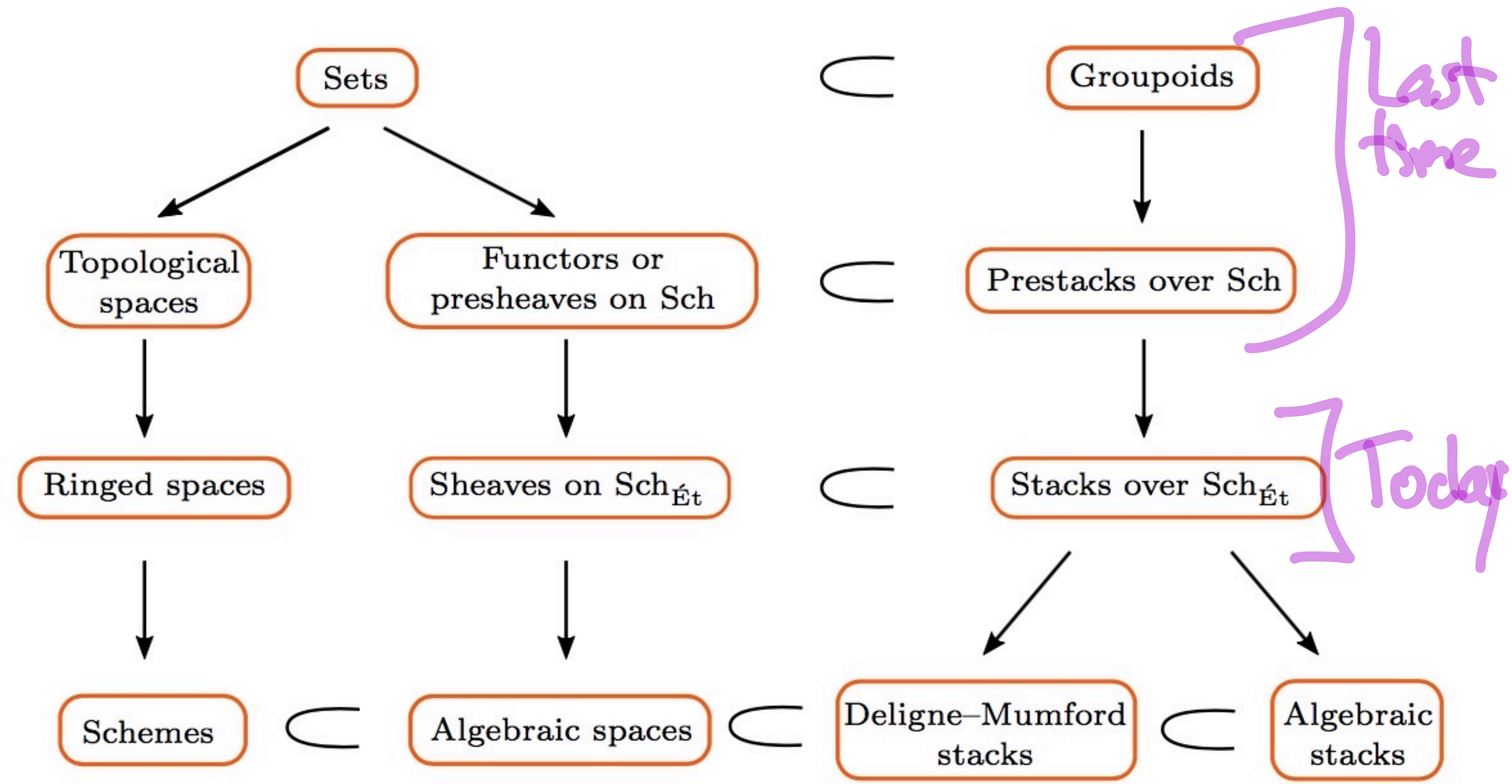
Lecture 4: Stacks



§0. Review

Let $\mathcal{S}$ be a category

<u>DEF</u> A <u>prestack over $\mathcal{S}$</u> is a functor

$$\begin{tikzcd} \mathcal{X} \arrow[d, "p"] \\ \mathcal{S} \end{tikzcd} \qquad \text{objects} \qquad \begin{tikzcd} a \arrow[r, "\alpha"] \arrow[d, mapsto] & b \arrow[d, mapsto] \\ p(a)=S \arrow[r, "f"] & T \end{tikzcd}$$

such that

① $\forall$ diagrams

$$\begin{tikzcd} \exists\, a \arrow[r, dashed] \arrow[d, mapsto] & b \arrow[d, mapsto] \\ S \arrow[r] & T \end{tikzcd} \qquad \text{pullback}$$

