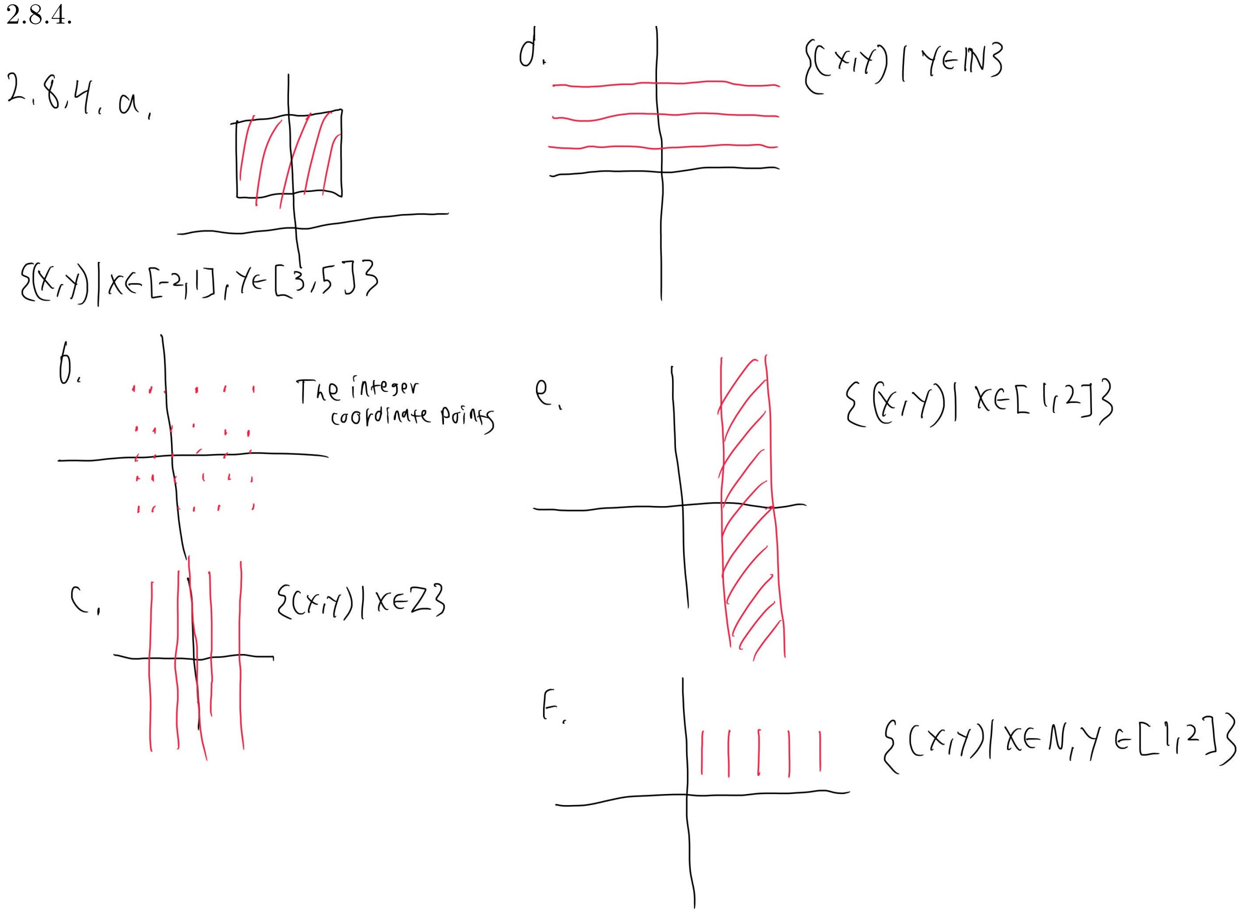
MATH 300 HW 4 Key

2.8.4.



