COURSE SUMMARY FOR MATH 508,
WINTER QUARTER 2017:
ADVANCED COMMUTATIVE ALGEBRA

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WEEK 1, JAN 4, 6: DIMENSION

Lecture 1: Introduction to dimension.
- Define Krull dimension of a ring $A$.
- Discuss dimension 0 rings. Recall Artinian rings and various equivalences.
- Prove that a PID has dimension 1.
- Prove that if $I \subset R$ is a nilpotent ideal, then $\dim R/I = \dim R$.

Lecture 2: Conservation of dimension under integral extensions.
- Prove that if $R \to S$ is an integral ring extension, then $\dim R = \dim S$.
- Define the codimension (or height) of a prime $p \subset A$, denoted as $\text{codim} p$, as the supremum of the lengths $k$ of strictly descending chains
  $p = p_0 \supset p_1 \supset \cdots \supset p_k$
  of prime ideals. Note that $\text{codim} p = \dim A_p$.
- Prove: if $\phi: R \to S$ is an integral ring homomorphism, then $\dim I = \dim \phi^{-1}(I)$.
- Discuss codimension 0 primes (i.e. minimal primes).
- Prove: if $R$ is Noetherian and $f \in R$ is a non-unit, then any prime $p \subseteq (f)$ has $\text{codim}(p) = 0$.

WEEK 2, JAN 9, 11 (JAN 13 CANCELLED): KRULL’S HAUPFTEIDEALSATZ AND CONSEQUENCES

Lecture 3: Krull’s Hauptidealsatz.
- State and prove Krull’s Principal Ideal Theorem (a.k.a. Krull’s Hauptidealsatz): if $A$ is a Noetherian ring and $f \in A$ is not a unit, then $\text{height}(f) \leq 1$; that is, for every prime ideal $p$ containing $f$, $\text{height } p \leq 1$.
- State and prove the following generalization of Krull’s Principal Ideal Theorem: if $A$ is a Noetherian ring and $I = (x_1, \ldots, x_n) \subset A$ is a proper ideal. Then $\text{height } I \leq n$; that is $\text{height } p \leq n$ for every prime ideal $p$ containing $I$ (or equivalently, for every prime ideal which is minimal among prime ideals containing $I$).
- Prove corollary: $\dim k[x_1, \ldots, x_n] = n$. 
Macaulay2 Tutorial. (Evening of Jan 9)

Lecture 4: System of parameters.

• Prove the converse theorem to Krull’s principal ideal theorem: if $A$ is a Noetherian ring and $I \subset A$ is a proper ideal of height $n$. Then there exist $x_1, \ldots, x_n \in I$ such that $\text{height}(x_1, \ldots, x_i) = i$ for $i = 1, \ldots, n$.

• Reinterpret dimension: if $(R, \mathfrak{m})$ is a Noetherian local ring, then $\dim R$ is the smallest number $n$ such that there exists $x_1, \ldots, x_m \in \mathfrak{m}$ with $R/(x_1, \ldots, x_m)$ Artinian. Such a sequence $x_1, \ldots, x_m \in \mathfrak{m}$ is called a system of parameters for $R$.

• Prove corollary: if $(R, \mathfrak{m}) \to (S, \mathfrak{n})$ is a local homomorphism of local Noetherian rings, then $\dim S \leq \dim R + \dim S/\mathfrak{m}S$.

• Prove corollary: if $R$ is a Noetherian ring, then $\dim R[x] = \dim R + 1$.

Week 3, Jan 18, 20 (Jan 16 MLK Holiday): Flatness

Lecture 5: Basics on flatness.

• Review of $\text{Tor}$. Key properties: short exact sequences induce long exact sequences of $\text{Tor}$ groups, $\text{Tor}_i(P, M) = 0$ for $P$ projective and $i > 0$, compatibility with localization, $\text{Tor}_i(M, N) = \text{Tor}_i(N, M)$ and thus can be computed as a derived functor in either the first or second term.

