SOME INCOMPLETE LECTURE NOTES ON BRIDGELAND STABILITY

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Contents

1. Lecture 1 (March 26): Stability conditions on abelian categories

This course is completely independent of the last quarter's course on the moduli of semistable vector bundles on a curve. But nevertheless, the moduli of semistable vector bundles serves as strong motivation for Bridgeland stability. For this reason, we will start with a quick recap of last quarter.

1.1. Quick recap.

Let C be a smooth, projective and connected curve over an algebraically closed field k.

Definition 1.1. A coherent sheaf F on C is *semistable* if for all subsheaves $F' \subset F$, we have that

$$\frac{\deg(F')}{\mathrm{rk}(F')} \leq \frac{\deg(F)}{\mathrm{rk}(F)}.$$

Remark 1.2. We call the ratio

$$\mu(F) := \frac{\deg(F)}{\mathrm{rk}(F)}$$

the *slope of* F. If F is torsion, $\mu(F) = +\infty$ and F is automatically semistable. If $\mathrm{rk}(F) > 0$ and F is semistable, then F is necessarily torsion free as otherwise the torsion subsheaf $F_{\mathrm{tors}} \subset F$ would destabilize F. In other words, we see that

$$F \text{ is semistable} \Longrightarrow F \text{ is torsion or torsion free.}$$

We recall the following two facts:

(Fact A) (Harder–Narasimhan filtrations) For any coherent sheaf F on C, there is a unique filtration (called the *Harder–Narasimhan filtration*)

$$0 = F_0 \subset F_1 \subset \cdots \subset F_{m-1} \subset F_m = F$$

with F_i/F_{i-1} semistable of slope $\mu_i = \mu(F_i/F_{i-1})$ such that the slopes $\mu_1 > \mu_2 > \cdots > \mu_m$ are strictly decreasing.

(Fact B) For any r, d with $r \geq 0$, there exists a non-empty irreducible projective variety $M_C^{\mathrm{ss}}(r, d)$ such that there is a bijection

$$k\text{-points of } M_C^{\mathrm{ss}}(r,d) \longleftrightarrow \text{S-equivalence classes of semistable vector bundles of rank } r \text{ and degree } d$$

1.2. **What about surfaces?** Do Facts A and B still hold if we replace the curve C with a smooth projective surface X? To approach this question, one first needs to settle on what stability means. The slope of a vector bundle on a curve involved the two basic invariants–the degree (i.e. 1st Chern class) and the rank (i.e., the 0th Chern class). For a vector bundle E on a surface, there are three fundamental invariants, namely the three Chern classes. Perhaps the most direct analogue of the slope then would be to fix an ample line bundle H on X and define

$$\mu(F) = \frac{H \cdot c_1(F)}{\mathrm{rk}(F)},$$

Then define F to be *semistable* by requiring that for all subsheaves $F' \subset F$, there is an inequality $\mu(F') \leq \mu(F)$ (just as in Definition 1.1).

With this definition, one can see that Fact A still holds but B is considerably more delicate. Donaldson constructed a moduli space using gauge theory of certain ASD connections on a topological vector bundle, and there is an algebraic construction of this space due to Jun Li. But in some sense, the construction of this moduli space is far from satisfactory; in particular, the equivalence relation in the moduli space is not just S-equivalence. On the other hand, there is another notion of semistability due to Gieseker also depending on a choice of ample line bundle H. In this case, both Facts A and B hold. We will discuss slope stability and Gieseker stability for surfaces in much greater detail later in the course.

In any case, one sees with surfaces that the notion of semistability (and therefore the corresponding moduil spaces) depend on a choice of ample line bundle H. Moreover, for a fixed H, there are still several notions for what semistability could mean. Bridgeland stability will allow us to systematically study all possible notions of stability.

1.3. **Vague goals.** In this course, our goals are vaguely to determine which other abelian categories have "nice" notions of stability satisfying Facts A and B? We will focus on the following (still vague) three questions:

(1) For a fixed abelian category $\mathcal{A}$, what is the space of all stability conditions? We will see that with the right definitions and suitable hypotheses, then this space carries the structure of a complex manifold!
(2) For a fixed stability condition and fixed invariants, is there is a moduli space of semistable objects?
(3) Can (2) and (3) be used to say anything interesting or geometric about the original category?

Throughout the course, our focus will be on addressing these goals with examples and applications in mind.

1.4. **Stability condition on abelian categories.** Let $\mathcal{A}$ be an abelian category. Define the *Grothendieck group* of $\mathcal{A}$ as

$$K(\mathcal{A}) := \text{free abelian group generated by } \mathrm{Ob}(\mathcal{A}) \,/\, \begin{array}{l}\text{subgroup generated by } [E]-[E']-[E''] \\ \text{where } 0 \to E' \to E \to E'' \to 0 \text{ is a} \\ \text{short exact sequence}\end{array}$$

Throughout the course, we will denote by $\overline{\mathbb{H}} \subset \mathbb{C}$ as the subset consisting of complex numbers z such that $\Im(z) > 0$ or z lies on the strictly negative real axis. A picture helps here. In other words, $\overline{\mathbb{H}}$ consists of the positive rays spanned by angles in $(0, \pi]$.

Definition 1.3. A *charge* on an abelian category $\mathcal{A}$ is an additive homomorphism

$$Z\colon K(\mathcal{A}) \to \mathbb{C}$$

such that for $0 \neq E \in \mathcal{A}$, $Z(E) \in \overline{\mathbb{H}}$. The *phase* of an object $0 \neq E \in \mathcal{A}$ is

$$\phi(E) := \frac{\arg(Z(E))}{\pi} \in (0, 1]$$

Remark 1.4. In other words, $Z(E)$ lies on the ray $\mathbb{R}_{>0}e^{i\pi\phi(E)}$ whose angle is the phase of E.

Remark 1.5. One could also define the *slope of* E as

$$\mu(E) := \frac{-\Re(Z(E))}{\Im(Z(E))}.$$

Trigonometry gives the relation

$$\tan(\pi\phi(E)) = \frac{-1}{\mu(E)}.$$

In particular, given a subobject $E' \subset E$, one has

$$\mu(E') \le \mu(E) \iff \phi(E') \le \phi(E). \tag{1.4.1}$$

Definition 1.6. Given a charge $Z\colon K(\mathcal{A}) \to \mathbb{C}$, we say that a nonzero object $E \in \mathcal{A}$ is *semistable* if for all subobjects $E' \subset E$, we have the inequality

$$\phi(E') \le \phi(E).$$

Remark 1.7. By (1.4.1), it is equivalent to consider the slope in defining semistability rather than the phase.

Definition 1.8. A *stability condition*[1] on an abelian category $\mathcal{A}$ is a charge $Z\colon K(\mathcal{A}) \to \mathbb{C}$ such that for all $0 \neq E \in \mathcal{A}$, there is a filtration (called the *Harder–Narasimhan filtration*)

$$0 = E_0 \subset E_1 \subset \cdots \subset E_{m-1} \subset E_m = F$$

such that E_i/E_{i-1} is semistable of phase $\phi_i = \phi(E_i/E_{i-1})$ and the phases $\phi_1 > \cdots > \phi_m$ are strictly decreasing.

Proposition 1.9. *If* $Z\colon K(\mathcal{A}) \to \mathbb{C}$ *is a stability condition on an abelian category* $\mathcal{A}$*, then Harder–Narasimhan filtrations are unique.*

[1] This is called a *stability function* by some authors.

Proof. The proof is completely formal (and not very enlightening) and identical to the argument for uniqueness of Harder–Narasimhan filtrations for coherent sheaves on smooth curves that we saw last quarter. □

Example 1.10. If C is a smooth, projective curve over a field k and $\mathcal{A} = \mathrm{Coh}(C)$ is the category of coherent sheaves, then

$$\begin{aligned} Z\colon K(\mathcal{A}) &\to \mathbb{C} \\ [E] &\mapsto -\deg(E) + i\,\mathrm{rk}(E) \end{aligned}$$

is a stability condition. Note that if the $\mathrm{rk}(E) = 0$, then E is torsion so that if $E \neq 0$, then $\deg(E) < 0$ so indeed $Z(E)$ always lies in $\mathbb{H}$.

1.5. Existence of Harder–Narasimhan filtrations.

Proposition 1.11. *Let $\mathcal{A}$ be an abelian category and $Z\colon K(\mathcal{A}) \to \mathbb{C}$ be a charge. Suppose that*

(a) $\mathcal{A}$ is Noetherian (i.e., every ascending chain of objects terminates); and
(b) the image of the imaginary part $K(\mathcal{A}) \to \mathbb{R}, [E] \mapsto \Im(Z(E))$ is discrete.

Then Z is a stability condition on $\mathcal{A}$.

Proof. We only need to show the existence of Harder–Narasimhan filtrations. The argument we saw last quarter for sheaves on a curve goes through unchanged. We sketched the argument again here inspired by [Bay05, Thm. 2.1.6]. □

1.6. **Bad news.** If X is a smooth projective surface, unfortunately the category $\mathrm{Coh}(X)$ does not have any natural stability conditions. (Here by natural, we mean that the charge $K(\mathcal{A}) \to \mathbb{C}$ factors through the Chern character $\mathrm{ch}\colon K(\mathcal{A}) \to \mathrm{H}^*(X, \mathbb{C})$.)

Bridgeland's idea (inspired by work of Douglas) was to look at other abelian categories (the so-called 'hearts of bounded t-structures')

$$\mathcal{A} \subset D^b(\mathrm{Coh}(X))$$

inside the bounded derived category of coherent sheaves on X and find stability conditions on $\mathcal{A}$. We will see (much later) that by varying $\mathcal{A}$ inside $D^b(\mathrm{Coh}(X))$, there are many stability conditions, so many in fact that they will form a complex manifold.

2. Lecture 2 (March 28): Quivers

TO BE ADDED

3. Lecture 3 (March 30): Stability conditions on quivers

TO BE ADDED

4. Lecture 4 (April 2): Derived categories

We have switched to two 80 minute lectures per week rather than three 50 minute lectures!!

For those completely unfamiliar with derived categories, I strongly recommend Richard Thomas's beautiful exposition [Tho01]. The motto of derived categories (as emphasized in [Tho01]) is "Complexes are good; cohomology of complexes are bad."

4.1. **Basic notions.** Even though most of you are likely familiar with these concepts, we'll define them precisely if only to set up our notation and conventions.

Let $\mathcal{A}$ be an abelian category. A *complex in* $\mathcal{A}$ is a sequence

$$A\colon \cdots \to A^{i-1} \xrightarrow{d^{i-1}} A^i \xrightarrow{d^i} A^{i+1} \to \cdots$$

such that $d^i \circ d^{i-1} = 0$. The *ith cohomology of* A is

$$\mathrm{H}^i(A) := \ker(d^i)/\operatorname{im}(d^{i-1}).$$

A map of complexes $f\colon A \to B$ is a sequence of maps $f^i\colon A^i \to B^i$ such that the obvious squares ($f^{i+1} \circ d^i = d^{i+1} \circ f^i$) commute. A map $f\colon A \to B$ is a *quasi-isomorphism* if the induced map $\mathrm{H}^i(A) \xrightarrow{\sim} \mathrm{H}^i(B)$ is an isomorphism for all i.

Example 4.1. The map of complexes

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z} \arrow[r, "2"] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & 0 \arrow[r] & \mathbb{Z}/2 \arrow[r] & 0 \end{tikzcd}$$

is a quasi-isomorphism but is not invertible as a map of complexes.

Example 4.2. If two complexes A, B are quasi-isomorphic, then necessarily $\mathrm{H}^i(A) \cong \mathrm{H}^i(B)$. But the converse is not true. For instance, $\mathbb{C}[x,y]^{\oplus 2} \xrightarrow{x,y} \mathbb{C}[x,y]$ has the same cohomology as $\mathbb{C}[x,y] \xrightarrow{0} \mathbb{C}$, but the two complexes are not quasi-isomorphic.

4.2. **The definition.** Let $\mathcal{A}$ be an abelian category. We define the *derived category of* $\mathcal{A}$ as the category $D^b(\mathcal{A})$ where objects are complexes

$$E\colon \cdots \to E^{i-1} \xrightarrow{d^{i-1}} E^i \xrightarrow{d^i} E^{i+1} \to \cdots$$

and morphisms are

$$\mathrm{Hom}_{D(\mathcal{A})}(E,F) = \left\{ \text{“roofs”} \quad \begin{tikzcd} & G \arrow[dl, "\text{qis}"'] \arrow[dr] & \\ E & & F \end{tikzcd} \;\middle|\; G \to E \text{ is a quasi-isomorphism} \right\} / \sim,$$

where two roofs $E \leftarrow G \to F$ is identified with $E \leftarrow G' \to F$ if and only if they are both dominated by a third roof $E \leftarrow G'' \to F$ as pictured here:

$$\begin{tikzcd} & & G'' \arrow[dl, "\text{qis}"'] \arrow[dr] & & \\ & G \arrow[dl, "\text{qis}"'] \arrow[drrr] & & G' \arrow[dlll, "\text{qis}"] \arrow[dr] & \\ E & & & & F. \end{tikzcd}$$

Example 4.3. In $D(\mathrm{Ab})$, the roof (where the superscripts below indicate the degrees of the terms in the complex)

$$[0 \to \mathbb{Z}/2^0 \to 0] \xleftarrow{\text{qis}} [\mathbb{Z}^{-1} \xrightarrow{2} \mathbb{Z}^0] \to [0 \to \mathbb{Z}^{-1} \to 0]$$

gives a non-zero map. In this case, the map on cohomology is the zero map.

We will not prove the following theorem nor will we worry about the set-theoretic hypotheses necessary for the construction of the derived category $D(\mathcal{A})$.

Theorem 4.4. *If $\mathcal{A}$ is an abelian category, the derived category $D(\mathcal{A})$ exists.*

Remark 4.5. While $D(\mathcal{A})$ is an additive category, it is *not* abelian.

4.3. **Shifting.** If A is a complex and $n \in \mathbb{Z}$, we define the shifted complex $A[n]$ by

$$A[n]^i = A^{n+i} \qquad d^i_{A[n]} = (-1)^i d_A.$$

Note that our conventions implies that the complex $A[1]$ is shifted to the left by one.

Shifting by n yields an additive equivalence $[n]\colon D(\mathcal{A}) \to D(\mathcal{A})$.

4.4. **Playing around with short exact sequences.** Consider a short exact sequence

$$0 \to A \to B \to C \to 0 \tag{4.4.1}$$

in $\mathcal{A}$. Observe that the complex $0 \to C \to 0$, with C in degree 0, is quasi-isomorphic to $0 \to A \to B \to 0$ with B in degree 0. The complex $0 \to A \to B \to 0$ projects onto the complex $0 \to A \to 0$. Summarizing we have maps (where the superscripts indicate the degrees)

$$[0 \to C^0 \to 0] \xleftarrow{\text{qis}} [0 \to A^{-1} \to B^0 \to 0] \to [0 \to A^{-1} \to 0 \to 0]$$

In other words, this gives a map $C \to A[1]$ in $D(\mathcal{A})$ where C and A are viewed in $D(\mathcal{A})$ as complexes supported in degree 0. The sequence $A \to B \to C \to A[1]$ will be an example of an exact triangle.

Suppose now that (4.4.1) does not split, that is, the corresponding class in $\operatorname{Ext}^1(C, A)$ is non-zero. Under the identification $\operatorname{Ext}^1(C, A) = \operatorname{Hom}_{D^b(\mathcal{A})}(C, A[1])$, the extension class in $\operatorname{Ext}^1(C, A)$ corresponds to the homomorphism $C \to A[1]$ constructed above. By shifting by -1, one can check that the composition $C[-1] \to A \to B$ is zero. In other words, we have a diagram

$$\begin{tikzcd} C[-1] \arrow[d, "\neq 0"] \arrow[dr, "0"] & & \\ A \arrow[r] & B \arrow[r] & C \end{tikzcd}$$

and we see that $A \to B$ is not a monomorphism in $D(\mathcal{A})$ and certainly $A \neq \ker(B \to C)$ in $D(\mathcal{A})$. While this doesn't prove that $D(\mathcal{A})$ is not abelian, it shows that perhaps your naive expectation for what the kernel of $B \to C$ is in $D(\mathcal{A})$ is in fact not a kernel.

4.5. **Mapping cones.** If $f\colon A \to B$ is a map of complexes in $\mathcal{A}$, then the *mapping cone of f* is the complex $C(f)$ defined by

$$\cdots \to C(f)^i = A^{i+1} \oplus B^i \xrightarrow{\begin{pmatrix} -d^{i+1} & 0 \\ f & d^i \end{pmatrix}} A^{i+2} \oplus B^{i+1} = C(f)^{i+1} \to \cdots$$

Observe that there is a map of complexes $C(f) \to A[1]$ defined by the projection $C(f)^i = A^{i+1} \oplus B^i \to A^{i+1} = A[1]^i$.

Example 4.6. Let $f\colon A \to B$ be a map of complexes where A, B are objects in $\mathcal{A}$ viewed as complexes supported in degree 0. Then $C(f) = [A^{-1} \xrightarrow{f} B^0]$. If $f\colon A \to B$ is injective, then $C(f) \cong \operatorname{coker}(f)$ and if $f\colon A \to B$ is surjective, then $C(f) \cong \ker(f)[1]$. So the cone construction embodies both the kernel and cokernel.

