Math 300: Last Lecture

Friday June 3, 2022

TODAY: REVIEW

① Exhibit truth table for

$$\underbrace{(P \Rightarrow \underbrace{\sim Q}_{})}_{} \Longrightarrow \underbrace{(P \vee Q)}_{}$$

P	Q	$\sim Q$	$P \Rightarrow \sim Q$	$P \vee Q$	$(P \Rightarrow \sim Q) \Rightarrow (P \vee Q)$
T	T	F	F	T	T
T	F	T	T	T	T
F	T	F	T	T	T
F	F	T	T	F	F

Follow-up: Simplify the expression, i.e. find a tautology with a simpler expression.

$$P \vee Q$$

② Determine whether True or False.

(a) $\forall x, \exists! y, \ 2x+3y=0$

(b) $\exists x, \forall y, \ x \leq y^2$

(c) $(x \geq 0) \Longrightarrow [\exists! y, \ y^4 = x]$

Tip: Write out expression using words.

(a) For all real numbers x, there is a unique soln y to $2x+3y=0$.

YES! Take $y = \underbrace{\frac{-2}{3}x}_{\text{unique}}$ ✓

$\boxed{\text{TRUE}}$

(b) Take $x=0$

$\forall y \quad 0 \leq y^2$ ✓

$\boxed{\text{TRUE}}$

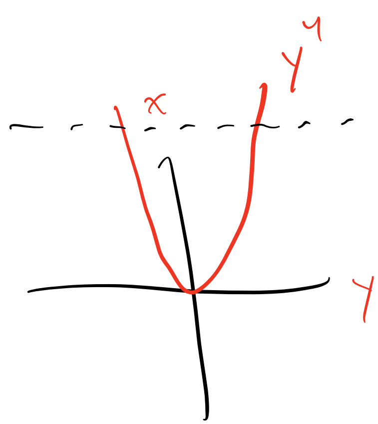
(c) For any $x \geq 0$, $y^4 = x$ has a unique soln.

unique pos. # y s.t. $y^4 = x$

Well, $\sqrt[4]{x}$ positive soln.

$-\sqrt[4]{x}$ is another soln

$\boxed{\text{FALSE}}$



(3) Prove that $\displaystyle\prod_{i=2}^{n}\left(1-\frac{1}{i^2}\right) = \frac{n+1}{2n}$

Notation: $\displaystyle\prod_{i=2}^{n}\left(1-\frac{1}{i^2}\right) = \left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\cdots\left(1-\frac{1}{n^2}\right)$

Pf (by induction)

Base case $n=2$: $1-\frac{1}{2^2} = \frac{2+1}{2\cdot 2} = \frac{3}{4}$ ✓

Inductive step: Assuming true for n, need to show true for $n+1$.

$$\prod_{i=2}^{n+1}\left(1-\frac{1}{i^2}\right) = \underbrace{\left(\prod_{i=2}^{n}\left(1-\frac{1}{i^2}\right)\right)}_{=\frac{n+1}{2n} \text{ by inductive hypothesis}}\cdot\left(1-\frac{1}{(n+1)^2}\right)$$

$$= \frac{n+1}{2n}\left(\underbrace{1-\frac{1}{(n+1)^2}}_{\frac{(n+1)^2-1}{(n+1)^2} = \frac{n^2+2n}{(n+1)^2} = \frac{n(n+2)}{(n+1)^2}}\right)$$

$$= \frac{\cancel{n+1}}{2\cancel{n}}\left(\frac{\cancel{n}(n+2)}{(n+1)^{\cancel{2}}}\right) = \frac{n+2}{2(n+1)}$$ ✓

(4) Consider functions $f: A\to B$, $g: B\to C$ and $h: C\to D$.

(a) Show that if $h\circ g\circ f$ is surjective, so is h.

(b) Show that the converse is false.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[rrr, bend right, "h\circ g\circ f"'] & B \arrow[r, "g"] & C \arrow[r, "h"] & D \end{tikzcd}$$

(a) Pick $d\in D$. Need to show $\exists c\ h(c)=d$

But $h\circ g\circ f$ is surjective, so $\exists a\in A$ with $(h\circ g\circ f)(a)=d \Rightarrow h(g(f(a)))=d$

Taking $c=g(f(a))$, get $h(c)=d$ ✓

(b) Need example where h is surjective but $h\circ g\circ f$ is not.

$$\begin{tikzcd} A=\{a\} \arrow[r] \arrow[rrr, bend right, "\text{not surjective}"'] & B=\{b_1,b_2\} \arrow[r, "\text{bij}"] & C=\{c_1,c_2\} \arrow[r, "\text{bij}", "\text{surjective}"'] & D=\{d_1,d_2\} \end{tikzcd}$$

$b_i \longmapsto c_i \longmapsto d_i$

(5) Show that

$$\{x \in \mathbb{R} \mid x > 0 \wedge 3x \in \mathbb{Z}\}$$

is countable

$$= \left\{ \frac{1}{3}, \frac{2}{3}, \frac{3}{3} = 1, \frac{4}{3}, \frac{5}{3}, \ldots \right\}$$

Define $\mathbb{N} \to \{x \in \mathbb{R} \mid x > 0 \wedge 3x \in \mathbb{Z}\}$

$$n \longmapsto \frac{n}{3}$$

This is bijective.

Reason:

↳ injective b/c $\frac{n}{3} = \frac{n'}{3} \Rightarrow n = n'$ ✓

surjective: if $x > 0$ & $3x = n \in \mathbb{Z}$

Then $n \mapsto \frac{n}{3} = x$ ✓

VARIANT $\{x \in \mathbb{R} \mid 3x \in \mathbb{Z}\}$

$$\left\{ \ldots, \frac{-2}{3}, \frac{-1}{3}, 0, \frac{1}{3}, \frac{2}{3}, \ldots \right\}$$

(6) True or False

(a) If A and B are countable, so is $A \cup B$.

(b) If A is uncountable and B is countable, $A - B$ is uncountable.

(c) $\mathbb{Q}$ is uncountable

(a) True

(b) Suppose B is countable.

Is it true: A uncountable $\Rightarrow A - B$ uncountable

Contrapositive: $A - B$ countable $\Rightarrow A$ countable ✓

Recall $A = (A - B) \cup B$

countable

TRUE

(c) $\mathbb{Q}$ countable

FALSE

⑦ How many positive divisors of 720 are there?

Prime factorization

$$720 = 72 \cdot 10 = 8 \cdot 9 \cdot 5 \cdot 2$$
$$= 2^4 \cdot 3^2 \cdot 5$$

Any positive divisor $d \mid 720$ is of the form

$$d = 2^a 3^b 5^c$$

where $a = 0, 1, -, 4$ ← 5 choices
$b = 0, 1, 2$ ← 3 choices
$c = 0, 1$ ← 2 choices

Allow $a=b=c=0 \to d=1$
$a=4, b=2, c=1 \to d=720$

$5 \cdot 3 \cdot 2 = \boxed{30}$ choices

⑧ We want to form a committee of 5 from a group of 12 people consisting of 4 teachers and 8 students. The committee must include 3 teachers. How many possibilities are there?

Every group has either 3 or 4 teachers.

↳ 3 teachers:

$$\binom{4}{3} \cdot \binom{8}{2} = 4 \cdot \frac{8 \cdot 7}{2}$$
$$= 4 \cdot 28$$
$$= 112$$

3 teachers out of 4 — 2 students from 8

↳ 4 teachers

$$\binom{4}{4}\binom{8}{1} = 1 \cdot 8 = 8$$

Total $= 112 + 8 = \boxed{120}$