• Examples of flat and non-flat modules

• Prove Going Down Theorem for flatness

• Prove: if $(R, \mathfrak{m}) \to (S, \mathfrak{n})$ is a flat local homomorphism of local Noetherian rings, then $\dim S = \dim R + \dim S/\mathfrak{m}S$.

Lecture 6: Homological characterization of flatness:

• Prove: Let $R$ be a ring. An $R$-module $M$ is flat if and only if $\text{Tor}_i^R(R/I, M) = 0$ for all finitely generated ideals $I \subset R$.

• Examples: flatness over the dual numbers, flatness over PIDs.

• Equational Criterion for Flatness: An $R$-module $M$ is flat if and only if the following condition is satisfies: For every relation $0 = \sum_i n_i m_i$ with $m_i \in M$ and $n_i \in R$, there exist elements $m_i' \in M$ and elements $a_{ij} \in R$ such that

$$\sum_j a_{ij} m_j' = m_i \text{ for all } i \text{ and } \sum_i a_{ij} n_i = 0 \text{ for all } j.$$  

Week 4, Jan 23, 25, 27: Artin–Rees Lemma, Krull’s Intersection Theorem, Local Criterion of Flatness

Lecture 7: flatness $\iff$ projective.

• Reinterpret equational criterion for flatness using commutative diagrams.
• Prove: if $M$ is finitely presented $R$-module, then $M$ is flat $\iff$ projective. If in addition $R$ is a local, then flat $\iff$ free $\iff$ projective.

• State and motivate simple version of Local Criterion for Flatness.

Lecture 8: Artin–Rees Lemma and Local Criterion for Flatness.

• Give motivation of the Artin–Rees Lemma.

• Prove Artin–Rees Lemma: Let $R$ be a Noetherian ring and $I \subset R$ an ideal. Let $M' \subset M$ be an inclusion of finitely generated $R$-modules. If $M = M_0 \supset M_1 \supset M_2 \supset \cdots$ is an $I$-stable filtration, so is $M' \supset M' \cap M_1 \supset M' \cap M_2 \supset \cdots$.

• Prove Krull’s Intersection Theorem: Let $R$ be a Noetherian ring, $I \subset R$ an ideal and $M$ a finitely generated $R$-module. Then there exists $x \in I$ such that

$$(1 - x) \bigcap_{k} I^k M = 0.$$  

In particular, if $(R, m)$ is local, then $\bigcap m^k M = 0$ and $\bigcap m^k = 0$. Or if $R$ is a Noetherian domain and $I$ is any ideal, then $\bigcap I^k = 0$.

• State general version of Local Criterion for Flatness: if $(R, m) \to (S, n)$ is a local homomorphism of local Noetherian rings and $M$ is a finitely generated $S$-module, then

$M$ is flat as an $S$-module $\iff$ $\text{Tor}_1^R(R/m, M) = 0$.

Lecture 9: Fibral Flatness Theorem.

• Finish proof of Local Criterion for Flatness.

• Prove the Fibral Flatness Theorem: Consider a local homomorphisms $(R, m) \to (S, n) \to (S', n')$ of local Noetherian rings. Let $M$ be a finitely generated $S'$-module which is flat over $R$. Then $M$ is flat over $S$ if and only if $M/mM$ is flat over $S/mS$.

• Discuss special case of the Fibral Flatness Theorem when $R = k[x]_{(x)}$.

Week 5, Jan 30, Feb 1, 3: Graded modules and completions

Lecture 10: Graded modules and flatness.

• Summary of flatness results.

• State Openness of Flatness and Grothendiecke’s generic freeness (without proof).

• Graded modules and Hilbert functions.

