\mathcal{O}}_{S,s} \arrow[r, "g"] \arrow[rr, bend left, "\widehat{\xi}"] \arrow[dr] & \operatorname{Spec} B \arrow[r, "\xi"] \arrow[d] & F \\
& \operatorname{Spec} \mathcal{O}_{S,s} &
\end{tikzcd}$$

where $\operatorname{Spec} B \to \operatorname{Spec} \mathcal{O}_{S,s}$ is smooth. We claim that we can find a commutative diagram

$$\begin{tikzcd} S' \arrow[r, hook] \arrow[dr] & \operatorname{Spec} B \arrow[d] \\ & \operatorname{Spec} \mathcal{O}_{S,s} \end{tikzcd} \tag{A.10.1}$$

where $S' \hookrightarrow \operatorname{Spec} B$ is a closed immersion, $(S', s') \to (\operatorname{Spec} \mathcal{O}_{S,s}, s)$ is étale, and the composition $\operatorname{Spec} \mathcal{O}_{S,s}/\mathfrak{m}_s^{N+1} \to S' \to \operatorname{Spec} B$ agrees with the restriction of $g \colon \operatorname{Spec} \widehat{\mathcal{O}}_{S,s} \to \operatorname{Spec} B$.[1] To see this, since $\Omega_{B/\mathcal{O}_{S,s}}$ is a locally free B-module, after replacing $\operatorname{Spec} B$ with an affine open neighborhood of $g(s)$, we may assume that $\Omega_{B/\mathcal{O}_{S,s}}$ is free with basis $db_1, \ldots, db_n$. This induces a homomorphism $\mathcal{O}_{S,s}[x_1, \ldots, x_n] \to B$ defined by $x_i \mapsto b_i$ and provides a factorization

$$\operatorname{Spec} B \to \mathbb{A}^n_{\mathcal{O}_{S,s}} \to \operatorname{Spec} \mathcal{O}_{S,s}$$

where $\operatorname{Spec} B \to \mathbb{A}^n_{\mathcal{O}_{S,s}}$ is étale. Choosing a lift of the composition

$$\begin{tikzcd} & & & \mathcal{O}_{S,s} \arrow[d] \\ \mathcal{O}_{S,s}[x_1, \ldots, x_n] \arrow[r] \arrow[urrr, dashed] & B \arrow[r] & \widehat{\mathcal{O}}_{S,s} \arrow[r] & \mathcal{O}_{S,s}/\mathfrak{m}_s^{N+1} \end{tikzcd}$$

defines a section $s \colon \operatorname{Spec} \mathcal{O}_{S,s} \to \mathbb{A}^n_{\mathcal{O}_{S,s}}$ and we define S' as the fibered product

$$\begin{tikzcd} S' \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\square"] & \operatorname{Spec} \mathcal{O}_{S,s} \arrow[d, hook, "s"] \\ \operatorname{Spec} B \arrow[r] & \mathbb{A}^n_{\mathcal{O}_{S,s}}. \end{tikzcd}$$

This gives the desired diagram (A.10.1), and the composition $\xi' \colon S' \to \operatorname{Spec} B \xrightarrow{\xi} F$ is an element which agrees with $\widehat{\xi}$ up to order N.

Finally, we must explain how to "smear out" the étale morphism $(S', s') \to (\operatorname{Spec} \mathcal{O}_{S,s}, s)$ and the element $\xi' \in F(S')$ to an étale morphism $(S'', s'') \to (S, s)$ and an element $\xi'' \in F(S'')$. Writing $\mathcal{O}_{S,s} = \operatorname{colim}_{g \notin \mathfrak{m}_s} A_g$, we may apply Propositions A.6.4, A.6.7 and B.4.4 (or a direct argument) to find an element $g \notin \mathfrak{m}_s$ and an affine scheme $S'' = \operatorname{Spec}(A_g[y_1, \ldots, y_n]/(f''_1, \ldots, f''_m))$ such that $S'' \times_{A_g} A_{\mathfrak{m}_s} \cong S'$ and such that $S'' \to \operatorname{Spec} A_g$ is étale. As F is limit preserving and $\Gamma(S'', \mathcal{O}_{S''}) = \operatorname{colim}_{h \notin \mathfrak{m}_s} A_{hg}[y_1, \ldots, y_n]/(f'_1, \ldots, f'_m)$, after replacing g with hg, we can find an element $\xi'' \in F(S'')$ restricting to ξ' and, in particular, agreeing with $\widehat{\xi}$ up to order N. $\square$

Exercise A.10.12 (Alternative formulations)**.** Let $(A, \mathfrak{m})$ be a henselian local G-ring.

[1] This is where the approximation occurs. It is not possible to find a morphism $S' \to \operatorname{Spec} B \to \operatorname{Spec} \mathcal{O}_{S,s}$ which is étale at a point s' over s such that the composition $\operatorname{Spec} \widehat{\mathcal{O}}_{S,s} \to S' \to \operatorname{Spec} B$ is equal to g.

(1) Let $F = \mathrm{Hom}(-, X)\colon \mathrm{Sch}/S \to \mathrm{Sets}$ be the contravariant functor of an affine scheme $X = \operatorname{Spec} A[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$ of finite type over A. Note that for an A-algebra B,

$$F(B) = \{a = (a_1, \ldots, a_n) \in B^{\oplus n} \mid f_i(a) = 0 \text{ for all } i\}.$$

If $\widehat{a} = (\widehat{a}_1, \ldots, \widehat{a}_n) \in \widehat{A}_{\mathfrak{m}}$ is a solution to the equations $f_1(x) = \cdots = f_m(x) = 0$, show that Artin Approximation implies that for every $N \geq 0$, there is a solution $a = (a_1, \ldots, a_n) \in A^{\oplus n}$ to the equations $f_1(x) = \cdots = f_m(x) = 0$ such that $a \cong \widehat{a} \mod \mathfrak{m}^{N+1}$.

(2) Show that (1) implies Artin Approximation.

Hint: Use that F is limit preserving to find a finitely generated A-subalgebra $B \subset \widehat{\mathcal{O}}_{S,s}$ and an element $\xi \in F(B)$ restricting to $\widehat{\xi}$.

A.10.3 A first application of Artin Approximation

The next corollary states an important fact which you may have taken for granted: if two schemes are formally isomorphic at two points, then they are isomorphic in the étale topology.

Corollary A.10.13. *Let X and Y be schemes of finite type over a scheme S and let $s \in S$ be a point such that $\mathcal{O}_{S,s}$ is a G-ring. If $x \in X$ and $y \in Y$ are points over s such that $\widehat{\mathcal{O}}_{X,x}$ and $\widehat{\mathcal{O}}_{Y,y}$ are isomorphic as $\mathcal{O}_S$-algebras, then there exists étale morphisms*

$$\begin{tikzcd} & (U,u) \arrow[dl] \arrow[dr] & \\ (X,x) & & (Y,y) \end{tikzcd} \tag{A.10.2}$$

inducing isomorphisms $\kappa(x) \xrightarrow{\sim} \kappa(u)$ and $\kappa(y) \xrightarrow{\sim} \kappa(u)$ on residue fields.

Proof. The functor

$$F\colon \mathrm{Sch}/X \to \mathrm{Sets}, \quad (T \to X) \mapsto \mathrm{Mor}(T, Y)$$

is limit preserving as it can be identified with the representable functor $\mathrm{Mor}_X(-, Y \times X)$ corresponding to the finite type morphism $Y \times X \to X$. The isomorphism $\widehat{\mathcal{O}}_{X,x} \cong \widehat{\mathcal{O}}_{Y,y}$ provides an element of $F(\operatorname{Spec} \widehat{\mathcal{O}}_{X,x})$. By applying Artin Approximation with $N = 1$, we obtain a diagram as in (A.10.2) with $U \to X$ étale at u with $\kappa(x) \xrightarrow{\sim} \kappa(u)$ and such that $\mathcal{O}_{Y,y}/\mathfrak{m}_y^2 \to \mathcal{O}_{U,u}/\mathfrak{m}_u^2$ is an isomorphism. Since $\widehat{\mathcal{O}}_{U,u}$ is abstractly isomorphic to $\widehat{\mathcal{O}}_{Y,y}$, Lemma A.10.15 implies that $\widehat{\mathcal{O}}_{Y,y} \to \widehat{\mathcal{O}}_{U,u}$ is an isomorphism and therefore that $(U, u) \to (Y, y)$ is étale. $\square$

Remark A.10.14. If $\phi\colon \widehat{\mathcal{O}}_{Y,y} \xrightarrow{\sim} \widehat{\mathcal{O}}_{X,x}$ is the specified isomorphism, it is not always possible to arrange that the induced diagram

$$\begin{tikzcd} & \widehat{\mathcal{O}}_{U,u} & \\ \widehat{\mathcal{O}}_{X,x} \arrow[ur] & & \widehat{\mathcal{O}}_{Y,y} \arrow[ul] \arrow[ll, "\phi"'] \end{tikzcd}$$

is commutative, but the proof (using Artin Approximation with a given $N \geq 1$) shows that we can arrange that the diagram commutes modulo $\mathfrak{m}_y^{N+1}$. See also [SP, Tag 0CAV].

Lemma A.10.15. *Let $(A, \mathfrak{m}_A) \to (B, \mathfrak{m}_B)$ be a local homomorphism of noetherian complete local rings. If $A/\mathfrak{m}_A^2 \to B/\mathfrak{m}_B^2$ is surjective, so is $A \to B$. If in addition $A = B$, then $A \to B$ is an isomorphism.*

Proof. This follows from the following version of Nakayama's lemma for noetherian complete local rings $(A, \mathfrak{m})$: if M is a (not-necessarily finitely generated) A-module such that $\bigcap_k \mathfrak{m}^k M = 0$ and $m_1, \ldots, m_n \in F$ generate $M/\mathfrak{m}M$, then $m_1, \ldots, m_n$ also generate M (see [Eis95, Exercise 7.2]). The final statement follows from the fact that a surjective endomorphism of a noetherian ring is an isomorphism. $\square$

Appendix B

Descent

It is hard to overstate the importance of descent in moduli theory. The central idea of descent is as simple as it is powerful. You already know that many properties of schemes and their morphisms can be checked on a Zariski-cover, and descent theory implies that they can also be checked on étale covers and often even faithfully flat covers. For example, if $Y' \to Y$ is étale and surjective, then a morphism $X \to Y$ is proper if and only if $X \times_Y Y' \to Y'$ is.

The applications of descent reach far beyond moduli theory. For instance, it can be used to reduce statements about schemes over a field k to the case when k is algebraically closed since $k \to \overline{k}$ is faithfully flat, or reduce statements over a noetherian local ring A to its completion $\widehat{A}$ since $A \to \widehat{A}$ is faithfully flat.

References: [BLR90, Ch.6], [Vis05], [Ols16, Ch. 4], [SP, Tag 0238], [EGA, §IV.2], and [SGA1, §VIII.7] (other descent results are scattered throughout EGA and SGA).

B.1 Descending quasi-coherent sheaves

Descent theory rests on the following algebraic fact.

Proposition B.1.1. *If $\phi\colon A \to B$ is a faithfully flat ring map, then the sequence*

$$0 \longrightarrow A \xrightarrow{\phi} B \underset{b \mapsto 1 \otimes b}{\overset{b \mapsto b \otimes 1}{\rightrightarrows}} B \otimes_A B$$

is exact. More generally, if M is an A-module, the sequence

$$0 \longrightarrow M \xrightarrow{m \mapsto m \otimes 1} M \otimes_A B \underset{m \otimes b \mapsto m \otimes 1 \otimes b}{\overset{m \otimes b \mapsto m \otimes b \otimes 1}{\rightrightarrows}} M \otimes_A B \otimes_A B \qquad \text{(B.1.1)}$$

is exact.

Proof. Note that $A \to B$ and $M \to M \otimes_A B$ are necessarily injective by Proposition A.2.17. Since $A \to B$ is faithfully flat, the sequence (B.1.1) is exact if and only if the sequence

$$M \otimes_A B \xrightarrow{m \otimes b' \mapsto m \otimes 1 \otimes b'} M \otimes_A B \otimes_A B \underset{m \otimes b \otimes b' \mapsto m \otimes 1 \otimes b \otimes b'}{\overset{m \otimes b \otimes b' \mapsto m \otimes b \otimes 1 \otimes b'}{\rightrightarrows}} M \otimes_A B \otimes_A B \otimes_A B$$

is exact. The above sequence can be rewritten as

$$M \otimes_A B \xrightarrow{x \mapsto x \otimes 1} (M \otimes_A B) \otimes_B (B \otimes_A B) \underset{x\otimes y \mapsto x \otimes 1 \otimes y}{\overset{x\otimes y \mapsto x\otimes y \otimes 1}{\rightrightarrows}} (M \otimes_A B) \otimes_B (B \otimes_A B) \otimes_B (B \otimes_A B)$$

which is precisely sequence (B.1.1) applied to ring $B \to B \otimes_A B$ given by $b \mapsto 1 \otimes b$ and the B-module $M \otimes_A B$. Since this ring map has a section $B \otimes_A B \to B$ given by $b \otimes b' \mapsto bb'$, we are reduced to proving the proposition when $\phi\colon A \to B$ has a section $s\colon B \to A$ with $s \circ \phi = \mathrm{id}_A$.

Let $x = \sum_i m_i \otimes b_i \in M \otimes_A B$ such that $\sum_i m_i \otimes b_i \otimes 1 = \sum_i m_i \otimes 1 \otimes b_i$. Applying $\mathrm{id}_M \otimes \mathrm{id}_B \otimes s\colon M \otimes_A B \otimes_A B \to M \otimes_A B \otimes_A A \cong M \otimes_A B$ to this identity shows that $x = \sum_i m_i \otimes \phi(s(b_i)) = \sum_i \phi(s(b_i)) m_i \otimes 1$ is in the image of $M \to M \otimes_A B$. $\square$

Exercise B.1.2. Denoting $(B/A)^{\otimes n}$ as the n-fold tensor product $B \otimes_A \cdots \otimes_A B$, show that the short exact sequence (B.1.1) extends to a long exact sequence $0 \to M \to M \otimes_A (B/A)^{\otimes 1} \to M \otimes_A (B/A)^{\otimes 2} \to \cdots$ with differentials

$$d\colon M \otimes_A (B/A)^{\otimes n} \to M \otimes_A (B/A)^{\otimes (n+1)}$$
$$m \otimes b_1 \otimes \cdots b_n \mapsto \sum_{i=0}^{n+1} (-1)^i m \otimes b_1 \otimes \cdots \otimes b_{i-1} \otimes 1 \otimes b_{i+1} \otimes \cdots b_n.$$

Proposition B.1.3. *Let $f\colon X \to Y$ be an fpqc morphism of schemes.*

(1) Let G and G' be quasi-coherent $\mathcal{O}_Y$-modules. Let p_1, p_2 denote the two projections $X \times_Y X \to X$ and q denote the composition $X \times_Y X \xrightarrow{p_i} X \xrightarrow{f} Y$. Then the sequence

$$\mathrm{Hom}_{\mathcal{O}_Y}(G, G') \xrightarrow{f^*} \mathrm{Hom}_{\mathcal{O}_X}(f^*G, f^*G') \underset{p_2^*}{\overset{p_1^*}{\rightrightarrows}} \mathrm{Hom}_{\mathcal{O}_{X \times_Y X}}(q^*G, q^*G')$$

is exact.

(2) Let F be a quasi-coherent $\mathcal{O}_X$-module and $\alpha\colon p_1^*F \to p_2^*F$ an isomorphism of $\mathcal{O}_{X\times_Y X}$-modules satisfying the cocycle condition $p_{23}^*\alpha \circ p_{12}^*\alpha = p_{13}^*\alpha$ on $X \times_Y X \times_X Y$. Then there exists a quasi-coherent $\mathcal{O}_Y$-module G and an isomorphism $\phi\colon F \to f^*G$ such that $p_1^*\phi = p_2^*\phi \circ \alpha$ on $X \times_Y X$. The data (F, ϕ) is unique up to unique isomorphism.

Remark B.1.4. The following diagram may be useful to internalize (2)

$$p_{23}^*\alpha \circ p_{12}^*\alpha = p_{13}^*\alpha \qquad p_1^*F \xrightarrow{\alpha} p_2^*F \qquad F \qquad G$$

$$X \times_Y X \times_Y X \underset{p_{23}}{\overset{p_{12},\, p_{13}}{\rightrightarrows}} X \times_Y X \underset{p_2}{\overset{p_1}{\rightrightarrows}} X \xrightarrow{f} Y$$

Keep in mind the special case that $\{U_i\}$ is an open covering of Y and $X = \coprod_i U_i$ in which case the above fiber products correspond to intersections.

The cocycle condition $p_{23}^*\alpha \circ p_{12}^*\alpha = p_{13}^*\alpha$ and the condition that $p_1^*\phi = p_2^*\phi \circ \alpha$ should be understood as the commutativity of

$$\begin{tikzcd} p_{12}^*p_1^*F \arrow[r, "p_{12}^*\alpha"] \arrow[d, equal] & p_{12}^*p_2^*F \arrow[r, equal] & p_{23}^*p_1^*F \arrow[d, "p_{23}^*\alpha"] \\ p_{13}^*p_1^*F \arrow[r, "p_{13}^*\alpha"] & p_{13}^*p_2^*F \arrow[r, equal] & p_{23}^*p_2^*F \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} p_1^*F \arrow[r, "p_1^*\phi"] \arrow[d, "\alpha"] & p_1^*f^*G \arrow[d, equal] \\ p_2^*F \arrow[r, "p_2^*\phi"] & p_2^*f^*G. \end{tikzcd}$$

Proposition B.1.3 can be reformulated as the statement that the category $\mathrm{QCoh}(Y)$ is equivalent to the *category of descent datum for* $X \to Y$, denoted by $\mathrm{QCoh}(X \to Y)$. Here the objects of $\mathrm{QCoh}(X \to Y)$ are pairs (F, α) consisting of a quasi-coherent $\mathcal{O}_X$-module F and an isomorphism $\alpha\colon p_1^*F \to p_2^*F$ satisfying the cocycle condition. A morphism $(F', \alpha') \to (F, \alpha)$ is a morphism $\beta\colon F' \to F$ such that $\alpha \circ p_1^*\beta = p_2^*\beta \circ \alpha'$.

Proof. If $X = \operatorname{Spec} B$ and $Y = \operatorname{Spec} A$ are affine, write $G = \widetilde{M}$ and $G' = \widetilde{M'}$. Proposition B.1.1 implies that $0 \to M' \to M' \otimes_A B \rightrightarrows M' \otimes_A B \otimes_A B$ is exact. Applying $\mathrm{Hom}_A(M, -)$ and using tensor-hom adjunction yields (1). For (2), writing $F = \widetilde{M}$, one defines N as the equalizers of two maps $M \rightrightarrows M \otimes_A B$ defined by $m \mapsto m \otimes 1$ and $m \mapsto \alpha(m \otimes 1)$.

The general case is handled by first reducing to the case that Y is affine. Since f is fpqc, Y is the image of a quasi-compact open subset $U \subset X$. By choosing a finite affine cover $\{U_i\}$ of U and replacing X with the affine scheme $\coprod_i U_i$, we have reduced to the case that X is affine. We leave the details to the reader. $\square$

Remark B.1.5. It turns out that effective descent for modules holds for a class of ring maps $A \to B$ larger than just faithfully flat maps. Namely, it holds for universally injective maps (see Definition A.2.20) and moreover the converse is true! More precisely, $A \to B$ is universally injective if and only if the functor

$$\begin{aligned} \mathrm{Mod}_A \quad &\to \{(N, \alpha) \mid \quad \begin{array}{l} N \in \mathrm{Mod}_B, \alpha\colon N \otimes_{B,p_1} (B \otimes_A B) \xrightarrow{\sim} N \otimes_{B,p_2} (B \otimes_A B) \\ \text{satisfying the cocycle condition } p_{23}^*\alpha \circ p_{12}^*\alpha = p_{13}^*\alpha\} \end{array} \\ M &\mapsto (M \otimes_A B, \mathrm{can}) \end{aligned}$$

to the category of descent data, is an equivalence of categories. See [Mes00] or [SP, Tag 08XA].

B.2 Descending morphisms

Proposition B.2.1. *Let $f\colon X \to Y$ be an fpqc morphism of schemes. If $g\colon X \to Z$ is a morphism to a scheme such that $p_1 \circ g = p_2 \circ g$ on $X \times_Y X$, then there exists a unique morphism $h\colon Y \to Z$ filling in the commutative diagram*

$$\begin{tikzcd} X \times_Y X \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & X \arrow[r, "f"] \arrow[dr, "g"'] & Y \arrow[d, dashed, "h"] \\ & & Z \end{tikzcd}$$

of solid arrows.

This result implies that every scheme is a sheaf in the fpqc topology; see Proposition 2.2.6.

B.3 Descending schemes

We will use the following notation: if $f\colon X \to Y$ and $W \to Y$ are morphisms of schemes, we denote f^*W as the fiber product $X \times_Y W$.

Proposition B.3.1 (Effective Descent)**.** *Let $f\colon X \to Y$ be an fpqc morphism of schemes. Let $\mathcal{P}$ be one of the following properties of morphisms of schemes: open immersion, closed immersion, locally closed immersion, affine, quasi-affine or separated and locally quasi-finite. If $Z \to X$ is $\mathcal{P}$-morphism of schemes and $\alpha\colon {p_1}^*(Z) \xrightarrow{\sim} p_2^*(Z)$ is an isomorphism over $X \times_Y X$ satisfying $p_{23}^*\alpha \circ p_{12}^*\alpha = p_{13}^*\alpha$, then there exist $\mathcal{P}$-morphism $W \to Y$ of schemes and an isomorphism $\phi\colon Z \to f^*(W)$ such that $p_1^*\phi = p_2^*\phi \circ \alpha$.*

Remark B.3.2. In the case of an open or closed immersion $Z \hookrightarrow X$, then the existence of α translates to the equality $p_1^{-1}(Z) = p_2^{-1}(Z)$ as subschemes of $X \times_Y X$ and there is no need for a cocycle condition.

It may be helpful to interpret the above statement using the diagram

$$\begin{tikzcd} p_{23}^*\alpha \circ p_{12}^*\alpha = p_{13}^*\alpha & p_1^*Z \arrow[r, "\alpha"] \arrow[d] & p_2^*Z \arrow[dl] & Z \arrow[r, dashed] \arrow[d] & W \arrow[d, dashed] \\ X \times_Y X \times_Y X \arrow[r, shift left=2, "p_{12}"] \arrow[r, "p_{13}"] \arrow[r, shift right=2, "p_{23}"'] & X \times_Y X \arrow[rr, shift left, "p_1"] \arrow[rr, shift right, "p_2"'] & & X \arrow[r, "f"] & Y. \end{tikzcd}$$

Proof. If $Z \hookrightarrow X$ is a closed immersion defined by an ideal sheaf $I_X \subset \mathcal{O}_X$, then one can apply Proposition A.6.8 to descend I_X to a quasi-coherent sheaf I_Y on Y and to descend the inclusion $I_X \hookrightarrow \mathcal{O}_X$ to an inclusion $I_Y \hookrightarrow \mathcal{O}_Y$. Then $W = V(I_Y) \hookrightarrow Y$ is the descended scheme. The case of open immersions can be handled by considering the reduced complement.

If $Z = \mathcal{S}\mathrm{pec}_X \mathcal{A}_X$ is affine over X, then Proposition A.6.8 allows us to first descend $\mathcal{A}_X$ to a quasi-coherent sheaf $\mathcal{A}_Y$ on Y and then the multiplication $\mathcal{A}_X \otimes_{\mathcal{O}_X} \mathcal{A}_X \to \mathcal{A}_X$ to a morphism $\mathcal{A}_Y \otimes_{\mathcal{O}_Y} \mathcal{A}_Y \to \mathcal{A}_Y$ which will necessarily satisfy the axioms making $\mathcal{A}_Y$ into a quasi-coherent $\mathcal{O}_Y$-algebra. Then one takes $W = \mathcal{S}\mathrm{pec}_Y \mathcal{A}_Y$. The case of quasi-affine morphisms is handled by combining the affine and open immersion cases.

If $Z \to X$ is separated and locally quasi-finite, then working locally on Z one reduces to the quasi-compact case in which case $Z \to X$ is quasi-affine by Proposition A.5.2. $\square$

We will often apply Effective Descent to show that a given sheaf in the big étale or fppf topology is representable by a scheme; see Proposition 2.2.11.

B.4 Descending properties

This is currently an incomplete list of the descent results needed.

B.4.1 Descending properties of morphisms

Proposition B.4.1 (Properties fpqc local on the target)**.** *Let $Y' \to Y$ be an fpqc morphism of schemes. Let $\mathcal{P}$ be one of the following properties of a morphism of schemes:*

(i) *isomorphism;*
(ii) *closed immersion;*
(iii) *open immersion;*
(iv) *surjective;*
(v) *proper;*
(vi) *flat;*
(vii) *smooth;*
(viii) *étale;*
(ix) *unramified;*
(x) *syntomic.*

Then $X \to Y$ has $\mathcal{P}$ if and only if $X \times_Y Y' \to Y'$ does.

Proposition B.4.2 (Properties local on the source)**.** *Let $X' \to X$ be an fppf morphism of schemes. Let $\mathcal{P}$ be one of the following properties of a morphism of schemes:*

(i) *surjective;*
(ii) *fppf;*
(iii) *smooth;*

Then $X \to Y$ has $\mathcal{P}$ if and only if $X' \to X \to Y$ does.

If $X' \to X$ is étale and surjective, then $X \to Y$ is étale if and only if $X' \to X \to Y$ is.

Proposition B.4.3 (Fpqc-local properties of quasi-coherent sheaves)**.** *Let $f\colon X \to Y$ be an fpqc morphism of schemes. Let $\mathcal{P} \in \{$finite type, finite presentation, flat, vector bundle, line bundle$\}$ be a property of quasi-coherent sheaves. If G is a quasi-coherent $\mathcal{O}_Y$-module, then G has $\mathcal{P}$ if and only if f^*G does. If X and Y are noetherian, then the same holds for the property of coherence.*

Proof. This reduces to the algebra statement: if $A \to B$ is a faithfully flat ring map, then an A-module M is finitely generated (resp. finitely presented, flat, locally free of finite rank) if and only if $M \otimes_A B$ is. The ($\Rightarrow$) implications are clear. Conversely, if $M \otimes_A B$ is finitely generated, then let $y_1, \ldots, y_m \in M \otimes_A B$ be generators and write $y_i = \sum x_i \otimes b_i$. Since $(x_1, \ldots, x_n)\colon A^n \to M$ base changes to a surjective map, it is surjective. Repeating this argument to the kernel, we see that the property of being finite presentation descends. For flatness, suppose that $M \otimes_A B$ is flat. By faithful flatness, the exactness of $M \otimes_A -$ is equivalent to the exactness of $(M \otimes_A B) \otimes_B (- \otimes_A B)$, which follows from the flatness of $A \to B$ and the flatness of the B-module $M \otimes_A B$. As being locally free of finite rank is equivalent to being finitely presented and flat, the final statement also follows.

See also [SP, Tag 05AY]. □

Proposition B.4.4 (Descending properties of schemes)**.** *Let $X \to Y$ be an fpqc morphism of schemes. Suppose X has one of the following properties: locally noetherian, quasi-compact, noetherian, integral, reduced, normal, and regular. Then Y has the same property.*

Proof. First note that quasi-compactness descends under any surjective map. Therefore, this reduces to the following statements in algebra: if $A \to B$ is a faithfully flat ring map and B is noetherian (resp. a domain, reduced, normal, or regular), then so is A. The map $A \to B$ is injective and $I = IB \cap A$ for every ideal $I \subset A$. For noetherianness, if $I_1 \subset I_2 \subset \cdots$ is an ascending chain of ideals, then since $I_1B \subset I_2B \subset \cdots$ terminates, so does $I_1 = I_1B \cap A \subset I_2 = I_2B \cap A \subset \cdots$. By injectivity of $A \to B$, the 'domain' and 'reduced' cases are clear.

For normality and regularity, we can assume that $A \to B$ is a local ring map. If B is a normal domain and a/b is integral over A where $a, b \in A$, then $a/b \in B$ as B is normal. This implies that a is the ideal of B generated by b, and thus $a\colon B \to B/bB$ is the zero map. As this map is the base change of $a\colon A \to A/bA$, faithful flatness implies that $a\colon A \to A/bA$ is the zero map and thus $a \in (b)$ and $a/b \in A$. For regularity, we will appeal to the fact that a noetherian local ring A of dimension d is regular if and only if every finitely generated A-module M has a resolution $0 \to A^{k_d} \to \cdots \to A^{k_1} \to A^{k_0} \to M \to 0$, moreover if this holds and

$$0 \to K \to \cdots \to A^{k_1} \to A^{k_0} \to M \to 0 \tag{B.4.1}$$

is any exact sequence, then K is free; see [Eis95, Thm. 19.12] and [SP, Tag 00OC]. If M is a finitely generated A-module, choose an exact sequence (B.4.1). Since B is regular, $K \otimes_A B$ is free. By Proposition B.4.3, K is free and thus A is regular.

See also [SP, Tags 033D, 034B and 06QL]. □

Remark B.4.5. For example, if A is a noetherian local ring, then the map $A \to \widehat{A}$ to is its completion is faithfully flat. If the completion $\widehat{A}$ is reduced (resp. normal, regular), then the above result implies that the same holds for A. While the converse holds for regularity, it does not hold in general for reducedness and normality. However, if A is essentially of finite type over a field (or more generally excellent), then A is reduced (resp. normal) if and only if $\widehat{A}$ is, and moreover in this case the normalization commutes with completion. See [SP, Tags 07NZ and 0C23].

The property of being locally noetherian is fppf-local, but the other properties are not. For instance, there are finite type $\Bbbk$-schemes which are non-reduced, non-normal, and non-regular but any such scheme is flat over $\Bbbk$. However, reducedness, normality, and regularity are smooth-local properties.

Proposition B.4.6 (Smooth local properties of schemes)**.** *Let $X \to Y$ be a smooth and surjective morphism of schemes. Let $\mathcal{P}$ be one of the following properties: locally noetherian, reduced, normal, and regularity. Then X has $\mathcal{P}$ if and only if Y has $\mathcal{P}$.*

Proof. The ($\Rightarrow$) implications follows from Proposition B.4.4. If Y is locally noetherian, then so is X by Hilbert's Basis Theorem. The remaining properties follow from the algebraic statement that if $A \to B$ is a smooth ring map and A is reduced (resp. normal, regular), then so is B [SP, Tags 033B, 033C and 036D]. See also [SP, Tag 034D]. □

The property of being a domain is however not smooth local (nor even étale local), e.g. there is a reducible étale neighborhood of the nodal cubic (see Example 0.5.2).

Proposition B.4.7 (Descending ampleness)**.** *Let $f\colon X \to Y$ be a morphism of schemes and L be a line bundle on X. If $Y' \to Y$ is an fpqc morphism of schemes, then L is relatively ample over Y if and only if the pullback of L to $X \times_Y Y'$ is relatively ample over Y'.*

Proof. See [SP, Tag 0D3C]. $\square$

Appendix C

Algebraic groups and actions

C.1 Group schemes and actions

C.1.1 Group schemes

Definition C.1.1. A *group scheme over a scheme* S is a morphism $\pi\colon G \to S$ of schemes together with a multiplication morphism $\mu\colon G \times_S G \to G$, an inverse morphism $\iota\colon G \to G$ and an identity morphism $e\colon S \to G$ (with each morphism over S) such that the following diagrams commute:

$$\begin{tikzcd} G \times_S G \times_S G \arrow[r, "\mathrm{id}_G \times \mu"] \arrow[d, "\mu \times \mathrm{id}_G"'] & G \times_S G \arrow[d, "\mu"] \\ G \times_S G \arrow[r, "\mu"] & G \end{tikzcd}$$

Associativity

$$\begin{tikzcd} G \arrow[r, "(\mathrm{id}_G, \iota)"] \arrow[d, "(\iota, \mathrm{id}_G)"'] \arrow[dr, "e \circ \pi"] & G \times_S G \arrow[d, "\mu"] \\ G \times_S G \arrow[r, "\mu"] & G \end{tikzcd}$$

Law of inverse

$$\begin{tikzcd} G \arrow[r, "(e \circ \pi, \mathrm{id}_G)"] \arrow[d, "(\mathrm{id}_G, e \circ \pi)"'] \arrow[dr, "\mathrm{id}_G"] & G \times_S G \arrow[d, "\mu"] \\ G \times_S G \arrow[r, "\mu"] & G \end{tikzcd}$$

Law of identity

For group schemes H and G over S, a *morphism of group schemes* is a morphism $\phi\colon H \to G$ schemes over S such that $\mu_G \circ (\phi \times \phi) = \phi \circ \mu_H$. A *(closed) subgroup* of G is a (closed) subscheme $H \subset G$ such that $H \to G \xrightarrow{\mu_G} G \times G$ factors through $H \times H$.

Remark C.1.2. If G and S are affine, then by reversing the arrows above gives $\Gamma(G, \mathcal{O}_G)$ the structure of a *Hopf algebra* over $\Gamma(S, \mathcal{O}_S)$.

Exercise C.1.3. Show that a group scheme over S is equivalently defined as a scheme G over S together with a factorization

$$\begin{tikzcd} \mathrm{Sch}/S \arrow[r, dashed] \arrow[dr, "\mathrm{Mor}_S(-, G)"'] & \mathrm{Gps} \arrow[d] \\ & \mathrm{Sets} \end{tikzcd}$$

where Gps $\to$ Sets is the forgetful functor.
(*We are not requiring that there exists a factorization; the factorization is part of the data. Indeed, the same scheme can have multiple structures as a group scheme, e.g.* $\mathbb{Z}/4$ *and* $\mathbb{Z}/2 \times \mathbb{Z}/2$ *over* $\mathbb{C}$.)

Example C.1.4. The following examples of group schemes are the most relevant for us. Let $S = \operatorname{Spec} R$ and V be a free R-module of finite rank:

(1) The *multiplicative group scheme over* R is $\mathbb{G}_{m,R} = \operatorname{Spec} R[t]_t$ with comultiplication $\mu^*\colon R[t]_t \to R[t]_t \otimes_R R[t']_{t'}$ given by $t \mapsto tt'$ while the *group scheme of nth roots of unity* is $\mu_{n,R}\colon \operatorname{Spec} R[t]/(t^n - 1)$ with comultiplication also defined by $t \mapsto tt'$.

(2) The *additive group scheme over* R is $\mathbb{G}_{a,R} = \operatorname{Spec} R[t]$ with comultiplication $\mu^*\colon R[t] \to R[t] \otimes_R R[t']$ given by $t \mapsto t + t'$.

(3) The *general linear group on* V is

$$\mathrm{GL}(V) = \operatorname{Spec}(\operatorname{Sym}^*(\operatorname{End}(V))_{\det})$$

with the comultiplication $\mu^*\colon \operatorname{Sym}^*(\operatorname{End}(V)) \to \operatorname{Sym}^*(\operatorname{End}(V)) \otimes_R \operatorname{Sym}^*(\operatorname{End}(V))$ defined as following: for a basis $v_1, \ldots, v_n$ of V, then for $i, j = 1, \ldots, n$, the endomorphisms $x_{ij}\colon V \to V$ defined by $v_i \mapsto v_j$ and $v_k \mapsto 0$ if $k \neq i$ define a basis of $\operatorname{End}(V)$, and we define $\mu^*(x_{ij}) = x_{i1}x'_{1j} + \cdots + x_{in}x'_{nj}$.

(4) The *special linear group on* V is $\mathrm{SL}(V)$ is the closed subgroup of $\mathrm{GL}(V)$ defined by $\det = 1$.

(5) The *projective linear group* PGL_n is the affine group scheme

$$\operatorname{Proj}(\operatorname{Sym}^*(\operatorname{End}(V)))_{\det}$$

with the comultiplication defined similarly to $\mathrm{GL}(V)$.

We write $\mathrm{GL}_{n,R} = \mathrm{GL}(R^n)$, $\mathrm{SL}_{n,R} = \mathrm{GL}(R^n)$ and $\mathrm{PGL}_{n,R} = \mathrm{PGL}(R^n)$. We often simply write $\mathbb{G}_m$, GL_n, SL_n and PGL_n when the base ring is understood.

Exercise C.1.5.

(a) Provide functorial descriptions of each of the group schemes above.

(b) Show that every abstract group G can be given the structure of a group scheme $\amalg_{g\in G} S$ over a base scheme S. Provide both explicit and functorial descriptions.

Proposition C.1.6.

(1) A group scheme $G \to S$ is separated (resp. quasi-separated) if and only if the identity section $S \to G$ is a closed immersion (resp. quasi-compact).

(2) A group scheme over a field is separated.

(3) A group scheme $G \to S$ of finite type is trivial if and only if the fiber G_s is trivial for each $s \in S$.

Proof. See [SP, Tags 047G and 047J]. For the last fact, observe that $G \to S$ is unramified since every fiber is. Therefore $\Omega_{G/S} = 0$ and the diagonal $G \to G \times_S G$ is an open immersion. It follows that the identity section $S \to G$ is a surjective open immersion, thus an isomorphism. See also [Con07, Thm. 2.2.5]. □

C.1.2 Group actions

Definition C.1.7. Let $G \to S$ be a group scheme with multiplication μ and identity e. An *action* of G on a scheme $X \xrightarrow{p} S$ is a morphism $\sigma\colon G \times_S X \to X$

over S such that the following diagrams commute:

$$
\begin{tikzcd}
G \times_S G \times_S X \arrow[r, "\mathrm{id}_G \times \sigma"] \arrow[d, "\mu \times \mathrm{id}_X"'] & G \times_S X \arrow[d, "\sigma"] \\
G \times_S X \arrow[r, "\sigma"] & G
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "(e \circ p, \mathrm{id}_X)"] \arrow[dr, "\mathrm{id}_X"'] & G \times_S X \arrow[d, "\sigma"] \\
& X
\end{tikzcd}
$$

Compatibility $\qquad$ Law of identity

If $X \to S$ and $Y \to S$ are schemes with actions of $G \to S$, a morphism $f\colon X \to Y$ of schemes over S is *G-equivariant* if $\sigma_Y \circ (\mathrm{id} \times f) = f \circ \sigma_X$, and is *$G$-invariant* if G-equivariant and Y has the trivial G-action.

Exercise C.1.8. Show that giving a group action of $G \to S$ on $X \to S$ is the same as giving an action of the functor $\mathrm{Mor}_S(-, G)\colon \mathrm{Sch}/S \to \mathrm{Gps}$ on the functor $\mathrm{Mor}_S(-, X)\colon \mathrm{Sch}/S \to \mathrm{Sets}$.
(*This requires first spelling out what it means for a functor to groups to act on a functor to sets.*)

C.1.3 Representations

Let $S = \operatorname{Spec} R$ be an affine scheme. Let $G \to S$ be a group scheme with multiplication μ and identity e. A *representation* (or *comodule*) of a group scheme G is an R-module V together with a homomorphism $\sigma\colon V \to \Gamma(G, \mathcal{O}_G) \otimes_R V$ (often referred to as a *coaction*) such that the following diagrams commute:

$$
\begin{tikzcd}
V \arrow[r, "\sigma"] \arrow[d, "\sigma"'] & \Gamma(G, \mathcal{O}_G) \otimes_R V \arrow[d, "\mathrm{id}_G \otimes \sigma"] \\
\Gamma(G, \mathcal{O}_G) \otimes_R V \arrow[r, "\mu^* \otimes \mathrm{id}_V"] & \Gamma(G, \mathcal{O}_G) \otimes_R \Gamma(G, \mathcal{O}_G) \otimes_R V
\end{tikzcd}
\qquad
\begin{tikzcd}
V \arrow[r, "\sigma"] \arrow[dr, "\mathrm{id}_V"'] & \Gamma(G, \mathcal{O}_G) \otimes_R V \arrow[d, "e^* \otimes \mathrm{id}_V"] \\
& V
\end{tikzcd}
\tag{C.1.1}
$$

Compatibility $\qquad$ Law of identity

Example C.1.9. Every R-module V can be viewed as a *trivial representation* with coaction $\sigma(v) = 1 \otimes v$. The comultiplication $\mu^*\colon \Gamma(G, \mathcal{O}_G) \to \Gamma(G, \mathcal{O}_G) \otimes_R \Gamma(G, \mathcal{O}_G)$ defines a representation $(\Gamma(G, \mathcal{O}_G), \mu^*)$ called the *regular representation*. The *standard representation* of $\mathrm{GL}_{n,R} = \operatorname{Spec} R[x_{ij}]_{\det}$ (or a subgroup scheme of $\mathrm{GL}_{n,R}$) is $V = R^n$ with coaction $\sigma\colon V \to \Gamma(\mathrm{GL}_{n,R}) \otimes_R V$ defined by $\sigma(e_i) = \sum_{j=1}^n x_{ij} \otimes e_j$ where $(e_1, \ldots, e_n)$ is the standard basis of V.

A representation V of G induces an action of G on $\mathbb{A}(V) = \operatorname{Spec} \operatorname{Sym}^* V$, which we refer to as a *linear action*. Morphisms of representations and subrepresentations are defined in the obvious way.

Exercise C.1.10. If $R = \Bbbk$ is a field and V is a finite dimensional vector space, show that giving V the structure as a representation is the same as giving a homomorphism $G \to \mathrm{GL}(V)$ of group schemes.

Example C.1.11 (Diagonalizable group schemes)**.** Let $\Bbbk$ be a field and A a finitely generated abelian group. We let $\Bbbk[A]$ be the free $\Bbbk$-module generated by elements of A. The $\Bbbk$-module $\Bbbk[A]$ has the structure of an $\Bbbk$-algebra with multiplication on generators induced from multiplication in A. The comultiplication $\Bbbk[A] \to$

$\Bbbk[A] \otimes_{\Bbbk} \Bbbk[A']$ defined by $a \mapsto a \otimes a'$ defines a group scheme $D_{\Bbbk}(A) = \operatorname{Spec} \Bbbk[A]$ over $\operatorname{Spec} \Bbbk$. A group scheme G over $\operatorname{Spec} \Bbbk$ is *diagonalizable* if $G \cong D_{\Bbbk}(A)$ for some A. (By allowing $\Bbbk$ to be a ring or even a scheme, one obtains the notion of *diagonalizable group scheme*).