Observe that the sequence $A \xrightarrow{f} B \to C(f)$ induces a long exact sequence of cohomology

$$\cdots \to \mathrm{H}^I(A) \to \mathrm{H}^i(B) \to \mathrm{H}^i(C(f)) \to \mathrm{H}^{i+1}(A) \to \cdots$$

This follows for instance by considering the short exact sequence $0 \to B \to C(f) \to A[1] \to 0$ of complexes and examining the usual long exact sequence in cohomology.

4.6. **Exact triangles.** We define a *triangle* as a sequence of maps $A \to B \to C \to A[1]$ such the composition of any two is zero. A triangle $A \to B \to C \to A[1]$ is *exact* (sometimes referred to as *distinguished*) if there exists a map of complexes $f\colon A' \to B'$ yielding a commutative diagram in $D(\mathcal{A})$

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & A[1] \arrow[d] \\
A' \arrow[r, "f"] & B' \arrow[r] & C(f) \arrow[r] & A'[1]
\end{tikzcd}$$

where all vertical maps are isomorphisms. Often, we will simply write $A \to B \to C$ as an exact triangle which can be expressed pictorially as

$$\begin{tikzcd}
A \arrow[rr] & & B \arrow[dl] \\
& C \arrow[ul, dashed, "{[1]}"] &
\end{tikzcd}$$

and sometimes we abbreviate an exact triangle as simply $A \to B \to C$.

4.7. **Bounded derived category.** We define $D^b(\mathcal{A}) \subset D(\mathcal{A})$ as the full subcategory consisting of complexes E such that the cohomology $\mathrm{H}^n(E) = 0$ for $n \gg 0$ and $n \ll 0$. We say that the cohomology of E is *bounded.* Moreover, we say that the complex E has *cohomology supported in an interval* $[a, b]$ if $\mathrm{H}^i(E) = 0$ for $i \notin [a, b]$.

Proposition 4.7. *Any object $E \in D^b(\mathcal{A})$ with cohomology supported in $[a, b]$ can be represented by a complex*

$$0 \to E^a \to E^{a+1} \to \cdots \to E^b \to 0.$$

We will establish this once we've introduced the truncation functors, which is an important concept on its own.

Definition 4.8. Let E be a complex in $\mathcal{A}$ and $a, b \in \mathbb{Z}$. Then we define the complex $\tau_{\geq a}E$ and the map $E \to \tau_{\geq a}E$ of complexes via the diagram

$$\begin{tikzcd}
E\colon \arrow[d] & \cdots \arrow[r] & E^{a-1} \arrow[r, "d^{a-1}"] \arrow[d] & E^a \arrow[r, "d^a"] \arrow[d] & E^{a+1} \arrow[r] \arrow[d] & \cdots \\
\tau_{\geq a}E\colon & 0 \arrow[r] & 0 \arrow[r] & \operatorname{coker}(d^{a-1}) \arrow[r] & E^{a+1} \arrow[r] & \cdots
\end{tikzcd}$$

Similarly, we we define the complex $\tau_{\leq b}E$ and the map $\tau_{\leq b}E \to E$ of complexes via the diagram

$$\begin{tikzcd}
\tau_{\leq b}E\colon \arrow[d] & \cdots \arrow[r] & E^{b-1} \arrow[r] \arrow[d] & \operatorname{coker}(d^b) \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \\
E\colon & \cdots \arrow[r] & E^{b-1} \arrow[r, "d^{b-1}"] & E^b \arrow[r, "d^b"] & E^{b+1} \arrow[r] & \cdots
\end{tikzcd}$$

Remark 4.9. One checks easily that $\mathrm{H}^i(E) \xrightarrow{\sim} \mathrm{H}^i(\tau_{\geq a}E)$ for $i \geq a$ and that $\mathrm{H}^i(\tau_{\leq b}E) \xrightarrow{\sim} \mathrm{H}^i(E)$ for $i \leq b$.

Remark 4.10. We will sometimes denote $\tau_{>a}$ as $\tau_{\geq a+1}$ and similarly $\tau_{<b}$ as $\tau_{\leq b-1}$.

Proof of Proposition 4.7. If E is a complex with cohomology supported in $[a, b]$, then we have quasi-isomorphisms

$$E \xleftarrow{\text{qis}} \tau_{\leq b}E \xrightarrow{\text{qis}} \tau_{\geq a}(\tau_{\leq b}E).$$

of complexes. This roof of complexes yields an isomorphism $E \to \tau_{\geq a}(\tau_{\leq b}E)$ in $D(\mathcal{A})$. □

Proposition 4.11. *For every object $E \in D^b(\mathcal{A})$, there exists a diagram (which we call a filtration of E)*

$$\begin{tikzcd}
0 = E_0 \arrow[rr] & & E_1 \arrow[r] \arrow[dl] & \cdots \arrow[r] & E_{m-1} \arrow[rr] & & E_m = E \arrow[dl] \\
& A_1[k_1] \arrow[ul, dashed, "{[1]}"] & & & & A_m[k_m] \arrow[ul, dashed, "{[1]}"] &
\end{tikzcd}$$

where the triangles are exact triangles, and where each $A_i \in \mathcal{A}$ and each k_i is an integer satisfying $k_1 > \cdots > k_m$. Explicitly, if the cohomology of E is supported in $[a, b]$, then then we may take $E_i = \tau_{\leq a+i-1}$ for $i = 0, \ldots, m$ with $m = b - a$.

Proof. TO BE ADDED □

4.8. **Summary of properties.** The bounded derived category $D^b(\mathcal{A})$ has the following properties:

(1) $D^b(\mathcal{A})$ is an additive category;
(2) There is a shift functor
$$[1]\colon D^b(\mathcal{A}) \to D^b(\mathcal{A}), \quad E \mapsto E[1]$$
where $E[1]^i = E^{i+1}$ and $d^i_{E[1]} = (-1)^i d^i_E$.
(3) There is a notion of an exact triangle $A \to B \to C \to A[1]$ in $D^b(\mathcal{A})$;
(4) The full subcategory $\mathcal{A} \subset D^b(\mathcal{A})$ consisting of complexes supported in degree 0 has the property that for every object $E \in D^b(\mathcal{A})$, there exists a filtration of E

$$\begin{tikzcd}
0 = E_0 \arrow[rr] & & E_1 \arrow[r] \arrow[dl] & \cdots \arrow[r] & E_{m-1} \arrow[rr] & & E_m = E \arrow[dl] \\
& A_1[k_1] \arrow[ul, dashed, "{[1]}"] & & & & A_m[k_m] \arrow[ul, dashed, "{[1]}"] &
\end{tikzcd}$$

where the triangles are exact triangles, and where each $A_i \in \mathcal{A}$ and each k_i is an integer satisfying $k_1 > \cdots > k_m$.

Properties (1)–(3) give $D^b(\mathcal{A})$ the structure of a **triangulated category** (see Definition 5.1) while property (4) says that $\mathcal{A} \subset D^b(\mathcal{A})$ is a **heart of a bounded t-structure** (see Definition 5.6).

4.9. **Vague correspondence.** There is a loose dictionary between properties/constructions in an abelian category $\mathcal{A}$ and analogous properties/constructions in the bounded derived category $D^b(\mathcal{A})$.

Properties of $\mathcal{A}$	**Properties of $D^b(\mathcal{A})$**
short exact sequences	exact triangles
kernels and cokernels	mapping cones
long exact sequences of Ext groups induced from short exact sequences	long exact sequences of Ext groups induced from an exact triangle (Proposition 5.4)
torsion pairs (Definition 7.10)	t-structures (Definition 5.12)

5. Lecture 5 (April 4): Triangulated categories and t-structures

5.1. Triangulated categories.

Definition 5.1. A *triangulated category* is the data of

- an additive category $\mathcal{D}$;
- an additive equivalence $T\colon \mathcal{D} \to \mathcal{D}$. **We will use the convention that** $A[1] := T(A)$ **for an object** $A \in \mathcal{D}$; and
- A set of triangles $\{A \to B \to C \to A[1]\}$ which we call *exact*;

such that the following axioms are satisfied:

(TR1) (i) $A \xrightarrow{\text{id}} A \to 0 \to A[1]$ is an exact triangle;

(ii) Suppose

$$\begin{tikzcd} A \arrow[r] \arrow[d, "\alpha"] & B \arrow[r] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & A[1] \arrow[d, "\alpha[1]"] \\ A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & A'[1] \end{tikzcd}$$

is a commutative diagram in $\mathcal{D}$ where the vertical maps α, β, γ are isomorphisms. If the top row is an exact triangle, then so is the bottom row.

(iii) Any morphism $f\colon A \to B$ in $\mathcal{D}$ can be completed to an exact triangle $A \to B \to C \to A[1]$;

(TR2) If $A \to B \to C \to A[1]$ is an exact triangle, so is $B \to C \to A[1] \to B[1]$.

(TR3) Given a commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "\alpha"] & B \arrow[r] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, dashed, "\gamma"] & A[1] \arrow[d, "\alpha[1]"] \\ A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & A'[1] \end{tikzcd}$$

of solid arrows in $\mathcal{D}$, there exists an arrow $\gamma\colon C \to C'$ making the above diagram commute. **Warning: We do not assume that there is a unique such arrow!**

(TR4) (Octahedral axiom) For morphisms $A \xrightarrow{f} B \xrightarrow{g} C$, consider the diagram

$$\begin{tikzcd}
& & C(f) \arrow[ddr, dashed] & \\
& B \arrow[ur] \arrow[dr, "g"] & & \\
A \arrow[ur, "f"] \arrow[rr, "g\circ f"] & & C \arrow[r] \arrow[dr] & C(g\circ f) \arrow[d, dashed] \\
& & & C(g)
\end{tikzcd}$$

of solid arrows where $A \xrightarrow{f} B \to C(f) \to A[1]$, $A \xrightarrow{g\circ f} C \to C(g\circ f) \to A[1]$ and $B \xrightarrow{g} C \to C(g) \to B[1]$ are exact triangles (which exist by TR3). Then there exists dotted arrows in the diagram above such that $C(f) \to C(g\circ f) \to C(g) \to C(f)[1]$ is an exact triangle.

Moreover, we require the following commutativity relations: (a) $C \to C(g)$ agrees with $C \to C(g\circ f) \to C(g)$ (i.e. the lower triangle commutes), (b) $B \to C \to C(g\circ f)$ agrees with $B \to C(f) \to C(g\circ f)$ (i.e. the square above commutes), (c) $C(f) \to A[1]$ agrees with the composition $C(f) \to C(g\circ f) \to A[1]$, and (d) $C(g\circ f) \to A[1] \to B[1]$ agrees with $C(g\circ f) \to C(g) \to B[1]$. *(One can view the objects in the above diagram as vertices of a octahedron where four of the faces correspond to exact triangles and the other four faces are required to commute (see* [Wei94, p. 375].

Remark 5.2. One should view the octahedral axiom as a parallel for the following familiar property for objects in an abelian category: for an inclusion of objects $A \subset B \subset C$ in an abelian category $\mathcal{A}$, then the three short exact sequences form solid arrows in the diagram

$$\begin{tikzcd}
& & B/A \arrow[ddr, dashed] & \\
& B \arrow[ur] \arrow[dr] & & \\
A \arrow[ur] \arrow[rr] & & C \arrow[r] \arrow[dr] & C/A \arrow[d, dashed] \\
& & & C/B
\end{tikzcd}$$

and the dotted arrows form a short exact sequence $0 \to B/A \to C/A \to C/B \to 0$ which is simply the "Fourth Isomorphism Theorem."

We won't prove the following theorem—see [Wei94, Chapter 10].

Theorem 5.3. *If $\mathcal{A}$ is an abelian category, the bounded derived category $D^b(\mathcal{A})$, together with the shift functor $[1]\colon D^b(\mathcal{A}) \to D^b(\mathcal{A})$ and exact triangles as defined above, is triangulated.*

Proposition 5.4. *Let $\mathcal{D}$ be a triangulated category and $A \to B \to C \to A[1]$ is an exact triangle. For all objects $X \in \mathcal{D}$, there are exact sequences*

$$\mathrm{Hom}(X, A) \to \mathrm{Hom}(X, B) \to \mathrm{Hom}(X, C)$$
$$\mathrm{Hom}(C, X) \to \mathrm{Hom}(B, X) \to \mathrm{Hom}(A, X)$$

of abelian groups.

Remark 5.5. For objects $X, Y \in \mathcal{D}$, we can define

$$\mathrm{Ext}^i(X, Y) = \mathrm{Hom}(X, Y[i]).$$

With this convention, applying the proposition to shifts (using axiom TR2) of the original triangle, we obtain long exact sequences

$$\cdots \to \mathrm{Ext}^i(X, A) \to \mathrm{Ext}^i(X, B) \to \mathrm{Ext}^i(X, C) \to \mathrm{Ext}^{i+1}(X, A) \to \cdots$$

and

$$\cdots \to \mathrm{Ext}^i(C, X) \to \mathrm{Ext}^i(B, X) \to \mathrm{Ext}^i(A, X) \to \mathrm{Ext}^{i+1}(C, X) \to \cdots$$

Proof. We will establish that $\mathrm{Hom}(X, A) \to \mathrm{Hom}(X, B) \to \mathrm{Hom}(X, C)$ is exact and leave the second sequence as an exercise. We first show that the composition $A \to B \to C$ is zero, which clearly implies that the composition $\mathrm{Hom}(X, A) \to \mathrm{Hom}(X, B) \to \mathrm{Hom}(X, C)$ is zero. Consider the commutative diagram

$$\begin{tikzcd}
A \arrow[r, "\mathrm{id}"] \arrow[d, "\mathrm{id}"] & A \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d, dashed] & A[1] \arrow[d, "\mathrm{id}"] \\
A \arrow[r] & B \arrow[r] & C \arrow[r] & A[1]
\end{tikzcd}$$

of solid arrows. By Axiom TR3, there exists a morphism $0 \to C$ such the diagram of dotted arrows commute. But this means that $A \to B \to C$ is zero.

Let $g\colon X \to B$ be a morphism such that the composition $X \xrightarrow{g} B \to C$ is zero. Consider the commutative diagram

$$\begin{tikzcd}
X \arrow[r, "\mathrm{id}"] \arrow[d, dashed, "f"] & X \arrow[r] \arrow[d, "g"] & 0 \arrow[r] \arrow[d] & X[1] \arrow[d, dashed, "f[1]"] \\
A \arrow[r] & B \arrow[r] & C \arrow[r] & A[1]
\end{tikzcd}$$

of solid arrows. Apply TR2 and TR3 to obtain a morphism $f\colon X \to A$ completing the above diagram. The commutativity implies that f maps to g under $\mathrm{Hom}(X, A) \to \mathrm{Hom}(X, B)$. $\square$

5.2. **Hearts of bounded t-structures.** We will now introduce the notion of a heart of a bounded t-structure inspired by Property (4) in §4.8 concerning $\mathcal{A} \subset D^b(\mathcal{A})$.

Definition 5.6. A *heart of a bounded t-structure* on a triangulated category $\mathcal{D}$ is an additive full subcategory $\mathcal{A} \subset \mathcal{D}$ such that

(1) For all $A, B \in \mathcal{A}$, we have $\mathrm{Ext}^i(A, B) = 0$ for $i < 0$. (This is equivalent to requiring that $\mathrm{Hom}(A[i], B[j]) = 0$ for $i > j$.)

(2) For all $E \in \mathcal{D}$, there exists a diagram (which we call a *filtration of* E)

(5.2.1)
$$\begin{tikzcd}
0 = E_0 \arrow[rr] & & E_1 \arrow[dl] \arrow[r] & \cdots \arrow[r] & E_{m-1} \arrow[rr] & & E_m = E \arrow[dl] \\
& A_1[k_1] \arrow[ul, dashed, "{[1]}"] & & & & A_m[k_m] \arrow[ul, dashed, "{[1]}"] &
\end{tikzcd}$$

where the triangles are exact triangles, and where each $A_i \in \mathcal{A}$ and each k_i is an integer satisfying $k_1 > \cdots > k_m$.

Remark 5.7. We will see examples later of hearts $\mathcal{A} \subset \mathcal{D}$ such that $D^b(\mathcal{A}) \neq \mathcal{D}$.

Given the above definition, we can already define the central notion in the class: a stability condition on a triangulated category.

Definition 5.8. A *stability condition* on a triangulated category $\mathcal{D}$ is a pair $(\mathcal{A}, Z)$ where

- $\mathcal{A} \subset \mathcal{D}$ of a bounded t-structure; and
- $Z\colon K(\mathcal{A}) \to \mathbb{C}$ is a stability condition on $\mathcal{A}$.

Remark 5.9. For this to make sense, we need to establish that the heart $\mathcal{A}$ is an abelian category. We will establish this shortly.

Remark 5.10. There are conflicting conventions in the literature regarding the terminology of a 'stability condition.' Some authors call the above notion a pre-stability condition and reserve the term 'stability condition' for when additional finiteness hypotheses are satisfied.

Although we already have enough terminology to begin discussing examples of stability conditions, it will be useful to systematically discuss properties of hearts and their relation to t-structures. Namely, we will show that a heart determines a unique 't-structure'. Similarly, we will show a stability condition on $\mathcal{D}$ uniquely determines a 'slicing.'

