Math 504: Modern Algebra, Fall Quarter 2017
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Homework 4
Due: Monday, October 23

Problem 3.1. Let H, K be groups and $\varphi\colon K \to \mathrm{Aut}(H)$ be a group homomorphism.

(1) If $\lambda \in \mathrm{Aut}(K)$, let $\varphi \circ \lambda\colon K \to \mathrm{Aut}(H)$ be defined by $k \mapsto \varphi(\lambda(k))$. Show that $H \rtimes_{\varphi} K \cong H \rtimes_{\varphi\circ\lambda} K$.
(2) If $\psi \in \mathrm{Aut}(H)$, let $\psi\varphi\psi^{-1}\colon K \to \mathrm{Aut}(H)$ be defined by $k \mapsto \psi\varphi(k)\psi^{-1}$. Show that $H \rtimes_{\varphi} K \cong H \rtimes_{\psi\varphi\psi^{-1}} K$.

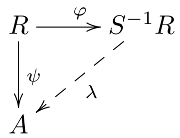
Problem 3.2 (Chinese Remainder Theorem)**.**

(1) Let R be a commutative ring. Suppose that $I_1, \dots, I_n$ are ideals such that $I_j + I_k = R$ for each $j \neq k$. Show that there is an isomorphism $R/(I_1 \cdots I_k) \cong R/I_1 \times \cdots \times R/I_k$ of rings.
(2) Let n be an integer with prime factorization $n = p_1^{a_1} \cdots p_k^{a_k}$. Show that there is an isomorphism $\mathbb{Z}/n \cong \mathbb{Z}/p_1^{a_1} \times \cdots \times \mathbb{Z}/p_k^{a_k}$ of rings.
(3) Show that a commutative ring R is isomorphic to a direct product $R_1 \times R_2$ of nonzero rings if and only if there are nonzero elements $e_1, e_2 \in R$ satisfying $e_1^2 = e_1$, $e_2^2 = e_2$, $e_1 e_2 = 0$ and $e_1 + e_2 = 1$.

Problem 3.3 (Localization)**.** Let R be a commutative ring. Let $S \subset R$ be a *multiplicative subset*; that is, S is a subset containing 1 and satisfies: if $x, y \in S$, then $xy \in S$. Define

$$S^{-1}R = R \times S/\sim$$

where $\sim$ is the equivalence relation on $R \times S$ defined by $(r, s) \sim (r', s')$ if there exists $t \in S$ such that $t(rs' - r's) = 0$. If one defines $(r, s) + (r', s') := (s'r + r's, ss')$ and $(r, s) \cdot (r', s') = (rr', ss')$, then $S^{-1}R$ is a commutative ring and $\phi\colon R \to S^{-1}R, r \mapsto (r, 1)$ is a ring homomorphism such that $\phi(S) \subset (S^{-1}R)^{\times}$. Show that $S^{-1}R$ satisfies the following universal property: for any commutative ring A and ring homomorphism $\psi\colon R \to A$ such that $\psi(S) \subset A^{\times}$, there exists a unique ring homomorphism $\lambda\colon S^{-1}R \to A$ such that



commutes.

Problem 3.4. Let R be a commutative ring.

(1) If $I \subset R$ is an ideal, show that there is a bijection between the prime ideals of R/I and the prime ideals of R containing I.
(2) If $S \subset R$ is a multiplicative subset, show that there is a bijection between the prime ideals of $S^{-1}R$ and the prime ideals of R not meeting S.
(3) If one replaces each instance of the word 'prime' with 'maximal' in statements (1) and (2) above, are the analogous statements still true? In each case, either prove the statement or provide a counterexample.

Problem 3.5. The *center* of a ring R is the subring of R containing elements $r \in R$ such that $rx = xr$ for all $x \in R$.

(1) Show that if R is a division ring, then the center of R is a field.
(2) Let R be a non-zero ring. What is the center of the matrix ring $M_n(R)$?

Problem 3.6. If R is a commutative ring, define the *formal power series ring* $R[[x]]$ to be the set of elements of the form $\sum_{n=0}^{\infty} a_n x^n$ where $a_n \in R$. Addition and multiplication are defined in the natural way; that is,

$$\left(\sum_{n=0}^{\infty} a_n x^n\right) + \left(\sum_{n=0}^{\infty} b_n x^n\right) := \sum_{n=0}^{\infty} c_n x^n$$

$$\left(\sum_{n=0}^{\infty} a_n x^n\right) \cdot \left(\sum_{n=0}^{\infty} b_n x^n\right) := \sum_{n=0}^{\infty} \big(\sum_{k=0}^{n} a_k b_{n-k}\big) x^n$$

(1) Show that $1 + x \in R[[x]]$ is a unit.
(2) Show that if R is an integral domain, so is $R[[x]]$.