The group scheme $D_{\Bbbk}(\mathbb{Z}^r) = \mathbb{G}_{m,\Bbbk}^r$ is the r-dimensional split torus while $D_{\Bbbk}(\mathbb{Z}/n) = \mu_n = \operatorname{Spec} \Bbbk[t]/(t^n - 1)$ is the group of nth roots of unity. The classification of finitely generated abelian groups implies that every diagonalizable group scheme is a product of $\mathbb{G}_m^r \times \mu_{n_1} \times \cdots \times \mu_{n_k}$.

A group scheme $G \to S$ is of *multiplicative type* if it becomes diagonalizable after étale locally on S.

Exercise C.1.12. Describe $D_{\Bbbk}(A)$ as a functor $\mathrm{Sch}/\Bbbk \to \mathrm{Gps}$.

Proposition C.1.13. *Let $G = D_{\Bbbk}(A)$ be a diagonalizable group scheme over a field $\Bbbk$. Every representation of G is a direct sum of one-dimensional representations. In particular, G is linearly reductive.*

Proof. Let $G = D_{\Bbbk}(A)$ and let V be a free representation of G with coaction $\sigma \colon V \to \Bbbk[A] \otimes_{\Bbbk} V$. Each $a \in A$, defines a one-dimensional representation $W_a = A$ of $D_{\Bbbk}(A)$ defined by the coaction $W_a \to \Bbbk[A] \otimes_{\Bbbk} W_a$ defined by $1 \mapsto a \otimes 1$. For $a \in A$, the subspace

$$V_a := \{v \in V \mid \sigma(v) = a \otimes v\}$$

is isomorphic to $W_a \otimes V_a$ as G-representations, where V_a is viewed as the trivial representation (so that in the case V_a is finite dimensional, $V_a \cong W_a^{\dim V_a}$). (Note that when $a = 0$, W_a is the trivial one-dimensional representation and $V^G = V_0$.)

Then $V \cong \oplus_{a \in A} V_a$ as G-representations. The details are left to the reader. See also [Mil17, Thm 12.30] and [$\mathrm{SGA3_I}$, 5.3.3]. □

C.2 Principal G-bundles

The concept of a principal G-bundle is an algebraic formulation of a topological fiber bundle $P \to T$ where G acts freely and transitively on P with quotient $T = P/G$.

C.2.1 Definition and equivalences

Definition C.2.1. Let $G \to S$ be an fppf affine group scheme. A *principal G-bundle over an S-scheme T* is a scheme P with an action of G via $\sigma \colon G \times_S P \to P$ such that $P \to T$ is a G-invariant fppf morphism and

$$(\sigma, p_2) \colon G \times_S P \to P \times_T P, \qquad (g, p) \mapsto (gp, p)$$

is an isomorphism.

Morphisms of principal G-bundles are G-equivariant morphisms of schemes. A principal G-bundle $P \to T$ is *trivial* if there is an G-equivariant isomorphism $P \cong G \times_S T$, where G acts on $G \times_S T$ via multiplication on the first factor.

A principal G-bundle $P \to T$ are examples of *G-torsors* (Definition 6.2.12) over $(\mathrm{Sch}/T)_{\mathrm{fppf}}$ by viewing P as a sheaf in the big fppf topology over T (see Example 6.2.16). In these notes, we will always distinguish between these two notions but in conversation or the literature, they are often conflated.

Exercise C.2.2. Show that $P \to T$ is a principal G-bundle over an S-scheme T if and only if $P \to T$ is a principal $G \times_S T$-bundle.

Exercise C.2.3. Show that a morphism of principal G-bundles is necessarily an isomorphism.

Principal G-bundles as fppf-locally trivial; if G is smooth, they are étale-locally trivial.

Proposition C.2.4. *Let $G \to S$ be an fppf group scheme and $P \to T$ be a G-equivariant morphism of S-schemes where T has the trivial action. Then $P \to T$ is a principal G-bundle if and only if there exists an fppf morphism $T' \to T$ such that $P \times_T T'$ is the trivial principal G-bundle over T'. Moreover, if $G \to S$ is smooth, then we can arrange that $T' \to T$ is surjective and étale.*

Proof. The ($\Rightarrow$) direction follows from the definition by taking $T' = P \to T$. For ($\Leftarrow$), after base changing $G \to S$ by $T \to S$, we assume that G is defined over T (see Exercise C.2.2). Let $G_{T'}$ and $P_{T'}$ be the base changes of G and P along $T' \to T$. The base change of the action map $(\sigma, p_2)\colon G \times_T P \to P \times_T P$ along $T' \to T$ is the action map $G_{T'} \times_{T'} P_{T'} \to P_{T'} \times_{T'} P_{T'}$ of $G_{T'}$ acting on $P_{T'}$ over T'. Since $P_{T'}$ is trivial, this latter action map is an isomorphism. Since the property of being an isomorphism descends under fppf morphisms (Proposition B.4.1), we conclude that $(\sigma, p_2)\colon G \times_T P \to P \times_T P$ is an isomorphism.

If G is smooth, then $T' = P \to S$ is a surjective smooth morphism such that $P_{T'}$ is trivial. Since there is a section of $T' \to S$ after a surjective étale morphism $S' \to S$ (Corollary A.3.6), $P_{S'}$ is also trivial. □

Proposition C.2.5 (Effective Descent for Principal G-bundles)**.** *Let $G \to S$ be an fppf affine group scheme. Let $f\colon X \to Y$ be an fpqc morphism of schemes over S. If $P \to X$ is a principal G-bundle and $\alpha\colon {p_1}^*(P) \xrightarrow{\sim} p_2^*(P)$ is an isomorphism of principal G-bundles over $X \times_Y X$ satisfying $p_{12}^*\alpha \circ p_{23}^*\alpha = p_{13}^*\alpha$, then there exists a principal G-bundle $Q \to Y$ and an isomorphism $\phi\colon P \to f^*(Q)$ of principal G-bundles such that $p_1^*\phi = p_2^*\phi \circ \alpha$.*

Proof. By Effective Descent (Proposition B.3.1) for affine morphisms, there is a scheme Q affine over Y and an isomorphism $\phi\colon P \to f^*(Q)$ of schemes. By applying descent for morphisms (Proposition A.6.4), we can descend the action $G \times_S P \to P$ to an action $G \times_S Q \to Q$ giving Q the structure of a principal G-bundle and making $\phi\colon P \to f^*(Q)$ a G-equivariant isomorphism. □

C.2.2 Examples of principal G-bundles

Exercise C.2.6. Let L/K be a finite Galois extension and $G = \mathrm{Gal}(L/K)$ be its Galois group viewed as a finite group scheme over $\operatorname{Spec} K$. Show that $\operatorname{Spec} L \to \operatorname{Spec} K$ is a principal G-bundle.

Exercise C.2.7. If T is a scheme, show that there there is an equivalence of categories

$$\{\text{line bundles on } T\} \xrightarrow{\sim} \{\text{principal } \mathbb{G}_m\text{-bundles on } T\}$$
$$L \mapsto \mathbb{A}(L) \setminus T$$

between the *groupoids* of line bundles on T (where the only morphisms allowed are isomorphisms) and $\mathbb{G}_m$-torsors on T. If L is a line bundle (i.e. invertible

$\mathcal{O}_T$-module), then $\mathbb{A}(L)$ denotes the total space $\operatorname{Spec}\operatorname{Sym}^* L^\vee$ of L and $T \subset \mathbb{A}(L)$ denotes the image of the zero section $T \to \mathbb{A}(L)$.

Exercise C.2.8. If T is a scheme and $d \geq 1$, show that there there is an equivalence of *groupoids*

$$\begin{aligned} \{\text{finite, étale, and degree } d \text{ covers of } T\ \} &\xrightarrow{\sim} \{\text{principal } S_d\text{-bundles over } T\} \\ (Y \to T) &\mapsto (\underbrace{Y \times_T \cdots \times_T Y}_{d \text{ times}} \setminus \Delta \to T) \\ (P/S_{d-1} \to T) &\leftarrow\!\shortmid (P \to T). \end{aligned}$$

For the rightward map, the symmetric group S_d acts on the d-fold fiber product $Y \times_T \cdots \times_T Y$ by permutation and Δ denotes the S_d-equivariant closed locus of d-tuples where at least two points coincide. Alternatively, $Y \times_T \cdots \times_T Y \setminus \Delta$ can be identified with the scheme $\underline{\mathrm{Isom}}_T(T \times \{1, \ldots, d\}, Y)$ parameterizing isomorphisms of the trivial finite étale cover of degree d and Y. For the leftward map, $S_{d-1} \subset S_d$ denotes the subgroup of permutations fixing the dth index and P/S_{d-1} denotes the quotient scheme of this free action (see Exercise 4.2.8).

Exercise C.2.9.

(a) Show that the standard projection $\mathbb{A}^{n+1} \setminus 0 \to \mathbb{P}^n$ is a principal $\mathbb{G}_m$-bundle.

(b) For each line bundle $\mathcal{O}(d)$ on $\mathbb{P}^n$, explicitly determine the corresponding principal $\mathbb{G}_m$-bundle. In particular, which $\mathcal{O}(d)$ correspond to the principal $\mathbb{G}_m$-bundle of (a)?

Exercise C.2.10. Let $G \to S$ be a smooth affine group scheme. Let $P \to T$ and $Q \to T$ be principal G-bundles. Show that the functor

$$\underline{\mathrm{Isom}}_T(P, Q) \colon \operatorname{Sch}/T \to \operatorname{Sets},$$

assigning a T-scheme T' to the set of isomorphisms of the principal G-bundles $P \times_T T'$ and $Q \times_T T'$, is representable by a scheme which is also a principal G-bundle over T.

Exercise C.2.11 (Principal GL_n-bundles)**.** Let T be a scheme.

(a) If E is a vector bundle over T of rank n, the *frame bundle* $\mathrm{Frame}_T(E)$ is defined as the functor $\underline{\mathrm{Isom}}_T(\mathcal{O}_T^n, E)$ on Sch/T, i.e

$$\begin{aligned} \mathrm{Frame}_T(E) \colon \operatorname{Sch}/T &\to \operatorname{Sets} \\ (T' \to T) &\mapsto \{\text{trivializations } \alpha \colon \mathcal{O}_{T'}^n \xrightarrow{\sim} f^* E\}. \end{aligned}$$

Show that $\mathrm{Frame}_T(E)$ is representable by a scheme and that $\mathrm{Frame}_T(E) \to T$ is a principal GL_n-bundle.

(b) If $P \to T$ is a principal GL_n-bundle, then define $P \times^{\mathrm{GL}_n} \mathbb{A}^n := (P \times \mathbb{A}^n)/\mathrm{GL}_n$ where GL_n acts diagonally via its given action on P and the standard action on $\mathbb{A}^n$. (The action is free and the quotient $(P \times \mathbb{A}^n)/\mathrm{GL}_n$ can be interpreted as the sheafification of the quotient presheaf $\operatorname{Sch}/T \to \operatorname{Sets}$ taking $T \mapsto (P \times \mathbb{A}^n)(T)/\mathrm{GL}_n(T)$ in the big Zariski (or big étale) topology or equivalently as the algebraic space quotient (Corollary 4.4.11)). Show that $(P \times \mathbb{A}^n)/\mathrm{GL}_n$ is representable by scheme and is the total space of a vector bundle over T. *Hint: Use Effective Descent for Principal G-bundles (C.2.5).*

(c) Conclude that

$$\begin{aligned} \{\text{vector bundles over } T\} &\to \{\text{principal } \mathrm{GL}_n\text{-bundles over } T\} \\ E &\mapsto \mathrm{Frame}_T(E) \\ (P \times \mathbb{A}^n)/\,\mathrm{GL}_n &\leftarrow\!\shortmid P \end{aligned}$$

defines an equivalence of categories between the *groupoids* of vector bundles over T and principal GL_n-bundles over T.

Exercise C.2.12 (Principal SL_n-bundles)**.** Show that the groupoid of principal SL_n-bundles over a scheme T is equivalent to the groupoid of pairs (V, α) where V is a vector bundle on T of rank n and $\alpha\colon \mathcal{O}_T \xrightarrow{\sim} \det V$ is a trivialization and a morphism $(V', \alpha') \to (V, \alpha)$ of pairs is an isomorphism $\phi\colon V' \to V$ such that $\alpha' = \alpha \circ \det \phi$.

Exercise C.2.13 (Brauer–Severi schemes)**.** A morphism $X \to T$ of schemes is a *Brauer–Severi scheme of relative dimension* r if there exists an étale cover $T' \to T$ and an isomorphism $X \times_T T' \cong \mathbb{P}^r_{T'}$. An example of a non-trivial Brauer-Severi scheme is $\operatorname{Proj} \mathbb{R}[x, y, z]/(x^2 + y^2 + z^2) \to \operatorname{Spec} \mathbb{R}$. Show that

$$\begin{aligned} \{\text{Brauer–Severi schemes of rel. dim. } r \text{ over } T\} &\to \{\text{principal } \mathrm{PGL}_r\text{-bundles over } T\} \\ X &\mapsto \underline{\mathrm{Isom}}_T(\mathbb{P}^r_T, X) \\ (P \times \mathbb{P}^r)/\,\mathrm{PGL}_r &\leftarrow\!\shortmid P \end{aligned}$$

defines an equivalence of groupoids.

Exercise C.2.14. Let $X \to S$ be a proper, flat, and finitely presented morphism of schemes. Assume that for every geometric point $\operatorname{Spec} \Bbbk \to S$, the geometric fiber $X \times_S \Bbbk$ is isomorphic to $\mathbb{P}^1_{\Bbbk}$. Show that $X \to S$ is a Brauer–Severi scheme of relative dimension 1 following one of the approaches below.

Approach 1 (local-to-global): Show that for every point $s \in S$, there is a finite and separable field extension $\kappa(s) \to K$ such that $X \times_S K \cong \mathbb{P}^1_K$. Show that there an étale neighborhood $(S', s') \to (S, s)$ such that $K \cong \kappa(s')$ over $\kappa(s)$. Assuming now that $X \times_s \kappa(s) \cong \mathbb{P}^1_{\kappa(s)}$, use deformation theory (Proposition D.2.6) to show that there are compatible isomorphisms $X \times_S \mathcal{O}_{S,s}/\mathfrak{m}_s^n \cong \mathbb{P}^1_{\mathcal{O}_{S,s}/\mathfrak{m}_s^n}$ for $n > 0$. Use Grothendieck's Existence Theorem (D.4.4) to show that $X \times_S \widehat{\mathcal{O}}_{S,s} \cong \mathbb{P}^1_{\widehat{\mathcal{O}}_{S,s}}$. Apply Artin Approximation (A.10.9) to show that there is an étale neighborhood $(S', s') \to (S, s)$ such that $X \times_S S' \cong \mathbb{P}^1_{S'}$.

Approach 2 (direct): Assuming that there is a section $\sigma\colon S \to X$ of $\pi\colon X \to S$, show that every point $s \in S$ has an open neighborhood $U \subset S$ such that $X \times_S U \cong \mathbb{P}^1_U$. Letting L be the line bundle on X corresponding to the Cartier divisor σ, use Cohomology and Base Change (A.7.5) to show that $\mathcal{E} := \pi_* \mathcal{L}$ is a rank 2 vector bundle on S, that $\pi^* \mathcal{E} \twoheadrightarrow \mathcal{L}$ is surjective, and that $X \cong \mathbb{P}(\mathcal{E})$ over S. Conclude by choosing an open neighborhood of $s \in S$ where $\mathcal{E}$ is trivial. Returning to the general case, show that there is an effective divisor D associated to $\Omega^\vee_{X/S}$ such that $D \to S$ is étale. Reduce to the case where $X \to S$ has a section by base changing by $D \to S$. See also [Har77, Prop. 25.3 and Exer. 25.2].

Exercise C.2.15 (Azumaya algebras)**.** An *Azumaya algebra of rank* r^2 over a noetherian scheme T is a (possibly non-commutative) associative $\mathcal{O}_T$-algebra

A which is coherent as an $\mathcal{O}_T$-module and such that there is an étale covering $T' \to T$ where $A \otimes_{\mathcal{O}_T} \mathcal{O}_{T'}$ is isomorphic to the matrix algebra $M_r(\mathcal{O}_T)$; see [Mil80, §IV.2]. An Azumaya algebra over a field $\Bbbk$ is a central simple algebra (i.e. a finite dimensional associative $\Bbbk$-algebra which is simple and whose center is $\Bbbk$); the quarternions is an example of a central simple algebra over $\mathbb{R}$.

Let A be an Azumaya algebra over a noetherian scheme T of rank r^2.

(a) Show that the sheaf $P_A := \underline{\mathrm{Isom}}_T(M_r(\mathcal{O}_T), A)$ defines a PGL_n-torsor.

(b) Show that the map $A \mapsto \underline{\mathrm{Isom}}_T(M_r(\mathcal{O}_T), A)$ defines a bijection between Azumaya algebras of rank r^2 and PGL_n-torsors.

The preceding two exercises yield a bijection between Azumaya algebras and Brauer–Severi schemes. Over a field $\Bbbk$, the bijection is defined by assigning a central simple algebra A of rank r^2 to the $\Bbbk$-subscheme $X \subset \mathrm{Gr}(r, A)$ classifying right ideals of A.

Exercise C.2.16 (Orthogonal group)**.** Let $\Bbbk$ be a field with $\mathrm{char}(\Bbbk) \neq 2$, and let V be an n dimensional vector space with a non-degenerate quadratic form q. Let $O(q) \subset \mathrm{GL}(V)$ be the subgroup of invertible matrices preserving the quadratic form. If $q = x_1^2 + \cdots + x_n^2$ is the diagonalized quadratic form, then $O_n = O(q)$ is the set of orthogonal matrices A (i.e. $AA^\top = I$).

Show that there is a bijection between principal $O(q)$-bundles over a $\Bbbk$-scheme T and vector bundles of rank n on T with a non-degenerate quadratic form.

C.3 Algebraic groups

An *algebraic group over a field* $\Bbbk$ is a group scheme G of finite type over $\Bbbk$.

A proper algebraic group over a field $\Bbbk$ is necessarily projective and has a commutative group law. It is called an *abelian variety*; examples include an elliptic curve (E, p). Chevalley's structure theorem (see [Che60] or [Con02]) states that a smooth and connected algebraic group G over a perfect field $\Bbbk$ has a unique smooth, connected and normal subgroup $N \triangleleft G$ such that G/N is an abelian variety.

C.3.1 Affine algebraic groups

An affine algebraic group over a field $\Bbbk$ is often referred to as a *linear algebraic group*.

Algebraic Group Facts C.3.1. Let G be an affine algebraic group over a field $\Bbbk$.

(1) Every representation V of G is a union of its finite dimensional subrepresentations.

(2) There exists a finite dimensional representation V and a closed immersion $G \hookrightarrow \mathrm{GL}(V)$ of group schemes.

(3) If $\mathrm{char}(\Bbbk) = 0$, then G is smooth.

(4) If $\Bbbk$ is perfect, then G is reduced if and only if G is geometrically reduced.

(5) There exists a canonical subgroup scheme $G^0 \subset G$ such that G^0 is the connected component of the identity element. Moreover, G^0 is geometrically irreducible and quasi-compact.

(6) Every algebraic subgroup $H \subset G$ is closed.

(7) If G acts on a finite type $\Bbbk$-scheme U and $u \in U$ is a closed point, the orbit Gu, defined set-theoretically as the image of $G \to U, g \mapsto g \cdot u$, is open in its closure $\overline{Gu}$.

(8) If $H \subset G$ is a subgroup, there is a representation V of G and a $\Bbbk$-point $x \in \mathbb{P}(V)$ whose stabilizer is H. In particular, there is a locally closed immersion $G/H \hookrightarrow \mathbb{P}(V)$, defined by $g \mapsto gx$, and G/H is quasi-projective.

A subgroup $T \subset G$ of an affine algebraic group over a field $\Bbbk$ is called a *torus* if $T_{\overline{\Bbbk}} \cong \mathbb{G}^n_{m,\overline{\Bbbk}}$ and a *maximal torus* if it not contained in a larger torus.

(8) G contains a maximal torus T such that $T_{\Bbbk'} \subset G_{\Bbbk'}$ is a maximal torus for every field extension $\Bbbk \to \Bbbk'$.

(9) If $\Bbbk$ is algebraically closed, all maximal tori are conjugate.

See [Bor91, Hum75, Spr98, Wat79, Mil17] and [SP, Tags 047J and 0BF6].

We will repeatedly use the following simple consequence of C.3.1(1).

Lemma C.3.2. *Let G be an affine algebraic group over a field $\Bbbk$. Let X be an affine scheme of finite type over $\Bbbk$ with an action of G.*

(1) There exists a G-equivariant closed immersion $X \hookrightarrow \mathbb{A}(V)$ where V is a finite dimensional G-representation.

(2) For every G-invariant closed subscheme $Z \subset X$, there exists a G-equivariant morphism $f \colon X \to \mathbb{A}(W)$, where W is a finite dimensional G-representation, such that $f^{-1}(0) = Z$.

Proof. Write $X = \operatorname{Spec} A$ and let $f_1, \ldots, f_n$ be $\Bbbk$-algebra generators. By C.3.1(1) there is a finite dimensional G-invariant subspace $V \subset A$ containing each f_i. The surjection $\operatorname{Sym}^* V \to A$ induces a G-equivariant embedding $X \hookrightarrow \mathbb{A}(V)$. For (2), let $Z = \operatorname{Spec} A/I$ and let $g_1, \ldots, g_m \in I$ be generators. Letting $W \subset I$ be a finite dimensional G-invariant subspace containing each g_i, we see that $f \colon X \to \mathbb{A}(W)$ is a G-equivariant map with $f^{-1}(0) = Z$. □

C.3.2 Line bundles with G-actions

If G is an algebraic group over a field $\Bbbk$ acting on a $\Bbbk$-scheme U via $\sigma \colon G \times U \to U$, then a *line bundle with a G-action* or a *G-linearization* is a line bundle L on U together with an isomorphism $\alpha \colon \sigma^* L \xrightarrow{\sim} p_2^* L$ satisfying the cocycle condition: the diagram

$$\begin{tikzcd}
& (\sigma \circ (\mathrm{id}_G \times \sigma))^* L \arrow[r, "(\mathrm{id}_G \times \sigma)^* \alpha"] & (p_2 \circ (\mathrm{id}_G \times \sigma))^* L \arrow[dr, equal] & \\
(\sigma \circ (\mu \times \mathrm{id}_U))^* L \arrow[ur, equal] \arrow[dr, "(\mu \times \mathrm{id}_U)^* \alpha"'] & & & (\sigma \circ p_{23})^* L \arrow[dl, "p_{23}^* \alpha"] \\
& (p_2 \circ (\mu \times \mathrm{id}_U))^* L \arrow[r, equal] & (p_2 \circ p_{23})^* L &
\end{tikzcd}$$

commutes where $\mu \colon G \times G \to G$ denotes multiplication. When U is projective, a very ample line bundle L with a G-action corresponds to a finite dimensional G-representation $V = \mathrm{H}^0(U, L)$ and a G-equivariant closed immersion $U \hookrightarrow \mathbb{P}(V)$.

Theorem C.3.3 (Sumihiro's Theorem on Linearizations)**.** *Let G be a connected, smooth, and affine algebraic group over an algebraically closed field. Let U be a normal scheme over $\Bbbk$ with an action of G.*

(1) If L is a line bundle on U, then there exists an integer $n > 0$ such that $L^{\otimes n}$ admits a G-action.

(2) If U is a quasi-projective, there exists a locally closed embedding $U \hookrightarrow \mathbb{P}(V)$ where V is a finite dimensional G-representation.

(3) Every point $u \in U$ has a G-invariant quasi-projective open neighborhood.

Proof. For (1), see [Sum74, Thm. 1], [Sum75, Lem. 1.2], and [KKLV89, Prop. 2.4]. Part (2) is a direct consequence of (1). For (3), see [Sum74, Lem. 8] and [Sum75, Thm. 3.8]. □

When G is a torus, then we have the stronger result that U has a G-invariant affine cover.

Theorem C.3.4 (Sumihiro's Theorem on Torus Actions)**.** *Let U be a normal scheme over an algebraically closed field $\Bbbk$ with an action of a torus T. Then any point $u \in U$ has a T-invariant affine open neighborhood.*

Proof. See [Sum74, Cor. 2] and [Sum75, Cor. 3.11]. □

Remark C.3.5. Theorems C.3.3 and C.3.4 can fail if U is not normal, e.g. the plane nodal cubic curve has a $\mathbb{G}_m$-action and no $\mathbb{G}_m$-invariant neighborhood of the origin can be embedded $\mathbb{G}_m$-equivariantly into projective space.

C.3.3 One-parameter subgroups

Let G be a smooth affine algebraic group over an algebraically closed field $\Bbbk$. A *one-parameter subgroup* is a homomorphism $\lambda \colon \mathbb{G}_m \to G$ of algebraic groups (which is not necessarily a subgroup). We define the subgroups:

$$\begin{array}{rll} C_\lambda = & \{g \in G \mid \lambda(t)g = g\lambda(t) \text{ for all } t\} & \text{(centralizer)} \\ P_\lambda = & \{g \in G \mid \lim_{t\to 0} \lambda(t)g\lambda(t)^{-1} \text{ exists}\} & \text{(parabolic)} \\ U_\lambda = & \{g \in G \mid \lim_{t\to 0} \lambda(t)g\lambda(t)^{-1} = 1\} & \text{(unipotent).} \end{array}$$

Functorially, for a $\Bbbk$-algebra R, $C_\lambda(R)$ (resp. $P_\lambda(R)$, $U_\lambda(R)$) consist of elements $g \in G(R)$ such that $\lambda_R = g^{-1}\lambda_R g$ (resp. $\lim_{t\to 0} \lambda(t)g\lambda(t)^{-1}$ exists, $\lim_{t\to 0} \lambda(t)g\lambda(t)^{-1} = 1$). The one-parameter subgroup λ induces a $\mathbb{G}_m$ action on G via conjugation: $t \cdot g := \lambda(t)g\lambda(t)^{-1}$. Under this action, C_λ is the fixed locus while P_λ is the attractor locus G_λ^+ as defined in §6.6.9.

The subgroup C_λ is the centralizer of λ. When G is reductive, the subgroup P_λ is parabolic (i.e. G/P_λ is projective) and U_λ is the unipotent radical of P_λ. There is a homomorphism $P_\lambda \to C_\lambda$ defined by $g \mapsto \lim_{t\to 0} \lambda(t)g\lambda(t)^{-1}$ which is the identity on C_λ yielding a split short exact sequence

$$1 \to U_\lambda \to P_\lambda \to C_\lambda \to 1.$$

Example C.3.6. Let $\lambda \colon \mathbb{G}_m \to \mathrm{GL}_n$ be a one-parameter subgroup. After a change of basis, we can assume that $\lambda(t) = \mathrm{diag}(t^{\lambda_1}, \cdots, t^{\lambda_n})$ with $\lambda_1 \leq \cdots \leq \lambda_n$. Given $(g_{ij}) \in \mathrm{GL}_n$, one has that

$$\lambda(t)\big(g_{ij}\big)\lambda(t)^{-1} = \big(t^{\lambda_i - \lambda_j} g_{ij}\big).$$

If $n_1, \ldots, n_s$ are integers with $\sum_i n_i = n$ such that

$$\lambda_1 = \cdots = \lambda_{n_1} < \lambda_{n_1+1} = \cdots = \lambda_{n_1+n_2} < \cdots < \lambda_{n-n_s+1} = \cdots = \lambda_n,$$

then $C_\lambda = \mathrm{GL}_{n_1} \times \cdots \times \mathrm{GL}_{n_s}$ is the subgroup of block diagonal matrices while P_λ is the subgroup of block upper triangular matrices.

For example, if $\lambda(t) = (t^{-1}, t^2, t^2, t^7)$, then

$$U_\lambda = \begin{pmatrix} 1 & * & * & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad P_\lambda = \begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & 0 & 0 & * \end{pmatrix}, \quad C_\lambda = \begin{pmatrix} * & 0 & 0 & 0 \\ 0 & * & * & 0 \\ 0 & * & * & 0 \\ 0 & 0 & 0 & * \end{pmatrix}.$$

We record the following properties of parabolic subgroups.

Proposition C.3.7. *Let G be a reductive algebraic group over an algebraically closed field $\Bbbk$.*

(a) If $\lambda, \lambda' \colon \mathbb{G}_m \to G$ are one-parameter subgroups, the intersection $P_\lambda \cap P_{\lambda'}$ of two parabolic subgroups contains a maximal torus of G.

(b) The unipotent radical U_λ acts freely and transitively on the set of one-parameter subgroups of P_λ which are conjugate (under P_λ) to λ.

(c) For a one-parameter subgroup $\lambda \colon \mathbb{G}_m \to G$, $N_G(P_\lambda) = P_\lambda$.

C.4 Reductivity

We denote by $\mathrm{Rep}(G)$ the category of representations of an algebraic group G. If V is a G-representation with coaction $\sigma \colon V \to \Gamma(G, \mathcal{O}_G) \otimes V$, then the *invariants* are $V^G := \{v \in V \mid \sigma(v) = 1 \otimes v\}$. A representation V of G is *irreducible* if every subrepresentation $W \subset V$ is either 0 or V.

C.4.1 Linear reductive groups

There are various notions of reductivity but the one most central to this book is linear reductivity.

Definition C.4.1. An affine algebraic group G over a field $\Bbbk$ is *linearly reductive* if the functor $\mathrm{Rep}(G) \to \mathrm{Vect}_\Bbbk$, taking a G-representation V to its G-invariants V^G, is exact.

Proposition C.4.2. *Let G be an affine algebraic group over a field $\Bbbk$. The following are equivalent:*

(1) G is linearly reductive;

(1′) The functor $\mathrm{Rep}^{\mathrm{fd}}(G) \to \mathrm{Vect}_\Bbbk$, $V \mapsto V^G$, on the category of finite dimensional representations is exact;

(2) Every G-representation (resp. finite dimensional G-representation) is a direct sum of irreducible representations.

(3) Given a G-representation (resp. finite dimensional G-representation) V and a G-invariant subspace $W \subset V$, there exists a G-invariant subspace $W' \subset V$ such that $V = W \oplus W'$.

(4) For every finite dimensional representation V and fixed $\Bbbk$-point $x \in \mathbb{P}(V)^G$, there exists a G-invariant linear function $f \in \Gamma(\mathbb{P}(V), \mathcal{O}(1))^G$ such that $f(x) \neq 0$.

Proof. TO ADD □

Remark C.4.3. In the notation introduced in §6.3, G is linearly reductive if and only if $\mathbf{B}G \to \operatorname{Spec} k$ is cohomologically affine or equivalently a good moduli space.

It is not hard to see that for a field extension $\Bbbk \to \Bbbk'$, G is linearly reductive if and only if $G_{\Bbbk'}$ is and that linearly reductive groups are closed under extension Linearly reductive algebraic group are also closed under extension. See Lemma 6.3.15 and Proposition 6.3.17.

Proposition C.4.4 (Maschke's Theorem)**.** *Let G be a finite group whose order is prime to* $\operatorname{char}(\Bbbk)$*. Then G is linearly reductive.*

Proof. If V is a G-representation, averaging over translates gives a G-equivariant $\Bbbk$-linear

$$R_V : V \to V^G, \qquad v \mapsto \frac{1}{|G|} \sum_{g \in G} g \cdot v, \tag{C.4.1}$$

which is on the identity on V^G and compatible with a map $f : V \to W$ of G-representations, i.e. $R_W \circ f = f \circ R_V$. It follows that a surjection $V \to W$ of G-representations induces a surjection $V^G \to W^G$ on invariants. □

Example C.4.5. In characteristic p, there is a 2-dimensional representation V of $G = \mathbb{Z}/p$ where a generator acts via the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. The surjection $V \to \Bbbk$ onto the first component is a surjection of G-representations. The induced map $V^G \to \Bbbk$ on invariants is the zero-map. Note however that the element $e_1^p \in \operatorname{Sym}^p V$ is G-invariant and maps to 1. Geometrically, this gives an action of G on $\mathbb{A}^2 = \mathbb{A}(V)$ where $(1, 0)$ is G-invariant; the invariant hypersurface x^p doesn't contain p but there is no such hyperplane.

Remark C.4.6 (Reynolds operator)**.** The map (C.4.1) is called a *Reynolds operator* for the action of G on V. If G is linearly reductive, the canonical projections $R_V : V \to V^G$ are Reynold operators, i.e. $\Bbbk$-linear maps which are the identity on V^G and compatible with maps of G-representations. For an action of G on a $\Bbbk$-scheme $\operatorname{Spec} A$ with dual action $A \to \Gamma(G, \mathcal{O}_G) \otimes A$, there is a projection $R_A : A \to A^G$. This is not a ring map but since multiplication $A^G \otimes A \to A$ is map of G-representations commuting with the Reynold operators, we have that

$$R_A(xy) = x R_A(y) \quad \text{for } x \in A^G,\ y \in A.$$

This is called the *Reynolds identity* and shows that $A \to A^G$ is an A^G-algebra homomorphism.

In Remark 6.3.9, the Reynolds operator was applied to show that A^G is finitely generated whenever A is. It can also be used to show that $\operatorname{Spec} A \to \operatorname{Spec} A^G$ separates G-orbits and has the properties of affine GIT quotients (Corollary 6.3.7); see [GIT, §1.2]. As we will see in the proof of Theorem C.4.7, one technique to prove that a given group is linearly reductive is to construct Reynolds operators.

C.4.2 Reductive groups

An affine algebraic group G over an algebraically closed field $\Bbbk$ is called *reductive* if every smooth, connected, unipotent and normal subgroup of G is trivial. [1] Over $\mathbb{C}$, an affine algebraic group is reductive if and only if it is the complexification of any maximal compact subgroup [Hoc65, XVII.5] Reductive groups are a particular nice class of algebraic groups appearing in many branches of mathematics and can be completely classified in terms of their root datum.

For a smooth affine algebraic group G, there are subgroup $R(G)$ and $R_u(G)$ of G, called the *radical* and *unipotent radical*, which are maximal among connected, normal and solvable (resp. connected normal unipotent) subgroups. Over an algebraically closed field $\Bbbk$, G is reductive if $R_u(G)$ is trivial and called *semisimple* if $R(G)$ is trivial. For a reductive group, the center $Z(G)$ of a reductive group G is diagonalizable containing the radical $R(G) \subset Z(G)$ as its largest subtorus, and the quotient $G/R(G)$ is semisimple. For a smooth affine algebraic group G, the quotient $G/R_u(G)$ is reductive. Over an arbitrary field $\Bbbk$, G is called reductive if $G_{\overline{\Bbbk}}$ is. The unipotent radical $R_u(G)$ commutes with separable field extensions and so over a perfect field $\Bbbk$ G is reductive if and only if $R_u(G)$ is trivial. See [Bor91, Hum75, Spr98, Mil17].

The classical algebraic groups of GL_n, PGL_n, SL_n, or SP_{2n} are reductive in every characteristic. We develop GIT in this book for actions by linearly reductive groups and it is therefore convenient to know that these classical groups are linearly reductive in characteristic 0.

Theorem C.4.7. *In characteristic* 0*, a reductive algebraic group is linearly reductive. The converse is true in every characteristic for smooth algebraic groups.*

Proof. In [Hil90], Hilbert established the linearly reductivity for $\mathrm{SL}_{n,\mathbb{C}}$ and $\mathrm{GL}_{n,\mathbb{C}}$ using a explicit differential operator well-known to 19th century invariant theorists: the Ω*-process*. Write $\Gamma(\mathrm{GL}_{n,\mathbb{C}}, \mathcal{O}_{\mathrm{GL}_{n,\mathbb{C}}}) = \mathbb{C}[X_{ij}]_{\det}$. Let V be a finite dimensional G-representation such that the diagonal matrices act with weight k, and let $\sigma\colon \mathbb{C}[V] \to \mathbb{C}[X_{ij}]_{\det} \otimes \mathbb{C}[V]$ be the dual action on the coordinate ring $\mathbb{C}[V]$ of $\mathbb{A}(V^\vee)$. The differential operator

$$\Omega := \det\big(\frac{\partial}{\partial X_{ij}}\big)$$

acts linearly on $\mathbb{C}[X_{ij}]_{\det}$ and also on $\mathbb{C}[X_{ij}]_{\det} \otimes \mathbb{C}[V]$. One checks that the map

$$V \to V^{\mathrm{GL}_n}, \quad f \mapsto \frac{1}{\Omega^k\big(\det(X_{ij})^k\big)}\Omega^k\big(\det(X_{ij})^k\sigma(f)\big)$$

defines a Reynolds operator. As with the averaging operator (C.4.1) in Maschke's Theorem (C.4.4), this shows that GL_n is linearly reductive and a variant of the argument shows that SL_n is also linearly reductive. The argument is algebraic and works over every field of characteristic 0. See also [Stu08, §4.3], [Dol03, §2.1] and [DK15, §4.5.3].

Extending an integral procedure developed by Hurwitz and Schur and ideas of Cartan, Weyl [Wey26, Wey25] showed that every reductive algebraic group over $\mathbb{C}$

[1] Sometimes G is also assumed to be connected. For a *reductive group scheme* $G \to S$, there is no such ambiguity in the literature: G is smooth and affine over S with connected and reductive geometric fibers [SGA3$_{\mathrm{III}}$, Exp. XIX, Defn. 2.7].

is linearly reductive. The technique is now referred to as 'Weyl's unitarian trick'. A Lie group G has a measure μ, called the *left Haar measure*. When G is compact, this measure is finite, and for a finite dimensional G-representation V, averaging gives $\Bbbk$-linear map

$$V \to V^G, \quad v \mapsto \frac{1}{\int_G d\mu(g)} \int_G (g \cdot v) d\mu(g)$$

constant on V^G and compatible with maps of G-representations. This is a Reynolds operator (Remark C.4.6) exactly as the averaging map in Maschke's Theorem (C.4.4) and implies that $V \mapsto V^G$ is exact. For every reductive algebraic group G over $\mathbb{C}$, there is a real Lie subgroup $K \subset G(\mathbb{C})$ which is dense in the Zariski topology and compact in the Euclidean topology; for $\mathrm{GL}_{n,\mathbb{C}}$, $K = U_n$ is the subgroup of unitary matrices (hence the name 'unitarian trick'). Then for a finite dimensional G-representation V, there is an identification $V^K = V^G$ and since the functor taking K-invariant is exact, so is the functor taking G-invariants. See also [Dol03, §3.2], [Bum13, Thm. 14.3].[2]

There is also an algebraic argument using the *Casimir operator*. First, one reduces to the case that G is semisimple because every reductive group is an extension of a torus by a semisimple group. Given a representation $\rho\colon \mathfrak{g} \to V$ of the Lie algebra, there is a symmetric bilinear form on $\mathfrak{g}$ defined by $\langle x, y \rangle = \mathrm{Tr}(\rho(x) \circ \rho(y))$. Letting $\{e_i\}$ be a basis of $\mathfrak{g}$ and $\{e_i'\}$ be a dual basis with respect to $\langle -, - \rangle$, the *Casimir operator* is the $\mathfrak{g}$-endomorphism $c_V := \sum_{i=1}^n \rho(e_i) \circ \rho(e_i')$ on V. To show that G is linearly reductive, it suffices to find a complement of any codimension 1 irreducible subspace $W \subset V$. As G is semisimple, G acts trivial on V/W and therefore so does $\mathfrak{g}$. It follows that $\mathfrak{g}$ takes V into W and therefore so does c_V, i.e. $c_V(V) \subset W$. On the other hand, since W is irreducible, c_V acts on W by multiplication by a scalar (Schur's lemma). It follows that $\ker(c_V) \subset V$ is a complement of W. See also [Mil17, Thm. 22.42], [Muk03, §4.3], [Hum78, §6.2] and [DK15, §4.5.2].

For the converse, we need to show that for a linearly reductive group G, the unipotent radical $R_u(G)$ is trivial. Since $G/R_u(G)$ is affine, Matsushima's Theorem (6.3.19) implies that $R_u(G)$ is linearly reductive. However, a non-trivial unipotent group is not linearly reductive. Indeed, it suffices to show this for $\mathbb{G}_a$. Let $V = \Bbbk^2$ be the two-dimensional representation given by $t \mapsto \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$. The projection $V \to \Bbbk, (x, y) \mapsto x$ is a surjection of $\mathbb{G}_a$-representations with no complement, i.e. there is no invariant $(x, y) \in V^{\mathbb{G}_a}$ with $x \neq 0$ (in fact there is no invariant $f \in (\mathrm{Sym}^d V)^{\mathbb{G}_a}$ with $d > 0$ and $f(1, 0)$ nonzero. See also [NM64]. $\square$

Example C.4.8. The algebraic groups such as GL_n, PGL_n, SL_n, or SP_{2n} are not linearly reductive in characteristic p. For example, in characteristic 2, consider the action of SL_2 acts on the space $V = \mathrm{Sym}^2(\Bbbk^\vee) = \{Ax^2 + Bxy + Cy^2\}$ of degree 2 binary forms. The subspace W consisting of squares L^2 of linear forms is a GL_2-invariant subspace with no complement; the quotient $V \to V/W = \mathbb{C}$ is

[2] This analytic argument suffices to show the linear reductivity of a reductive group G over every characteristic 0 field by the limit methods of §A.6 (or by using the classification theorem of reductive groups): there is a subfield $\Bbbk' \subset \Bbbk$ of finite transcendence degree over $\mathbb{Q}$ and a group scheme $G' \to \operatorname{Spec} \Bbbk'$ such that $G'_{\Bbbk} = G$. Choosing an embedding $\Bbbk' \hookrightarrow \mathbb{C}$, and using the fact that both the notions of reductivity and linear reductivity are insensitive to separable field extensions, we see that the linear reductivity of $G'_{\mathbb{C}}$ implies the linear reductivity of G.

given by $(A, B, C) \mapsto B$. While there is no invariant *linear* function not vanishing at $(0, 1, 0)$, the discriminant $\Delta = B^2 \in \operatorname{Sym}^2 V^\vee$ is an invariant function nonzero at $(0, 1, 0)$ (verifying the geometric reductivity condition given below).

