Definitions.

- An *algebraic space* is a sheaf X on $\mathrm{Sch}_{\text{Ét}}$ such that there exist a scheme U and a surjective étale morphism $U \to X$ representable by schemes.
- A *Deligne–Mumford stack* is a stack $\mathcal{X}$ over $\mathrm{Sch}_{\text{Ét}}$ such that there exist a scheme U and a surjective, étale and representable morphism $U \to \mathcal{X}$.
- An *algebraic stack* is a stack $\mathcal{X}$ over $\mathrm{Sch}_{\text{Ét}}$ such that there exist a scheme U and a surjective, smooth and representable morphism $U \to \mathcal{X}$.

Last time

Theorem (Representability of the Diagonal).

(1) The diagonal of an algebraic space is repr. by schemes.

(2) The diagonal of an algebraic stack is representable.

Theorem (Algebraicity of quotients).

(1) $R \rightrightarrows U$ *smooth groupoid of schemes* $\implies$ $[U/R]$ *is an alg stack.*

(2) $R \rightrightarrows U$ *étale groupoid of schemes* $\implies$ $[U/R]$ *is a DM stack.*

(3) $R \rightrightarrows U$ *étale equiv. relationss of schemes* $\implies$ U/R *is an alg space.*

Today

① dimension

② tangent spaces

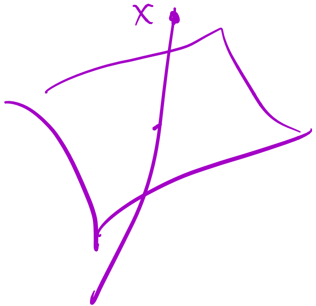
③ residual gerbes

§1 DIMENSION

Recall If X is a scheme,

$\dim X =$ Krull dim of top. space $|X|$

$x \in X \to \dim_x X = \min_{x \in U \subset X} \dim U$



Define $\dim \mathcal{X}$ using presentation

$U \xrightarrow{sm} \mathcal{X}$

“$\dim \mathcal{X} = \dim U - \operatorname{reldim}(U \to \mathcal{X})$”

Definitions.

(1) Let X be a noetherian algebraic space and $x \in |X|$. Choose an étale presentation $(U, u) \to (X, x)$ and define

$$\dim_x X := \dim_u U \in \mathbb{Z}_{\geq 0} \cup \infty$$

Well-defined b/c étale maps preserve dim.

(2) Let $\mathcal{X}$ be an algebraic stack and $x \in |\mathcal{X}|$. Choose a smooth presentation $(U, u) \to (\mathcal{X}, x)$ and let $s, t \colon R \rightrightarrows U$ be the smooth groupoid. Define

$$\dim_x \mathcal{X} := \dim_u U - \dim_{e(u)} R_u \in \mathbb{Z} \cup \infty$$

where R_u is the fiber of $s \colon R \to U$ over u and $e \colon U \to R$ denotes the identity morphism in the groupoid.

$$\begin{tikzcd}
 & e(u) \in R_u \arrow[r] \arrow[ddd] & R \arrow[rr, "s"] \arrow[ddd, "t"'] & & U \arrow[ll, bend right, "e"'] \arrow[r, phantom, "\ni"] \arrow[ddd] & u \arrow[ddd, mapsto] \\
t^{-1}(u) \arrow[ur, equal] & & & e(u) \arrow[d, mapsto] & u \arrow[l, mapsto] & \\
 & & & u & & \\
 & \operatorname{Spec} k(u) \arrow[r] & U \arrow[rr] & & \mathcal{X} \arrow[r, phantom, "\ni"] & x
\end{tikzcd}$$

(3) If $\mathcal{X}$ is a noetherian algebraic space or stack, define

$$\dim \mathcal{X} = \sup_{x \in |\mathcal{X}|} \dim_x \mathcal{X} \in \mathbb{Z} \cup \infty.$$

Setup

$$\begin{tikzcd} R_u \arrow[r] \arrow[d, "sm"] & R \arrow[r, "s"] \arrow[d, "t"] & U \arrow[l, bend right, "e"'] \arrow[d, "sm"] & u \arrow[d, mapsto] \\ \operatorname{Spec} k(u) \arrow[r] & U \arrow[r] & \mathcal{X} & x \end{tikzcd}$$

$\dim_x \mathcal{X} :=$
$\dim_u U - \dim_{e(u)} R_u$

Prop $\dim_x \mathcal{X}$ is well-defined.