• Prove: Let $R = \bigoplus_{d \geq 0} R_d$ be a graded ring which is finitely generated as an $R_0$-algebra by elements of degree 1. Assume $R_0$ is a local Noetherian domain. Let $M$ be a finitely generated $R$-modules. Then $M$ is flat/$R_0$ if and only if for $p \in \text{Spec} R_0$, the Hilbert function $H_M \otimes_{R_0} k(p) := \dim_{k(p)} M_d \otimes_{R_0} k(p)$ is independent of $p$. 
Lecture 11: Completions.
- Definition of the completion of a ring (and module) with respect to an ideal.
- Arithmetic and geometric examples.
- Show that if $R$ is Noetherian and $I \subset R$ is an ideal, then $M \otimes_R \hat{R} \to \hat{M}$ is an isomorphism for all finitely generated modules $M$.
- Conclude that $R \to \hat{R}$ is flat.

Lecture 12: Completions continued.
- Show that if $R$ is Noetherian, then so is the completion $\hat{R}$ of $R$ along an ideal $I$.
- Conclude that $R$ Noetherian $\Rightarrow R[[x]]$ Noetherian.
- Mention Hensel’s Lemma.
- Mention Cohen’s Structure Theorem.

Week 6, Feb 8, 10 (Feb 6 Snow Day): regular sequences and Koszul complexes

Lecture 13: regular sequences.
- Introduce regular sequences: we say $x_1, \ldots, x_n \in R$ that is a $M$-regular sequence if $x_i$ is a non-zero divisor on $M/(x_1, \ldots, x_{i-1})M$ for $i = 1, \ldots, n$ and that $M \neq (x_1, \ldots, x_n)M$.
- Give examples.
- Prove: Let $R$ be a ring and $x_1, \ldots, x_n$ be a regular sequence. Set $I = (x_1, \ldots, x_n)$. Show that the natural homomorphism

$$R/I[y_1, \ldots, y_n] \to \text{Gr}_I R, \quad y_i \mapsto x_i \in I/I^2,$$

is an isomorphism. In particular, $I/I^2$ is a free $R/I$-module of rank $n$.
- State more general version (which has the same proof) when $M$ is an $R$-module and $x_1, \ldots, x_n$ is a $M$-regular sequence.

Lecture 14: Koszul complex.
- Finish proof of proposition from last class.
- Introduce alternating products.
- Give concrete definition of the Koszul complex: if $R$ is a ring, $M$ is an $R$-module and $x = (x_1, \ldots, x_n) \in R^n$, then $K(x; M) = K(x_1, \ldots, x_n; M)_\bullet$ is the chain complex of $R$-modules where $K(x; M)_k = \wedge^k(R^n)$ and the differential $\wedge^k(R^n) \to \wedge^{k-1}(R^n)$ is defined by $e_{i_1} \wedge \cdots \wedge e_{i_k} \mapsto \sum_{j=1}^k (-1)^j x_{i_j} e_{i_1} \wedge \cdots \wedge \hat{e}_{i_j} \wedge \cdots \wedge e_{i_k}$.
- Write down examples in the case $n = 1, 2, 3$.

Week 7, Feb 13, 15, 17: Koszul homology, depth and regularity

Lecture 15: Koszul complexes and regular sequences.
- Reintrepret the Koszul complex as a tensor product of complexes.
• Define Koszul homology: if $R$ is a ring, $M$ is an $R$-module and $x = (x_1, \ldots, x_n) \in R^n$, then $H_i(x; M) := H_i(K(x; M))$.

• Prove: Let $R$ be a ring, $M$ be an $R$-module and $x = (x_1, \ldots, x_n) \in R^n$. If $x$ is an $M$-regular sequence, then $H_i(x; M) = 0$ for $i \geq 1$.

• Show that the converse is true if $(R, m)$ is a local Noetherian ring, $M$ is a finitely generated $R$-module and $x_1, \ldots, x_n \in m$.

**Lecture 16: depth.**

• Finish proof of converse from previous lecture.