Theorem C.4.9 (Matsushima's Theorem)**.** *Let G be a reductive group over a field $\Bbbk$. Then a subgroup $H \subset G$ is reductive if and only if G/H is affine.*

Proof. See Proposition 6.3.19 for a proof when G is linearly reductive. The general case can be proven in a similar way relying on the a generalization of Serre's Criterion for Affineness: an algebraic space U, satisfying the property that for a surjection $\mathcal{A} \to \mathcal{B}$ of $\mathcal{O}_U$-algebras every global section of $\mathcal{B}$ has a positive power that lifts, is affine. See also [Mat60], [BB63], [Ric77], [FS82], [Alp13, Thm. 12.5] and [Alp14, Thm 9.4.1]. □

C.4.3 Geometrically reductive groups

An affine algebraic group G is called *geometrically reductive* (or sometimes called *semi-reductive*) if for every surjection $V \to W$ of G-representations and $w \in W^G$, there exists $n > 0$ such that w^{p^n} is in the image of $\operatorname{Sym}^{p^n} V \to \operatorname{Sym}^{p^n} W$. It suffices to take V finite dimensional and W the trivial representation in which case the condition translates to a geometric property that for a fixed $\Bbbk$-point $x \in \mathbb{P}(V)^G$, there is an invariant homogenous polynomial $f \in \Gamma(\mathbb{P}(V), \mathcal{O}(p^n))^G$ for $n > 0$ with $f(x) \neq 0$ (analogous to Proposition C.4.2(4) except that f need not be linear).

Geometrically reductive groups appear in the context of GIT (§6.6) as their defining property can be used to show that the quotient morphisms $\operatorname{Spec} A \to \operatorname{Spec} A^G$ have desirable properties (e.g. separates closed orbits and A^G is finitely generated). In an effort to extend GIT to actions by reductive groups such as SL_n and GL_n in positive characteristic, Mumford conjectured in [GIT, preface] a reductive group is geometrically reductive. This conjecture was resolved by Haboush [Hab75]; see also [SS11], [Ses69] and [Oda64]. Conversely, a smooth geometrically reductive group is reductive. In fact, an affine algebraic group G is geometrically reductive if and only if G_{red} is reductive.

On the other hand, a linearly reductive group is clearly geometrically reductive. The converse is true in characteristic 0 [Alp14, Lem. 9.2.8].

We thus have the implications:

$$\begin{tikzcd}
\text{linearly reductive} \arrow[r, Rightarrow] & \text{geometrically reductive} \arrow[l, Rightarrow, bend right, "\text{char}=0"'] \arrow[r, Rightarrow, bend left, "G \text{ smooth}"] & \text{reductive.} \arrow[l, Rightarrow]
\end{tikzcd}$$

A smooth algebraic group G in characteristic p is linearly reductive if and only if the connected component G^0 is a torus and the order of G/G^0 is prime to p [Nag62]. Every finite (possibly non-reduced) group scheme G is geometrically reductive but is linearly reductive if and only if G^0 is diagonalizable and G/G^0 has order prime to p [HR15, Thm. 1.2]. A commutative algebraic group G is reductive if and only if its diagonalizable.

We also point out that a smooth algebraic group G satisfying the property that A^G is finitely generated for every coaction on a finitely generated $\Bbbk$-algebra, then G is necessarily reductive.

Appendix D

Deformation Theory

Deformation theory is the study of the local geometry of a moduli space $\mathcal{M}$ near an object $E_0 \in \mathcal{M}(\Bbbk)$. We focus primarily on the following three deformation problems:

(A) Embedded deformations $Z_0 \subset X$ of a closed subscheme Z_0 in a fixed projective scheme X over $\Bbbk$. Here the moduli problem is the Hilbert functor $\mathrm{Hilb}^P(X)$ and $E_0 = [Z_0 \subset X] \in \mathrm{Hilb}^P(X)(\Bbbk)$.

(B) Deformations of a scheme E_0 over $\Bbbk$. In this section, the main example for us is when E_0 is a smooth curve in which case the moduli problem is $\mathscr{M}_g$ and $[E_0] \in \mathscr{M}_g(\Bbbk)$.

(C) Deformations of a coherent sheaf E_0 on a fixed projective scheme C over $\Bbbk$. The main example for us is when C is a smooth curve and E_0 is a vector bundle in which case the moduli problem is $\mathcal{B}\mathrm{un}_C$ and $[E_0] \in \mathcal{B}\mathrm{un}_C(\Bbbk)$.

In this chapter, we sketch the local-to-global approach of deformation theory by zooming in around $E_0 \in \mathcal{M}(\Bbbk)$ and studying successively first order neighborhoods of $\mathcal{M}$ at E_0, higher order deformations of E_0, formal neighborhoods of E_0 and eventually étale or smooth neighborhoods of E_0.

(1) A *first order deformation* of E_0 is an object $E \in \mathcal{M}(\Bbbk[\epsilon])$ over the dual numbers $\Bbbk[\epsilon] := \Bbbk[\epsilon]/(\epsilon^2)$ together with an isomorphism $\alpha \colon E_0 \to E|_{\operatorname{Spec} \Bbbk}$, or in other words a commutative diagram

$$\begin{tikzcd} \operatorname{Spec} \Bbbk \arrow[r, hook] \arrow[dr, "{[E_0]}"'] & \operatorname{Spec} \Bbbk[\epsilon] \arrow[d, dashed, "{[E]}"] \\ & \mathcal{M} \end{tikzcd}$$

allowing us to view E as a tangent vector of $\mathcal{M}$ at E_0. We classify first order deformations of Problems (A)–(C) in §D.1.

(2) Given a surjection $A' \twoheadrightarrow A$ of artinian local $\Bbbk$-algebras with residue field $\Bbbk$ and an object object $E \in \mathcal{M}(A)$ with an isomorphism $E_0 \to E|_{\operatorname{Spec} \Bbbk}$, a *deformation of E over A'* is an object $E' \in \mathcal{M}(A')$ with an isomorphism

$\alpha\colon E \to E'|_{\operatorname{Spec} A}$. Pictorially, this corresponds to a commutative diagram

$$\begin{tikzcd} \operatorname{Spec} \Bbbk \arrow[r, hook] \arrow[drr, "{[E_0]}"'] & \operatorname{Spec} A \arrow[r, hook] \arrow[dr, "{[E]}"] & \operatorname{Spec} A' \arrow[d, dashed, "{[E']}"] \\ & & \mathcal{M}. \end{tikzcd}$$

For Problems (A)–(C), we determine when a deformation E' of E over A' exists and we classify them in §D.2

(3) Given a noetherian complete local $\Bbbk$-algebra $(R, \mathfrak{m})$, a *formal deformation of* E_0 *over* R is a compatible collection of deformations $E_n \in \mathcal{M}(R/\mathfrak{m}^{n+1})$ of E_0, and a formal deformation $\{E_n\}$ is *versal* if every other deformation factors through one of the E_n (see Definition D.3.5 for a precise definition). Rim–Schlessinger's Criteria (Theorem D.3.11) provides criteria for the existence of a versal deformation $\{E_n\}$ of E_0, and in §D.3 we verify the criteria for the Problems (A)–(C).

(4) A formal deformation $\{E_n\}$ over $(R, \mathfrak{m})$ is *effective* if there exists an object $\widehat{E} \in \mathcal{M}(R)$ extending the $\{E_n\}$, or in other words there exists a commutative diagram

$$\begin{tikzcd} \operatorname{Spec} R/\mathfrak{m} \arrow[r, hook] \arrow[drrrr, "{[E_0]}"'] & \operatorname{Spec} R/\mathfrak{m}^2 \arrow[r, hook] \arrow[drrr, "{[E_1]}"] & \operatorname{Spec} R/\mathfrak{m}^3 \arrow[r, hook] \arrow[drr, "{[E_2]}"] & \cdots \arrow[r, hook] & \operatorname{Spec} R \arrow[d, dashed, "{[\widehat{E}]}"] \\ & & & & \mathcal{M}. \end{tikzcd}$$

In §D.4, we show how Grothendieck's Existence Theorem (D.4.4) implies that formal deformations are effective for Problems (A)–(C).

(5) In §D.5, we take a detour from the local-to-global approach to provide a glimpse into the role of the cotangent complex in deformation theory.

(6) Given an effective versal formal deformation $\widehat{E}$ over R, Artin Algebraization (Theorem D.6.6) ensures the existence of a *finite type* $\Bbbk$-scheme U with a point $u \in U(\Bbbk)$ and an object $E \in \mathcal{M}(U)$ such that $R \cong \widehat{\mathcal{O}}_{U,u}$ and $\widehat{E}|_{\operatorname{Spec} R/\mathfrak{m}^{n+1}} \cong E|_{\operatorname{Spec} R/\mathfrak{m}^{n+1}}$ for all n.

(7) Artin's Axioms for Algebraicity (Theorems D.7.1 and D.7.4) provides criteria to verify the algebraicity of a moduli problem $\mathcal{M}$. Namely, it provides conditions to ensure that the morphism $[E]\colon U \to \mathcal{M}$ constructed above is a smooth morphism in an open neighborhood of E_0.

In this chapter, $\Bbbk$ will denote an algebraically closed field. In §D.3, §D.6 and §D.7, we work over the category of $\Bbbk$-schemes for convenience but the results hold more generally.

D.1 First order deformations

Denote the *dual numbers* by $\Bbbk[\epsilon] := \Bbbk[\epsilon]/(\epsilon^2)$.

D.1.1 First order embedded deformations

Definition D.1.1. Let X be a projective scheme over a $\Bbbk$ and $Z_0 \subset X$ be a closed subscheme. A *first order deformation of* $Z_0 \subset X$ is a closed subscheme

$Z \subset X_{\Bbbk[\epsilon]} := X \times_{\Bbbk} \Bbbk[\epsilon]$ flat over $\Bbbk[\epsilon]$ such that $Z_0 = Z \times_{\Bbbk[\epsilon]} \Bbbk$. Pictorially, a first order deformation is a filling of the diagram

$$\begin{tikzcd}
 & X \arrow[rr, hook] \arrow[dd] & & X_{\Bbbk[\epsilon]} \arrow[dd] \\
Z_0 \arrow[ur, hook, "\mathrm{cl}"] \arrow[rr, hook, dashed] \arrow[dr] & & Z \arrow[ur, hook, dashed, "\mathrm{cl}"] \arrow[dr, dashed, "\mathrm{flat}"] & \\
 & \operatorname{Spec} \Bbbk \arrow[rr, hook] & & \operatorname{Spec} \Bbbk[\epsilon].
\end{tikzcd}$$

with a scheme Z and dotted arrows making the diagram cartesian.

We say that $Z \subset X_{\Bbbk[\epsilon]}$ is *trivial* if $Z = Z_0 \times_{\Bbbk} \Bbbk[\epsilon]$.

Remark D.1.2. Since Z_0 and the central fiber $Z \times_{\Bbbk[\epsilon]} \Bbbk$ of Z are embedded in X, it makes sense to require that they are equal.

Remark D.1.3. The closed subscheme $Z_0 \subset X$ defines a $\Bbbk$-point $[Z_0 \subset X] \in \operatorname{Hilb}^P(X)$ of the Hilbert scheme where P is the Hilbert polynomial of Z_0 with respect to a fixed ample line bundle on X. A first order deformation corresponds to a commutative diagram

$$\begin{tikzcd}
\operatorname{Spec} \Bbbk \arrow[r, "{[Z_0 \subset X]}"] \arrow[d, hook] & \operatorname{Hilb}^P(X) \\
\operatorname{Spec} \Bbbk[\epsilon], \arrow[ur, "{[Z \subset X_{\Bbbk[\epsilon]}]}"'] &
\end{tikzcd}$$

or in other words a tangent vector $[Z \subset X_{\Bbbk[\epsilon]}] \in T_{\operatorname{Hilb}^P(X),[Z_0 \subset X]}$.

Proposition D.1.4. *Let X be a projective scheme over a $\Bbbk$ and $Z_0 \subset X$ be a closed subscheme defined by a sheaf of ideals $I_0 \subset \mathcal{O}_X$. There is a bijection*

$$\{\textit{first order deformations } Z \subset X_{\Bbbk[\epsilon]}\} \cong \mathrm{H}^0(Z_0, N_{Z_0/X})$$

where $N_{Z_0/X} = \mathscr{H}om_{\mathcal{O}_{Z_0}}(I_0/I_0^2, \mathcal{O}_{Z_0})$ is the normal sheaf. Under this correspondence, the trivial deformation corresponds to $0 \in \mathrm{H}^0(Z_0, N_{Z_0/X})$.

Remark D.1.5. In light of Remark D.1.3, this proposition gives a bijection $T_{\operatorname{Hilb}^P(X),[Z_0 \subset X]} \cong \mathrm{H}^0(Z_0, N_{Z_0/X})$.

Proof. We sketch the bijection and point the reader to [Har10, Prop. 2.3] and [Ser06, Prop. 3.2.1] for details. After reducing to the affine case $X = \operatorname{Spec} B$ and $Z_0 = \operatorname{Spec} B/I_0$, we need to show that the set of first order deformations is bijective to

$$\mathrm{H}^0(Z_0, N_{Z_0/X}) \cong \operatorname{Hom}_{B/I_0}(I_0/I_0^2, B/I_0) \cong \operatorname{Hom}_B(I_0, B/I_0).$$

Given a first order deformation $Z = \operatorname{Spec} B[\epsilon]/I$, the flatness of Z over $\Bbbk[\epsilon]$ ensures that tensoring the exact sequence $0 \to I \to B[\epsilon] \to B[\epsilon]/I \to 0$ of $\Bbbk[\epsilon]$-modules with $\Bbbk = \Bbbk[\epsilon]/(\epsilon)$ yields an exact sequence $0 \to I_0 \to B \to B/I_0 \to 0$. We define $\alpha\colon I_0 \to B/I_0$ as follows: for $x_0 \in I_0$, choose a preimage $x = a + b\epsilon \in I$ and set

$\alpha(x_0) := \bar{b} \in B/I_0$. Conversely, given a B-module homomorphism $\alpha\colon I_0 \to B/I_0$, we define

$$I = \{a + b\epsilon \mid a \in I_0, b \in B \text{ such that } \bar{b} = \alpha(a) \in B/I_0\} \subset B[\epsilon].$$

Then $(B[\epsilon]/I) \otimes_{\Bbbk[\epsilon]} \Bbbk = B/I_0$. To see that $B[\epsilon]/I$ is flat over $\Bbbk[\epsilon]$, we need to check that the map $B/I_0 \xrightarrow{\epsilon} B[\epsilon]/I$ is injective (see Remark A.2.7): given $b \in B$ with $\epsilon b \in I$, then $b \in I_0$ by the definition of I. Thus $Z = \operatorname{Spec} B[\epsilon]/I$ defines a first order deformation of Z_0. □

D.1.2 Locally trivial first order deformations of schemes

Definition D.1.6. Let X_0 be a scheme over a $\Bbbk$. A *first order deformation of* X_0 is a scheme X flat over $\Bbbk[\epsilon]$ together with an isomorphism $\alpha\colon X_0 \to X \times_{\Bbbk[\epsilon]} \Bbbk$, or in other words a cartesian diagram

$$\begin{tikzcd} X_0 \arrow[r, hook, dashed] \arrow[d] \arrow[dr, phantom, "\square"] & X \arrow[d, dashed, "\text{flat}"] \\ \operatorname{Spec} \Bbbk \arrow[r, hook] & \operatorname{Spec} \Bbbk[\epsilon]. \end{tikzcd} \tag{D.1.1}$$

A *morphism of first order deformations* (X, α) *and* (X', α') is a morphism $\beta\colon X \to X'$ of schemes over $\Bbbk[\epsilon]$ such that $(\beta \times_{\Bbbk[\epsilon]} \Bbbk) \circ \alpha = \alpha'$, or in other words considering X and X' in cartesian diagrams (D.1.1), we require the restriction of β to central fiber X_0 to be the identity.

We say that X is *trivial* if X is isomorphic as first order deformations to $X \times_{\Bbbk} \Bbbk[\epsilon]$, and *locally trivial* if there exists a Zariski-cover $X = \bigcup_i U_i$ such that U_i is a trivial first order deformation of $U_i \times_{\Bbbk[\epsilon]} \Bbbk \subset X_0$.

Everymorphism of deformations is necessarily an isomorphism. This is a consequence of the following algebraic fact.

Lemma D.1.7. *Let A be a ring, $\mathfrak{m} \subset A$ be a nilpotent ideal (e.g. $(A, \mathfrak{m})$ is an artinian local ring) and $M \to N$ be a homomorphism of A-modules. Assume that N is flat over A. If $M/\mathfrak{m}M \to N/\mathfrak{m}N$ is an isomorphism, then so is $M \to N$.*

Proof. The right exact sequence $M \to N \to C \to 0$ becomes $M/\mathfrak{m}M \to N/\mathfrak{m}N \to C/\mathfrak{m}C \to 0$ after modding out by $\mathfrak{m}$, and we see that $C = \mathfrak{m}C$. As $\mathfrak{m}^n = 0$ for some n, we obtain that $C = \mathfrak{m}C = \mathfrak{m}^2C = \cdots = \mathfrak{m}^nC = 0$. Considering now the exact sequence $0 \to K \to M \to N \to 0$, the flatness of N implies that we get an exact sequence $0 \to K/\mathfrak{m}K \to M/\mathfrak{m}M \to N/\mathfrak{m}N \to 0$. Thus $K = \mathfrak{m}K = \cdots = \mathfrak{m}^nK = 0$ and we see that $M \to N$ is an isomorphism. □

Proposition D.1.8. *Every first order deformation of a smooth affine scheme X_0 over $\Bbbk$ is trivial. In other words, X_0 is rigid.*

Proof. Let X be a first order deformation of X_0. Since $X_0 \to \operatorname{Spec} \Bbbk$ is smooth, we may apply the Infinitesimal Lifting Criterion for Smoothness (A.3.1) to construct a lift $X \to X_0$ making the diagram

$$\begin{tikzcd} X_0 \arrow[r, "\text{id}"] \arrow[d, hook] & X_0 \arrow[d, "\text{smooth}"] \\ X \arrow[r] \arrow[ur, dashed] & \operatorname{Spec} \Bbbk \end{tikzcd}$$

commute. This induces a morphism $X \to X_0 \times_{\Bbbk} \Bbbk[\epsilon]$ over $\Bbbk[\epsilon]$ which restricts to the identity on X_0, and thus is an isomorphism by Lemma D.1.7.

See also [Har77, Exer. II.8.7]. $\square$

Remark D.1.9. If X_0 is not smooth or affine, then first order deformations are not necessarily trivial. For example, if $X_0 = \operatorname{Spec} \Bbbk[x,y]/(xy)$ is the nodal affine plane curve, then $X = \operatorname{Spec} \Bbbk[x,y,\epsilon]/(xy-\epsilon) \to \operatorname{Spec} \Bbbk[\epsilon]$ is a non-trivial first order deformation.

On the other hand, consider an elliptic curve $E_0 = V(y^2z - x(x-z)(x-2z)) \subset \mathbb{P}^2$ is a elliptic curve over $\Bbbk$ with $\operatorname{char}(\Bbbk) \neq 2, 3$. It is easy to write down global deformations by deforming the coefficients of the defining equations: $\mathcal{E} = V(y^2z - (x-\lambda z)(x-z)(x-2z)) \subset \mathbb{P}^2 \times \mathbb{A}^1$ (where $\mathbb{A}^1$ has coordinate λ) defines a flat projective morphism $\mathcal{E} \to \mathbb{A}^1$ such the central fiber $\mathcal{E}_0$ is isomorphic to E_0. Restricting $\mathcal{E}$ to the family $E := \mathcal{E} \times_{\mathbb{A}^1} \operatorname{Spec} \Bbbk[\lambda]/\lambda^2$ over the dual numbers defines a *non-trivial* first order deformation. For an affine open $U_0 \subset E_0$ and setting $U \subset E$ to be the open subscheme with the same topological space as U_0, then there is an isomorphism $U \xrightarrow{\sim} U_0 \times_{\Bbbk} \Bbbk[\epsilon]$. These local isomorphism however do not glue to a global isomorphism $E \xrightarrow{\sim} E_0 \times_{\Bbbk} \Bbbk[\epsilon]$. Since every deformation of a smooth scheme is obtained by gluing together trivial deformations, we need to understand automorphisms of trivial deformations in order to classify global deformations.

Lemma D.1.10. *If $X_0 = \operatorname{Spec} A$ is an affine scheme over $\Bbbk$ and $X = \operatorname{Spec} A[\epsilon]$ is the trivial first order deformation, then there are identifications*

$$\{\textit{automorphisms } X \to X \textit{ of first order defs}\} \cong \operatorname{Der}_{\Bbbk}(A, A) \cong \operatorname{Hom}_A(\Omega_{A/\Bbbk}, A).$$

Proof. The second equivalence is given by the universal property of the module of differentials. An automorphism of the trivial first order deformation corresponds to a $\Bbbk[\epsilon]$-algebra isomorphism $\phi \colon A \oplus A\epsilon \to A \oplus A\epsilon$ which is the identity modulo ϵ. Therefore, ϕ is determined by the images $\phi(a) = a + d(a)\epsilon$ of elements $a \in A$ where $d \colon A \to A$ is $\Bbbk$-linear map. Since ϕ is a ring homomorphism, for elements $a, a' \in A$, we must have that $aa' + d(aa')\epsilon = (a + d(a)\epsilon)(a' + d(a')\epsilon) = aa' + (ad(a') + a'd(a))\epsilon$ and we see that $d \colon A \to A$ is a $\Bbbk$-derivation. $\square$

For a scheme X_0 over $\Bbbk$, let $\operatorname{Def}(X_0)$ and $\operatorname{Def}^{\mathrm{lt}}(X_0)$ denote the sets of isomorphism classes of first order and locally trivial first order deformations.

Proposition D.1.11. *For a scheme X_0 of finite type over $\Bbbk$ with affine diagonal, there is a bijection*

$$\operatorname{Def}^{\mathrm{lt}}(X_0) \cong \mathrm{H}^1(X_0, T_{X_0}),$$

where $T_{X_0} = \mathscr{H}om_{\mathcal{O}_{X_0}}(\Omega_{X_0/\Bbbk}, \mathcal{O}_{X_0})$. The trivial deformation corresponds to $0 \in \mathrm{H}^1(X_0, T_{X_0})$.

In particular, if in addition X_0 is smooth over $\Bbbk$, then there is a bijection

$$\operatorname{Def}(X_0) \cong \mathrm{H}^1(X_0, T_{X_0}).$$

Proof. Let $X \to \operatorname{Spec} \Bbbk[\epsilon]$ be a locally trivial first order deformation of X_0. Choose an affine cover $\{U_i\}$ of X_0 and isomorphisms $\phi_{ij} := \phi_i \colon U_i \times_{\Bbbk} \Bbbk[\epsilon] \xrightarrow{\sim} X|_{U_i}$, where $X|_{U_i} \subset X$ denotes the open subscheme with the same topological space as U_i. Letting $U_{ij} = U_i \cap U_j$, we have automorphisms $\phi_j^{-1}|_{U_{ij} \times_{\Bbbk} \Bbbk[\epsilon]} \circ \phi_i|_{U_{ij} \times_{\Bbbk} \Bbbk[\epsilon]}$ of the trivial deformation $U_{ij} \times_{\Bbbk} \Bbbk[\epsilon]$ which by Lemma D.1.10 corresponds to elements $\phi_{ij} \in \mathrm{Hom}_{\mathcal{O}_{U_{ij}}}(\Omega_{U_{ij}/\Bbbk}, \mathcal{O}_{U_{ij}})$. Since $\phi_{ij} \circ \phi_{jk} = \phi_{ik}$ on $U_{ijk} := U_i \cap U_j \cap U_k$, we have that $\phi_{ij} + \phi_{jk} = \phi_{ik} \in T_{X_0}(U_{ijk})$. Recall that $\mathrm{H}^1(X_0, T_{X_0})$ can be computed using the Céch complex

$$0 \longrightarrow \bigoplus_i T_{X_0}(U_i) \xrightarrow{d_0} \bigoplus_{i,j} T_{X_0}(U_{ij}) \xrightarrow{\quad d_1 \quad} \bigoplus_{i,j,k} T_{X_0}(U_{ijk})$$
$$(s_{ij}) \longmapsto (s_{ij}|_{U_{ijk}} - s_{ik}|_{U_{ijk}} + s_{jk}|_{U_{ijk}})_{ijk}$$

As $\{\phi_{ij}\} \in \bigoplus_{i,j} T_{X_0}(U_{ij})$ is in the kernel of d_1, it defines an element of $\mathrm{H}^1(X_0, T_{X_0}) = \ker(d_1)/\operatorname{im}(d_0)$. Conversely, given an element of $\mathrm{H}^1(X_0, T_{X_0})$ and a choice of representative $\{\phi_{ij}\} \in \ker(d_1)$, then viewing each ϕ_{ij} as an automorphism ϕ_{ij} of the trivial deformation of U_{ij}, we may glue together the trivial deformations $U_i \times_{\Bbbk} \Bbbk[\epsilon]$ along $U_{ij} \times_{\Bbbk} \Bbbk[\epsilon]$ via ϕ_{ij} to construct a global first order deformation X of X_0.

For the final statement, observe that since X_0 is smooth over $\Bbbk$, every first order deformation is locally trivial by Proposition D.1.8. See also [Har77, Exer. III.4.10 and Ex. III.9.13.2]. ☐

Example D.1.12. If C is a smooth projective curve of genus $g \geq 2$, then we've computed that

$$T_{\mathscr{M}_g,[C]} = \mathrm{H}^1(C, T_C) \overset{\mathrm{SD}}{=} \mathrm{H}^0(C, \Omega_{C/\Bbbk}^{\otimes 2})$$

which by Riemann–Roch is a $3g-3$ dimensional vector space.

Exercise D.1.13. Use the Euler exact sequence to show that $\mathrm{H}^1(\mathbb{P}^n, T_{\mathbb{P}^n}) = 0$ and conclude that every first order deformation of $\mathbb{P}^n$ is trivial, i.e. $\mathbb{P}^n$ is rigid.

D.1.3 First order deformations of vector bundles and coherent sheaves

Definition D.1.14. Let X be a projective scheme over $\Bbbk$ and E_0 be a coherent sheaf. A *first order deformation of* E_0 is a pair (E, α) where E is a coherent sheaf on $X \times_{\Bbbk} \Bbbk[\epsilon]$ flat over $\Bbbk[\epsilon]$ and $\alpha \colon E_0 \xrightarrow{\sim} E|_X$ is an isomorphism. Pictorially, we have

$$\begin{tikzcd} E_0 \arrow[d, dash] & E \arrow[d, dash, "\text{flat}/\Bbbk[\epsilon]"] \\ X \arrow[r, hook] & X_{\Bbbk[\epsilon]}. \end{tikzcd}$$

A *morphism* $(E, \alpha) \to (E', \alpha')$ *of first order deformations* is a morphism $\beta \colon E \to E'$ (equivalently an isomorphism by Lemma D.1.7) of coherent sheaves on X' such that $\alpha' = \beta|_X \circ \alpha$.

We say that (E, α) is *trivial* if it isomorphic as first order deformations to $(p^* E_0, \mathrm{id})$ where $p \colon X_{\Bbbk[\epsilon]} \to X$.

Proposition D.1.15. *Let X be a scheme over $\Bbbk$ and E_0 be a coherent sheaf. There is a bijection*

$$\{\textit{first order deformations } (E, \alpha) \textit{ of } E_0\}/\sim \;\; \cong \mathrm{Ext}^1_{\mathcal{O}_X}(E_0, E_0)$$

Under this correspondence, the trivial deformation corresponds to $0 \in \mathrm{Ext}^1_{\mathcal{O}_X}(E_0, E_0)$.

If in addition E_0 is a vector bundle (resp. line bundle), then the set of isomorphism classes of first order deformations of E_0 is bijective to $\mathrm{H}^1(X, \mathscr{E}nd_{\mathcal{O}_X}(E_0))$ (resp. $\mathrm{H}^1(X, \mathcal{O}_X)$).

Proof. If (E, α) is a first order deformation then since E is flat over $\Bbbk[\epsilon]$, we may tensor the exact sequence $0 \to \Bbbk \xrightarrow{\epsilon} \Bbbk[\epsilon] \to \Bbbk \to 0$ of $\Bbbk[\epsilon]$-modules with E to obtain an exact sequence $0 \to E_0 \xrightarrow{\epsilon} E \to E_0 \to 0$ (after identifying $E \otimes_{\Bbbk[\epsilon]} \Bbbk$ with E_0 via α). Since $\mathrm{Ext}^1_{\mathcal{O}_X}(E_0, E_0)$ parameterizes isomorphism classes of extensions [Har77, Exer. III.6.1], we have constructed an element of $\mathrm{Ext}^1_{\mathcal{O}_X}(E_0, E_0)$. Conversely, given an exact sequence $0 \to E_0 \xrightarrow{\alpha} E \to E_0 \to 0$, then E is a coherent sheaf on $X_{\Bbbk[\epsilon]}$ and is flat over $\Bbbk[\epsilon]$ by the flatness criterion over the dual numbers (see Remark A.2.7). The restriction $E|_X$ is isomorphic to E_0 via α.

See also [Har10, Thm. 2.7]. $\square$

Remark D.1.16. The classifications of Propositions D.2.2, D.2.6 and D.2.15 give vector space structures to the set of isomorphism classes of first order deformations. The vector space structures can also be witnessed as a consequences of Rim–Schlessinger's homogeneity condition; see Lemma D.3.13.

D.2 Higher order deformations and obstructions

Let $\mathcal{M}$ be a moduli problem and $E \in \mathcal{M}(A)$ be an object defined over a ring A. If $A' \twoheadrightarrow A$ is a surjection of rings with square-zero kernel, in this section we address the following two questions:

(1) Does E deform to an object $E' \in \mathcal{M}(A')$?

(2) If so, can we classify all such deformations?

Pictorially, we have:

$$\begin{tikzcd} E \arrow[d, dash] & E' \arrow[d, dash, dashed] \\ \mathrm{Spec}\, A \arrow[r, hook] & \mathrm{Spec}\, A'. \end{tikzcd}$$

where $\mathrm{Spec}\, A \hookrightarrow \mathrm{Spec}\, A'$ is a closed immersion of schemes with the same topological space. Note that since $J = \ker(A' \to A)$ is square-zero, $J = J/J^2$ is naturally a module over $A = A'/J$. In other words, Question (1) is asking whether there is some "obstruction" to the existence of a deformation E' while (2) seeks to classify all higher order deformations given that there is no obstruction.

An interesting case is when A and A' are local artinian algebras with residue field $\Bbbk$ and the kernel $J = \ker(A' \to A)$ satisfies $\mathfrak{m}_{A'} J = 0$ (which implies that $J^2 = 0$). In this case, $J = J/\mathfrak{m}_{A'} J$ is naturally a vector space over $\Bbbk = A'/\mathfrak{m}_{A'}$. Setting $E_0 := E|_\Bbbk \in \mathcal{M}(\Bbbk)$, we can view E as a deformation over E_0 over A and we are attempting to classify the higher order deformations over A'. If there are no obstructions to deforming, then the Infinitesimal Lifting Criterion for Smoothness (3.7.1) implies that $\mathcal{M}$ is smooth at $[E_0]$.

The previous section studied the specific case when $A = \Bbbk$ and $A' = \Bbbk[\epsilon]$ in which case deformations of an object $E_0 \in \mathcal{M}(\Bbbk)$ over A' correspond to first order deformations. In this case, the obstruction vanishes as there is always the trivial deformation (i.e. the pullback of E_0 along $\mathrm{Spec}\, \Bbbk[\epsilon] \to \mathrm{Spec}\, \Bbbk$). Other examples of $A' \to A$ to keep in mind are $\Bbbk[x]/x^{n+1} \twoheadrightarrow \Bbbk[x]/x^n$ and $\mathbb{Z}/p^{n+1} \twoheadrightarrow \mathbb{Z}/p^n$ where we inductively attempt to deform E_0 over the nilpotent thickenings $\operatorname{Spec} \Bbbk[x]/x^{n+1} \hookrightarrow \mathbb{A}^1$ and $\operatorname{Spec} \mathbb{Z}/p^{n+1} \hookrightarrow \operatorname{Spec} \mathbb{Z}$.

D.2.1 Higher order embedded deformations

Definition D.2.1. Let $A' \twoheadrightarrow A$ be a surjection of noetherian rings with square-zero kernel. Let X' be a scheme over A' and set $X := X' \times_{A'} A$. Let $Z \subset X$ be a closed subscheme flat over A. A *deformation of $Z \subset X$ over A'* is a closed subscheme $Z' \subset X'$ flat over A' such that $Z' \times_{A'} A = Z$ as closed subschemes of X. Pictorially, a deformation is a filling of the cartesian diagram

$$\begin{tikzcd}
 & X \arrow[rr, hook] \arrow[dd] & & X' \arrow[dd] \\
Z \arrow[ur, hook, "\mathrm{cl}"] \arrow[dr, "\mathrm{flat}"'] \arrow[rr, dashed] & & Z' \arrow[ur, hook, dashed, "\mathrm{cl}"] \arrow[dr, dashed, "\mathrm{flat}"] & \\
 & \operatorname{Spec} A \arrow[rr, hook] & & \operatorname{Spec} A'.
\end{tikzcd}$$

The formulation of the next proposition uses the following notion: a *torsor of a group G* is a transitive and free action of G on a set.

Proposition D.2.2. *Let $A' \twoheadrightarrow A$ be a surjection of noetherian rings with square-zero kernel J. Let X' be a scheme over A' with affine diagonal (e.g. separated) and $Z \subset X := X' \times_{\Bbbk} A$ be a closed subscheme flat over A defined by a sheaf of ideals $I \subset \mathcal{O}_X$. Then*

(1) If there exists a deformation $Z' \subset X'$ of $Z \subset X$ over A', then the set of such deformations is a torsor under $\mathrm{H}^0(Z, N_{Z/X} \otimes_A J) = \operatorname{Ext}^0_{\mathcal{O}_Z}(I/I^2, J)$.

(2) There exists an element $\mathrm{ob}_Z \in \operatorname{Ext}^1_{\mathcal{O}_Z}(I/I^2, J)$ (depending on Z and $A' \twoheadrightarrow A$) such that there exists a deformation of $Z \subset X$ over A' if and only if $\mathrm{ob}_Z = 0$.

Remark D.2.3. An interesting example is when $X = X_0 \times_{\Bbbk} A$ and $X' = X_0 \times_{\Bbbk} A'$ are the base changes of a $\Bbbk$-scheme X_0. If the closed subscheme $Z \subset X$ has constant Hilbert polynomial P (i.e. for each $s \in \operatorname{Spec} A$, the Hilbert polynomial of $Z_s \subset X_0 \times_{\Bbbk} \kappa(s)$, with respect to a fixed ample line bundle on X_0, is independent of s), then we have an object $[Z \subset X] \in \operatorname{Hilb}^P(X_0)(A)$ of the Hilbert scheme. In this case, a deformation of $Z \subset X$ over A' is an object $[Z' \subset X'] \in \operatorname{Hilb}^P(X_0)(A')$ which restricts to $[Z \subset X]$. Note that when $A' \twoheadrightarrow A$ is a surjection of local artinian $\Bbbk$-algebras with $\mathfrak{m}_{A'} J = 0$, then there is an identification $\mathrm{H}^0(Z, N_{Z/X} \otimes_A J) = \mathrm{H}^0(Z_0, N_{Z_0/X_0} \otimes_{\Bbbk} J)$ where $Z_0 = Z \times_A \Bbbk$.

Remark D.2.4. In the case that deformations of $Z \subset X$ over A' exist Zariski-locally on X, then there is an obstruction element $\mathrm{ob}_Z \in H^1(Z, N_{Z/X} \otimes_A J)$.

Proof. Suppose first that $X' = \operatorname{Spec} B'$, $X = \operatorname{Spec} B$ where $B = B' \otimes_{A'} A$ and $Z = \operatorname{Spec} B/I$. If there exists a deformation $Z' = \operatorname{Spec} B'/I'$, then there is an exact diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & I \otimes_A J \arrow[r] \arrow[d] & I' \arrow[r] \arrow[d] & I \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & B \otimes_A J \arrow[r] \arrow[d] & B' \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & (B/I) \otimes_A J \arrow[r] \arrow[d] & B'/I' \arrow[r] \arrow[d] & B/I \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

The exactness of the bottom row (resp. middle row) is equivalent to the flatness of B'/I' (resp. B') over A' by the Local Criterion of Flatness (Corollary A.2.6) while the exactness of the left column follows from the flatness of B/I over A. Conversely, an exact diagram defines an deformation $Z' = \operatorname{Spec} B'/I'$.

We will define an action $\operatorname{Hom}_B(I, (B/I) \otimes_A J)$ on the set of deformations as follows: given $\phi \in \operatorname{Hom}_B(I, (B/I) \otimes_A J)$ and a deformation $Z' = \operatorname{Spec} B'/I'$, define $I'' \subset B'$ as the set of elements $x'' \in B'$ such that its image $\overline{x}'' \in B$ lies in I and such that a lifting $x' \in I'$ of $\overline{x}'' \in I$ satisfies $x'' - x' = \phi(\overline{x''}) \in (B/I) \otimes_A J$ (noting that this condition is independent of the choice of lifting x'). One checks that $\operatorname{Spec} B'/I''$ is another deformation.

On the other hand, given two deformations defined by ideals I' and I'', we define $\phi \colon I \to (B/I) \otimes_A J$ by $\phi(x) = \overline{x' - x''}$ where $x' \in I'$ and $x'' \in I''$ are lifts of x (which forces $x' - x'' \in B \otimes_A J$). One checks that this is a B-module homomorphism providing an inverse to the above construction. We have natural identifications

$$\operatorname{Hom}_B(I, (B/I) \otimes_A J) = \operatorname{Hom}_{B/I}(I/I^2, B/I \otimes_A J) = \mathrm{H}^0(Z, N_{Z/X} \otimes_A J).$$

The above constructions globalize to X and establishes (1).

For (2), we will assume that there is an open cover $\{U_i\}$ of X such that there exists deformations $Z'_i \subset X' \cap U_i$ of $Z \cap U_i \subset X \cap U_i$ (noting that X and X' are homeomorphic). On $U_{ij} = U_i \cap U_j$, the two deformations $Z'_i|_{U_{ij}}$ and $Z'_j|_{U_{ij}}$ defines an element $\phi_{ij} \in \mathrm{H}^0(U_{ij}, N_{Z/X} \otimes_A J)$ which in turn defines a Cěch 1-cocycle $(\phi_{ij}) \in \mathrm{H}^1(X, N_{Z/X} \otimes_A J)$. We leave the reader to check that the vanishing of (ϕ_{ij}) characterizes whether there is a deformation of $Z \subset X$ over A'.

See also [Har10, Thm. 6.2]. □

D.2.2 Higher order deformations of schemes

In this chapter, we discuss higher order deformations and obstructions for smooth schemes and for local complete intersections.

Definition D.2.5. Let $A' \twoheadrightarrow A$ be a surjection of noetherian rings with square-zero kernel and $X \to \operatorname{Spec} A$ be a flat morphism of schemes. A *deformation of $X \to \operatorname{Spec} A$ over A'* is a flat morphism $X' \to \operatorname{Spec} A'$ together with an isomorphism $\alpha\colon X \xrightarrow{\sim} X' \times_{A'} A$ over A, or in other words a cartesian diagram

$$\begin{tikzcd} X \arrow[r, hook, dashed] \arrow[d, "\text{flat}"] & X' \arrow[d, dashed, "\text{flat}"] \\ \operatorname{Spec} A \arrow[r, hook] & \operatorname{Spec} A'. \end{tikzcd} \tag{D.2.1}$$

A *morphism of deformations over A'* is a morphism of schemes over A' restricting to the identity on X. By Lemma D.1.7, every morphism of deformations is an isomorphism.

Proposition D.2.6 (Higher Order Deformations of Smooth Schemes). *Let $A' \twoheadrightarrow A$ be a surjection of noetherian rings with square-zero kernel J. If $X \to \operatorname{Spec} A$ is a smooth morphism of schemes where X has affine diagonal (e.g. separated), then*

1. *The group of automorphisms of a deformation $X' \to \operatorname{Spec} A'$ of $X \to \operatorname{Spec} A$ over A' is bijective to $\mathrm{H}^0(X, T_{X/A} \otimes_A J)$.*
2. *If there exists a deformation of $X \to \operatorname{Spec} A$ over A', then the set of isomorphism classes of all such deformations is a torsor under $\mathrm{H}^1(X, T_{X/A} \otimes_A J)$.*
3. *There is an element $\mathrm{ob}_X \in \mathrm{H}^2(X, T_{X/A} \otimes_A J)$ with the property that there exists a deformation of $X \to \operatorname{Spec} A$ over A' if and only if $\mathrm{ob}_X = 0$.*

Remark D.2.7. If A and A' are local artinian rings with residue field $\Bbbk$ such that $\mathfrak{m}_{A'} J = 0$ and we set $X_0 := X \times_A \Bbbk$, then automorphisms, deformations and obstructions are classified by $\mathrm{H}^i(X_0, T_{X_0} \otimes_{\Bbbk} J)$ for $i = 0, 1, 2$.

Proof. When $X = \operatorname{Spec} A$ is an affine scheme, the same argument of Lemma D.1.10 shows that group of automorphisms of X' is identified with $\operatorname{Hom}_A(\Omega_{A/\Bbbk}, J) = \mathrm{H}^0(X, T_{X/A} \otimes J)$. Since $T_{X/A} \otimes J$ and the assignment of an open subscheme U to the group of automorphisms of the first order deformation $X'|_U$ are sheaves, Part (1) follows.

