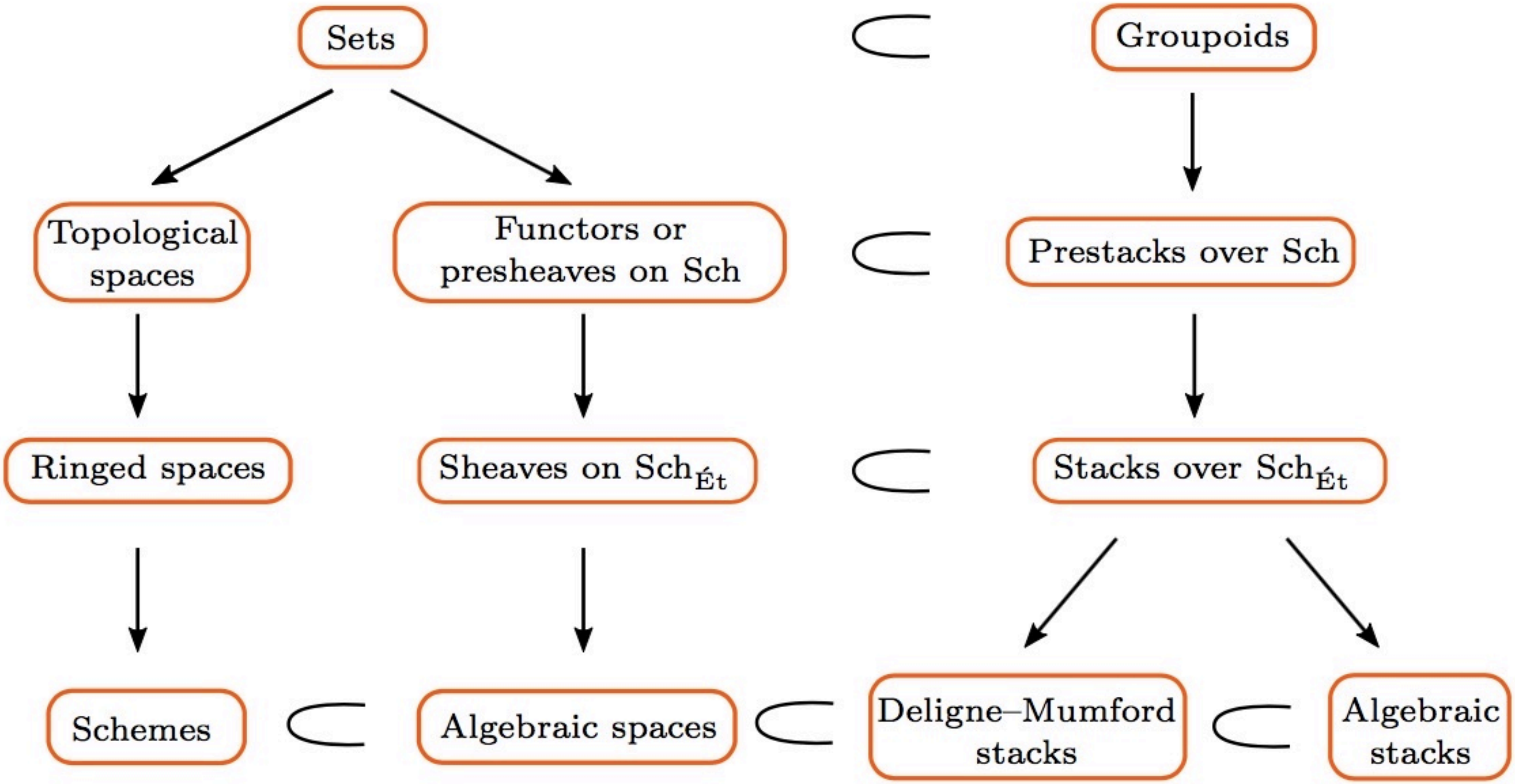
# LECTURE 5 : Algebraic spaces and stacks



