LECTURE 9 : Characterization of DM stacks

Thm. *The moduli space $\overline{\mathcal{M}}_g$ of stable curves of genus $g \geq 2$ is a smooth, proper and irreducible Deligne–Mumford stack of dimension $3g-3$ which admits a projective coarse moduli space.*

After today : $\mathcal{M}_g$ is a Deligne-Mumford stack which is smooth over Spec $\mathbb{Z}$ of rel. dim $3g-3$.

§0. Recap: 3 new concepts

(i) Dimension

Def. Let $\mathcal{X}$ be an algebraic stack and $x \in |\mathcal{X}|$. Choose a smooth presentation $(U, u) \to (\mathcal{X}, x)$ and let $s, t \colon R \rightrightarrows U$ be the smooth groupoid. Define

$$\dim_x \mathcal{X} := \dim_u U - \dim_{e(u)} \boxed{R_u} \in \mathbb{Z} \cup \infty$$

where R_u is the fiber of $s \colon R \to U$ over u and $e \colon U \to R$ denotes the identity morphism in the groupoid.

Essentially,

"$\dim \mathcal{X} = \dim U - \underbrace{\operatorname{rel\,dim}(U \to \mathcal{X})}_{\text{dim of a fiber}}$"

Well-defined b/c dim is well-behaved for a smooth morphism $X \to Y$, $x \mapsto y$

$$\dim_x X = \dim_y Y + \dim X_y$$

(ii) Tangent space

Def. Let $\mathcal{X}$ be an alg stack and $x \in \mathcal{X}(k)$. The *Zariksi tangent space* is defined as the set

$$T_{\mathcal{X},x} := \left\{ \text{2-commutative diagrams} \quad \begin{tikzcd} \operatorname{Spec} k \arrow[d, hook] \arrow[dr, "x"] & \\ \operatorname{Spec} k[\epsilon] \arrow[r, "\tau"'] \arrow[ur, phantom, "\alpha"] & \mathcal{X} \end{tikzcd} \right\} \Big/ \sim$$

where $(\tau, \alpha) \sim (\tau', \alpha')$ if $\exists$ iso $\beta \colon \tau \xrightarrow{\sim} \tau'$ in $\mathcal{X}(k[\epsilon])$ compatible with α and α', i.e. $\alpha' = \beta|_{\operatorname{Spec} k} \circ \alpha$.

Fact: $T_{\mathcal{X},x}$ is a k-vector space

Example: $\mathcal{M}_{g,n} \to \operatorname{Spec} k$, $\operatorname{Spec} k \xrightarrow{[C]} \mathcal{M}_{g,n}$

dim $3g-3$

inf. def. theory $\Rightarrow$

$$T_{\mathcal{M}_g,[C]} = H^1(C, T_C)$$

(iii) Residual gerbes

Clarification:

Defn

① An alg. stack $\mathcal{X}$ is quasi-compact if $|\mathcal{X}|$ is.

$\Longleftrightarrow \exists \operatorname{Spec} A \xrightarrow{sm} \mathcal{X}$ presentation

② A map $\mathcal{X} \to \mathcal{Y}$ is quasi-compact

$\forall \operatorname{Spec} A \to \mathcal{Y}$, $\mathcal{X} \times_{\mathcal{Y}} \operatorname{Spec} A$ is quasi-compact

③ $\mathcal{X}$ noetherian

$\Longleftrightarrow \mathcal{X}$ loc. noeth &

$\mathcal{X}, \Delta_{\mathcal{X}}, \Delta_{\Delta_{\mathcal{X}}}$ q. compact

④ $x \in |\mathcal{X}|$ is f. type if $\exists$ representative $\operatorname{Spec} k \to \mathcal{X}$ of x which is loc. of f. type

Rmk $\mathcal{X}$ noeth

$\operatorname{Spec} k \to \mathcal{X}$ loc. f. type

$\not\Rightarrow$ f. type

(iii) Residual gerbes

Def. • Let $\mathcal{X}$ be an alg stack and $x \in |\mathcal{X}|$.

• Choose a smooth presentation $(U, u) \to (\mathcal{X}, x)$.