Let $\{U_i\}$ be an affine cover of X. Part (2) follows from a similar argument to Proposition D.1.11: Fix a deformation $X' \to \operatorname{Spec} A$ of X. For every other deformation $X'' \to \operatorname{Spec} A$, we know that over each affine U_i, there is an isomorphism $\phi_i\colon X'|_{U_i} \to X''|_{U_i}$ and we let $\phi_{ij} = \phi_j^{-1}|_{X'|_{U_{ij}}} \circ \phi_i|_{X'|_{U_{ij}}}$ viewed as an element of $\mathrm{H}^0(U_i, T_{X/A} \otimes J)$. The Céch 1-cycle (ϕ_{ij}) defines an element in $\mathrm{H}^1(X, T_{X/A} \otimes J)$.

For (3), again using that U_i is affine, we can choose a deformation $U_i' \to \operatorname{Spec} A'$ of U_i. We can also choose isomorphisms $\phi_{ij}\colon U_i'|_{U_{ij}} \to U_j'|_{U_{ij}}$. This defines gluing data for a deformation X' if $\phi_{jk} \circ \phi_{ij} = \phi_{ik}$ on the triple intersections U_{ijk}. The automorphism $\Psi_{ijk} = \phi_{ik}^{-1} \circ \phi_{jk} \circ \phi_{ij}$ restricts to the identity on U_{ijk} and thus defines an element of $\mathrm{H}^0(U_{ijk}, T_{X/A} \otimes J)$. Consider the Céch complex with $F = T_{X/A} \otimes J$

$$\begin{array}{ccccc} \bigoplus_{i,j} F(U_{ij}) & \xrightarrow{d_1} & \bigoplus_{i,j,k} F(U_{ijk}) & \xrightarrow{d_2} & \bigoplus_{i,j,k,l} F(U_{ijkl}) \\ (s_{ij}) & \longmapsto & (s_{ij} - s_{ik} + s_{jk})_{ijk} & & \\ & & (s_{ijk}) & \longmapsto & (s_{ijk} - s_{ijl} + s_{ikl} - s_{jkl})_{ijkl}. \end{array}$$

One checks that $d_2(\Psi_{ijk}) = 0$ and that if ϕ'_{ij} is a different choice of isomorphisms then the corresponding element (Ψ'_{ijk}) differs from (Ψ_{ijk}) by an element in the image of d_1. Thus (Ψ_{ijk}) is a well-defined element of $\mathrm{H}^2(X, T_{X/A} \otimes J)$.

See also [Har10, Cor. 10.3]. $\square$

Exercise D.2.8 (Interpretation of deformations and obstruction using gerbes)**.** With the hypotheses of Proposition D.2.6, consider the category $\mathcal{G}$ over Sch/X whose objects over $S \to X$ are cartesian diagrams

$$\begin{tikzcd} S \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & S' \arrow[d] \\ \operatorname{Spec} A \arrow[r, hook] & \operatorname{Spec} A' \end{tikzcd}$$

where $S \to \operatorname{Spec} A$ is the composition $S \to X \to \operatorname{Spec} A$. A morphism $(S \to X, S \hookrightarrow S' \to \operatorname{Spec} A') \to (T \to X, T \hookrightarrow T' \to \operatorname{Spec} A')$ is the data of a morphism $\phi\colon S' \to T'$ over A' such that ϕ restricts to a morphism $S \to T$ over X.

(a) Show that $\mathcal{G}$ is a gerbe banded by the sheaf of groups $T_{X/A} \otimes_A J$ on X. (*Hint: Use Lemma D.1.7 to show it is a prestack. See Definition 6.2.21 for the definition of a banded gerbe.*)

(b) Give an alternate proof of Proposition D.2.6. (*Hint:* For part (3), use Exercise 6.2.30.)

Exercise D.2.9 (Deformations of principal G-bundles)**.** Let G be a smooth affine algebraic group over a field $\Bbbk$ with Lie algebra $\mathfrak{g}$. Let $X \hookrightarrow X'$ be a closed immersion of finite type $\Bbbk$-schemes defined by a square-zero sheaf of ideals J and assume that X has affine diagonal. If $P \to X$ is a principal G-bundle, one can define deformations over X' and automorphisms of deformations analogous to the case of smooth morphisms. Show that

(1) The group of automorphisms of a deformation $P' \to X'$ of $P \to X$ is bijective to $\mathrm{H}^0(X, \mathfrak{g} \otimes J)$.

(2) If there exists a deformation of over X', then the set of isomorphism classes of all such deformations is a torsor under $\mathrm{H}^1(X, \mathfrak{g} \otimes J)$.

(3) There is an element $\mathrm{ob}_X \in \mathrm{H}^2(X, \mathfrak{g} \otimes J)$ with the property that there exists a deformation over X' if and only if $\mathrm{ob}_X = 0$.

Example D.2.10 (Abelian varieties)**.** If X_0 is an abelian variety over $\mathbb{C}$ of dimension n, then it turns out that deforming X_0 as an abstract scheme is equivalent to deforming it as an abelian variety, and that obstructions to deforming X_0 as an abelian variety also live in $\mathrm{H}^2(X_0, T_{X_0})$. Using that $\Omega_{X_0} = \mathcal{O}_{X_0}^n$ is trivial and the Hodge symmetries, we see that $\mathrm{H}^2(X_0, T_{X_0}) = \mathrm{H}^2(X_0, \mathcal{O}_{X_0})^{\oplus n} = \mathrm{H}^0(X_0, \bigwedge^2 \mathcal{O}_{X_0}^n)^{\oplus n}$ is non-zero. Nevertheless, Grothendieck and Mumford showed that given every deformation problem as in (D.2.1), the obstruction $\mathrm{ob}_X \in \mathrm{H}^2(X, T_{X/A} \otimes_A J)$ vanishes! This shows that abelian varieties are unobstructed and their moduli is formally smooth. See [Oor71].

Proposition D.2.11 (Higher Order Deformations of Complete Intersections)**.** *Let X_0 be a scheme of finite type over a field $\Bbbk$ such that X_0 is generically smooth and a local complete intersection. Let $A' \twoheadrightarrow A$ be a surjection of local noetherian rings with residue field $\Bbbk$. Assume that the kernel $J := \ker(A' \to A)$ satisfies $\mathfrak{m}_{A'} J = 0$. If $X \to \operatorname{Spec} A$ is a flat morphism of schemes with central fiber $X \times_A \Bbbk \cong X_0$, then*

(1) The group of automorphisms of a deformation $X' \to \operatorname{Spec} A'$ of $X \to \operatorname{Spec} A$ over A' is bijective to $\mathrm{Ext}^0_{\mathcal{O}_{X_0}}(\Omega_{X_0}, J)$.

(2) *If there exists a deformation of $X \to \operatorname{Spec} A$ over A', then the set of isomorphism classes of all such deformations is a torsor under $\operatorname{Ext}^1_{\mathcal{O}_{X_0}}(\Omega_{X_0}, J)$.*

(3) *There is an element $\mathrm{ob}_X \in \operatorname{Ext}^2_{\mathcal{O}_{X_0}}(\Omega_{X_0}, J)$ with the property that there exists a deformation of $X \to \operatorname{Spec} A$ over A' if and only if $\mathrm{ob}_X = 0$.*

Proof. See [Vis97, Thm. 4.4] for an explicit argument. Alternatively, since X_0 is generically smooth and a local complete intersection, the cotangent complex L_{X_0} is quasi-isomorphic to Ω_{X_0} (see Example D.5.11) and thus the result follows from the fact that the cotangent complex controls deformations (Theorem D.5.10). $\square$

Exercise D.2.12.

(1) Show that under the bijection $\operatorname{Def}(\Bbbk[\![x,y]\!]/(xy)) \cong \Bbbk$ an element $t \in \Bbbk$ corresponds to the first order deformation $\operatorname{Spec} \Bbbk[\![x,y,\epsilon]\!]/(xy - t\epsilon)$.

(2) Classify first order deformations of the A_k-singularity $\Bbbk[\![x,y]\!]/(y^2 - x^{k+1})$.

Exercise D.2.13 (Higher Order Deformations of Morphisms)**.** Let $\Bbbk$ be a field and $A' \twoheadrightarrow A$ be a surjection of local artinian rings with residue field $\Bbbk$. Let $f \colon X \to Y$ be a morphism of schemes over A. A *deformation of $f \colon X \to Y$ over A'* is a morphism $f' \colon X' \to Y'$ of schemes over $\operatorname{Spec} A'$ together with isomorphisms $\alpha' \colon X \xrightarrow{\sim} X' \times_{A'} A$ and $\beta' \colon Y \xrightarrow{\sim} Y' \times_{A'} A$ such that both X' and Y' are flat over A' and such that the base change of f' to A is equal to f under the isomorphisms α and β. In other words, a deformation is a cartesian diagram

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d, "f"] \arrow[dr, phantom, "\square"] & X' \arrow[d, "f'"] \\ Y \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & Y' \arrow[d] \\ \operatorname{Spec} A \arrow[r, hook] & \operatorname{Spec} A'. \end{tikzcd}$$

A *morphism* of deformations $(X' \to Y', \alpha', \beta') \to (X'' \to Y'', \alpha'', \beta'')$ consists of morphisms $X' \to X''$ and $Y' \to Y''$ over A' compatible with the given isomorphisms.

Assume that X and Y are proper A, and that $f_*\mathcal{O}_X = \mathcal{O}_Y$ and $\mathrm{R}^1 f_* \mathcal{O}_X = 0$. Show that the functor taking a deformation $f' \colon X' \to Y'$ of $f \colon X \to Y$ over A' to the deformation X' over X over A' induces an isomorphism of categories.

Hint: Given a deformation X' over X, define Y' as the ringed space $(Y, f_*\mathcal{O}_{X'})$ (using that X and X' have the same topological space). Use the conditions of f and the flatness of X' over A' to show that Y' is a scheme flat over A'. See also [Vak06, §5.3], [Ran89, Thm. 3.3], and [SP, Tag 0E3X]. (For additional properties of deformations of morphisms, see [Ser06, §3.4].)

D.2.3 Higher order deformations of vector bundles

Definition D.2.14. Let $A' \twoheadrightarrow A$ be a surjection of noetherian rings with square-zero kernel J. Let $X' \to \operatorname{Spec} A'$ be finite type morphism of schemes and set $X := X' \times_{A'} A$. Given a coherent sheaf E on X flat over A, a *deformation of E*

over $A' \to A$ is a pair (E', α) where E' is a coherent sheaf on X' flat over A' and $\alpha\colon E \to E'|_X$ is an isomorphism. Pictorially, we have

$$\begin{tikzcd} E \arrow[d, dash, "\text{flat}/A"] & E' \arrow[d, dash, dashed, "\text{flat}/A'"] \\ X \arrow[r, hook] & X'. \end{tikzcd}$$

A *morphism* $(E, \alpha) \to (E', \alpha')$ *of deformations* is a morphism $\beta\colon E \to E'$ of coherent sheaves on $X_{A'}$ such that $\alpha' = \beta|_X \circ \alpha$. By Lemma D.1.7, every morphism of deformations is an isomorphism.

Proposition D.2.15. *Let $A' \twoheadrightarrow A$ be a surjection of noetherian rings with square-zero kernel J. Let $X' \to \operatorname{Spec} A'$ be a flat and finite type morphism of schemes such that X' has affine diagonal (e.g. separated) and set $X := X' \times_{A'} A$. Let E be a vector bundle on X over A.*

(1) The group of automorphisms of a deformation E' of E over A' is bijective to $\mathrm{H}^0(X, \mathscr{E}nd_{\mathcal{O}_X}(E) \otimes_A J)$.

(2) If there exists a deformation of E over A', then the set of isomorphism classes of all such deformations is a torsor under $\mathrm{H}^1(X, \mathscr{E}nd_{\mathcal{O}_X}(E) \otimes_A J)$.

(3) There is an element $\mathrm{ob}_E \in \mathrm{H}^2(X, \mathscr{E}nd_{\mathcal{O}_X}(E) \otimes_A J)$ with the property that there exists a deformation of E over A' if and only if $\mathrm{ob}_E = 0$.

Remark D.2.16. If X and X' are base changes of a finite type $\Bbbk$-scheme X_0 with affine diagonal, and A and A' are local artinian rings with residue field $\Bbbk$ such that $\mathfrak{m}_{A'} J = 0$, then automorphisms, deformations and obstructions are classified by $\mathrm{H}^2(X_0, \mathscr{E}nd_{\mathcal{O}_{X_0}}(E_0) \otimes_{\Bbbk} J)$ for $i = 0, 1, 2$ where $E_0 = E|_{X_0}$.

Proof. See [Har10, Thm. 7.1]. □

Exercise D.2.17. Give an alternative proof of Proposition D.2.15 using the technique outlined in Exercise D.2.8.

D.3 Versal formal deformations and Rim–Schlessinger's Criteria

D.3.1 Functors of artin rings

For an algebraically closed field $\Bbbk$, let $\mathrm{Art}_{\Bbbk}$ denote the category of artinian local $\Bbbk$-algebras with residue field $\Bbbk$. The opposite category $\mathrm{Art}_{\Bbbk}^{\mathrm{op}}$ is equivalent to the category of local artinian $\Bbbk$-schemes (S, s) with $\kappa(s) = \Bbbk$.

Definition D.3.1. We say that a covariant functor $F\colon \mathrm{Art}_{\Bbbk} \to \mathrm{Sets}$ is *pro-representable* if there exists a noetherian complete local $\Bbbk$-algebra R such that for all $A \in \mathrm{Art}_{\Bbbk}$, there is a isomorphism $F \xrightarrow{\sim} h_R$ where $h_R := \mathrm{Hom}_{\Bbbk-\mathrm{alg}}(R, -)$.

Remark D.3.2. If $F\colon \mathrm{Sch}/\Bbbk \to \mathrm{Sets}$ is a contravariant functor and $x_0 \in F(\Bbbk)$, then we can consider the induced functor of artin rings

$$F_{x_0}\colon\ \mathrm{Art}_{\Bbbk} \to \mathrm{Sets}, \quad A \mapsto \{x \in F(A) \mid x|_{\Bbbk} = x_0 \in F(\Bbbk)\}$$

where $x|_{\Bbbk}$ denotes the image of x under $F(A) \to F(A/\mathfrak{m}_A)$. If F is representable by a scheme X and $x \in X$ is the $\Bbbk$-point corresponding to x_0, then F_{x_0} is pro-representable by $\widehat{\mathcal{O}}_{X,x}$.

Exercise D.3.3. Provide an example of a non-representable contravariant functor $F\colon \mathrm{Sch}/\Bbbk \to \mathrm{Sets}$ and an object $x_0 \in F(\Bbbk)$ such that F_{x_0} is pro-representable.

Many functors of artin rings are not pro-representable. For example, if C_0 is a smooth connected projective curve with a non-trivial automorphism group, then the covariant functor $F_{C_0}\colon \mathrm{Art}_{\Bbbk} \to \mathrm{Sets}$ where $F_{C_0}(A)$ consists of isomorphism classes of smooth proper families of curves $\mathcal{C} \to \operatorname{Spec} A$ such that $\mathcal{C} \times_A A/\mathfrak{m}_A$ is isomorphic to C_0, is not pro-representable. Nevertheless many moduli functors admit *versal* deformations.

Remark D.3.4. To work over a more general base (e.g. of mixed characteristic), one can consider instead the following setup: let Λ be a noetherian complete local ring with residual field $\Bbbk$ (not necessarily algebraically closed) and Art_Λ be the category of artinian local Λ-algebras $(A, \mathfrak{m})$ with an identification $\Bbbk \xrightarrow{\sim} A/\mathfrak{m}$. Rim–Schlessinger's Criteria (Theorem D.3.11) holds after replacing $\mathrm{Art}_{\Bbbk}$ with Art_Λ. More generally, one can consider the setup where $A \to \Bbbk$ is a finite morphism to a field, not assumed to be the residue field.

Setting $\Lambda = \Bbbk$ recovers our setup but in many applications it is often useful to take Λ to be a ring of Witt vectors, e.g. $\Lambda = \mathbb{Z}_p$. In this way, one can consider deforming an object E_0 over $\mathbb{F}_p$ inductively along extensions $\mathbb{Z}/p^{n+1} \twoheadrightarrow \mathbb{Z}/p^n$ with the hope of applying Rim–Schlessinger's Criteria (Theorem D.3.11) and Grothendieck's Existence Theorem (D.4.4) to deform E_0 to an object $\widehat{E}$ over the *characteristic zero* ring $\mathbb{Z}_p$; see Section D.4.1

D.3.2 Versal deformations

As it's important to keep track of automorphisms, we will present Rim–Schlessinger's Criteria, a generalization of Schlessinger's Criterion from functors to prestacks. Therefore we will formulate the definition of versality for prestacks $\mathcal{X}$ over $\mathrm{Art}_{\Bbbk}^{\mathrm{op}}$. We will assume that $\mathcal{X}(\Bbbk)$ is equivalent to a set consisting of a single object, i.e. there is a unique morphism between any two objects in $\mathcal{X}(\Bbbk)$.

Definition D.3.5. Let $\mathcal{X}$ be a prestack over $\mathrm{Art}_{\Bbbk}^{\mathrm{op}}$ such that the groupoid $\mathcal{X}(\Bbbk)$ is equivalent to the set $\{x_0\}$.

(1) A *formal deformation* $(R, \{x_n\})$ *of* x_0 is the data of a noetherian complete local $\Bbbk$-algebra $(R, \mathfrak{m}_R)$ together with objects $x_n \in \mathcal{X}(R/\mathfrak{m}_R^{n+1})$ and morphisms $x_{n-1} \to x_n$ over $\operatorname{Spec} R/\mathfrak{m}_R^n \to \operatorname{Spec} R/\mathfrak{m}_R^{n+1}$, or in other words an element of $\varprojlim \mathcal{X}(R/\mathfrak{m}^n)$. When $\mathcal{X} = F$ is a covariant functor $\mathrm{Art}_{\Bbbk} \to \mathrm{Sets}$, a formal deformation is a compatible sequence of elements $x_n \in F(R/\mathfrak{m}_R^{n+1})$.

(2) A formal deformation $(R, \{x_n\})$ is *versal* if for every surjection $A \twoheadrightarrow A_0$ in $\mathrm{Art}_{\Bbbk}$ with $\mathfrak{m}_A^{n+1} = 0$, object $\eta \in \mathcal{X}(A)$ and $\Bbbk$-algebra homomorphism $\phi_0\colon R/\mathfrak{m}_R^{n+1} \to A_0$ with an isomorphism $\alpha_0\colon x_n|_{A_0} \xrightarrow{\sim} \eta|_{A_0}$ in $\mathcal{X}(A_0)$, there exists a $\Bbbk$-algebra homomorphism $\phi\colon R/\mathfrak{m}_R^{n+1} \to A$ and an isomorphism $\alpha\colon x_n|_A \xrightarrow{\sim} \eta$ in $\mathcal{X}(A)$ such that ϕ_0 is the composition $R/\mathfrak{m}_R^{n+1} \xrightarrow{\phi} A \twoheadrightarrow A_0$ $\alpha|_{A_0} = \alpha_0$.

(3) A versal formal deformation $(R, \{x_n\})$ is *miniversal* (or *a pro-representable hull*) if the induced map $\mathrm{Hom}_{\Bbbk\text{-alg}}(R, \Bbbk[\epsilon]) \to \mathcal{X}(\Bbbk[\epsilon])/\sim$ on isomorphism classes, defined by $(R \to R/\mathfrak{m}_R^2 \xrightarrow{\phi} \Bbbk[\epsilon]) \mapsto \phi(x_1)$, is bijective.

Remark D.3.6. The deformation $x_n \in \mathcal{X}(R/\mathfrak{m}_R^{n+1})$ can be viewed via Yoneda's 2-Lemma as a morphism $\operatorname{Spec} R/\mathfrak{m}_R^{n+1} \to \mathcal{X}$ or more precisely as $h_{R/\mathfrak{m}_R^{n+1}} \to \mathcal{X}$. Likewise, we can view a formal deformation as a morphism $\{x_n\} \colon h_R \to \mathcal{X}$ where $h_R = \mathrm{Hom}_{\Bbbk\text{-alg}}(R, -)$ (see Exercise D.3.8). With this terminology, $\{x_n\}$ is versal if there exists a lift for every commutative diagram

$$\begin{tikzcd} \operatorname{Spec} A_0 \arrow[r] \arrow[d, hook] & h_R \arrow[d, "\{x_n\}"] \\ \operatorname{Spec} A \arrow[r, "\eta"] \arrow[ur, dashed] & \mathcal{X} \end{tikzcd} \tag{D.3.1}$$

of solid arrows where $A \twoheadrightarrow A_0$ is a surjection in $\mathrm{Art}_\Bbbk$. In this way, we see that a versal formal deformation corresponds to the Infinitesimal Lifting Criterion for Smoothness (see Smooth Equivalences A.3.1(2) and Theorem 3.7.1) with respect to artinian local $\Bbbk$-algebras. Meanwhile a miniversal deformation is a versal formal deformation inducing an isomorphism on tangent spaces $h_R(\Bbbk[\epsilon]) \to \mathcal{X}(\Bbbk[\epsilon])/\sim$.

Remark D.3.7. The condition of versality can be checked on surjections $A \twoheadrightarrow A_0$ with $\ker(A \to A_0) \cong \Bbbk$. Indeed, the kernel of every surjection $A \twoheadrightarrow A_0$ in $\mathrm{Art}_\Bbbk$ is a finite dimensional $\Bbbk$-vector space and $A \to A_0$ can be factored into surjections where each kernel is one-dimensional.

Exercise D.3.8. Let R be a noetherian complete local $\Bbbk$-algebra and let $h_R = \mathrm{Hom}_{\Bbbk\text{-alg}}(R, -)$ be the covariant functor $\mathrm{Art}_\Bbbk \to \mathrm{Sets}$ which we can also view as a prestack over $\mathrm{Art}_\Bbbk^{\mathrm{op}}$. If $\mathcal{X}$ is a prestack over $\mathrm{Art}_\Bbbk^{\mathrm{op}}$, show that giving a formal deformation $(R, \{x_n\})$ is equivalent to giving a morphism $h_R \to \mathcal{X}$ of prestacks.

Remark D.3.9. If F is pro-representable by R, then letting $x_n \in F(R/\mathfrak{m}_R^{n+1})$ correspond to the surjection $(R \twoheadrightarrow R/\mathfrak{m}_R^{n+1}) \in h_R(R/\mathfrak{m}_R^{n+1})$, it is easy to see that $\{x_n\}$ is a versal formal deformation. In this case, there is a unique lift in (D.3.1)

Remark D.3.10 (Global prestacks to local deformation prestacks)**.** If $\mathcal{X}$ is a prestack over $\mathrm{Sch}/\Bbbk$ and $x_0 \in \mathcal{X}(\Bbbk)$, we can consider the *local deformation prestack* $\mathcal{X}_{x_0}$ *at* x_0 as the prestack of morphisms $x_0 \to x$ over $\mathrm{Art}_\Bbbk^{\mathrm{op}}$ where a morphism $(x_0 \xrightarrow{\alpha} x) \to (x_0 \xrightarrow{\alpha'} x')$ is a morphism $\beta \colon x \to x'$ such that $\alpha' = \alpha \circ \beta$. In other words, an object of $\mathcal{X}_{x_0}$ is a pair (x, α) where $x \in \mathcal{X}(A)$ and $\alpha \colon x_0 \to x|_\Bbbk$ is an isomorphism. Note that the fiber category $\mathcal{X}_{x_0}(\Bbbk)$ is equivalent to the set $\{x_0 \xrightarrow{\mathrm{id}} x_0\}$.

If $\mathcal{X}$ is algebraic with a smooth presentation $U \to \mathcal{X}$ from a scheme and $u \in U(\Bbbk)$ is a point mapping to x_0, then we may set $x_n \in \mathcal{X}(\mathcal{O}_{U,u}/\mathfrak{m}_u^{n+1})$ to be the composition $\operatorname{Spec} \mathcal{O}_{U,u}/\mathfrak{m}_u^{n+1} \hookrightarrow U \to \mathcal{X}$. Then $\{x_n\}$ is a versal formal deformation.

On the other hand, if $\mathcal{X}$ is not yet known to be algebraic, one can sometimes verify the existence of versal formal deformation via Rim–Schlessinger's Criteria (Theorem D.3.11) as a first step to verifying the algebraicity of $\mathcal{X}$ via Artin's Axioms for Algebraicity (Theorem D.7.1).

D.3.3 Rim–Schlessinger's Criteria

Rim–Schlessinger's Criteria provides necessary and sufficient conditions for a prestack $\mathcal{X}$ over $\mathrm{Art}_{\Bbbk}^{\mathrm{op}}$ or covariant functor $F\colon \mathrm{Art}_{\Bbbk} \to \mathrm{Sets}$ to admit a versal formal deformation.

Theorem D.3.11 (Rim–Schlessinger's Criteria)**.** *Let $\mathcal{X}$ be a prestack over $\mathrm{Art}_{\Bbbk}^{\mathrm{op}}$ such that the groupoid $\mathcal{X}(\Bbbk)$ is equivalent to the set $\{x_0\}$. For morphisms $B_0 \to A_0$ and $A \to A_0$ in $\mathrm{Art}_{\Bbbk}$, consider the natural functor*

$$\mathcal{X}(B_0 \times_{A_0} A) \to \mathcal{X}(B_0) \times_{\mathcal{X}(A_0)} \mathcal{X}(A) \tag{D.3.2}$$

Then $\mathcal{X}$ admits a miniversal formal deformation if and only if

(RS$_1$) *the functor (D.3.2) is essentially surjective whenever $A \twoheadrightarrow A_0$ is surjection with kernel $\Bbbk$;*

(RS$_2$) *the map (D.3.2) is essentially surjective when $A_0 = \Bbbk$ and $A = \Bbbk[\epsilon]$, and given two commutative diagrams*

$$\begin{tikzcd} x_0 \arrow[r] \arrow[d] & y_0 \arrow[d, "\alpha_1"'] \arrow[dr, "\alpha_2"] & \\ x \arrow[r] \arrow[rr, bend right] & y \arrow[r, dashed, "\beta"] & y' \end{tikzcd} \quad \textit{over} \quad \begin{tikzcd} \operatorname{Spec} \Bbbk \arrow[r, hook] \arrow[d] & \operatorname{Spec} B_0 \arrow[d] \\ \operatorname{Spec} \Bbbk[\epsilon] \arrow[r, hook] & \operatorname{Spec}(\Bbbk[\epsilon] \times_{\Bbbk} B_0) \end{tikzcd}$$

there exists an isomorphism $\beta\colon y \to y'$ in $\mathcal{X}(\Bbbk[\epsilon] \times_{\Bbbk} B_0)$ such that $\alpha_2 = \beta \circ \alpha_1$.

(RS$_3$) $\dim_{\Bbbk} T_{\mathcal{X}} < \infty$ *where* $T_{\mathcal{X}} := \mathcal{X}(\Bbbk[\epsilon])/\!\sim$.

Moreover, $\mathcal{X}$ is prorepresentable if and only if $\mathcal{X}$ is equivalent to a functor and

(RS$_4$) *the map (D.3.2) is an equivalence whenever $A \twoheadrightarrow A_0$ is a surjection with kernel $\Bbbk$.*

Conditions (RS$_2$)–(RS$_3$) (sometimes referred to as *semi-homogeneity*) may be difficult to parse[1] but in practice it is almost always just as easy to verify the stronger condition (RS$_4$) (called *homogeneity*), and in fact the even stronger condition (RS$_4^*$) (called *strong homogeneity*) introduced in §D.3.4.

Remark D.3.12 (Schlessinger's Criteria)**.** When $\mathcal{X}$ is a covariant functor $F\colon \mathrm{Art}_{\Bbbk} \to \mathrm{Sets}$ with $F(\Bbbk) = \{x_0\}$, then (RS$_1$)–(RS$_4$) translate into Schlessinger's conditions as introduced in [Sch68]:

(H$_1$) the map (D.3.2) is surjective whenever $A \twoheadrightarrow A_0$ is a surjection with kernel $\Bbbk$;

(H$_2$) the map (D.3.2) is bijective when $A_0 = \Bbbk$ and $A = \Bbbk[\epsilon]$;

(H$_3$) $\dim_{\Bbbk} F(\Bbbk[\epsilon]) < \infty$; and

(H$_4$) the map (D.3.2) is bijective whenever $A \twoheadrightarrow A_0$ is a surjection with kernel $\Bbbk$.

The functor F admits a miniversal formal deformation if (H$_1$)–(H$_3$) hold and is pro-representable if (H$_3$)–(H$_4$) hold.

If $\mathcal{X}$ satisfies (RS$_1$)–(RS$_3$), then the functor $F_{\mathcal{X}}\colon \mathrm{Sch}/\Bbbk \to \mathrm{Sets}$ parameterizing isomorphism classes of objects satisfies (H$_1$)–(H$_3$) but the converse does not always hold. On the other hand, the essential surjectivity of $\mathcal{X}(B_0 \times_{A_0} A) \to \mathcal{X}(B_0) \times_{\mathcal{X}(A_0)}$

[1] The second part of (RS$_2$) is slightly stronger than requiring that two objects in $\mathcal{X}(\Bbbk[\epsilon] \times_{\Bbbk} B_0)$ are isomorphic if and only if their images are. Note that (RS$_2$) does not require the isomorphism $\beta\colon y \to y'$ to be compatible with the given morphisms $x \to y$ and $x \to y'$.

$\mathcal{X}(A)$ implies the surjectivity of $F_{\mathcal{X}}(B_0 \times_{A_0} A) \to F_{\mathcal{X}}(B_0) \times_{F_{\mathcal{X}}(A_0)} F_{\mathcal{X}}(A)$ and the fully faithfulness for $\mathcal{X}$ implies the injectivity of $F_{\mathcal{X}}$ as long as $\mathrm{Aut}_{\mathcal{X}(B_0)}(y_0) \to \mathrm{Aut}_{\mathcal{X}(A_0)}(y_0|_{A_0})$ is surjective for an object $y_0 \in \mathcal{X}(B_0)$. This latter condition holds in the case when $F_{\mathcal{X}}(A_0)$ is a set, e.g. when $A_0 = \Bbbk$. If $\mathcal{X}$ is the local deformation prestack arising from an object $x_0 \in \widetilde{\mathcal{X}}(\Bbbk)$ of an algebraic stack $\widetilde{\mathcal{X}}$ over $\mathrm{Sch}/\Bbbk$ as in Remark D.3.10, then the surjectivity condition on automorphisms translates to the inertia stack $I_{\mathcal{X}} \to \mathcal{X}$ being smooth at $e(x_0)$, where $e\colon \mathcal{X} \to I_{\mathcal{X}}$ is the identity section, by the Infinitesimal Lifting Criterion for Smoothness (Theorem 3.7.1).

While the existence of a miniversal formal deformation of $F_{\mathcal{X}}$ suffices for many applications, for moduli problems with automorphisms it is more natural to ask for the existence of a miniversal formal deformation of $\mathcal{X}$ and this generality is needed for some applications, e.g. Artin's Algebraization (Theorem D.6.6) and Artin's Axioms for Algebraicity (Theorem D.7.4).

Before proceeding to the proof, we first show that (RS_1)–(RS_2) yield natural structures on sets of deformations. In particular, they induce a vector space structure on the tangent space $T_{\mathcal{X}} = \mathcal{X}(\Bbbk[\epsilon])/{\sim}$ which allows us to make sense of condition (RS_3).

Lemma D.3.13. *Let $\mathcal{X}$ be a prestack over $\mathrm{Art}_{\Bbbk}^{\mathrm{op}}$ such that the groupoid $\mathcal{X}(\Bbbk)$ is equivalent to the set $\{x_0\}$, and let $F_{\mathcal{X}}\colon \mathrm{Art}_{\Bbbk} \to \mathrm{Sets}$ be the covariant functor assigning $A \in \mathrm{Art}_{\Bbbk}$ to the set of isomorphism classes $\mathcal{X}(A)/\sim$. Assume that Condition (RS_2) holds for $\mathcal{X}$.*

(1) The tangent space $T_{\mathcal{X}} = F_{\mathcal{X}}(\Bbbk[\epsilon])$ has a natural structure of a $\Bbbk$-vector space. More generally, for every finite dimensional $\Bbbk$-vector space V, denoting $\Bbbk[V]$ as the $\Bbbk$-algebra $\Bbbk \oplus V$ defined by $V^2 = 0$, the set $F_{\mathcal{X}}(\Bbbk[V])$ has a natural structure of a $\Bbbk$-vector space and there is a functorial bijection $F_{\mathcal{X}}(\Bbbk[V]) = T_{\mathcal{X}} \otimes_{\Bbbk} V$.

(2) Consider a surjection $B \twoheadrightarrow A$ in $\mathrm{Art}_{\Bbbk}$ with square-zero kernel I and an element $x \in \mathcal{X}(A)$, and let $\mathrm{Lift}_x(B)$ be the set of morphisms $x \to y$ over $\operatorname{Spec} A \to \operatorname{Spec} B$ where $x \xrightarrow{\alpha} y$ is declared equivalent to $x \xrightarrow{\alpha'} y'$ if there is an isomorphism $\beta\colon y \to y'$ such that $\alpha' = \beta \circ \alpha$. There is an action of $T_{\mathcal{X}} \otimes I$ on $\mathrm{Lift}_x(B)$ which is functorial in $\mathcal{X}$. Assuming $\mathrm{Lift}_x(B)$ is non-empty, this action is transitive if Condition (RS_1) holds for $\mathcal{X}$ and free and transitive (i.e. $\mathrm{Lift}_x(B)$ is a torsor under $T_{\mathcal{X}} \otimes I$) if Condition (RS_4) holds for $\mathcal{X}$.

Proof. We first note if V is a finite dimensional vector space, then $\Bbbk[V] = \Bbbk[\epsilon] \times_{\Bbbk} \cdots \times_{\Bbbk} \Bbbk[\epsilon]$ and by applying (RS_2) inductively, we see that the statement of Condition (RS_2) also holds for $A_0 = \Bbbk$ and $A = \Bbbk[V]$. For $B_0 \in \mathrm{Art}_{\Bbbk}$, the first part of (RS_2) implies that $F_{\mathcal{X}}(B_0 \times_{\Bbbk} \Bbbk[V]) \xrightarrow{\sim} F_{\mathcal{X}}(B_0) \times F_{\mathcal{X}}(\Bbbk[V])$ is a bijection. In particular, $F_{\mathcal{X}}(\Bbbk[V] \times_{\Bbbk} \Bbbk[W]) \xrightarrow{\sim} F_{\mathcal{X}}(\Bbbk[V]) \times F_{\mathcal{X}}(\Bbbk[W])$ is bijective for every pair of finite dimensional vector spaces, or in other words the functor $V \mapsto F_{\mathcal{X}}(\Bbbk[V])$ commutes with finite products.

The vector space structure of $T_{\mathcal{X}} = F_{\mathcal{X}}(\Bbbk[\epsilon])$ follows from the bijectivity of

$$F_{\mathcal{X}}(\Bbbk[\epsilon] \times_{\Bbbk} \Bbbk[\epsilon']) \xrightarrow{\sim} F_{\mathcal{X}}(\Bbbk[\epsilon]) \times F_{\mathcal{X}}(\Bbbk[\epsilon']). \tag{D.3.3}$$

Indeed, addition of $\tau_1, \tau_2 \in F_{\mathcal{X}}(\Bbbk[\epsilon])$ is defined by using the above identification to view $(\tau_1, \tau_2) \in F_{\mathcal{X}}(\Bbbk[\epsilon] \times_{\Bbbk} \Bbbk[\epsilon'])$ and then taking its image under $F(\Bbbk[\epsilon] \times_{\Bbbk} \Bbbk[\epsilon']) \to F(\Bbbk[\epsilon])$ induced by the ring map $\Bbbk[\epsilon] \times_{\Bbbk} \Bbbk[\epsilon'] \to \Bbbk[\epsilon]$ taking $(\epsilon, 0)$ and $(0, \epsilon')$ to ϵ.

Scalar multiplication of $c \in \Bbbk$ on $\tau \in F_{\mathcal{X}}(\Bbbk[\epsilon])$ is defined by taking the image of τ under $F_{\mathcal{X}}(\Bbbk[\epsilon]) \to F_{\mathcal{X}}(\Bbbk[\epsilon])$ induced by the map $\Bbbk[\epsilon] \to \Bbbk[\epsilon]$ taking ϵ to $c\epsilon$.

The same argument gives $F_{\mathcal{X}}(\Bbbk[V])$ the structure of a vector space such that the assignment $V \mapsto F_{\mathcal{X}}(\Bbbk[V])$ is a $\Bbbk$-linear functor $\mathrm{Vect}_{\Bbbk}^{\mathrm{fd}} \to \mathrm{Vect}_{\Bbbk}$ defined on finitely dimensional $\Bbbk$-vector spaces. The natural map

$$F_{\mathcal{X}}(\Bbbk[\epsilon]) \times \mathrm{Hom}_{\Bbbk}(\Bbbk[\epsilon], \Bbbk[V]) \to F_{\mathcal{X}}(\Bbbk[V]), \qquad (\tau, \phi) \mapsto \phi^* \tau$$

is $\Bbbk$-bilinear and under the equivalences $T_{\mathcal{X}} = F_{\mathcal{X}}(\Bbbk[\epsilon])$ and $V = \mathrm{Hom}_{\Bbbk}(\Bbbk[\epsilon], \Bbbk[V])$ corresponds to a linear map $T_{\mathcal{X}} \otimes V \to F_{\mathcal{X}}(\Bbbk[V])$, which is an isomorphism. This finishes the proof of (1).

For (2), we observe that the natural map

$$B \times_A B \to \Bbbk[I] \times_{\Bbbk} B, \qquad (b_1, b_2) \mapsto (\overline{b_1} + b_2 - b_1, b_1)$$

is an isomorphism. We therefore have a diagram

$$\mathcal{X}(\Bbbk[I]) \times \mathcal{X}(B) \twoheadleftarrow \mathcal{X}(\Bbbk[I] \times_{\Bbbk} B) \cong \mathcal{X}(B \times_A B) \xrightarrow{p_1^*} \mathcal{X}(B) \tag{D.3.4}$$

where the left functor is essentially surjective by the first part of (RS_2). Given $\tau \in T_{\mathcal{X}} \otimes I = F_{\mathcal{X}}(\Bbbk[I])$ (with a choice of representative in $\mathcal{X}(\Bbbk[I])$) and $(x \xrightarrow{\alpha} y) \in \mathrm{Lift}_x(B)$, we would like to define $\tau \cdot (x \xrightarrow{\alpha} y)$ as the image under p_1^* of a choice of preimage of (τ, y). To see that this is well-defined, consider two elements $z, z' \in \mathcal{X}(\Bbbk[I] \times_{\Bbbk} B)$ whose images in $\mathcal{X}(\Bbbk[I]) \times \mathcal{X}(B)$ are isomorphic to (τ, y). This yields a diagram

$$\begin{tikzcd} x_0 \arrow[r] \arrow[d] & y \arrow[d, "\alpha_1"'] \arrow[dr, "\alpha_2"] & \\ \tau \arrow[r] \arrow[rr, bend right] & z \arrow[r, dashed, "\beta"] & z' \end{tikzcd} \quad \text{over} \quad \begin{tikzcd} \operatorname{Spec} \Bbbk \arrow[r, hook] \arrow[d] & \operatorname{Spec} B \arrow[d] \\ \operatorname{Spec} \Bbbk[I] \arrow[r, hook] & \operatorname{Spec}(\Bbbk[I] \times_{\Bbbk} B) \end{tikzcd}$$

and by the second part of (RS_2), there exists a dotted arrow β such that $\alpha_2 = \beta \circ \alpha_1$. Therefore choices of pullbacks $p_1^* z$ and $p_1^* z'$ in $\mathcal{X}(B)$ defines the same element in $\mathrm{Lift}_x(B)$. If (RS_1) holds, then the statement of Condition (RS_1) also holds for every surjection $A \mapsto A_0$ (since we may factor it as a composition of surjections whose kernels are $\Bbbk$). Therefore if (RS_1) holds (resp. (RS_4) holds), then $\mathcal{X}(B \times_A B) \to \mathcal{X}(B) \times_{\mathcal{X}(A)} \mathcal{X}(B)$ is essentially surjective (resp. an equivalence) and we see that the action is transitive (resp. free and transitive). $\square$

Proof Theorem D.3.11. The details of the necessity of these conditions are left to the reader. We will establish the sufficiency. The tangent space $T_{\mathcal{X}} := \mathcal{X}(\Bbbk[\epsilon])/\!\sim$ has the structure of a vector space by Lemma D.3.13(1) and is finite dimensional by (RS_2). Let $N = \dim_{\Bbbk} T_{\mathcal{X}}$ with basis $x_1, \ldots, x_N$ and define $S = \Bbbk[\![x_1, \ldots, x_N]\!]$. We will construct inductively a decreasing sequence of ideals $J_0 \supset J_1 \supset \cdots$ and objects $\eta_n \in \mathcal{X}(S/J_n)$ together with morphisms $\eta_n \to \eta_{n+1}$ over $\operatorname{Spec} S/J_n \hookrightarrow \operatorname{Spec} S/J_{n+1}$. We set $J_0 = \mathfrak{m}_S$ and $\eta_0 = x_0 \in \mathcal{X}(\Bbbk)$. We also set $J_1 = \mathfrak{m}_S^2$ so that $S/J_1 \cong \Bbbk[T_{\mathcal{X}}]$. Using the bijection $F_{\mathcal{X}}(\Bbbk[T_{\mathcal{X}}]) \cong T_{\mathcal{X}} \otimes_{\Bbbk} T_{\mathcal{X}}$ of Lemma D.3.13(1), the element $\sum_i x_i \otimes x_i$ defines an isomorphism class of an object $\eta_1 \in \mathcal{X}(S/J_1)$ such that the induced map $\operatorname{Spec} S/J_1 \to \mathcal{X}$ induces a bijection on tangent spaces. By construction, we have a morphism $\eta_0 \to \eta_1$ over $\operatorname{Spec} \Bbbk \hookrightarrow \operatorname{Spec} S/J_1$.

