may 20

Recap

$\{\det_n\} \in VP = \{p\text{-computable}\}$
$\{\text{perm}_n\} \in VNP = \{p\text{-definable sequences}\}$

(with $VP \subseteq VNP$)

Def^n

a determinantal expression of size m of a polynomial $f(x_1,\dots,x_n)$ is an $m\times m$ matrix

$$M = \begin{pmatrix} m_{11}, & \cdots, & m_{1m} \\ \vdots & & \vdots \\ m_{m1}, & \cdots, & m_{mm} \end{pmatrix}$$

where each M_{ij} is affine linear in x_i's

$$f(x_1,\dots,x_n) = \det_m(M)$$

explicitly,

$$M_{ij} = a_{ij_1}x_1 + \dots + a_{ij_n}x_n + b_{ij}$$

constants in $\mathbb{k}$

if $\text{char}(k)=2$ $(\mathbb{Z}/2)$ then $\text{perm}_n = \det_n$

Def^n

the determinantal complexity of f is
$dc(f)$ = smallest m such that $\exists$ det. expression of f of size m.

examples

① $xy - z^2 = \det\begin{pmatrix} x & z \\ z & y\end{pmatrix}$

② $x^2 + yz + xz = \det\begin{pmatrix} x & y & 1 \\ 0 & x & 1 \\ z & 0 & 1\end{pmatrix}$

$\downarrow$

$dc(\ ) \le 3$

OR

$= \det\begin{pmatrix} x & -z \\ x+y & x\end{pmatrix}$

$\hookrightarrow dc(\ ) = 2$

observation

if $\deg(f) = d$, then $dc(f) \ge d$

THM

let $f(x_1, \dots, x_n)$ be a polynomial

Then $dc(f) \le \underbrace{C_e(f)}_{\text{expression}} + 2$

in particular, any poly has a det. expr.

PROOF

we will give a procedure

$\{$arithmetic tree$\}$ $\longrightarrow$ $\{$directed labelled graph G with distinguished vertices s ("some") and t ("target") such that every cycle cover has same sign$\}$

tree gives an expression for f $\longrightarrow$ G such that $\det(G) = f$

ex:

for constants...

$c \in k$

$$\begin{tikzcd} c \arrow[d, dash] \\ c \end{tikzcd} \qquad \begin{tikzcd} s \arrow[d, "c"] \\ t \end{tikzcd} \qquad \det = -c$$

x_i

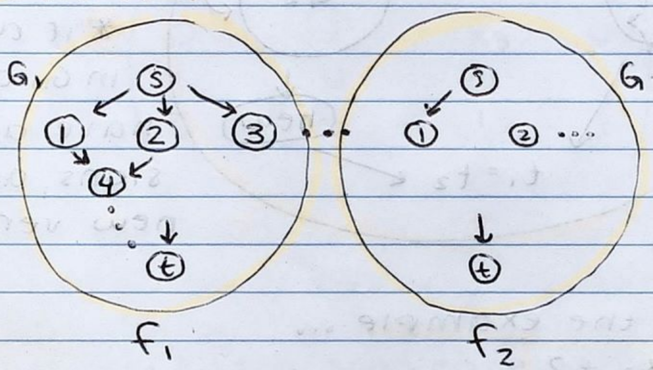
$$\begin{tikzcd} x_i \arrow[d, dash] \\ x_i \end{tikzcd} \qquad \begin{tikzcd} s \arrow[d, "x_i"] \\ t \end{tikzcd} \qquad \det = -x_i$$

suppose G_1 and G_2 directed labelled graph such that all cycle covers have the same sign such that $f_1 = \det G_1$