$S = p(a)$

② $\forall$ diagrams

$$\begin{tikzcd} a' \arrow[rr, bend left] \arrow[r, dashed, "\exists!"] \arrow[d, mapsto] & a \arrow[r] \arrow[d, mapsto] & b \arrow[d, mapsto] \\ S' \arrow[r] \arrow[rr, bend left] & S \arrow[r] & T \end{tikzcd}$$

univ. prop

The <u>fiber category</u> $\mathcal{X}(S)$ has object over S & morphism over id_S

<u>Exer</u>: $\mathcal{X}(S)$ is a groupoid.

Examples

① Schemes are prestacks

If X is a scheme,

$$\begin{tikzcd} Sch/X \arrow[d] \\ Sch \end{tikzcd} \quad \text{prestack} \qquad \text{objects} \quad \begin{tikzcd} S \arrow[r] \arrow[d, mapsto] & X \\ S \end{tikzcd}$$

② $\boxed{\mathcal{M}_g}$ objects $\begin{array}{c} C \\ \downarrow \\ S \end{array}$ sm. fam of gen g curves

$$Mor\left(\begin{array}{c} C \\ \downarrow \\ S \end{array}, \begin{array}{c} C' \\ \downarrow \\ S' \end{array} \right) = \left\{ \text{cart. diag} \begin{tikzcd} C \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & C' \arrow[d] \\ S \arrow[r] & S' \end{tikzcd} \right\}$$

$$\begin{tikzcd} \mathcal{M}_g \arrow[d] \\ Sch \end{tikzcd} \quad \text{prestack}$$

MORPHISMS

Def A <u>morphism</u> $f : \mathcal{X} \to \mathcal{Y}$ of prestacks is a functor s.t.

$$\begin{tikzcd} \mathcal{X} \arrow[rr, "f"] \arrow[dr, "p_{\mathcal{X}}"'] & & \mathcal{Y} \arrow[dl, "p_{\mathcal{Y}}"] \\ & S & \end{tikzcd}$$

strictly commutes

Def A 2-morphism $\alpha : f \to g$ between morphism is a nat. trans

Notation

$$\begin{tikzcd} \mathcal{X} \arrow[r, bend left, "f"{name=U}] \arrow[r, bend right, "g"'{name=D}] & \mathcal{Y} \arrow[from=U, to=D, phantom, "\Downarrow\alpha"] \end{tikzcd}$$

2-comm. diagrams

$$\begin{tikzcd} \mathcal{X}' \arrow[r, "g'"] \arrow[d, "f'"'] \arrow[dr, phantom, "\alpha \swarrow"] & \mathcal{X} \arrow[d, "f"] \\ \mathcal{Y}' \arrow[r, "g"] & \mathcal{Y} \end{tikzcd} \qquad \begin{tikzcd} f \circ g' \arrow[d, "\alpha"', "\wr"] \\ g \circ f' \end{tikzcd}$$

The 2-Yoneda Lemma. *Let $\mathcal{X}$ be a prestack over a category $\mathcal{S}$ and $S \in \mathcal{S}$. The functor*

$$\mathrm{MOR}(S, \mathcal{X}) \to \mathcal{X}(S), \qquad f \mapsto f_S(\mathrm{id}_S)$$

is an equivalence of categories.

Upshot

a map $S \xrightarrow{a} \mathcal{X} \longleftrightarrow$ object $a \in \mathcal{X}(S)$

⚠ Conflate the two

§2. Fiber Products of Prestacks

Consider morphisms of prestacks over $\mathcal{S} \supset S$

$$\begin{tikzcd}
\mathcal{X}\times_{\mathcal{Y}}\mathcal{Y}' \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\alpha\,\Swarrow"] & \mathcal{Y}' \arrow[d, "g"] \\
\mathcal{X} \arrow[r, "f"] & \mathcal{Y}
\end{tikzcd}$$

Construction of $\mathcal{X}\times_{\mathcal{Y}}\mathcal{Y}'$.