## §0. Recap

Let $\mathcal{S}$ be a site

**Def 1** A prestack over $\mathcal{S}$ is a functor

$$\begin{tikzcd}
\mathcal{X} \arrow[d, "p"] & \text{object} & a \arrow[r] \arrow[d, mapsto] & b \arrow[d, mapsto] \\
\mathcal{S} & & p(a)=S \arrow[r] & T
\end{tikzcd}$$

such that

① pullbacks exist

② Any map in $\mathcal{X}$ satisfies a univ. property

**Def 2** A prestack $\mathcal{X}$ over $\mathcal{S}$ is a stack if

① morphism glue uniquely with respect to covers $\{S_i \to S\}$

$$\begin{tikzcd}
a|_{S_i} \arrow[rr] \arrow[rrr, bend left, "\phi_i"] \arrow[dr, mapsto] & & a \arrow[r, dashed, "\exists!"] \arrow[d, mapsto] & b \arrow[dl, mapsto] \\
& S_i \arrow[r] & S & \text{cover}
\end{tikzcd}$$

$$\phi_i|_{S_{ij}} = \phi_j|_{S_{ij}}$$

② Objects glue with respect to covers

$$\begin{tikzcd}
a_i|_{S_{ij}} \arrow[r, "\alpha_{ij}"] \arrow[d, mapsto] & a_j|_{S_{ij}} \arrow[dl, mapsto] & a_i \arrow[rr, dashed, "\exists"] \arrow[d, mapsto] & & a \arrow[d, dashed, mapsto] \\
\coprod S_{ij} \arrow[rr, shift left] \arrow[rr, shift right] & & \coprod S_i \arrow[rr] & & S
\end{tikzcd}$$

$$\alpha_{jk} \circ \alpha_{ij} = \alpha_{ik}$$

# Examples

① $X$ scheme $\rightsquigarrow$ stack $\mathcal{X}_X$ over $Sch_{Ét}$

objects: maps $S \to X$

② stack

$$\begin{tikzcd} QCoh \arrow[d] \\ Sch \end{tikzcd}$$

objects $(\underset{\text{scheme}}{\underset{\uparrow}{S}}, \underset{\text{q. coh.}}{\underset{\uparrow}{\mathcal{F}}})$

③ $M_g$ objects $C \to S$ sm. family of curves

morphisms: cart. diag

Know: All stacks over $Sch_{Zar}$

Étale descent $\Rightarrow$ stacks over $Sch_{Ét}$

# §1. Summary of descent

**Key algebra fact:** If $\phi \colon A \to B$ is a faithfully flat ring map and $M$ is an $A$-module, the sequence

$$\begin{tikzcd} M \arrow[r, "m \mapsto m\otimes 1"] & M \otimes_A B \arrow[r, shift left, "m\otimes b \mapsto m\otimes b\otimes 1"] \arrow[r, shift right, "m\otimes b \mapsto m\otimes 1\otimes b"'] & M \otimes_A B \otimes_A B \end{tikzcd}$$

is exact.

can. isom between 2 pullbacks of $\mathcal{F}'$

$\mathcal{F}' := \mathcal{F}_{S'}$

$\cancel{\mathcal{F} = \widetilde{M}}$

$$\begin{tikzcd} S' \times_S S' \arrow[r, shift left] \arrow[r, shift right] & S' = \operatorname{Spec} B \arrow[r] & \operatorname{Spec} A = S \end{tikzcd}$$

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Fix an étale cover $\{S_i \to S\}$

**Descending quasi-coherent sheaves:**

- Morphisms: Given $\mathcal{F}, \mathcal{G} \in \mathrm{QCoh}(S)$,

$$\left\{ \begin{matrix} \text{maps } \mathcal{F}|_{S_i} \xrightarrow{\phi_i} \mathcal{G}|_{S_i} \\ \text{s.t. } \phi_i|_{S_{ij}} = \phi_j|_{S_{ij}} \end{matrix} \right\} \longleftrightarrow \{\mathcal{F} \xrightarrow{\phi} \mathcal{G}\}$$