5.3. **t-structures.** Let $\mathcal{A} \subset \mathcal{D}$ be a heart of a bounded t-structure. Define the following additive full subcategories of $\mathcal{D}$

$$\mathcal{D}_{>0} := \{E \in \mathcal{D} \mid \text{ in the filtration (5.2.1) of } E, \text{ all } k_i > 0\}$$
$$\mathcal{D}_{\leq 0} := \{E \in \mathcal{D} \mid \text{ in the filtration (5.2.1) of } E, \text{ all } k_i \leq 0\}$$

Proposition 5.11. *Let $\mathcal{A} \subset \mathcal{D}$ be a heart of a bounded t-structure. The pair $(\mathcal{D}_{>0}, \mathcal{D}_{\leq 0})$ of additive full subcategories satisfy:*

(1) $\mathcal{D}_{>0}[1] \subset \mathcal{D}_{>0}$ (in other words, for all $E \in \mathcal{D}_{>0}$, we have $E[1] \in \mathcal{D}_{>0}$);
(2) For all objects $E \in \mathcal{D}$, there exists an exact triangle
$$E_{>0} \to E \to E_{\leq 0}$$
where $E_{>0} \in \mathcal{D}_{>0}$ and $E_{\leq 0} \in \mathcal{D}_{\leq 0}$; and
(3) $\mathrm{Hom}_{\mathcal{D}}(\mathcal{D}_{>0}, \mathcal{D}_{\leq 0}) = 0$ (in other words, for all $E \in \mathcal{D}_{>0}$ and $F \in \mathcal{D}_{\leq 0}$, then $\mathrm{Hom}_{\mathcal{D}}(E, F) = 0$).

Proof. Property (1) is clear. For (2), let $E \in \mathcal{D}$ and consider a filtration (5.2.1) of E. Let $i = 0, \ldots, m$ be the integer such that $k_1 > \cdots > k_i > 0 \geq k_{i+1} > \cdots > k_m$. Clearly, $E_i \in \mathcal{D}_{>0}$. Complete the morphism $E_i \to E$ to an exact triangle
$$E_i \to E \to Q.$$

We need to show that $Q \in \mathcal{D}_{>0}$. To do this, we will show use the octahedral axiom TR4 to show that the filtration $E_i \to E_{i+1} \to \cdots \to E_m$ induces a filtration $0 = Q_i \to Q_{i+1} \to \cdots \to Q_m = Q$ with the same factors $A_i[k_i]$.

Extend $E_i \to E_{m-1}$ to an exact triangle $E_i \to E_{m-1} \to Q_{m-1}$ and consider the diagram

$$\begin{tikzcd}
 & & Q_{m-1} \arrow[ddr, dashed] & \\
 & E_{m-1} \arrow[ur] \arrow[dr] & & \\
E_i \arrow[ur] \arrow[rr] & & E_m \arrow[r] \arrow[dr] & Q_m \arrow[d, dashed] \\
 & & & A_m[k_m].
\end{tikzcd}$$

The octahedral axiom implies that the dotted arrows above from an exact triangle $Q_{m-1} \to Q_m \to A_m[k_m]$. We may inductively apply this argument to obtain exact triangles $Q_{j-1} \to Q_j \to A_j[k_j]$ for $j = i, \ldots, m$. Observe that this process ends with the exact triangle $E_i \xrightarrow{\mathrm{id}} E_i \to Q_i = 0$. We obtain the filtration

$$\begin{tikzcd}
0 = Q_i \arrow[rr] & & Q_{i+1} \arrow[r] \arrow[dl] & \cdots \arrow[r] & Q_{m-1} \arrow[rr] & & Q_m = Q \arrow[dl] \\
 & A_{i+1}[k_{i+1}] \arrow[ul, dashed, "{[1]}"] & & & & A_m[k_m] \arrow[ul, dashed, "{[1]}"] &
\end{tikzcd}$$

with $0 \geq k_{i+1} > \cdots > k_m$. This establishes that $Q \in \mathcal{D}_{>0}$.

To establish (3), we first show that for $A \in \mathcal{A}$, $E_{\leq 0} \in \mathcal{D}_{\leq 0}$ and $i > 0$, we have that $\mathrm{Hom}(A[i], E_{\leq 0}) = 0$. Choose a filtration (5.2.1) of $E_{\leq 0}$ with $0 \geq k_1 > \cdots > k_m$. Applying Proposition 5.4 to the last triangle, we obtain an exact sequence

$$\mathrm{Hom}(A[i], E_{m-1}) \to \mathrm{Hom}(A[i], E_m) \to \cancel{\mathrm{Hom}(A[i], A_m[k_m])}$$

where the last group vanishes since $i > k_m$. By inductively applying this argument, we see that $\mathrm{Hom}(A[i], E_m) = 0$.

To show the statement in general, we can choose a filtration (5.2.1) of $E_{>0}$ with $k_1 > \cdots > k_m > 0$ and again applying Proposition 5.4 to the last exact triangle, we have an exact sequence

$$\mathrm{Hom}(E_{m-1}, E_{\leq 0}) \to \mathrm{Hom}(E_m, E_{\leq 0}) \to \cancel{\mathrm{Hom}(A_m[k_m], E_{\leq 0})}$$

where the last group vanishes. Induction implies that $\mathrm{Hom}(E_m, E_{\leq 0})$ □

Definition 5.12. A *t-structure* on a triangulated category $\mathcal{D}$ is a pair of additive full subcategories $(\mathcal{D}_{>0}, \mathcal{D}_{\leq 0})$ satisfying Properties (1)–(3) in Proposition 5.11.

The *heart* of a t-structure $(\mathcal{D}_{>0}, \mathcal{D}_{\leq 0})$ is

$$\mathcal{A} := \mathcal{D}_{>0} \cap \mathcal{D}_{\leq 0}[1].$$

The t-structure $(\mathcal{D}_{>0}, \mathcal{D}_{\leq 0})$ is *bounded* if for every $E \in \mathcal{D}$, we have that $E \in \mathcal{D}_{\leq 0}[n] \cap \mathcal{D}_{>0}[-n]$ for some integer n.

Remark 5.13. This definition may look familiar to those that have seen the concept of a semi-orthogonal decomposition $\mathcal{D} = \langle \mathcal{A}, \mathcal{B} \rangle$ of a triangulated category $\mathcal{D}$. The only difference (but this is a big difference!) is that in a semi-orthogonal decomposition, the subcategories $\mathcal{A}, \mathcal{B} \subset \mathcal{D}$ are required to be triangulated (and thus closed under arbitrary shifts) and not just additive.

We introduce the following convention for a t-structure $(\mathcal{D}_{>0}, \mathcal{D}_{\leq 0})$. For each integer k, we define

$$\mathcal{D}_{\leq k} := \mathcal{D}_{\leq 0}[-k]$$
$$\mathcal{D}_{>k} := \mathcal{D}_{>0}[-k].$$

We will also set $\mathcal{D}_{<k} := \mathcal{D}_{\leq k-1}$ and $\mathcal{D}_{\geq k} = \mathcal{D}_{>k-1}$. With this convention, the heart $\mathcal{A}$ is $\mathcal{D}_{>0} \cap \mathcal{D}_{<0}$ and boundedness is the requirement that any object E is in $\mathcal{D}_{\leq -n} \cap \mathcal{D}_{>n}$ for some integer n. Note also that if $(\mathcal{D}_{>0}, \mathcal{D}_{\leq 0})$ is a t-structure, then so is $(\mathcal{D}_{>k}, \mathcal{D}_{\leq k})$ for any integer k.

Example 5.14. Let $\mathcal{A}$ be an abelian category and $\mathcal{D}^b(\mathcal{A})$ be its bounded derived category. Then the inclusion $\mathcal{A} \subset \mathcal{D}^b(\mathcal{A})$ consisting of complexes supported in degree 0 is a heart of a bounded t-structure (Property (4) in Section 4.8). In this case,

$$\mathcal{D}_{>0} := \{E \in \mathcal{D} \mid \mathrm{H}^i(E) = 0 \text{ for } i > 0\}$$
$$\mathcal{D}_{\leq 0} := \{E \in \mathcal{D} \mid \mathrm{H}^i(E) = 0 \text{ for } i \leq 0\}.$$

This is called the *standard t-structure*. In other words, $\mathcal{D}_{>0}$ consists of complexes $0 \to \cdots \to E^{-2} \to E^{-1} \to E^0 \to 0$ supported in non-positive degrees and $\mathcal{D}_{\leq 0}$ consists of complexes $0 \to E^1 \to E^2 \to \cdots \to 0$ supported in positive degrees. Similarly, $\mathcal{D}_{>k}$ consists of complexes with vanishing cohomology in degrees $> k$.

Proposition 5.11 says that the heart uniquely determines a bounded t-structure. On the other hand, the next proposition will show that the heart of a bounded t-structure is indeed a heart of a bounded t-structure as defined in Definition 5.6. In other words, our notation for 'hearts' is consistent.

Proposition 5.15. *Let $(\mathcal{D}_{>0}, \mathcal{D}_{\leq 0})$ be a bounded t-structure with heart $\mathcal{A} = \mathcal{D}_{>0} \cap \mathcal{D}_{\leq 0}[1]$. Then $\mathcal{A}$ satisfies conditions (1)–(2) in Definition 5.6. In other words, $\mathcal{A}$ is a heart of a bounded t-structure as defined in Definition 5.6.*

Proof. Exercise. $\square$

6. Lecture 6 (April 9): Properties of t-structures

6.1. More on t-structures.

Proposition 6.1. *Let $(\mathcal{D}_{>0}, \mathcal{D}_{\leq 0})$ be a bounded t-structure on a triangulated category $\mathcal{D}$. There is a well-defined functor*

$$\mathcal{D} \to \mathcal{D}_{>0}, \qquad E \mapsto E_{>0}$$

which is right adjoint to the inclusion $\mathcal{D}_{>0} \subset \mathcal{D}$ and a well-defined functor

$$\mathcal{D} \to \mathcal{D}_{\leq 0}, \qquad E \mapsto E_{\leq 0}$$

which is left adjoint to the inclusion $\mathcal{D}_{\leq 0} \subset \mathcal{D}$.

Proof. Let $f\colon E \to F$ be a morphism in $\mathcal{D}$ and choose exact triangles

$$\begin{tikzcd} E_{>0} \arrow[r] \arrow[d, dashed, "\alpha"] & E \arrow[r] \arrow[d, "f"] & E_{\leq 0} \arrow[d, dashed, "\beta"] \\ F_{>0} \arrow[r] & F \arrow[r] & F_{\leq 0} \end{tikzcd}$$

fitting into a diagram of solid arrows. We would like to show that their are unique dotted arrows α and β above making the diagram commute. In particular, this would imply that the decomposition $E_{>0} \to E \to E_{\leq 0}$ of an object E in $\mathcal{D}$ is unique up to unique isomorphism. Moreover, it would establish that assignments $E \mapsto E_{>0}$ and $E \mapsto E_{\leq 0}$ are functorial.

Since $\mathrm{Hom}(E_{>0}, -)$ is a homological functor (Proposition 5.4), we have an exact sequence

$$\cancel{\mathrm{Hom}(E_{>0}, F_{\leq 0}[-1])} \to \mathrm{Hom}(E_{>0}, F_{>0}) \to \mathrm{Hom}(E_{>0}, F) \to \cancel{\mathrm{Hom}(E_{>0}, F_{\leq 0})}$$

of abelian groups. But $\mathrm{Hom}(E_{>0}, F_{\leq 0}) = 0$ as there are no nonzero homomorphisms from any object in $\mathcal{D}_{>0}$ to any object in $\mathcal{D}_{\leq 0}$. Moreover, $\mathrm{Hom}(E_{>0}, F_{\leq 0}[-1]) = 0$ as $E_{>0} \in \mathcal{D}_{>0}$ and $F_{\leq 0}[-1] \in \mathcal{D}_{\leq 0}$. Thus the composition $E_{>0} \to E \xrightarrow{f} F$ is induced by a unique morphism $\alpha\colon E_{>0} \to F_{>0}$. The isomorphism $\mathrm{Hom}(E_{>0}, F_{>0}) \xrightarrow{\sim} \mathrm{Hom}(E_{>0}, F)$ establishes that $E \mapsto E_{>0}$ is right adjoint to the inclusion functor. A similar argument implies the existence and uniqueness of β as well as the left adjointness of $E \mapsto E_{\leq 0}$. $\square$

6.2. **Right and left orthogonal complements.** We show here that one of the subcategories in a t-structure $(\mathcal{D}_{>0}, \mathcal{D}_{\leq 0})$ uniquely determines the other.

Proposition 6.2. *Let $(\mathcal{D}_{>0}, \mathcal{D}_{\leq 0})$ be a bounded t-structure on a triangulated category $\mathcal{D}$ with heart $\mathcal{A}$, Then*

$$\begin{aligned} \mathcal{D}_{\leq 0} &= \{D \mid \mathrm{Hom}(E, D) = 0 \textit{ for all } E \in \mathcal{D}_{>0}\} \\ &= \{D \mid \mathrm{Hom}(A[k], D) = 0 \textit{ for all } A \in \mathcal{A} \textit{ and } k \geq 0\} \end{aligned}$$

and similarly

$$\begin{aligned} \mathcal{D}_{>0} &= \{D \mid \mathrm{Hom}(D, E) = 0 \textit{ for all } E \in \mathcal{D}_{\leq 0}\} \\ &= \{D \mid \mathrm{Hom}(D, A[k]) = 0 \textit{ for all } A \in \mathcal{A} \textit{ and } k < 0\}. \end{aligned}$$

Remark 6.3. For a subcategory $\mathcal{E} \subset \mathcal{D}$, one often defines the right orthogonal complement as $\mathcal{E}^{\perp} = \{D \mid \mathrm{Hom}(E, D) = 0 \text{ for all } E \in \mathcal{E}\}$ and the left orthogonal complement as ${}^{\perp}\mathcal{E} = \{D \mid \mathrm{Hom}(D, E) = 0 \text{ for all } E \in \mathcal{E}\}$. With this notation, the above proposition implies that $\mathcal{D}_{\leq 0} = \mathcal{D}_{>0}^{\perp}$ and $\mathcal{D}_{>0} =^{\perp} \mathcal{D}_{\leq 0}$.

Proof. Exercise. $\square$

6.3. **Abelianness of a heart.**

Proposition 6.4. *The heart of a bounded t-structure on a triangulated category is abelian.*

Proof. Let $\mathcal{D}$ be a triangulated category with t-structure $(\mathcal{D}_{>0}, \mathcal{D}_{\leq 0})$ and heart $\mathcal{A}$. We will show only that kernels exist in $\mathcal{A}$. Let $f\colon A \to B$ be a morphism in $\mathcal{A}$. We first complete f to an exact triangle

$$A \xrightarrow{f} B \to C \to A[1]$$

where $C \in \mathcal{D}$. Using the t-structure $(\mathcal{D}_{\geq 0}, \mathcal{D}_{<0})$, we obtain another exact triangle

$$C_{\geq 0} \to C \to C_{<0}$$

with $C_{\geq 0} \in \mathcal{D}_{\geq 0}$ and $C_{<0} \in \mathcal{D}_{<0}$. We claim that $C_{<0} \in \mathcal{A}$. To show this, it suffices to show that $C_{<0} \in \mathcal{D}_{>0}$ as $\mathcal{A} = \mathcal{D}_{<0} \cap \mathcal{D}_{>0}$. By applying Proposition 6.2, we are reduced to showing that

$$\mathrm{Hom}(C_{<0}, T[k]) = 0$$

for all $T \in \mathcal{A}$ and $k < 0$. Note that $T[k] \in \mathcal{D}_{\leq 0}$ The exact triangles $C \to C_{<0} \to C_{\geq 0}[1]$ and $B \to C \to A[1]$ yields a diagram

$$\begin{tikzcd}
 & \cancel{\operatorname{Hom}(C_{\geq 0}[1], T[k])} \arrow[d] & \\
 & \operatorname{Hom}(C_{<0}, T[k]) \arrow[d] & \\
\cancel{\operatorname{Hom}(A[1], T[k])} \arrow[r] & \operatorname{Hom}(C, T[k]) \arrow[r] & \cancel{\operatorname{Hom}(B, T[k])}
\end{tikzcd}$$

where both the row and column are exact sequences of abelian groups by Proposition 5.4. As $k < 0$, the above terms with a red line are all zero. This implies that $\operatorname{Hom}(C, T[k]) = 0$ and thus $\operatorname{Hom}(C_{<0}, T[k]) = 0$.

Finally, we show that the composition $B \to C \to C_{<0}$ is the cokernel of $f \colon A \to B$. First, observe that the composition $A \xrightarrow{f} B \to C_{<0}$ is zero since $A \to B \to C$ is zero. Consider a morphism $B \to Q$ such that the composition $A \xrightarrow{f} B \to Q$ is zero. Consider the diagram

$$\begin{tikzcd}
 & C_{\geq 0} \arrow[dr] & & \\
A \arrow[r, "f"] \arrow[ddrr, "0"'] & B \arrow[r] \arrow[ddr] & C \arrow[dr] \arrow[dd, dashed, "\alpha"] & \\
 & & & C_{<0} \arrow[dl, dashed, "\beta"] \\
 & & Q &
\end{tikzcd}$$

Since $\operatorname{Hom}(-, Q)$ is a cohomological functor (Proposition 5.4), there exists a morphism $\alpha \colon C \to Q$ restricting to the given morphism $B \to Q$. As $\operatorname{Hom}(C_{\geq 0}, Q) = 0$ since $Q \in \mathcal{D}_{<0}$, there exists the dotted arrow $\beta \colon C_{<0} \to Q$ restricting to α, which establishes that $B \to Q$ factors through the $B \to C_{<0}$.