- Objects = triples (x, y', γ)

  $x \in \mathcal{X}$, $y' \in \mathcal{Y}'$ over same S — in $\mathcal{Y}$: $f(x) \xrightarrow{\gamma} g(y')$ over ids

- A map $(x_1, y_1', \gamma_1) \to (x_2, y_2', \gamma_2)$ is a triple (q, χ, γ')

  $x_1 \xrightarrow{\chi} x_2$ $\quad S_1 \xrightarrow{q} S_2$

  $y_1' \xrightarrow{\chi'} y_2'$

  such that

$$\begin{tikzcd}
f(x_1) \arrow[r, "f(\chi)"] \arrow[d, "\gamma_1"] & f(x_2) \arrow[d, "\gamma_2"] \\
g(y_1') \arrow[r, "g(\gamma')"] & g(y_2')
\end{tikzcd}$$

  commutes.

$\Rightarrow (\mathcal{X}\times_{\mathcal{Y}}\mathcal{Y}')(S) = \mathcal{X}(S) \times_{\mathcal{Y}(S)} \mathcal{Y}'(S)$

fib. prod. of groupoids

Theorem. *The prestack $\mathcal{X}\times_{\mathcal{Y}}\mathcal{Y}'$ satisfies the following universal property: for any 2-commutative diagram*

$$\begin{tikzcd}
\mathcal{T} \arrow[drr, bend left, "q_2"] \arrow[ddr, bend right, "q_1"'] \arrow[dr, dashed, "\exists h"] & & \\
& \mathcal{X}\times_{\mathcal{Y}}\mathcal{Y}' \arrow[r, "p_2"] \arrow[d, "p_1"] \arrow[dr, phantom, "\alpha\Nearrow"] & \mathcal{Y}' \arrow[d, "g"] \\
& \mathcal{X} \arrow[r, "f"] & \mathcal{Y},
\end{tikzcd}$$

there exist a morphism $h\colon \mathcal{T} \to \mathcal{X}\times_{\mathcal{Y}}\mathcal{Y}'$ and 2-isomorphisms $\beta\colon q_1 \to p_1 \circ h$ and $\gamma\colon q_2 \to p_2 \circ h$ such that

$$\boxed{\begin{tikzcd}
f\circ q_1 \arrow[r, "f(\beta)"] \arrow[d, "\tau"] & f\circ p_1 \circ h \arrow[d, "\alpha\circ h"] \\
g\circ q_2 \arrow[r, "g(\gamma)"] & g\circ p_2 \circ h
\end{tikzcd}}$$

commutes.

$g\circ p_2 \xleftarrow{\alpha} f\circ p_1$ 2-morphism

For object (x, y', γ)

$\alpha_{(x,y',\gamma)}\colon f(x) \xrightarrow{\gamma} g(y')$

Exer Let $\boxed{\mathcal{M}_{g,1}}$ be the prestack

object: $\begin{array}{c} C \\ \downarrow \uparrow \sigma \\ S \end{array}$ sm. fam of gen g curves with section σ

morphism = cart. diagrams of families compatible w/ section

$$\begin{tikzcd}
C \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{M}_{g,1} \arrow[d, "\text{univ. fam.}"] & \left[\begin{array}{c} C \\ \downarrow\uparrow\sigma \\ S \end{array}\right] \arrow[d] \\
S \arrow[r] & \mathcal{M}_g & \left[\begin{array}{c} C \\ \downarrow \\ S \end{array}\right]
\end{tikzcd}$$

Fix map $\xleftrightarrow{\text{2-Yon}}$ family of curves $C \to S$

Exer Let $\mathcal{X}$ be a prestack

Let $a, b : T \to \mathcal{X}$

Define presheaf

$$\underline{\mathrm{Isom}}_T(a,b) : Sch/T \to Sets$$

$$(S \xrightarrow{f} T) \mapsto \mathrm{Isom}_{\mathcal{X}(S)}(f^*a, f^*b)$$

Show

$$\begin{tikzcd}
\underline{\mathrm{Isom}}_T(a,b) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & T \arrow[d, "(a,b)"] \\
\mathcal{X} \arrow[r, "\Delta"] & \mathcal{X} \times \mathcal{X}
\end{tikzcd}$$