- Objects:

$$\left\{ \begin{matrix} \mathcal{F}_i \in \mathrm{QCoh}(S_i) \ \&\ \mathcal{F}_i|_{S_{ij}} \overset{\alpha_{ij}}{\xrightarrow{\sim}} \mathcal{F}_j|_{S_{ij}} \\ \text{s.t. } \alpha_{jk} \circ \alpha_{ij} = \alpha_{ik} \end{matrix} \right\} \longleftrightarrow \{\mathcal{F} \in \mathrm{QCoh}(S)\}$$

$\Downarrow$

QCoh stack over $\mathrm{Sch}_{Et}$

**Descending affine morphisms:**

$$\left\{ \begin{matrix} X_i \xrightarrow{\text{aff}} S_i \ \&\ X_i|_{S_{ij}} \overset{\alpha_{ij}}{\xrightarrow{\sim}} X_j|_{S_{ij}} \\ \text{s.t. } \alpha_{jk} \circ \alpha_{ij} = \alpha_{ik} \end{matrix} \right\} \longleftrightarrow \{X \xrightarrow{\text{aff}} S\}$$

$X_i|_{S_{ij}} \xrightarrow{\alpha_{ij}} X_i|_{S_{ij}}$

$$\begin{tikzcd} & X_i \arrow[d, "\text{aff.}"] \arrow[r, dashed] & X \arrow[d, dashed, "\text{aff}"] \\ S_{ij} \arrow[r, shift left] \arrow[r, shift right] & S_i \arrow[r] & S \end{tikzcd}$$

Last time

$$\begin{tikzcd} C_i \arrow[r, hook] \arrow[d, "\pi"] & \mathbb{P}(\pi_* \Omega^{\otimes 3}_{C/S}) \\ S_i \end{tikzcd}$$

Special case:

① closed immersions

$X_i|_{S_{ij}} = X_j|_{S_{ij}}$

$$\begin{tikzcd} & & X_i \arrow[d, hook] & \\ \coprod S_{ij} \arrow[rr, shift left] \arrow[rr, shift right] & & \coprod S_i \arrow[r] & \boxed{S} \end{tikzcd}$$

② principal G-bundles

Fix $G \to T$ smooth & affine gp scheme

**Def** A $G$-bundle over $T$ is a map $P \to T$ & $G$ action on $P$ over $T$

$\exists \{T_i \xrightarrow{\text{ét}} T\}$

$$\begin{tikzcd} G \times T_i \cong P \times_T T_i \arrow[r] \arrow[d] & P \arrow[d] \\ T_i \arrow[r] & T \end{tikzcd}$$

### Descending $G$-bundles:

$$\boxed{\left\{\begin{matrix} X_i \xrightarrow{G\text{-bdl}} S_i \ \& \ X_i|_{S_{ij}} \overset{\alpha_{ij}}{\tilde{\to}} X_j|_{S_{ij}} \\ \text{s.t. } \alpha_{jk}\circ\alpha_{ij} = \alpha_{ik} \end{matrix}\right\} \longleftrightarrow \{X \xrightarrow{G\text{-bdl}} S\}}$$

<u>Key pt</u>: G-bdl P→T is nec. affine

More generally, we have descent for

- $\boxed{\text{quasi-affine morphism}}$

Ex: separated & q.finite

Zariski Main Thm ⇒ q.affine

Even more generally (for later reference)

### Descending separated and locally q.finite maps:

$$\left\{\begin{matrix} X_i \xrightarrow[\text{loc q.fin}]{\text{sep}} S_i \ \& \ X_i|_{S_{ij}} \overset{\alpha_{ij}}{\tilde{\to}} X_j|_{S_{ij}} \\ \text{s.t. } \alpha_{jk}\circ\alpha_{ij} = \alpha_{ik} \end{matrix}\right\} \longleftrightarrow \{X \xrightarrow[\text{loc q.fin}]{\text{sep}} S\}$$