The *residual gerbe of x* is the substack $\mathcal{G}_x \subset \mathcal{X}$ defined as the stackification of the full subcategory $\mathcal{G}_x^{\mathrm{pre}} \subset \mathcal{X}$ of objects $a \in \mathcal{X}$ over S which factor as $a\colon S \to \operatorname{Spec} \kappa(u) \to \mathcal{X}$.

Essentially, the residual gerbe $\mathcal{G}_x \subset \mathcal{X}$ is the smallest substack containing x.

In other words,

$$\begin{tikzcd}
\operatorname{Spec} k \arrow[rr, "x"] \arrow[dr, "\text{epi}"'] \arrow[ddr, bend right] & & \mathcal{X} \\
& \mathcal{G}_x \arrow[ur, "\text{mono}"'] & \\
& \mathcal{Z} \arrow[u, dashed, "\exists!"] \arrow[uur, bend right] &
\end{tikzcd}$$

Thm. • *Let $\mathcal{X}$ be a noetherian algebraic stack.*

• *Let $x \in |\mathcal{X}|$ be a finite type point with smooth stabilizer.*

Then $\mathcal{G}_x$ is an alg. stack and $\mathcal{G}_x \hookrightarrow \mathcal{X}$ is a loc. closed imm.

Moreover, if $(U, u) \to (\mathcal{X}, x)$ is a smooth morphism from a scheme U, then

$$\begin{tikzcd}
O(u) \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & U \arrow[d] \\
\mathcal{G}_x \arrow[r, hook] & \mathcal{X}
\end{tikzcd}$$

where $O(u)$ is the orbit $s(t^{-1}(u))$ of the induced groupoid $s, t\colon R := U \times_{\mathcal{X}} U \rightrightarrows U$.

If $\mathcal{X} = [U/G]$, then $O(u)$ is normal G-orbit

If $\mathcal{X}$ f.type $/k$ and $x \in \mathcal{X}(k)$

then $\mathcal{G}_x = BG_x \longrightarrow \mathcal{X}$

§1. Miniversal presentations

Theorem (Existence of Miniversal Presentations). *Let $\mathcal{X}$ be a noetherian algebraic stack and $x \in |\mathcal{X}|$ a finite type point with smooth stabilizer G_x.*
$\implies$ *$\exists$ smooth morphism $(U,u) \to (\mathcal{X},x)$ from a scheme of relative dimension $\dim G_x$ s.t.*

$$\begin{tikzcd} \operatorname{Spec} \kappa(u) \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & U \arrow[d] \\ \mathcal{G}_x \arrow[r, hook] & \mathcal{X}. \end{tikzcd}$$

In particular, if G_x is finite and reduced, there is an étale morphism $(U,u) \to (\mathcal{X},x)$ from a scheme.

Defn We say $\mathcal{U} \xrightarrow{sm} \mathcal{X}$, $u \mapsto x$ is miniversal at u if $T_{\mathcal{U},u} \to T_{\mathcal{X},x}$ ism. of tangent spaces.

Later: We will see that $\mathcal{U} \to \mathcal{X}$ in the thm is miniversal.

Ex: $\mathbb{G}_m \curvearrowright \mathbb{A}^2$, $t \cdot (x,y) = (tx, ty)$

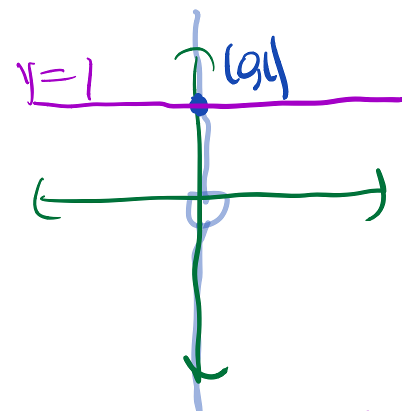
$$\begin{tikzcd} \operatorname{Spec} k \arrow[r, hook, "0"] \arrow[d] & \mathbb{A}^2 \arrow[d] \\ B\mathbb{G}_m \arrow[r, hook, "0"] & {[\mathbb{A}^2/\mathbb{G}_m]} \end{tikzcd}$$

res. gerbe at 0

$\implies \mathbb{A}^2 \to [\mathbb{A}^2/\mathbb{G}_m]$ is miniversal at 0

But it is not miniversal at $(0,1)$.

