TODAY: Groupoids and prestacks

$$
\begin{tikzcd}
\text{Sets} \arrow[d] \arrow[dr] \arrow[rrr, phantom, "\subset"] & & & \text{Groupoids} \arrow[d] & \\
\text{Topological spaces} \arrow[d] & \text{Functors or presheaves on Sch} \arrow[d] \arrow[rr, phantom, "\subset"] & & \text{Prestacks over Sch} \arrow[d] & \\
\text{Ringed spaces} \arrow[d] & \text{Sheaves on } \mathrm{Sch}_{\text{Ét}} \arrow[d] \arrow[rr, phantom, "\subset"] & & \text{Stacks over } \mathrm{Sch}_{\text{Ét}} \arrow[dl] \arrow[dr] & \\
\text{Schemes} \arrow[r, phantom, "\subset"] & \text{Algebraic spaces} \arrow[r, phantom, "\subset"] & \text{Deligne–Mumford stacks} \arrow[rr, phantom, "\subset"] & & \text{Algebraic stacks}
\end{tikzcd}
$$

Last time

Today

§0 Last time

① A *site* is the data of

- category S
- For each $U \in S$, a set $\mathrm{Cov}(U)$ of coverings

A covering of U is a set of maps $\{U_i \to U\}_{i \in I}$

Require

- identity is a cover
- covers pullback
- covers compose

Ex: Big étale site $\boxed{Sch_{Et}}$

category $= Sch$

A covering is $\{U_i \xrightarrow{et} U\}$ s.t. $\coprod U_i \twoheadrightarrow U$

Special coverings $\{U' \xrightarrow{et} U\}$

$\{U_i \to U\} \sim \{\coprod U_i \to U\}$

② A *presheaf* on S is contra. functor $F: S \to Sets$

A *sheaf* is a presheaf F s.t. $\forall \{U_i \to U\}$ covers

$$0 \to F(U) \to \prod_i F(U_i) \rightrightarrows \prod F(\underbrace{U_{ij}}_{U_i \times_U U_j})$$

exact

Exer A presheaf F is a sheaf on Sch_{Et} $\Leftrightarrow$

① sheaf big Zariski site Sch_{Zar}

② $\forall\ U' \xrightarrow{et} U$ of affines

$$0 \to F(U) \to F(U') \rightrightarrows F(U' \times_U U')$$

Prop X scheme

$h_X = \mathrm{Mor}(-, X) : Sch \to Set$

sheaf in Sch_{Et}

Moduli perspective

Let $F: Sch \to Sets$ be a moduli functor

functor + viewing $F(S) = \{\text{families of objects}/S\}$

F is a sheaf $\iff$ families glue uniquely in étale top

Example

$F_{M_g} : Sch \to Sets$

$S \mapsto \left\{ \begin{matrix} C \\ \downarrow \\ S \end{matrix} \ \text{sm. family of genus } g \text{ curves} \right\}$

Not representable

Not sheaf!

$\boxed{\text{repr} \Rightarrow \text{sheaf}}$

Why we care about representable functors?

Suppose $\exists$ scheme M repr. F_{M_g}

$$Mor(S, M) \cong F_{M_g}(S)$$

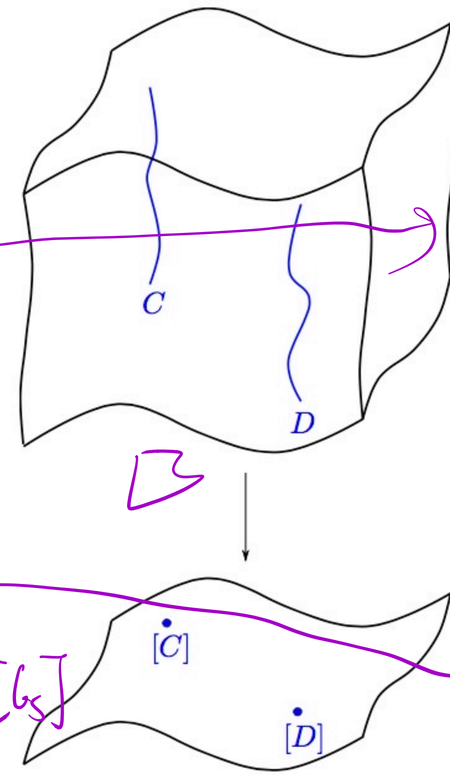
$S = M$ $\qquad id_M \longleftrightarrow \begin{matrix} \mathcal{U} \\ \downarrow \\ M \end{matrix}$ univ. family