### Consequence of descent:

- Let $\mathcal{P}$ be a property of morphisms: open imm, closed imm, affine, quasi-affine, or sep & loc. q.finite, G-torsor
- Let $S' \overset{\text{ét}}{\twoheadrightarrow} S$ be étale surjection of schemes.
- Let $F' \to$ ~~$F$~~ $S$ be a map of ~~pre~~sheaves

scheme sheaf

$$\begin{tikzcd} F' \arrow[r] \arrow[d, "\mathcal{P}"'] \arrow[dr, phantom, "\square"] & F \arrow[d, "\textcircled{\mathcal{P}}"] \\ S' \arrow[r, two heads, "\text{ét}"] & S. \end{tikzcd}$$

⇒ F scheme

scheme scheme

$$\boxed{\begin{pmatrix} F' \text{ is a scheme } \& \\ F' \to S' \text{ has } \mathcal{P} \end{pmatrix} \Longleftrightarrow \begin{pmatrix} F \text{ is a scheme } \& \\ F \to S \text{ has } \mathcal{P} \end{pmatrix}}$$

# §2. Hilbert schemes

**Thm (Grothendieck)**

- Let $X \xrightarrow{\mathcal{O}_X(1)} T$ proj. map of noeth. schemes
- Let $P \in \mathbb{Q}[z]$ poly.

The functor

$$Sch/T \longrightarrow Sets$$

$$(S \to T) \longmapsto \left\{ \begin{tikzcd} Z \arrow[r, hook, "cl."] \arrow[dr, "\text{flat}\ \text{fin pres}"'] & X \times_T S \arrow[d] \\ & S \end{tikzcd} \;\middle|\; \begin{array}{l} \forall s \in S \\ \text{Hilb poly}(Z_s) = P \end{array} \right\}$$

is represented by a scheme $Hilb^P(X/T)$ projective over $T$.

# §3. Main definitions

2 definitions:
- representable by schemes
- representable

**Def.** A map $F \to G$ of presheaves/prestacks over Sch is *representable* ~~by schemes~~ if for all maps $S \to G$ from a scheme, $F \times_G S$ is a ~~scheme~~. alg. space

$$\begin{tikzcd} F \times_G S \arrow[r, "P"] \arrow[d] & S \text{ scheme} \arrow[d] \\ F \arrow[r] & G \end{tikzcd}$$

**Def.** Let $\mathcal{P}$ be a property of maps of schemes (e.g. surjective or étale). A map $F \to G$ representable by schemes *has property* $\mathcal{P}$ if for all $S \to G$ from a scheme, $F \times_G S \to S$ has $\mathcal{P}$.

**Def.** Let $\mathcal{P}$ be a property of maps of schemes étale-local on the source (e.g. surjective, étale or smooth). A representable map $F \to G$ *has property* $\mathcal{P}$ if for all $S \to G$ from a scheme and all étale presentations $U \to F \times_G \cancel{G}\, S$, the composition $U \to F \times_G S \to S$ has $\mathcal{P}$.

If $X' \xrightarrow{\text{ét}} X$ (surjective) $X \to Y$ $P \iff X' \to X \to Y$ has $P$

$$\begin{tikzcd} U \arrow[r] & F \times_G S \arrow[r] \arrow[d] & S \text{ scheme} \arrow[d] \\ & F \arrow[r, "\text{rep}"] & G \end{tikzcd}$$

alg. space

**DEF** An alg. space is a sheaf $X$ on $\text{Sch}_{\text{Ét}}$ such that $\exists$ scheme $U$ & $U \to X$ repn by schemes, étale & sur.

