MATH 300 HW 6 Key

3.2.13.a Because $|a_i - b_i| \geq 0$ for all i by definition of absolute value, we have $d(s_1, s_2) = \sum_{i=1}^{n} |a_i - b_i| \geq 0$ because it is a sum of non-negative terms.

b. We have $d(s_1, s_2) = 0 \leftrightarrow |a_i - b_i| = 0$ for all $i \leftrightarrow a_i = b_i$ for all $i \leftrightarrow s_1 = s_2$.

c. We have

$$d(s_1, s_2) = \sum_{i=1}^{n} |a_i - b_i| = \sum_{i=1}^{n} |b_i - a_i| = d(s_2, s_1)$$

d. We will begin with the base case $n = 1$. If $d(s_1, s_3) = 0$ then the inequality $0 \leq d(s_1, s_2) + d(s_2, s_3)$ immediately follows by part a. If $d(s_1, s_3) = 1$ then $s_1 \neq s_3$. It follows that we cannot have both $s_1 = s_2$ and $s_2 = s_3$ and so either $d(s_1, s_2) = 1$ or $d(s_2, s_3) = 1$ and the inequality holds. This proves the base case.

We then assume that the statement holds for strings of some length $k \geq 1$ and let s_1, s_2, s_3 be strings of length $k + 1$. We let s_i' be the strings s_i with last character removed. Then

$$\begin{aligned} d(s_1, s_3) &= \sum_{i=1}^{k+1} |a_i - c_i| = \sum_{i=1}^{k} |a_i - c_i| + |a_{k+1} - c_{k+1}| = d(s_1', s_3') + d(a_{k+1}, c_{k+1}) \\ &\leq d(s_1', s_2') + d(s_2', s_3') + d(a_{k+1}, b_{k+1}) + d(b_{k+1}, c_{k+1}) = d(s_1, s_2) + d(s_2, s_3) \end{aligned}$$

which finishes the proof.

3.2.16.a. We define $a_1 = 1$ and $a_n = a_{n-1} + 2$ for $n > 1$.
b. We define $a_1 = 1$, $a_2 = 1$, and $a_n = a_{n-2} + 2$ for $n > 2$.
c. We define $a_1 = 2$ and $a_n = a_{n-1} + (n - 1)$ for $n > 1$.
d. We define $a_1 = 1$, $a_2 = 2$, and $a_n = a_{n-1} + a_{n-2}$ for $n > 2$.

3.3.2.a. $f = g \circ h$ where $h(x) = x^2 + \cos(x)$ and $g(x) = x^3$.
b. $f = g \circ h$ where $h(x) = x^5 - 7x$ and $g(x) = x + 1$.
c. $f = g \circ h$ where $h(x) = 2x$ and $g(x) = \frac{x^2}{4} 10^x$.
d. $f = g \circ h$ where $h(x) = x - 1$ and $g(x) = \frac{5}{(x+1)\sqrt{x}}$.

3.2.7.a. We will prove via the contrapositive. Suppose f is not injective. Then there exists $x \neq y \in A$ such that $f(x) = f(y) = a$ for some $a \in B$. Then $(g \circ f)(x) = g(a) = (g \circ f)(y)$ and so $g \circ f$ is not injective.

b. Suppose $g \circ f$ is surjective. Then for all $c \in C$ there exists $a \in A$ such that $(g \circ f)(a) = c$. Then $f(a) = b \in B$ and so $g(b) = c$. It follows that g is surjective.

3.2.8.a. Let $A = \{1\}$ and $B = C = \{1, 2, 3\}$. We define $f : A \to B$ by $f(a) = 1$ and $g : B \to C$ by $g(b) = 1$. Then $g(1) = g(2)$ and so g is not injective but $g \circ f$ is injective because the domain is a singleton.

b. Let $A = C = \{1\}$ and $B = \mathbb{R}$ and define $f : A \to B$ by $f(x) = 0$ and define $g : B \to C$ by $g(x) = 1$. Then $g \circ f$ is surjective because the codomain is a singleton but f is not surjective.

