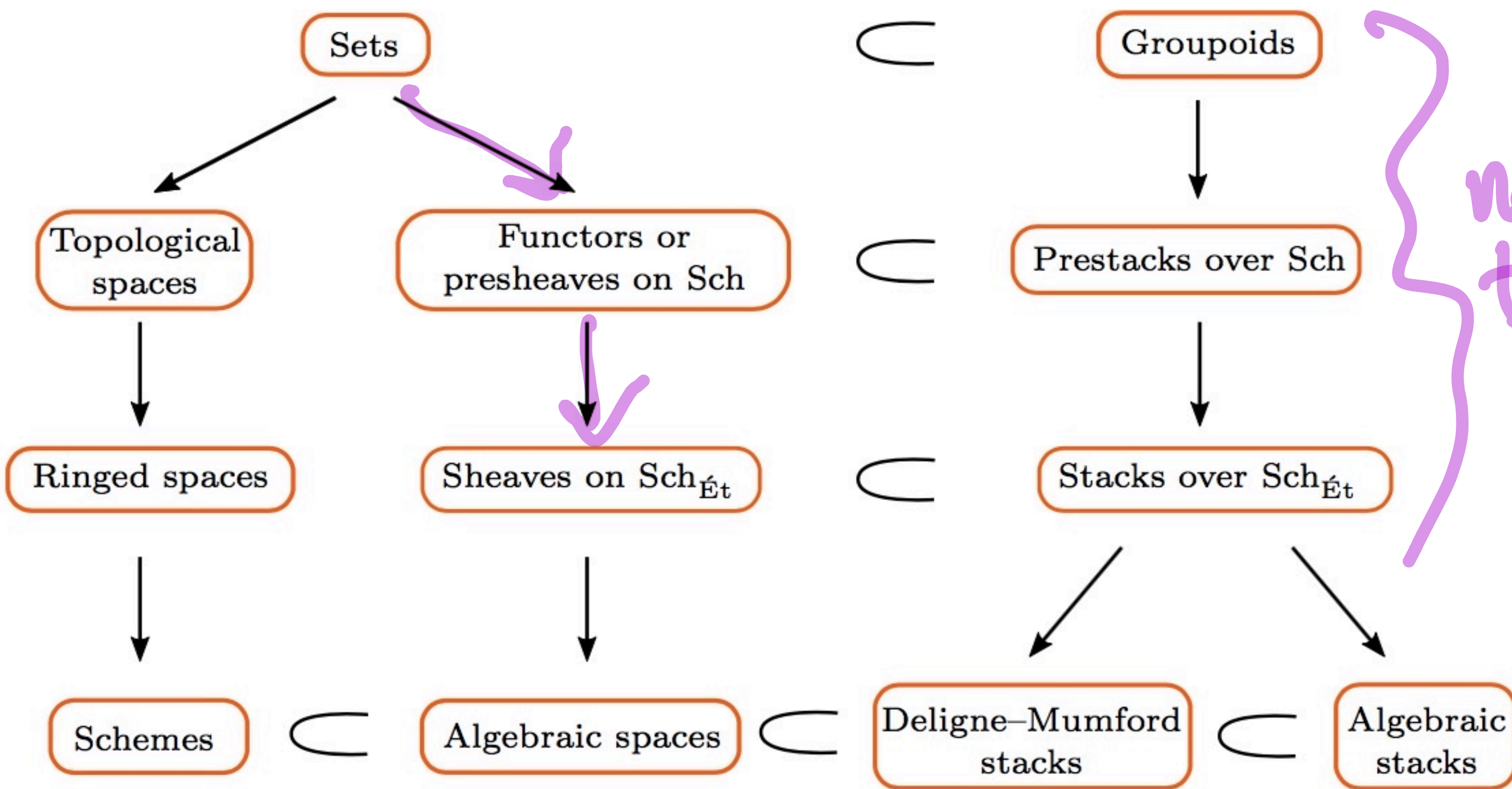
Lecture 2: Sites and sheaves



next time

Stacks references

- LMB
- Knutson
- Olsson *

Course note

Stacks project

Halpern-Leistner

Intro to moduli theory

§1. Motivation: étale topology

What is an étale morphism?

alg covering space

Definition. For a morphism $f\colon X \to Y$ of schemes of finite type over $\mathbb{C}$, the following are equivalent:

- f is étale;

  flat & fibers X_y smooth dim $=0$

- f is smooth of relative dimension 0;
- f is flat and unramified;

  $X_y = f^{-1}(y) = \bigsqcup \operatorname{Spec} \mathbb{C}$, $y \in Y(\mathbb{C})$

- f is flat and $\Omega_{X/Y} = 0$;

  completion

- $\forall x \in X(\mathbb{C})$, the map $\widehat{\mathcal{O}}_{Y,f(x)} \xrightarrow{\sim} \widehat{\mathcal{O}}_{X,x}$ is an iso;
- for any $A \twoheadrightarrow A_0$ of artinian $\mathbb{C}$-algebras, any commutative diagram

lifting criterion for étaleness

$\operatorname{Spec} A_0 = \operatorname{Spec} \mathbb{C}$, $\operatorname{Spec} A = \operatorname{Spec} \mathbb{C}[\varepsilon]/\varepsilon^2$