Suppose we've constructed J_n and $\eta_{n-1} \to \eta_n$. We claim that the set of ideals

$$\begin{aligned}\Sigma = \{J \subset S \mid \mathfrak{m}_S J_n \subset J \subset J_n \text{ and there exists } \eta_n \to \eta \\ \text{over } \operatorname{Spec} S/J_n \hookrightarrow \operatorname{Spec} S/J\}\end{aligned} \tag{D.3.5}$$

has a minimal element. Indeed, it is non-empty since $J_n \in \Sigma$ and given $J, K \in \Sigma$, we must check that $J \cap K \in \Sigma$. To achieve this, choose an ideal $J' \subset S$ satisfying $J \subset J' \subset I$ with $J \cap K = J' \cap K$ and $J' + K = I$. Then $A/(J' \cap K) \cong A/J' \times_{A/I} A/K$. Letting $\eta_J \in \mathcal{X}(S/J)$ and $\eta_K \in \mathcal{X}(S/K)$ be the objects corresponding to J and K, the data of $(\eta_J|_{S/J'} \leftarrow \eta_n \to \eta_K)$ defines an object of $\mathcal{X}(S/J') \times_{\mathcal{X}(S/I)} \mathcal{X}(S/K)$. The functor $\mathcal{X}(A/(J \cap K)) \to \mathcal{X}(S/J') \times_{\mathcal{X}(S/I)} \mathcal{X}(S/K)$ is essentially surjective by (RS$_1$) and the existence of preimage of $(\eta_J|_{S/J'} \leftarrow \eta_n \to \eta_K)$ shows that $J \cap K \in \Sigma$.

Setting $J = \bigcap_n J_n$, then $R = S/J$ is a noetherian complete local $\Bbbk$-algebra with ideals $I_n := J_n/J$. Since $\mathfrak{m}_S J_n \subset J_{n+1}$, we have that $\mathfrak{m}_R^{n+1} \subset I_n$ and thus $\xi_n := \eta_n|_{R/\mathfrak{m}_R^{n+1}}$ defines a formal deformation of x_0 over R.

We must check that $\xi := \{\xi_n\}$ is versal. Suppose $B \twoheadrightarrow A$ is a surjection in $\mathrm{Art}_{\Bbbk}$ with kernel $\Bbbk$ and that we have a diagram

$$\begin{tikzcd} x \arrow[r] \arrow[d] & \xi \\ y & \end{tikzcd} \quad \text{over} \quad \begin{tikzcd} \operatorname{Spec} A \arrow[r, "g"] \arrow[d, hook] & h_R \\ \operatorname{Spec} B. \arrow[ur, dashed, "\widetilde{g}"'] & \end{tikzcd}$$

We need to construct a morphism $y \to \xi$ extending $x \to \xi$. We claim that it suffices to construct a morphism $\widetilde{g}\colon \operatorname{Spec} B \to h_R$ (i.e. a ring map $R \to B$) extending g. Since $h_R(\Bbbk[\epsilon]) \to T_{\mathcal{X}}$ is bijective, Lemma D.3.13(2) implies that there are actions of $T_{\mathcal{X}}$ on the sets $\mathrm{Lift}_x(B)$ and $\mathrm{Lift}_g(B)$ of isomorphism classes of lifts of x and g to objects in $\mathcal{X}(B)$ and $h_R(B)$ which are compatible with the map $\mathrm{Lift}_g(B) \to \mathrm{Lift}_x(B)$ where $\widetilde{g} \mapsto \widetilde{g}^*\xi$. Thus, we can find $\tau \in T_{\mathcal{X}}$ such that $y = \tau \cdot (\widetilde{g}^*\xi) = (\tau \cdot \widetilde{g})^*\xi$. This gives an arrow $y \to \xi$ over $\tau \cdot \widetilde{g}\colon \operatorname{Spec} B \to h_R$.

To construct $\widetilde{g}$, choose n such that $R \to A$ factors as $R \to R/I_n = S/J_n \to A$. It suffices to show that $\operatorname{Spec} A \to \operatorname{Spec} S/J_n$ extends to a map $\operatorname{Spec} B \to \operatorname{Spec} S/J_{n+1}$ and for this, it suffices to show the existence of a dotted arrow making the diagram

$$\begin{tikzcd} \operatorname{Spec} A \arrow[r] \arrow[d, hook] & \operatorname{Spec} S/J_n \arrow[d, hook] \arrow[ddr, hook] & \\ \operatorname{Spec} B \arrow[r] & \operatorname{Spec} B \times_A (S/J_n) \arrow[dr, dashed] & \\ & & \operatorname{Spec} S/J_{n+1} \end{tikzcd}$$

commutative. As $S = \Bbbk[\![x_1, \ldots, x_n]\!]$, we may choose an extension $S \to B$ of $S \to S/J_n \to A$. Then $B \times_A (S/J_n) = S/K$ where K is the kernel of the induced map $S \to B \times_A (S/J_n)$. The kernel K lies in the set of ideals defined in (D.3.5): the inclusion $K \subset J_n$ is clear, the inclusion $\mathfrak{m}_S J_n \subset K$ is implied by the equality $\ker(B \to A) = \Bbbk$, and the existence of $\eta_n \to \eta$ over $\operatorname{Spec} S/J_n \hookrightarrow \operatorname{Spec} S/K$ follows from applying (RS$_1$) to the above square. Thus $J_{n+1} \subset K$ and we have a ring map $S/J_{n+1} \to S/K = B \times_A (S/J_n)$ inducing the desired dotted arrow.

Finally, we must show that if $\mathcal{X}$ is equivalent to a functor F and (RS$_4$) holds, then F is prorepresentable by $\xi = \{\xi_n\}$. Given a surjection $B \to A$ with kernel $\Bbbk$ and $x \in F(A)$, it suffices to show the existence of a *unique* lift in every diagram

$$\begin{tikzcd} \operatorname{Spec} A \arrow[r, "g"] \arrow[d, hook] & h_R \arrow[d] \\ \operatorname{Spec} B \arrow[r] \arrow[ur, dashed] & F. \end{tikzcd}$$

This holds because the map $\mathrm{Lift}_g(B) \to \mathrm{Lift}_x(B)$ is bijective by Lemma D.3.13(2) as both are torsors under $T_{\mathcal{X}}$.

See also [Sch68, Thm. 2.11], [SGA7-I, Thm. VI.1.11] and [SP, Tag 06IX], where the result is established more generally for prestacks over the category Art_Λ introduced in Remark D.3.4. ☐

D.3.4 Verifying Rim–Schlessinger's Conditions

Consider the following *strong homogeneity* condition:

(RS$_4^*$) $\mathcal{X}(B_0 \times_{A_0} A) \to \mathcal{X}(B_0) \times_{\mathcal{X}(A_0)} \mathcal{X}(A)$ is an equivalence for every map $B_0 \to A_0$ and surjection $A \twoheadrightarrow A_0$ of rings with square-zero kernel (where the rings are not necessarily local artinian);

If $\mathcal{X}$ is a prestack over $(\mathrm{Sch}/\Bbbk)$ satisfying (RS$_4^*$), then the local deformation prestack $\mathcal{X}_{x_0}$ at x_0 (see Remark D.3.10) is easily checked to satisfy (RS$_4$). On the other hand, it turns out that every algebraic stack satisfies (RS$_4^*$); see [SP, Tag 07WN]. In other words, the Ferrand pushout $\operatorname{Spec}(B_0 \times_{A_0} A)$ is a pushout in the category of algebraic stacks. Condition (RS$_4^*$) will appear in our second version of Artin's Axioms for Algebraicity (Theorem D.7.4) as it will be useful to verify openness of versality (in addition to implying (RS$_2$)–(RS$_3$) ensuring the existence of versal formal deformations).

For a moduli problem $\mathcal{M}$, it is often possible to verify (RS$_4^*$) (and thus (RS$_4$) as well as (RS$_1$)–(RS$_2$)) as a consequence of Proposition A.8.5: for a ring map $B_0 \to A_0$ and surjection $A \twoheadrightarrow A_0$, the functor $\mathrm{Mod}(B_0 \times_{A_0} A) \to \mathrm{Mod}(B_0) \times_{\mathrm{Mod}(A_0)} \mathrm{Mod}(A)$ restricts to an equivalence on flat modules. When B_0, A_0 and A are artinian, there is an elementary argument for this fact since flatness translates to freeness for modules over an artinian ring (Proposition A.2.3).

We say that a prestack $\mathcal{X}$ over $\mathrm{Sch}/\Bbbk$ *admits formal versal deformations* if for every $\Bbbk$-point x_0, the local deformation prestack $\mathcal{X}_{x_0}$ (Remark D.3.10) admits a formal versal deformation.

Proposition D.3.14. *Each of the moduli problems* $\mathrm{Hilb}^P(X)$, $\mathscr{M}_g$ *(with* $g \geq 2$*) and* $\mathcal{B}\mathrm{un}_C$ *over* $\Bbbk$ *satisfy* (RS$_3$) *and* (RS$_4^*$)*, and therefore admit formal versal deformations.*

Proof. To check (RS$_3$) for objects $[Z_0 \subset X]$, C_0 and E_0 of $\mathcal{X} = \mathrm{Hilb}^P(X)$, $\mathscr{M}_g$ and $\mathcal{B}\mathrm{un}_C$ defined over $\Bbbk$, we have identifications of the tangent spaces $T_{\mathcal{X}}$ with the *finite dimensional* $\Bbbk$-vector spaces $\mathrm{H}^0(Z_0, N_{Z_0/X})$, $H^1(C_0, T_{C_0})$ and $\mathrm{H}^1(X, \mathscr{E}nd_{\mathcal{O}_X}(E_0))$ by Propositions D.1.4, D.1.11 and D.1.15.

For (RS$_4^*$), let $B_0 \to A_0$ be a ring map and $A \twoheadrightarrow A_0$ be a surjection with square-zero kernel. Set $B = B_0 \times_{A_0} A$. For $\mathrm{Hilb}^P(X)$, Corollary A.8.6(1)–(2) implies that the diagram

$$\begin{tikzcd} X_{A_0} \arrow[r, hook] \arrow[d] & X_A \arrow[d] \\ X_{B_0} \arrow[r, hook] & X_B \end{tikzcd}$$

is a pushout and that the functor

$$\mathrm{QCoh}(X_B) \to \mathrm{QCoh}(X_{B_0}) \times_{\mathrm{QCoh}(X_{A_0})} \mathrm{QCoh}(X_A) \tag{D.3.6}$$

restricts to an equivalence between the full subcategory of finitely presented $\mathcal{O}_{X_B}$-modules flat over B and the fiber product of the full subcategories of finitely presented $\mathcal{O}$-modules flat over B_0 and A. This implies the desired equivalence $\mathrm{Hilb}^P(X)(B) \to \mathrm{Hilb}^P(X)(B_0) \times_{\mathrm{Hilb}^P(X)(A_0)} \mathrm{Hilb}^P(X)(A)$ between closed subschemes flat over the base.

For $\mathscr{M}_g$, the essential surjectivity of $\mathscr{M}_g(B) \to \mathscr{M}_g(B_0) \times_{\mathscr{M}_g(A_0)} \mathscr{M}_g(A)$ translates into the existence of an extension

$$\begin{tikzcd}[row sep=small] & \mathcal{C}_0 \arrow[rr, hook] \arrow[dl] \arrow[dd] & & \mathcal{C} \arrow[dl, dashed] \arrow[dd] \\ \mathcal{D}_0 \arrow[rr, dashed] \arrow[dd] & & \mathcal{D} \arrow[dd, dashed] & \\ & \operatorname{Spec} A_0 \arrow[rr, hook] \arrow[dl] & & \operatorname{Spec} A \arrow[dl] \\ \operatorname{Spec} B_0 \arrow[rr, hook] & & \operatorname{Spec} B & \end{tikzcd}$$

of smooth families of curves. The existence of $\mathcal{D}$ as a pushout of top face follows from Theorem A.8.1. The fact that $\mathcal{D}$ is smooth over B follows from Corollary A.8.6(2). The properness of $\mathcal{D} \to \operatorname{Spec} B$ follows from the properness of $\mathcal{D}_0 \to \operatorname{Spec} B_0$. The fully faithfulness translates to the bijectivity of

$$\mathrm{Aut}(\mathcal{D}/B) \to \mathrm{Aut}(\mathcal{D}_0/B_0) \times_{\mathrm{Aut}(\mathcal{C}_0/A_0)} \mathrm{Aut}(\mathcal{C}/A)$$

and follows direct from the fact that $\mathcal{D}$ is a pushout of the top face. Alternatively, one can replicate the above argument for $\mathrm{Hilb}^P(X)$ using the tricanonical embedding.

For $\mathcal{B}\mathrm{un}_C$, Corollary A.8.6(1) implies that the functor (D.3.6) restricts to an equivalence on finitely presented $\mathcal{O}$-modules flat over the base and therefore also on vector bundles. □

D.4 Effective formal deformations and Grothendieck's Existence Theorem

We often would like to know when a formal deformation is effective.

Definition D.4.1. Let $\mathcal{X}$ be a prestack (or functor) over $(\mathrm{Sch}/\Bbbk)$. Let $x_0 \in \mathcal{X}(\Bbbk)$ and consider a formal deformation $(R, \{x_n\})$ of x_0 (or more precisely a formal deformation of the deformation stack $\mathcal{X}_{x_0}$ at x_0 as defined in Remark D.3.10). We say that $\{x_n\}$ is *effective* if there exists an object $\widehat{x} \in \mathcal{X}(R)$ and compatible isomorphisms $x_n \xrightarrow{\sim} \widehat{x}|_{\operatorname{Spec} R/\mathfrak{m}^{n+1}}$.

Remark D.4.2. A formal deformation $(R, \{x_n\})$ is effective if it is in the essential image of the natural functor $\mathcal{X}(R) \to \varprojlim \mathcal{X}(R/\mathfrak{m}^n)$ or in other words if there exists a dotted arrow making the diagram

$$\begin{tikzcd}
\operatorname{Spec} R/\mathfrak{m} \arrow[r, hook] \arrow[drrrr, "x_0"'] & \operatorname{Spec} R/\mathfrak{m}^2 \arrow[r, hook] \arrow[drrr, "x_1"] & \operatorname{Spec} R/\mathfrak{m}^3 \arrow[r, hook] \arrow[drr, "x_2"] & \cdots \arrow[r, hook] & \operatorname{Spec} R \arrow[d, dashed, "\widehat{x}"] \\
 & & & & \mathcal{X}
\end{tikzcd}$$

commutative.

Example D.4.3. If $F\colon \mathrm{Sch}/\Bbbk \to \mathrm{Sets}$ is a contravariant functor representable by a scheme X over $\Bbbk$, then every formal deformation $(R, \{x_n\})$ is effective. Indeed, x_n corresponds to a morphism $\operatorname{Spec} R/\mathfrak{m}^{n+1} \to X$ with image $x \in X(\Bbbk)$ and thus to a $\Bbbk$-algebra homomorphism $\phi_n\colon \widehat{\mathcal{O}}_{X,x} \to R/\mathfrak{m}^{n+1}$. By taking the inverse image of ϕ_n, we have a local homomorphism $\widehat{\mathcal{O}}_{X,x} \to R$ which in turn defines a morphism $\widehat{x}\colon \operatorname{Spec} R \to X$ extending $\{x_n\}$.

Grothendieck's Existence Theorem—sometimes referred to as Formal GAGA—can often be applied to show that formal deformations are effective.

Theorem D.4.4 (Grothendieck's Existence Theorem)**.** *Let $X \to \operatorname{Spec} R$ be a proper morphism of schemes where $(R, \mathfrak{m})$ is a noetherian complete local ring. Set $X_n := X \times_R R/\mathfrak{m}^{n+1}$ The functor*

$$\operatorname{Coh}(X) \to \varprojlim \operatorname{Coh}(X_n), \qquad E \mapsto \{E_n\}, \tag{D.4.1}$$

where E_n is the pullback of E along $X_n \to X$, is an equivalence of categories.

Proof. See [EGA, III.5.1.4], [FGI+05, Thm. 8.4.2] and [SP, Tag 088E]. □

Remark D.4.5. The essential surjectivity of (D.4.1) translates to an extension of the diagram

$$\begin{tikzcd}
E_0 \arrow[d, dash] & E_1 \arrow[d, dash] & E_2 \arrow[d, dash] & & E \arrow[d, dash, dashed] \\
X_0 \arrow[r, hook] \arrow[d] & X_1 \arrow[r, hook] \arrow[d] & X_2 \arrow[r, hook] \arrow[d] & \cdots \arrow[r, hook] & X \arrow[d] \\
\operatorname{Spec} R/\mathfrak{m} \arrow[r, hook] & \operatorname{Spec} R/\mathfrak{m}^2 \arrow[r, hook] & \operatorname{Spec} R/\mathfrak{m}^3 \arrow[r, hook] & \cdots \arrow[r, hook] & \operatorname{Spec} R
\end{tikzcd}$$

while the fully faithfulness of (D.4.1) translates to the bijectivity of the natural map $\operatorname{Hom}_{\mathcal{O}_X}(E, F) \to \varprojlim \operatorname{Hom}_{\mathcal{O}_{X_n}}(E_n, F_n)$ for coherent sheaves E and F on X.

Using the language of formal schemes and setting $\widehat{X} = X \times_{\operatorname{Spec} R} \operatorname{Spf} R$ to be the $\mathfrak{m}$-adic completion of X, then Grothendieck's Existence Theorem asserts that the functor $\operatorname{Coh}(X) \to \operatorname{Coh}(\widehat{X})$, defined by $E \mapsto \widehat{E}$, is an equivalence.

Corollary D.4.6. *Let $(R, \mathfrak{m})$ be a noetherian complete local ring and $X_n \to \operatorname{Spec} R/\mathfrak{m}^{n+1}$ be a sequence of proper morphisms such that $X_n \times_{R/\mathfrak{m}^{n+1}} R/\mathfrak{m}^n \cong X_{n-1}$. If L_n is a compatible sequence of line bundles on X_n such that L_0 is ample, then there exists a projective morphism $X \to \operatorname{Spec} R$ and an ample line bundle L on X and compatible isomorphisms $X_n \cong X \times_R R/\mathfrak{m}^{n+1}$ and $L_n \xrightarrow{\sim} L|_{X_n}$.*

Remark D.4.7. It follows that there an extension in the cartesian diagram

$$\begin{tikzcd}
X_0 \arrow[r, hook] \arrow[d] & X_1 \arrow[r, hook] \arrow[d] & X_2 \arrow[r, hook] \arrow[d] & \cdots \arrow[r, hook, dashed] & X \arrow[d, dashed] \\
\mathrm{Spec}\, R/\mathfrak{m} \arrow[r, hook] & \mathrm{Spec}\, R/\mathfrak{m}^2 \arrow[r, hook] & \mathrm{Spec}\, R/\mathfrak{m}^3 \arrow[r, hook] & \cdots \arrow[r, hook] & \mathrm{Spec}\, R
\end{tikzcd}$$

such that X is projective over R. We say that the formal deformation $\{X_n \to \mathrm{Spec}\, R/\mathfrak{m}^{n+1}\}$ of X_0 is *effective* (which is sometimes referred to as *algebraizable*.

Proof. We sketch how this follows from Grothendieck's Existence Theorem. Consider the finitely generated graded $\mathbb{k}$-algebra $B = \bigoplus \mathfrak{m}^n/\mathfrak{m}^{n+1}$ and the quasi-coherent graded $\mathcal{O}_{X_0}$-algebra $\mathcal{A} = B \otimes_{\mathbb{k}} \mathcal{O}_{X_0}$. By applying Serre's vanishing theorem to $\mathrm{Spec}_{X_0} \mathcal{A}$ and the ample line bundle $L_0 \otimes_{\mathcal{O}_{X_0}} \mathcal{O}_{X_0'}$, we see that $\mathrm{H}^1(X_0, \mathcal{A} \otimes L_0^{\otimes d}) = 0$ for $d \gg 0$. We have a closed immersion $X_0 \hookrightarrow \mathbb{P}^N$ defined by a basis $s_{0,0}, \dots, s_{0,N}$ of $\mathrm{H}^0(X_0, L_0^{\otimes d})$. Noting that $\mathfrak{m}^n \mathcal{O}_{X_{n+1}}/\mathfrak{m}^{n+1}\mathcal{O}_{X_{n+1}}$ is identified with $\ker(\mathcal{O}_{X_{n+1}} \to \mathcal{O}_{X_n})$, we may tensor the corresponding short exact sequence by $L_{n+1}^{\otimes d}$ to obtain a short exact sequence

$$0 \to (\mathfrak{m}^n \mathcal{O}_{X_{n+1}}/\mathfrak{m}^{n+1}\mathcal{O}_{X_{n+1}}) \otimes L_0^{\otimes d} \to L_{n+1}^{\otimes d} \to L_n^{\otimes d} \to 0,$$

where we've used that that $(\mathfrak{m}^n \mathcal{O}_{X_{n+1}}/\mathfrak{m}^{n+1}\mathcal{O}_{X_{n+1}}) \otimes L_{n+1}^{\otimes d}$ is supported on X_0 along with the identifications $L_{n+1} \otimes \mathcal{O}_{X_m} \cong L_m$ for $m \leq n$. The vanishing of $\mathrm{H}^1(X_0, \mathcal{A} \otimes L_0^{\otimes d})$ implies that we may lift the sections $s_{0,0}, \dots, s_{0,N}$ inductively to compatible sections $s_{n,0}, \dots, s_{n,N}$ of $\mathrm{H}^0(X_n, L_n^{\otimes d})$. By Nakayama's Lemma, the induced morphisms $X_n \hookrightarrow \mathbb{P}^N_{R/\mathfrak{m}^{n+1}}$ are closed immersions giving a commutative diagram

$$\begin{tikzcd}
& \mathbb{P}^N \arrow[rr, hook] \arrow[dd] & & \mathbb{P}^n_{R/\mathfrak{m}^2_R} \arrow[rr, hook] \arrow[dd] & & \cdots \arrow[r, hook] & \mathbb{P}^N_R \arrow[dd] \\
X_0 \arrow[ur, hook, "\mathrm{cl}"] \arrow[rr, hook] \arrow[dr] & & X_1 \arrow[ur, hook, "\mathrm{cl}"] \arrow[rr, hook] \arrow[dr] & & \cdots \arrow[r, hook, dashed] & X \arrow[ur, hook, dashed, "\mathrm{cl}"] \arrow[dr, dashed] & \\
& \mathrm{Spec}\, \mathbb{k} \arrow[rr, hook] & & \mathrm{Spec}\, R/\mathfrak{m}^2 \arrow[rr, hook] & & \cdots \arrow[r, hook] & \mathrm{Spec}\, R
\end{tikzcd}$$

Grothendieck's Existence Theorem (D.4.4) gives an equivalence $\mathrm{Coh}(\mathbb{P}^N_R) \to \varprojlim \mathrm{Coh}(\mathbb{P}^N_{R/\mathfrak{m}^{n+1}})$. Essential surjectivity gives a coherent sheaf E on $\mathbb{P}^N_R$ extending $\{\mathcal{O}_{X_n}\}$ and full faithfulness gives a surjection $\mathcal{O}_{\mathbb{P}^N_R} \to E$ extending $\mathcal{O}_{\mathbb{P}^N_{R/\mathfrak{m}^{n+1}}} \to \mathcal{O}_{X_n}$. We take $X \subset \mathbb{P}^N_R$ to be the closed subscheme defined by $\ker(\mathcal{O}_{\mathbb{P}^N_R} \to E)$.

See also [EGA, III.5.4.5], [FGI+05, Thm. 8.4.10] and [SP, Tag 089A]. □

Remark D.4.8. Suppose that X is flat over R and that we are only given an ample line bundle L_0 on X_0 (and not the line bundles L_n). Then the obstruction to deforming L_{n-1} to L_n is an element $\mathrm{ob}_{L_{n-1}} \in \mathrm{H}^2(X, \mathcal{O}_X \otimes_{\mathbb{k}} \mathfrak{m}^n)$ by Proposition D.2.15. If these cohomology groups vanish (e.g. if X is of dimension 1), then there exists compatible extensions L_n and thus the formal deformation $\{X_n \to \mathrm{Spec}\, R/\mathfrak{m}^{n+1}\}$ are effective.

Without the existence of deformations L_n of L_0, it is not necessarily true that formal deformations are effective. For instance, there is a projective K3 surface

(X_0, L_0) and a first order deformation $X_1 \to \operatorname{Spec} \Bbbk[\epsilon]$ which is not projective (so L_0 does not deform to X_1), and a formal deformation which is not effective; see [Har10, Ex. 21.2.1]. Similarly formal deformations of abelian varieties may not be effective. Note that for both K3 surfaces and abelian varieties, Rim–Schlessinger's Criteria applies to construct versal formal deformations.

Corollary D.4.9. *For each of the moduli problems* $\mathrm{Hilb}^P(X)$, $\mathscr{M}_g$ *(with* $g \geq 2$*) and* $\mathcal{B}\mathrm{un}_C$ *over* $\Bbbk$*, every formal deformation is effective. In particular, there exist effective versal formal deformations.*

Proof. For $\mathrm{Hilb}^P(X)$, we show the effectivity of a formal deformation $\{Z_n \subset X_{R/\mathfrak{m}^{n+1}}\}$ by following the argument at the end of the proof of Corollary D.4.6 (with $X_n \subset \mathbb{P}^N_{R/\mathfrak{m}^{n+1}}$ replaced with $Z_n \subset X_{R/\mathfrak{m}^{n+1}}$): Grothendieck's Existence Theorem (D.4.4) implies the existence of a coherent sheaf E on X_R extending $\{\mathcal{O}_{Z_n}\}$ and a surjection $\mathcal{O}_{X_R} \to E$ extending $\{\mathcal{O}_{X_n} \to \mathcal{O}_{Z_n}\}$, and we take $Z \subset X_R$ defined by $\ker(\mathcal{O}_{X_R} \to E)$.

For $\mathscr{M}_g$, the effectivity of a formal deformation $\{C_n \to \operatorname{Spec} R/\mathfrak{m}^{n+1}\}$ follows from Corollary D.4.6 by taking L_n be the ample bundle $\Omega_{C_n/(R/\mathfrak{m}^{n+1})}$, or by taking L_0 to be any ample bundle on C_0 and using Proposition D.2.15 and the vanishing of $\mathrm{H}^2(C_0, \mathcal{O}_{C_0})$ to inductively deform L_0 to a compatible sequence of line bundle L_n on C_n.

For $\mathcal{B}\mathrm{un}_C$, the effectivity of a formal deformation of vector bundles E_n on $C_{R/\mathfrak{m}^{n+1}}$ follows directly from Grothendieck's Existence Theorem (D.4.4) noting that the coherent extension is necessarily a vector bundle.

The last statement follows from the existence of versal formal deformations of these moduli problems (Proposition D.3.14). □

Exercise D.4.10. If $\mathcal{X}$ is an algebraic stack locally of finite type over $\Bbbk$ and $(R, \mathfrak{m})$ is a noetherian complete local ring with residue field $\Bbbk$, show that the functor

$$\mathcal{X}(R) \to \varprojlim \mathcal{X}(R/\mathfrak{m}^{n+1})$$

is an equivalence of categories. In particular, every formal deformation is effective.

D.4.1 Lifting to characteristic 0

One striking application of deformation theory is to "lift" a smooth variety X_0 over a field $\Bbbk$ of $\mathrm{char}(\Bbbk) = p$ to characteristic 0. We say that X_0 is *liftable to characteristic* 0 if there exists a noetherian complete local ring $(R, \mathfrak{m})$ of characteristic 0 such that $R/\mathfrak{m} = \Bbbk$ and a smooth scheme $X \to \operatorname{Spec} R$ such that $X_0 \cong X \times_R \Bbbk$.[2] One can hope to then use characteristic 0 techniques (e.g. Hodge theory) on X and deduce properties of X_0. The strategy to lift a variety X_0 is to inductively deform X_0 to smooth schemes X_n over $R/\mathfrak{m}^{n+1}$ and then apply Grothendieck's Existence Theorem to effective the formal deformation. Note however that to achieve this, we must work over a mixed characteristic base as in Remark D.3.4 rather than over a fixed field $\Bbbk$.

Smooth curves are liftable as obstructions to deforming both the curve and the ample line bundle both vanish. Serre produced an example of a non-liftable

[2]There are some variants to this definition, e.g. when R is already given as a complete DVR with residue field $\Bbbk$.

projective threefold (see [Har10, Thm. 22.4]) which Mumford extended to a non-liftable projective surface (see [FGI+05, Cor. 8.6.7]). On the other hand, Mumford showed that principally polarized abelian varieties are liftable [Mum69] while Deligne showed that K3 surfaces are liftable [Del81]. These examples are quite interesting as in both cases, formal deformations are not necessarily effective (see Remark D.4.8) and additional techniques are needed.

D.5 Cotangent complex

In this chapter, we summarize properties of the cotangent complex of a morphism of schemes as introduced in [Ill71] globalizing work of André [And67] and Quillen [Qui68, Qui70] on the cotangent complex of a ring homomorphism. One advantage of the cotangent complex is that it allows us to describe the deformations and obstruction of singular schemes; see Theorem D.5.10.

D.5.1 Properties of the cotangent complex

Theorem D.5.1. *For every morphism $f\colon X \to Y$ of schemes (resp. finite type morphism of noetherian schemes), there exists a complex*

$$L_{X/Y}\colon \ \cdots \to L^{-1}_{X/Y} \to L^0_{X/Y} \to 0$$

of flat $\mathcal{O}_X$-modules with quasi-coherent (resp. coherent) cohomology, whose image in $D^-_{\mathrm{QCoh}}(\mathcal{O}_X)$ (resp. $D^-_{\mathrm{Coh}}(\mathcal{O}_X)$) is also denoted by $L_{X/Y}$. It satisfies the following properties:

(1) $\mathrm{H}^0(X, L_{X/Y}) \cong \Omega_{X/Y}$*;*

(2) f is smooth if and only if f is locally of finite presentation and $L_{X/Y}$ is a perfect complex supported in degree 0. In this case $L_{X/Y}$ is quasi-isomorphic to the complex where the vector bundle $\Omega_{X/Y}$ sits in degree 0;

(3) If f is flat and finitely presented, then f is syntomic if and only if $L_{X/Y}$ is a perfect complex supported in degrees $[-1, 0]$. Explicitly, if f factors as a local complete intersection $X \hookrightarrow \widetilde{Y}$ defined by a sheaf of ideals I and a smooth morphism $\widetilde{Y} \to Y$, then $L_{X/Y}$ is quasi-isomorphic to $0 \to I/I^2 \xrightarrow{d} \Omega_{X/Y} \to 0$ (with $\Omega_{X/Y}$ in degree 0);

(4) If

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

is a cartesian diagram with either f or g flat (or more generally f and g are tor-independent), then there is a quasi-isomorphism $g'^* L_{X/Y} \to L_{X'/Y'}$. (Note that without any flatness condition $g'^*\Omega_{X/Y} \cong \Omega_{X'/Y'}$.)

(5) If $X \xrightarrow{f} Y \to Z$ is a composition of morphisms of schemes, then there is an exact triangle in $D^-_{\mathrm{QCoh}}(\mathcal{O}_X)$

$$f^* L_{Y/Z} \to L_{X/Z} \to L_{X/Y} \to f^* L_{Y/Z}[1].$$

This induces a long exact sequence on cohomology

$$\begin{tikzcd}
 & \cdots \arrow[r] & \mathrm{H}^{-2}(L_{X/Y}) \arrow[dll] \\
\mathrm{H}^{-1}(f^*L_{X/Z}) \arrow[r] & \mathrm{H}^{-1}(L_{X/Z}) \arrow[r] & \mathrm{H}^{-1}(L_{X/Y}) \arrow[dll] \\
f^*\Omega_{Y/Z} \arrow[r] & \Omega_{X/Z} \arrow[r] & \Omega_{X/Y} \arrow[r] & 0
\end{tikzcd}$$

extending the usual right exact sequence on differentials [Har77, II.8.12]. (Note that if f is smooth, then $H^{-1}(L_{X/Y}) = 0$ and $f^*\Omega_{Y/Z} \to \Omega_{X/Z}$ is injective.)

Proof. See [Ill71, II.1.2.3], [SP, Tag 08T2] for the definition of the cotangent complex of a morphism of schemes (and more generally for morphisms of ringed topoi). For (1)–(5), see [Ill71, II.1.2.4.2, II.3.1.2, II.3.2.6, II.2.2.3 and II.2.1.2] and [SP, Tags 08UV, 0D0N, 0FK3, 08QQ and 08T4] (noting that [SP, Tag 08RB] relates the *naive cotangent complex* $NL_{X/Y}$ to $L_{X/Y}$). □

D.5.2 Truncations of the cotangent complex

The definition of the cotangent complex relies on simplicial techniques and we won't attempt an exposition here. We will however give an explicit description of its truncation, which often suffice for applications.

First, if $X \to Y$ factors as a closed immersion $X \hookrightarrow P$ defined by a sheaf of ideals I and a smooth morphism $P \to Y$, then the truncation $\tau_{\geq -1}(L_{X/Y})$ of $L_{X/Y}$ in degrees $[-1,0]$ is quasi-isomorphic to $0 \to I/I^2 \xrightarrow{d} \Omega_{X/Y} \to 0$ (with $\Omega_{X/Y}$ in degree 0). In the case that $X \to Y$ is smooth or syntomic, then $X \hookrightarrow \widetilde{Y}$ is a regular immersion, I/I^2 is a vector bundle and $L_{X/Y}$ is quasi-isomorphic to $\tau_{\geq -1}(L_{X/Y})$ (Theorem D.5.1(3)).

For a morphism $X = \operatorname{Spec} A \to \operatorname{Spec} B = Y$ of affine schemes, Lichtenbaum–Schlessinger [LS67] offer an explicit description of $\tau_{\geq -2}(L_{X/Y})$. Choose a polynomial ring $P = B[x_i]$ (with possibly infinitely many generators) and a surjection $P \twoheadrightarrow A$ as B-algebras with kernel I. Choose a free P-module $F = \oplus_{\lambda \in \Lambda} P$ and a surjection $p \colon F \twoheadrightarrow I$ of P-modules with kernel $K = \ker(p)$. Let $K' \subset K$ be the submodule generated by $p(x)y - p(y)x$ for $x, y \in F$. Then the truncation $\tau_{\geq -2}(L_{X/Y})$ (or rather $\tau_{\geq -2}(L_{B/A})$) is quasi-isomorphic to the complex of A-modules

$$K/K' \to F \otimes_P A \to \Omega_{P/B} \otimes_P A \tag{D.5.1}$$

with the last term in degree 0; see [SP, Tag 09CG].

One defines the T^i *functors* on the category of A-modules by

$$T^i(A/B, -) := \mathrm{H}^i(\operatorname{Hom}_A(L_{A/B}, -)),$$

which can be used for instance to describe deformations of schemes (see Example D.5.11). See also [LS67, §2.3] and [Har10, §1.3].

D.5.3 Extensions of algebras and schemes

Definition D.5.2. An *extension* of a ring homomorphism $R \to A$ by an A-module J is an exact sequence of R-modules

$$0 \to J \to A' \to A \to 0$$

where $A' \to A$ is an R-algebra homomorphism and $J \subset A'$ is an ideal with $J^2 = 0$. (Note that since $J^2 = 0$, $J = J/J^2$ is a module over $A = A'/J$.) The *trivial extension* is $A[J] := A \oplus J$ where multiplication is defined by $J^2 = 0$.

A morphism of extensions is a morphism of short exact sequences which is the identity on J and A. By the five lemma, a morphism of extensions is necessarily an isomorphism. We let $\underline{\mathrm{Exal}}_R(A, J)$ be the groupoid of extensions of $R \to A$ by J, and $\mathrm{Exal}_R(A, J)$ the set of isomorphism classes.

Remark D.5.3. Geometrically, an extension is a commutative diagram of schemes

$$\begin{tikzcd}
\operatorname{Spec} A \arrow[r, hook] \arrow[d] & \operatorname{Spec} A' \arrow[dl] \\
\operatorname{Spec} R &
\end{tikzcd}$$

such that $J \cong \ker(A' \to A)$ and $J^2 = 0$.

The set of extensions $\mathrm{Exal}_R(A, J)$ is functorial with respect to A and J:

(a) Given a map $B \to A$ of R-algebras, there is a map $\mathrm{Exal}_R(A, J) \to \mathrm{Exal}_R(B, J)$ given by mapping a complex $0 \to J \to A' \to A \to 0$ to $0 \to J \to A' \times_A B \to B \to 0$.

(b) Given an A-module map $\alpha\colon J \to J$, there is a map $\alpha_*\colon \mathrm{Exal}_R(A, J) \to \mathrm{Exal}_R(A, J)$ given by mapping a complex $0 \to J \to A' \to A \to 0$ to $0 \to J \to (A' \oplus J)/\{(-x, \alpha(x)), x \in J\} \to A \to 0$.

(c) Given modules J and K, the natural map $(p_{1,*}, p_{2,*})\colon \mathrm{Exal}_R(A, J \oplus K) \to \mathrm{Exal}_R(A, J) \oplus \mathrm{Exal}_R(A, K)$, induced from (b) by the projections $p_1\colon J \oplus K \to J$ and $p_2\colon J \oplus K \to K$, is a bijection.

Moreover, $\mathrm{Exal}_R(A, J)$ naturally has the structure of an A-module: scalar multiplication by $x \in A$ is defined using (b) with $x\colon J \to J$ and addition is defined by $\mathrm{Exal}_R(A, J) \times \mathrm{Exal}_R(A, J) \cong \mathrm{Exal}_R(A, J \oplus J) \xrightarrow{\Sigma_*} \mathrm{Exal}_R(A, J)$ using the bijection in (c) and the map Σ_* of (b) where $\Sigma\colon J \oplus J \to J$ is addition. The maps (a)–(c) are in fact maps of A-modules. See [Ill71, §III.1.1] for details.

Proposition D.5.4. *Let R be a ring.*

(1) Given a R-algebra A and an exact sequence $0 \to J' \to J \to J'' \to 0$ of A-modules, there is an exact sequence

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Der}_R(A, J') \arrow[r] & \mathrm{Der}_R(A, J) \arrow[r] & \mathrm{Der}_R(A, J'') \arrow[dll, rounded corners] \\
& \mathrm{Exal}_R(A, J) \arrow[r] & \mathrm{Exal}_R(A, J) \arrow[r] & \mathrm{Exal}_R(A, J'')
\end{tikzcd}$$

of A-modules.

(2) Given a homomorphism $B \to A$ of R-algebras, there is an exact sequence

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Der}_B(A, J) \arrow[r] & \mathrm{Der}_R(A, J) \arrow[r] & \mathrm{Der}_R(B, J) \arrow[dll, rounded corners] \\
& \mathrm{Exal}_B(A, J) \arrow[r] & \mathrm{Exal}_R(A, J) \arrow[r] & \mathrm{Exal}_R(B, J)
\end{tikzcd}$$

of A-modules.

Proof. See [EGA, 0.20.2.3] and [Ill71, III.1.2.4.3, III.1.2.5.4]. □

Remark D.5.5. The top row of (D.5.4) can be realized using the right exact sequence $\Omega_{B/R} \otimes_B A \to \Omega_{A/R} \to \Omega_{A/B} \to 0$. Namely, apply $\mathrm{Hom}_A(-, J)$ and use the identities $\mathrm{Hom}_A(\Omega_{A/B}, J) = \mathrm{Der}_B(A, J)$, $\mathrm{Hom}_A(\Omega_{A/R}, J) = \mathrm{Der}_R(A, J)$ and $\mathrm{Hom}_A(\Omega_{B/R} \otimes_B A, J) = \mathrm{Hom}_B(\Omega_{B/R}, J) = \mathrm{Der}_R(B, J)$.

The cotangent complex can be applied to extend these sequences to long exact sequences; see Remark D.5.8.

The definition of Exal extends naturally to schemes (and more generally to ringed topoi).

Definition D.5.6. An *extension* of a morphism $X \to S$ of schemes by a quasi-coherent $\mathcal{O}_X$-module J is a short exact sequence

$$0 \to J \to \mathcal{O}_{X'} \to \mathcal{O}_X \to 0$$

where $X \hookrightarrow X'$ is a closed immersion of schemes defined by sheaf of ideals $J \subset \mathcal{O}_{X'}$ with $J^2 = 0$. (Note that the condition $J^2 = 0$ implies that the $J \subset \mathcal{O}_{X'}$ is naturally a $\mathcal{O}_X$-module.) The *trivial extension* is $X[J] := (X, \mathcal{O}_X \oplus J)$ where the ring structure is is defined by $J^2 = 0$.

A *morphism of extensions* is a morphism of short exact sequences which is the identity on J and $\mathcal{O}_X$. We let $\underline{\mathrm{Exal}}_S(X, J)$ be the category of extensions of $X \to S$ by J, and $\mathrm{Exal}_S(X, J)$ be the set of isomorphism classes.

The set $\mathrm{Exal}_S(X, J)$ is naturally an $\mathcal{O}_X$-module and is functorial in X and J. In fact, the groupoid $\underline{\mathrm{Exal}}_S(X, J)$ is a *Picard category*, and the prestack over Sch/S whose fiber category over $f\colon T \to S$ is $\underline{\mathrm{Exal}}_T(X_T, f^*J)$ is a *Picard stack*; see [Ill71, III.1.1.5] and [SGA4, XVIII.1.4].

D.5.4 The cotangent complex and deformation theory

Theorem D.5.7. *If $X \to Y$ is a morphism of schemes and J is a quasi-coherent $\mathcal{O}_Y$-module, there is a natural isomorphism*

$$\mathrm{Exal}_Y(X, J) \cong \mathrm{Ext}^1_{\mathcal{O}_X}(L_{X/Y}, J).$$

Proof. See [Ill71, III.1.2.3]. □

Remark D.5.8. This identification allows us to use the cotangent complex to extend the 6-term left exact sequences of Proposition D.5.4 to long exact sequences. Namely, applying $\mathrm{Hom}_{\mathcal{O}_X}(L_{X/Y}, -)$ to the exact sequence $0 \to J' \to J \to J''$ extends D.5.4(1) and applying $\mathrm{Hom}_{\mathcal{O}_X}(-, J)$ to the exact triangle $f^*L_{Y/Z} \to L_{X/Z} \to L_{X/Y}$ extends D.5.4(2).