$$\begin{tikzcd} \{(0,y) \mid y \neq 0\} \arrow[r, hook] \arrow[d] & \mathbb{A}^2 \arrow[d] \\ \operatorname{Spec} k \arrow[r, "{(0,1)}"] & {[\mathbb{A}^2/\mathbb{G}_m]} \end{tikzcd}$$



$$\begin{tikzcd} \operatorname{Spec} k[x] = \mathbb{A}^1 \arrow[r, "y=1"] & \mathbb{A}^2 \arrow[r] & {[\mathbb{A}^2/\mathbb{G}_m]} \arrow[d, hook] \\ & & \mathbb{P}^1 \end{tikzcd}$$

is miniversal at 1

Theorem (Existence of Miniversal Presentations).
Let $\mathcal{X}$ be a noetherian algebraic stack and $x \in |\mathcal{X}|$ a finite type point with smooth stabilizer G_x.
$\implies$ *$\exists$ smooth morphism $(U,u) \to (\mathcal{X},x)$ from a scheme of relative dimension $\dim G_x$ s.t.*

$$\begin{tikzcd} \operatorname{Spec} \kappa(u) \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & U \arrow[d] \\ \mathcal{G}_x \arrow[r, hook] & \mathcal{X}. \end{tikzcd}$$

In particular, if G_x is finite and reduced, there is an étale morphism $(U,u) \to (\mathcal{X},x)$ from a scheme.

Proof Choose $U \to \mathcal{X}$ smooth pres

$$\begin{tikzcd} \mathcal{O}(u) \arrow[r, hook, "\text{loc. cl.}"] \arrow[d] & U \ni u \arrow[d] \\ \mathcal{G}_x \arrow[r, hook] & \mathcal{X} \ni x \end{tikzcd}$$

dim $= c$

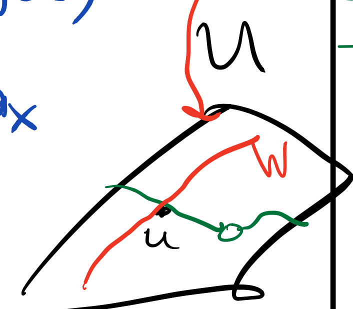
dim $= -\dim G_x$

Let $n = \operatorname{rel\,dim}(U \to \mathcal{X})$

$c = n - \dim G_x$

If $c = 0$, we win

slice transverse to the orbit



$\mathcal{O}(u)$ is a smooth scheme of dim c

$\Rightarrow \exists$ reg. sequence $f_1, \dots, f_c \in m_u \subset \mathcal{O}_{\mathcal{O}(u),u}$ generating max'l ideal

After shrinking U, can assume $f_1, \dots, f_c \in \Gamma(U)$.

Define $W = V(f_1, \dots, f_c) \subseteq U$

By design, $W \cap \mathcal{O}(u) = \{u\}$

Inductive apply loc crit of flatness

Fact $(A, m_A) \to (B, m_B)$ flat local ring hom of local noeth rings.

Let $f \in m_A$ s.t. $f \otimes 1 \in A \otimes_B B/m_B$ non-zero divisor.

$\Rightarrow A \to B \to B/f$ flat

$$\begin{tikzcd} \operatorname{Spec} \kappa(u) \arrow[r, hook] \arrow[d, "\text{smooth}"] & W \arrow[r, hook] \arrow[d, "\text{flat}"] & U \arrow[dl] \\ \mathcal{G}_x \arrow[r, hook] & \mathcal{X} & \end{tikzcd}$$

Shrink W so that $W \to \mathcal{X}$ smooth

Theorem (Existence of Miniversal Presentations). *Let $\mathcal{X}$ be a noetherian algebraic stack and $x \in |\mathcal{X}|$ a finite type point with smooth stabilizer G_x.*
$\implies$ *$\exists$ smooth morphism $(U,u) \to (\mathcal{X},x)$ from a scheme of relative dimension $\dim G_x$ s.t.*

$$\begin{tikzcd} \operatorname{Spec}\kappa(u) \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & U \arrow[d] \\ \mathcal{G}_x \arrow[r, hook] & \mathcal{X}. \end{tikzcd}$$

In particular, if G_x is finite and reduced, there is an étale morphism $(U,u) \to (\mathcal{X},x)$ from a scheme.

