Math 300, Monday 5/16

Defn: A set S is countable if either S is finite or $S \approx \mathbb{N}$.

↳ A set S is uncountable if S is not countable

↳ A set S is countably infinite if S is countable and not finite

Last time:

↳ $\mathbb{Z}$ countable

FACTS

① $\left.\begin{array}{l} A \text{ finite} \\ B \text{ countable} \end{array}\right\} \Rightarrow A \cup B$ countable

② $\left.\begin{array}{l} B \text{ countable} \\ A \subseteq B \end{array}\right\} \Rightarrow A$ countable

③ A, B countable $\Rightarrow A \cup B$ countable

Example: $\mathbb{Z} = \mathbb{Z}_- \cup \{0\} \cup \mathbb{Z}_+$

bijective to $\mathbb{N}$

Facts ① and ③ $\Rightarrow$ $\mathbb{Z}$ countable

④ A, B countable $\Rightarrow A \times B$ countable

Example: $\mathbb{N} \times \mathbb{N}$ is countable

Reason: Fact 4

Example $\mathbb{Q}$ is countable

Pf: $\mathbb{Q} = \mathbb{Q}_- \cup \{0\} \cup \mathbb{Q}_+$

neg. positives

Suffices to show $\mathbb{Q}_+$ countable because $\mathbb{Q}_- \approx \mathbb{Q}_+$ and fact ① & ③

<u>Goal</u>: $\mathbb{Q}_+$ is countable

$\mathbb{Q}_+ = \left\{ \frac{a}{b} \;\middle|\; a, b \in \mathbb{N} \text{ s.t. } a \text{ and } b \text{ have no common divisors} \right\}$

$\frac{1}{3} = \frac{2}{6} = \frac{3}{9}$

Define

$f : \mathbb{Q}_+ \to \mathbb{N} \times \mathbb{N}$ injection.

$\frac{a}{b} \mapsto (a, b)$

$\Rightarrow \mathbb{Q}_+ \subseteq \underbrace{\mathbb{N} \times \mathbb{N}}_{\text{countable}}$

By Fact 2, $\mathbb{Q}_+$ countable

Since $\mathbb{Q}_+$ is infinite

$\Rightarrow \mathbb{Q}_+ \simeq \mathbb{N}$ $\blacksquare$

<u>Defn</u> Given sets A and B, we say $\underline{\#A \leq \#B}$ if $\exists f : A \to B$ injection.

↳ Say $\#A = \#B$ or $A \approx B$ if $\exists f : A \to B$ bijection

<u>Properties</u>

1) $\#A \leq \#B$, $\#B \leq \#C$

$\Rightarrow \#A \leq \#C$

<u>Reason</u>: $A \xrightarrow[\text{injection}]{f} B \xrightarrow[\text{injection}]{g} C$

$\Rightarrow g \circ f : A \to C$ injection ✓

Defn Given sets A and B, we say $\#A \le \#B$ if $\exists f : A \to B$ injection.

Fact ① $\#A \le \#B$, $\#B \le \#C$

$\Rightarrow \#A \le \#C$

Schröder–Bernstein Thm

If $\#A \le \#B$ and $\#B \le \#A$, then $\#A = \#B$.

Surprising fact:

It says that if

$$\left.\begin{array}{l}\exists \text{ injections } A \to B \\ \text{and } \quad B \to A\end{array}\right\} \Rightarrow \exists \text{ bijection } A \to B$$

Skip the proof

Example $(0,1) \approx [0,1]$

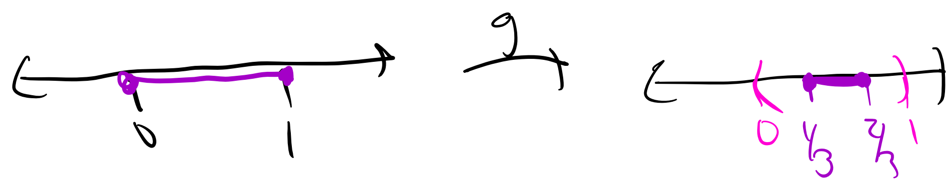
$\{x \in \mathbb{R} \mid 0 < x < 1\}$ $\quad$ $\{x \in \mathbb{R} \mid 0 \le x \le 1\}$

Injection 1:

$$f : (0,1) \to [0,1]$$
$$x \mapsto x$$

Injection 2

$$g : [0,1] \to (0,1) \quad \text{injection}$$
$$x \mapsto \frac{x}{3} + \frac{1}{3}$$



Schröder–Bernstein

$\Rightarrow (0,1) \approx [0,1]$

Cantor's Thm #1

For any set S, $\#S < \#P(S)$

Defn: For sets A and B, we say $\#A < \#B$ if

(1) $\#A \leq \#B$

(2) $\#A \not\approx \#B$ (i.e. $\nexists$ bijection $A \to B$)

Recall also

$$P(S) = \{A \subseteq S \text{ subset}\}$$

Rmk: Easy if S is finite

Reason $\#P(S) = 2^{\#S}$

and $2^{\#S} > \#S$

Pf: We can assume $S \neq \emptyset$.

(If $S = \emptyset$, $P(S) = \{\emptyset\} \not\approx S$)

(1) Show $\#S \leq \#P(S)$

Define $f : S \longrightarrow P(S)$ injection ✓

$x \longmapsto \{x\}$

(2) Show $\#S \not\approx \#P(S)$

Pf by contradiction: assume

$\exists\, g : S \longrightarrow P(S)$ bijection.

Consider

$$E = \{x \in S \mid \underset{\text{element of } S}{x} \notin \underset{\text{subset of } S}{g(x)}\} \subseteq S$$

$\Rightarrow E \in P(S)$

Since g is a bijection, $\exists\, z \in S$ s.t. $E = g(z)$.

Ques: Is $z \in E$?

$z \in E \iff z \notin g(z) = E$, contradiction