• State the theorem: Let $R$ be a Noetherian ring and $M$ be a finitely generated $R$-module. Let $x = (x_1, \ldots, x_n) \in R^n$ and set $I = (x_1, \ldots, x_n) \subset R$. Assume $IM \neq M$. Let $d$ be the smallest integer such that $H_{n-d}(x; M) \neq 0$. Then any maximal $M$-regular sequence in $I$ has length $d$.

• Define the depth of $I$ on $M$, denoted by depth$(I, M)$, as this smallest integer $d$.

• Give examples.

• Begin proof of theorem.

**Lecture 17: depth and regular local rings.**

• Finish proof characterizing depth.

• State (but do not prove): Let $R$ be a Noetherian ring, $I \subset R$ be an ideal and $M$ be a finitely generated graded $R$-module. Assume $I + \text{Ann}(M) \neq R$. Then depth$(I, M)$ is the smallest $i$ such that $\text{Ext}^i_R(R/I, M) \neq 0$.

• Define a regular local ring $(R, m)$ as a Noetherian local ring such that $\dim R = \dim_{R/m} \frac{m}{m^2}$.

• Give a few examples: e.g., say when $k[x_1, \ldots, x_n]/(x_1, \ldots, x_n)$ is regular.

• Prove that a regular local ring is a domain.

**Week 8, Feb 22, 24 (Feb 20 President’s Day): Free resolutions and projective dimension**

**Lecture 17: minimal free resolutions and projective dimension.**

• Show that if $(R, m)$ is a regular local ring and $x_1, \ldots, x_k$ are elements of $m$ which are linearly independent in $m/m^2$, then $x_1, \ldots, x_k$ is a regular sequence and $R/(x_1, \ldots, x_k)$ is a regular local ring. Such a sequence whose length is equal to $\dim R$ is called a regular system of parameters.

• Introduce the projective dimension of a module $M$, denoted by $\text{pd} M$, as the smallest length of a projective resolution of $M$.

• Introduce the global dimension of a ring $R$, denoted by $\text{gl dim} M$, as the supremum of $\text{pd} M$ over finitely generated $R$-modules $M$. 
• Define a minimal free resolution of an \( R \)-module \( M \) as a free resolution
\[
\cdots \rightarrow F_k \xrightarrow{d_k} F_{k-1} \rightarrow \cdots \rightarrow F_0 \rightarrow M
\]
such that \( \text{im}(d_k) \subset \text{m}F_{k-1} \).

• Show: if \((R, m)\) is a local Noetherian ring and \( M \) is a finitely generated \( R \)-module, then \( \text{pd} M \) is equal to length of any minimal free resolution and is also characterized by the smallest \( i \) such that \( \text{Tor}^R_{i+1}(R/m, M) = 0 \). Conclude that \( \text{gl dim} R = \text{pd} R/m \)

Lecture 18: Auslander–Buchsbaum Theorem.
• Recall notions and results from previous lecture.
• Show that if \((R, m)\) is a regular local ring, then \( \dim R = \text{gl dim} R/m \). Discuss examples showing that this is not true if \( R \) is not regular.
• Discuss (without proof) graded analogues: minimal graded free resolutions, graded Betti numbers and namely Hilbert’s Syzygy Theorem (any finitely generated graded module over the polynomial ring \( k[x_1, \ldots, x_n] \) has a finite free graded resolutions of length \( \leq n \).
• Recall the notion of depth and prove that if \( R \) is a Noetherian ring and \( I \subset R \) is an ideal, then \( \text{depth}(I, R) \leq \text{codim} I \).
• Prove the Auslander-Buchsbaum Theorem: Let \((R, m)\) be a Noetherian local ring and \( M \neq 0 \) be a finitely generated \( R \)-module with \( \text{pd} M < \infty \). Then
\[
\text{pd} M = \text{depth}(m, R) - \text{depth}(m, M).
\]

Week 9, Feb 27, Mar 1 (Mar 3 cancelled): The Auslander–Buchsbaum–Serre Theorem and Cohen–Macaulay rings

Lecture 19: Homological characterization of regular rings.
• Prove the Auslander–Buchsbaum–Serre Theorem: If \((R, m)\) is a Noetherian local ring, the following are equivalent:
  (i) \( R \) is regular.
  (ii) \( \text{gl dim} R < \infty \).
  (iii) \( \text{pd} R/m < \infty \).