Corollary (Equiv. characterizations of DM stacks). *Let $\mathcal{X}$ be a noetherian alg. stack. The following are equiv:*

(1) $\mathcal{X}$ is DM;

(2) every point of $\mathcal{X}$ has a finite, reduced stabilizer;

(3) the diagonal $\mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is unramified.

(2) $\Longleftrightarrow$ (3) ess. defn of unramified

unramified = loc. of f.type & discrete & reduced fibers

(2) $\Longrightarrow$ (1) by Thm.

(1) $\Longrightarrow$ (2): $\mathcal{X}$ is DM

$\exists$ $U \xrightarrow{\text{ét}} \mathcal{X}$
scheme

$$\begin{tikzcd} & & U \\ R \arrow[r, "\text{unramified}"] \arrow[d] \arrow[urr, "\text{étale}"] \arrow[dr, phantom, "\square"] & U \times U \arrow[d] \arrow[ur] & \\ \mathcal{X} \arrow[r] & \mathcal{X} \times \mathcal{X} & \end{tikzcd}$$

Cor Let $\mathcal{X}$ noeth DM stack.
Assume $\Delta_{\mathcal{X}}$ repr by schemes. TFAE

(1) $\mathcal{X}$ alg. space

(2) every pt has trivial stab.

(3) $\mathcal{X} \longrightarrow \mathcal{X} \times \mathcal{X}$ mono

($\Longleftrightarrow$ $\mathcal{X}$ is a sheaf)

Fact: $G \longrightarrow S$ f.type group scheme is trivial $\Longleftrightarrow$ fibers are trivial

FACT If C is a smooth, proj curve/k, then $\mathrm{Aut}(C)$ is finite & reduced. $\boxed{g \geq 2}$

Reason:

① $\mathrm{Aut}(C)$ is an alg. group.

Use Hilbert scheme

$$C \xrightarrow{\alpha \atop \sim} C \quad \longmapsto \quad C \xrightarrow{\Gamma_\alpha} C \times C$$

② Inf. def theory

Lie alg. $T_e \mathrm{Aut}(C) = H^0(C, T_C)$
$= 0$

Cor: $\mathcal{M}_g$ is a DM stack of f. type over $\mathbb{Z}$ w/ affine diagonal.

§2 Smoothness

Theorem (Formal Lifting Criteria). *Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Then f is smooth if and only if f is locally of finite presentation and for every diagram*

$$\begin{tikzcd} \operatorname{Spec} A_0 \arrow[r] \arrow[d, hook] & \mathcal{X} \arrow[d, "f"] \\ \operatorname{Spec} A \arrow[r] \arrow[ur, dashed] & \mathcal{Y}, \end{tikzcd}$$

of solid arrows where $A \twoheadrightarrow A_0$ is a surjection of rings with nilpotent kernel, there exists a lifting.

If $\mathcal{X}$ and $\mathcal{Y}$ are noetherian, then it suffices to consider diagrams where A and A_0 are local artinian rings.