When $X = \mathrm{Spec}\, A \to \mathrm{Spec}\, B = Y$ is a morphism of affine schemes, using the T^i functors of §D.5.2, the above equivalence translates to $\mathrm{Exal}_B(A, J) = T^1(A/B, J)$. This can be established using the explicit description of the Lichtenbaum–Schlessinger truncated cotangent complex (D.5.1); see [LS67, 4.2.2] and [Har10, Thm. 5.1]. The T^i functors can also be used to extend the 6-term sequences of Proposition D.5.4 to 9-term sequences; see [LS67, 2.3.5-6] and [Har10, Thms. 3.4-5].

Remark D.5.9. More generally, there is an equivalence between the groupoid $\underline{\mathrm{Exal}}_Y(X, J)$ and the groupoid obtained from the 2-term complex

$$[C^{-1} \xrightarrow{d} C^0] := \tau_{\leq 0}(\mathrm{RHom}_{\mathcal{O}_X}(\tau_{\geq -1}L_{X/Y}, J)[1])$$

where objects are elements of C^0 and $\mathrm{Mor}(c, c') = d^{-1}(c - c')$; see [Ill71, III.1.2.2].

Theorem D.5.10. *Consider the following deformation problem*

$$\begin{tikzcd} X \arrow[r, hook, dashed] \arrow[d, "f"] & X' \arrow[d, dashed, "f'"] \\ Y \arrow[r, hook, "i"] & Y' \end{tikzcd}$$

where $f\colon X \to Y$ is a morphism of schemes and $i\colon Y \hookrightarrow Y'$ is a closed immersion of schemes defined by an ideal sheaf $J \subset \mathcal{O}_{Y'}$ with $J^2 = 0$. A deformation *is a morphism $f'\colon X' \to Y'$ making the above diagram cartesian and a morphism of deformations is a morphism over Y' restricting to the identity on X.*

(1) The group of automorphisms of a deformation $f'\colon X' \to Y'$ is isomorphic to $\mathrm{Ext}^0_{\mathcal{O}_X}(L_{X/Y}, f^*J)$.

(2) If there exists a deformation, then the set of deformations is a torsor under $\mathrm{Ext}^1_{\mathcal{O}_X}(L_{X/Y}, f^*J)$.

(3) There exists an element $\mathrm{ob}_X \in \mathrm{Ext}^2_{\mathcal{O}_X}(L_{X/Y}, f^*J)$ with the property that there exists a deformation if and only if $\mathrm{ob}_X = 0$.

Proof. See [Ill71, III.2.1.7] and [SP, Tag 08UZ]. See also [LS67, 4.2.5] and [Har10, Thm. 10.1] for descriptions in the affine case using the truncated cotangent complex. □

Example D.5.11. As a reality check, let's first consider a smooth morphism $f\colon X \to \operatorname{Spec} A$ is smooth and a surjection $A' \twoheadrightarrow A$ of noetherian rings with square-zero kernel J. By the identification

$$\mathrm{Ext}^i_{\mathcal{O}_X}(L_{X/A}, f^*J) = \mathrm{H}^i(X, T_{X/A} \otimes_A J),$$

we recover Proposition D.2.6.

Second, let's consider a scheme X_0 locally of finite type over a field $\Bbbk$ which is generically smooth and a local complete intersection. In this case, every point of X_0 has an open neighborhood U such that $U = V(I) \subset \mathbb{A}^n$ where I is generated by a regular sequence. We always have a right exact sequence

$$I/I^2 \xrightarrow{d} \Omega_{\mathbb{A}^n}|_U \to \Omega_U \to 0, \tag{D.5.2}$$

and by properties of the cotangent complex (D.5.1(3)), we have that $L_U = [I/I^2 \to \Omega_{\mathbb{A}^n}|_U]$ and supported in degrees $[-1, 0]$. On the other hand, since U is generally smooth, $I/I^2 \to \Omega_{\mathbb{A}^n}|_U$ is generically injective. But as I/I^2 is a vector bundle (as I is generated by a regular sequence), it can have no torsion subsheaves. It follows that the sequence D.5.2 is also left exact, and that Ω_U is quasi-isomorphic to L_U. Thus, Ω_{X_0} is also quasi-isomorphic to L_{X_0}, and automorphisms, deformations, and obstructions are classified by $\mathrm{Ext}^i_{\mathcal{O}_X}(\Omega_{X_0}, J)$ recovering Proposition D.2.11.

One major advantage of the cotangent complex is that for $\mathrm{Ext}^i_{\mathcal{O}_X}(L_{X/A}, f^*J)$ for $i = 0, 1, 2$ classifies automorphisms, deformations and obstructions for an arbitrary morphism. Moreover, the truncated cotangent complex $\tau_{\geq -2}L_{X/A}$ suffices to compute automorphisms, deformations and obstructions; for instance when $X = \operatorname{Spec} B$ is affine, we get equivalent descriptions using the T^i functors $T^i(L_{B/A}, f^*J)$ as defined in §D.5.2.

Remark D.5.12. There are analogous results for other deformation problems. For instance, for the deformation problem

$$\begin{tikzcd}
X \arrow[r, hook] \arrow[dr, "f"] \arrow[ddr] & X' \arrow[dr, dashed] \arrow[ddr] & \\
 & Y \arrow[r, hook] \arrow[d] & Y' \arrow[d] \\
 & Z \arrow[r, hook] & Z',
\end{tikzcd}$$

where the horizontal morphisms are closed immersions defined by square-zero ideal sheaves J_X, J_Y and J_Z, then automorphisms, deformations and obstructions are classified by $\operatorname{Ext}^i_{\mathcal{O}_X}(f^*L_{Y/Z}, J_X)$ for $i = -1, 0, 1$ [Ill71, III.2.2.4]. An important special case is when $Y = Y'$ and $Z = Z'$.

D.6 Artin Algebraization

Artin Algebraization is a procedure to “algebraize" or extend an effective versal formal deformation $\xi \in \mathcal{M}(R)$ to an object $\eta \in \mathcal{M}(U)$ over a *finite type* $\Bbbk$-scheme U. In this section, we show how Artin Algebraization follows from Artin Approximation following the ideas of Conrad and de Jong [CJ02].

D.6.1 Limit preserving prestacks

Extending the definition of a limit preserving functor §A.10.2, we say that a prestack $\mathcal{X}$ over $\mathrm{Sch}/\Bbbk$ is *limit preserving* (or *locally of finite presentation*) if for every system B_λ of $\Bbbk$-algebras, the natural functor

$$\operatorname{colim} \mathcal{X}(B_\lambda) \to \mathcal{X}(\operatorname{colim} B_\lambda)$$

is an equivalence of categories. When $\mathcal{X}$ is an algebraic stack over $\Bbbk$, then this equivalent to the morphism $\mathcal{X} \to \operatorname{Spec} \Bbbk$ being locally of finite presentation; see Exercise 3.3.31).

Lemma D.6.1. *Each of the prestacks* $\operatorname{Hilb}^P(X)$, $\mathscr{M}_g$ *(with* $g \geq 2$*) and* $\mathcal{B}\mathrm{un}_C$ *over* $(Sch/\Bbbk)$ *are limit preserving.*

Proof. To add. $\square$

D.6.2 Conrad–de Jong Approximation

In Artin Approximation (Theorem A.10.9), the initial data is an object over a noetherian complete local $\Bbbk$-algebra $\widehat{\mathcal{O}}_{S,s}$ which is assumed to be the completion of a finitely generated $\Bbbk$-algebra at a maximal ideal. We will now see that a similar approximation result still holds if this latter hypothesis is dropped and one approximates *both* the complete local ring and the object.

Recall also that if $(A, \mathfrak{m})$ is a local ring and M is an A-module, then the *associated graded module of* M is defined as $\mathrm{Gr}_{\mathfrak{m}}(M) = \bigoplus_{n \geq 0} \mathfrak{m}^n M/\mathfrak{m}^{n+1} M$; it is a graded module over the graded ring $\mathrm{Gr}_{\mathfrak{m}}(A)$.

Theorem D.6.2 (Conrad–de Jong Approximation)**.** *Let $\mathcal{X}$ be a limit preserving prestack over* $\mathrm{Sch}/\Bbbk$*. Let $(R, \mathfrak{m}_R)$ be a noetherian complete local $\Bbbk$-algebra and let $\xi \in \mathcal{X}(R)$. Then for every integer $N \geq 0$, there exist*

(1) an affine scheme $\operatorname{Spec} A$ *of finite type over $\Bbbk$ and a $\Bbbk$-point $u \in \operatorname{Spec} A$,*

(2) an object $\eta \in \mathcal{X}(A)$,

(3) an isomorphism $\phi_{N+1} \colon R/\mathfrak{m}_R^{N+1} \cong A/\mathfrak{m}_u^{N+1}$,

(4) an isomorphism of $\xi|_{R/\mathfrak{m}_R^{N+1}}$ and $\eta|_{A/\mathfrak{m}_u^{N+1}}$ via ϕ_N, and

(5) an isomorphism $\mathrm{Gr}_{\mathfrak{m}_R}(R) \cong \mathrm{Gr}_{\mathfrak{m}_u}(A)$ of graded $\Bbbk$-algebras.

The proof of this theorem will proceed by simultaneously approximating equations and relations defining R and the object ξ. The statements (1)–(4) will be easily obtained as a consequence of Artin Approximation. A nice insight of Conrad and de Jong is that condition (5) can be ensured by Artin Approximation, and moreover that this condition suffices to imply the isomorphism of complete local $\Bbbk$-algebras in Artin Algebraization. As such, condition (5) takes the most work to establish.

We will need some preparatory results controlling the constant appearing in the Artin–Rees lemma.

Definition D.6.3 (Artin–Rees Condition)**.** Let $(A, \mathfrak{m})$ be a noetherian local ring. Let $\varphi \colon M \to N$ be a morphism of finite A-modules. Let $c \geq 0$ be an integer. We say that $(\mathsf{AR})_c$ *holds for* φ if

$$\varphi(M) \cap \mathfrak{m}^n N \subset \varphi(\mathfrak{m}^{n-c} M), \quad \forall n \geq c.$$

The Artin–Rees lemma implies that $(\mathsf{AR})_c$ holds for φ if c is sufficiently large; see [AM69, Prop. 10.9] or [Eis95, Lem. 5.1].

Lemma D.6.4. *Let $(A, \mathfrak{m})$ be a noetherian local ring. Let*

$$L \xrightarrow{\alpha} M \xrightarrow{\beta} N \qquad \text{and} \qquad L' \xrightarrow{\alpha'} M \xrightarrow{\beta'} N$$

be two complexes of finite A-modules. Let c be a positive integer. Assume that

(a) the first sequence is exact,

(b) the complexes are isomorphic modulo $\mathfrak{m}^{c+1}$, and

(c) $(\mathsf{AR})_c$ holds for α and β.

Then there exists an isomorphism $\mathrm{Gr}_{\mathfrak{m}}(\operatorname{coker} \beta) \to \mathrm{Gr}_{\mathfrak{m}}(\operatorname{coker} \beta')$ of graded $\mathrm{Gr}_{\mathfrak{m}}(A)$-modules.

Proof. The proof while technical is rather straightforward. First by taking free presentations of L and L', we can assume that $L = L'$. One shows that $(\mathsf{AR})_c$ holds for β' and that the second sequence is exact. Then one establishes the equality

$$\mathfrak{m}^{n+1} N + \beta(M) \cap \mathfrak{m}^n N = \mathfrak{m}^{n+1} N + \beta'(M) \cap \mathfrak{m}^n N$$

by using that $(\mathsf{AR})_c$ holds for β to show the containment "$\subset$" and then using $(\mathsf{AR})_c$ holds for β' to get the other containment. The statement then follows from the description $\mathrm{Gr}_{\mathfrak{m}}(\operatorname{coker} \beta)_n = \mathfrak{m}^n N/(\mathfrak{m}^{n+1} N + \beta(M) \cap \mathfrak{m}^n N)$ and the similar description of $\mathrm{Gr}_{\mathfrak{m}}(\operatorname{coker} \beta')_n$. For details, see [CJ02, §3] and [SP, Tag 07VF]. $\square$

Proof of Conrad–de Jong Approximation (Theorem D.6.2). Since $\mathcal{X}$ is limit preserving and R is the colimit of its finitely generated $\Bbbk$-subalgebras, there is an affine scheme $V = \operatorname{Spec} B$ of finite type over $\Bbbk$ and an object γ of $\mathcal{X}$ over V together with a 2-commutative diagram

$$\begin{tikzcd} \operatorname{Spec} R \arrow[r] \arrow[rr, bend left, "\xi"] & V \arrow[r, "\gamma"] & \mathcal{X}. \end{tikzcd}$$

Let $v \in V$ be the image of the maximal ideal $\mathfrak{m} \subset R$. After adding generators to the ring B if necessary, we can assume that the composition $\widehat{\mathcal{O}}_{V,v} \to R \to R/\mathfrak{m}^2$ is surjective. This implies that $\widehat{\mathcal{O}}_{V,v} \to R$ is surjective by Lemma A.10.15. The goal now is to simultaneously approximate over V the equations and relations defining the closed immersion $\operatorname{Spec} R \hookrightarrow \operatorname{Spec} \widehat{\mathcal{O}}_{V,v}$ and the object ξ. In order to accomplish this goal, we choose a resolution

$$\widehat{\mathcal{O}}_{V,v}^{\oplus r} \xrightarrow{\widehat{\alpha}} \widehat{\mathcal{O}}_{V,v}^{\oplus s} \xrightarrow{\widehat{\beta}} \widehat{\mathcal{O}}_{V,v} \to R \to 0 \tag{D.6.1}$$

as $\widehat{\mathcal{O}}_{V,v}$-modules and consider the functor

$$\begin{aligned} F\colon (\mathrm{Sch}/V) &\to \mathsf{Sets} \\ (T \to V) &\mapsto \left\{\text{complexes } \mathcal{O}_T^{\oplus r} \xrightarrow{\alpha} \mathcal{O}_T^{\oplus s} \xrightarrow{\beta} \mathcal{O}_T\right\}. \end{aligned}$$

It is not hard to check that this functor is limit preserving. The resolution in (D.6.1) yields an element of $F(\widehat{\mathcal{O}}_{V,v})$. Applying Artin Approximation (Theorem A.10.9) yields an étale morphism $(V' = \operatorname{Spec} B', v') \to (V, v)$ and an element

$$(B'^{\oplus r} \xrightarrow{\alpha'} B'^{\oplus s} \xrightarrow{\beta'} B') \in F(V') \tag{D.6.2}$$

such that α', β' are equal to $\widehat{\alpha}, \widehat{\beta}$ modulo $\mathfrak{m}^{N+1}$.

Let $U = \operatorname{Spec} A \hookrightarrow \operatorname{Spec} B' = V'$ be the closed subscheme defined by $\operatorname{im} \beta'$ and let $u = v' \in U$. Consider the composition

$$\eta\colon U \hookrightarrow V' \to V \xrightarrow{\gamma} \mathcal{X}$$

As $R = \operatorname{coker} \widehat{\beta}$ and $A = \operatorname{coker} \beta'$, we have an isomorphism $R/\mathfrak{m}^{N+1} \cong A/\mathfrak{m}_u^{N+1}$ together with an isomorphism of $\xi|_{R/\mathfrak{m}^{N+1}}$ and $\eta|_{A/\mathfrak{m}_u^{N+1}}$. This gives statements (1)–(4).

To establish (5), we need to show that there are isomorphisms $\mathfrak{m}^n/\mathfrak{m}^{n+1} \cong \mathfrak{m}_u^n/\mathfrak{m}_u^{n+1}$. For $n \leq N$, this is guaranteed by the isomorphism $R/\mathfrak{m}^{N+1} \cong A/\mathfrak{m}_u^{N+1}$. On the other hand, for $n \gg 0$, this can be seen to be a consequence of the Artin–Rees lemma. To handle the middle range of n, we need to control the constant appearing in the Artin–Rees lemma. First note that before we applied Artin Approximation, we could have increased N to ensure that $(\mathsf{AR})_N$ holds for $\widehat{\alpha}$ and $\widehat{\beta}$. We are thus free to assume this. Now statement (5) follows directly if we apply Lemma D.6.4 to the exact complex $\widehat{\mathcal{O}}_{V,v}^{\oplus r} \xrightarrow{\widehat{\alpha}} \widehat{\mathcal{O}}_{V,v}^{\oplus s} \xrightarrow{\widehat{\beta}} \widehat{\mathcal{O}}_{V,v}$ of (D.6.1) and the complex $\widehat{\mathcal{O}}_{V,v}^{\oplus r} \xrightarrow{\widehat{\alpha}'} \widehat{\mathcal{O}}_{V,v}^{\oplus s} \xrightarrow{\widehat{\beta}'} \widehat{\mathcal{O}}_{V,v}$ obtained by restricting (D.6.2) to $F(\widehat{\mathcal{O}}_{V,v})$.

See also [CJ02] and [SP, Tag 07XB]. $\square$

Exercise D.6.5. Show that Conrad–de Jong Approximation implies Artin Approximation.

D.6.3 Artin Algebraization

Artin Algebraization has a stronger conclusion than Artin Approximation or Conrad–de Jong Approximation in that no approximation is necessary. It guarantees the existence of an object η over a pointed affine scheme $(\operatorname{Spec} A, u)$ of finite type over $\Bbbk$ which agrees with the given effective formal deformation ξ to all orders. In order to ensure this, we need to impose that ξ is *versal at* u, i.e. that the restrictions $\xi_n = \xi|_{A/\mathfrak{m}_u^{n+1}}$ define a versal formal deformation $\{\xi_n\}$ over A (Definition D.3.5).

Theorem D.6.6 (Artin Algebraization)**.** *Let $\mathcal{X}$ be a limit preserving prestack over* $\mathrm{Sch}/\Bbbk$*. Let $(R, \mathfrak{m})$ be a noetherian complete local $\Bbbk$-algebra and $\xi \in \mathcal{X}(R)$ be an effective versal formal deformation. There exist*

(1) an affine scheme $\operatorname{Spec} A$ of finite type over $\Bbbk$ and a $\Bbbk$-point $u \in \operatorname{Spec} A$;

(2) an object $\eta \in \mathcal{X}(A)$;

(3) an isomorphism $\alpha\colon R \xrightarrow{\sim} \widehat{A}_{\mathfrak{m}_u}$ of $\Bbbk$-algebras; and

(4) a compatible family of isomorphisms $\xi|_{R/\mathfrak{m}^{n+1}} \cong \eta|_{A/\mathfrak{m}_u^{n+1}}$ (under the identification $R/\mathfrak{m}^{n+1} \cong A/\mathfrak{m}_u^{n+1}$) for $n \geq 0$.

Remark D.6.7. If $\mathcal{X}$ is an algebraic stack locally of finite type over $\Bbbk$, then there exists an isomorphism $\xi \cong \eta|_{\widehat{A}_{\mathfrak{m}_u}}$.

Remark D.6.8. In the case that R is known to be the completion of a finitely generated $\Bbbk$-algebra, this theorem can be viewed as an easy consequence of Artin Approximation. Indeed, one applies Artin Approximation with $N = 1$ and then uses versality to obtain compatible maps $R \to A/\mathfrak{m}_u^{n+1}$ and therefore a map $R \to \widehat{A}_{\mathfrak{m}_u}$ which is an isomorphism modulo $\mathfrak{m}^2$. As R and $\widehat{A}_{\mathfrak{m}_u}$ are abstractly isomorphic, the homomorphism $R \to \widehat{A}_{\mathfrak{m}_u}$ is an isomorphism (Lemma A.10.15) and the statement follows. The argument in the general case is analogous except we use Conrad–de Jong Approximation instead of Artin Approximation.

Proof of Artin Algebraization (Theorem D.6.6). Applying Conrad–de Jong Approximation (Theorem D.6.2) with $N = 1$, we obtain an affine scheme $\operatorname{Spec} A$ of finite type over $\Bbbk$ with a $\Bbbk$-point $u \in \operatorname{Spec} A$, an object $\eta \in \mathcal{X}(A)$, an isomorphism $\phi_2\colon \operatorname{Spec} A/\mathfrak{m}_u^2 \to \operatorname{Spec} R/\mathfrak{m}^2$, an isomorphism $\alpha_2\colon \xi|_{R/\mathfrak{m}^2} \to \eta|_{A/\mathfrak{m}_u^2}$, and an isomorphism $\mathrm{Gr}_{\mathfrak{m}}(R) \cong \mathrm{Gr}_{\mathfrak{m}_u}(A)$ of graded $\Bbbk$-algebras. We claim that ϕ_2 and α_2 can be extended inductively to a compatible family of morphisms $\phi_n\colon \operatorname{Spec} A/\mathfrak{m}_u^{n+1} \to \operatorname{Spec} R$ and isomorphisms $\alpha_n\colon \xi|_{A/\mathfrak{m}_u^{n+1}} \to \eta|_{A/\mathfrak{m}_u^{n+1}}$. Given ϕ_n and α_n, versality of ξ implies that there is a lift ϕ_{n+1} filling in the commutative diagram

$$\begin{tikzcd}
\operatorname{Spec} A/\mathfrak{m}_u^n \arrow[r, "\phi_n"] \arrow[d] & \operatorname{Spec} R \arrow[d, "\xi"] \\
\operatorname{Spec} A/\mathfrak{m}_u^{n+1} \arrow[r, "\eta|_{A/\mathfrak{m}_u^{n+1}}"'] \arrow[ur, dashed, "\phi_{n+1}"'] & \mathcal{X},
\end{tikzcd}$$

which establishes the claim. By taking the limit, we have a homomorphism $\widehat{\phi}\colon R \to \widehat{A}_{\mathfrak{m}_u}$ which is surjective since ϕ_2 is surjective (Lemma A.10.15). On the other hand, for each n the $\Bbbk$-vector spaces $\mathfrak{m}^N/\mathfrak{m}^{N+1}$ and $\mathfrak{m}_u^N/\mathfrak{m}_u^{N+1}$ have the same dimension. This implies that $\widehat{\phi}$ is an isomorphism.

See also [Art69b, Thm. 1.6] and [CJ02, §4], where the statement is established more generally when $\mathcal{X}$ is defined over a scheme S whose local rings are G-rings where it is required that $\operatorname{Spec} R/\mathfrak{m} \xrightarrow{\xi_0} \mathcal{X} \to S$ be of finite type. □

D.7 Artin's Axioms for Algebraicity

A spectacular application of Artin Algebraization is a criterion—which is often verifiable in practice—ensuring that a given stack is algebraic. This is called Artin's Axioms for Algebraicity and we provide two versions below Theorems D.7.1 and D.7.4. This foundational result was proved by Artin in the very same paper [Art74] where he introduced algebraic stacks.

The first version can be proved easily using Artin Algebraization.

Theorem D.7.1. *(Artin's Axioms for Algebraicity—first version) Let $\mathcal{X}$ be a stack over $\Bbbk$. Then $\mathcal{X}$ is an algebraic stack locally of finite type over $\Bbbk$ if and only if the following conditions hold:*

(1) (Limit preserving) The stack $\mathcal{X}$ is limit preserving over $\mathrm{Sch}/\Bbbk$*, i.e. for every system B_λ of $\Bbbk$-algebras, the functor*

$$\operatorname{colim} \mathcal{X}(B_\lambda) \to \mathcal{X}(\operatorname{colim} B_\lambda)$$

is an equivalence of categories.

(2) (Representability of the diagonal) The diagonal $\mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is representable.

(3) (Existence of versal formal deformations) Every $x_0 \in \mathcal{X}(\Bbbk)$ has a versal formal deformation $\{x_n\}$ over a noetherian complete local $\Bbbk$-algebra $(R, \mathfrak{m})$ with residue field $\Bbbk$.

(4) (Effectivity) For every noetherian complete local $\Bbbk$-algebra $(R, \mathfrak{m})$ with residue field $\Bbbk$, the natural functor

$$\mathcal{X}(\operatorname{Spec} R) \to \varprojlim \mathcal{X}(\operatorname{Spec} R/\mathfrak{m}^n)$$

is an equivalence of categories.

(5) (Openness of versality) For every morphism $U \to \mathcal{X}$ from a finite type $\Bbbk$-scheme which is versal at $u \in U(\Bbbk)$ (i.e. the formal deformation $\{\operatorname{Spec} \widehat{\mathcal{O}}_{U,u}/\mathfrak{m}_u^{n+1} \to \mathcal{X}\}$ is versal), there exists an open neighborhood V of u such that $U \to \mathcal{X}$ is versal at every $\Bbbk$-point of V.

Proof. We first note that for a representable and locally of finite type morphism $U \to \mathcal{X}$ from a finite type $\Bbbk$-scheme U, the Infinitesimal Lifting Criterion for Smoothness (Smooth Equivalences A.3.1, Theorem 3.7.1) implies that $U \to \mathcal{X}$ is smooth if and only if it is versal at all $\Bbbk$-points $u \in U$. Indeed, this is clear when $U \to \mathcal{X}$ is representable by schemes, and the general case follows as one can see that both properties are étale-local on U.

For ($\Rightarrow$), (1) holds by Exercise 3.3.31, (2) holds by Theorem 3.2.1 and (4) holds by Exercise D.4.10. If $U \to \mathcal{X}$ is a morphism from a finite type $\Bbbk$-scheme, then it is necessarily representable and locally of finite type. By using the above equivalence between versality and smoothness, (3) holds by choosing a smooth presentation $U \to \mathcal{X}$ and a preimage $u \in U(\Bbbk)$ of x_0 and taking the formal deformation $\{\operatorname{Spec} \mathcal{O}_{U,u}/\mathfrak{m}_u^{n+1} \to \mathcal{X}\}$, and (5) holds by openness of smoothness.

For the converse, we first note that representability of the diagonal, i.e. condition (2), implies that every morphism $U \to \mathcal{X}$ from a scheme U is representable and the limit preserving property (1) implies that $U \to \mathcal{X}$ is locally of finite type. For every object $x_0 \in \mathcal{X}(\Bbbk)$, we will construct a smooth morphism $U \to \mathcal{X}$ from a scheme and a preimage $u \in U(\Bbbk)$ of x_0. Conditions (3)–(4) guarantee that there exists an effective versal formal deformation $\widehat{x}\colon \operatorname{Spec} R \to \mathcal{X}$ of x_0 where $(R, \mathfrak{m})$ is a noetherian complete local $\Bbbk$-algebra with residue field $\Bbbk$. By Artin Algebraization (Theorem D.6.6), there exists a finite type $\Bbbk$-scheme U, a point $u \in U(\Bbbk)$, a morphism $p\colon U \to \mathcal{X}$, an isomorphism $R \cong \widehat{\mathcal{O}}_{U,u}$ and compatible isomorphisms $p|_{R/\mathfrak{m}^{n+1}} \xrightarrow{\sim} \widehat{x}|_{R/\mathfrak{m}^{n+1}}$. By (5), we can replace U with an open neighborhood of u so that $U \to \mathcal{X}$ is versal at every $\Bbbk$-point of U. By the equivalence in the first paragraph, we have obtained a smooth morphism $(U, u) \to (\mathcal{X}, x_0)$.

See also [Art74], [LMB, Cor. 10.11] and [SP, Tag 07Y4] where the result is established more generally. $\square$

Remark D.7.2. In practice, condition (1)–(4) are often easy to verify directly with (3) a consequence of Rim–Schlessinger's Criteria (Theorem D.3.11) and (4) a consequence of Grothendieck's Existence Theorem (D.4.4). Also note that (2) can sometimes be established by applying the theorem to the diagonal $\mathcal{X} \to \mathcal{X} \times \mathcal{X}$, i.e. to the Isom sheaves $\operatorname{Isom}_T(x, y)$ of objects $x, y \in \mathcal{X}(T)$ over a scheme T. In some cases Condition (5) can be checked directly while for more general moduli problems, it is often a consequence of a well-behaved deformation and obstruction theory as will be explained in the next section.

D.7.1 Refinements of Artin's Axioms

We state a refinement of Artin's Axioms for Algebraicity that is often easier to verify in practice. To formulate the statements, we will need a bit of notation. Let $\xi \in \mathcal{X}(A)$ be an object over a finitely generated $\Bbbk$-algebra A. Let M be a finite A-module and denote by $A[M]$ the ring $A \oplus M$ defined by $M^2 = 0$. Let $\operatorname{Def}_\xi(M)$ the set of isomorphism classes of diagrams

$$\begin{tikzcd} \operatorname{Spec} A \arrow[r, "\xi"] \arrow[d, hook] & \mathcal{X} \\ \operatorname{Spec} A[M], \arrow[ur, dashed, "\eta"] & \end{tikzcd}$$

where an isomorphism of two extensions $\eta, \eta'\colon \operatorname{Spec} A[M] \to \mathcal{X}$ is by definition an isomorphism $\eta \xrightarrow{\sim} \eta'$ in $\mathcal{X}(A[M])$ restricting to the identity on ξ. Let $\operatorname{Aut}_\xi(M)$ be the group of automorphisms of the trivial deformation $\xi'\colon \operatorname{Spec} A[M] \to \operatorname{Spec} A \to \mathcal{X}$. Note that when $\xi \in \mathcal{X}(\Bbbk)$, then $\operatorname{Def}_\xi(\Bbbk)$ is precisely the tangent space of $\mathcal{X}$ at ξ and is identified with $T\mathcal{X}_\xi = \mathcal{X}_\xi(\Bbbk[\epsilon])/\sim$ of the local deformation prestack at ξ while $\operatorname{Aut}_\xi(\Bbbk)$ is the group of infinitesimal automorphism of ξ and is identified with the kernel $\operatorname{Aut}_{\mathcal{X}(\Bbbk[\epsilon])}(\xi') \to \operatorname{Aut}_{\mathcal{X}(\Bbbk)}(\xi)$.

Lemma D.7.3. *Suppose that $\mathcal{X}$ is a prestack over* $\mathrm{Sch}/\Bbbk$ *satisfying the strong homogeneity condition* (RS_4^*). *Let $\xi \in \mathcal{X}(A)$ be an object over a finitely generated $\Bbbk$-algebra A.*

(1) For every A-module M, $\mathrm{Def}_\xi(M)$ and $\mathrm{Aut}_\xi(M)$ are naturally A-modules, and the functors

$$\mathrm{Aut}_\xi(-)\colon \mathrm{Mod}(A) \to \mathrm{Mod}(A)$$
$$\mathrm{Def}_\xi(-)\colon \mathrm{Mod}(A) \to \mathrm{Mod}(A)$$

are A-linear.

(2) Consider a surjection $B \twoheadrightarrow A$ in $\mathrm{Art}_{\Bbbk}$ with square-zero kernel I, and let $\mathrm{Lift}_\xi(B)$ be the set of morphisms $\xi \to \eta$ over $\mathrm{Spec}\, A \to \mathrm{Spec}\, B$ where $\xi \xrightarrow{\alpha} \eta$ is declared equivalent to $\xi \xrightarrow{\alpha'} \eta'$ if there is an isomorphism $\beta\colon \eta \to \eta'$ such that $\alpha' = \beta \circ \alpha$. There is an action of $\mathrm{Def}_\xi(I)$ on $\mathrm{Lift}_\xi(B)$ which is functorial in B and I. Assuming $\mathrm{Lift}_\xi(B)$ is non-empty, this action is free and transitive.

Proof. This can be established by arguing as in Lemma D.3.13. For instance, scalar multiplication by $x \in A$ is defined by pulling back along the morphism $\mathrm{Spec}\, A[M] \to \mathrm{Spec}\, A[M]$ induced by the A-algebra homomorphism $A[M] \to A[M], a+m \mapsto a+xm$. Condition (RS$_4^*$) implies that the functor $\mathcal{X}(A[M\oplus M]) \to \mathcal{X}(A[M]) \times_{\mathcal{X}(A)} \mathcal{X}(A[M])$ is an equivalence. Addition $M \oplus M \to M$ induces an A-algebra homomorphism $A[M \oplus M] \to A[M]$ and thus a functor

$$\mathcal{X}(A[M]) \times_{\mathcal{X}(A)} \mathcal{X}(A[M]) \cong \mathcal{X}(A[M \oplus M]) \to \mathcal{X}(A[M])$$

which defines addition on $\mathrm{Def}_\xi(M)$ and $\mathrm{Aut}_\xi(M)$. $\square$

Theorem D.7.4 (Artin's Axioms for Algebraicity—second version)**.** *A stack $\mathcal{X}$ over $(\mathrm{Sch}/\Bbbk)_{\text{ét}}$ is an algebraic stack locally of finite type over $\Bbbk$ if the following conditions hold:*

(AA$_1$) *(Limit preserving) The stack $\mathcal{X}$ is limit preserving;*

(AA$_2$) *(Representability of the diagonal) The diagonal $\mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is representable;*

(AA$_3$) *(Finiteness of tangent spaces) For every object $\xi\colon \mathrm{Spec}\,\Bbbk \to \mathcal{X}$, $\mathrm{Def}_\xi(\Bbbk)$ is a finite dimensional $\Bbbk$-vector space;*

(AA$_4$) *(Strong homogeneity) For every $\Bbbk$-algebra homomorphism $B_0 \to A_0$ and surjection $A \twoheadrightarrow A_0$ of $\Bbbk$-algebras with square-zero kernel, the functor*

$$\mathcal{X}(B_0 \times_{A_0} A) \to \mathcal{X}(B_0) \times_{\mathcal{X}(A_0)} \mathcal{X}(A)$$

is an equivalence, i.e. Condition (RS$_4^*$) holds;

(AA$_5$) *(Effectivity) For every noetherian complete local $\Bbbk$-algebra $(R, \mathfrak{m})$, the natural functor*

$$\mathcal{X}(\mathrm{Spec}\, R) \to \varprojlim \mathcal{X}(\mathrm{Spec}\, R/\mathfrak{m}^n)$$

is an equivalence of categories;

(AA$_6$) *(Existence of an obstruction theory) For every object $\xi \in \mathcal{X}(A)$ over a finitely generated $\Bbbk$-algebra A, there exists the following data*

(a) there is an A-linear functor

$$\mathrm{Ob}_\xi(-)\colon \mathrm{Mod}(A) \to \mathrm{Mod}(A),$$

and for every surjection $B \to A$ with square-zero kernel I, there is an element $\mathrm{ob}_\xi(B) \in \mathrm{Ob}_\xi(I)$ such that there is an extension

$$\begin{tikzcd} \operatorname{Spec} A \arrow[r, "\xi"] \arrow[d, hook] & \mathcal{X} \\ \operatorname{Spec} B \arrow[ur, dashed] & \end{tikzcd}$$

if and only if $\mathrm{ob}_\xi(B) = 0$, *and*

(b) *for every composition $B \to B' \to A$ of $\Bbbk$-algebras such that $B \twoheadrightarrow A$ and $B' \twoheadrightarrow A$ are surjective with square-zero kernels I and I', the image of $\mathrm{ob}_\xi(B)$ under $\mathrm{Ob}_\xi(I) \to \mathrm{Ob}_\xi(I')$ is $\mathrm{ob}_\xi(B')$; and*

(AA_7) *(Coherent deformation theory) For every object $\xi \in \mathcal{X}(A)$ over a $\Bbbk$-algebra A, the functors $\mathrm{Def}_\xi(-)$ and $\mathrm{Ob}_\xi(-)$ commute with products.*

Moreover (AA_2) can be removed if we replace (AA_3) and (AA_7) with:

($\mathrm{AA}_{3'}$) *For every object $\xi\colon \operatorname{Spec}\Bbbk \to \mathcal{X}$, $\mathrm{Aut}_\xi(\Bbbk)$ and $\mathrm{Def}_\xi(\Bbbk)$ are finite dimensional $\Bbbk$-vector spaces; and*

($\mathrm{AA}_{7'}$) *For every object $\xi \in \mathcal{X}(A)$ over a $\Bbbk$-algebra A, the functors $\mathrm{Aut}_\xi(-)$, $\mathrm{Def}_\xi(-)$ and $\mathrm{Ob}_\xi(-)$ commute with products.*

Proof. Conditions (AA_3)–(AA_4) above allow us to apply Rim–Schlessinger's Criteria (Theorem D.3.11) to deduce the existence of versal formal deformations, i.e. Condition D.7.1(3) holds. It remains to check openness of versality, i.e. Condition D.7.1(5), in order to apply the first version (Theorem D.7.1) to establish this version.

Let $\xi_0\colon U_0 \to \mathcal{X}$ be a morphism from an affine scheme $U_0 = \operatorname{Spec} B_0$ of finite type over $\Bbbk$ which is versal at a point $u_0 \in U_0(\Bbbk)$. By (AA_1)–(AA_2), the morphism $\xi_0\colon U_0 \to \mathcal{X}$ is representable and locally of finite type. Let $\Sigma = \{u \in U_0(\Bbbk) \mid \xi_0\colon U_0 \to \mathcal{X} \text{ is not versal at } u\}$. If openness of versality does not hold, then $u_0 \in \overline{\Sigma}$ and there exists a countably infinite subset $\Sigma' = \{u_1, u_2, \dots\} \subset \Sigma$ of distinct points with $u_0 \in \overline{\Sigma'}$.

<u>*Step 1.*</u> We claim that there exists a commutative diagram

$$\begin{tikzcd} U_0 \arrow[r, hook] \arrow[d, "\xi_0"'] & U_1 \arrow[r, hook] \arrow[dl, "\xi_1"] & U_2 \arrow[r, hook] \arrow[dll, "\xi_2"] & \cdots \\ \mathcal{X} & & & \end{tikzcd}$$

where each closed immersion $U_{n-1} \hookrightarrow U_n$ is defined by a short exact sequence

$$0 \to \kappa(u_n) \to \mathcal{O}_{U_n} \to \mathcal{O}_{U_{n-1}} \to 0,$$

and for each n and open neighborhood $W \subset U_n$ of u_n, the restriction $\xi_n|_W$ is not the trivial deformation of $\xi_0|_{W \cap U_0}$, i.e. there is no morphism $r\colon \xi_n|_W \to \xi_0|_{W\cap U_0}$ such that $\xi_n|_W \xrightarrow{r} \xi_0|_{W\cap U_0} \to \xi_n|_W$ is the identity. Note that for each $m \geq n$, $U_n \hookrightarrow U_m$ is a closed immersion which is square-zero (i.e. $\ker(\mathcal{O}_{U_m} \to \mathcal{O}_{U_n})$ is square-zero). We will inductively construct $U_n = \operatorname{Spec} B_n$ and $\xi_n \in \mathcal{X}(U_n)$, Since $\xi_0\colon U_0 \to \mathcal{X}$ and $\xi_{n-1}\colon U_{n-1} \to \mathcal{X}$ are isomorphic in an open neighborhood of

u_n, the morphism $\xi_{n-1}\colon U_{n-1} \to \mathcal{X}$ is also not versal at u_n. By definition of versality (using Remark D.3.7) there exists a surjection $A \to A_0$ in $\mathrm{Art}_{\Bbbk}$ with $\ker(A \to A_0) = \Bbbk$ and a commutative diagram

$$\begin{tikzcd} \operatorname{Spec} A_0 \arrow[r] \arrow[d, hook] & U_{n-1} \arrow[d, "\xi_{n-1}"] \\ \operatorname{Spec} A \arrow[r] \arrow[ur, dashed, "\nexists"] & \mathcal{X}, \end{tikzcd} \tag{D.7.1}$$

such that u_n is the image of $\operatorname{Spec} A_0 \to U_{n-1}$, which does not admit a lift $\operatorname{Spec} A \to U_{n-1}$. Using strong homogeneity (AA$_4$), there exists an extension of the commutative diagram

$$\begin{tikzcd} \operatorname{Spec} A_0 \arrow[r] \arrow[d, hook] & U_{n-1} = \operatorname{Spec} B_{n-1} \arrow[d, hook] \arrow[ddr, bend left, "\xi_{n-1}"] & \\ \operatorname{Spec} A \arrow[r] \arrow[drr, bend right] & U_n = \operatorname{Spec}(A \times_{A_0} B_{n-1}) \arrow[dr, dashed, "\xi_n"] & \\ & & \mathcal{X} \end{tikzcd}$$

yielding an object ξ_n over $U_n = \operatorname{Spec} B_n$ with $B_n := A \times_{A_0} B_{n-1}$. If ξ_n were the trivial deformation of ξ_0 in an open neighborhood of u_n, then $\operatorname{Spec} A \to \mathcal{X}$ would be the trivial deformation of $\operatorname{Spec} A_0$ contradicting the obstruction to a lift of (D.7.1). Finally note that $\ker(B_n \to B_{n-1}) = \Bbbk$ since $\ker(A \to A_0) = \Bbbk$. This establishes the claim.

<u>*Step 2.*</u> Letting $\widehat{B} = \varprojlim B_n$ and $\widehat{U} = \operatorname{Spec} \widehat{B}$, we claim that there exists an object $\widehat{\xi} \in \mathcal{X}(\widehat{U})$ extending each $\xi_n \in \mathcal{X}(U_n)$. Let $M_n = \ker(B_n \to B_0)$ (noting that $M_0 = 0$). Since $M_n^2 = 0$, we can view M_n as a B_0-module. The $\Bbbk$-algebra

$$\widetilde{B} := \{(b_0, b_1, \ldots) \in \prod_{n \geq 0} B_n \mid \text{the image of each } b_n \text{ under } B_n \to B_0 \text{ is } b_0\}$$

has the following properties:

- The surjective $\Bbbk$-algebra homomorphism $\widetilde{B} \to B_0$ defined by $(b_i) \mapsto b_0$ has kernel $M := \prod_{n \geq 0} M_n$;
- The map $\widetilde{B} \to B_0[M]$ defined by $(b_0, b_1, b_2, \ldots) \mapsto (b_0, b_1 - b_0, b_2 - b_1, b_3 - b_2, \ldots)$ is a surjective $\Bbbk$-algebra homomorphism with square-zero kernel;
- The composition $\widehat{B} \to \widetilde{B} \to B_0[M]$ induces a short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \ker(\widehat{B} \to B_0) \arrow[r] \arrow[d, equal] & \ker(\widetilde{B} \to B_0) \arrow[r] \arrow[d, equal] & \ker(B_0[M] \to B_0) \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \varprojlim_{n \geq 0} M_n \arrow[r] & \prod_{n \geq 0} M_n \arrow[r] & \prod_{n \geq 0} M_n \arrow[r] & 0 \\ & & (b_0, b_1, \ldots) \arrow[r, mapsto] & (b_1 - b_0, b_2 - b_1, \ldots) & \end{tikzcd}$$

- There is an identification $\widehat{B} = \widetilde{B} \times_{B_0[M]} B_0$.