2.8.9. Prove $(\cup_{i\in I}A_i) \times S = \cup_{i\in I}(A_i \times S)$.

We will first prove that $(\cup_{i\in I}A_i) \times S \subset \cup_{i\in I}(A_i \times S)$. Let $(a, b) \in (\cup_{i\in I}A_i) \times S$. Then $a \in \cup_{i\in I}A_i$ and $b \in S$. It follows that there exists $k \in I$ such that $a \in A_k$. Therefore $(a, b) \in A_k \times S$ and thus $(a, b) \in \cup_{i\in I}(A_i \times S)$. We conclude that $(\cup_{i\in I}A_i) \times S \subset \cup_{i\in I}(A_i \times S)$.

We will now prove that $(\cup_{i\in I}A_i) \times S \supset \cup_{i\in I}(A_i \times S)$. Let $(a, b) \in \cup_{i\in I}(A_i \times S)$. Then there exists $k \in I$ such that $(a, b \in A_k \times S$. It follows that $a \in A_k$ and $b \in S$. Therefore $a \in \cup_{i\in I}A_i$ and thus $(a, b) \in (\cup_{i\in I}A_i) \times S$. We conclude that $(\cup_{i\in I}A_i) \times S \supset \cup_{i\in I}(A_i \times S)$. We combine our two conclusions to obtain the desire equality $(\cup_{i\in I}A_i) \times S = \cup_{i\in I}(A_i \times S)$.

2.9.5.

a. We give the partitions

$$\begin{aligned}
\Pi_1 &= \{\{1\},\{2\},\{3\},\{4\},\{5\},\{6\}\}\\
\Pi_2 &= \{\{1,2\},\{3,4\},\{5\},\{6\}\}\\
\Pi_3 &= \{\{1,2\},\{3,4\},\{5,6\}\}\\
\Pi_4 &= \{\{1,2\},\{3,4,5,6\}\}
\end{aligned}$$

b. We give the partitions

$$\begin{aligned}
\Pi_1 &= \{\{1\},\{2\},\{3\},\{4\},\{n : n \in \mathbb{N} \text{ and } n > 4\}\}\\
\Pi_2 &= \{\{1,2\},\{3\},\{4\},\{n : n \in \mathbb{N} \text{ and } n > 4\}\}\\
\Pi_3 &= \{\{1,2,3,4\},\{n : n \in \mathbb{N} \text{ and } n > 4\}\}\\
\Pi_4 &= \{\mathbb{N}\}
\end{aligned}$$

c. For a set S, let Π be the partition consisting solely of singletons, i.e. Π consists solely of subsets containing a single element. Let Π' be any other partition of S. Let $A \in \Pi$ be any subset of S in the partition. By definition, $A = \{s\}$ for some $s \in S$. Since Π' is a partition of S, we have $\cup_{B \in \Pi'} B = S$. Therefore there exists $B \in \Pi'$ such that $s \in B$ and therefore $A = \{s\} \subset B$. We can then conclude that Π is finer than Π'. We conlcude that Π is the finest partition of S.

2.9.6.

a. $R = \{(1,2),(2,3)\}$ is not transitive, reflexive, or symmetric.

b. $R = \{(1,2),(2,3),(1,3),(1,1),(2,2),(3,3)\}$ is transitive and reflexive but not symmetric.

2.9.8. We have $s = \{1,2,3,4\}$ and $R = \{(1,2),(1,3),(2,3),(1,4),(4,1)\}$. The symmetric closure is $R' = \{(1,2),(2,1),(1,3),(3,1),(2,3),(3,2),(1,4),(4,1)\}$.

2.9.10.

a. R does not need to be symmetric, because there are some roads which are only one-way, indicating a non-symmetric relation. (Any answer with sufficient justification is fine here, due to ambiguity).

b. All cities for which it's possible to drive between would need to have a direct road constructed linking them together, if one does not already exist.

2.9.11.a.