A similar argument shows that $C_{\geq 0}[-1] \in \mathcal{A}$ and that the composition $C_{\geq 0}[-1] \to C[-1] \to A$ is the kernel of $f \colon A \to B$. $\square$

6.4. **Cohomology with respect to a heart.** Let $\mathcal{A} \subset \mathcal{D}$ be a heart of a bounded t-structure $(\mathcal{D}_{>0}, \mathcal{D}_{\leq 0})$. For any $E \in \mathcal{D}$, there is a filtration

$$\begin{tikzcd}
0 = E_0 \arrow[rr] & & E_1 \arrow[r] \arrow[dl] & \cdots \arrow[r] & E_{m-1} \arrow[rr] & & E_m = E \arrow[dl] \\
 & A_1[k_1] \arrow[ul, dashed, "{[1]}"] & & & & A_m[k_m] \arrow[ul, dashed, "{[1]}"] &
\end{tikzcd}$$

where each $A_i \in \mathcal{A}$ and $k_1 > \cdots > k_m$ are decreasing integers. Proposition 6.1 implies that these filtrations are unique and functorial. In particular, the factors A_i are unique and functorial so we can define:

Definition 6.5. The *kth cohomology of E with respect to $\mathcal{A}$ as*

$$\mathrm{H}^k_{\mathcal{A}}(E) = \begin{cases} A_i & \text{if } k_i = k \\ 0 & \text{otherwise.} \end{cases}$$

In other words, the nonzero cohomology groups of E are $\mathrm{H}^{k_i}(E) = A_i$.

Example 6.6. If $\mathcal{D} = D^b(\mathcal{A})$ is the bounded derived category of an abelian category $\mathcal{A}$ and $(\mathcal{D}_{>0}, \mathcal{D}_{\leq 0})$ is the standard t-structure (see Example 5.14) with heart $\mathcal{A}$, then $\mathrm{H}^k_{\mathcal{A}}(E)$ is simply the ordinary cohomology $\mathrm{H}^k(E)$ of a complex $E \in \mathcal{D}$.

Proposition 6.7. *The functor*

$$\mathrm{H}^k_{\mathcal{A}}(-)\colon \mathcal{D} \to \mathcal{A}, \qquad E \mapsto \mathrm{H}^k_{\mathcal{A}}(E)$$

is homological; that is if $A \to B \to C \to A[1]$ is an exact triangle, then

$$\mathrm{H}^k_{\mathcal{A}}(A) \to \mathrm{H}^k_{\mathcal{A}}(B) \to \mathrm{H}^k_{\mathcal{A}}(C)$$

is exact.

Remark 6.8. As in Remark 5.5, by applying the above proposition to shifted exact triangles, there are long exact sequences

$$\cdots \to \mathrm{H}^k_{\mathcal{A}}(A) \to \mathrm{H}^k_{\mathcal{A}}(B) \to \mathrm{H}^k_{\mathcal{A}}(C) \to \mathrm{H}^{k+1}_{\mathcal{A}}(C) \to \cdots$$

This fact should be viewed as a natural extension of the familiar fact that a short exact sequence of complexes induces a long exact sequence of cohomology.

Proof. Applying Proposition 6.1 to the t-structures $(\mathcal{D}_{>k}, \mathcal{D}_{\leq k})$ and $(\mathcal{D}_{\geq k}, \mathcal{D}_{<k})$ yields "truncation" functors

$$\tau_{>k}\colon \mathcal{D} \to \mathcal{D}_{>k} \qquad \text{and} \qquad \tau_{<k}\colon \mathcal{D} \to \mathcal{D}_{<k}$$

such that $\mathrm{H}^k_{\mathcal{A}}(E) = \tau_{>k}(\tau_{<k}(E))$ for $E \in \mathcal{D}$. Moreover, Proposition 6.1 yields an exact triangle

$$H^k_{\mathcal{A}}(A) \to H^k_{\mathcal{A}}(B) \to H^k_{\mathcal{A}}(C)$$

of objects in $\mathcal{A}$. The statement follows from appealing to the below lemma. □

Lemma 6.9. *Let $\mathcal{D}$ be a triangulated category with a heart $\mathcal{A}$ of a bounded t-structure. If $A \to B \to C \to A[1]$ is an exact triangle in $\mathcal{D}$, then $A \to B \to C$ is an exact sequence in $\mathcal{A}$.*

Proof. By Yoneda's lemma, it suffices to show that for each $X \in \mathcal{A}$,

$$\mathrm{Hom}(X, A) \to \mathrm{Hom}(X, B) \to \mathrm{Hom}(X, C)$$

is an exact sequence of abelian groups, but this follows from Proposition 5.4. □

6.5. **The Grothendieck group of a triangulated category.** We define the *Grothendieck group* of a triangulated category $\mathcal{D}$ to be

$$K(\mathcal{D}) = \mathbb{Z}[\mathrm{Ob}(\mathcal{D})]/\sim,$$

the free abelian group generated by the objects in $\mathcal{D}$ modulo the subgroup generated by the relations $[B] = [A] + [C]$ for exact triangles $A \to B \to C$.

Proposition 6.10. *If $\mathcal{A} \subset \mathcal{D}$ is the heart of a bounded t-structure, then the natural homomorphism $K(\mathcal{A}) \to K(\mathcal{D})$ is an isomorphism.*

Proof. We define the map

$$K(\mathcal{D}) \to K(\mathcal{A}), \qquad [E] \mapsto \sum_{k\in\mathbb{Z}} (-1)^k \operatorname{H}^k_{\mathcal{A}}(E).$$

First, the sum above is a finite sum since the t-structure is bounded. In order to check that this is well-defined, we need to show that $[B] - [A] - [C]$ maps to zero whenever $A \to B \to C$ is an exact triangle. By Proposition 6.7 and Remark 6.8, we have bounded long exact sequences

$$\cdots \to \operatorname{H}^k_{\mathcal{A}}(A) \to \operatorname{H}^k_{\mathcal{A}}(B) \to \operatorname{H}^k_{\mathcal{A}}(C) \to \operatorname{H}^{k+1}_{\mathcal{A}}(C) \to \cdots$$

Breaking this into short exact sequences, one can argue that this implies the equality

$$\sum_k (-1)^k [H^k_{\mathcal{A}}(B)] = \sum_k (-1)^k [H^k_{\mathcal{A}}(A)] + \sum_k (-1)^k [H^k_{\mathcal{A}}(C)]$$

in $K(\mathcal{A})$. The map $K(\mathcal{D}) \to K(\mathcal{A})$ is easily checked to be the inverse of the natural map $K(\mathcal{A}) \to K(\mathcal{D})$. □

7. Lecture 7 (April 11): Stability conditions, slicings and tilts

7.1. Stability conditions.

We begin by repeating Definition 7.1.

Definition 7.1. A *stability condition* on a triangulated category $\mathcal{D}$ is a pair $(\mathcal{A}, Z)$ where

- $\mathcal{A} \subset \mathcal{D}$ of a bounded t-structure; and
- $Z\colon K(\mathcal{A}) \to \mathbb{C}$ is a stability condition on $\mathcal{A}$.

By Proposition 6.4, the heart $\mathcal{A}$ is an abelian category. By Proposition 6.10, there is a natural isomorphism $K(\mathcal{A}) \xrightarrow{\sim} K(\mathcal{D})$ of Grothendieck groups. Recall that the definition of a stability condition on an abelian category $\mathcal{A}$ requires that for all $0 \neq A \in \mathcal{A}$, the charge $Z(A) \in \overline{\mathbb{H}}$, the subset of the complex numbers with either positive imaginary part or strictly negative real numbers. The phase of A is $\phi(A) = \frac{\arg(Z(A))}{\pi} \in (0, 1]$ and that A is called semistable if for all $0 \neq A' \subset A$, the inequality $\phi(A') \leq \phi(A)$ is satisfied. Moreover, we require that every object $0 \neq A \in \mathcal{A}$ admits a *Harder–Narasimhan filtration*

$$0 = A_0 \subset A_1 \cdots \subset A_{m-1} \subset A_m$$

where each factor A_i/A_{i-1} is semistable of phase ϕ_i, and $\phi_1 > \cdots > \phi_m$.

In a similar way to how we explored the properties of a heart of a bounded t-structure introduced in Definition 5.6 and were led naturally to the definition of a t-structure (Definition 5.12) giving an alternative perspective on hearts (Proposition 5.15), we now explore the structure of a stability condition on a triangulated category with the goal of giving an alternative description in terms of slicings (see Definition 7.5 and Proposition 7.6).

Definition 7.2. For a stability condition $(\mathcal{A}, Z)$ on a triangulated category $\mathcal{D}$, define for each $\phi \in (0, 1]$ the following additive full subcategory of $\mathcal{D}$

$$\mathcal{P}(\phi) = \{E \in \mathcal{A} \mid \ E \text{ is semistable of phase } \phi\} \cup \{0\},$$

and extend this definition for all $\phi \in \mathbb{R}$ by setting

$$\mathcal{P}(\phi + 1) := \mathcal{P}(\phi)[1].$$

Proposition 7.3. *Let $(\mathcal{A}, Z)$ be a a stability condition on on a triangulated category $\mathcal{D}$. Define the categories $\mathcal{P}(\phi)$ for $\phi \in \mathbb{R}$ using Definition 7.2.*

(1) $\mathrm{Hom}(\mathcal{P}(\phi_1), \mathcal{P}(\phi_2)) = 0$ *if* $\phi_1 > \phi_2$*;*
(2) $\mathcal{P}(\phi + 1) = \mathcal{P}(\phi)[1]$*; and*
(3) for all $E \in \mathcal{D}$*, there exists a filtration*

(7.1.1)

$$\begin{tikzcd}
0 = E_0 \arrow[rr] & & E_1 \arrow[r] \arrow[dl] & \cdots \arrow[r] & E_{m-1} \arrow[rr] & & E_m = E \arrow[dl] \\
& A_1 \arrow[ul, dashed, "{[1]}"] & & & & A_m \arrow[ul, dashed, "{[1]}"] &
\end{tikzcd}$$

where the triangles are exact triangles, and where each $A_i \in \mathcal{P}(\phi_i)$ *with* $\phi_1 > \cdots > \phi_m$ *decreasing real numbers.*

Remark 7.4. Property (3) should be viewed as an extension of the filtrations introduced in (5.2.1) as a requirement of a heart of a bounded t-structure. The difference here is that now we allow the factors to be indexed by decreasing real numbers (as opposed to integers).

Proof. Statement (2) is clear. We leave (1) as an exercise. We spell out some of the details for (3). Given $E \in \mathcal{D}$, we first use the heart of the bounded t-structure to obtain a filtration

$$\begin{tikzcd}
0 = E_0 \arrow[rr] & & E_1 \arrow[r] \arrow[dl] & \cdots \arrow[r] & E_{m-1} \arrow[rr] & & E_m = E \arrow[dl] \\
& A_1[k_1] \arrow[ul, dashed, "{[1]}"] & & & & A_m[k_m] \arrow[ul, dashed, "{[1]}"] &
\end{tikzcd}$$

with $A_i \in \mathcal{A}$ and $k_1 > \cdots > k_m$ decreasing integers. For each A_i, we take the Harder–Narasimhan filtration

$$0 = A_{i,0} \subset A_{i,1} \cdots \subset A_{i,m-1} \subset A_{i,n_i}$$

where each factor $A_{i,j}/A_{i,j-1} \in \mathcal{P}(\phi_{i,j})$ is semistable of phase $\phi_{i,j}$, and $\phi_{i,1} > \cdots > \phi_{i,n_i}$.

By using the octahedral axiom as in the proof of Proposition 5.11, we may combine these filtrations to a filtration

$$0 = E_{1,0} \to E_{1,1} \to \cdots \to E_{1,n_1} \to E_{2,1} \to \cdots \to E_{2,n_2} \to \cdots \to E_{m,n_m} = E$$

with factors $A_{1,1}[k_1], \cdots, A_{1,n_1}[k_1], A_{2,1}[k_2], \cdots, A_{2,n_2}[k_2], \cdots, A_{m,n_m}[k_m]$ which have decreasing phases $\phi_{1,1} + k_1 > \cdots > \phi_{1,n_1} + k_1 > \phi_{2,1} + k_2 > \cdots > \phi_{2,n_2} + k_2 > \cdots > \phi_{m,n_m} + k_m$. □

7.2. **Slicings.**

Definition 7.5. A *slicing* $\mathcal{P}$ of a triangulated category $\mathcal{D}$ is an additive full subcategory $\mathcal{P}(\phi) \subset \mathcal{D}$ for each $\phi \in \mathbb{R}$ satisfying properties (1)–(3) in Proposition 7.3.

We introduce the following conditions. For each interval $(a, b]$ where a and b are real numbers possibly $\pm\infty$, we define

$$\mathcal{P}((a, b]) := \{0 \neq E \in \mathcal{D} \mid \text{in the filtration (7.1.1) each } \phi_i \in (a, b]\}$$

This subcategory includes by definition the zero object. Similarly, one can define $\mathcal{P}(I)$ for other intervals $I = (a, b)$, $[a, b)$, or $[a, b]$.

As promised, slicing allows for an alternative definition of a stability condition.

Proposition 7.6. *Giving a stability condition $(\mathcal{A}, Z)$ on a triangulated category $\mathcal{D}$ is equivalent to giving a slicing $\mathcal{P}$ and a group homomorphism (which we will also refer to as a charge)*

$$Z\colon K(\mathcal{D}) \to \mathbb{C}$$

such that for all $E \in \mathcal{P}(\phi)$, we have that $\phi(E) \in \mathbb{R}_{>0}e^{i\pi\phi}$.

Proof. Proposition 7.3 produces a slicing $\mathcal{P}$ from a stability condition $(\mathcal{A}, Z)$. The last condition is clear for $E \in \mathcal{P}(\phi)$ with $\phi \in (0, 1]$, and is checked for all ϕ by using the fact that $Z([E[n]]) = (-1)^n Z([E]) = e^{\pi i n} Z([E])$ for each integer n.

Conversely, given a slicing $\mathcal{P}$ with charge Z, one checks that

$$\mathcal{A} = \mathcal{P}((0, 1])$$

is a heart of the bounded t-structure $(\mathcal{D}_{>0}, \mathcal{D}_{\leq 0}) = (\mathcal{P}(0, \infty), \mathcal{P}(-\infty, 0])$, and that $Z\colon K(\mathcal{D}) = K(\mathcal{A}) \to \mathbb{C}$ is a stability condition on $\mathcal{A}$. We leave the details to the reader. $\square$

Convention 7.7. *We will freely switch between thinking of a stability condition as a pair $(\mathcal{A}, Z)$ as introduced in Definition 7.1 where $\mathcal{A}$ is a heart and Z is a charge and as a pair $(\mathcal{P}, Z)$ where $\mathcal{P}$ is a slicing and Z is charge. In particular, we will use both the notation $(\mathcal{A}, Z)$ and $(\mathcal{P}, Z)$ for a stability condition.*

Example 7.8. Let C be a smooth, projective and connected curve over an algebraically closed field k. Let $D^b(C) := D^b(\mathrm{Coh}(C))$ be the bounded derived category of coherent $\mathcal{O}_C$-modules. Let $\mathrm{Coh}(C) \subset D^b(C)$ be the standard heart and

$$Z\colon K(\mathrm{Coh}(C)) \to \mathbb{C}, \qquad [E] \mapsto -\deg(E) + i\,\mathrm{rk}(E).$$

Then $(\mathrm{Coh}(C), Z)$ is a stability condition on $D^b(C)$. For $\phi \in (0, 1]$, we have that $\mathcal{P}(\phi)$ as the full subcategory of $\mathrm{Coh}(C)$ consisting of semistable coherent sheaves with phase ϕ. Note that $\mathcal{P}(1) \subset \mathrm{Coh}(C)$ is the subcategory of torsion sheaves while for $\phi \in (0, 1)$, $\mathcal{P}(\phi)$ consists of semistable vector bundles with phase ϕ (or equivalent slope μ where $\tan(\pi\phi) = -1/\mu$).

More generally, if $Z\colon K(\mathrm{Coh}(C)) \to \mathbb{C}$ is any stability condition, then the charge of a torsion sheaf $\mathcal{O}_D$ must lie on the negative real line. This is because tensoring with both $\mathcal{O}(D)$ and $\mathcal{O}(-D)$ must keep the charge in $\overline{\mathbb{H}}$. On the other hand, there is more freedom for the choice of the charge of a torsion free sheaves. In fact, if $z \in \mathbb{C}$ with $\Im(z) > 0$, then the charge

$$Z\colon K(\mathrm{Coh}(C)) \to \mathbb{C}, \qquad -\deg(E) + z\,\mathrm{rk}(E)$$

defines a stability condition on $D^b(C)$.