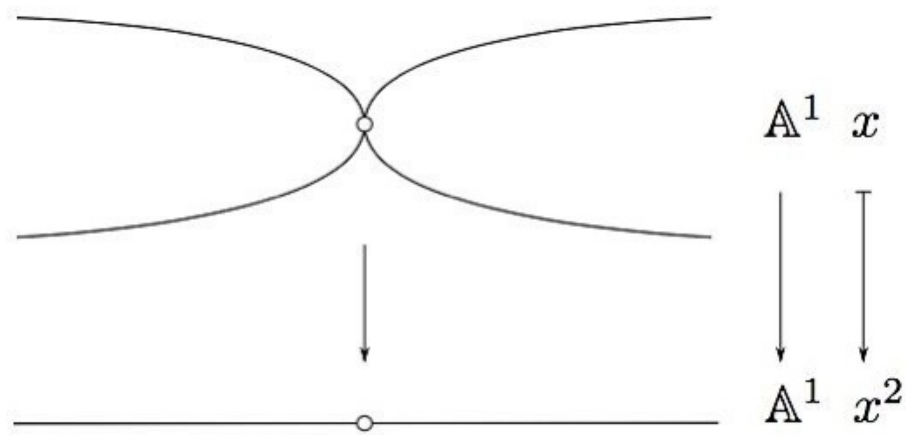
$$\begin{tikzcd} \operatorname{Spec} A_0 \arrow[r, "x"] \arrow[d, hook] & X \arrow[d, "f"] \\ \operatorname{Spec} A \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

$y \in Y$ & tangent vector

of solid arrows can be uniquely filled in;

- (assuming in addition that X and Y are smooth) $\forall x \in X(\mathbb{C})$, the map $T_{X,x} \to T_{Y,f(x)}$ is an iso.

Double cover of $\mathbb{A}^1 \setminus 0$



étale except at 0

K/L field ext

$\operatorname{Spec} L \to \operatorname{Spec} K$ étale $\Leftrightarrow$ $K \to L$ finite & separable

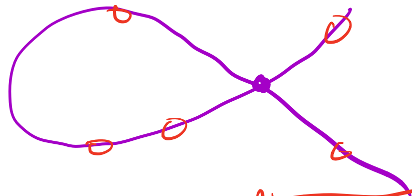
But why the étale topology?

Allows you to zoom in!!

Example 1 (nodes)

$C = V(y^2 - x^2(x-1)) \subset \mathbb{A}^2$

nodal cubic



irred in Zariski top

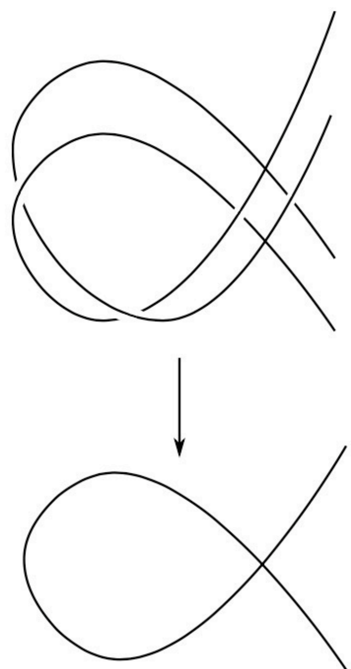
But if we adjoin "$\boxed{t = \sqrt{x-1}}$" then

$\underline{y^2 - x^2(x-1)} = (y-xt)(y+xt)$

factors

$C' = \operatorname{Spec} k[x,y,t]_t / (y^2 - x^2(x-1),\ \underline{t^2 - x+1}) \xrightarrow{\text{étale}} C$

reducible



Observe that the completion

$$\hat{\mathcal{O}}_{C,c} \cong k[[x,y]]/(y^2 - x^2(x+1))$$

$$= k[[x,y,t]]/(y-xt)(y+xt)$$

where $t = \sqrt{x+1} = 1 + \frac{1}{2}x - \frac{1}{4}x^2 + \cdots$

power series expansion

$\rightsquigarrow$ Spec $\hat{\mathcal{O}}_{C,c}$ reducible

Take

$F : Sch/C \to Sets$

$(C' \to C) \mapsto \left\{ \begin{array}{l} \text{decomp.} \\ C' = C'_1 \cup C'_2 \end{array} \right\}$

$\{$ idempotent in $\Gamma(C')\}$

We have an element

$\xi \in F(\hat{\mathcal{O}}_{C,c})$

$\rightsquigarrow$ Artin approx $(N=1)$ gives étale cover

Ex 2 Artin approximation

Principle: Alg properties that hold for the completion $\hat{\mathcal{O}}_{X,x}$ also hold in an étale nbhd of $x \in X$.

