WED JUNE 1 : Math 300

Today: Combinatorics Take II

"The art of counting"

Recall from last time:

We defined "n choose k"

$$\binom{n}{k} = \#\{\text{subsets } A \subset \{1, \dots, n\} \text{ of length } k\}$$

i.e. # ways to choose k elements (order doesn't matter) from n

Fact $\binom{n}{k} = \dfrac{n!}{k!\,(n-k)!}$

Two identities

(1) $\binom{n}{k} = \binom{n}{n-k}$

(2) $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

Pascal's Triangle

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

PASCAL'S TRIANGLE

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

5 10 10 5 1

By defn, each number is the sum of two previous.

$$\binom{0}{0}=1$$

$$\binom{1}{0}=1 \quad \binom{1}{1}=1$$

$$\binom{2}{0}=1 \quad \binom{2}{1}=2 \quad \binom{2}{2}=1$$

$$\binom{3}{0}=1 \quad \binom{3}{1}=3 \quad \binom{3}{2}=3 \quad \binom{3}{3}=1$$

$\vdots$

Reason that they are the same

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

	sum
1	1
1 1	2
1 2 1	4
1 3 3 1	8
1 4 6 4 1	16
1 5 10 10 5 1	32

FACT: $\sum_{k=0}^{n} \binom{n}{k} = 2^{k}$

Pf: Consider the power set

$$P(\{1,\dots,n\}) = \bigcup_{k=0}^{n} \{\text{subsets } A \subset \{1,\dots,n\} \text{ of size } k\}$$

by defn = set of all subsets

This defines a partition of $P(\{1,\dots,n\})$

That is, it is a disjoint union

$\Rightarrow$ The left hand side has the same number of elements as right side

$$\Rightarrow \quad 2^n = \sum_{k=0}^{n} \binom{n}{k}$$

$\#P(\{1,\dots,n\})$ $\qquad$ $\#\{A \subset \{1,\dots,n\} \text{ of size } k\}$

Consider taking $(x+y)^n$

$n=0 \quad (x+y)^0 = 1$

$(x+y)^1 \quad 1\cdot x + 1\cdot y$

$(x+y)^2 \quad 1x^2 + 2xy + 1y^2$

$(x+y)^3 \quad x^3 + 3x^2y + 3xy^2 + y^3$

$(x+y)^4 \quad x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$

FACT $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$

COR OF FACT: Take $x=y=1$

$$2^n = \sum_{k=0}^{n} \binom{n}{k}$$

so we give a 2nd proof of this.

PF 1 (by induction) $n=0,1$: ✓

Assume true for n.

by inductive hypothesis

$$(x+y)^{n+1} = (x+y)^n (x+y)$$

$$= \left(\sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k\right)(x+y)$$

$$= \sum_{k=0}^{n} \binom{n}{k} x^{n-k+1} y^k + \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^{k+1}$$

$$= \sum_{j=0}^{n+1} \underbrace{\left(\binom{n}{j} + \binom{n}{j-1}\right)}_{\binom{n+1}{j}} x^{n+1-j} y^j$$

$$= \sum_{j=0}^{n+1} \binom{n+1}{j} x^{n+1-j} y^j \quad ✓$$

PF 2 (by counting)

$$(x+y)^n = \underbrace{(x+y)(x+y)(x+y)(x+y)\cdots(x+y)}_{n \text{ times}}$$

$$= \sum_{k=0}^{n} ? \; x^{n-k} y^k$$

To get a term $x^{n-k}y^k$, need to choose y k's & $n-k$ x's.

$\rightsquigarrow$ # of ways $= \binom{n}{k}$

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k \quad ✓$$