Remark. To be explicit, a *lifting* of

$$\begin{tikzcd} \operatorname{Spec} A_0 \arrow[r, "x"] \arrow[d, "g"'] & \mathcal{X} \arrow[d, "f"] \arrow[dl, phantom, "\swarrow_{\alpha}"] \\ \operatorname{Spec} A \arrow[r, "y"'] & \mathcal{Y}, \end{tikzcd}$$

is a map $\operatorname{Spec} A \to \mathcal{X}$ and a 2-isomorphisms β and γ

$$\begin{tikzcd} \operatorname{Spec} A_0 \arrow[r, "x"] \arrow[d, "g"'] & \mathcal{X} \arrow[d, "f"] \\ \operatorname{Spec} A \arrow[r, "y"'] \arrow[ur, "\widetilde{x}"] & \mathcal{Y}, \end{tikzcd}$$

such that

$$\begin{tikzcd} & f \circ x \arrow[dl, "\alpha"'] & \\ y \circ g & & f \circ \widetilde{x} \circ g \arrow[ul, "f(\beta)"'] \arrow[ll, "g^*\gamma"] \end{tikzcd}$$

Rmk $\mathcal{X} \to \mathcal{Y}$ is smooth $\iff$

$\forall$ diagrams

$\mathcal{X}(A) \to \mathcal{X}(A_0) \times_{\mathcal{Y}(A_0)} \mathcal{Y}(A)$

ess. surjective

Skip proof

- Similar criteria

  étale $\iff$ $\exists!$ lifting

  unramified $\iff$ $\exists$ at most 1 lifting

Prop Let $\mathcal{X}$ noeth alg. stack. Let $x \in |\mathcal{X}|$ f.type pt with smooth stab.

Suppose

$$\begin{tikzcd} \operatorname{Spec} k(u) \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] \arrow[dr, bend left, "f(u)"] & U \arrow[d, "\text{smooth}"] \\ \mathcal{G}_x \arrow[r, hook] & \mathcal{X} \end{tikzcd} \qquad \begin{matrix} u \\ \downarrow \\ x \end{matrix}$$

Then $T_{U,u} \twoheadrightarrow T_{\mathcal{X},f(u)}$ as $k(u)$-vect spaces

i.e. $U \to \mathcal{X}$ miniversal at u

Pf: Formal lift. exist $\quad k = k(u)$

$$\begin{tikzcd} \operatorname{Spec} k \arrow[r, "u"] \arrow[d] & U \arrow[d, "\text{smooth}"] \\ \operatorname{Spec} k[\varepsilon] \arrow[r] \arrow[ur, dotted, "\exists"] & \mathcal{X} \end{tikzcd} \qquad \begin{matrix} T_{U,u} \\ \downarrow \\ T_{\mathcal{X},f(u)} \end{matrix}$$

Injectivity

$$\begin{matrix} (\operatorname{Spec} k[\varepsilon] \xrightarrow{\tau} U) & \in & T_{U,u} \\ \downarrow & & \downarrow \\ 0 & & T_{\mathcal{X},f(u)} \end{matrix}$$

By defn of res. gerbe

$$\begin{tikzcd} \operatorname{Spec} k[\varepsilon] \arrow[r, "\tau"] \arrow[d] \arrow[dr, dashed] & U \arrow[d, "\text{smooth"] \\ & \operatorname{Spec} k(u) \arrow[u] \arrow[dl] \arrow[d, phantom, "\square"] \\ \mathcal{G}_x \arrow[r, hook] & \mathcal{X} \end{tikzcd}$$

$\Longrightarrow \tau = 0.$

$$\mathbb{A}^1 \to [\mathbb{A}^1/\mathbb{G}_m]$$

0

$B\mathbb{G}_m$

Cor $\mathcal{X}$ noeth. alg. stack smooth $/k$.

- Let $x \in \mathcal{X}(k)$ with smooth stab.

$$\Rightarrow \dim_x \mathcal{X} = \dim T_{\mathcal{X},x} - \dim G_x$$

Proof Take miniversal presentation

$$\begin{tikzcd}
\operatorname{Spec} k(u) \arrow[r] \arrow[d, "\substack{\text{rel dim} \\ = \dim G_x}"'] & U \arrow[d] \arrow[dr, "sm"] & \\
G_x \arrow[r, hook] & \mathcal{X} \arrow[r, "sm"] & \operatorname{Spec} k
\end{tikzcd}$$

$$\dim_x \mathcal{X} = \dim_u U - \dim G_x$$
$$= \dim T_{U,u} - \dim G_x$$
$$= \dim T_{\mathcal{X},x} - \dim G_x \quad \checkmark$$