Keep in mind that $K(C) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathrm{Pic}^0(C)$ where the map onto the first and second $\mathbb{Z}$ is given by the rank and the degree. Later we will try to characterize all stability conditions (which we will call numerical) where the charge $Z\colon K(C) \to \mathbb{C}$ factors through $(\deg, \mathrm{rk})\colon K(C) \to \mathbb{Z}^2$. If $C \not\cong \mathbb{P}^1$, then $\mathrm{Pic}^0(C)$ is an infinitely generated abelian group (corresponding to the k-points of an abelian variety) so there are conceivably many other charges on $K(C)$ yielding stability conditions,

In a similar spirit to Proposition 6.4, we have the following result

Proposition 7.9. *Let $(\mathcal{P}, Z)$ be a stability condition on a triangulated category $\mathcal{D}$. For each $\phi \in \mathbb{R}$, the category $\mathcal{P}(\phi)$ is abelian.*

Proof. We may assume that $\phi \in (0,1]$ so that $\mathcal{P}(\phi)$ is contained in the heart $\mathcal{A} = \mathcal{P}((0,1])$. Let $f\colon E \to F$ be a morphism in $\mathcal{P}(\phi)$. By Proposition 6.4, we know that $\ker(f), \operatorname{coker}(f) \in \mathcal{A}$. We need to show that $\ker(f), \operatorname{coker}(f)$ are semistable of phase ϕ. We have short exact sequences

$$0 \to \ker(f) \to E \to \operatorname{im}(f) \to 0 \qquad \text{and} \qquad 0 \to \operatorname{im}(f) \to F \to \operatorname{coker}(F) \to 0.$$

Consider the Harder–Narasimhan filtration of $\operatorname{im}(f)$ in $\mathcal{A}$

$$0 = I_0 \subset I_1 \subset \cdots \subset I_{m-1} \subset I_m = \operatorname{im}(f)$$

with semistable quotients I_i/I_{i-1} of decreasing phases $\varphi_1 > \cdots > \varphi_m$. Since $I_1 \subset \operatorname{im}(f) \subset F$ and F is semistable, we have $\phi \geq \varphi_1$. On the other hand, we have the quotient $E \to \operatorname{im}(f) \to I_m/I_{m-1}$ and semistability of E implies that $\varphi_m \geq \phi$. Combining these two statements, we see that $\phi = \varphi_1 = \varphi_m$ so that $m = 1$; therefore $\operatorname{im}(f)$ is semistable of phase ϕ. But then $\ker(f)$ has phase ϕ and any subobject of $\ker(f)$ with larger phase would destabilize E. Likewise, $\operatorname{coker}(f)$ has phase ϕ and any quotient of $\operatorname{coker}(f)$ with smaller phase would destabilize F. □

7.3. **Torsion pairs and tilting.** To construct more interesting stability conditions, we need the ability to construct new hearts of bounded t-structures. To this end, it will be extremely useful to construct a new heart $\mathcal{A}^\#$ (which we will call the tilted heart) from the data of a heart $\mathcal{A}$ and a torsion pair in $\mathcal{A}$. A torsion pair in an abelian category is analogous to the notion of a t-structure on a triangulated category.

Definition 7.10. A *torsion pair* in an abelian category $\mathcal{A}$ is a pair $(\mathcal{T}, \mathcal{F})$ of additive full subcategories satisfying

1. $\operatorname{Hom}(T, F) = 0$ for all $T \in \mathcal{T}$ and $F \in \mathcal{F}$, and
2. For all $E \in \mathcal{A}$, there exists an exact sequence
$$0 \to T_E \to E \to F_E \to 0$$
with $T_E \in \mathcal{T}$ and $F_E \in \mathcal{F}$.

Analogous to Proposition 6.1, we have uniqueness and functoriality of the factorization in Condition (2) above.

Proposition 7.11. *Let $(\mathcal{T}, \mathcal{F})$ be a torsion pair in an abelian category $\mathcal{A}$. There is a well-defined functor*

$$\mathcal{A} \to \mathcal{T}, \qquad E \mapsto T_E$$

which is right adjoint to the inclusion $\mathcal{T} \subset \mathcal{A}$ and a well-defined functor

$$\mathcal{A} \to \mathcal{F}, \qquad E \mapsto F_E$$

which is left adjoint to the inclusion $\mathcal{F} \subset \mathcal{A}$.

Proof. Exercise. □

Definition 7.12. Given a torsion pair $(\mathcal{T}, \mathcal{F})$ in $\mathcal{A}$, the *tilt of $\mathcal{A}$ with respect to* $(\mathcal{T}, \mathcal{F})$ is the full subcategory of $D^b(\mathcal{A})$

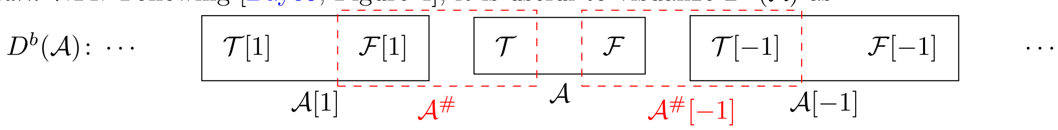
$$\mathcal{A}^\# = \left\{ (E^{-1} \xrightarrow{d} E^0) \in D^b(\mathcal{A}) \mid \ker(d) \in \mathcal{F} \text{ and } \operatorname{coker}(d) \in \mathcal{T} \right\}.$$

Proposition 7.13. *$\mathcal{A}^{\#}$ is the heart of the bounded t-structure $(\mathcal{D}^{\#}_{>0}, \mathcal{D}^{\#}_{\leq 0})$ defined by*

$$\mathcal{D}^{\#}_{>0} = \left\{E\colon \cdots \to E^{-2} \to E^{-1} \xrightarrow{d^{-1}} E^0 \to 0 \middle|\ \operatorname{coker}(d^{-1}) \in \mathcal{T}\right\}$$
$$\mathcal{D}^{\#}_{\leq 0} = \left\{E\colon 0 \to E^0 \xrightarrow{d^0} E^1 \to E^2 \to \cdots \middle|\ \ker(d^0) \in \mathcal{F}\right\}.$$

Proof. Exercise. □

Remark 7.14. Following [Bay05, Figure 4], it is useful to visualize $D^b(\mathcal{A})$ as



where every object has a finite filtration with terms going from left to right, and there are no morphisms in $D^b(\mathcal{A})$ from the left to the right. The tilted heart $\mathcal{A}^{\#}$ is the dotted red box above containing $\mathcal{F}[1]$ and $\mathcal{T}$.

Example 7.15. If X is a Noetherian scheme and $\mathcal{A} = \mathrm{Coh}(X)$, then the pair $(\mathcal{T}, \mathcal{F})$ consisting of torsion sheaves and torsion-free sheaves is a torsion pair. Indeed, any coherent sheaf $\mathcal{F}$ has a unique factorization

$$0 \to \mathcal{F}_{\mathrm{tors}} \to \mathcal{F} \to \mathcal{F}/\mathcal{F}_{\mathrm{tors}} \to 0$$

where $\mathcal{F}_{\mathrm{tors}} \subset \mathcal{F}$ is the torsion subsheaf.

Example 7.16. Let $(Z, \mathcal{A})$ be a stability condition on a triangulated category $D^b(\mathcal{A})$ with slicing $\mathcal{P}$. Let $\alpha \in (0,1]$. Then $\mathcal{T}_\alpha = \mathcal{P}((\alpha, 1])$ and $\mathcal{F}_\alpha = \mathcal{P}((0, \alpha])$ is a torsion pair in the heart $\mathcal{A}$. The tilt of $\mathcal{A}$ with respect to $(\mathcal{T}_\alpha, \mathcal{F}_\alpha)$ is the heart

$$\mathcal{A}^{\#} = \mathcal{P}((\alpha, 1+\alpha]).$$

A picture is useful here! (See [Huy11, Figure before Remark 1.19].) The tilt $\mathcal{A}^{\#}$ is obtained by rotation the x-axis counter-clockwise by $\pi\alpha$.

We define the charge on $K(\mathcal{A}^{\#})$ as

$$Z^{\#}\colon K(\mathcal{A}^{\#}) \to \mathbb{C}, \qquad [E] \mapsto e^{\pi\alpha} Z(E).$$

One checks that $(\mathcal{A}^{\#}, Z^{\#})$ is also a stability condition on $D^b(\mathcal{A})$.

Later we will reinterpret this stability condition in terms of the action of the universal cover $\widetilde{\mathrm{GL}}_2^+(\mathbb{R})$ of GL_2^+ on the space of stability conditions.

8. Lecture 8 (April 16): Quiver GIT—Lecture by Chi-yu Cheng

This lecture established the following result from [Kin94]:

Theorem 8.1. *For each quiver $Q = (Q_0, Q_1, s, t)$ and $\theta \in \mathbb{Z}^{Q_0}$, let $\sigma = (\mathrm{Rep}^{\mathrm{fd}}(Q), Z)$ be the stability condition on $D^b(Q)$ with standard heart and with charge*

$$Z\colon K(Q) \to \mathbb{C}, \qquad [V = (V_i)_{i \in Q_0}] \mapsto \sum_{i \in Q_0} (\dim V_i)(\theta_i + i).$$

For each dimension vector $d \in \mathbb{Z}^{Q_0}$, there is a quasi-projective variety $M_Q^{\sigma-\mathrm{ss}}(d)$ parameterizing S-equivalence classes of σ-semistable quiver representations V with $\dim V = d$. *Moreover, if Q has no self-loops, then $M_Q^{\sigma-\mathrm{ss}}(d)$ is projective.*

The moduli space $M_Q^{\sigma-\mathrm{ss}}(d)$ is constructed using GIT as follows. First, one lets W_i be a vector space of dimension d_i for each $i \in Q_0$ and one defines the *representation space*

$$\mathcal{R}_Q(d) := \prod_{\alpha \in Q_1} \mathrm{Hom}(W_{s(\alpha)}, W_{t(\alpha)})$$

which inherits an action of

$$\mathrm{GL}_d := \prod_{i \in Q_0} \mathrm{GL}(W_i)$$

where $g = (g_i)$ acts on $f = (f_\alpha)$ via $(g \cdot f)_\alpha = g_{t(\alpha)} f_\alpha g_{s(\alpha)}^{-1}$. One can define the character

$$\chi\colon \mathrm{GL}_d \to \mathbb{G}_m, \qquad (g_i) \mapsto \prod_{i \in Q_0} \det(g_i).$$

Any point of the representation space $\mathcal{R}_Q(d)$ may be considered as a representation of the quiver Q, and two points yield isomorphic quiver representations if and only if they are in the same GL_d-orbit. Therefore, to construct the moduli space $M_Q^{\sigma-\mathrm{ss}}(d)$, we take the GL_d-quotient of $\mathcal{R}_Q(d)$. In order to make this work, one needs to show that a point in $\mathcal{R}_Q(d)$ is GIT-semistable with respect to the character χ if and only if the corresponding quiver representation is σ-semistable. (Note that both χ and σ depend on the choice of θ.) This last fact is established via an elegant calculation using the Hilbert–Mumford criterion.

9. Lecture 9 (April 18): Examples of stability conditions

Recall that we have two ways to think about a stability condition on a triangulated category $\mathcal{D}$: (1) a pair $(\mathcal{A}, Z)$ consisting of a heart $\mathcal{A}$ of a bounded t-structure and a charge $K(\mathcal{A}) \cong K(\mathcal{D}) \to \mathbb{C}$ defining a stability condition on the category $\mathcal{A}$ (which is necessarily abelian), and (2) a pair $(\mathcal{P}, Z)$ where $\mathcal{P}$ is a slicing of $\mathcal{D}$ and $Z\colon K(\mathcal{D}) \to \mathbb{C}$ is a charge such that for $E \neq 0 \in \mathcal{P}(\phi)$, the charge satisfies $Z(E) \in \mathbb{R}_{>0} e^{\pi i \phi}$.

Recall that there are various filtrations floating around in this theory:

- If $\mathcal{A}$ is an abelian category and $Z\colon K(\mathcal{A}) \to \mathbb{C}$ is a stability condition on $\mathcal{A}$, then any object $A \in \mathcal{A}$ has a filtration $0 = A_0 \subset A_1 \subset \cdots \subset A_{m-1} \subset A_m$ where the factors A_i/A_{i-1} are semistable of strictly decreasing phases between $(0, 1]$.
- If $\mathcal{A} \subset \mathcal{D}$ is a heart of a bounded t-structure, then every object in $E \in \mathcal{D}$ has a filtration (5.2.1) where the factors are in shifts $A_i[k_i]$ of $\mathcal{A}$ with the k_i's strictly decreasing integers.
- A slicing $\mathcal{P}$ of a triangulated category $\mathcal{D}$ is a combination of the above two filtrations: every object $E \in \mathcal{D}$ has a filtration (7.1.1) where the factors A_i are semistable of phases $\phi_i \in \mathbb{R}$ which are strictly decreasing.

Recall also that tilting by a torsion pair $(\mathcal{T}, \mathcal{F})$ on an abelian category $\mathcal{A}$ yields a new heart $\mathcal{A}^{\#} \subset D^b(\mathcal{A})$ (Proposition 7.13).

9.1. **Space of stability conditions.** Let $\mathcal{D}$ be a triangulated category. Let $\mathrm{Stab}(\mathcal{D})$ be the set of stability conditions on $\mathcal{D}$. One of the goals of this course is to show that $\mathrm{Stab}(\mathcal{D})$ has the structure of complex manifold. More precisely, let us fix a group homomorphism $K(\mathcal{D}) \to \Lambda$ to a lattice $\Lambda \cong \mathbb{Z}^r$. Define the set $\mathrm{Stab}_\Lambda^{\mathrm{lf}}(\mathcal{D})$ of "locally finite" (to be defined later) stability conditions $\sigma = (\mathcal{A}, Z)$

where the charge $Z\colon K(\mathcal{D}) \to \mathbb{C}$ factors through $K(\mathcal{D}) \to \Lambda$. We will show that $\mathrm{Stab}^{\mathrm{lf}}_{\Lambda}(\mathcal{D})$ is a complex manifold and moreover that the forgetful morphism

$$\mathrm{Stab}^{\mathrm{lf}}_{\Lambda}(\mathcal{D}) \to \mathrm{Hom}_{\mathrm{gp}}(\Lambda, \mathbb{C}) \cong \mathbb{C}^r$$

is a local homeomorphism.

In this course, we are mainly interested in the following three questions:

Question 9.1.

(1) *For certain triangulated categories $\mathcal{D}$ (such as the bounded derived category of coherent sheaves on certain projective varieties or finite dimensional representations of certain quivers), what is the space* $\mathrm{Stab}(\mathcal{D})$*? In the case of a projective variety, we are interested in studying* $\mathrm{Stab}^{\mathrm{lf}}_{\Lambda}(\mathcal{D})$ *where* $\Lambda = H^*(X, \mathbb{C})$ *and* $K(X) \to \Lambda$ *is the Chern character.*

(2) *For a fixed stability condition* $\sigma = (\mathcal{A}, Z) \in \mathrm{Stab}(\mathcal{D})$ *and fixed invariants* α*, what can we say about the moduli space* $M^{\sigma-\mathrm{ss}}$ *of* σ*-semistable objects in* $\mathcal{A}$ *with invariants* α*? In many cases, this will be a projective variety.*

(3) *Can we use our understanding of (1) and (2) to say anything interesting about the original category?*

9.2. **Moduli of quiver representations.** Question 9.1(2) has a very satisfactory answer for certain stability conditions on the bounded derived category $D^b(Q)$ of finite dimensional representations of a quiver Q. Namely, if $Q = (Q_0, Q_1, s, t)$ is a quiver, then we've seen that if we choose any tuple of complex numbers $\theta = (\theta_i) \in \overline{\mathbb{H}}^{Q_0}$, then

$$Z_\theta\colon K(Q) \to \mathbb{C}, \qquad V = (V_i) \mapsto \sum_i (\dim V_i)\theta_i$$

defines a stability condition $\sigma_\theta = (\mathrm{Rep}^{\mathrm{fd}}(Q), Z_\theta)$ on $D^b(Q)$ with the standard heart $\mathrm{Rep}^{\mathrm{fd}}(Q)$. In other words, this gives us a patch

$$\overline{\mathbb{H}}^n \subset \mathrm{Stab}(Q).$$

We've seen last time (Theorem 8.1) that for certain choices of $\theta \in \mathbb{H}^{Q_0}$ and any dimension vector $d \in \mathbb{Z}^{Q_0}$, there is quasi-projective variety $M_Q^{\sigma_\theta-\mathrm{ss}}(d)$ parameterizing σ_θ-semistable quiver representations with dimension vector d up to S-equivalence. Moreover, if Q has no self-loops then $M_Q^{\sigma_\theta-\mathrm{ss}}(d)$ is projective.

Recall also that by Gabriel's theorem implies that if Q is a quiver of ADE-type, then for any fixed dimension vector $d \in \mathbb{Z}^{Q_0}$, then there are only finitely many isomorphism classes of quiver representations of dimension d. Thus, the moduli space $M_Q^{\sigma_\theta-\mathrm{ss}}(d)$ with respect to any choice z is just a point.