Since the lift $\xi_n \in \mathcal{X}(B_n)$ of ξ_0 exists for each n, $\mathrm{ob}_\xi(B_n) = 0 \in \mathrm{Ob}_\xi(M_n)$. By (AA$_6$)(b), the element $\mathrm{ob}_\xi(\widetilde{B})$ maps to $\mathrm{ob}_\xi(B_n)$ under $\mathrm{Ob}_\xi(M) \to \mathrm{Ob}_\xi(M_n)$. By (AA$_7$), the map $\mathrm{Ob}_\xi(M) \hookrightarrow \prod_n \mathrm{Ob}_\xi(M_n)$ is injective[3] and thus $\mathrm{ob}_\xi(\widetilde{B}) = 0 \in \mathrm{Ob}_\xi(M)$ which shows that there exists a lift $\widetilde{\xi} \in \mathcal{X}(\widetilde{B})$ of ξ_0.

The restrictions $\widetilde{\xi}|_{B_n}$ are not necessarily isomorphic to ξ_n. However, we may use the free and transitive action $\mathrm{Def}_\xi(M_n) = \mathrm{Lift}_\xi(B_0[M_n])$ on the non-empty set of liftings $\mathrm{Lift}_\xi(\widetilde{B_n})$ to find elements $t_n \in \mathrm{Def}_\xi(M_n)$ such that $\xi_n = t_n \cdot \widetilde{\xi}|_{B_n}$ (Lemma D.7.3). Since $\mathrm{Def}_\xi(M) \xrightarrow{\sim} \prod_n \mathrm{Def}_\xi(M_n)$ by (AA$_7$), there exists $\widetilde{t} \in \mathrm{Def}_\xi(M)$ mapping to (t_n). After replacing $\widetilde{\xi}$ with $\widetilde{t} \cdot \widetilde{\xi}$, we can arrange that $\widetilde{\xi}|_{B_n}$ and ξ_n are isomorphic for each n.

We now show that each restriction $\widetilde{\xi}|_{B_0[M_n]} \in \mathrm{Def}_\xi(M_n)$ under the composition $\widetilde{B} \to B_0[M] \to B_0[M_n]$ is the trivial deformation. Indeed, the map $M = \ker(\widetilde{B} \to B_0) \to \ker(B_0[M_n] \to B_0) = M_n$ induces a map $\mathrm{Def}_\xi(M) \to \mathrm{Def}_\xi(M_n)$ on deformation modules which under the identification $\mathrm{Def}_\xi(M) \xrightarrow{\sim} \prod_n \mathrm{Def}_\xi(M_n)$ of (AA$_7$) sends an element $(\eta_0, \eta_1, \ldots)$ to $(\eta_{n+1}|_{B_n} - \eta_n)$. The ring map $\widetilde{B} \to B_0[M_n]$ also induces a map $\mathrm{Lift}_\xi(\widetilde{B}) \to \mathrm{Lift}_\xi(B_0[M_n])$ which is equivariant with respect to $\mathrm{Def}_\xi(M) \to \mathrm{Def}_\xi(M_n)$. It follows that the image of $\widetilde{\xi}$ in $\mathrm{Lift}_\xi(B_0[M_n]) = \mathrm{Def}_\xi(M_n)$ is $\xi_{n+1}|_{B_n} - \xi_n = 0$.

The existence of $\widehat{\xi} \in \mathcal{X}(\widehat{B})$ extending $(\xi_n) \in \varprojlim \mathcal{X}(B_n)$ now follows from applying the identity $\widehat{B} = \widetilde{B} \times_{B_0[M]} B_0$ and strong homogeneity (AA$_4$) to the diagram

$$\begin{tikzcd}
\operatorname{Spec} B_0[M] \arrow[r] \arrow[d, hook] & \operatorname{Spec} B_0 \arrow[d, hook] \arrow[ddr, bend left, "\xi_0"] & \\
\operatorname{Spec} \widetilde{B} \arrow[r] \arrow[drr, bend right, "\widetilde{\xi}"'] & \operatorname{Spec} \widehat{B} \arrow[dr, dashed, "\widehat{\xi}"] & \\
& & \mathcal{X}
\end{tikzcd}$$

Step 3. We now use the versality of $\xi_0 \colon U_0 \to \mathcal{X}$ at u_0 to arrive at a contradiction. Since $\mathcal{X}$ is limit preserving (AA$_1$), there exists a finitely generated $\Bbbk$-subalgebra $B' \subset \widehat{B}$ and an object $\xi' \in \mathcal{X}(B')$ together with an isomorphism $\widehat{\xi} \xrightarrow{\sim} \xi|_{\widehat{B}}$. After possibly enlarging B', we may assume that the composition $B' \hookrightarrow \widehat{B} \to B_0$ is surjective. There is thus a closed immersion $U_0 \hookrightarrow U' := \operatorname{Spec} B'$ and we can consider the commutative diagram

$$\begin{tikzcd}
U_0 \arrow[dr, "i"] \arrow[drr, hook, bend left] \arrow[ddr, bend right, "\mathrm{id}"'] & & \\
& U_0 \times_{\mathcal{X}} U' \arrow[r] \arrow[d] & U' = \operatorname{Spec} B' \arrow[d, "\xi'"] \\
& U_0 \arrow[r, "\xi_0"] & \mathcal{X}.
\end{tikzcd}$$

[3]The hypotheses of (AA$_7$) can be weakened to only require the injectivity of $\mathrm{Ob}_\xi(M) \hookrightarrow \prod_n \mathrm{Ob}_\xi(M_n)$ although in practice one usually verifies that this map is bijective.

where the fiber product $U_0 \times_{\mathcal{X}} U'$ is an algebraic space locally of finite type over $\Bbbk$. Since $\xi_0 \colon U_0 \to \mathcal{X}$ is versal at u_0, it follows from the artinian version of the Infinitesimal Lifting Criterion for Smoothness (Smooth Equivalences A.3.1) that $U_0 \times_{\mathcal{X}} U' \to U'$ is smooth at $i(u_0)$. After replacing U_0 with an open affine neighborhood of u_0, U' with the corresponding open and $\{u_1, u_2, \ldots\}$ with an infinite subsequence contained in this open, we can arrange that $U_0 \times_{\mathcal{X}} U' \to U'$ is smooth. The non-artinian version of the Infinitesimal Lifting Criterion for Smoothness implies the section of $U_0 \times_{\mathcal{X}} U' \to U'$ over U_0 extends to a global section $U' \to U_0 \times_{\mathcal{X}} U'$. This implies that ξ' is the trivial deformation of ξ_0, contradicting our choice of $\xi' \colon U' \to \mathcal{X}$.

Our exposition follows [SP, Tag 0CYF] and [Hal17, Thm. A]. See also [Art74, Thm. 5.3] and [HR19a, Main Thm.]. □

Remark D.7.5. The converse of the theorem also holds. For the necessity of the conditions, we only need to check (AA$_3$), (AA$_4$), (AA$_6$) and (AA$_7$). Condition (AA$_3$) (finiteness of the tangent spaces) holds as $\mathcal{X}$ is of finite type over $\Bbbk$. The strong homogeneity condition (AA$_4$) holds by [SP, Tag 07WN]. Condition (AA$_6$) (existence of an obstruction theory) follows from the existence of a cotangent complex $L_{\mathcal{X}/\Bbbk}$ for $\mathcal{X}$ satisfying properties analogous to Theorem D.5.1; see [Ols06]. If $\xi\colon \operatorname{Spec} A \to \mathcal{X}$ is a morphism from a finitely generated $\Bbbk$-algebra A and I is an A-module, then we set $\mathrm{Ob}_\xi(I) := \mathrm{Ext}^1_A(\xi^* L_{\mathcal{X}/\Bbbk}, I)$. Property (AA$_6$)(b) holds as a consequence of [Ols06, Thm. 1.5], a generalization of [Ill71, III.2.2.4] (which was discussed in Remark D.5.12) from morphisms of schemes to representable morphisms of algebraic stacks. Finally, Condition (AA$_7$) ($\mathrm{Def}_\xi(-)$ and $\mathrm{Ob}_\xi(-)$ commutes with products) follows from cohomology and base change.

D.7.2 Verifying Artin's Axioms

Theorem D.7.6. *Each of the stacks* $\mathrm{Hilb}^P(X)$*,* $\mathscr{M}_g$ *(with* $g \geq 2$*) and* $\mathcal{B}\mathrm{un}_C$ *over* $(Sch/\Bbbk)_{\text{ét}}$ *are algebraic stacks locally of finite type over* $\Bbbk$*.*

Proof. We check condition the conditions of Theorem D.7.4. Condition (AA$_1$) (limit preserving) was verified in Lemma D.6.1. For (AA$_{3'}$), the finite dimensionality of the vector spaces $\mathrm{Def}_\xi(\Bbbk)$ and $\mathrm{Aut}_\xi(\Bbbk)$ for an object $\xi \in \mathcal{X}(\Bbbk)$ can be identified with:

- $\mathrm{H}^0(Z, N_{Z/X})$ and $\{0\}$ for $\xi = [Z \subset X] \in \mathrm{Hilb}^P(X)(\Bbbk)$ (Proposition D.1.4),
- $\mathrm{H}^1(C, T_C)$ and $\mathrm{H}^0(C, T_C)$ for $\xi = [C] \in \mathscr{M}_g(\Bbbk)$ (Lemma D.1.10 and Proposition D.1.11) and
- $\mathrm{Ext}^1_{\mathcal{O}_C}(E, E)$ and $\mathrm{Ext}^0_{\mathcal{O}_C}(E, E)$ for $\xi = [E] \in \mathcal{B}\mathrm{un}_C(\Bbbk)$ (Proposition D.1.15).

Condition (AA$_4$) (the strong homogeneity condition of (RS$_4^*$)) was checked in Proposition D.3.14. Condition (AA$_5$) (effectivity) was checked in Corollary D.4.9 as a consequence of Grothendieck's Existence Theorem. For Condition (AA$_6$), we define obstruction theories as follows: for a finitely generated $\Bbbk$-algebra A and an A-module M, we set

- $\mathrm{Ob}_\xi(M) := \mathrm{Ext}^1_{\mathcal{O}_Z}(I_Z/I_Z^2, M)$ for $\xi = [Z \subset X_A] \in \mathrm{Hilb}^P(X)(A)$ where $I_Z \subset \mathcal{O}_{X_A}$ is the sheaf of ideal defining Z.
- $\mathrm{Ob}_\xi(M) := \mathrm{H}^2(\mathcal{C}, T_{\mathcal{C}/A} \otimes_A M) = 0$ for $\xi = [\mathcal{C} \to \operatorname{Spec} A] \in \mathscr{M}_g(A)$, and

- $\mathrm{Ob}_\xi(M) := \mathrm{H}^2(C_A, \mathscr{E}nd_{\mathcal{O}_{C_A}}(E) \otimes_A M) = 0$ for $\xi = [E] \in \mathcal{B}\mathrm{un}_C(A)$.

Property (AA$_6$)(a) holds for these obstruction theories as a consequence of Propositions D.2.2, D.2.6 and D.2.15; these results also show that $\mathrm{Aut}_\xi(M)$ and $\mathrm{Def}_\xi(M)$ are identified with the analogous cohomology groups. Condition (AA$_{7'}$) ($\mathrm{Aut}_\xi(-)$, $\mathrm{Def}_\xi(-)$ and $\mathrm{Ob}_\xi(-)$ commutes with products) for follows from (Corollary D.7.7). □

Corollary D.7.7. *Let $X \to \operatorname{Spec} A$ be a proper morphism of noetherian schemes. Let E and F be coherent sheaves on X with F flat over A. Then the functors*

$$\begin{aligned} \mathrm{H}^i(X, F \otimes_A -)\colon \ \mathrm{Mod}(A) \to \mathrm{Mod}(A) \qquad &\textit{and} \\ \mathrm{Ext}^i_{\mathcal{O}_X}(E, F \otimes_A -)\colon \ \mathrm{Mod}(A) \to \mathrm{Mod}(A) \qquad & \end{aligned}$$

commute with products.

Proof. Since F is flat over A, there is a perfect complex $K^\bullet$ of A-modules such that $\mathrm{H}^i(X, F \otimes_A -) \cong \mathrm{H}^i(K^\bullet \otimes_A -)$ (Theorem A.7.1). Write $K^d = A^{\oplus r_d}$. For every set of A-modules $\{M_\alpha\}$ we have an identification of complexes

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \prod_\alpha M_\alpha^{\oplus r_0} & \longrightarrow & \prod_\alpha M_\alpha^{\oplus r_1} & \longrightarrow & \cdots & \longrightarrow & \prod_\alpha M_\alpha^{\oplus r_n} & \longrightarrow & 0 \\ & & \| & & \| & & & & \| & & \\ 0 & \longrightarrow & (\prod_\alpha M_\alpha)^{\oplus r_0} & \longrightarrow & (\prod_\alpha M_\alpha)^{\oplus r_1} & \longrightarrow & \cdots & \longrightarrow & (\prod_\alpha M_\alpha)^{\oplus r_n} & \longrightarrow & 0. \end{array}$$

The top row is the product of the complexes $K^\bullet \otimes_A M_\alpha$ and its cohomology is identified with $\prod_\alpha \mathrm{H}^i(X, F \otimes_A M_\alpha)$ while the bottom row is $K^\bullet \otimes_A (\prod_\alpha M_\alpha)$ with cohomology groups $\mathrm{H}^i(X, F \otimes_A (\prod_\alpha M_\alpha))$. For the second statement, one needs to apply alternative versions of cohomology and base change (see [EGA, III.7.7.5], [SP, Tag 08JR] and [Hal14, Thm. E]). □

Appendix E

Birational Geometry

E.1 Birational geometry of surfaces

By a *surface*, we mean an integral scheme X of pure dimension 2 which is either of finite type over an algebraically closed field $\Bbbk$ or of finite type over a complete DVR R with algebraically closed residue field $\Bbbk$. In the latter case, we say that X is smooth (resp. projective), we mean that the structure morphism $X \to \operatorname{Spec} R$ is smooth (resp. projective), and a curve $X_0 \subset X$ is by definition a component in the central fiber.

Theorem E.1.1 (Minimal Resolutions)**.** *Let X be a surface. There exists a unique projective birational morphism $\pi \colon \widetilde{X} \to X$ from a smooth surface such that every other resolution $Y \to X$ factors as $Y \to \widetilde{X} \to X$ (or equivalently such that $K_{\widetilde{X}} \cdot E \geq 0$ for every π-exceptional curve E).*

Proof. See [Kol07, Thm. 2.16]. □

Theorem E.1.2 (Embedded Resolutions of Curves in Surfaces)**.** *Let X be a surface and $X_0 \subset X$ be a curve. There is a finite sequence of blow-ups at reduced points of X_0 yielding a projective birational morphism $\widetilde{X} \to X$ such that $\widetilde{X}$ is smooth and such that the preimage $\widetilde{X}_0$ of X_0 has set-theoretic normal crossings, i.e. $(\widetilde{X}_0)_{\mathrm{red}}$ is nodal.*

Proof. See [Har77, Thm V.3.9] and [Kol07, Thm. 1.47]. □

Theorem E.1.3 (Structure Theorem of Birational Morphisms of Surfaces)**.** *Everyprojective birational morphism $f \colon X \to Y$ of smooth surfaces is the composition of blowing up smooth points.*

Proof. See [Har77, Thm V.5.5] and [Kol07, Thm 2.13]. □

Theorem E.1.4 (Hodge Index Theorem for Exceptional Curves)**.** *Let $f \colon X \to Y$ be a projective and generically finite morphism of surfaces with X smooth and Y quasi-projective. Let $E_1, \ldots, E_n$ be the exceptional curves. Then the intersection form matrix $(E_i \cdot E_j)$ is negative-definite. In particular, $E_i^2 < 0$ for each i.*

Proof. See [Kol07, Thm 2.12]. □

Theorem E.1.5 (Castelnuovo's Contraction Theorem)**.** *Let X be a smooth projective surface and E a smooth rational curve with $E^2 = -1$. Then there is a projective morphism $X \to Y$ to a smooth surface and a point $y \in Y$ such that $f^{-1}(y) = E$ and $X \setminus E \to Y \setminus \{y\}$ is an isomorphism.*

Proof. See [Har77, Thm. V.5.7] and [Kol07, Thm 2.14]. □

Remark E.1.6. If $E^2 < -1$, then E can still be contracted to a point but the surface may be singular.

One can show that the process of repeatedly contracting smooth rational -1 curves in a smooth projective surface terminates (see [Har77, Thm 5.8]). Thus by applying Castelnuovo's Contractibility Criterion a finite number of times, one obtains:

Corollary E.1.7 (Existence of Minimal Models)**.** *A smooth surface X admits a projective birational morphism $X \to X_{\text{min}}$ to a smooth surface such that every projective birational morphism $X_{\text{min}} \to Y$ to a smooth surface is an isomorphism. In particular, X_{min} has no smooth rational -1 curves.*

E.2 Positivity

The standard reference for this material is [Laz04a, Laz04b].

E.2.1 Ample line bundles

Let X be a proper scheme over an algebraically closed field $\Bbbk$. A line bundle L on X is *ample* if for some $m > 0$, $L^{\otimes m}$ is very ample, i.e. defines a closed embedding $|L^{\otimes m}|: X \hookrightarrow \mathbb{P}^N$ into projective space. Ampleness can be equivalently characterized by any of the following conditions:

- $L^{\otimes m}$ is very ample for $m \gg 0$,
- for every $x \in X$, there exists a section $s \in \Gamma(X, L)$ such that $X_s = \{s \neq 0\}$ is affine and contains x,
- for every coherent sheaf F, the tensor product $F \otimes L^{\otimes m}$ is base point free for $m \gg 0$, or
- for every coherent sheaf F on X, the cohomology groups $\mathrm{H}^i(X, F \otimes L^{\otimes m}) = 0$ vanish for $i > 0$ and $m \gg 0$.

See [Har77, §II.7, III.5.3] or [SP, Tags 01PR and 0B5U].

Proposition E.2.1 (Openness of Ampleness)**.** *Let $f\colon X \to Y$ be a proper, flat, and finitely presented morphism of schemes, and L be a line bundle on X. If for some $y \in Y$, the restriction L_y of y to the fiber X_y is ample (resp. very ample and $\mathrm{H}^i(X_y, L_y) = 0$ for $i > 0$), then there exists an open neighborhood $U \subset S$ of s such that the restriction L_U on X_U is relatively ample (resp. relatively very ample) over U. In particular, for all $u \in U$, L_u is ample (resp. very ample) on X_u.*

Proof. If L_y is ample on X_y, then for $n \gg 0$, $L_y^{\otimes n}$ is very ample and $\mathrm{H}^i(X_y, L_y^{\otimes n}) = 0$ for $i > 0$. It therefore suffices to handle the very ample case. By Cohomology and Base Change (A.7.5), after replacing Y with an open neighborhood of y, f_*L is a vector bundle and the comparison map $f_*L \otimes \kappa(t) \to \mathrm{H}^i(X_t, L_t)$ is an isomorphism for $t \in Y$. By further replacing Y with an affine open neighborhood, we can arrange that $\mathrm{H}^0(X, L)$ is freely generated by sections $s_0, \ldots, s_n$ that restrict to a basis in $\mathrm{H}^0(X_y, L_y)$. The vanishing locus $V := V(s_0, \ldots, s_n) \subset X$ is closed and disjoint from X_y. By replacing Y with an affine open neighborhood of s contained in $Y \setminus f(V)$, we may assume that the sections s_i generate L, and that they define a morphism $g\colon X \to \mathbb{P}^n_Y$ over Y restricting to a closed immersion $g_y\colon X_y \hookrightarrow \mathbb{P}^n_{\kappa(y)}$. By upper semi-continuity of fiber dimension, there is a closed locus $Z \subset \mathbb{P}^n_Y$ consisting of points z such that $\dim g^{-1}(z) > 0$. Since Z is disjoint from $\mathbb{P}^n_{\kappa(y)}$, we may shrink Y further so that $g\colon X \to \mathbb{P}^n_Y$ is quasi-finite, and hence finite as g is proper. The cokernel $\mathcal{O}_{\mathbb{P}^n_Y} \to g_*\mathcal{O}_X$ is coherent and its support is a closed subscheme of $\mathbb{P}^n$ disjoint from $\mathbb{P}^n_{\kappa(y)}$. By shrinking Y further, we may arrange that $g\colon X \to \mathbb{P}^n_Y$ is a closed immersion and hence $L = g^*\mathcal{O}_{\mathbb{P}^n_Y}(1)$ is very ample.

See also [Laz04a, Thm. 1.2.17, Thm. 1.7.8], [EGA, III_1.4.7.1, IV_3.9.6.4], [KM98, Prop. 1.41] and [SP, Tags 0D3A and 0D3D]; the openness of ampleness holds without assuming flatness of $X \to Y$. □

We also recall that ampleness can be checked on finite covers.

Proposition E.2.2. *Let $f\colon X \to Y$ be a finite morphism of noetherian schemes and L be a line bundle on Y. If L is ample, then so is f^*L. If f is surjective, then the converse is true.*

Proof. See [Har77, Exer III.5.7]. □

Remark E.2.3. As an immediate consequence, we see that a line bundle L on X is ample if and only if its restriction $L_{(X_i)_{\mathrm{red}}}$ to the reduced subscheme of each irreducible component X_i is ample.

E.2.2 Nef line bundles

A line bundle L on a proper scheme X over a field $\Bbbk$ is *nef* if

$$\int_C c_1(L) \geq 0$$

for every irreducible curve. Here $\int_C c_1(L)$ is the same number as $C\dot{L}$ or $\deg L|_C$.

Proposition E.2.4 (Openness of Nefness)**.** *Let X be a proper and flat scheme over a DVR R and L be a line bundle on X. Let $0, \eta \in \operatorname{Spec} R$ be the closed and generic points. If $L|_{X_0}$ is nef, then so is $L|_{X_\eta}$.*

Proof. To be added. □

Remark E.2.5. For proper, flat, and surjective morphisms $X \to S$, it is shown in [Laz04a, Prop 1.4.14] that if $L|_{X_s}$ is ample for a point $s \in S$, then there exists a countable union $B \subset S$ of proper subschemes not containing s such that $L|_{X_t}$ is nef for every $t \in S \setminus B$. It is not known whether there exists an open subset $s \in U \subset S$ with $L|_{X_t}$ nef for $t \in S$.

Proposition E.2.6. *Let $f : X \to Y$ be a proper morphism of noetherian schemes, and let L be a line bundle on Y. If L is nef, then so is f^*L. If f is surjective, then the converse is true.*

Theorem E.2.7 (Kleiman's Theorem)**.** *If L is a line bundle on a proper scheme X over a field $\Bbbk$, then L is nef if and only if for every irreducible subvariety $Z \subset X$ of dimension k,*

$$\int_Z c_1(L)^k \geq 0.$$

Proof. See [Laz04a, Thm. 1.4.9], [Kol96, Thm. 2.17], or the original source [Kle66]. □

Remark E.2.8 (Ample and nef cones)**.** It's also worthwhile to keep in mind that ample and nef line bundles generate cones $\mathrm{Amp}(X), \mathrm{Nef}(X) \subset N^1(X)_{\mathbb{R}}$, called the *ample cone* and *nef cone*. As a consequence of Kleiman's theorem, one can show that for a projective variety, the nef cone is the closure of the ample cone, and that the ample cone is the interior of the nef cone; see [Laz04a, Thm. 1.4.23].

E.2.3 Effective, base point free, and semiample line bundles

We have the following notions for a line bundle L on X:

- L is *effective* if $\Gamma(X, L) \neq 0$,
- L is *base point free* (or *globally generated*) if for every $x \in X$, there exists $s \in \Gamma(X, L)$ with $s(x) \neq 0$, or equivalent the linear series $|L|$ defines a morphism $X \to \mathbb{P}^{\mathrm{h}^0(X,L)-1}$, and
- L is *semiample* if for some $m > 0$, $L^{\otimes m}$ is base point free.

A semiample line bundle L is necessarily nef; indeed if for some $m > 0$, $L^{\otimes m}$ defines a morphism $f : X \to \mathbb{P}^N$ with $f^*\mathcal{O}(1) \cong L^{\otimes m}$, then the projection formula implies that $\int_C c_1(L^{\otimes m}) = \int_{f(C)} c_1(\mathcal{O}(1)) \geq 0$. We thus have the implications

$$\text{base point free} \implies \text{semiample} \implies \text{nef}.$$

E.2.4 Big line bundles

A line bundle L on a normal variety X is *big* if for some $m > 0$, the rational map $\phi_m : X \overset{|L^{\otimes m}|}{\dashrightarrow} \mathbb{P}^N$ is birational onto its image for some $m > 0$. For a possibly non-normal variety X, we say a line bundle L is big if its pullback to the normalization is big.

Proposition E.2.9 (Kodaira's Lemma)**.** *Let X be a projective variety and L be a big line bundle. If E is an effective line bundle, then for m sufficiently divisible, $L^{\otimes m} \otimes E^\vee$ is effective.*

Proof. See [Laz04a, Prop. 2.2.6]. □

Proposition E.2.10 (Equivalences of Bigness)**.** *We have the following equivalences for a line bundle $L = \mathcal{O}_X(D)$ on an irreducible variety:*

L *is big* $\iff$ $\dim \operatorname{im} \phi_m = \dim X$ *for m sufficiently large*
$\iff$ *there exists a constant C such that* $\mathrm{h}^0(X, L^{\otimes m}) \geq C \cdot m^{\dim X}$ *for m sufficiently large*
$\iff$ *for every ample divisor A on X, there exists a positive integer $m > 0$ and an effective divisor N on X such that $mD = A + N$ (linear equivalence).*
$\iff$ *there exists an ample divisor A on X, a positive integer $m > 0$, and an effective divisor N on X such that $mD \equiv A + N$ (numerical equivalence).*

Proof. See [Laz04a, §2.2] for details; the last three equivalences follow from Kodaira's lemma. □

As a consequence of Proposition E.2.10, we see that up scaling (i.e. taking positive tensor powers), a big line bundle is the same as the sum of an ample and effective line bundle. In particular, the sum of a big and effective line bundle is also big. To summarize,

$$\text{big} \overset{\text{up to scaling}}{\Longleftrightarrow} \text{ample} + \text{effective}$$

$$\text{big} + \text{effective} \implies \text{big}.$$

Proposition E.2.11. *Let $f \colon X \to Y$ be a generically quasi-finite and proper morphism of varieties such that $f_* \mathcal{O}_X = \mathcal{O}_Y$ (e.g. f is a proper birational morphism of normal varieties). For a line bundle L on Y, L is big if and only if $f^* L$ is big.*

Proof. The projection formula

$$f_* f^* L^{\otimes m} \cong L^{\otimes m} \otimes f_* \mathcal{O}_X \cong L^{\otimes m}$$

implies that $\Gamma(Y, f^* L^{\otimes m}) = \Gamma(X, L^{\otimes m})$. Since X and Y have the same dimension, the result follows from the above equivalences of bigness. □

Theorem E.2.12 (Asymptotic Riemann-Roch)**.** *Let X be a proper scheme over a field $\Bbbk$ of dimension n, and let L be a nef line bundle on X. Then the Euler characteristic $\chi(X, L^{\otimes m})$ is a polynomial of degree $\leq n$ in m*

$$\mathrm{h}^0(X, L^{\otimes m}) = \frac{(c_1(L)^n)}{n!} m^n + O(m^{n-1}).$$

Remark E.2.13. See [Laz04a, Cor. 1.4.41] for a proof in the projective case and [Kol96, Thm. VI.2.15] in general.

This immediately yields the following useful characterization of bigness for nef line bundles.

Corollary E.2.14. *On a proper scheme of dimension n, a nef line bundle L is big if and only if $c_1(L)^n > 0$.*

Remark E.2.15 (Big and pseudo-effective cones)**.** Big and effective divisors generate cones $\mathrm{Big}(X), \mathrm{Eff}(X) \subset N^1(X)_{\mathbb{R}}$, called the *big cone* and *effective cone*. The closure $\overline{\mathrm{Eff}}(X)$ of $\mathrm{Eff}(X)$ is called the *pseudo-effective cone*. The big cone $\mathrm{Big}(X)$ is the interior of $\overline{\mathrm{Eff}}(X)$, and $\overline{\mathrm{Eff}}(X) = \overline{\mathrm{Big}(X)}$ [Laz04a, Thm. 2.2.6].

In particular, we have the implication:

$$\text{big} + \text{nef} \Rightarrow \text{big}.$$

E.2.5 Ampleness criteria

We review general techniques here to show that a line bundle L on a proper scheme X is ample. Perhaps the first strategy to keep in mind is that if L is semiample and strictly nef, then L is ample.

Lemma E.2.16. *Let X be a proper scheme. If L is a semiample line bundle and $\int_T c_1(L) = \deg L|_C > 0$ for all curves T, then L is ample.*

Proof. For some $m > 0$, $L^{\otimes m}$ defines a morphism $f\colon X \to \mathbb{P}^N$ which does not contract any curves. It follows that $f\colon X \to \mathbb{P}^N$ is a proper and quasi-finite morphism of schemes, thus finite. Therefore, $L^{\otimes m} = f^*\mathcal{O}(1)$ is ample. □

See also Lemma 5.8.1 for a similar property of algebraic spaces and Deligne–Mumford stacks.

Remark E.2.17. The semiampleness condition can be very challenging to verify in practice. However, there are powerful base point free theorems in birational geometry stemming either from vanishing theorems or analytic methods that can reduce semiampleness to bigness and nefness. For instance, Kawamata's base point free theorem states that if (X, Δ) is a proper klt pair with Δ effective and D is a nef Cartier divisor such that $aD - K_X - \Delta$ is nef and big for some $a > 0$, then D is semiample [KM98, Thm. 3.3]. One can contrast this result with the Abundance Conjecture that states that if (X, Δ) is a proper log canonical pair with Δ effective, then the nefness of $K_X + \Delta$ implies semiampleness [KM98, Conj. 3.12].

Alternatively, it is a classical result of Zariski and Wilson that if X is a normal projective variety and D is a nef and big divisor, then D is semiample if and only if its graded section ring $\bigoplus_n \Gamma(X, \mathcal{O}_X(nD))$ is finitely generated; see [Laz04a, Thm. 2.3.15]. While [BCHM10] can sometimes be applied to verify the finite generation, this result already presumes the projectivity of X; nevertheless, this can be applied for instance to show that a given birational model of X is projective.

In positive characteristic, Keel's theorem provides another technique: on a projective variety X, a nef line bundle L is semiample if and only if the restriction of L to the *exceptional locus* E is semiample, where the exceptional locus E is defined as the union of irreducible subvarieties the $Z \subset X$ satisfying $L^{\dim Z} \cdot Z = 0$ [Kee99].

E.2.6 Numerical criteria for ampleness

The Nakai–Moishezon Criterion[1] for ampleness provides a convenient method to establish projectivity. We state the criteria for proper schemes but this is extended to proper algebraic spaces in Theorem 5.8.4.

[1] This is also known as the Nakai Criterion or the Nakai–Moishezon–Kleiman Criterion. See [Laz04a, §1.2.B] for a historical account and further references.

Theorem E.2.18 (Nakai–Moishezon Criterion)**.** *If X is a proper scheme, a line bundle L is ample if and only if for all irreducible closed subvarieties $Z \subset X$, $c_1(L)^{\dim Z} \cdot Z > 0$*

Remark E.2.19. Using Corollary E.2.14, the Nakai–Moishezon Criterion translates to:

$$L \text{ is ample } \iff L|_Z \text{ is big for all irreducible closed subvarieties } Z \subset X.$$

Proof. Let $n = \dim X$. First, if L is very ample, then for some $m > 0$, $L^{\otimes m}$ is very ample and $m^n c_1(L)^{\dim Z} \cdot Z = c_1(L^{\otimes m})^{\dim Z} \cdot Z > 0$ as its the degree of Z under the projective embedding defined by $L^{\otimes m}$. To show the converse, we follow the proofs of [Laz04a, Thm. 1.2.23], [Kol96, Thm VI.2.18], and [Har77, Thm. V.1.10] (surface case). Since we already know that L is nef, it suffices to show that L is semiample (Lemma E.2.16).

First, by Proposition E.2.2, we may assume that X is a normal variety and we write $L = \mathcal{O}_X(D)$ for a divisor D. Since D is big on X, some positive multiple mD is effective; replacing D by mD, there exists a non-zero section $s \in \mathrm{H}^0(X, \mathcal{O}_X(D))$. In particular, $\mathcal{O}_X(D)$ is base point free away from the support of D. We aim to show that for $m \gg 0$, $\mathcal{O}_X(mD)$ is also base point free on D.

By induction on $n = \dim X$, we can assume that $\mathcal{O}_X(D)|_D$ is ample; the base case for the induction is $n = 1$, where a line bundle is ample if and only if it has positive degree. Consider the exact sequence

$$0 \to \mathcal{O}_X((m-1)D) \to \mathcal{O}_X(mD) \to \mathcal{O}_D(mD) \to 0.$$

For $m \gg 0$, $\mathcal{O}_D(mD)$ is base point free and $\mathrm{H}^1(X, \mathcal{O}_D(mD) = 0$. It follows that $\mathrm{H}^1(X, \mathcal{O}_X((m-1)D)) \twoheadrightarrow \mathrm{H}^1(X, \mathcal{O}_X(mD))$ is surjective but since each vector space is finitely generated, we see that these surjections eventually become isomorphisms for $m \gg 0$. Thus, for $m \gg 0$, $\mathrm{H}^0(X, \mathcal{O}_X(mD)) \to \mathrm{H}^0(D, \mathcal{O}_D(mD))$ is surjective and $\mathcal{O}_X(mD)$ is base point free on D. □

We use this criterion to establish Kollar's Ampleness Criteria (Theorem 5.8.5), which we in turn apply to establish the projectivity of $\overline{M}_g$. The following two additional numerical criteria for ampleness will not be used in these notes but are included to offer a more complete treatment.

Theorem E.2.20 (Kleiman's Criterion)**.** *If X is a projective scheme, a divisor D is ample if and only if for all $C \in \overline{\mathrm{NE}(X)}$, $D \cdot C > 0$.*

Remark E.2.21. See [?], [Kol96, Thm. VI.2.19], and [Laz04a, Thm. 1.4.23]. Note that it is not enough to check that $D \cdot C$ for only irreducible curves $C \subset X$; one must check it on curve classes in the closure $\overline{\mathrm{NE}(X)}$ of the effective cone of curves. See [Har66a, p.50-56] for a counterexample due to Mumford.

Kleiman's Criterion also holds for $\mathbb{Q}$-factorial (e.g. smooth) proper schemes but is not known in general for proper schemes or algebraic spaces.

Theorem E.2.22 (Sesahdri's criterion)**.** *If X is a proper scheme, a line bundle L is ample if and only if there exists an $\epsilon > 0$ such that for every point $x \in X$ and every irreducible curve $C \subset X$, $c_1(L) \cdot C > \epsilon \operatorname{mult}_x(C)$, where $\operatorname{mult}_x(C)$ denotes the multiplicity of C at x.*

Remark E.2.23. See [Laz04a, Thm. 1.4.13] or [Kol96, Thm. 2.18] for a proof. This criterion also holds for proper algebraic spaces; see [Cor93].

E.2.7 Nef vector bundles

In Kollár's Criterion (Theorem 5.8.5), nefness of vector bundles plays an essential role:

Definition E.2.24. A vector bundle E on a scheme X is called *nef* (or *semipositive*) if for every map $f\colon C \to X$ from a proper curve, every quotient line bundle of $f^*E \twoheadrightarrow L$ has nonnegative degree.

We note that when E is a line bundle, then this is clearly equivalent to the usual notion of nefness: for all proper curves $C \subset X$, $\deg L|_C \geq 0$.

Proposition E.2.25. *Let E be a vector bundle on a proper scheme X. Then the following are equivalent:*

E is nef	$\iff$	*for every map $f\colon C \to X$ from a proper curve, every quotient vector bundle of $f^*E \twoheadrightarrow W$ has nonnegative degree;*
	$\iff$	*$\mathcal{O}_{\mathbb{P}(E)}(1)$ is nef on $\mathbb{P}(E) \to X$.*

Remark E.2.26. See [Har66a] or [Laz04b, Ch. 6] for details. There is a similar notion of an ample vector bundle (which we won't need in these notes) where one defines a vector bundle E to be ample if $\mathcal{O}_{\mathbb{P}(E)}(1)$ is ample on $\mathbb{P}(E)$. This notion also has some nice equivalences. If X is an irreducible projective variety and E is base point free, then E is ample if and only if for every map $f\colon C \to X$ from a proper curve, every quotient line bundle of $f^*E \twoheadrightarrow L$ is non-trivial. There are also cohomological characterizations of ampleness for vector bundles in the same spirit as their line bundle counterparts. Moreover, nefness of E can then be characterized as requiring that for every map $f\colon C \to X$ from a proper curve and for every ample line bundle H on C, the vector bundle $H \otimes f^*E$ is ample.

Proposition E.2.27.

(1) Quotients and extensions of nef vector bundles are nef.

(2) If E is nef, then so is $\bigwedge^k E$ and $\operatorname{Sym}^k E$ for $k \geq 0$.

Proof. Part (1) follows from the definition of nefness. Part (2): to be added. □

As a consequence of Proposition E.2.4 and the first equivalence of Proposition E.2.25, we obtain that nefness is open in a proper flat family over a DVR:

Proposition E.2.28 (Openness of Nefness)**.** *Let X be a proper and flat scheme over a DVR R and E be a vector bundle on X. Let $0, \eta \in \operatorname{Spec} R$ be the closed and generic points. If $E|_{X_0}$ is nef, then so is $E|_{X_\eta}$.*

E.3 Vanishing theorems

Kollár's argument for the projectivity of $\overline{M}_g$ makes use of the following vanishing theorem in positive characteristic due to Ekedahl [Eke88]. The characteristic zero version is due to Bombieri [Bom73].

Theorem E.3.1 (Bombieri–Ekedahl vanishing)**.** *Let S be a smooth projective surface over $\Bbbk$ which is minimal and of general type. If $\operatorname{char}(\Bbbk) \neq 2$, then $\mathrm{H}^1(S, K_S^{\otimes -n}) = 0$ for all $n \geq 1$. If $\operatorname{char}(\Bbbk) = 2$, then $\mathrm{h}^1(S, K_S^{\otimes -n}) \leq 1$ for all $n \geq 2$.*

Bibliography

[AB83] M. F. Atiyah and R. Bott, *The Yang-Mills equations over Riemann surfaces*, Philos. Trans. Roy. Soc. London Ser. A **308** (1983), no. 1505, 523–615.

[ACG11] Enrico Arbarello, Maurizio Cornalba, and Phillip A. Griffiths, *Geometry of algebraic curves. Volume II*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 268, Springer, Heidelberg, 2011, With a contribution by Joseph Daniel Harris.

[ACV03] Dan Abramovich, Alessio Corti, and Angelo Vistoli, *Twisted bundles and admissible covers*, vol. 31, 2003, Special issue in honor of Steven L. Kleiman, pp. 3547–3618.

[AGV08] Dan Abramovich, Tom Graber, and Angelo Vistoli, *Gromov-Witten theory of Deligne-Mumford stacks*, Amer. J. Math. **130** (2008), no. 5, 1337–1398.

[AHHL21] Jarod Alper, Jochen Heinloth, and Daniel Halpern-Leistner, *Cartan-Iwahori-Matsumoto decompositions for reductive groups*, Pure Appl. Math. Q. **17** (2021), no. 2, 593–604.

[AHLH18] Jarod Alper, Daniel Halpern-Leistner, and Jochen Heinloth, *Existence of moduli spaces for algebraic stacks*, 2018.

[AHLHR22] Jarod Alper, Daniel Halpern-Leistner, Jack Hall, and David Rydh, *Artin algebraization for pairs with applications to the local structure of stacks and ferrand pushouts*, 2022.

[AHR19] Jarod Alper, Jack Hall, and David Rydh, *The étale local structure of algebraic stacks*, 2019.

[AHR20] ———, *A Luna étale slice theorem for algebraic stacks*, Ann. of Math. (2) **191** (2020), no. 3, 675–738.

[AJP16] Norbert A'Campo, Lizhen Ji, and Athanase Papadopoulos, *On the early history of moduli and teichmüller spaces*, 2016, arXiv:1602.07208.

[AK80] Allen B. Altman and Steven L. Kleiman, *Compactifying the Picard scheme*, Adv. in Math. **35** (1980), no. 1, 50–112.

[AK10] Valery Alexeev and Allen Knutson, *Complete moduli spaces of branch-varieties*, J. Reine Angew. Math. **639** (2010), 39–71.

[Alp13] Jarod Alper, *Good moduli spaces for Artin stacks*, Ann. Inst. Fourier (Grenoble) **63** (2013), no. 6, 2349–2402.

[Alp14] J. Alper, *Adequate moduli spaces and geometrically reductive group schemes*, Algebr. Geom. **1** (2014), no. 4, 489–531.

[AM69] M. F. Atiyah and I. G. Macdonald, *Introduction to commutative algebra*, Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1969.

[And67] Michel André, *Méthode simpliciale en algèbre homologique et algèbre commutative*, Lecture Notes in Mathematics, Vol. 32, Springer-Verlag, Berlin-New York, 1967.

[AOV08] D. Abramovich, M. Olsson, and A. Vistoli, *Tame stacks in positive characteristic*, Ann. Inst. Fourier (Grenoble) **58** (2008), no. 4, 1057–1091.

[Art62] M. Artin, *Grothendieck topolgoies*, Notes from a seminar of M. Artin, Harvard U., 1962.

[Art69a] ———, *Algebraic approximation of structures over complete local rings*, Inst. Hautes Études Sci. Publ. Math. (1969), no. 36, 23–58.

[Art69b] ———, *Algebraization of formal moduli. I*, Global Analysis (Papers in Honor of K. Kodaira), Univ. Tokyo Press, Tokyo, 1969, pp. 21–71.

[Art74] ———, *Versal deformations and algebraic stacks*, Invent. Math. **27** (1974), 165–189.