univ. ~~fami~~ families $\Rightarrow$

$$\begin{tikzcd} C \arrow[r] \arrow[d] & \mathcal{U} \arrow[d] \\ S \arrow[r] & M \end{tikzcd}$$

$C_s \to s$

$s \mapsto [C_s]$



§1 Groupoids

Definition. A *groupoid* is a category $\mathcal{C}$ where every morphism is an isomorphism.

Ex 1 $\mathcal{M}_g(\mathbb{C}) :=$ category

object = sm, conn, proj curve/$\mathbb{C}$ of genus g

$\mathrm{Mor}(C, C') := \mathrm{Isom}_{\mathrm{Sch}/\mathbb{C}}(C, C')$

Ex 2 (sets) Σ set

Define $\mathcal{C}_\Sigma$ category

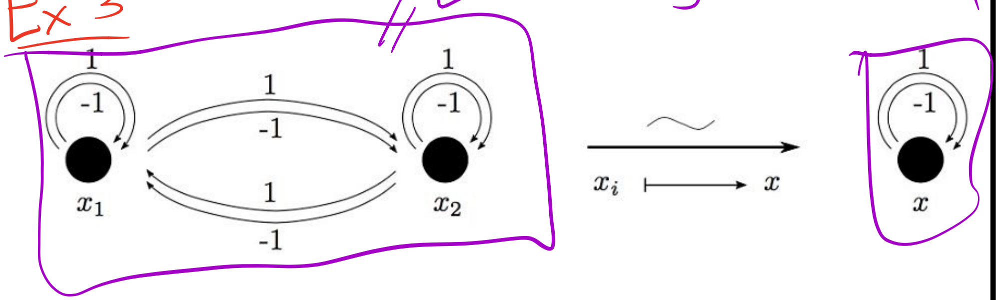
objects = elements $x \in \Sigma$

only have id_x maps

Equivalences of groupoids are equiv. of categories

Ex 3

$[(\mathbb{Z}/2)/(\mathbb{Z}/4)]$ $B(\mathbb{Z}/2)$



Ex 4

$\simeq$ set

equiv. to a set

Ex 5 (moduli groupoid of orbits)

Let G group $\curvearrowright$ Σ set

Define $[\Sigma/G]$ quotient groupoid

objects = elements $x \in \Sigma$

$\mathrm{Mor}(x, x') = \{ g \in G \mid x' = gx \}$

Exer: Show $[\Sigma/G]$ equiv. to a set $\iff$ $G \curvearrowright \Sigma$ free

Ex 6 (classifying stacks)

Special case $\Sigma = *$ $\rightsquigarrow$ BG category w/ one object $*$

$\mathrm{Mor}(*, *) = G$

Ex 7

$\mathbb{Z}/4 \to \mathbb{Z}/2 \rightsquigarrow \mathbb{Z}/4 \curvearrowright \mathbb{Z}/2$

Ex 8 $FB :=$ cat. of finite set & bijections

$$FB = \coprod_{n \geq 0} BS_n$$

Fiber products of groupoids

Consider

$$\begin{tikzcd} \mathcal{C} \times_{\mathcal{D}} \mathcal{D}' \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\square"] & \mathcal{D}' \arrow[d, "g"] \\ \mathcal{C} \arrow[r, "f"] & \mathcal{D} \end{tikzcd}$$

cartesian

groupoids

Define $\boxed{\mathcal{C} \times_{\mathcal{D}} \mathcal{D}'}$ as category of triples (c, d', α)

