MATH 300 - Introduction to Mathematical Reasoning, Spring 2022 - Homework 5 Solution

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Section 2.10 Q2

Solution:

Proof. We prove by the induction. For the basis step observe that when $n = 1$,

$$1^3 = \left(\frac{1 * (1+1)}{2}\right)^2 = 1.$$

For the induction step assume that

$$1^3 + 2^3 + \cdots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$$

Then we have

$$\begin{aligned}
1^3 + 2^3 + \cdots + n^3 + (n+1)^3 &= \left(\frac{n(n+1)}{2}\right)^2 + (n+1)^3 \\
&= [n+1]^2 \left(\frac{n^2}{4} + (n+1)\right) \\
&= [n+1]^2 \left(\frac{n^2+4n+4}{4}\right) \\
&= [n+1]^2 \left(\frac{n+2}{2}\right)^2 \\
&= \left(\frac{(n+1)(n+2)}{2}\right)^2.
\end{aligned}$$

□

Section 2.10 Q4

Solution:

Proof. We use an induction argument. When $n = 0$, $n^4 - 4n^2 = 0$ is divisible by 3. Now suppose that $n^4 - 4n^2$ is divisible by 3 and consider $(n+1)^4 - 4(n+1)^2$.

We have

$$\begin{aligned}(n+1)^4 - 4(n+1)^2 &= n^4 + 4n^3 + 6n^2 + 4n + 1 - 4n^2 - 8n - 4 \\ &= n^4 - 4n^2 + 4n^3 + 6n^2 - 4n - 3\end{aligned}$$

□

Section 2.10 Q6

Solution:

Proof. Basis step: $(n = 2)$. $\frac{1}{1\cdot 2} = \frac{1}{2}$.

Induction step.

Suppose that $\dfrac{1}{1\cdot 2} + \dfrac{1}{2\cdot 3} + \cdots + \dfrac{1}{(n-1)n} = 1 - \dfrac{1}{n}$.

Then we have

$$\begin{aligned}\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{(n-1)n} + \frac{1}{n(n+1)} &= 1 - \frac{1}{n} + \frac{1}{n(n+1)} \\ &= 1 - \frac{1}{n} + \frac{1}{n} - \frac{1}{n+1} \\ &= 1 - \frac{1}{n+1}\end{aligned}$$

They key step is the identity

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}.$$

This can be derived by the method of partial fractions or also just by carrying out the sum of fractions on the right. □

Section 2.10 Q8

Solution: Although it may seem true at first glance there is one major flaw. $P(1) \implies P(2)$ does not hold. What's the problem? The first k elements in this case is just the first one, while the last k elements is just the last one. These two subsets are disjoint! There is no overlap between the sets and therefore no reason why the first horse should have the same color as the last one.

Moreover, $P(2)$ is false. Just take a set of two horses with different colors.

Section 3.1 Q4

Solution:

(a) $f(x) = 1, \quad g(x) = \dfrac{x-5}{x-5}$.

The domain of f is $\mathbb{R}$. The domain of g however is $\mathbb{R} - \{5\}$. Since the domains are different the functions are different. It is worth noting that f restricted to the domain of g is equal to g.

(b) $f(x) = \sqrt{x}, \quad g(x) = \sqrt{|x|}$.

The domain of f is $\{x \in \mathbb{R} : x \geq 0\}$. On the other hand g is defined for all $\mathbb{R}$. Once again the functions are different.

(c) $f(x) = |x|, \quad g(x) = \sqrt{x^2}$.

This time the functions agree on domain ($\mathbb{R}$) and values. Therefore, they are the same.

(d) $f(x) = x^2 - x - 6, \quad g(x) = (x-4)(x+3) + 6$.

Both of these functions are polynomials defined on the entire set $\mathbb{R}$. By expanding out g we can check that it is equal to f.

(e) $f(x) = x^2, \quad g(x) = \begin{cases} x^2 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$.

Both functions are defined on the real line. However, they differ on any irrational point. For instance, $f(\sqrt{2}) = 2$ while $g(\sqrt{2}) = 0$.