[AW71] Michael Artin and Gayn Winters, *Degenerate fibres and stable reduction of curves*, Topology **10** (1971), 373–383.

[BB63] A. Białynicki-Birula, *On homogeneous affine spaces of linear algebraic groups*, Amer. J. Math. **85** (1963), 577–582.

[BB73] ———, *Some theorems on actions of algebraic groups*, Ann. of Math. (2) **98** (1973), 480–497.

[BCHM10] Caucher Birkar, Paolo Cascini, Christopher D. Hacon, and James McKernan, *Existence of minimal models for varieties of log general type*, J. Amer. Math. Soc. **23** (2010), no. 2, 405–468.

[Beh04] K. Behrend, *Cohomology of stacks*, Intersection theory and moduli, ICTP Lect. Notes, XIX, Abdus Salam Int. Cent. Theoret. Phys., Trieste, 2004, pp. 249–294.

[Beh14] ———, *Introduction to algebraic stacks*, Moduli spaces, London Math. Soc. Lecture Note Ser., vol. 411, Cambridge Univ. Press, Cambridge, 2014, pp. 1–131.

[BH93] Winfried Bruns and Jürgen Herzog, *Cohen-Macaulay rings*, Cambridge Studies in Advanced Mathematics, vol. 39, Cambridge University Press, Cambridge, 1993.

[BHL17] Bhargav Bhatt and Daniel Halpern-Leistner, *Tannaka duality revisited*, Adv. Math. **316** (2017), 576–612.

[Bir09] George D. Birkhoff, *Singular points of ordinary linear differential equations*, Trans. Amer. Math. Soc. **10** (1909), no. 4, 436–470.

[Bir76] A. Biał ynicki Birula, *Some properties of the decompositions of algebraic varieties determined by actions of a torus*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **24** (1976), no. 9, 667–674.

[BLR90] Siegfried Bosch, Werner Lütkebohmert, and Michel Raynaud, *Néron models*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 21, Springer-Verlag, Berlin, 1990.

[Bom73] E. Bombieri, *Canonical models of surfaces of general type*, Inst. Hautes Études Sci. Publ. Math. (1973), no. 42, 171–219.

[Bor91] Armand Borel, *Linear algebraic groups*, second ed., Graduate Texts in Mathematics, vol. 126, Springer-Verlag, New York, 1991.

[Bri97] M. Brion, *Equivariant Chow groups for torus actions*, Transform. Groups **2** (1997), no. 3, 225–267.

[BS03] Holger Brenner and Stefan Schröer, *Ample families, multihomogeneous spectra, and algebraization of formal schemes*, Pacific J. Math. **208** (2003), no. 2, 209–230.

[BT72] F. Bruhat and J. Tits, *Groupes réductifs sur un corps local*, Inst. Hautes Études Sci. Publ. Math. (1972), no. 41, 5–251.

[Bum13] Daniel Bump, *Lie groups*, second ed., Graduate Texts in Mathematics, vol. 225, Springer, New York, 2013.

[CH88] Maurizio Cornalba and Joe Harris, *Divisor classes associated to families of stable varieties, with applications to the moduli space of curves*, Ann. Sci. École Norm. Sup. (4) **21** (1988), no. 3, 455–475.

[Che60] C. Chevalley, *Une démonstration d'un théorème sur les groupes algébriques*, J. Math. Pures Appl. (9) **39** (1960), 307–317.

[CJ02] B. Conrad and A. J. de Jong, *Approximation of versal deformations*, J. Algebra **255** (2002), no. 2, 489–515.

[Cle73] A. Clebsch, *Zur Theorie der Riemann'schen Fläche*, Math. Ann. **6** (1873), no. 2, 216–230.

[CLS11] David A. Cox, John B. Little, and Henry K. Schenck, *Toric varieties*, Graduate Studies in Mathematics, vol. 124, American Mathematical Society, Providence, RI, 2011.

[Con02] Brian Conrad, *A modern proof of Chevalley's theorem on algebraic groups*, J. Ramanujan Math. Soc. **17** (2002), no. 1, 1–18.

[Con05a] ———, *Formal GAGA for Artin stacks*, https://math.stanford.edu/~conrad/papers/formalgaga.pdf, 2005.

[Con05b] Brian Conrad, *Keel-mori theorem via stacks*, 2005.

[Con07] Brian Conrad, *Arithmetic moduli of generalized elliptic curves*, J. Inst. Math. Jussieu **6** (2007), no. 2, 209–278.

[Cor93] M. D. T. Cornalba, *On the projectivity of the moduli spaces of curves*, J. Reine Angew. Math. **443** (1993), 11–20.

[Cox95] David A. Cox, *The homogeneous coordinate ring of a toric variety*, J. Algebraic Geom. **4** (1995), no. 1, 17–50.

[CP21] Giulio Codogni and Zsolt Patakfalvi, *Positivity of the CM line bundle for families of K-stable klt Fano varieties*, Invent. Math. **223** (2021), no. 3, 811–894.

[CS79] James B. Carrell and Andrew John Sommese, $\mathbf{C}^*$*-actions*, Math. Scand. **43** (1978/79), no. 1, 49–59.

[DC71] Jean A. Dieudonné and James B. Carrell, *Invariant theory, old and new*, Academic Press, New York-London, 1971.

[Del81] P. Deligne, *Relèvement des surfaces $K3$ en caractéristique nulle*, Algebraic surfaces (Orsay, 1976–78), Lecture Notes in Math., vol. 868, Springer, Berlin-New York, 1981, Prepared for publication by Luc Illusie, pp. 58–79.

[Del85] Pierre Deligne, *Le lemme de Gabber*, no. 127, 1985, Seminar on arithmetic bundles: the Mordell conjecture (Paris, 1983/84), pp. 131–150.

[DH98] Igor V. Dolgachev and Yi Hu, *Variation of geometric invariant theory quotients*, Inst. Hautes Études Sci. Publ. Math. (1998), no. 87, 5–56, With an appendix by Nicolas Ressayre.

[dJ03] Aise Johan de Jong, *A result of gabber*, preprint, `https://www.math.columbia.edu/~dejong/papers/2-gabber.pdf`, 2003.

[DK15] Harm Derksen and Gregor Kemper, *Computational invariant theory*, enlarged ed., Encyclopaedia of Mathematical Sciences, vol. 130, Springer, Heidelberg, 2015, With two appendices by Vladimir L. Popov, and an addendum by Norbert A'Campo and Popov, Invariant Theory and Algebraic Transformation Groups, VIII.

[DM69] P. Deligne and D. Mumford, *The irreducibility of the space of curves of given genus*, Inst. Hautes Études Sci. Publ. Math. (1969), no. 36, 75–109.

[DMOS82] Pierre Deligne, James S. Milne, Arthur Ogus, and Kuang-yen Shih, *Hodge cycles, motives, and Shimura varieties*, Lecture Notes in Mathematics, vol. 900, Springer-Verlag, Berlin-New York, 1982.

[Dol03] Igor Dolgachev, *Lectures on invariant theory*, London Mathematical Society Lecture Note Series, vol. 296, Cambridge University Press, Cambridge, 2003.

[Dri13] V. Drinfeld, *On algebraic spaces with an action of $\mathbb{G}_m$*, 2013, arXiv:1308.2604.

[EG84] David Eisenbud and Shiro Goto, *Linear free resolutions and minimal multiplicity*, J. Algebra **88** (1984), no. 1, 89–133.

[EG98] Dan Edidin and William Graham, *Equivariant intersection theory*, Invent. Math. **131** (1998), no. 3, 595–634.

[EGA] A. Grothendieck, *Éléments de géométrie algébrique*, I.H.E.S. Publ. Math. **4, 8, 11, 17, 20, 24, 28, 32** (1960, 1961, 1961, 1963, 1964, 1965, 1966, 1967).

[EHKV01] Dan Edidin, Brendan Hassett, Andrew Kresch, and Angelo Vistoli, *Brauer groups and quotient stacks*, Amer. J. Math. **123** (2001), no. 4, 761–777.

[EHM92] Philippe Ellia, André Hirschowitz, and Emilia Mezzetti, *On the number of irreducible components of the Hilbert scheme of smooth space curves*, Internat. J. Math. **3** (1992), no. 6, 799–807.

[Eis95] D. Eisenbud, *Commutative algebra*, Graduate Texts in Mathematics, vol. 150, Springer-Verlag, New York, 1995, With a view toward algebraic geometry.

[Eke88] Torsten Ekedahl, *Canonical models of surfaces of general type in positive characteristic*, Inst. Hautes Études Sci. Publ. Math. (1988), no. 67, 97–144.

[ER21] Dan Edidin and David Rydh, *Canonical reduction of stabilizers for Artin stacks with good moduli spaces*, Duke Math. J. **170** (2021), no. 5, 827–880.

[ES14] Torsten Ekedahl and Roy Skjelnes, *Recovering the good component of the Hilbert scheme*, Ann. of Math. (2) **179** (2014), no. 3, 805–841.

[ESm87] Geir Ellingsrud and Stein Arild Strø mme, *On the homology of the Hilbert scheme of points in the plane*, Invent. Math. **87** (1987), no. 2, 343–352.

[Fal03] Gerd Faltings, *Finiteness of coherent cohomology for proper fppf stacks*, J. Algebraic Geom. **12** (2003), no. 2, 357–366.

[Fer03] D. Ferrand, *Conducteur, descente et pincement*, Bull. Soc. Math. France **131** (2003), no. 4, 553–585.

[FGA$_{IV}$] Alexander Grothendieck, *Techniques de construction et théorèmes d'existence en géométrie algébrique. IV. Les schémas de Hilbert*, Séminaire Bourbaki, Vol. 6, Soc. Math. France, Paris, 1995, pp. Exp. No. 221, 249–276.

[FGI$^+$05] Barbara Fantechi, Lothar Göttsche, Luc Illusie, Steven L. Kleiman, Nitin Nitsure, and Angelo Vistoli, *Fundamental algebraic geometry*, Mathematical Surveys and Monographs, vol. 123, American Mathematical Society, Providence, RI, 2005, Grothendieck's FGA explained.

[Fog68] John Fogarty, *Algebraic families on an algebraic surface*, Amer. J. Math. **90** (1968), 511–521.

[FS82] Walter Ricardo Ferrer Santos, *A note on reductive groups*, An. Acad. Brasil. Ciênc. **54** (1982), no. 3, 469–471.

[Ful69] William Fulton, *Hurwitz schemes and irreducibility of moduli of algebraic curves*, Ann. of Math. (2) **90** (1969), 542–575.

[Ful93] ———, *Introduction to toric varieties*, Annals of Mathematics Studies, vol. 131, Princeton University Press, Princeton, NJ, 1993, The William H. Roever Lectures in Geometry.

[Ful98] ———, *Intersection theory*, second ed., Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 2, Springer-Verlag, Berlin, 1998.

[Ful10] Damiano Fulghesu, *The stack of rational curves*, Comm. Algebra **38** (2010), no. 7, 2405–2417.

[Gab62] Pierre Gabriel, *Des catégories abéliennes*, Bull. Soc. Math. France **90** (1962), 323–448.

[GH78] Phillip Griffiths and Joseph Harris, *Principles of algebraic geometry*, Wiley-Interscience [John Wiley & Sons], New York, 1978, Pure and Applied Mathematics.

[Gie82] D. Gieseker, *Lectures on moduli of curves*, Tata Institute of Fundamental Research Lectures on Mathematics and Physics, vol. 69, Published for the Tata Institute of Fundamental Research, Bombay; Springer-Verlag, Berlin-New York, 1982.

[Gir71] Jean Giraud, *Cohomologie non abélienne*, Springer-Verlag, Berlin-New York, 1971, Die Grundlehren der mathematischen Wissenschaften, Band 179.

[GIT] David Mumford, *Geometric invariant theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Neue Folge, Band 34, Springer-Verlag, Berlin-New York, 1965.

[GLP83] L. Gruson, R. Lazarsfeld, and C. Peskine, *On a theorem of Castelnuovo, and the equations defining space curves*, Invent. Math. **72** (1983), no. 3, 491–506.

[God58] Roger Godement, *Topologie algébrique et théorie des faisceaux*, Actualités Scientifiques et Industrielles [Current Scientific and Industrial Topics], No. 1252, Hermann, Paris, 1958, Publ. Math. Univ. Strasbourg. No. 13.

[Got78] Gerd Gotzmann, *Eine Bedingung für die Flachheit und das Hilbertpolynom eines graduierten Ringes*, Math. Z. **158** (1978), no. 1, 61–70.

[Gre89] Mark Green, *Restrictions of linear series to hyperplanes, and some results of Macaulay and Gotzmann*, Algebraic curves and projective geometry (Trento, 1988), Lecture Notes in Math., vol. 1389, Springer, Berlin, 1989, pp. 76–86.

[Gre98] Mark L. Green, *Generic initial ideals*, Six lectures on commutative algebra (Bellaterra, 1996), Progr. Math., vol. 166, Birkhäuser, Basel, 1998, pp. 119–186.

[Gro57a] A. Grothendieck, *Sur la classification des fibrés holomorphes sur la sphère de Riemann*, Amer. J. Math. **79** (1957), 121–138.

[Gro57b] Alexander Grothendieck, *Sur quelques points d'algèbre homologique*, Tohoku Math. J. (2) **9** (1957), 119–221.

[Gro61] ______, *Techniques de construction en géométrie analytique. x. construction de l'espace de teichmüller*, Séminaire Henri Cartan **13** (1960-1961), no. 2 (fr), talk:17.

[Gro68] ______, *Le groupe de Brauer I, II, III*, Dix exposés sur la cohomologie des schémas, Adv. Stud. Pure Math., vol. 3, North-Holland, Amsterdam, 1968, pp. 46–188.

[Gro17] Philipp Gross, *Tensor generators on schemes and stacks*, Algebr. Geom. **4** (2017), no. 4, 501–522.

[Göt90] Lothar Göttsche, *The Betti numbers of the Hilbert scheme of points on a smooth projective surface*, Math. Ann. **286** (1990), no. 1-3, 193–207.

[Göt94] ______, *Hilbert schemes of zero-dimensional subschemes of smooth varieties*, Lecture Notes in Mathematics, vol. 1572, Springer-Verlag, Berlin, 1994.

[Hab75] W. J. Haboush, *Reductive groups are geometrically reductive*, Ann. of Math. (2) **102** (1975), no. 1, 67–83.

[Hai98] Mark Haiman, *t, q-Catalan numbers and the Hilbert scheme*, vol. 193, 1998, Selected papers in honor of Adriano Garsia (Taormina, 1994), pp. 201–224.

[Hal13] J. Hall, *Moduli of singular curves,* appendix to *Towards a classification of modular compactifications of $M_{g,n}$ by D. Smyth*, Invent. Math. **192** (2013), no. 2, 459–503.

[Hal14] ______, *Cohomology and base change for algebraic stacks*, Math. Z. **278** (2014), no. 1-2, 401–429.

[Hal17] Jack Hall, *Openness of versality via coherent functors*, J. Reine Angew. Math. **722** (2017), 137–182.

[Har66a] Robin Hartshorne, *Ample vector bundles*, Inst. Hautes Études Sci. Publ. Math. (1966), no. 29, 63–94.

[Har66b] ______, *Connectedness of the Hilbert scheme*, Inst. Hautes Études Sci. Publ. Math. (1966), no. 29, 5–48.

[Har66c] ______, *Residues and duality*, Lecture notes of a seminar on the work of A. Grothendieck, given at Harvard 1963/64. With an appendix by P. Deligne. Lecture Notes in Mathematics, No. 20, Springer-Verlag, Berlin-New York, 1966.

[Har77] ———, *Algebraic geometry*, Springer-Verlag, New York-Heidelberg, 1977, Graduate Texts in Mathematics, No. 52.

[Har10] ———, *Deformation theory*, Graduate Texts in Mathematics, vol. 257, Springer, New York, 2010.

[Has03] Brendan Hassett, *Moduli spaces of weighted pointed stable curves*, Adv. Math. **173** (2003), no. 2, 316–352.

[Hes78] Wim H. Hesselink, *Uniform instability in reductive groups*, J. Reine Angew. Math. **303(304)** (1978), 74–96.

[Hes81] ———, *Concentration under actions of algebraic groups*, Paul Dubreil and Marie-Paule Malliavin Algebra Seminar, 33rd Year (Paris, 1980), Lecture Notes in Math., vol. 867, Springer, Berlin, 1981, pp. 55–89.

[Hil90] David Hilbert, *Ueber die Theorie der algebraischen Formen*, Math. Ann. **36** (1890), no. 4, 473–534.

[Hil93] ———, *Ueber die vollen Invariantensysteme*, Math. Ann. **42** (1893), no. 3, 313–373.

[Hil93] ———, *Theory of algebraic invariants*, Cambridge University Press, Cambridge, 1993, Translated from the German and with a preface by Reinhard C. Laubenbacher, Edited and with an introduction by Bernd Sturmfels.

[HK00] Yi Hu and Sean Keel, *Mori dream spaces and GIT*, vol. 48, 2000, Dedicated to William Fulton on the occasion of his 60th birthday, pp. 331–348.

[HL10] D. Huybrechts and M. Lehn, *The geometry of moduli spaces of sheaves*, second ed., Cambridge Mathematical Library, Cambridge University Press, Cambridge, 2010.

[HL14] Daniel Halpern-Leistner, *On the structure of instability in moduli theory*, 2014.

[HM82] Joe Harris and David Mumford, *On the Kodaira dimension of the moduli space of curves*, Invent. Math. **67** (1982), no. 1, 23–88, With an appendix by William Fulton.

[HM98] Joe Harris and Ian Morrison, *Moduli of curves*, Graduate Texts in Mathematics, vol. 187, Springer-Verlag, New York, 1998.

[Hoc65] G. Hochschild, *The structure of Lie groups*, Holden-Day, Inc., San Francisco-London-Amsterdam, 1965.

[HR14] Jack Hall and David Rydh, *The Hilbert stack*, Adv. Math. **253** (2014), 194–233.

[HR15] ———, *Algebraic groups and compact generation of their derived categories of representations*, Indiana Univ. Math. J. **64** (2015), no. 6, 1903–1923.

[HR19a] ———, *Artin's criteria for algebraicity revisited*, Algebra Number Theory **13** (2019), no. 4, 749–796.

[HR19b] ———, *Coherent Tannaka duality and algebraicity of Hom-stacks*, Algebra Number Theory **13** (2019), no. 7, 1633–1675.

[Hum75] James E. Humphreys, *Linear algebraic groups*, Graduate Texts in Mathematics, No. 21, Springer-Verlag, New York-Heidelberg, 1975.

[Hum78] ———, *Introduction to Lie algebras and representation theory*, Graduate Texts in Mathematics, vol. 9, Springer-Verlag, New York-Berlin, 1978, Second printing, revised.

[Hur91] A. Hurwitz, *Ueber Riemann'sche Flächen mit gegebenen Verzweigungspunkten*, Math. Ann. **39** (1891), no. 1, 1–60.

[Ill71] L. Illusie, *Complexe cotangent et déformations. I*, Lecture Notes in Mathematics, Vol. 239, Springer-Verlag, Berlin, 1971.

[IM65] N. Iwahori and H. Matsumoto, *On some Bruhat decomposition and the structure of the Hecke rings of p-adic Chevalley groups*, Inst. Hautes Études Sci. Publ. Math. (1965), no. 25, 5–48.

[Ive72] Birger Iversen, *A fixed point formula for action of tori on algebraic varieties*, Invent. Math. **16** (1972), 229–236.

[JP13] Lizhen Ji and Athanase Papadopoulos, *Historical development of Teichmüller theory*, Arch. Hist. Exact Sci. **67** (2013), no. 2, 119–147.

[JS21] Joachim Jelisiejew and Łukasz Sienkiewicz, *Białynicki-Birula decomposition for reductive groups in positive characteristic*, J. Math. Pures Appl. (9) **152** (2021), 189–210.

[Kee99] Seán Keel, *Basepoint freeness for nef and big line bundles in positive characteristic*, Ann. of Math. (2) **149** (1999), no. 1, 253–286.

[Kem78] George R. Kempf, *Instability in invariant theory*, Ann. of Math. (2) **108** (1978), no. 2, 299–316.

[Kin94] A. D. King, *Moduli of representations of finite-dimensional algebras*, Quart. J. Math. Oxford Ser. (2) **45** (1994), no. 180, 515–530.

[Kir84] Frances Clare Kirwan, *Cohomology of quotients in symplectic and algebraic geometry*, Mathematical Notes, vol. 31, Princeton University Press, Princeton, NJ, 1984.

[Kir85] ———, *Partial desingularisations of quotients of nonsingular varieties and their Betti numbers*, Ann. of Math. (2) **122** (1985), no. 1, 41–85.

[KKLV89] Friedrich Knop, Hanspeter Kraft, Domingo Luna, and Thierry Vust, *Local properties of algebraic group actions*, Algebraische Transformationsgruppen und Invariantentheorie, DMV Sem., vol. 13, Birkhäuser, Basel, 1989, pp. 63–75.

[KKV89] Friedrich Knop, Hanspeter Kraft, and Thierry Vust, *The Picard group of a G-variety*, Algebraische Transformationsgruppen und Invariantentheorie, DMV Sem., vol. 13, Birkhäuser, Basel, 1989, pp. 77–87.

[Kle66] Steven L. Kleiman, *Toward a numerical theory of ampleness*, Ann. of Math. (2) **84** (1966), 293–344.

[KM97] S. Keel and S. Mori, *Quotients by groupoids*, Ann. of Math. (2) **145** (1997), no. 1, 193–213.

[KM98] János Kollár and Shigefumi Mori, *Birational geometry of algebraic varieties*, Cambridge Tracts in Mathematics, vol. 134, Cambridge University Press, Cambridge, 1998, With the collaboration of C. H. Clemens and A. Corti, Translated from the 1998 Japanese original.

[Knu71] Donald Knutson, *Algebraic spaces*, Lecture Notes in Mathematics, Vol. 203, Springer-Verlag, Berlin-New York, 1971.

[Kol90] János Kollár, *Projectivity of complete moduli*, J. Differential Geom. **32** (1990), no. 1, 235–268.

[Kol96] ———, *Rational curves on algebraic varieties*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 32, Springer-Verlag, Berlin, 1996.

[Kol07] ———, *Lectures on resolution of singularities*, Annals of Mathematics Studies, vol. 166, Princeton University Press, Princeton, NJ, 2007.

[Kol18] ———, *Mumford's influence on the moduli theory of algebraic varieties*, 2018, arXiv:1809.10723.

[Kon82] Jerzy Konarski, *A pathological example of an action of k^**, Group actions and vector fields (Vancouver, B.C., 1981), Lecture Notes in Math., vol. 956, Springer, Berlin, 1982, pp. 72–78.

[KP17] Sándor J. Kovács and Zsolt Patakfalvi, *Projectivity of the moduli space of stable log-varieties and subadditivity of log-Kodaira dimension*, J. Amer. Math. Soc. **30** (2017), no. 4, 959–1021.

[Kra84] Hanspeter Kraft, *Geometrische Methoden in der Invariantentheorie*, Aspects of Mathematics, D1, Friedr. Vieweg & Sohn, Braunschweig, 1984.

[Kre99] Andrew Kresch, *Cycle groups for Artin stacks*, Invent. Math. **138** (1999), no. 3, 495–536.

[Kre13] ———, *Flattening stratification and the stack of partial stabilizations of prestable curves*, Bull. Lond. Math. Soc. **45** (2013), no. 1, 93–102.

[Lam99] T. Y. Lam, *Lectures on modules and rings*, Graduate Texts in Mathematics, vol. 189, Springer-Verlag, New York, 1999.

[Laz69] Daniel Lazard, *Autour de la platitude*, Bull. Soc. Math. France **97** (1969), 81–128.

[Laz04a] Robert Lazarsfeld, *Positivity in algebraic geometry. I*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 48, Springer-Verlag, Berlin, 2004, Classical setting: line bundles and linear series.

[Laz04b] ———, *Positivity in algebraic geometry. II*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 49, Springer-Verlag, Berlin, 2004, Positivity for vector bundles, and multiplier ideals.

[Liu02] Qing Liu, *Algebraic geometry and arithmetic curves*, Oxford Graduate Texts in Mathematics, vol. 6, Oxford University Press, Oxford, 2002, Translated from the French by Reinie Erné, Oxford Science Publications.

[LMB] G. Laumon and L. Moret-Bailly, *Champs algébriques*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge., vol. 39, Springer-Verlag, Berlin, 2000.

[LS67] S. Lichtenbaum and M. Schlessinger, *The cotangent complex of a morphism*, Trans. Amer. Math. Soc. **128** (1967), 41–70.

[Lun73] Domingo Luna, *Slices étales*, Sur les groupes algébriques, 1973, pp. 81–105. Bull. Soc. Math. France, Paris, Mémoire 33.

[Lur04] Jacob Lurie, *Coherent tannaka duality for geometric stacks*, 2004.

[LXZ21] Yuchen Liu, Chenyang Xu, and Ziquan Zhuang, *Finite generation for valuations computing stability thresholds and applications to k-stability*, 2021, arXiv:2102.09405.

[Mac71] Saunders MacLane, *Categories for the working mathematician*, Graduate Texts in Mathematics, Vol. 5, Springer-Verlag, New York-Berlin, 1971.

[Mac07] Diane Maclagan, *Notes on hilbert schemes*, 2007.

[Man39] W. Mangler, *Die Klassen von topologischen Abbildungen einer geschlossenen Fläche auf sich*, Math. Z. **44** (1939), no. 1, 541–554.

[Mat60] Yozô Matsushima, *Espaces homogènes de Stein des groupes de Lie complexes*, Nagoya Math. J. **16** (1960), 205–218.

[Mat89] Hideyuki Matsumura, *Commutative ring theory*, second ed., Cambridge Studies in Advanced Mathematics, vol. 8, Cambridge University Press, Cambridge, 1989, Translated from the Japanese by M. Reid.

[Mes00] Bachuki Mesablishvili, *Pure morphisms of commutative rings are effective descent morphisms for modules—a new proof*, Theory Appl. Categ. **7** (2000), No. 3, 38–42.

[MFK94] D. Mumford, J. Fogarty, and F. Kirwan, *Geometric invariant theory*, third ed., Ergebnisse der Mathematik und ihrer Grenzgebiete (2) [Results in Mathematics and Related Areas (2)], vol. 34, Springer-Verlag, Berlin, 1994.

[Mil80] James S. Milne, *Étale cohomology*, Princeton Mathematical Series, vol. 33, Princeton University Press, Princeton, N.J., 1980.

[Mil17] J. S. Milne, *Algebraic groups*, Cambridge Studies in Advanced Mathematics, vol. 170, Cambridge University Press, Cambridge, 2017, The theory of group schemes of finite type over a field.

[Mir95] Rick Miranda, *Algebraic curves and Riemann surfaces*, Graduate Studies in Mathematics, vol. 5, American Mathematical Society, Providence, RI, 1995.

[MM64] Alan Meyer and David Mumford, *Further comments on boundary points*, Lecture notes prepared in connection with the Summer Institute on Algebraic Geometry held at Woods Hole, MA, American Mathematical Society, 1964 (unpublished).

[Muk03] Shigeru Mukai, *An introduction to invariants and moduli*, Cambridge Studies in Advanced Mathematics, vol. 81, Cambridge University Press, Cambridge, 2003, Translated from the 1998 and 2000 Japanese editions by W. M. Oxbury.

[Mum62] David Mumford, *Further pathologies in algebraic geometry*, Amer. J. Math. **84** (1962), 642–648.

[Mum65] ———, *Picard groups of moduli problems*, Arithmetical Algebraic Geometry (Proc. Conf. Purdue Univ., 1963), Harper & Row, New York, 1965, pp. 33–81.

[Mum66] ———, *Lectures on curves on an algebraic surface*, Annals of Mathematics Studies, No. 59, Princeton University Press, Princeton, N.J., 1966, With a section by G. M. Bergman.

[Mum69] ———, *Bi-extensions of formal groups*, Algebraic Geometry (Internat. Colloq., Tata Inst. Fund. Res., Bombay, 1968), Oxford Univ. Press, London, 1969, pp. 307–322.

[Mum70] ———, *Abelian varieties*, Tata Institute of Fundamental Research Studies in Mathematics, No. 5, Published for the Tata Institute of Fundamental Research, Bombay; Oxford University Press, London, 1970.

[Mum77] ———, *Stability of projective varieties*, Enseign. Math. (2) **23** (1977), no. 1-2, 39–110.

[Nag62] Masayoshi Nagata, *Local rings*, Interscience Tracts in Pure and Applied Mathematics, No. 13, Interscience Publishers a division of John Wiley & Sons New York-London, 1962.

[Nag62] ———, *Complete reducibility of rational representations of a matrix group*, J. Math. Kyoto Univ. **1** (1961/62), 87–99.

[Nag64] ———, *Invariants of a group in an affine ring*, J. Math. Kyoto Univ. **3** (1963/64), 369–377.

[Nak97] Hiraku Nakajima, *Heisenberg algebra and Hilbert schemes of points on projective surfaces*, Ann. of Math. (2) **145** (1997), no. 2, 379–388.

[Nak99a] ———, *Lectures on Hilbert schemes of points on surfaces*, University Lecture Series, vol. 18, American Mathematical Society, Providence, RI, 1999.

[Nak99b] ———, *Lectures on Hilbert schemes of points on surfaces*, University Lecture Series, vol. 18, American Mathematical Society, Providence, RI, 1999.

[Nér64] André Néron, *Modèles minimaux des variétés abéliennes sur les corps locaux et globaux*, Inst. Hautes Études Sci. Publ.Math. No. **21** (1964), 128.

[Nes84] Linda Ness, *A stratification of the null cone via the moment map*, Amer. J. Math. **106** (1984), no. 6, 1281–1329, With an appendix by David Mumford.

[New78] P. E. Newstead, *Introduction to moduli problems and orbit spaces*, Tata Institute of Fundamental Research Lectures on Mathematics and Physics, vol. 51, Tata Institute of Fundamental Research, Bombay; Narosa Publishing House, New Delhi, 1978.

[NM64] Masayoshi Nagata and Takehiko Miyata, *Note on semi-reductive groups*, J. Math. Kyoto Univ. **3** (1963/64), 379–382.

[Oda64] Tadao Oda, *On Mumford's conjecture concerning reducible rational representations of algebraic linear groups*, J. Math. Kyoto Univ. **3** (1963/64), 275–286.

[Ols05] Martin C. Olsson, *On proper coverings of Artin stacks*, Adv. Math. **198** (2005), no. 1, 93–106.

[Ols06] M. Olsson, *Deformation theory of representable morphisms of algebraic stacks*, Math. Z. **253** (2006), no. 1, 25–62.

[Ols16] Martin Olsson, *Algebraic spaces and stacks*, American Mathematical Society Colloquium Publications, vol. 62, American Mathematical Society, Providence, RI, 2016.

[Oor71] Frans Oort, *Finite group schemes, local moduli for abelian varieties, and lifting problems*, Compositio Math. **23** (1971), 265–296.

[Pop85] Dorin Popescu, *General Néron desingularization*, Nagoya Math. J. **100** (1985), 97–126.

[Pop86] D. Popescu, *General Néron desingularization and approximation*, Nagoya Math. J. **104** (1986), 85–115.

[Pop90] Dorin Popescu, *Letter to the editor: "General Néron desingularization and approximation"*, Nagoya Math. J. **118** (1990), 45–53.

[PS85] Ragni Piene and Michael Schlessinger, *On the Hilbert scheme compactification of the space of twisted cubics*, Amer. J. Math. **107** (1985), no. 4, 761–774.

[Qui68] Daniel Quillen, *Homology of commutative rings*, Unpublished, 1968, pp. 1–81.

[Qui70] ———, *On the (co-) homology of commutative rings*, Applications of Categorical Algebra (Proc. Sympos. Pure Math., Vol. XVII, New York, 1968), Amer. Math. Soc., Providence, R.I., 1970, pp. 65–87.

[Ran89] Ziv Ran, *Deformations of maps*, Algebraic curves and projective geometry (Trento, 1988), Lecture Notes in Math., vol. 1389, Springer, Berlin, 1989, pp. 246–253.

[Ray70] Michel Raynaud, *Faisceaux amples sur les schémas en groupes et les espaces homogènes*, Lecture Notes in Mathematics, Vol. 119, Springer-Verlag, Berlin-New York, 1970.

[Rei89] Zinovy Reichstein, *Stability and equivariant maps*, Invent. Math. **96** (1989), no. 2, 349–383.

[RG71] Michel Raynaud and Laurent Gruson, *Critères de platitude et de ivité. Techniques de "platification" d'un module*, Invent. Math. **13** (1971), 1–89.

[Ric77] R. W. Richardson, *Affine coset spaces of reductive algebraic groups*, Bull. London Math. Soc. **9** (1977), no. 1, 38–41.

[Rie57] B. Riemann, *Theorie der Abel'schen Functionen*, J. Reine Angew. Math. **54** (1857), 115–155.

[Rom05] Matthieu Romagny, *Group actions on stacks and applications*, Michigan Math. J. **53** (2005), no. 1, 209–236.

[Ryd13] David Rydh, *Existence and properties of geometric quotients*, J. Algebraic Geom. **22** (2013), no. 4, 629–669.

[Ryd15] ———, *Noetherian approximation of algebraic spaces and stacks*, J. Algebra **422** (2015), 105–147.

[Ryd20] ———, *A generalization of luna's fundamental lemma for stacks with good moduli spaces*, 2020.

[Sai87] Takeshi Saito, *Vanishing cycles and geometry of curves over a discrete valuation ring*, Amer. J. Math. **109** (1987), no. 6, 1043–1085.

[Sch68] Michael Schlessinger, *Functors of Artin rings*, Trans. Amer. Math. Soc. **130** (1968), 208–222.

[Ser88] Jean-Pierre Serre, *Algebraic groups and class fields*, Graduate Texts in Mathematics, vol. 117, Springer-Verlag, New York, 1988, Translated from the French.

[Ser06] E. Sernesi, *Deformations of algebraic schemes*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 334, Springer-Verlag, Berlin, 2006.

[Ses69] C. S. Seshadri, *Mumford's conjecture for* GL(2) *and applications*, Algebraic Geometry (Internat. Colloq., Tata Inst. Fund. Res., Bombay, 1968), Oxford Univ. Press, London, 1969, pp. 347–371.

[Ses72] ______, *Quotient spaces modulo reductive algebraic groups*, Ann. of Math. (2) **95** (1972), 511–556; errata, ibid. (2) 96 (1972), 599.

[Ses77] ______, *Geometric reductivity over arbitrary base*, Advances in Math. **26** (1977), no. 3, 225–274.

[SGA1] A. Grothendieck and M. Raynaud, *Revêtements étales et groupe fondamental*, Séminaire de Géométrie Algébrique, I.H.E.S., 1963.

[SGA3$_{\mathrm{I}}$] *Schémas en groupes. I: Propriétés générales des schémas en groupes*, Séminaire de Géométrie Algébrique du Bois Marie 1962/64 (SGA 3). Dirigé par M. Demazure et A. Grothendieck. Lecture Notes in Mathematics, Vol. 151, Springer-Verlag, Berlin, 1970.

[SGA3$_{\mathrm{III}}$] *Schémas en groupes. III: Structure des schémas en groupes réductifs*, Séminaire de Géométrie Algébrique du Bois Marie 1962/64 (SGA 3). Dirigé par M. Demazure et A. Grothendieck. Lecture Notes in Mathematics, Vol. 153, Springer-Verlag, Berlin, 1970.

[SGA4] *Théorie des topos et cohomologie étale des schémas.*, Lecture Notes in Mathematics, Vol. 269, 270, 305, Springer-Verlag, Berlin, 1972, Séminaire de Géométrie Algébrique du Bois-Marie 1963–1964 (SGA 4), Dirigé par M. Artin, A. Grothendieck, et J. L. Verdier. Avec la collaboration de N. Bourbaki, P. Deligne et B. Saint-Donat.

[SGA4$\frac{1}{2}$] P. Deligne, *Cohomologie étale*, Lecture Notes in Mathematics, Vol. 569, Springer-Verlag, Berlin, 1977.

[SGA6] *Théorie des intersections et théorème de Riemann-Roch*, Lecture Notes in Mathematics, Vol. 225, Springer-Verlag, Berlin-New York, 1971, Séminaire de Géométrie Algébrique du Bois-Marie 1966–1967 (SGA 6), Dirigé par P. Berthelot, A. Grothendieck et L. Illusie. Avec la collaboration de D. Ferrand, J. P. Jouanolou, O. Jussila, S. Kleiman, M. Raynaud et J. P. Serre.

[SGA7-I] Alexander Grothendieck, Michel Raynaud, and Dock Sang Rim, *Groupes de monodromie en géométrie algébrique. I*, Lecture Notes in Mathematics, Vol. 288, Springer-Verlag, 1972, Séminaire de Géométrie Algébrique du Bois-Marie 1967–1969 (SGA 7 I).

[SGA7-II] Pierre Deligne and Nicholas Katz, *Groupes de monodromie en géométrie algébrique. II*, Lecture Notes in Mathematics, Vol 340, Springer-Verlag, 1973, Séminaire de Géométrie Algébrique du Bois-Marie 1967–1969 (SGA 7, II).

[Sil09] Joseph H. Silverman, *The arithmetic of elliptic curves*, second ed., Graduate Texts in Mathematics, vol. 106, Springer, Dordrecht, 2009.

[Som82] A. J. Sommese, *Some examples of* $\mathbf{C}^*$ *actions*, Group actions and vector fields (Vancouver, B.C., 1981), Lecture Notes in Math., vol. 956, Springer, Berlin, 1982, pp. 118–124.

[SP] The Stacks Project Authors, *Stacks Project*, http://stacks.math.columbia.edu, 2022.

[Spr77] T. A. Springer, *Invariant theory*, Lecture Notes in Mathematics, Vol. 585, Springer-Verlag, Berlin-New York, 1977.

[Spr98] ———, *Linear algebraic groups*, second ed., Progress in Mathematics, vol. 9, Birkhäuser Boston, Inc., Boston, MA, 1998.

[SR72] Neantro Saavedra Rivano, *Catégories tannakiennes*, Bull. Soc. Math. France **100** (1972), 417–430.

[SS11] Pramathanath Sastry and C. S. Seshadri, *Geometric reductivity—a quotient space approach*, J. Ramanujan Math. Soc. **26** (2011), no. 4, 415–477.

[SS20] Roy Skjelnes and Gregory G. Smith, *Smooth hilbert schemes: their classification and geometry*, 2020, arXiv:2008.08938.

[Stu08] Bernd Sturmfels, *Algorithms in invariant theory*, second ed., Texts and Monographs in Symbolic Computation, SpringerWienNewYork, Vienna, 2008.

[Sum74] Hideyasu Sumihiro, *Equivariant completion*, J. Math. Kyoto Univ. **14** (1974), 1–28.

[Sum75] H. Sumihiro, *Equivariant completion. II*, J. Math. Kyoto Univ. **15** (1975), no. 3, 573–605.

[Swa98] Richard G. Swan, *Néron-Popescu desingularization*, Algebra and geometry (Taipei, 1995), Lect. Algebra Geom., vol. 2, Int. Press, Cambridge, MA, 1998, pp. 135–192.

[Tha96] Michael Thaddeus, *Geometric invariant theory and flips*, J. Amer. Math. Soc. **9** (1996), no. 3, 691–723.

[Tot99] Burt Totaro, *The Chow ring of a classifying space*, Algebraic K-theory (Seattle, WA, 1997), Proc. Sympos. Pure Math., vol. 67, Amer. Math. Soc., Providence, RI, 1999, pp. 249–281.

[Tot04] B. Totaro, *The resolution property for schemes and stacks*, J. Reine Angew. Math. **577** (2004), 1–22.

[TT90] R. W. Thomason and Thomas Trobaugh, *Higher algebraic K-theory of schemes and of derived categories*, The Grothendieck Festschrift, Vol. III, Progr. Math., vol. 88, Birkhäuser Boston, Boston, MA, 1990, pp. 247–435.

[Vak06] Ravi Vakil, *Murphy's law in algebraic geometry: badly-behaved deformation spaces*, Invent. Math. **164** (2006), no. 3, 569–590.

[Vak17] ———, *The rising sea: Foundations of algebraic geometry*, math216.wordpress.com, 2017.

[Vie95] Eckart Viehweg, *Quasi-projective moduli for polarized manifolds*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 30, Springer-Verlag, Berlin, 1995.

[Vis89] Angelo Vistoli, *Intersection theory on algebraic stacks and on their moduli spaces*, Invent. Math. **97** (1989), no. 3, 613–670.

[Vis91] ———, *The Hilbert stack and the theory of moduli of families*, Geometry Seminars, 1988–1991 (Italian) (Bologna, 1988–1991), Univ. Stud. Bologna, Bologna, 1991, pp. 175–181.

[Vis97] ———, *The deformation theory of local complete intersections*, 1997.

[Vis05] ———, *Grothendieck topologies, fibered categories and descent theory*, Fundamental algebraic geometry, Math. Surveys Monogr., vol. 123, Amer. Math. Soc., Providence, RI, 2005, pp. 1–104.

[Wat79] William C. Waterhouse, *Introduction to affine group schemes*, Graduate Texts in Mathematics, vol. 66, Springer-Verlag, New York-Berlin, 1979.

[Wei58] André Weil, *On the moduli of riemann surfaces*, To Emil Artin on his sixtieth birthday. Unpublished manuscript; Collected Papers, Vol. II (1958), 381–389.

[Wei94] Charles A. Weibel, *An introduction to homological algebra*, Cambridge Studies in Advanced Mathematics, vol. 38, Cambridge University Press, Cambridge, 1994.

[Wey25] H. Weyl, *Theorie der Darstellung kontinuierlicher halb-einfacher Gruppen durch lineare Transformationen. I*, Math. Z. **23** (1925), no. 1, 271–309.

[Wey26] ———, *Theorie der Darstellung kontinuierlicher halb-einfacher Gruppen durch lineare Transformationen. II*, Math. Z. **24** (1926), no. 1, 328–376.

[XZ20] Chenyang Xu and Ziquan Zhuang, *On positivity of the CM line bundle on K-moduli spaces*, Ann. of Math. (2) **192** (2020), no. 3, 1005–1068.