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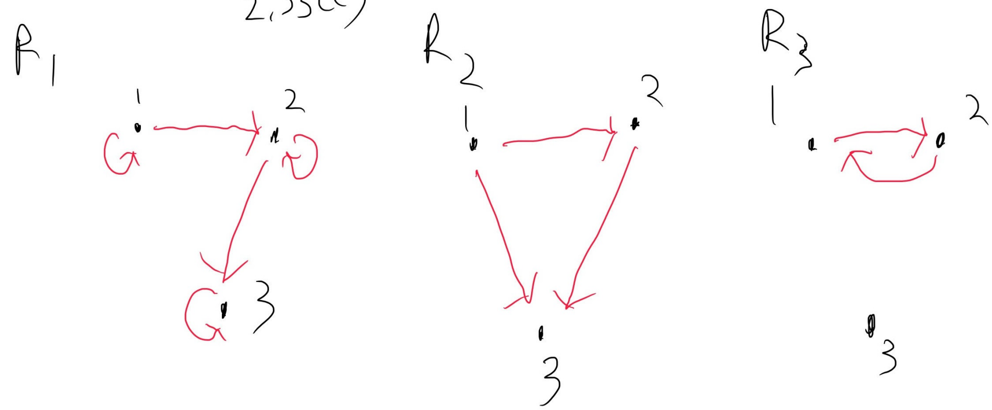
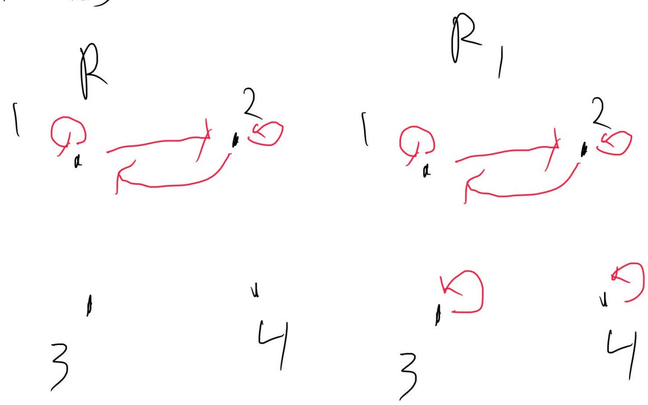
2.53(c)



2.56(b)



b. For every directed edge between vertices there is an edge between them pointing in the opposite direction as well.

c. There is a directed edge from each vertex to itself.

d. For each vertex there is a directed edge connecting it to each vertex it is possible to reach via following directed edges.

2.9.12.a. For every directed edge connecting two vertices, draw an identical edge between them with opposite direction if one does not already exist.

b. Draw a directed edge from each vertex to itself, if one does not already exist.

c. For each vertex, draw a directed edge from it to every vertex it can reach by only following connected edges in the indicated direction, assuming one does not already exist.

d. Erase the arrow on each directed edge and replace it with one pointing in the opposite direction.

2.9.16.a. Let $A = \{1, 2, 3\}$. Then it is neither true that $\{1\} \subset \{3\}$ nor that $\{3\} \subset \{1\}$ and so it is not a total ordering.

b. It is neither true that $5|7$ nor that $7|5$ and so the relation is not a total ordering.

c. Let $\bar{x} = (x_1, .., x_n)$ and $\bar{y} = (y_1, .., y_m)$ be words with ith letters x_i and y_i respectively, which x_i, y_i belong to the set of letters of the alphabet. Let R be the relation on the alphabet defined by xRy if and only if x comes before y alphabetically. Then we define the desired relation L by

$$\bar{x}L\bar{y} \iff (\exists k \leq n, m \text{ such that } x_kRy_k \text{ and } x_i = y_i \ \forall i < k) \text{ OR } (n < m \text{ and } x_i = y_i \ \forall i \leq n)$$

2.9.20.a. No. It is clearly not transitive.

b. Yes. It is transitive, reflexive, and symmetric. It partitions the set of solid colored cars into equal color subsets.

c. No. It is not symmetric e.g. $3|9$ but $9 \not| 3$.

d. Yes. It partitions the real numbers by absolute value.

e. No. It is not transitive.

f. Yes. It partitions $\mathbb{R} \times \mathbb{R}$ into horizontal lines.