Comments about grading. It is true that f is continuous while g is not, and therefore they cannot be equal. However, continuity is a high level concept and at this point we are interested in establishing more elemental properties about functions. This is why I took some points off for people that used continuity as an argument without proving that g is not continuous (or that f is continuous for that matter). In fact, whether these functions are continuous or not depends on the "topology" that you choose for $\mathbb{R}$. You will hear more about this if you continue on your math journey.

Side question on that note: Can you prove that $\sqrt{2}$ is irrational?

Section 3.1 Q6

Solution:

(a) **Reflexive**: $f(a) = f(a)$.

Symmetric: $f(a) = f(b) \iff f(b) = f(a)$.

Transitive: If $f(a) = f(b)$ and $f(b) = f(c)$ then $f(a) = f(b) = f(c)$.

(b) If x is a car of a given color then $[x]$ is the set of all car with that same color.

(c) $A = B = \mathbb{R}$, $f(x) = x^2$: $[x] = \{-x, x\}$.

(d) $A = B = \mathbb{R}$, $f(x) = |x|$: $[x] = \{-x, x\}$.

(e) $A = \mathbb{R} \times \mathbb{R}$, $B = P(\mathbb{R})$ (the *power set* of $\mathbb{R}$), $f(x, y) = \{x, y\}$: $[(x, y)] = \{(x, y), (y, x)\}$.

(f) $A = \mathbb{R} \times \mathbb{R}$, $B = \mathbb{R}$, $f(x, y) = x + y$: $[(x, y)] = \{(x', y') \in \mathbb{R} \times \mathbb{R} : x' + y' = x + y\}$. This is a line of slope -1 passing through the point (x, y).

(g) $A = \mathbb{R} \times \mathbb{R}$, $B = \mathbb{R}$, $f(x, y) = x^2 + y^2$: $[(x, y)] = \{(x', y') \in \mathbb{R} \times \mathbb{R} : (x')^2 + (y')^2 = x^2 + y^2\}$. This is a circle centered at the origin and with radius equal to $\sqrt{x^2 + y^2}$.

Section 3.1 Q7

Solution:

(a) $\chi_{\emptyset}(s) = 0$ for all $s \in S$.

$\chi_S(s) = 1$ for all $s \in S$.

(b) $\chi_{A'}(s) = 1 - \chi_A(s)$. This follows because for every $s \in S'$ it is either in A or A' and it cannot be in both at the same time.

(c) *Proof.* Observe that χ_A and χ_B have the same domain which is S, and the same codomain $\{0, 1\}$. Now suppose that $A = B$, then for every $s \in S$, $s \in A \iff s \in B$ and equivalently $s \notin A \iff s \notin B$. Then, $\chi_A(s) = 1 \iff \chi_B(s) = 1$ and $\chi_A(s) = 0 \iff \chi_B(s) = 0$. Since these are the only two values in the codomain this shows that $\chi_A = \chi_B$.

Now suppose that $\chi_A = \chi_B$, then $\chi_A(s) = 1 \iff \chi_B(s) = 1$. But we also have the equivalences $s \in A \iff \chi_A(s) = 1$ and $s \in B \iff \chi_B(s) = 1$. By transitivy and symmetry of logical equivalences we conclude that $s \in A \iff s \in B$ and since this is for arbitrary $s \in S$ we conclude that $A = B$. $\square$

(d) *Proof.*

$$\begin{aligned}
\chi_{A\cap B}(s) = 1 &\iff s \in A \cap B \\
&\iff s \in A \wedge s \in B \\
&\iff \chi_A(s) = 1 \wedge \chi_B(s) = 1 \\
&\iff \chi_A(s)\chi_A(s) = 1
\end{aligned}$$

For the last equivalence observe that if any of the factors is zero then the whole product is zero. This forces both factors to be one.

Suppose that $s \in A \cup B$. We must work three cases $s \in A - B$, $s \in B - A$ or $s \in A \cap B$. Observe that by what we just prove we can replace $\chi_A(s)\chi_A(s)$ by $\chi_{A\cap B}(s)$.