§3. Stacks

Definition. A *stack over a site* $\mathcal{S}$ is a prestack $\mathcal{X}\to\mathcal{S}$ such that for all coverings $\{S_i\to S\}$ of $S\in\mathcal{S}$:

(1) (morphisms glue) For $a,b\in\mathcal{X}$ over S and maps $\phi_i\colon a|_{S_i}\to b$ s.t. $\phi_i|_{S_{ij}}=\phi_j|_{S_{ij}}$

Axiom 2 $\Leftrightarrow$ $\underline{\mathrm{Isom}}_T(a,b)$ sheaves

$$\begin{tikzcd} & a|_{S_i} \arrow[dr] \arrow[drr, bend left, "\phi_i"] & & \\ a|_{S_{ij}} \arrow[ur] \arrow[dr] & & a \arrow[r, dashed, "\exists!"] & b \\ & a|_{S_j} \arrow[ur] \arrow[urr, bend right, "\phi_j"'] & & \end{tikzcd} \quad \text{over} \quad \begin{tikzcd} & S_i \arrow[dr] & \\ S_{ij} \arrow[ur] \arrow[dr] & & S, \\ & S_j \arrow[ur] & \end{tikzcd}$$

there exists a unique map $\phi\colon a\to b$ with $\phi|_{S_i}=\phi_i$.

(2) (objects glue) For $a_i\in\mathcal{X}$ over S_i and isos $\alpha_{ij}\colon a_i|_{S_{ij}}\to a_j|_{S_{ij}}$ with $\boxed{\alpha_{ij}|_{S_{ijk}}\circ\alpha_{jk}|_{S_{ijk}}=\alpha_{ik}|_{S_{ijk}}}$ cocycle

$$\begin{tikzcd} & a_i & \\ a_i|_{S_{ij}} \arrow[r, "\alpha_{ij}"] & a_j|_{S_{ij}} \arrow[u] \arrow[d] & a \arrow[ul, dashed] \arrow[dl, dashed] \arrow[l, phantom, "\exists"] \\ & a_j & \end{tikzcd} \quad \text{over} \quad \begin{tikzcd} & S_i \arrow[dr] & \\ S_{ij} \arrow[ur] \arrow[dr] & & S \\ & S_j \arrow[ur] & \end{tikzcd}$$

there exists $a\in\mathcal{X}$ over S and isos $\phi_i\colon a|_{S_i}\to a_i$ s.t. $\alpha_{ij}\circ\phi_i|_{S_{ij}}=\phi_j|_{S_{ij}}$.

Equiv.,

$$\mathcal{X}(S)\to\prod_i\mathcal{X}(S_i)\rightrightarrows\prod_{i,j}\mathcal{X}(S_{ij})\rightrightarrows\prod_{i,j,k}\mathcal{X}(S_{ijk})$$

"exact"

① Sheaves are stacks

$F\colon\mathcal{S}\to\mathrm{Sets}$ presheaf

$\rightsquigarrow \mathcal{X}_F$ prestack $\quad \mathcal{X}_F(S)=F(S)$

F sheaf $\Leftrightarrow$ $\mathcal{X}_F$ stack

② Stack of q.coh sheaves over Sch_{Et}

Define

$$\begin{array}{c} \mathrm{QCoh} \\ \downarrow \\ \mathrm{Sch} \end{array}$$

objects: (S,F) $\quad$ S scheme, $F\in\mathrm{QCoh}(S)$

$\mathrm{Mor}((S,F),(S',F'))=\{S\xrightarrow{f}S',\ F'\to f_*F \text{ s.t. } f^*F'\xrightarrow{\sim}F\}$

Know: QCoh stack over Sch_{Zar}

Gluing sheaves [Hartshorne Exer. II.1.15 & 22]
Let $\{S_i\}$ be a Zariski-open cover of a scheme S.