Call $U \to X$ étale presentation

**DEF** A Deligne Mumford stack is a stack $\mathfrak{X}$ over $\text{Sch}_{\text{Ét}}$ such that $\exists$ scheme $U$ & $U \to \mathfrak{X}$ representable, étale & surjective.

← étale pres.

**DEF** An algebraic stack is a stack $\mathfrak{X}$ over $\text{Sch}_{\text{Ét}}$ such that $\exists$ scheme $U$ & $U \to \mathfrak{X}$ representable, smooth & surjective.

← smooth pres.

# History

- Alg. spaces were introduced by Artin & Knutson ~69, 71
  (quasi-compact assumption on $\Delta_X$)
- DM stacks were introduced by Deligne & Mumford '69
  (called then algebraic stacks)
  (assume $\Delta$ repr by schemes)
- Algebraic stacks were introduced by Artin '74
  (called then algebraic stacks)
  (assumed loc. f.type excellent Dedekind domain)

# Terminology

⚠ Warning:

Our defns are not standard

- usually has representability cond. on $\Delta_{\mathcal{X}} : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$
- we will show it is equivalent.

⚠ Warning:

Different authors have different hypotheses on $\Delta$

We follow Olsson & Stacks project

We use $Sch_{Ét}$, not $Sch_{fppf}$

## Morphisms

- Maps of alg. spaces are maps of sheaves
- Maps of Deligne-Mumford or algebraic stacks are map of stacks over $Sch_{Et}$

**Exer:** Show fiber products exist for alg. spaces, DM stack & alg stacks

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**Exer:** Show that any stack over $(Sch/S)_{Et}$ can be viewed as a stack over $Sch_{Et}$

# §4. Algebraicity of Quotient Stacks

Setup (Think $T = \operatorname{Spec} k$)

- $G \to T$ smooth & affine group scheme
- (think: $U$ scheme)
  $U$ alg. space with action of $G$
  defined analogous to schemes
- $[U/G]$ denotes stack over $(Sch/T)_{ét}$

objects

$$\begin{tikzcd} P \arrow[r, "G\text{-equiv}"] \arrow[d, "G\text{-bdl}"] & U \\ S & \end{tikzcd}$$

**THM** $[U/G]$ is an alg. stack over $T$

Moreover, $U \to [U/G]$ is a $G$-torsor and in particular repn, surjective & smooth & affine.

smooth pres.

top, bottom, back & front are cartesian

**PROOF** Set $\mathfrak{X} = [U/G]$

Let $S \to \mathfrak{X}$ be a map from a scheme

Consider

$$\begin{tikzcd} U_S \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & U \arrow[d] \\ S \arrow[r] & \mathfrak{X} = [U/G] \end{tikzcd}$$

~~U scheme~~ ~~ét~~ alg space

Goal: $U_S \to S$ is a $G$-torsor

(in part, we want to show $U_S$ scheme affine/$S$)

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Since $[U/G] = ([U/G]^{pre})^{st}$

$$\begin{tikzcd} \exists\ S' \arrow[r] \arrow[d, "ét"'] & U \arrow[d] \arrow[dl, phantom, "\Swarrow"] \\ S \arrow[r] & \mathfrak{X} \end{tikzcd}$$

Form cube

$$\begin{tikzcd} & U_{S'} \arrow[rr, "G\text{-bdl}"] \arrow[dl] & & S' \arrow[dl] \arrow[dd, "ét"] \\ G \times U \arrow[rr, "G\text{-bdl}"] \arrow[dd] & & U \arrow[dd] & \\ & U_S \arrow[rr, "G\text{-bdl}"] \arrow[dl] & & S \arrow[dl] \\ U \arrow[rr] & & \mathfrak{X} & \end{tikzcd}$$

scheme

scheme

sheaf

**Cor** If $G$ is a finite group acting freely on an alg. space $X$, the quotient sheaf $X/G$ is an alg. space.

here $X \longrightarrow X/G$ étale pres.