4.1.2. Let L and M be lines. Let $(a_1, a_2) \neq (b_2, b_2) \in L$ and $(c_1, c_2) \neq (d_2, d_2) \in M$. We then define 'direction vectors' $(l_1, l_2) = (b_1 - a_1, b_2 - a_2)$ and $(m_1, m_2) = (d_1 - c_1, d_2 - c_2)$ for L and M respectively.

We then define functions $f : \mathbb{R} \to L$ and $g : \mathbb{R} \to M$ by

$$f(t) = (a_1, a_2) + t(l_1, l_2) \qquad g(t) = (c_1, c_2) + t(m_1, m_2)$$

Since these are each bijections, we have $L \approx \mathbb{R}$ and $M \approx \mathbb{R}$ and so $L \approx M$.

4.1.4.a. We define a bijection $f : A \times B \to B \times A$ by $f(a, b) = (b, a)$.

b. If $A \approx C$ and $B \approx D$ then there exists bijections $f : A \to C$ and $g : B \to D$. We define a bijection $h : A \times B \to C \times D$ by $h(a, b) = (f(a), g(b))$.

c. We define a bijection $f : (A \times B) \times C \to A \times (B \times C)$ by

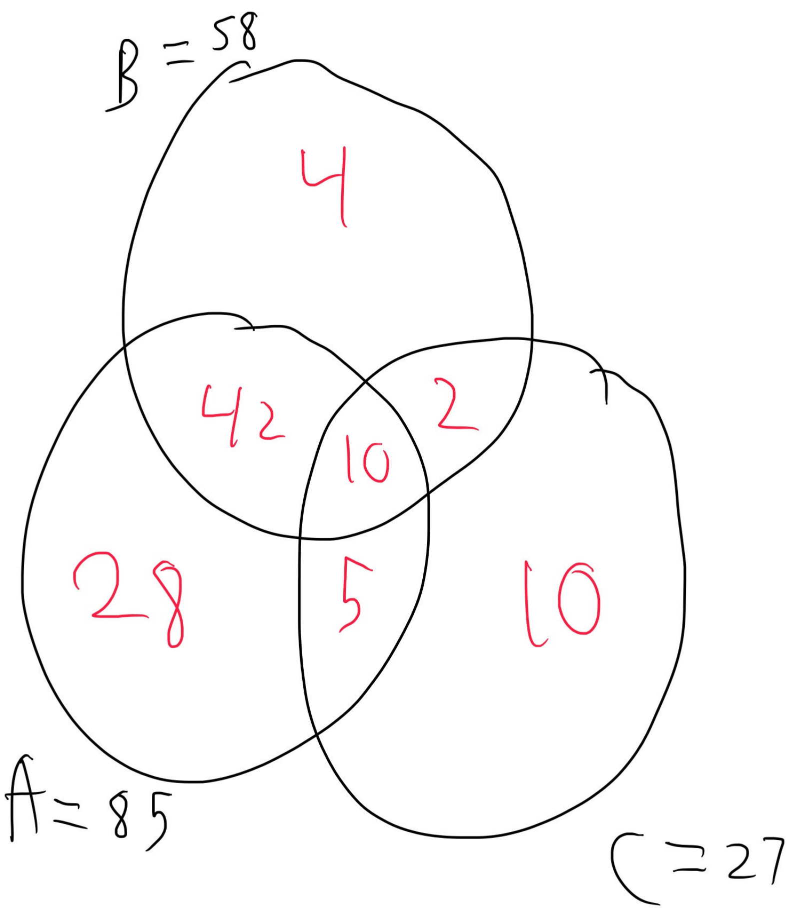
$$f((a, b), c) = (a, (b, c))$$

d. We define a bijection $f : \{w\} \times A$ by $f(w, a) = a$.

4.1.6. We define a bijection $f : [-2, 8] \to [3, 5]$ by $f(x) = \frac{x+17}{5}$.

4.1.9. We have

$$\#(A \cup B \cup C) = \#A + \#B + \#C - \#(A \cap B) - \#(A \cap C) - \#(B \cap C) + \#(A \cap B \cap C)$$



4.1.10.
a. 14 people.
b. 12 people.
c. 10 people.