We will use

- **FACT** Let $X \xrightarrow{f} Y$ be a smooth map of noeth schemes
$$\begin{tikzcd} x \arrow[r, mapsto] & y \end{tikzcd}$$
Then $\dim_x X = \dim_y Y + \dim_x X_y$

Pf

- Let $U' \to \mathcal{X}$ be another presentation $\overset{u' \mapsto x}{}$
with groupoid $R' \rightrightarrows U'$
- Consider $U'' = U \times_{\mathcal{X}} U'$ $\ni$ $\begin{tikzcd} u'' \arrow[r, mapsto] \arrow[d, mapsto] & u' \\ u & \end{tikzcd}$
- By symmetry, suffices to show

$$(*) \quad \dim_u U - \dim_{e(u)} R_u = \dim_{u''} U'' - \dim_{e''(u'')} R''_{u''}$$

PICTURE

$$\begin{tikzcd} R'_{u'} = U''_u \arrow[r] \arrow[d] & U'' \arrow[r] \arrow[d] & U' \arrow[d] \\ \operatorname{Spec} k(u) \arrow[r] & U \arrow[r] & \mathcal{X} \end{tikzcd}$$

- Apply **FACT** to $U'' \to U$

$$\dim_{u''} U'' = \dim_u U + \dim_{u''} U''_u$$
$$= \dim_u U + \dim_{e'(u')} R'_{u'}$$

By substituting into $(*)$, suffices to show

$$(**) \quad \dim_{e''(u'')} R''_{u''} = \dim_{e(u)} R_u + \dim_{e'(u')} R'_{u'}$$

For simplicity, assume u, u', u'' have same residue field k

$$\begin{tikzcd} & R''_{u''} \arrow[rr] \arrow[dl] \arrow[dd] & & R'_{u'} \arrow[dl] \arrow[dd] \\ U'' \arrow[rr] \arrow[dd] & & U' \arrow[dd] & \\ & R_u \arrow[rr] \arrow[dl] & & \operatorname{Spec} k \arrow[dl, "x"] \\ U \arrow[rr] & & \mathcal{X} & \end{tikzcd}$$

cart. cube

back square ✓

Example

① G smooth, affine alg. group$/k$

U k-scheme w/ G action

$$\boxed{\dim [U/G] = \dim U - \dim G}$$

② $\dim BG = -\dim G$

③ $\dim [\mathbb{A}^1/\mathbb{G}_m] = 0$

④ $\dim [\mathbb{A}^2/\mathbb{G}_m] = 1$

$\mathbb{P}^1$

$B\mathbb{G}_m$

dim -1

⑤ $$\mathcal{M}_g = [\underset{\substack{\uparrow \\ \text{loc. closed in Hilb}}}{H'}/PGL_n]$$

$$\dim \mathcal{M}_g = \dim H' - \dim PGL_n$$

Later

§2. Tangent spaces

Def. Let $\mathcal{X}$ be an alg stack and $x \in \mathcal{X}(k)$. The *Zariksi tangent space* is defined as the set

$$T_{\mathcal{X},x} := \left\{ \text{2-commutative diagrams} \begin{tikzcd} \operatorname{Spec} k \arrow[d, hook] \arrow[dr, "x"] \arrow[dr, bend left=50, "x"] & \\ \operatorname{Spec} k[\epsilon] \arrow[r, "\tau"] \arrow[ur, phantom, "\overset{\alpha}{\Leftarrow}"] & \mathcal{X} \end{tikzcd} \right\} \Big/ \sim$$

Set

where $(\tau, \alpha) \sim (\tau', \alpha')$ if $\exists$ iso $\beta \colon \tau \xrightarrow{\sim} \tau'$ in $\mathcal{X}(k[\epsilon])$ compatible with α and α', i.e. $\alpha' = \beta|_{\operatorname{Spec} k} \circ \alpha$.

- *Scalar multiplication:* For $c \in k$ on $(\tau, \alpha) \in T_{\mathcal{X},x}$, $c \cdot (\tau, \alpha)$ is defined as the composition

$$\operatorname{Spec} k[\epsilon] \xrightarrow{\epsilon \mapsto c\epsilon} \operatorname{Spec} k[\epsilon] \xrightarrow{\tau} \mathcal{X}$$

with the same 2-isomorphism α.

