MATH 300 HW 8 Key

4.2.5.a. Since $A - B \subset A$ we define the injective function $f : A - B \to A$ by $f(a) = a$. Thus $\#(A - B) \leq \#A$.

b. Suppose $\#A \leq \#B$. Then there exists an injection $f : A \to B$. We define an injective function $f_C : A \times C \to B \times C$ by $f_C(a, c) = f_C(f(a), c)$.

c. Not necessarily true. Let $A = \mathbb{R}$ and $B = \emptyset$. Then $A - B = \mathbb{R}$ and there does not exist any functions $f : \mathbb{R} \to \emptyset$ and thus it is not true that $\#(A - B) \leq \#B$.

4.3.10. Let F be the family of intervals. We pick a random interval to denote as I'. We define a surjection $f : \mathbb{Q} \to F$ by $f(q) = I'$ if $q \notin \cup_{I \in F} I$ and $f(q) = I_q$ otherwise, where I_q is the unique interval in F containing q. Since every interval in F contains at least one rational number, this must be a surjection. It follows that $\#F \leq \#\mathbb{Q} = \#\mathbb{N}$ and thus F is countable.

Common pitfalls to avoid for this proof: We cannot simply count our intervals going left to right. For example, we might have a family

$$F = \{[\frac{1}{2n+1}, \frac{1}{2n}] \ : \ n \in \mathbb{N}\} \cup \{[1, 2], [-2, -1]\}$$

in which case the majority of straightforward counting schemes would fail since there are an infinite number of intervals squeezed into a compact space.

Alternatively, we could have defined an injection $f : F \to \mathbb{Q}$. This would work fine, but we have to be careful that this function is well-defined; that is, that there is a clearly defined and unique output for each input. This could be as simple as saying: "For each interval I we pick a rational number it contains and denote it q_I. We then define $f : F \to \mathbb{Q}$ by $f(I) = q_I$."

4.3.11. We have $g : \mathbb{N} \times \mathbb{N}$ by $g(m, n) = 2^{m-1}(2n - 1)$. We first prove this is an injection. Suppose $g(a, b) = g(c, d)$. Then

$$2^{a-1}(2b - 1) = 2^{c-1}(2d - 1)$$

Since $2b - 1$ and $2d - 1$ are both odd, it follows by the Fundamental Theorem of Arithmetic that $a = c$. We then divide both sides by 2^{a-1} and get

$$(2b - 1) = (2d - 1)$$

and therefore $b = d$ as well. Thus g is injective.

We now prove surjectivity. Let $n \in NN$ have prime factorization $n = 2^{a_0} p_1^{a_1} \ldots p_k^{a_k}$. Then $p_1^{a_1} \ldots p_k^{a_k}$ is odd so we can write it as $p_1^{a_1} \ldots p_k^{a_k} = 2b - 1$ for some unique b. Then $g(a_0 + 1, b) = n$, so g is surjective. Then g is a bijection so the statement follows.

4.3.12.a. We know that $\sum_{i=1}^{\infty} \frac{1}{2^i} = 1$ and so we can conclude that the "total lengths" is 1 and so $\mathbb{Q}$ is geometrically less than or equal to the size of an integral of length 1.

b. Let $\epsilon > 0$. We know that $\sum_{i=1}^{\infty} \frac{\epsilon}{2^i} = \epsilon$ and so we can conclude that the "total lengths" is ϵ and so $\mathbb{Q}$ is geometrically less than or equal to the size of an integral of length ϵ. Since this holds for all $\epsilon > 0$, we can conclude that $\mathbb{Q}$ is smaller than any interval of nonzero length, or in other words, has length "0".

c. Since these intervals will overlap, the "total" length we calculate is actually an overestimate.

4.3.14. We define a circle by $C = \{(x, y) \ : \ x^2 + y^2 = \sqrt{2}\}$. Let $x, y \in \mathbb{Q}$. Then $x^2 + y^2 \in \mathbb{Q}$ and so $x^2 + y^2 \neq \sqrt{2}$, since $\sqrt{2} \notin \mathbb{Q}$, and thus $(x, y) \notin C$.

4.3.15. We define a line

$$L = \{(x, y) \ : \ y = -x + \sqrt{2}\} = \{(x, y) \ : \ y + x = \sqrt{2}\}$$

Suppose $x, y \in \mathbb{Q}$. Then $x + y \in \mathbb{Q}$ and, since $\sqrt{2} \notin \mathbb{Q}$, we must have $x + y \neq \sqrt{2}$ and thus $(x, y) \notin L$.

7. Let n be a positive integer. We begin by observing that

$$n^2 + 1 = n^2 + 2n + 1 - 2n = (n+1)^2 - 2n$$

Suppose $n + 1 | n^2 + 1$. Then we can write

$$(n+1)^2 - 2n = (n+1)(n + 1 - \frac{2n}{n+1})$$

and therefore $n+1|2n$. Therefore there exists $k \in \mathbb{N}$ such that $(n+1)k = 2n$. This implies $kn + k = 2n$ and as such we must have $k < 2$. It follows that $k = 1$. This leaves us with $n + 1 = 2n$, so we conclude that $n = 1$. To summarize, $n + 1|n^2 + 1 \rightarrow n = 1$.

To conclude the proof, we examine the case $n = 1$. In this case we have $n+1 = 2$ and $n^2+1 = 2$ and so the property does indeed hold. We conclude that $n + 1|n^2 + 1$ if and only if $n = 1$.

8. We wish to find all positive integers n not equal to 3 such that $n - 3|n^3 - 3$. We begin by factoring

$$n^3 - 3 = (n-3)(n^2 + 3n + 9) + 24$$

and therefore $n-3$ divides n^3-3 if and only if $n-3$ divides 24. When we exclude $n = 3$ we find that this is equivalent to saying $n \in \{1, 2, 4, 5, 6, 7, 9, 11, 15, 27\}$.