MATH 300 - Introduction to Mathematical Reasoning, Spring 2022 - Homework 7 Solution

Written by Jonathan Andrés Niño Cortés

Section 4.1 Q16

Solution:

(a) Since there are three digits each with two possible choices the total number of codewords is $2 \times 2 \times 2 = 8$.

(b) Suppose by contradiction that such a set of five codewords exists. Consider the first digit of the codewords. There are two possible values and 5 codewords so by the generalized pigeonhole principle there must be at least three digits with the same value.

Now, consider the subset formed by these three codewords. Using the pigeonhole principle we conclude that 2 of them must coincide on their second digit as well. But then this two codewords have the same first and second digit. Their distance is at most one and this contradicts our assumption on the minimum distance in these codewords.

Section 4.1 Q21

Solution:

(a) If $\#A = m$ and $\#B = n$ then there is a correspondence between functions from B to A and $n-$tuples of elements of A.

Therefore, $A^B \approx \underbrace{A \times \ldots \times A}_{n \text{ times}}$.

By Theorem 4.21 the cardinality of this set is m^n.

(b) For an injection to exists we must have $\#B \leq \#A$. Therefore, when $n > m$ this number is zero.

When $n \leq m$, the image of an injection is a subset of A that has the same cardinality as B. Now we will use the **Product rule** extensively.

We know that there are $\frac{m!}{n!(m-n)!}$ possible choices . Now, let $b_1, \ldots, b_n$ be a enumeration of the elements of B. For a fixed set S of size n on A, we must choose an element a_{i_1} such that $f(b_1) = a_{i_1}$. There are n possible choices. Next for $f(b_2)$ we must choose an element in $S - \{a_{i_1}\}$ and we denote this choice by a_{i_2}. Recursively for a_{i_k} we must choose an element in $S - \{a_{i_1, \ldots a_{i_{k-1}}}\}$ so there are $n - k + 1$.

By the product rule the total number of injections is

$$\frac{m!}{n!(m-n)!} \times n \times n-1 \times \ldots \times 2 \times 1 = \frac{m!}{(m-n)!}.$$

(c)

$$(A \times B)^C \approx A^C \times B^C.$$

We show that there is a bijection between these sets. The elements on the left are functions $f : C \to A \times B$. While the elements on the right consists of pairs of functions (g, h) such that $g : C \to A$ and $h : C \to B$.

Given a function f as above we can construct a pair of functions (g, h) as follows. Suppose that for $c \in C$, $f(c) = (a, b)$. Then define $g(c) = a$ and $h(c) = b$.

This construction can be reversed. If you have a pair (g, h) then define $f(c) = (g(c), h(c))$.

In other words the function is invertible. Therefore, it gives a bijection between this two sets.

Section 4.1 Q22

Solution:

(a) Let $a_1, \ldots, a_n$ be a set of elements of A such that $A = \bigsqcup_{i=1}^{n} [a_i]$. Suppose by contradiction that all $[a_i]$ are finite. Then by Corollary 4.15 $\#A = \sum_{i=1}^{n} \#[a_i] < \infty$. But this contradicts the fact that A is not finite. Therefore, at least one of the $[a_i]$ must be infinite.

(b) There were multiple examples on previous homeworks of relations on infinite sets such that each class . To give one was one.

For instance, let $A = \mathbb{R}$ and consider the relation $x \sim y$ if $x^2 = y^2$. Then all equivalence classes have at most two elements with the exception of $[0]$ which just has one.

The reason why this does not contradict part (a) is because there are infinite equivalence classes.

Section 4.2 Q2

Solution: This statement is false. Consider the case where $A = B = \mathbb{Z}$ and $f(a) = g(a) = 2a$. This functions are injective but not surjective. Yet clearly, A and B have the same cardinality.

Section 4.2 Q4

Solution:

(a) Consider the sequence $\{1/2, 1/3, 1/4, \ldots, \}$.