**Upshot:**

# §5. Algebraicity of $\mathcal{M}_g$

**THM** If $g \geq 2$, $\mathcal{M}_g$ is algebraic.

**PF**
- Know $\mathcal{M}_g$ stack over $Sch_{Et}$
- Strategy: $\mathcal{M}_g \cong [H/G]$

where $H \subset$ Hilb lo. closed param. 3-can. embedded sm. curves

Consider

$$\begin{tikzcd} \mathcal{C} \arrow[r, hook] \arrow[d, "\pi"', "\text{sm family}"] & \mathbb{P}(\underbrace{\pi_* \Omega^{\otimes 3}_{\mathcal{C}/S}}_{\text{rank } 5(g-1)}) \arrow[dl] \\ S & \end{tikzcd}$$

On a fiber $s \in S$

$$\begin{tikzcd} \mathcal{C}_s \arrow[r, hook] \arrow[d] & \mathbb{P}^{5g-6} \arrow[dl] \\ \operatorname{Spec} k(s) & \end{tikzcd}$$

Hilb. poly by Riemann-Roch

$P(n) = (6n-1)(g-1)$

Define $H := \text{Hilb}_P(\mathbb{P}^{5g-6})$

proj over $\mathbb{Z}$

$$\begin{tikzcd} \mathcal{C} \arrow[r, hook] \arrow[d, "\text{univ. family}"'] & \mathbb{P}^{5g-6} \times H \arrow[dl] \\ H & \end{tikzcd}$$

Idea: Look at locus in $H$ where of smooth curves tri-canonically embedded

lo. closed

**Claim:** $\exists!\, H' \hookrightarrow H = \text{Hilb}_p(\mathbb{P}^{5g-6})$ containing $h \in H$ s.t.

(a) $\mathcal{C}_h$ is smooth & geom. connected

(b) Setting $\mathcal{C}' = \mathcal{C}|_{H'}$, $\Omega^{\otimes 3}_{\mathcal{C}'/H'}$ and $\mathcal{O}_{\mathcal{C}'}(1)$ differ by the pullback of a line bundle on $H'$.

Over $h \in H$, $\Omega^{\otimes 3}_{\mathcal{C}_h} \cong \mathcal{O}_{\mathcal{C}_h}(1)$

(c) $\mathcal{C}_h \hookrightarrow \mathbb{P}^{5g-6}$ is embedded via $|\Omega^{\otimes 3}_{\mathcal{C}_h}|$

complete linear series

$$\Gamma(\mathbb{P}^{5g-6}, \mathcal{O}(1)) \xrightarrow{\sim} \Gamma(\mathcal{C}_h, \Omega^{\otimes 3}_{\mathcal{C}_h})$$

**Claim:** $\exists!\, H' \hookrightarrow H = \mathrm{Hilb}_p(\mathbb{P}^{5g-6})$ containing $h \in H$ s.t.

(a) $\mathcal{C}_h$ is smooth & geom. connected

(b) Setting $\mathcal{C}' = \mathcal{C}|_{H'}$, $\Omega^{\otimes 3}_{\mathcal{C}'/H'}$ and $\mathcal{O}_{\mathcal{C}'}(1)$ differ by the pullback of a line bundle on $H'$.

(c) $\mathcal{C}_h \hookrightarrow \mathbb{P}^{5g-6}$ is embedded via $|\Omega^{\otimes 3}_{\mathcal{C}_h}|$

Pf of claim:

$$\begin{tikzcd} \mathcal{C} \arrow[r, hook] \arrow[d, "\pi"', "\text{proper}"] & \mathbb{P}^{5g-6} \times H \\ H & \end{tikzcd}$$

(a) (i) $\{h \in H \mid \mathcal{C}_h \text{ smooth}\} \subset H$ open

(ii) Stein fact.

$$\begin{tikzcd} \mathcal{C} \arrow[r] \arrow[dr] & \operatorname{Spec} \underline{\pi_* \mathcal{O}_{\mathcal{C}}} \arrow[d] \\ & H \end{tikzcd}$$

coh.

