

WED JUNE 1 : Math 300

Today: Combinatorics Take II
"The art of counting"

Recall from last time:

We defined "n choose k"

$$\binom{n}{k} = \# \left\{ \begin{array}{l} \text{subsets } A \subseteq \{1, \dots, n\} \\ \text{of length } k \end{array} \right\}$$

i.e. # ways to choose k elements
(order doesn't matter) from n

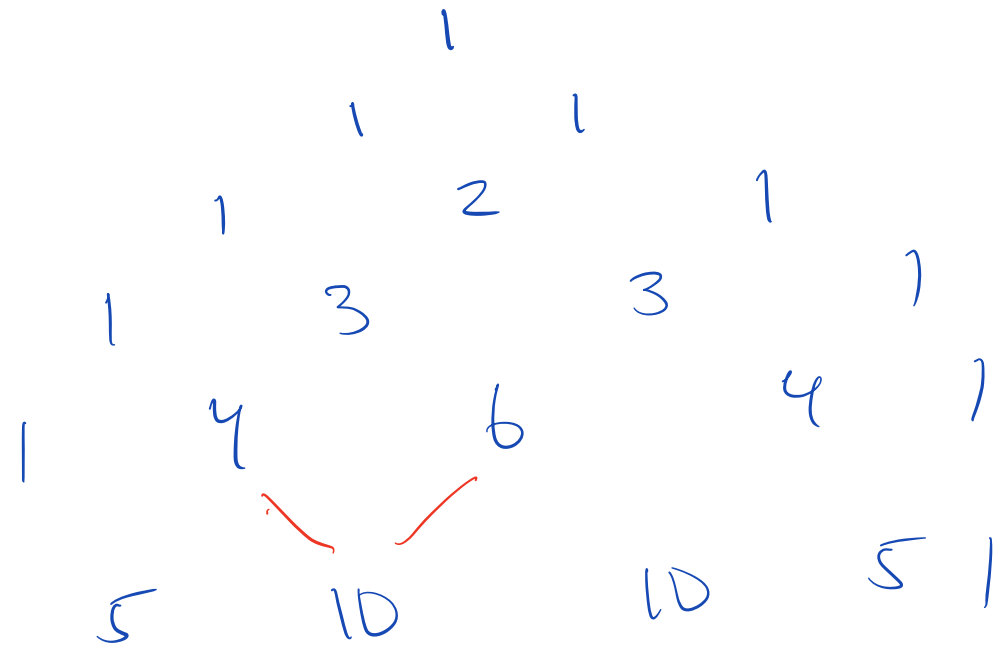
FACT $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