Here are some more interesting examples:

Example 9.2. Let Q be the quiver with one vertex and one loop. For an integer d, the representation space is

$$\mathcal{R}_Q(d) = \mathrm{Hom}(k^d, k^d) = \mathrm{Mat}_{d,d}$$

The action of GL_d on $\mathcal{R}_Q(d)$ is given by conjugation. Let $\theta \in \overline{\mathbb{H}}$ and $\sigma_\theta \in \mathrm{Stab}(Q)$ be the corresponding stability condition. In this case, every quiver representation is semistable. The GIT quotient is

$$\mathrm{Mat}_{d,d} \to \mathrm{Mat}_{d,d} /\!\!/ \mathrm{GL}_d \cong \mathbb{A}^d$$

which takes a matrix A to the coefficients of its characteristic polynomial. In other words, two different matrices are identified in the GIT quotient if the elementary symmetric functions evaluated at the generalized eigenvalues are equal. In particular, a matrix M in Jordan block form is S-equivalent to the matrix M_0 where the 1's above the diagonal in M are set to be 0. Indeed, one can see that $M_0 \in \overline{\mathrm{GL}_n \cdot M}$ explicitly here as a degeneration coming from a one-parameter subgroup $\lambda\colon \mathbb{G}_m \to \mathrm{GL}_n$ so that $M_0 = \lim_{t\to 0} \lambda(t)M$.

In this case, we see that the moduli space $M^{\sigma_\theta-\mathrm{ss}} = \mathbb{A}^d$ is not projective.

Example 9.3. Let Q be the Kronecker quiver with two vertices 1 and 2 and $n+1$ arrows from 1 to 2. Let us first consider the case of dimension vector $d = (1,1)$. In this case, the representation space

$$\mathcal{R}_(d) = k^{d+1}$$

which has an action of $\mathbb{G}_m^2$. The diagonal $\mathbb{G}_m$ acts trivially so we might as well consider the action of the quotient $\mathbb{G}_m$. This action is given by the standard scaling action $t \cdot (x_0, \ldots, x_d) = (tx_0, \ldots, tx_d)$. Let $\theta = (\theta_1, \theta_2) \in \overline{\mathbb{H}}^2$ and $\sigma_\theta \in \mathrm{Stab}(Q)$ the corresponding stability condition. Let S_1 and S_2 be the simple quiver representations with dimension vectors $(1,0)$ and $(0,1)$, respectively. If V is a Q-representation corresonding to $x = (x_0, \ldots, x_d) \in k^{d+1}$, then S_1 is always a subrepresentation of V while S_2 is a subrepresentation if and only if each $x_i = 0$. We have three cases:

- $\arg(\theta_1) < \arg(\theta_2)$: there are no semistable representations since $S_2 \subset V$ always destabilizes a quiver representation V; clearly $Z(S_2) = \theta_2$ has larger phase then $Z(V) = \theta_1 + \theta_2$. In this case, the moduli space is empty;
- $\arg(\theta_1) = \arg(\theta_2)$: every representation is strictly semistable and they are all S-equivalent; In this case, the moduli space is a point;
- $\arg(\theta_1) > \arg(\theta_2)$: the semistable representations correspond to tuples $x = (x_0, \ldots, x_n)$ such that not all the x_i are zero. The moduli space is $\overline{M}_Q^{\sigma_\theta-\mathrm{ss}} = (\mathbb{A}^{n+1} \setminus 0)/\mathbb{G}_m \cong \mathbb{P}^n$.

Example 9.4. Let Q again be the Kronecker quiver with two vertices 1 and 2 and $n+1$ arrows. This time consider the dimension vector $d = (1,2)$. The representation space is

$$\mathcal{R}_Q(d) = \mathrm{Hom}(k, k^2)^{\oplus(n+1)} = \mathrm{Mat}_{2,n+1}$$

and the group $\mathrm{GL}(d) = \mathbb{G}_m \times \mathrm{GL}_2$. Since the diagonal $\mathbb{G}_m$ acts trivially, we can simply consider the action of GL_2. We can consider a matrix

$$\begin{pmatrix} x_0 & \cdots & x_n \\ y_0 & \cdots & y_n \end{pmatrix}$$

as a quiver representation. Choose again $z = (\theta_1, \theta_2) \in \overline{\mathbb{H}}$ and let's consider only the case of $\arg(\theta_1) > \arg(\theta_2)$ as otherwise the moduli space will be empty or a point. In this case, a quiver representation V will be semistable if and only if there is no subrepresentation $V' \subset V$ where the dimension vector of V' is $(1,1)$. This is equivalent to saying that the vectors x and y are linearly independent. Let $\sigma \subset \mathrm{Mat}_{2,n}$ be the matrices that are not full rank. Thus, the moduli space is

$$\overline{M}_Q^{\sigma_\theta-\mathrm{ss}} = (\mathrm{Mat}_{2,n} \setminus \Sigma)/\mathbb{G}_m \cong \mathrm{Gr}(2, k^{n+1})$$

More generally, if we consider the dimension vector $d = (1, d_2)$ with the same choices for θ, we will have

$$\overline{M}_Q^{\sigma_\theta-\mathrm{ss}} \cong \mathrm{Gr}(d_2, k^{n+1}).$$

9.3. **A tilting of** $\mathrm{Coh}(\mathbb{P}^1)$**.** Let $(\mathrm{Coh}(\mathbb{P}^1), Z)$ be the stability condition on $D^b(\mathbb{P}^1)$ defined by the charge

$$Z\colon K(\mathrm{Coh}(\mathbb{P}^1)), \qquad [E] \mapsto -\deg(E) + i\,\mathrm{rk}(E).$$

Using Example 7.16, we can tilt this stability condition by rotating counter-clockwise by $\pi/4$. That is, if we set $\alpha = 1/4$, then the new heart $\mathcal{A}^\#$ contains both $\mathcal{O}$ and $\mathcal{O}(-1)[1]$.

In fact, by Beilinson, if we set $G = \mathcal{O} \oplus \mathcal{O}(1)$ and $E = \mathrm{End}(G, G)$, then

$$\begin{aligned} \mathrm{RHom}(G, -)\colon D^b(\mathbb{P}^1) &\to D^b(\mathrm{Mod}^{\mathrm{fg}}(E)) \\ F &\mapsto \mathrm{RHom}(G, F) \end{aligned} \tag{9.3.1}$$

gives an equivalence of triangulated categories between bounded complexes on $\mathbb{P}^1$ and bounded complexes of finitely generated left modules over E. This is an instance of the more general fact.

Proposition 9.5. [Bon89, Theorem 6.2] *If X is a smooth projective variety over $\mathbb{C}$ and $D^b(X)$ is generated by a strong exceptional collection $(E_0, \dots, E_n)$*[2] *then if set $G = \bigoplus_i E_i$ and $E = \mathrm{End}(G)$, then*

$$\begin{aligned} \mathrm{RHom}(G, -)\colon D^b(X) &\to D^b(\mathrm{Mod}^{\mathrm{fg}}(E)) \\ F &\mapsto \mathrm{RHom}(G, F) \end{aligned}$$

is an equivalence of categories.

Moreover, if Q is the Kronecker quiver with two vertices and two arrows from the first vertex to the second, we have an equivalence of $D^b(\mathrm{Mod}^{\mathrm{fg}}(E))$ with the derived category of finite dimensional Q-representations $D^b(Q) = D^b(\mathrm{Rep}^{\mathrm{fd}}(Q))$.

Let S_1 and S_2 be the simple representations of Q with dimension vectors $(1, 0)$ and $(0, 1)$ respectively. One can check that $\mathrm{RHom}(G, -)$ sends $\mathcal{O}$ to S_1 and $\mathcal{O}(-1)[1]$ to S_2.

In other words, the above equivalence (9.3.1) yields an identification of the heart $\mathcal{A}^\# \subset D^b(\mathbb{P}^1)$ with the heart $\mathrm{Rep}^{\mathrm{fd}}(Q) \subset D^b(Q)$.

10. Lecture 10 (April 23): More examples

10.1. **Another application of quiver GIT.** Continuing from the last lecture, one can use Theorem 8.1 to give an intrinsic GIT construction of the moduli space of semistable vector bundles over a smooth projective curve. This construction is based on the beautiful paper [ACK09]. This construction is more functorial and involves much simpler semistability calculations than the GIT construction of the moduli space we saw last quarter. DETAILS TO BE ADDED LATER

[2] An exception collection is a tuple of objects $(E_1, \dots, E_n)$ such that

$$\mathrm{Ext}^n(E_i, E_j) = \begin{cases} 0 & \text{if } n \neq 0 \text{ or } i < j \\ k & \text{otherwise.} \end{cases}$$

It generates the triangulated category $\mathcal{D}$ if the small triangulated subcategory of D containing the objects E_i is all of $\mathcal{D}$.

10.2. **The action of $\widetilde{\mathrm{GL}}_2^+(\mathbb{R})$.** Let $\mathrm{GL}_2^+(\mathbb{R})$ be the group of 2×2 matrices with positive determinant. The universal cover of $\widetilde{\mathrm{GL}}_2^+(\mathbb{R})$ of $\mathrm{GL}_2^+(\mathbb{R})$ has the following description:

$$\begin{aligned}\widetilde{\mathrm{GL}}_2^+(\mathbb{R}) = \{(A, f) \ \mid A \in \mathrm{GL}_2^+(\mathbb{R}), f : \mathbb{R} \to \mathbb{R} \text{ increasing such that} \\ Ae^{i\pi\theta} \in \mathbb{R}_{>0} e^{i\pi f(\theta)} \text{ and } f(\theta + 1) = f(\theta) + 1\}\end{aligned}$$

There is a right action of the group $\widetilde{\mathrm{GL}}_2^+(\mathbb{R})$ on $\mathrm{Stab}(\mathcal{D})$ as follows: given $(\mathcal{P}, Z) \in \mathrm{Stab}(\mathcal{D})$ and $(A, f) \in \widetilde{\mathrm{GL}}_2^+(\mathbb{R})$, then

$$(\mathcal{P}, Z) \cdot (A, f) = (\mathcal{P}', A^{-1} \circ Z) \qquad \text{where } \mathcal{P}'(\phi) = \mathcal{P}(f(\phi)))$$

The heart of the new bounded t-structure is $\mathcal{A}^{\#} = \mathcal{P}((f(0), f(1)])$.

For instance, for each $\alpha \in \mathbb{R}$, we have an element $R_\alpha = (e^{i\pi\alpha}, x \mapsto x + \alpha) \in \widetilde{\mathrm{GL}}_2^+(\mathbb{R})$. The heart of new stability condition $(\mathcal{A}, Z) \cdot R_\alpha$ is $\mathcal{P}((\alpha, \alpha + 1])$. For $\alpha \in (0, 1]$, this is a tilt of the original heart $\mathcal{A} = \mathcal{P}((0, 1])$ by the torsion pair $(\mathcal{T}_\alpha, \mathcal{F}_\alpha) = \big(\mathcal{P}((\alpha, 1]), \mathcal{P}((0, \alpha])\big)$. If α is an even integer, then the charge doesn't change and the new heart is $\mathcal{A}[\alpha]$. If α is an odd integer, then the new charge is $-Z$ and the new heart is $\mathcal{A}[\alpha]$.

10.3. **A few homological results.** The following two propositions will be essential in all of future attempts to compute the space of stability conditions.

Proposition 10.1. *If $\mathcal{A}_1, \mathcal{A}_2$ are both hearts of bounded t-structures on a triangulated category $\mathcal{D}$ and $\mathcal{A}_1 \subset \mathcal{A}_2$, then $\mathcal{A}_1 = \mathcal{A}_2$.*

Proof. Let $E \in \mathcal{A}_2$. The heart $\mathcal{A}_1$ induces a filtration

$$\begin{tikzcd}
0 = E_0 \arrow[rr] & & E_1 \arrow[dl] \arrow[r] & \cdots \arrow[r] & E_{m-1} \arrow[rr] & & E_m = E \arrow[dl] \\
& A_1[k_1] \arrow[ul, dashed, "{[1]}"] & & & & A_m[k_m] \arrow[ul, dashed, "{[1]}"] &
\end{tikzcd}$$

with each $A_i \in \mathcal{A}_1$ and $k_1 > \cdots > k_m$. This is also a filtration where the factors are in shifts of $\mathcal{A}_2$. But there is also the trivial filtration

$$\begin{tikzcd}
0 \arrow[rr] & & E \arrow[dl] \\
& E[0] \arrow[ul, dashed, "{[1]}"] &
\end{tikzcd}$$

so by uniqueness of filtrations induced by heart $\mathcal{A}_2$, we must have $m = 1$ and that $E = A_m \in \mathcal{A}_1$. □

Definition 10.2. An abelian category $\mathcal{A}$ has *cohomological dimension* $\leq n$ if for all $A, B \in \mathcal{A}$

$$\mathrm{Ext}^i(A, B) = 0 \quad \text{for } i > n.$$

Remark 10.3. Note that $\mathcal{A}$ is semisimple (i.e. every short exact sequence splits) if and only if $\mathcal{A}$ has cohomological dimension 0.

Proposition 10.4. *If $\mathcal{A}$ is an abelian category with cohomological dimension ≤ 1, then any object $E \in D^b(\mathcal{A})$ is isomorphic to the direct sum $\bigoplus_{n \in \mathbb{Z}} \mathrm{H}^n(E)[-n]$ of its cohomology.*

Remark 10.5. In other words, in $D^b(\mathcal{A})$, any object E (with cohomology in the interval $[a,b]$) is isomorphic to the complex

$$0 \to \mathrm{H}^a(E)[-a] \xrightarrow{0} \mathrm{H}^{a+1}(E)[-a+1] \xrightarrow{0} \cdots \xrightarrow{0} \mathrm{H}^b(E)[b] \to 0$$

Proof. Suppose that the cohomology of E lives in the interval $[a, a+k]$. We will argue by induction on k where the base case $k=0$ is clear. If $k>0$, consider the exact triangle obtained by truncating

$$\tau_{\leq a}E \to E \to \tau_{>a}E. \tag{10.3.1}$$

Note that $\tau_{\leq a}E \cong \mathrm{H}^a(E)[-a]$. By the inductive hypothesis, we have that $\tau_{>a}E \cong \bigoplus_{n=a+1}^{a+k} \mathrm{H}^n(E)[-n]$. The exact triangle (10.3.1) splits if and only if the morphism $\tau_{>a}E \to \tau_{\leq a}E[1]$ is zero. We compute that

$$\begin{aligned}\mathrm{Hom}_{D^b(\mathcal{A})}(\tau_{>a}E, \tau_{\leq a}E[1]) &= \mathrm{Hom}_{D^b(\mathcal{A})}\left(\bigoplus_{n=a+1}^{a+k} \mathrm{H}^n(E)[-n], \mathrm{H}^a(E)[1-a]\right) \\ &= \bigoplus_{n=a+1}^{a+k} \mathrm{Ext}_{\mathcal{A}}^{n-a+1}\left(\mathrm{H}^a(E), \mathrm{H}^n(E)\right) \\ &= 0\end{aligned}$$

because for each $n \geq a+1$, the index $n-a+1 \geq 2$ so that the Ext group vanishes because $\mathcal{A}$ has cohomological dimension ≤ 1. $\square$

10.4. **A simple example.** Although some might argue that this example is completely trivial, it is nonetheless exhibits a basic feature of the space of stability conditions. Let $\mathcal{A} = \mathrm{Vect}_k^{\mathrm{fd}}$ be the category of finite dimensional vector spaces over a field k. Alternatively, one can view $\mathcal{A}$ as the category $\mathrm{Coh}(\mathrm{Spec}\, k)$ of coherent sheaves on a point or as the category $\mathrm{Rep}^{\mathrm{fd}}(A_1)$ of finite dimensional representations of the A_1-quiver. Note that $\mathcal{A}$ is a semisimple category and has a unique simple object $S=k$, which is the one dimensional vector space.

Proposition 10.6. *If $(\mathcal{A}', Z)$ is a stability condition on $D^b(\mathcal{A})$, then S is semisimple.*

Proof. If S is not semistable, there exists an an exact triangle (induced by say part of the Harder–Narasimhan filtration of S)

$$A \to S \to B \tag{10.4.1}$$

with $A, B \in D^b(\mathcal{A})$ nonzero and $\mathrm{Ext}^i(A,B) = 0$ for $i \leq 0$. Since $\mathcal{A}$ is semisimple, we may apply Proposition 10.4 to write

$$A = \bigoplus_{n\in\mathbb{Z}} A_n[-n] \qquad \text{and} \qquad B = \bigoplus_{n\in\mathbb{Z}} B_n[-n]$$

where $A_n = H^n(A)$ and $B_n = H^n(B)$. The exact triangle (10.4.1) gives a long exact sequence of cohomology

$$0 \to B_{-1} \to A_0 \to S \to B_0 \to A_1 \to 0 \tag{10.4.2}$$

and $B_{i-1} \xrightarrow{\sim} A_i$ for $i \neq 0, 1$. Given any morphism $\alpha\colon A_i \to B_{i-1}$, one has the factorization

$$A \twoheadrightarrow A_i[-i] \xrightarrow{\alpha[-i]} B_{i-1}[-i] \hookrightarrow B[-1].$$

This shows that there are no non-zero maps $\alpha\colon A_i \to B_{i-1}$ as we've expressed $\mathrm{Hom}(A_i, B_{i-1}) \subset \mathrm{Hom}(A, B[-1]) = \mathrm{Ext}^{-1}(A, B) = 0$, In particular, for $i \neq 0, 1$, the isomorphism $A_i \xrightarrow{\sim} B_{i-1}$ must be zero so $A_i = B_{i-1} = 0$.