We can write (0,1) as $(0,1) - \{1/2, 1/3, 1/4, \ldots, \} \cup \{1/2, 1/3, 1/4, \ldots, \}$ and [0,1] as $[0,1] - \{0, 1, 1/2, 1/3, 1/4, \ldots, \} \cup \{0, 11/2, 1/3, 1/4, \ldots, \}$

Now observe that $(0,1) - \{1/2, 1/3, 1/4, \ldots, \} = [0,1] - \{0, 1, 1/2, 1/3, 1/4, \ldots, \}$. This is the "otherwise" in the definition of h and we see that when restricting to this set h is just the identity.

Therefore, we only need to show that h restricted to $\{0, 11/2, 1/3, 1/4, \ldots, \}$ gives a bijection with $\{1/2, 1/3, 1/4, \ldots, \}$.

But this is the case because h is shifting the elements in this sequence by two and then mapping 0 and 1 to $1/2$ and $1/3$ respectively.

(b) If we consider the sequence $\frac{1}{n}$ what the function is doing is shifting the elements two places to the right and filling the two resulting spaces with 0 and 1.

Section 4.2 Q6

Solution: Since $\#A \leq \#B$ there exists an injective function $f : A \hookrightarrow B$. Now define the function $f' : P(A) \to P(B)$ in the following way.

For every $S \in P(A)$ let $f'(S) = \{f(x) : x \in S\}$. Then we can show that f' is injective. Take $S, T \in P(A)$ such that $f'(S) = f'(T)$. By this set equality we can do the following argument.

$$\begin{aligned} x \in S &\implies f(x) \in f'(S) \\ &\implies f(x) \in f'(T) \\ &\implies \exists y \in T : f(x) = f(y) \\ \text{(injectivity of } f) &\implies x = y \\ &\implies x \in T \end{aligned}$$

By symmetry (i.e. replacing S by T) we get the converse. Therefore, $S = T$ and we conclude that f' is injective.

Section 4.2 Q8

Solution:

(a) Suppose that A is a set such that $\#A = n$. We know that there is a correspondence between subsets of A and indicator functions. $f : A \to \{0, 1\}$. By Exercise 2.21.a we conclude that the cardinality of $P(A) = 2^n$.

Finally, by Cantor's Theorem we know that for any set $n = \#A < \#P(A) = 2^n$.

(b) We prove this by induction. First observe that the statement is true for $n = 0$ since $0 < 2^0 = 1$. Now, suppose that the statement is true for k that is $k < 2^k$. Since $k \geq 1$ we have $k + 1 \leq k + k = 2k$. On the other hand $2k < 2 \cdot 2^k = 2^{k+1}$. Therefore, $k + 1 \leq 2k < 2^{k+1}$.

Section 4.3 Q2

Solution: To create a countably infinite amount of the space in the hotel, ask the current resident of room n to move to the $2n$ position. This effectively frees all odd numbered rooms and now there is space available for the new hosts.

Section 4.3 Q4

Solution: This is true. A simple argument is that $A - B \subseteq A$. Therefore there exists an injection $i : A - B \to A$ given by set inclusion. Therefore, $\#(A - B) \leq \#A$. It follows that if A is countable then $(A - B)$ must be countable as well.

Section 4.3 Q7

Solution:

(a) We can write
$$\mathbb{N} \times \mathbb{N} = \bigcup_{i \in \mathbb{N}} \{i\} \times \mathbb{N}.$$
Each $\{i\} \times \mathbb{N}$ is in bijection with $\mathbb{N}$ (see Exercise 4.1.4d). Thus we have decompose this as a countably infinite union of countably infinite sets.

(b) Let $A = \bigcup_{i \in \mathbb{N}} A_i$ where $A_i \approx \mathbb{N}$.

Let f_i be the bijection between$\mathbb{N}$ and A_i. Then we can construct a bijection g between $\mathbb{N} \times \mathbb{N}$ and A. Define $g(i, j) = f_i(j)$.

Then we can see that $\{i\} \times \mathbb{N} \approx A_i$ and therefore $\mathbb{N} \times \mathbb{N} \approx A$.