Two identities

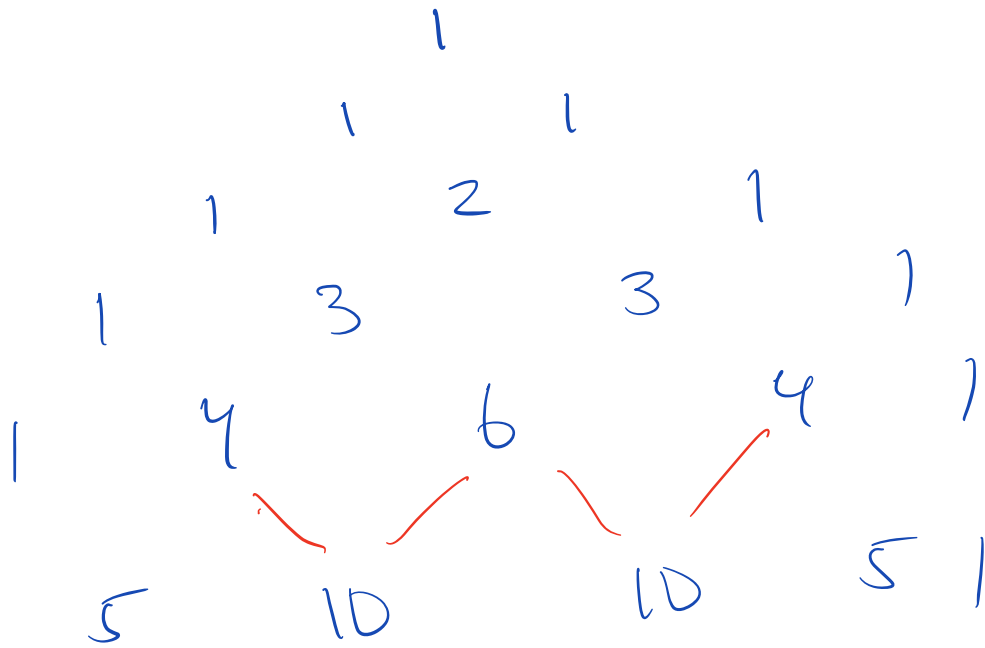
$$(1) \binom{n}{k} = \binom{n}{n-k}$$

$$(2) \binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

PASCAL'S TRIANGLE



PASCAL'S TRIANGLE



By defn, each number is the sum of two previous.

$$\begin{array}{cccc} & & \binom{0}{0} = 1 & & \\ & & \binom{1}{0} = 1 & & \binom{1}{1} = 1 \\ & & \binom{2}{0} = 1 & & \binom{2}{1} = 2 & & \binom{2}{2} = 1 \\ & & \binom{3}{0} = 1 & & \binom{3}{1} = 3 & & \binom{3}{2} = 3 & & \binom{3}{3} = 1 \end{array}$$

Reason that they are the same

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

								1
			1					2
		1	2	1				4
	1	3	3	1				8
	1	4	6	4	1			16
1	5	10	10	5	1			32

sum
↷

FACT: $\sum_{k=0}^n \binom{n}{k} = 2^n$

PF: Consider the power set

$$P(\{1, \dots, n\}) = \bigcup_{k=0}^n \left\{ \text{subsets } A \subseteq \{1, \dots, n\} \text{ of size } k \right\}$$

↑
by defn = set of all subsets

This defines a partition of $P(\{1, \dots, n\})$

That is, it is a disjoint union

⇒ The left hand side has the

same number of elements as right side

$$\Rightarrow 2^n = \sum_{k=0}^n \binom{n}{k}$$

↑
$P(\{1, \dots, n\})$

↑
{Ac{1,...,n} of size k}

Consider taking $(x+y)^n$

$n=0 \quad (x+y)^0 = 1$

$(x+y)^1 = 1 \cdot x + 1 \cdot y$

$(x+y)^2 = 1x^2 + 2xy + 1y^2$

$(x+y)^3 = 1x^3 + 3x^2y + 3xy^2 + 1y^3$

$(x+y)^4 = 1x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + 1y^4$

FACT $(x+y)^n = \sum_{k=0}^n \binom{n}{k} x^{n-k} y^k$

COE OF FACT: Take $x=y=1$

$2^n = \sum_{k=0}^n \binom{n}{k}$ so we give a 2nd proof of this.

PF 1 (by induction) $n=0,1$: ✓

Assume true for n .

$$(x+y)^{n+1} = (x+y)^n (x+y)$$

by inductive hypothesis

$$= \left(\sum_{k=0}^n \binom{n}{k} x^{n-k} y^k \right) (x+y)$$

$$= \sum_{k=0}^n \binom{n}{k} x^{n-k+1} y^k + \sum_{k=0}^n \binom{n}{k} x^{n-k} y^{k+1}$$

$$= \sum_{j=0}^{n+1} \underbrace{\left(\binom{n}{j} + \binom{n}{j-1} \right)}_{\binom{n+1}{j}} x^{n+1-j} y^j$$

$$= \sum_{j=0}^{n+1} \binom{n+1}{j} x^{n+1-j} y^j \quad \checkmark$$

PF 2 (by counting)

$$(x+y)^n = \underbrace{(x+y)(x+y)(x+y)(x+y)\dots(x+y)}_{n \text{ times}}$$

$$= \sum_{k=0}^n \binom{n}{k} x^{n-k} y^k$$

To get a term $x^{n-k} y^k$, need to choose y k 's & $n-k$ x 's.

→ # of ways = $\binom{n}{k}$

$$(x+y)^n = \sum_{k=0}^n \binom{n}{k} x^{n-k} y^k \quad \checkmark$$