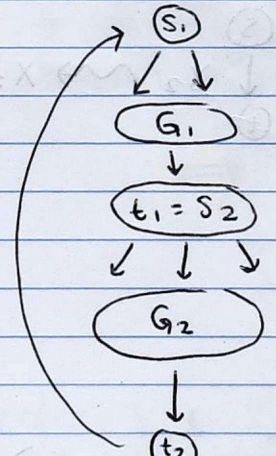
$$f_2 = \det G_2$$

EXAMPLE: $f = (x_1 + x_2)x_3 + 2$

(a) $f_1 * f_2$ — produce a graph of product of f_1 and f_2



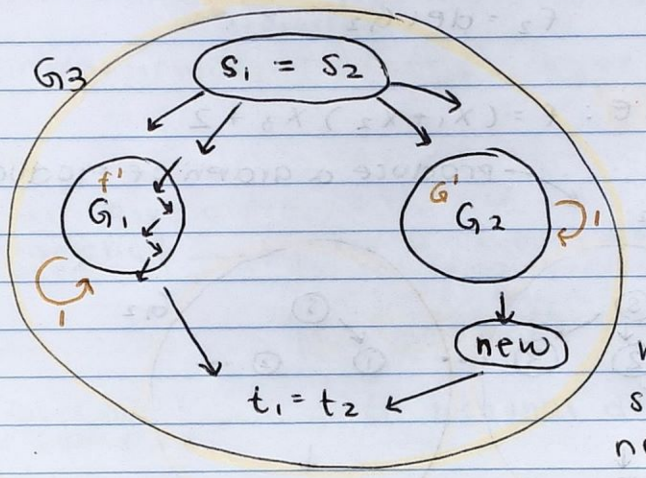
$\det M_1 = f_1$
$\det M_2 = f_2$ $\rightarrow \begin{pmatrix} M_1 & \\ & M_2 \end{pmatrix}$
$\det(M_1 M_2)$



* any cycle cover in G_3 is the union of a cycle cover in G_1 and G_2 and its weight is the product of weights

$\Longrightarrow$ after adding a 1 from $t \rightarrow s$,
$\det(G_3) = \pm f_1 f_2$

(b) now we want to produce a new graph whose cycle covers are either in G_1 or G_2. so its sum instead of product.



* if cycle covers in G_1 and G_2 have different signs, add a new vertex!

so back to the example ...

$\boxed{f = (x_1 + x_2)x_3 + 2}$

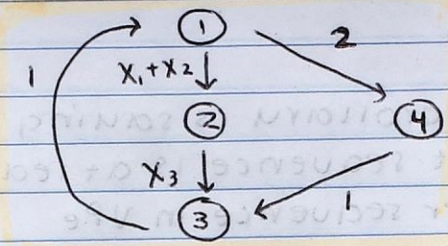
$$\begin{tikzcd} s \arrow[d, "x_1"] \\ t \end{tikzcd} \rightsquigarrow x_1 \qquad \begin{tikzcd} s \arrow[d, "x_2"] \\ t \end{tikzcd} \rightsquigarrow x_2$$

$$\begin{tikzcd} s \arrow[d, "x_1 + x_2"] \\ t \end{tikzcd} \qquad \begin{tikzcd} s \arrow[d, "x_3"] \\ t \end{tikzcd}$$

$$\begin{tikzcd} s \arrow[d, "x_1 + x_2"] \\ 1 \arrow[d, "x_3"] \\ t \end{tikzcd} \qquad \begin{tikzcd} s \arrow[d, "2"] \\ t \end{tikzcd}$$

$\det(G) = (x_1 + x_2)x_3 + 2$

G:



we have 4 vertices, so...

$$M = \begin{pmatrix} 0 & x_1 + x_2 & 0 & 2 \\ 0 & 1 & x_3 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

prop

if $\{f_n\}$ is a seauence of polynomials,

1) $\deg(f_n)$ is polu bounded
2) # variabies of f is polu bounded
3) $dc(f_n)$ is poly bounded

$\Longrightarrow \{f_n\} \in VP$

reasoning
$\{f_n\}$ is a p-projection of $\{\det_n\}$

cor

$\{\det_n\} \in VP_e$ - hard
That is, any seauence $\{f_n\} \in VP_e$ is a p-projection of $\{\det_n\}$

$$\begin{array}{ccc} & & VP_e \\ & & \cap \\ \{det_n\} & \in & VP \\ & & \cap \\ \{perm_n\} & \in & VNP \end{array}$$

SUMMARY

what this corollaru is sauing is that the determinant seauence is atleast as hard as any other seauence in VP_e

and what we like to show is that the permanent is VNP complete which is as hard as every other thing in VNP

so what we need to show, just like the cook-levin theorem where we showed every language was poly. reducible to SAT, that any seauence in VNP is poly. reducible, that is a p-projection of the permanent.