Axiom 1

- Let F, G sheaves on S. Maps $F|_{S_i}\xrightarrow{\phi_i}G|_{S_i}$ with $\phi_i|_{S_{ij}}=\phi_j|_{S_{ij}}$ glue uniquely to $F\to G$.
- Sheaves F_i on S_i with isos $F_i|_{S_{ij}}\xrightarrow{\alpha_{ij}}F_j|_{S_{ij}}$ with $\alpha_{ij}|_{S_{ijk}}\circ\alpha_{jk}|_{S_{ijk}}=\alpha_{ik}|_{S_{ijk}}$ glue to F on S.

étale descent for q.coh $\Longrightarrow$ QCoh stack over Sch_{Et}

Definition. A *stack over a site* $\mathcal{S}$ is a prestack $\mathcal{X} \to \mathcal{S}$ such that for all coverings $\{S_i \to S\}$ of $S \in \mathcal{S}$:

(1) (morphisms glue) For $a, b \in \mathcal{X}$ over S and maps $\phi_i \colon a|_{S_i} \to b$ s.t. $\phi_i|_{S_{ij}} = \phi_j|_{S_{ij}}$

$$\begin{tikzcd}
 & a|_{S_i} \arrow[dr] \arrow[drr, bend left, "\phi_i"] & & \\
a|_{S_{ij}} \arrow[ur] \arrow[dr] & & a \arrow[r, dashed] & b \\
 & a|_{S_j} \arrow[ur] \arrow[urr, bend right, "\phi_j"'] & &
\end{tikzcd}
\quad \text{over} \quad
\begin{tikzcd}
 & S_i \arrow[dr] & \\
S_{ij} \arrow[ur] \arrow[dr] & & S, \\
 & S_j \arrow[ur] &
\end{tikzcd}$$

there exists a unique map $\phi \colon a \to b$ with $\phi|_{S_i} = \phi_i$.

(2) (objects glue) For $a_i \in \mathcal{X}$ over S_i and isos $\alpha_{ij} \colon a_i|_{S_{ij}} \to a_j|_{S_{ij}}$ with $\alpha_{ij}|_{S_{ijk}} \circ \alpha_{jk}|_{S_{ijk}} = \alpha_{ik}|_{S_{ijk}}$

$$\begin{tikzcd}
 & & a_i \arrow[dr, dashed] & \\
a_i|_{S_{ij}} \arrow[r, "\alpha_{ij}"] & a_j|_{S_{ij}} \arrow[ur] \arrow[dr] & & a \\
 & & a_j \arrow[ur, dashed] &
\end{tikzcd}
\quad \text{over} \quad
\begin{tikzcd}
 & S_i \arrow[dr] & \\
S_{ij} \arrow[ur] \arrow[dr] & & S \\
 & S_j \arrow[ur] &
\end{tikzcd}$$

there exists $a \in \mathcal{X}$ over S and isos $\phi_i \colon a|_{S_i} \to a_i$ s.t. $\alpha_{ij} \circ \phi_i|_{S_{ij}} = \phi_j|_{S_{ij}}$.

Recall

> **Gluing schemes** [Hartshorne Exer. II.2.12]
>
> • Schemes X_i with opens U_{ij} and isos $U_{ij} \xrightarrow{\alpha_{ij}} U_{ji}$ with $\alpha_{ij} \circ \alpha_{jk} = \alpha_{ik}$ glue to a scheme X.

③ Stack of families over Sch_{Zar}

Define object = $X \to S$ map

$\mathcal{M}_g \subset$ Fam $\to$ Sch

$(X \to S) \mapsto S$

$$\mathrm{Mor}\left(\begin{matrix} X \\ \downarrow \\ S \end{matrix}, \begin{matrix} Y \\ \downarrow \\ T \end{matrix}\right) = \left\{ \begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & Y \arrow[d] \\ S \arrow[r] & T \end{tikzcd} \right\}$$

Prop: Fam stack over Sch_{Zar}

Axiom 1: morphisms glue

$$\begin{tikzcd}
X_a|_{S_i} \arrow[r, hook] \arrow[d] \arrow[rr, bend left] & X_a \arrow[r, dashed, "\exists!"] \arrow[d] & X_b \arrow[dl] \\
S_i \arrow[r, hook, "op"] & S &
\end{tikzcd}$$

Not a stack over Sch_{Et}

Axiom 2

$$\begin{tikzcd}
X_i|_{S_{ij}} \arrow[r] \arrow[d] & X_j|_{S_{ij}} \arrow[dl] & X_i \arrow[r, hook] \arrow[d] & X \arrow[d, dashed] \\
S_{ij} \arrow[rr, hook] & & S_i \arrow[r, hook] & S
\end{tikzcd}$$

(4) $\mathcal{M}_g$

Def. A *family of smooth curves of genus g* is a smooth, proper morphism $\mathcal{C} \to S$ of schemes s.t. every geom. fiber $\mathcal{C}_{\overline{\kappa(s)}}$ is a connected curve of genus g.