- $c \in \mathcal{C}$
- $d' \in \mathcal{D}'$
- $\alpha : f(c) \to g(d')$

$\mathrm{Mor}((c_1, d_1', \alpha_1), (c_2, d_2', \alpha_2)) =$

$$\left\{ \begin{array}{c} c_1 \xrightarrow{\beta} c_2 \\ d_1' \xrightarrow{\gamma} d_2' \end{array} \;\middle|\; \begin{tikzcd} f(c_1) \arrow[r, "f(\beta)"] \arrow[d, "\alpha_1"'] \arrow[dr, phantom, "\circlearrowleft"] & f(c_2) \arrow[d, "\alpha_2"] \\ g(d_1') \arrow[r, "g(\gamma)"] & g(d_2') \end{tikzcd} \right\}$$

Ques: What is the univ. property?

Examples

①
$$\begin{tikzcd} G \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & * \arrow[d] \\ * \arrow[r] & BG \end{tikzcd}$$

② group $G \curvearrowright$ set Σ

Let $x \in \Sigma$

orbit Gx

stab G_x

Exer

$$\begin{tikzcd} Gx \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \Sigma \arrow[d] \\ BG_x \arrow[r] & {[\Sigma/G]} \end{tikzcd}$$

$* \longmapsto x$

§2. Motivation of prestacks

Specifying a moduli ~~functor~~ prestack: requires specifying

(1) families of objects;
(2) ~~when~~ how two families of objects are isomorphic; and
(3) and how families pull back under morphisms.

Remember equivalences

First attempt   Consider maps

$$\mathrm{Sch} \xrightarrow{F} \underbrace{\text{Groupoids}}_{\text{cat of cats}} \quad \text{"2 cat"}$$

How can we write down a map?

- For every scheme S, $F(S)$ groupoid
- $\forall S \xrightarrow{f} T$, $f^*: F(T) \to F(S)$
- $S \xrightarrow{f} T \xrightarrow{g} U$

$$g^* \circ f^* \underset{\psi_{f,g}}{\simeq} (g \circ f)^* \quad \text{isom of functor}$$

Is this enough?

Should be a compatibility cond of $\psi_{f,g}$ for

$$S \to T \to U \to V \ldots$$

pseudofunctor or lax...

Instead, we build a massive category $\mathcal{X}$ with all this data

$$\begin{array}{ccc} \mathcal{X} & & \text{object } (S, a) \\ \downarrow & & \downarrow \\ \mathrm{Sch} & & S \end{array}$$

scheme S, $a \in F(S)$

$$\text{"}\mathcal{X} = \bigcup_{S \in \mathrm{Sch}} \underline{F(S)}\text{"}$$

§3 Prestacks

$\mathcal{S}$ = cat.

Let $p\colon \mathcal{X} \to \mathcal{S}$ be a functor of categories.

$$\begin{tikzcd} \mathcal{X} \arrow[d, "p"] \\ \mathcal{S} \end{tikzcd} \qquad \begin{tikzcd} a \arrow[r, "\alpha"] \arrow[d, mapsto] & b \arrow[d, mapsto] \\ S \arrow[r, "f"] & T \end{tikzcd}$$

f^*b

a, b objects of $\mathcal{X}$

objects of $\mathcal{S}$

a is over S

α is over f

Def. A functor $p\colon \mathcal{X} \to \mathcal{S}$ is a *prestack over a* $\mathcal{S}$ if

(1) (pullbacks exist) for any diagram

$$\begin{tikzcd} a \arrow[r, dashed] \arrow[d, dashed, mapsto] & b \arrow[d, mapsto] \\ S \arrow[r] & T \end{tikzcd}$$

a'

of solid arrows, $\exists\ a \to b$ over $S \to T$;

(2) (universal property for pullbacks) for any diagram

$$\begin{tikzcd} a \arrow[r, dashed] \arrow[rr, bend left] \arrow[d, mapsto] & b \arrow[r] \arrow[d, mapsto] & c \arrow[d, mapsto] \\ R \arrow[r] & S \arrow[r] & T \end{tikzcd}$$

of solid arrows, $\exists$ unique arrow $a \to b$ over $R \to S$ filling in the diagram.