Since S is simple, (10.4.2) breaks into either of two sequences:

(1) a short exact sequence $0 \to B_{-1} \to A_0 \to S \to 0$ and an isomorphism $B_0 \xrightarrow{\sim} A_1$, which by the above observation implies that $B_0 = A_1 = 0$. As $\mathcal{A}$ is semisimple, we have that $A_0 = B_{-1} \oplus S$. The surjection $A_0 \twoheadrightarrow B_{-1}$ is nonzero contradicting $\mathrm{Hom}(A_0, B_{-1}) = 0$; or

(2) a short exact sequence $0 \to S \to B_0 \to A_1 \to 0$ and an isomorphism $B_{-1} \xrightarrow{\sim} A_0$, which again implies that $B_{-1} = A_0 = 0$. Writing $B_0 = A_1 \oplus S$ gives a non-zero inclusion $A_1 \hookrightarrow B_0$ contradicting $\mathrm{Hom}(A_0, B_{-1}) = 0$.

We've shown there can't exist a triangle (10.4.1) with $A, B \neq 0$. □

Corollary 10.7. *If $(\mathcal{A}', Z)$ is a stability condition on $D^b(\mathcal{A})$, then there exists a unique integer k such that $\mathcal{A}' = \mathcal{A}[k]$.*

Proof. Indeed, let k be the unique integer such that $S[k] \in \mathcal{A}'$. This gives an inclusion of hearts $\mathcal{A}[k] \subset \mathcal{A}'$, which must be an equality by Proposition 10.1. □

We conclude that a stability condition $(\mathcal{A}', Z) \in \mathrm{Stab}(D^b(\mathcal{A}))$ is defined by two pieces of data:

(1) An integer $k \in \mathbb{Z}$ such that $\mathcal{A}' = \mathcal{A}[k]$; and
(2) The charge $\theta := Z(S) \in \mathbb{C}$ such that $(-1)^k Z(S) \in \overline{\mathbb{H}}$.

Let $\log\colon \mathrm{H} \to \mathbb{C}$ be the inverse of the exponential whose image is the set of complex numbers whose imaginary part is in $(0, \pi]$. Then we can define

$$\begin{aligned} &\mathrm{Stab}(D^b(A_1)) \to \mathbb{C} \\ &\qquad (\mathcal{A}, Z) \mapsto \log((-1)^k\theta) - \pi i k. \end{aligned}$$

One checks that this is bijective (easy) and continuous (here is where you need a '-' sign). Moreover, we have a commutative diagram

$$\begin{tikzcd} \mathrm{Stab}(D^b(\mathcal{A})) \arrow[r, "\sim"] \arrow[d] & \mathbb{C} \arrow[d, "\exp"] \\ \mathrm{Hom}_{\mathrm{gp}}(K(\mathcal{A}), \mathbb{C}) \arrow[r, "\sim"] & \mathbb{C} \end{tikzcd}$$

where the vertical left arrow is the forgetful functor $(\mathcal{A}, Z) \mapsto Z$ and the bottom arrow is the map $Z \mapsto Z(S)$. We make the following observations:

- The image of the forgetful functor $\mathrm{Stab}(D^b(\mathcal{A})) \to \mathrm{Hom}_{\mathrm{gp}}(K(\mathcal{A}), \mathbb{C}) \cong \mathbb{C}$ is $\mathbb{C}^*$;
- There are identifications $\mathrm{Aut}(D^b(\mathcal{A})) \cong \pi_1(\mathbb{C}^*) \cong \mathbb{Z}$ acting naturally on the simply connected space of stability conditions $\mathrm{Stab}(D^b(\mathcal{A})) \cong \mathbb{C} \cong \widetilde{\mathbb{C}^*}$. Here $\mathrm{Stab}(D^b(\mathcal{A}))$ is the universal cover of $\mathbb{C}^*$ and the action of $\mathrm{Aut}(D^b(\mathcal{A}))$ by shifts corresponds to deck transformations.
- The group $\widetilde{\mathrm{GL}}_2^+(\mathbb{R})$ acts transitively on $\mathrm{Stab}(D^b(\mathcal{A}))$. The subgroup $\mathbb{Z} \subset \widetilde{\mathrm{GL}}_2^+(\mathbb{R})$ consisting of pairs $(\mathrm{id}, \phi \mapsto \phi + 2n)$ (where n is an integer) acts on $\mathrm{Stab}(D^b(\mathcal{A}))$ by preserving the charge and by shifting a heart $\mathcal{A}'$ by $\mathcal{A}'[2n]$.

11. Lecture 11 (April 25): The A_2-quiver

For abelian categories $\mathcal{A}$, we would like to understand the covering

$$\begin{tikzcd} \mathrm{Stab}(D^b(\mathcal{A})) \arrow[d] & \circlearrowleft & \widetilde{\mathrm{GL}}_2^+(\mathbb{R}) \arrow[d] \\ \mathrm{Hom}(K(\mathcal{A}), \mathbb{C}) & \circlearrowleft & \mathrm{GL}_2^+(\mathbb{R}) \end{tikzcd}$$

which is equivariant with respect to $\widetilde{\mathrm{GL}}_2^+(\mathbb{R}) \to \mathrm{GL}_2^+(\mathbb{R})$. Note that $\mathrm{GL}_2^+(\mathbb{R})$ deformation retracts onto $\mathrm{SO}_2(\mathbb{R}) \cong S^1$ (using Gram-Schmidt to write every matrix A uniquely as a product $A = QR$ where $Q \in \mathrm{SO}_2(\mathbb{R})$ and R is upper triangular with 1's along the diagonal). Therefore, $\pi_1(\mathrm{GL}_2^+(\mathbb{R})) \cong \mathbb{Z}$.

Recall that $\mathrm{Stab}(D^b(\mathcal{A}))$ also has a left action by the group $\mathrm{Aut}(D^b(\mathcal{A}))$ of autoequivalences of $D^b(\mathcal{A})$. This action commutes with the $\mathrm{GL}_2^+(\mathbb{R})$ action.

Today, our goal to understand this picture when $\mathcal{A}$ is the category of finite dimensional representations of the A_2-quiver.

11.1. The A_2-quiver.

Set $\mathcal{A} = \mathrm{Rep}^{\mathrm{fd}}(A_2)$. We will denote $D^b(A_2) := D^b(\mathcal{A})$. An object $V \in \mathcal{A}$ corresponds to a pair of vector spaces V_1, V_2 and a linear transformation $f\colon V_1 \to V_2$. We write this as $V = [V_1 \xrightarrow{f} V_2]$.

There are two simple objects $S_1 = [k \to 0]$ and $S_2 = [0 \to k]$, and an indecomposable object $T = [k \xrightarrow{\mathrm{id}} k]$ which sits in an exact sequence

$$0 \to S_2 \to T \to S_1 \to 0. \tag{11.1.1}$$

Any object $[V_1 \xrightarrow{f} V_2]$ decomposes as

$$\begin{bmatrix} V_1 \\ \downarrow \\ V_2 \end{bmatrix} = \begin{bmatrix} \ker(f) \\ \downarrow \\ 0 \end{bmatrix} \oplus \begin{bmatrix} [V_1/\ker(f)] \\ \downarrow \\ \mathrm{im}(f) \end{bmatrix} \oplus \begin{bmatrix} 0 \\ \downarrow \\ \mathrm{coker}(f) \end{bmatrix} \tag{11.1.2}$$

or in other words $S_1^{\oplus a} \oplus T^{\oplus b} \oplus S_2^{\oplus c}$ where $a = \dim\ker(f)$, $b = \mathrm{rk}(f) = \dim\mathrm{im}(f)$, and $c = \dim\mathrm{coker}(f)$.

For $V = [V_1 \xrightarrow{f} V_2]$, we have that

$$\begin{array}{lll} \mathrm{Hom}(S_1, V) = \ker(f) & \mathrm{Hom}(S_2, V) = V_2 & \mathrm{Hom}(T, V) = V_1 \\ \mathrm{Hom}(V, S_1) = V_1^\vee & \mathrm{Hom}(V, S_2) = \mathrm{coker}(f^\vee) & \mathrm{Hom}(V, T) = V_2^\vee \end{array} \tag{11.1.3}$$

We see that S_1 is injective, S_2 is projective and T is both injective and projective. Note that this exact sequence (11.1.1) is both an injective resolution of S_2 and a projective resolution of S_1. Using (11.1.1), we can compute that

$$\mathrm{Ext}^1(S_1, V) = \mathrm{coker}(f) \qquad \text{and} \qquad \mathrm{Ext}^1(V, S_2) = \ker(f^\vee) \tag{11.1.4}$$

while higher Ext's vanish. Indeed, applying $\mathrm{Hom}(-, V)$ to $S_2 \to T$ yields $\mathrm{Hom}(T_1, V) = V_1 \xrightarrow{f} V_2 = \mathrm{Hom}(S_2, V)$ and the 0th cohomoloy is $\ker(f)$ while the 1st cohomology is $\mathrm{coker}(f)$. The second statement is similar. Moreover, given an exact sequence $0 \to V' \to V \to V'' \to 0$ in $\mathcal{A}$, the long exact sequence induced by the derived functors $\mathrm{Ext}^i(S_1, -)$ is precisely the snake lemma applied to the morphism from the short exact sequences.

For instance, $\mathrm{Ext}^1(S_1, S_2) = k$ is generated by the extension (11.1.1).

The above calculations establish that the abelian category $\mathcal{A}$ has cohomological dimension 1.

11.2. **Stability functions on the standard heart.** For $\theta_1, \theta_2 \in \overline{\mathbb{H}}$, group homomorphism

$$Z\colon K(A_2) \to \mathbb{C}, \qquad [V] \mapsto (\dim V_1)\theta_1 + (\dim V_2)\theta_2$$

defined a stability condition on $\mathcal{A}$. We break the analysis into three cases according to the phases $\phi_1 = \phi(S_1)$ and $\phi_2 = \phi(S_2)$.

Case 1: $\phi_1 < \phi_2$. In this case, $S_2 \subset T$ is destabilizing and the Harder–Narasimhan filtration of T is $0 \to S_2 \to T \to S_1 \to 0$. Then $\mathcal{P}(\phi_1) = \langle S_1 \rangle$ and $\mathcal{P}(\phi_2) = \langle S_2 \rangle$. Our picture of the slicing on $D^b(A_2)$ is

$$\cdots \quad \underbrace{\boxed{\langle S_2[1] \rangle \qquad \langle S_1[1] \rangle}}_{\mathcal{A}[1]} \quad \underbrace{\boxed{\langle S_2 \rangle \qquad \langle S_1 \rangle}}_{\mathcal{A}} \quad \cdots$$

Case 2: $\phi_1 = \phi_2$. Everything in $\mathcal{A}$ is semistable of phase ϕ_1; that is, $\mathcal{P}(\phi_1) = \mathcal{A}$.

Case 3: $\phi_1 > \phi_2$. In this case, T is semistable. We have $\mathcal{P}(\phi_1) = \langle S_1 \rangle$, $\mathcal{P}(\phi') = \langle T \rangle$ where $\phi_1 > \phi' > \phi_2$ is the angle of $\theta_1 + \theta_2$, and $\mathcal{P}(\phi_2) = \langle S_2 \rangle$. Our picture of the slicing is

$$\cdots \quad \underbrace{\boxed{\langle S_1[1] \rangle \qquad \langle T[1] \rangle \qquad \langle S_2[1] \rangle}}_{\mathcal{A}[1]} \quad \underbrace{\boxed{\langle S_1 \rangle \qquad \langle T \rangle \qquad \langle S_2 \rangle}}_{\mathcal{A}} \quad \cdots$$

11.3. **Orbits of the stability conditions on the standard heart.** We now consider the orbits of the stability conditions on the standard heart introduced in §11.2 under the action of $\widetilde{\mathrm{GL}}_2^+(\mathbb{R})$. We will focus on identifying the new hearts in each of the three cases.

Case 1: $\phi_1 < \phi_2$. Let $(A, f) \in \widetilde{\mathrm{GL}}_2^+(\mathbb{R})$ and $(\mathcal{P}', Z') = (\mathcal{P}, Z) \cdot (A, f)$. The new heart is $\mathcal{P}'((0,1]] = \mathcal{P}((f(0), f(1)])$. If $f(0) \in [n - 1 + \phi_2, n + \phi_1)$ for an integer n, then then $\mathcal{P}'((0,1]) = \mathcal{A}[n] = \langle S_1[n], S_0[n] \rangle$. Otherwise, if $f(0) \in [n + \phi_1, n + \phi_2)$, then $\mathcal{P}'((0,1]) = \langle S_0[n+1], S_1[n] \rangle$. In the latter case, our picture looks like

$$\cdots \qquad \langle S_2[1] \rangle \qquad \underbrace{\boxed{\langle S_1[1] \rangle \qquad \langle S_2 \rangle}}_{\mathcal{A}^{\#}} \qquad \langle S_1 \rangle \qquad \cdots$$

where the red box is our new heart. This is obtained for instance by rotation counterclockwise by R_α for an angle α such that ϕ_2 just crosses the negative real axis. This heart is the tilt of the standard heart $\mathcal{A}$ with respect to the torsion pair $\mathcal{T} = \langle S_1 \rangle$ and $\mathcal{F} = \langle S_0 \rangle$. The factorization of an element $[V_1 \to V_2] \in \mathcal{A}$ induced from this torsion pair is

$$0 \to \begin{bmatrix} 0 \\ \downarrow \\ V_2 \end{bmatrix} \to \begin{bmatrix} V_1 \\ \downarrow \\ V_2 \end{bmatrix} \to \begin{bmatrix} V_1 \\ \downarrow \\ 0 \end{bmatrix} \to 0.$$

Explicitly, as $\mathcal{A}$ has cohomological dimension 1, we may write the new heart as

$$\mathcal{A}^{\#} = \left\{ \begin{bmatrix} V \\ \downarrow \\ 0 \end{bmatrix}^{-1} \xrightarrow{0} \begin{bmatrix} 0 \\ \downarrow \\ W \end{bmatrix}^{0} \right\}$$

In other words, every object $\mathcal{A}^{\#}$ is isomorphic in $D^b(A_2)$ to $S_1[1]^a \oplus S_2^b$ for some a, b. By computing $\mathrm{Hom}(S_1[1]^a \oplus S_2^b, S_1[1]^{a'} \oplus S_2^{b'}) \cong \mathrm{Hom}(k^a, k^{a'}) \oplus \mathrm{Hom}(k^b, k^{b'})$.

We see that $\mathcal{A}^{\#}$ is simply the category of pairs of vector spaces (or equivalently $\mathbb{Z}/2$-graded vector spaces). This is a semisimple category such that $D^b(\mathcal{A}^{\#}) \not\cong D^b(\mathcal{A})$. For instance, the object T is not in $D^b(\mathcal{A}^{\#})$.

Case 2: $\phi_1 = \phi_2$. In this case, clearly the heart obtained by acting by any element of $\widetilde{\mathrm{GL}}_2^+(\mathbb{R})$ is simply a shift of the standard heart.

Case 3: $\phi_1 > \phi_2$. Acting by an element of $\widetilde{\mathrm{GL}}_2^+(\mathbb{R})$ on the stability condition from §11.2 on the standard heart yields a stability condition whose heart is one of the following three possibilities:

(a) $\mathcal{A}[n] = \langle S_1[n], T[n], S_2[n] \rangle$,
(b) $\langle S_2[n+1], S_1[n], T[n] \rangle$, or
(c) $\langle T[n+1], S_2[n+1], S_1[n] \rangle$.

Case (b) is the nth shift of tilt of the standard heart with respect to the torsion pair $\mathcal{T} = \langle S_1, T \rangle$ and $\mathcal{F} = \langle S_2 \rangle$. The factorization with respect to this torsion pair of an element $[V_1 \xrightarrow{f} V_2] \in \mathcal{A}$ is

$$0 \to \begin{bmatrix} V_1 \\ \downarrow \\ \mathrm{im}(f) \end{bmatrix} \to \begin{bmatrix} V_1 \\ \downarrow \\ V_2 \end{bmatrix} \to \begin{bmatrix} 0 \\ \downarrow \\ \mathrm{coker}(f) \end{bmatrix} \to 0.$$

Elements in the tilted heart $\mathcal{A}^{\#}$ can be written as

$$\left\{ \begin{bmatrix} V_1 & \\ \downarrow & \text{surjective} \\ V_2 & \end{bmatrix}^{-1} \xrightarrow{0} \begin{bmatrix} 0 \\ \downarrow \\ W_2 \end{bmatrix}^{0} \right\}.$$

Case (c) is the nth shift of tilt of the standard heart with respect to the torsion pair $\mathcal{T} = \langle S_0 \rangle$ and $\mathcal{F} = \langle T, S_2 \rangle$. The decomposition of an element $[V_1 \xrightarrow{f} V_2] \in \mathcal{A}$ is

$$0 \to \begin{bmatrix} \ker(f) \\ \downarrow \\ 0 \end{bmatrix} \to \begin{bmatrix} V_1 \\ \downarrow \\ V_2 \end{bmatrix} \to \begin{bmatrix} V_1/\ker(f) \\ \downarrow \\ V_2 \end{bmatrix} \to 0.$$

Elements in the tilt can be written as

$$\left\{ \begin{bmatrix} V_1 \\ \downarrow \\ 0 \end{bmatrix}^{-1} \xrightarrow{0} \begin{bmatrix} W_1 & \\ \downarrow & \text{injective} \\ W_2 & \end{bmatrix}^{0} \right\}$$

11.4. **A reflection functor on $D^b(A_2)$.** Consider the equivalence

$$\begin{aligned} \Sigma \colon D^b(A_2) &\to D^b(A_2) \\ S_1 &\mapsto S_2[1] \\ T &\mapsto S_1 \\ S_2 &\mapsto T \end{aligned}$$

(This is clearly essentially surjective and one checks easily its fully faithful by our explicit understanding of the Homs between the shifts of the three indecomposable objects.) Observe that Σ^3 is the shift functor [1].