Prop. *Let $\mathcal{C} \to S$ be a family of smooth curves of genus $g \geq 2$. Then for $k \geq 3$, $\Omega_{\mathcal{C}/S}^{\otimes k}$ is relatively very ample and $\pi_*(\Omega_{\mathcal{C}/S}^{\otimes k})$ is a vector bundle of rank $(2k-1)(g-1)$.*

$E \Rightarrow$

$$\begin{tikzcd} \mathcal{C} \arrow[r, hook] \arrow[dr] & \mathbb{P}(E) \arrow[d] \\ & S \end{tikzcd}$$

Prop ($g \geq 2$) $\mathcal{M}_g$ is a stack over Sch_{Et}

Axiom 2

$$\begin{tikzcd} & \mathbb{P} & \mathcal{C}_j|_{S_{ij}} & \mathbb{P}(E_i) \arrow[r, "\acute{e}t"] & \mathbb{P}(E) \\ \mathcal{C}_{ij} = \mathcal{C}_i|_{S_{ij}} \arrow[ur] \arrow[urr] \arrow[rr, hook] \arrow[d, "E_i|_{S_{ij}} \to E_j|_{S_{ij}}"] & & \mathcal{C}_i \arrow[ur] \arrow[r, hook] \arrow[d, "\pi_i"] & \mathcal{C} \arrow[ur] \arrow[d] & \\ \mathcal{C}_{ij} \arrow[rr] & & S_i \arrow[r] & S & E_i, E \end{tikzcd}$$

Set $E_i := \pi_{i,*}\Omega^{\otimes 3}_{\mathcal{C}_i/S_i}$

Know $E_i|_{S_{ij}} \cong \pi_{ij,*}(\Omega^{\otimes 3}_{\mathcal{C}_{ij}/S_{ij}}) \cong E_j|_{S_{ij}}$

Prop (Effective Descent for Closed Immersions). *Let $\{X_i \to X\}$ be an étale cover. Closed subschemes $Z_i \subset X_i$ with $Z_i|_{X_{ij}} = Z_j|_{X_{ij}}$ glue to a closed subscheme $Z \subset X$.*

Étale descent $\Longrightarrow$ $\mathcal{C} \to S$ sm. fam.

Étale descent $\Longrightarrow$ vect. bdl E on S

(5) Quotient stack $[X/G]$

Setup: $\boxed{G \to S \text{ smooth, affine group scheme}}$

$X \to S$ scheme

$G \curvearrowright X$ functorially $\forall T \to S$

$G(T) \curvearrowright X(T)$

Last time: Defined prestack $[X/G]^{pre}$ over Sch/S

$$[X/G]^{pre}(T) = [X(T)/G(T)]$$

Exer: Show it is not a stack on $(Sch/S)_{et}$

$\{T_i \xrightarrow{et} T\}$ object over T_i

$T_i \to X$

$\rightsquigarrow$

$$\begin{array}{ccc} G \times T_i & \xrightarrow{G\text{-equiv}} & X \\ \downarrow{\scriptstyle \text{trivial}} & & \\ T_i & & \end{array}$$

$\rightsquigarrow$ glue to

$$\begin{array}{c} P \\ \downarrow{\scriptstyle G\text{-bdl}} \\ T \end{array}$$

Def. A *principal G-bundle over T* is a scheme $\boxed{\mathcal{P}}$ with a G-action s.t. $\mathcal{P} \to T$ is G-invariant and $\exists \{T_i \xrightarrow{\text{ét}} T\}$

$$\begin{tikzcd} G\times T_i \cong T_i \times_T \mathcal{P} \arrow[r, "G\text{-equiv.}"] \arrow[d] & \mathcal{P} \circlearrowleft G \arrow[d, "G\text{-inv.}"] \\ T_i \arrow[r] & T \end{tikzcd}$$

affine

$S = \operatorname{Spec} k$,
G alg. group

Define $[X/G]$ over Sch/S

objects
$$\begin{tikzcd} P \arrow[r, "f"] \arrow[d, "G\text{-bdl}"] & X \quad G\text{-equiv} \\ T \end{tikzcd}$$