$\to$ Every arrow in $\mathcal{X}$ is a pullback

⚠ Warning

Not standard terminology

usually called "cat fib. in groupoids"

⚠ Abuse of notation

① Often only write $\mathcal{X}$, not $\mathcal{X} \to \mathcal{S}$

② Often don't spell out composition law in $\mathcal{X}$.

Notation:

- Write $\boxed{f^*b}$ or $\boxed{b|_S}$ choice of ~~as~~ pullback
- For $S \in \mathcal{S}$, the $\boxed{\text{fiber category}}$

$\boxed{\mathcal{X}(S)}$ = objects of $\mathcal{X}$ over S

morphisms are over id_S

Exer: $\mathcal{X} \to \mathcal{S}$ prestack

$\Longrightarrow \mathcal{X}(S)$ groupoid $\forall S \in \mathcal{S}$

EXAMPES

$\boxed{\mathcal{S} = Sch}$

① **Presheaves:** $F : Sch \to Sets$ presheaf

Build $\mathcal{X}_F$ as follows:

- objects: pairs (S, a) $\quad$ S scheme, $a \in F(S)$
- $Mor((S,a),(T,b)) = \{ f : S \to T \mid a = f^* b \}$

$$\begin{tikzcd} \mathcal{X}_F \arrow[d, "p"] & (S,a) \arrow[d, mapsto] \\ Sch & S \end{tikzcd}$$

Conflate F & $\mathcal{X}_F$

② **Schemes** Let X be a scheme

Apply previous construction to

$h_X = Mor(-, X) : Sch \to Set$

Build

$$\begin{tikzcd} \mathcal{X}_X \arrow[d] \\ Sch \end{tikzcd}$$

Object = map $S \to X$

$Mor(S \to X, T \to X) =$

$$\left\{ \begin{tikzcd} S \arrow[rr] \arrow[dr] & & T \arrow[dl] \\ & X & \end{tikzcd} \right\}$$

③ $\mathcal{M}_g$ prestack

objects: $\begin{tikzcd} C \arrow[d] \\ S \end{tikzcd}$ fam. of smooth curves of genus g

$$Mor\left(\begin{tikzcd} C \arrow[d] \\ S \end{tikzcd}, \begin{tikzcd} C' \arrow[d] \\ S' \end{tikzcd} \right) = \left\{ \begin{tikzcd} C \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & C' \arrow[d] \\ S \arrow[r] & S' \end{tikzcd} \right\}$$

Exer: $[X/G]^{pre}(T) = [X(T)/G(T)]$ quot. groupoid

④ $\boxed{Bun(C)}$ for C sm, conn, proj curve$/k$

- objects: pairs $(S, \mathcal{F})$ $\quad$ $\mathcal{F}$ vect. bdl on $C \times S$
- $Mor((S, \mathcal{F}), (S', \mathcal{F}')) = \left\{ \begin{array}{l} S \to S' \\ (f \times id)^* \mathcal{F}' \xrightarrow[\text{isom}]{\alpha} \mathcal{F} \end{array} \right\}$ choices

Better: $\mathcal{F}' \to (f \times id)_* \mathcal{F}$ s.t. adjunction is iso

⑤ Quotient prestack $[X/G]^{pre}$ / (Sch/S)

$G \to S$ group scheme $\curvearrowright$ $X \to S$

Object map $T \to X$ over S

$Mor(T \to X, T' \to X) =$

$\left\{ \begin{array}{l} T \to T' \\ g \in G(T) \end{array} \;\middle|\; (T \to T' \to X) = g(T \to X) \right\}$