Lecture 20: Cohen–Macaulay rings.
• Prove the following corollaries of the Auslander–Buchsbaum–Serre Theorem:
  − If \( R \) is a regular local ring, then so is \( R_p \) for every prime ideal \( p \).
  − \( k[x_1, \ldots, x_n] \) is a regular ring (i.e., all localizations at prime ideals are regular local rings).
• Define a Noetherian local ring \((R, m)\) to be Cohen–Macaulay if \( \text{depth}(m, R) = \dim R \).
Show that the following examples of Cohen–Macaulay rings: (1) regular local rings, (2) local Artinian rings, and (3) local Noetherian dimension 1 reduced rings.

Prove: If \((R, \mathfrak{m})\) is a Cohen–Macaulay local ring and \(p\) is an associated prime, then \(p\) is a minimal prime and \(\dim R = \dim R/p\). (In other words, Cohen–Macaulay rings can have no embedded primes (i.e. associated but not minimal primes) and is equidimensional.)

Give some examples of rings that are not Cohen–Macaulay: \(k[x, y]/(x^2, xy)\), \(k[x, y, z]/(xz, yz)\), ...

**Week 10, Mar 6, 8, 10: Cohen–Macaulay, Normal, Complete Intersections and Gorenstein rings**

**Lecture 21: Properties of Cohen-Macaulay rings and Miracle Flatness.**

- Prove: If \((R, \mathfrak{m})\) is a Cohen–Macaulay local ring, then for any ideal \(I \subset R\), we have \(\text{depth}(I, R) = \dim R - \dim R/I = \text{codim} I\). (In particular, the defining property of being Cohen–Macaulay holds for all ideals. Also, in the homework, we will see that Cohen–Macaulay rings satisfy a stronger dimension condition known as catenary.)

- Prove: Let \((R, \mathfrak{m})\) be a Cohen–Macaulay local ring. Then \(x_1, \ldots, x_n \in \mathfrak{m}\) is a regular sequence if and only if \(\dim R/(x_1, \ldots, x_n) = \dim R - n\). In other words, if \((R, \mathfrak{m})\) is a Cohen–Macaulay local ring, then any system of parameters is a regular sequence.

- Prove Miracle Flatness: Let \((R, \mathfrak{m}) \to (S, \mathfrak{n})\) be a local homomorphism of Noetherian local rings. Suppose that \(R\) is regular and \(S\) is Cohen–Macaulay. Then \(R \to S\) is flat if and only if \(\dim S = \dim R + \dim S/mS\).

**Lecture 22: Complete Intersections and Normal rings.**

- Define a Noetherian local ring \((R, \mathfrak{m})\) to be a complete intersection if the completion \(\hat{R}\) is the quotient of a regular local ring by a regular sequence. Observe that any regular local ring modulo a regular sequence is a complete intersection.

- Show that any complete intersection local ring is Cohen–Macaulay. Give an example of a Cohen–Macaulay ring (e.g. \(k[x, y]/(x, y)^2\)) which is not a complete intersection.

- Given a Noetherian local ring \((R, \mathfrak{m})\) with residue field \(k = R/\mathfrak{m}\) and minimal generators \(x_1, \ldots, x_n\) of \(\mathfrak{m}\), define the invariants \(\epsilon_i(R) = \dim_k H_i(x_1, \ldots, x_n)\) (the Koszul homology). The number of minimal generators is called the embedding dimension of \(R\) is denoted \(\text{emb } \dim(R)\).