- *Addition:* Use the equivalence $(\tau_1, \alpha_1), (\tau_2, \alpha_2)$

$$\boxed{\mathcal{X}(k[\epsilon_1] \times_k k[\epsilon_2]) \to \mathcal{X}(k[\epsilon_1]) \times_{\mathcal{X}(k)} \mathcal{X}(k[\epsilon_2])}$$

Not easy

$$\begin{tikzcd} \operatorname{Spec} k \arrow[r, hook] \arrow[d, hook] & \operatorname{Spec} k[\epsilon_1] \arrow[d] \arrow[dr, bend left] & \\ \operatorname{Spec} k[\epsilon_2] \arrow[r] \arrow[rr, bend right] \arrow[ur, phantom, "\Leftarrow"] & \operatorname{Spec} k[\epsilon_1] \times_k k[\epsilon_2] \arrow[r, dashed] & \mathcal{X} \end{tikzcd}$$

pushout in cat. of alg. stacks

Define $(\tau_1, \alpha_1) + (\tau_2, \alpha_2)$ as the composition

$$\operatorname{Spec} k[\epsilon] \to \operatorname{Spec}(k[\epsilon_1] \times_k k[\epsilon_2]) \to \mathcal{X}$$

$\epsilon \mapsfrom (\epsilon_1, 0)$

$\epsilon \mapsfrom (0, \epsilon_2)$

Prop. *If $\mathcal{X}$ is an algebraic stack with affine diagonal and $x \in \mathcal{X}(k)$, then $T_{\mathcal{X},x}$ is naturally a k-vector space.*

Exer: $T_{\mathcal{X},x}$ is G_x-repn

(set-theoretically)

$g \in G_x(k)$

$(\tau, \alpha) \in T_{\mathcal{X},x}$

$\underline{g} \cdot (\tau, \alpha) = (\tau, g \circ \alpha)$

Examples

① G smooth & affine alg. group

$$\operatorname{Spec} k \xrightarrow{x} BG$$

$$T_{BG,x} = 0$$

②

$$\begin{tikzcd}
 & \mathbb{A}^1 \ni 0 \arrow[d] & & \operatorname{Spec} k \arrow[dll, "0"'] \arrow[d] \\
\operatorname{Spec} k \arrow[r, hook, "1", "\text{open}"'] & {[\mathbb{A}^1/\mathbb{G}_m]} \ni x & & B\mathbb{G}_m \arrow[ll, hook']
\end{tikzcd}$$

smooth of dim $= 0$

$G_1 = \{1\} \curvearrowright T_{x,1} = 0$

$G_0 = \mathbb{G}_m \curvearrowright T_{x,0} = 1$

③ $B\mu_p$ over k of char $= p$

$$\operatorname{Spec} k \xrightarrow{x} B\mu_p$$

← Don't know it's algebraic

$$T_{B\mu_p,x} = 1$$

$B\mu_p = [\mathbb{G}_m/\mathbb{G}_m]$

↑

dim $= 0$

$$1 \to \mu_p \to \mathbb{G}_m \to \mathbb{G}_m \to 1$$

$$t \mapsto t^p$$

④ $\mathcal{M}_g$ $g \geq 2$

Fix $\operatorname{Spec} k \xrightarrow{[C]} \mathcal{M}_g$ where C sm. proj curve

By defn,

$$T_{\mathcal{M}_g,[C]} = \left\{ \begin{tikzcd}
\operatorname{Spec} k \arrow[d] \arrow[dr, "{[C]}"] & \\
\operatorname{Spec} k[\varepsilon] \arrow[r, "\tau"'] & \mathcal{M}_g
\end{tikzcd} \right\} \Big/ \sim$$

$$= \left\{ \begin{tikzcd}
C_\varepsilon \arrow[d] \\
0 \in \operatorname{Spec} k[\varepsilon]
\end{tikzcd} \ \& \ C_{\varepsilon_0} \cong C \right\} \Big/ \sim$$

Fact from inf. def. theory

$$= H^1(C, T_C)$$

$$h^1(C, T_C) \overset{SD}{=} h^0(C, \Omega_C^{\otimes 2}) \overset{RR}{=} 2(2g-2) - (g-1)$$

$$= 3g - 3$$

§3. Residual gerbes

Recall If X is a scheme & $x \in X$, then $\exists$ residue field $k(x)$ and a monomorphism $\operatorname{Spec} k(x) \xrightarrow{\text{mono}} X$

Goal: Analogous construction

Given $x \in |\mathcal{X}|$, want to consider the smallest substack of $\mathcal{X}$ containing x.

Def We say $x \in |\mathcal{X}|$ is of *finite type* if $\exists$ representative of x

$\operatorname{Spec} k \to \mathcal{X}$ locally of f.type

*mistake in lecture.

If $\mathcal{X}$ noeth,

$\operatorname{Spec} k \xrightarrow{x} \mathcal{X}$ loc. f.type $\Longleftrightarrow$ $\operatorname{Spec} k \to \mathcal{X}$ f.type

FACT X scheme

$x \in X$ f.type $\Longleftrightarrow$ $x \in X$ loc. closed

Ex: R DVR, $K = \operatorname{Frac}(R)$

$$\operatorname{Spec} K \xhookrightarrow[\text{f.type}]{\text{op.}} \operatorname{Spec} R$$

For schemes f.type$/k$, any k-point is closed.