$\boxed{G(T) \curvearrowright X(T)}$

§4. Morphisms of prestacks

Def. (1) A *morphism of prestacks* $f\colon \mathcal{X} \to \mathcal{Y}$ is a functor $f\colon \mathcal{X} \to \mathcal{Y}$ such that the diagram

$$\begin{tikzcd}
a \in \mathcal{X} \arrow[rr, "f"] \arrow[dr, "p_{\mathcal{X}}"'] & & \mathcal{Y} \arrow[dl, "p_{\mathcal{Y}}"] \\
& \mathcal{S} &
\end{tikzcd}$$

strictly commutes. i.e. $P_{\mathcal{X}}(a) = P_{\mathcal{Y}}(f(a))$

(2) If $f, g\colon \mathcal{X} \to \mathcal{Y}$ are morphisms of prestacks, a *2-morphism*

$$\begin{tikzcd}
\mathcal{X} \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathcal{Y}.
\end{tikzcd}$$

α 2-arrow

α nat. trans.

is a natural transformation $\alpha\colon f \to g$ such that for every object $a \in \mathcal{X}$, the morphism $\alpha_a\colon f(a) \to g(a)$ in $\mathcal{Y}$ is over the identity in $\mathcal{S}$.

(3) Define the category $\boxed{\mathrm{MOR}(\mathcal{X}, \mathcal{Y})}$ as

$$\text{objects} = \text{morphisms of presetacks}$$
$$\text{morphisms} = \text{2-morphisms}$$

(4) We say that a diagram

$$\begin{tikzcd}
\mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}' \arrow[r, "f'"] \arrow[d, "g'"'] \arrow[dr, phantom, "\swarrow\alpha"] & \mathcal{Y}' \arrow[d, "g"] \\
\mathcal{X} \arrow[r, "f"] & \mathcal{Y}
\end{tikzcd}$$

together with a 2-isomorphism $\alpha\colon g \circ f' \xrightarrow{\sim} f \circ g'$ is *2-commutative.*

(5) An *isomorphism* is a map $f\colon \mathcal{X} \to \mathcal{Y}$ with an inverse, i.e. $\exists g\colon \mathcal{Y} \to \mathcal{X}$ and 2-isos $g \circ f \xrightarrow{\sim} \mathrm{id}_{\mathcal{X}}$ and $f \circ g \xrightarrow{\sim} \mathrm{id}_{\mathcal{Y}}$.

Exer: $\mathrm{MOR}(\mathcal{X}, \mathcal{Y})$ groupoid

Exer $\mathcal{X} \xrightarrow{f} \mathcal{Y}$ iso $\Longleftrightarrow$ $\mathcal{X}(S) \to \mathcal{Y}(S)$ iso $\forall S$

The 2-Yoneda Lemma. *Let $\mathcal{X}$ be a prestack over a category $\mathcal{S}$ and $S \in \mathcal{S}$. The functor*

$$\mathrm{MOR}(S, \mathcal{X}) \to \mathcal{X}(S), \qquad f \mapsto f_S(\mathrm{id}_S)$$

is an equivalence of categories.

Rmk: If $f\colon S \to \mathcal{X}$ morphism
view as a prestack

$\forall$ scheme T

$f_T\colon \mathrm{Mor}(T, S) \to \mathcal{X}(T)$

$T = S$ $\quad \mathrm{id}_S \mapsto f_S(\mathrm{id}_S) \in \mathcal{X}(S)$

Construct inverse $\quad$ Let $a \in \mathcal{X}(S)$

For each $T \xrightarrow{g} S$, choose g^*a

Define $f\colon S \to \mathcal{X}$ by defining $f_T\colon \mathrm{Mor}(T, S) \to \mathcal{X}(T)$

$g \mapsto g^*a$

Need to define it on morphisms ....