Special case
$BG = [S/G]$

$$\operatorname{Mor}\left(\begin{tikzcd} P \arrow[r, "f"] \arrow[d] & X \\ T \end{tikzcd}, \begin{tikzcd} P' \arrow[r, "f'"] \arrow[d] & X \\ T' \end{tikzcd} \right) = \left\{ \begin{tikzcd} P \arrow[r] \arrow[d] \arrow[rr, bend left, "f"] \arrow[dr, phantom, "\square"] & P' \arrow[r, "f'"] \arrow[d] & X \\ T \arrow[r] & T' \end{tikzcd} \right\}$$

$$\begin{tikzcd} \operatorname{Spec} k \arrow[d] \\ BG \end{tikzcd}$$

Prop $[X/G]$ stack$/(Sch/k)_{Et}$

§4. Fiber products

Easy lemma

$$\begin{tikzcd} & \mathcal{X} \arrow[d] \\ \mathcal{Y}' \arrow[r] & \mathcal{Y} \end{tikzcd}$$

stacks over site $\mathcal{S}$

$\Rightarrow \mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}'$ is a stack over $\mathcal{S}$

Exer:

① $$\begin{tikzcd} G\times X \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X \arrow[d, "p"] \\ X \arrow[r] & {[X/G]} \end{tikzcd}$$

$$\begin{tikzcd} S \arrow[d, "\text{univ } G\text{-bdl}"] \\ BG \end{tikzcd}$$

② $$\begin{tikzcd} G\times X \arrow[r] \arrow[d] & X\times X \arrow[d] \\ {[X/G]} \arrow[r] & {[X/G]\times[X/G]} \end{tikzcd}$$

$\ni P \xrightarrow{f} X$

③ For any G-bdl $(P \to T)$

$$\begin{tikzcd} P \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, "\square"] & X \arrow[d, "\text{univ. } G\text{-bdl}"] \\ T \arrow[r, "\exists"] & {[X/G]} \end{tikzcd}$$

§5. Stackification

Prop (Stackification). *For a prestack $\mathcal{X}$ over a site $\mathcal{S}$, there exists a morphism $\mathcal{X} \to \mathcal{X}^{\mathrm{st}}$ to a stack such that for any stack $\mathcal{Y}$ over $\mathcal{S}$, the functor*

$$\boxed{\mathrm{MOR}(\mathcal{X}^{\mathrm{st}}, \mathcal{Y}) \to \mathrm{MOR}(\mathcal{X}, \mathcal{Y})}$$

is an equivalence of categories.

Sketch. 2-step

① Construct $\mathcal{X} \to \mathcal{X}_1$

prestack where morphism glue uniq.

$\mathrm{Ob}\,\mathcal{X}_1 = \mathrm{Ob}\,\mathcal{X}$

A map $a \to b$ in $\mathcal{X}_1$ over $\boxed{S \xrightarrow{f} T}$ is by defn an element of

$$\big(\underline{\mathrm{Isom}}_S(a, f^*b)\big)^{sh}(S)$$

presheaf over Sch/S

$\boxed{\text{2-step}}$

$$\begin{tikzcd} \mathcal{X} \arrow[r] \arrow[dr] & \mathcal{X}^{st} \arrow[d, dashed, "\exists"] \arrow[dl, phantom, "\Leftarrow"] \\ & \mathcal{Y} \end{tikzcd}$$

stack

② Construct $\mathcal{X}_1 \to \mathcal{X}^{st}$

Formally define object of $\mathcal{X}^{st}$ as

$$\{\{S_i \xrightarrow{et} S\},\ a_i,\ \phi_{ij} : a_i|_{S_{ij}} \to a_j|_{S_{ij}}\}$$

objects over S_i satisfy cocycle

Exer: $\big([X/G]^{pre}\big)^{st} \cong [X/G]$