Not true for alg. stacks

$$\operatorname{Spec} \mathbb{Q} \hookrightarrow [\mathbb{A}^1_{\mathbb{Q}} / \mathbb{G}_m]$$

not closed

Def.

- Let $\mathcal{X}$ be an algebraic stack and $x \in |\mathcal{X}|$.
- Choose a smooth presentation $(U, u) \to (\mathcal{X}, x)$.

The *residual gerbe of x* is the substack $\mathcal{G}_x \subset \mathcal{X}$ defined as the stackification of the full subcategory $\boxed{\mathcal{G}_x^{\mathrm{pre}} \subset \mathcal{X}}$ of objects $a \in \mathcal{X}$ over S which factor as $a\colon S \to \operatorname{Spec} \kappa(u) \to \mathcal{X}$.

$$\begin{tikzcd}
 & \operatorname{Spec}\kappa(u) \arrow[r] & U \arrow[d] \\
S \arrow[ur, "\exists"] \arrow[r] & \mathcal{G}_x^{pre} \arrow[r] & \mathcal{X}
\end{tikzcd}$$

$\boxed{\text{Fact } \mathcal{X} \text{ noeth } \mathcal{G}_x \text{ indep. of presentation}}$

$x\ \operatorname{Spec} k \longrightarrow \mathcal{X} \quad \mathcal{G}_x =$

Thm.

- *Let $\mathcal{X}$ be a noetherian algebraic stack.*
- *Let $x \in |\mathcal{X}|$ be a finite type point with smooth and affine stabilizer.*

Then $\mathcal{G}_x$ is an alg. stack and $\boxed{\mathcal{G}_x \hookrightarrow \mathcal{X} \text{ is a loc. closed imm.}}$

Moreover, if $(U, u) \to (\mathcal{X}, x)$ is a smooth morphism from a scheme U, then

$$\begin{tikzcd}
O(u) \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & U \arrow[d] \\
\mathcal{G}_x \arrow[r, hook] & \mathcal{X}
\end{tikzcd}$$

where $O(u)$ is the orbit $s(t^{-1}(u))$ of the induced groupoid $s, t\colon R := U \times_{\mathcal{X}} U \rightrightarrows U$.

$$O(u) = s(t^{-1}(u)) \overset{\text{set}}{=} \{v \mid \exists\, v \xrightarrow{\rho} u \text{ in } R\}$$

$\rightsquigarrow$ Give $O(u)$ reduced scheme structure

Compare: $G \curvearrowright U$ f.type/k

Fact: Any G-orbit of $u \in U(k)$ is locally closed.

$$G \to U, \quad g \mapsto g \cdot u$$

Ex 1

$$\begin{tikzcd}
\mathbb{A}^1 \setminus 0 \arrow[r] \arrow[d] & \mathbb{A}^1 \arrow[d] & \operatorname{Spec} k \arrow[l, "0"'] \arrow[d] \\
\mathcal{G}_1 = \operatorname{Spec} k \arrow[r, "1"] & {[\mathbb{A}^1/\mathbb{G}_m]} & B\mathbb{G}_m = \mathcal{G}_0 \arrow[l, "0"']
\end{tikzcd}$$

In general, $\mathcal{G}_x$ is a gerbe over the residue field $k(x)$

$$\begin{tikzcd}
BG_{x'} \arrow[r] \arrow[d] & \mathcal{G}_x \arrow[r, hook] \arrow[d] & \mathcal{X} \\
\operatorname{Spec} k' \arrow[r, "e)"] \arrow[urr, "x'"] & \operatorname{Spec} k(x) &
\end{tikzcd}$$

Thm. • *Let $\mathcal{X}$ be a noetherian algebraic stack.*

- *Let $x \in |\mathcal{X}|$ be a finite type point with smooth and affine stabilizer.*

Then $\mathcal{G}_x$ is an alg. stack and $\mathcal{G}_x \hookrightarrow \mathcal{X}$ is a loc. closed imm.

Moreover, if $(U, u) \to (\mathcal{X}, x)$ is a smooth morphism from a scheme U, then

$$\begin{tikzcd}
O(u) \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & U \arrow[d] \\
\mathcal{G}_x \arrow[r, hook] & \mathcal{X}
\end{tikzcd}$$

where $O(u)$ is the orbit $s(t^{-1}(u))$ of the induced groupoid $s, t \colon R := U \times_{\mathcal{X}} U \rightrightarrows U$.