So if $s \in A - B$ then we have

$$\chi_A(s) + \chi_B(s) - \chi_{A\cap B}(s) = 1 + 0 - 0 = 1$$

If $s \in B - A$ then we have

$$\chi_A(s) + \chi_B(s) - \chi_{A\cap B}(s) = 0 + 1 - 0 = 1$$

and finally if $s \in A \cap B$ we have

$$\chi_A(s) + \chi_B(s) - \chi_{A\cap B}(s) = 1 + 1 - 1 = 1$$

On the other hand if $s \notin A \cup B$ then it follows that $s \notin A$, $s \notin B$ and $s \notin A \cap B$. Therefore,

$$\chi_A(s) + \chi_B(s) - \chi_{A\cap B}(s) = 0 + 0 - 0 = 0$$

□

Section 3.2 Q4 Let S and T be sets with three elements and two elements, respectively. In each case state the answer and justify briefly.

Solution:

(a) How many functions are there from S to T?

For each point in the domain you have to choose one of the points in the codomain. The answer is 2^3.

(b) How many injections are there from S to T?

None. The reason is that there are more elements in S than in T. Then necessarily two elements in S would get mapped to the same element in T. This is sometimes refer to as the **Pigeonhole principle**.

(c) How many injections are there from T to S?

For the first element in T we have three options to choose from. Once this choice is made and we move on to the second element we have two options left since we cannot choose the same element of S as before. The number of injections is therefore $3 \cdot 2 = 6$.

(d) How many surjections are there from S to T?

One way we can come up with this is getting the number of functions that are not surjections. In this case is simple because there are only two elements. The only non surjective functions are the constants, i.e. the ones that send all of the elements of S to the same element in T. There are only two of those. Then the number of surjective functions is equal to the number of functions minus 2. That is $2^3 - 2 = 6$.

(e) How many surjections are there from T to S?

None. Since the number of elements in the domain is less than those in the codomain, for any function necessarily there is going to be an element in S that one get assign as the image of an element of T.

(f) Guess the answers to (a) and (b) if S has m elements and T has n elements.

The first one is n^m. The second one is $n!/(n-m)!$ if $m \leq n$ and 0 otherwise.

Side question: Thus this cover the case where S or T are the empty set?

Section 3.2 Q6

Solution:

(a) f is surjective and the restriction $f|_C$ is surjective.

Let $A = \{1,2\}$, $B = \{1\}$ and $C = \{1\}$ and take f equal to the constant function. That is $f(1) = 1$ and $f(2) = 1$.

(b) f is surjective but the restriction $f|_C$ is not surjective.

Let $A = \{1,2,3\}$, $B = \{1,2\}$ and $C = \{1\}$ and take f equal to the identity. That is $f(1) = 1$ and $f(2) = 2$.

(c) f is injective and the restriction $f|_C$ is injective.

Take the same example as (c).

(d) f is injective but the restriction $f|_C$ is not injective.

This is not possible. We can show this by checking that for any C, if f is injective then $f|_C$ is injective. Take two elements $a, b \in C$ and suppose that $f|_C(a) = f|_C(b)$. Then $f(a) = f(b)$ and since we are assuming that f is injective we conclude that $a = b$.

Section 3.2 Q10

Solution: The map f is not injective. To see this consider the line $L = \{(0,y) : y \in \mathbb{R}\}$. For any point $P = (0,y)$, the line segment OP is contained in L. Therefore $f(P) = L$ for all $P \in L \cap \Pi^*$.

There are many valid choices for S such that $f|_S$ is a bijection. For instance take the line $S = \{(x,1) : x \in \mathbb{R}\} \cup \{(1,0)\}$.

To see that this is surjective take any line L that passes through the origin. Any line can be defined as $\{(x,y) : ax + by = 0\}$, $(a,b) \neq (0,0)$. Then observe that if $a \neq 0$ then the point $P = (-b/a, 1)$ belongs to $L \cap S$. If $a = 0$ then $P = (1,0) \in L \cap S$. In both cases we get that $f|_S(P) = L$.

To see it is injective suppose that two points in S, (x,y) and (x',y') get map into the same line $L = \{(x,y) : ax + by = 0\}$. Then

$$ax + by = ax' + by' = 0$$

If $a = 0$ then the previous equation becomes $bx = by' = 0$ so that $y = y' = 0$. Then the only point in S whose y coordinate is zero is (1,0) so $(x,y) = (x',y') = (1,0)$.

If $a \neq 0$, then (1,0) does not satisfy the equation $ax + by = 0$. So to belong to S we must have $y = y' = 1$. But then

$$ax + b = ax' + b = 0$$

from which $x = x' = -b/a$ follows.