Remark 11.1. This auto-equivalence Σ^{-1} can be viewed as a reflection functor between the quiver A_2 and its opposite

$$A_2 \colon 1 \to 2 \qquad \text{and} \qquad A_2^{\mathrm{opp}} \colon 1 \leftarrow 2$$

To see this, first consider the functor on abelian categories

$$F \colon \operatorname{Rep}^{\mathrm{fd}}(A_2) \to \operatorname{Rep}^{\mathrm{fd}}(A_2^{\mathrm{opp}})$$
$$[V_1 \xrightarrow{f} V_2] \mapsto [V_1 \leftarrow \ker(f)]$$

The functor F is left-exact and its right derived functor

$$RF \colon D^b(A_2) \to D^b(A_2^{\mathrm{opp}}),$$
$$\begin{bmatrix} V_1 \\ \downarrow \\ V_2 \end{bmatrix}^0 \mapsto \left(\begin{bmatrix} V_1 \\ \uparrow \\ \ker(f) \end{bmatrix}^0 \xrightarrow{0} \begin{bmatrix} 0 \\ \uparrow \\ \operatorname{coker}(f) \end{bmatrix}^1 \right)$$

Let S_1', S_2', T' be the corresponding objects in $\operatorname{Rep}(A_2^{\mathrm{opp}})$. Then one checks that $S_1 \mapsto T'$, $S_2[1] \mapsto S_1'$ and $T \mapsto S_2'$.

It turns out that Σ is an example of a Serre functor. Recall that an exact functor $\Sigma \colon \mathcal{D} \to \mathcal{D}$ of triangulated categories is called a *Serre functor* if for all objects $A, B \in \mathcal{D}$, there is a natural bifunctorial identifications

$$\operatorname{Hom}_{\mathcal{D}}(A, B) = \operatorname{Hom}_{\mathcal{D}}(B, \Sigma(A))^\vee. \tag{11.4.1}$$

For example, if X is a smooth projective variety, then $\Sigma = - \otimes \omega_X[n]$ is a Serre functor. This simply expresses Serre duality. For instance, if $A = \mathcal{O}_X$ and B is a coherent $\mathcal{O}_X$-module, then (11.4.1) reads $\Gamma(X, B) = \operatorname{Ext}^n(B, \omega)^\vee$.

Proposition 11.2. *$\Sigma \colon D^b(A_2) \to D^b(A_2)$ is a Serre functor.*

Proof. This follows by examining the formulas (11.1.3) and (11.1.4). For instance, for an A_2-representation $V = [V_1 \xrightarrow{f} V_2]$, we have

$$\operatorname{Hom}(S_1, V) = \ker(f) = \ker(f^\vee)^\vee = \operatorname{Hom}(V, S_2[1])^\vee$$
$$\operatorname{Hom}(S_1, V[1]) = \operatorname{coker}(f) = \operatorname{coker}(f^\vee)^\vee = \operatorname{Hom}(V[1], S_2[1])$$

$\square$

For later use, we will use the following general fact

Lemma 11.3. *Let $\Sigma \colon \mathcal{D} \to \mathcal{D}$ be a Serre functor on a triangulated category $\mathcal{D}$. Then Σ commutes with any autoequivlance.*

Remark 11.4. See §11.5 for the precise definition of autoequivalences.

Proof. Let $F \colon \mathcal{D} \to \mathcal{D}$ be an exact equivalence. We need to give a natural isomorphism $\Sigma(F(E)) = F(\Sigma(E))$ for each object $E \in \mathcal{D}$. By Yoneda's lemma (and since F is an equivalence), it suffices to check that $\operatorname{Hom}(F(X), \Sigma(F(E))) = \operatorname{Hom}(F(X), F(\Sigma(E))$ for all $X \in \mathcal{D}$. We compute

$$\begin{aligned} \operatorname{Hom}(F(X), \Sigma(F(E))) &= \operatorname{Hom}(F(E), F(X)) \\ &= \operatorname{Hom}(E, X) \\ &= \operatorname{Hom}(X, \Sigma(E)) \\ &= \operatorname{Hom}(F(X), F(\Sigma(E))) \end{aligned}$$

$\square$

11.5. **The group** $\operatorname{Aut}(D^b(A_2))$**.** An *autoequivalence* of a triangulated category $\mathcal{D}$ is an equivalence $F\colon \mathcal{D} \to \mathcal{D}$ of additive categories such that F sends exact triangles to exact triangles and F commutes with the shift functor [1]. The group $\operatorname{Aut}(\mathcal{D})$ denotes the set of autoequivalences modulo natural isomorphisms where the group law is composition.

Proposition 11.5. *We have an identification* $\operatorname{Aut}(D^b(A_2)) \cong \mathbb{Z}$ *generated by* Σ*.*

Proof. Let $F\colon D^b(A_2) \to D^b(A_2)$ be an autoequivalence. The only objects $E \in \mathcal{D}$ with $\operatorname{End}(E) \cong k$ are shifts of S_1, T and S_2. (This follows using Proposition 10.4 and the decomposition (11.1.2) to write every object in $\mathcal{D}$ as a direct sum of shifts of S_1's, T's and S_2's.) Since F is fully faithful, $\operatorname{Hom}(F(S_1), F(S_1)) = k$ and therefore $F(S_1)$ is a shift of S_1, T or S_2. Therefore, after applying a suitable (possibly negative) power of Σ, we may assume that $F(S_1) = S_1$. But Lemma 11.3 implies that F commutes with Σ. This means

$$F(S_2[1]) = F(\Sigma(S_1)) = \Sigma(F(S_1)) = S_2[1]$$

which implies that $F(S_2) = F(S_2)$. One sees similarly that $F(T) = T$.[3] $\square$

12. Lecture 12 (April 30): More on the A_2-quiver

As before, $\mathcal{A} = \operatorname{Rep}^{\mathrm{fd}}(A_2)$. Last time we saw several hearts on $D^b(A_2)$ that arose from the $\widetilde{\mathrm{GL}}_2^+$-action on a stability condition on the standard heart. These arose as natural tilts of the standard heart. These hearts were

- $\mathcal{A}$, the standard heart;
- $\Sigma(\mathcal{A}) = \langle S_2[1], S_1, T\rangle$, the tilt of the standard heart with respect to the torsion pair $(\langle S_1, T\rangle, \langle S_2\rangle)$;
- $\Sigma^2(\mathcal{A}) = \langle T[1], S_2[1], S_1\rangle$, the tilt of the standard heart with respect to the torsion pair $(\langle S_1\rangle, \langle T, S_2\rangle)$;
- $\langle S_1[1], S_2\rangle$, the tilt of the standard heart with respect to the torsion pair $(\langle S_2\rangle, \langle S_1\rangle)$.

12.1. **Are there any other hearts?** Yes, there are! From the heart $\langle S_1[1], S_2\rangle$ arising from the $\widetilde{\mathrm{GL}}_2^+(\mathbb{R})$-orbit of the stability condition in Case (1), one is led to suspect that perhaps

$$\mathcal{A}_a = \langle S_1[a], S_2\rangle$$

is the heart of a bounded t-structure for all $a > 0$.

Lemma 12.1. *For $a > 0$, the full subcategory $\langle S_1[a], S_2\rangle \subset D^b(A_2)$ is the heart of the bounded t-structure defined by*

$$\mathcal{D}_{>0} = \left\{ \cdots \to V^{-a} \to S_2^{\oplus d_{-a+1}} \to \cdots \to S_2^{\oplus d_0} \to 0 \to \cdots \right\}$$

with $S_2^{\oplus d_0}$ in degree 0 and

$$\mathcal{D}_{\leq 0} = \left\{ 0^{-a} \to S_1^{\oplus e_{-a+1}} \to \cdots \to S_1^{\oplus e_{-1}} \to S_1^{\oplus e_0} \to V^1 \cdots \right\}$$

with $S_1^{\oplus d_0}$ in degree 0.

Proof. Exercise. $\square$

Remark 12.2. For $a > 0$, the heart $\mathcal{A}_a$ is semisimple.

[3] We thank Alex Voet for pointing out both that Σ is a Serre functor and that this can be used with Lemma 11.3 to give a clean argument that any autoequivalence F with $F(S_1) = S_1$ must be the identity.

12.2. **Summary of hearts.** We have the following list of hearts:

($a = 0$)
- $\mathcal{A} = \langle S_1, S_2 \rangle$
- $\Sigma(\mathcal{A}) = \langle S_2[1], S_1, T \rangle$
- $\Sigma^2(\mathcal{A}) = \langle T[1], S_2[1], S_1 \rangle$

($a > 0$)
- $\mathcal{A}_a := \langle S_1[a], S_2 \rangle$
- $\Sigma(\mathcal{A}_a) = \langle S_2[a+1], T \rangle$
- $\Sigma^2(\mathcal{A}_a) = \langle T[a+1], S_1 \rangle$

Are these all the possible hearts?

12.3. **Two of three lemma.**

Proposition 12.3. *For any stability condition $(Z, \mathcal{P})$ on $D^b(A_2)$, at least two of the objects $\{S_0, S_1, T\}$ are semistable.*

Proof. If all three are semistable, then we are done. Otherwise, by possibly applying Σ or Σ^2, we can assume that S_1 is not semistable. In this case, there exists a triangle

$$A \to S_1 \to B \tag{12.3.1}$$

in $D^b(A_2)$ with $A, B \neq 0$, B semistable, and $\mathrm{Ext}^i(A, B) = 0$ for $i \leq 0$. As $\mathcal{A} = \mathrm{Rep}^{\mathrm{fd}}(A_2)$ has cohomological dimension 1, Proposition 10.4 implies that we can write $A = \oplus_i A_i[-i]$ and $B = \oplus B_i[-i]$. As in the proof of Proposition 10.6, we have the following key identity:

$$\mathrm{Hom}(A_i, B_{i-1}) = 0 \qquad \text{for all } i. \tag{12.3.2}$$

Indeed,

$$\mathrm{Hom}_{\mathcal{A}}(A_i, B_{i-1}) = \mathrm{Hom}_{D^b(A_2)}(A_i[-i], B_{i-1}[-(i-1)][-1]) \subset \mathrm{Ext}^{-1}(A, B) = 0.$$

as $A_i[-i]$ and $B_{i-1}[-(i-1)]$ are direct summands of A and B, respectively. (In fact, the same argument shows that $\mathrm{Ext}^1_{\mathcal{A}}(A_i, B_{i-1}) = 0$.)

The triangle $A \to S_1 \to B$ induces the long exact sequence

$$0 \to B_{-1} \to A_0 \to S_1 \to B_0 \to A_1 \to 0$$

and for $i \neq 0, 1$ isomorphisms $B_{i-1} \xrightarrow{\sim} A_i$ which imply by the above remark that $A_i = B_{i-1} = 0$. Since S_1 is simple, we have either:

(a) $B_{-1} \xrightarrow{\sim} A_0$ (and thus both are zero by (12.3.2)) and $0 \to S_1 \to B_0 \to A_1 \to 0$, or

(b) $B_0 \xrightarrow{\sim} A_1$ (and thus both are zero by (12.3.2)) and $0 \to B_{-1} \to A_0 \to S_1 \to 0$.

In case (a), the short exact sequence splits as S_1 is injective. Thus, there is an inclusion $A_1 \hookrightarrow B_0 = S_0 \oplus A_1$, which contradicts $A_1 \neq 0$ and (12.3.2).

In case (b), we can write each B_{-1} and A_0 as a direct sum of S_1's, T's and S_2's. We first note that B_{-1} and A_0 have no S_1's; otherwise, there would exist a non-zero map $A_0 \to B_{-1}$. Applying $\mathrm{Hom}(T, -)$ to the short exact sequence and using that T is projective yields

$$0 \to \mathrm{Hom}(T, B_{-1}) \to \mathrm{Hom}(T, A_0) \to k \to 0$$

which implies that the number of T's in A_0 is greater than in B_{-1}. It follows that B_{-1} has no T's while A_0 has one. Therefore $B_{-1} = S_2^b$ and $A_0 = S_2^a \oplus T$. We must have that $a = 0$ and $b = 1$. We conclude that $B = B_{-1}[1] = S_2[1]$ and that the exact triangle (12.3.1) is $T \to S_1 \to S_2[1]$. In any case, we see that S_2 is semistable since B is.

It remains to show that T is also semistable. If not, there exists an exact triangle

$$A \to T \to B \tag{12.3.3}$$

in $D^b(A_2)$ with B semistable and $\operatorname{Ext}^i(A,B) = 0$ for $i \le 0$. Again, we have that for $i \ne 0, 1$, $A_i = B_{i-1} = 0$ and an exact sequence

$$0 \to B_{-1} \to A_0 \to T \to B_0 \to A_1 \to 0$$

We note that $A_0 \to T$ and $T \to B_0$ must be non-zero as otherwise, we obtain short exact sequences which since T is both injective and projective would give non-zero maps $A_0 \to B_{-1}$ or $A_1 \to B_0$. Thus, we have short exact sequences

$$0 \to B_{-1} \to A_0 \to S_2 \to 0 \qquad \text{and} \qquad 0 \to S_1 \to B_0 \to A_1 \to 0.$$

We see that B_{-1} and A_0 must have the same number of T's and S_1's, both necessarily being zero. By looking at the number of S_2's, we conclude that $B_{-1} = 0$ and $A_0 = S_2$. Similarly, we argue that $B_0 = S_1$ and $A_1 = 0$. Thus, the exact triangle (12.3.3) is $S_2 \to T \to S_1$ and $S_1 = B$ is semistable, a contradiction. □

Corollary 12.4. *The hearts in §12.2 are all possibles hearts of bounded t-structures arising from stability conditions on $D^b(A_2)$.*

Proof. Let $(\mathcal{P}, Z) \in \operatorname{Stab}(D^b(A_2))$ be a stability condition with heart $\mathcal{A}' = \mathcal{P}((0,1])$. After applying the autoequivalence Σ a certain number of times, we may assume that $S_1 \in \mathcal{A}'$ is semistable of phase ϕ_1. We know that either S_2 or T is semistable.

If S_2 is semistable of phase ϕ_2, then since $\operatorname{Ext}^1(S_2, S_1) = \operatorname{Hom}(S_2, S_1[1]) \ne 0$, we must have that the phase $\phi(S_2) \le \phi(S_1[1]) = \phi(S_1) + 1$. This means that for some integer $a \ge 0$, we have that $S_1, S_2[1-a] \in \mathcal{A}'$. If $a = 0$, then $T = \ker(S_1 \to S_2[1])$ is also in $\mathcal{A}'$. By Proposition 10.1, this implies that $\mathcal{A}' = \langle S_2[1], S_1, T \rangle = \Sigma(\mathcal{A})$. If $a > 0$, then $\mathcal{A}' = \mathcal{A}_a[-a]$.

If T is semistable of phase ϕ_T, then since $\operatorname{Hom}(T, S_1) \ne 0$, we have that $\phi_T \le \phi_1$. This gives $S_1, T[a] \in \mathcal{A}'$ for some integer $a \ge 0$. If $a = 0$, then $S_2 = \ker(T \to S_1) \in \mathcal{A}'$ so $\mathcal{A}' = \mathcal{A}$ is the standard heart. If $a > 0$, then we have $\mathcal{A}' = \langle T[a], S_1 \rangle = \Sigma^2(\mathcal{A}_{a-1})$. □

13. Lecture 13 (May 2)

Let C be a smooth, projective and connected curve of positive genus. Let $\operatorname{Stab}^{\mathrm{num}}(C)$ be the space of stability conditions $\sigma = (\mathcal{P}, Z)$ such that (1) Z factors as $K(C) \xrightarrow{(\deg, \mathrm{rk})} \mathbb{Z}^2 \to \mathbb{C}$ and (2) for all $\phi \in \mathbb{R}$, the abelian category $\mathcal{P}(\phi)$ is of finite length. We began the proof of

Theorem 13.1. $\operatorname{Stab}^{\mathrm{num}}(C) \cong \widetilde{\mathrm{GL}}_2^+$.

We followed the exposition in [Huy11].

14. Lecture 14 (May 7)

Finished the proof of Theorem 13.1 and began discussing the topology of the space of stability conditions on any triangulated category.

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