geom con fibers

$\lambda: \mathcal{O}_H \to \pi_* \mathcal{O}_{\mathcal{C}}$

$\{h \in H \mid \mathcal{C}_h \text{ geom. conn}\} = \{h \in H \mid \lambda \text{ is isom. over } h\}$

$= H \setminus (\mathrm{Supp}(\ker \lambda) \cup \mathrm{Supp}(\operatorname{coker} \lambda))$ ✓

**Thm.**

- *Let* $f\colon \overset{L}{X} \to Y$ *be a flat, proper map of noeth schemes with geom integral fibers.*
- *Let* $L$ *be a line bundle on* $X$.
- *Assume:* $f_* \mathcal{O}_X = \mathcal{O}_Y$ *and holds after base change.*

$\implies$ $\exists$ *closed subscheme* $Z \hookrightarrow Y$ *s.t.* $T \to Y$ *factors through* $Z$ *if and only if* $L|_{X_T}$ *is the pullback of a line bundle on* $T$.

Equiv: Separatedness of

$\mathrm{Pic}_{X/Y}: \mathrm{Sch}/Y \to \mathrm{Set}$

$(S \to Y) \mapsto \mathrm{Pic}(X \times_Y S)/\mathrm{Pic}(S)$

$\implies$ (b) ✓

(c) $H^0(\mathbb{P}^{5g-6}, \mathcal{O}(1)) \to \pi_* \Omega^{\otimes 3}_{\mathcal{C}/H}$

Locus in $H$ where this is an isom. ✓

**Claim:** $\exists!\, H' \hookrightarrow H = \mathrm{Hilb}_p(\mathbb{P}^{5g-6})$ containing $h \in H$ s.t.

(a) $\mathcal{C}_h$ is smooth & geom. connected

(b) Setting $\mathcal{C}' = \mathcal{C}|_{H'}$, $\Omega^{\otimes 3}_{\mathcal{C}'/H'}$ and $\mathcal{O}_{\mathcal{C}'}(1)$ differ by the pullback of a line bundle on $H'$.

(c) $\mathcal{C}_h \hookrightarrow \mathbb{P}^{5g-6}$ is embedded via $|\Omega^{\otimes 3}_{\mathcal{C}_h}|$

∃ functorial action of $PGL_{5g-5} = \underline{Aut}(\mathbb{P}^{5g-6})$ on $H$ and $H' \subset H$ is invariant

<u>GOAL:</u> $\mathcal{M}_g \cong [H'_3/PGL_{5g-5}]$

④ <u>Ess. surjective:</u>

Suffices to show that $\forall\, C \xrightarrow{fam} S$ $\exists\, \{S_i \to S\}$ s.t. $C_{S_i} \to S_i$ is in <u>the image</u>

$$\begin{tikzcd} \pi_*\Omega^{\otimes 3}_{C/S} & C \arrow[d, "\pi"] \\ & S \end{tikzcd}$$

Choose Zariski-cover where $\pi_*\Omega^{\otimes 3}_{C/S}$ is trivialization

① Construct map

$$\begin{tikzcd} {[H'/PGL_{5g-6}]^{pre}} \arrow[r] & \mathcal{M}_g \\ (S \to H') \arrow[r, mapsto] & \left(\begin{array}{c} C \\ \downarrow \\ S \end{array}\right) \end{tikzcd}$$

$$\begin{tikzcd} C \arrow[r, hook] \arrow[d] & \mathbb{P}^{5g-6} \times S \\ S \end{tikzcd}$$

② This is fully faithful

<u>Reason:</u> An aut of $C \to S$ ⇝ aut $\Omega^{\otimes 3}_{C/S}$ ⇝ aut of proj. space

③ By univ. property of stackification

$\exists\, [H'/PGL_{5g-6}] \to \mathcal{M}_g$

Also fully faithful