PF OF SPECIAL CASE: $\mathcal{X}$ f.type/k, $x \in \mathcal{X}(k)$

Step 1: $\exists\ BG_x \xrightarrow[\text{mono}]{\text{f.type}} \mathcal{X}$

Recall BG_x^{pre} prestack where fibers cat

$BG_x^{pre}(S)$ = cat. w/ one object, morphism = $G_x(S)$

$BG_x^{pre} \to \mathcal{X}$ where

$BG_x^{pre}(S) \to \mathcal{X}(S)$

$* \longmapsto (S \to \operatorname{Spec} k \xrightarrow{x} \mathcal{X})$

$$\begin{tikzcd}
\exists & BG_x^{pre} \arrow[r] \arrow[d] & \mathcal{X} & \text{mono of prestack} \\
\operatorname{Spec} k \arrow[r, "\text{smooth}"'] \arrow[urr, "x", "\text{f.type}"'] & BG_x \arrow[ur, "\text{f.type}"', "\text{mono of stacks}"] & &
\end{tikzcd}$$

Step 2 Can assume $BG_x \xrightarrow[\substack{\text{flat} \\ \text{sur}}]{\substack{\text{f.type} \\ \text{mono}}} \mathcal{X}$

Can assume $BG_x \to \mathcal{X}$ dense image

replace $\mathcal{X}$ with smallest closed substack containing BG_x

Since

$\operatorname{Spec} k \to BG_x \to \mathcal{X}$ dense image

gen flatness $\Longrightarrow \operatorname{Spec} k \to \mathcal{X}$ flat

$\Longrightarrow BG_x \to \mathcal{X}$ flat

Since image is open, can we assume surjective

STEP 3 $BG_x \xrightarrow{\sim} \mathcal{X}$

alg. stack + sheaf

Don't know yet it's an alg. space

$$\begin{tikzcd} O \arrow[r, "\text{f.type, mono}", "\text{flat}"'] \arrow[d] & U \arrow[d] \\ BG_x \arrow[r, "\text{f.type, mono}", "\text{flat}"'] & \mathcal{X} \end{tikzcd}$$

Can assume U affine.

$\Longrightarrow$ Δ_O affine

Let's assume for a moment

O, U are sheaves in Sch_{fppf}

$\Longrightarrow$ Suffices to show $\forall T \to U$

$$\begin{tikzcd} \exists\, T' \arrow[r, two heads, "\text{fppf}"] \arrow[d] & T \arrow[d] \\ O \arrow[r] & U \end{tikzcd}$$

scheme

$$\begin{tikzcd} T' = \tilde{O}_T \arrow[r, two heads, "\text{sm}"] \arrow[d] \arrow[rr, bend left, two heads, "\text{fppf}"] & O_T \arrow[r, two heads, "\text{fppf}"] \arrow[d] & T \arrow[d] \\ \exists\, \tilde{O} \arrow[r, two heads, "\text{smooth}"] & O \arrow[r] & U \end{tikzcd}$$

sch

sch

schemes

alg stack + sheaf

Missing ingredient

Thm An alg. space X is a sheaf on Sch_{fppf}

Extension: $\mathcal{X}$ alg stack + sheaf

Assume $\Delta_{\mathcal{X}}$ repn by scheme

Then $\mathcal{X}$ sheaf on Sch_{fppf}

Sketch Let $X \to X^f$ sheafification in Sch_{fppf}

① $X \to X^f$ injective

② Sections of X^f lift étale-locally to X.

Let $T \to X^f$

Choose $U \xrightarrow{ét} X$

Can assume X, T, U are q. compact

$$\begin{tikzcd}
 & U_{T'} \arrow[rr, "\Rightarrow ét", "\substack{q.cpt \\ q.affine}"'] \arrow[d] & & T' \arrow[dl] \arrow[d, "fppf"] & \\
\text{sheaf} \arrow[r] & U_T \arrow[r] \arrow[d] & X_T \arrow[r] \arrow[d] & T \arrow[d] & \text{scheme} \\
 & U \arrow[r, "ét"] & X \arrow[r] & X^f &
\end{tikzcd}$$

Need to show: U_T scheme & $U_T \xrightarrow[qc]{ét} T$

By defn of sheafification, $\exists\, T' \xrightarrow{fppf} T$ & $T' \to X$ over $T \to X^f$

Since $X \to X^f$ mono

$U_{T'} := T' \times_{X^f} U = T' \times_X U$

Descent for q. affine morphs $\Rightarrow$

U_T scheme & $U_T \xrightarrow{ét} T$