- State: \(R\) is a complete intersection if and only if \(\dim R = \text{emb } \dim R - \epsilon_1(R)\). Give several examples of both when this holds and doesn’t.
• Recall that a domain $R$ is called normal if it is integrally closed in its fraction field.
• For a Noetherian ring $R$, introduce Serre’s properties:
  $(R_i)$ for all $p \in \text{Spec } R$ with $\text{codim}(p) \leq i$, $R_p$ is regular.
  $(S_i)$ for all $p \in \text{Spec } R$, $\text{depth } R_p \geq \min(\text{codim}(p), i)$.
Note that: $R$ is regular if and only if $(R_i)$ holds for all $i$ and $R$ is Cohen–Macaulay if and only if $(S_i)$ holds for all $i$.
• Reinterpret the Conditions $(S_0)$, $(S_1)$ and $(S_2)$. State that $R$ is reduced if and only if $(R_0)$ and $(S_1)$ hold.
• State Serre’s Normality Criterion: Let $R$ be a Noetherian ring. Then $R$ is normal if and only if $(R_1)$ and $(S_2)$ hold. (Prove the $\Leftarrow$ implication.)
• Mention Algebraic Hartog’s: If $R$ is a normal Noetherian domain, then $R = \bigcap_{\text{codim}(p) = 1} R_p$.

**Lecture 23: Gorenstein rings.**

• Recall the notion of an injective module and an injective resolution. If $R$ is a ring, define the injective dimension of an $R$-module, denoted by $\text{inj dim}_R M$, as the smallest length of an injective resolution of $M$.
• State (and explain some of the implications in) the following lemma: If $R$ is a ring and $M$ is an $R$-module, then the following are equivalent:
  (i) $\text{inj dim}_R M \leq n$.
  (ii) $\text{Ext}^{n+1}_R(N, M) = 0$ for all $R$-modules $N$.
  (iii) $\text{Ext}^{n+1}_R(R/I, M) = 0$ for all ideals $I \subset R$.
If, in addition, $(R, m)$ is local and $M$ is finitely generated, then the above is also equivalent to:
  (iv) $\text{Ext}^{n+1}_R(R/p, M) = 0$ for all $p \in \text{Spec } R$.
If, in addition, $(R, m)$ is a Noetherian local ring and $M$ is finitely generated, then the above is also equivalent to:
  (iv) $\text{Ext}^{n+1}_R(R/m, M) = 0$.
• Conclude: $(R, m)$ is a Noetherian local and $M$ is a finitely generated $R$-module, then $\text{inj dim}_R M$ is the largest $i$ such that $\text{Ext}_R^i(R/m, M) \neq 0$.
• State: If $(R, m)$ is a Noetherian local and $M$ is a finitely generated $R$-module with $\text{inj dim}_R M < \infty$, then $\dim M \leq \text{inj dim}_R M = \text{depth}(m, R)$.
• Compare the above characterizations and properties of injective dimension with what we’ve seen for projective dimension.
• Define a Noetherian local ring $(R, m)$ to be Gorenstein if $\text{inj dim}_R R < \infty$.
• Give the following equivalences (explaining some of the implications): Let $(R, m)$ is a Noetherian local ring of dimension $n$ with residue field $k = R/m$, then the following are equivalent
(i) $R$ is Gorenstein.
(ii) $\text{inj dim}_R R = n$.
(iii) $\text{Ext}_R^i(k, M) = 0$ for some $i > n$.
(iv) $\text{Ext}_R^i(k, M) = \begin{cases} 0 & i \neq n \\ k & i = n. \end{cases}$
(v) $R$ is Cohen–Macaulay and $\text{Ext}_R^n(k, R) = k$.
(vi) There exists a regular sequence $x_1, \ldots, x_n \in m$ such that $R/(x_1, \ldots, x_n)$
is Gorenstein and dimension 0.

• Prove: Let $(R, m)$ be a Noetherian local ring. Then

regular $\implies$ complete intersection $\implies$ Gorenstein $\implies$ Cohen–Macaulay

Give examples showing that each implication is strict.