Field k

$\dim_{[C]} \mathcal{M}_{g,k} =$

$\dim T_{\mathcal{M}_{g,k},[C]} = 3g-3$

Ex) $\mathcal{M}_g \to \operatorname{Spec} \mathbb{Z}$ smooth of rel dim $3g-3$

$g \geq 2$

Smoothness

Assume A, A_0 local rings artinian, $\ker(A \to A_0) = k$

$$\begin{tikzcd}
\operatorname{Spec} k \arrow[d] \arrow[drr, bend left, "{[C]}"] & & \\
\operatorname{Spec} A_0 \arrow[rr] \arrow[d] & & \mathcal{M}_g \arrow[d] \\
\operatorname{Spec} A \arrow[rr] \arrow[urr, dashed, "\exists"] & & \operatorname{Spec} \mathbb{Z}
\end{tikzcd}$$

$$\begin{tikzcd}
C \arrow[r] \arrow[d] & C_0 \arrow[r, hook, dashed] \arrow[d] & \tilde{C} \arrow[d, dashed] \\
\operatorname{Spec} k \arrow[r, hook] & \operatorname{Spec} A_0 \arrow[r, hook] & \operatorname{Spec} A
\end{tikzcd}$$

inf. def theory

$\exists\, ob \in H^2(C, T_C) = 0$

$ob = 0 \iff \exists\, \tilde{C} \to \operatorname{Spec} A$ extension

§3. Properness

Definition. For algebraic stacks $\mathcal{X}$ and $\mathcal{Y}$, define:

(1) A morphism $\mathcal{X} \to \mathcal{Y}$ is *universally closed* if for every morphism $\mathcal{Y}' \to \mathcal{Y}$ of algebraic stacks, the morphism $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}' \to \mathcal{Y}'$ induces a closed map $|\mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}'| \to |\mathcal{Y}|$.

(2) A representable morphism $\mathcal{X} \to \mathcal{Y}$ is *separated* if the diagonal $\mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ (which is repr. by schemes) is proper.

(3) A representable morphism $\mathcal{X} \to \mathcal{Y}$ is *proper* if it is universally closed, separated and of finite type.

(4) A morphism $\mathcal{X} \to \mathcal{Y}$ of algebraic stacks is *separated* if the diagonal $\mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ (which is representable) is proper.

(5) A morphism $\mathcal{X} \to \mathcal{Y}$ of algebraic stacks is *proper* if it is universally closed, separated and of finite type.

If X scheme,

Δ_X closed imm $\iff$ Δ_X proper

Ex: BG, G finite grp

Δ_{BG} is finite but not closed imm

Theorem (Val. Criteria for Univ. Closed/Proper/Separated). *Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a finite type morphism of algebraic stacks and consider a 2-commutative diagram*

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] \arrow[dr, phantom, "\swarrow\alpha"] & \mathcal{X} \arrow[d, "f"] \\ \operatorname{Spec} R \arrow[r] & \mathcal{Y} \end{tikzcd} \qquad (\star)$$

where R is a valuation ring with fraction field K. Then

(1) f is universally closed $\iff$ $\forall$ diagrams $(\star)$, $\exists$ an extension $R \to R'$ of valuation rings and $K \to K'$ of fraction fields together with a lifting

$$\begin{tikzcd} \operatorname{Spec} K' \arrow[r] \arrow[d] & \operatorname{Spec} K \arrow[r] \arrow[d] & \mathcal{X} \arrow[d, "f"] \\ \operatorname{Spec} R' \arrow[r] \arrow[urr, dashed] & \operatorname{Spec} R \arrow[r] & \mathcal{Y}. \end{tikzcd}$$

(2) f is separated $\iff$ any 2 liftings of $(\star)$ are isomorphic.

(3) f is proper $\iff$ every diagram $(\star)$ has a lifting after an extension $R \to R'$ and any 2 liftings are isomorphic.

Moreover, if $f\colon \mathcal{X} \to \mathcal{Y}$ is a finite type morphism of noetherian algebraic stacks, than it suffices to consider DVRs R and extensions such that $K \to K'$ is of finite transcendence degree.