Theorem (Artin Approximation).

excellent

$\forall$ $\mathcal{O}_{S,s} \xrightarrow{\text{regular}} \hat{\mathcal{O}}_{S,s}$

flat + geom. reg fibers

- *Let S be an exc. scheme (e.g. finite type / k or $\mathbb{Z}$).* field
- *Let F: Sch $/S \to$ Sets be a limit preserving functor.* contra
- *Let $\widehat{\xi} \in F(\operatorname{Spec} \widehat{\mathcal{O}}_{S,s})$ where $s \in S$ is a point.*

For any integer $N \geq 0$, there exist a residually-trivial étale morphism

$$(S', s') \to (S, s) \qquad \textit{and} \qquad \xi' \in F(S')$$

such that the restrictions of $\widehat{\xi}$ and ξ' to $\operatorname{Spec}(\mathcal{O}_{S,s}/\mathfrak{m}_s^{N+1})$ are equal.

Neron-Popescu approx

If $A \to B$ reg. ring hom of noeth. rings then $B = \operatorname{colim} B_\lambda$ of $A \xrightarrow{\text{smooth}} B_\lambda$

$\longleftrightarrow$ $\forall$ direct systems B_λ of $\mathcal{O}_S$-algebras

$\underset{\to}{\operatorname{colim}} F(\operatorname{Spec} B_\lambda) \xrightarrow{\sim} F(\underset{\to}{\operatorname{colim}} B_\lambda)$

Finiteness { ξ_0 $\exists \xi_\lambda$ Big!

$\operatorname{Spec} \underset{\to}{\operatorname{colim}} B_\lambda \to \operatorname{Spec} B_\lambda$

Ex: If $X \to S$ is a scheme

$\operatorname{Mor}_S(-, X) : \mathrm{Sch}/S \to \mathrm{Sets}$

limit pres $\iff$ $X \to S$ loc. finite pres

Ex 3 Étale cohomology

C sm conn proj curve

$\rightsquigarrow H^1(C, \mathbb{Z}/n) = 0$

BUT $H^1(C_{ét}, \mathbb{Z}/n) = (\mathbb{Z}/n)^{2g}$

..., Weil conjectures, ...

Neron model

Ex 4 : Descent theory

Recall that any good property P of morphisms of schemes satisfies

- stable under composition
- stable under base change
- Zariski-local on target :

if $\{Y_i\}$ is open cover $Y = \bigcup Y_i$

$$\begin{tikzcd} f^{-1}(Y_i) \arrow[r] \arrow[d, "f_i"] & X \arrow[d, "f"] \\ Y_i \arrow[r] & Y \end{tikzcd}$$

f has $P \iff$ each f_i has P

Same is true in étale topology

if $\{Y_i \xrightarrow{ét} Y\}$ étale cover

§2 Sites

We expand the notion of a topology.

Definition. A *Grothendieck topology* on a category $\mathcal{S}$ consists of the following data: for each object $U \in \mathcal{S}$, there is a set $\mathrm{Cov}(U)$ consisting of *coverings* of U, i.e. collections of morphisms $\{U_i \to U\}_{i \in I}$ in $\mathcal{S}$.

We require that:

(1) $U' \xrightarrow{\sim} U$ iso $\implies$ $(U' \to U) \in \mathrm{Cov}(U)$.

(2) For a map $V \to U$,

$$\{U_i \to U\}_{i\in I} \in \mathrm{Cov}(U) \implies \{U_i \times_U V \to V\}_{i \in I} \in \mathrm{Cov}(V)$$

(in part., the fiber products $U_i \times_U V$ exist in $\mathcal{S}$)

(3) comp. of covers

$$\begin{matrix} \{U_i \to U\}_{i\in I} \in \mathrm{Cov}(U) \\ \{U_{ij} \to U_i\}_{j \in J_i} \in \mathrm{Cov}(U_i) \end{matrix} \implies \begin{matrix} \{U_{ij} \to U_i \to U\}_{i \in I, j \in J_i} \\ \in \mathrm{Cov}(U) \end{matrix}$$

A *site* is a category $\mathcal{S}$ with a Grothendieck topology.

Ex 1 (top spaces)

Let X be top. space

$\mathcal{S} = \mathrm{Op}(X) :=$ category of open subsets U
cat

$$\mathrm{Mor}(V,U) = \begin{cases} * & \text{if } V \subset U \\ \phi & \text{otherwise} \end{cases}$$

coverings $\left\{ \begin{array}{l} \text{For each } U \in \mathrm{Op}(X), \\ \mathrm{Cov}(U) = \{ \{U_i \to U\} \mid \bigcup U_i = U \} \end{array} \right.$

Ex 2 (small étale site)

Let X be a scheme

$X_{et} :=$ category of schemes étale$/X$

object is $U \xrightarrow{et} X$

$$\mathrm{Cov}(U \to X) = \{ \{U_i \to U\} \mid \bigsqcup U_i \twoheadrightarrow U \}$$

Ex 3 (big étale site)

$\boxed{Sch}$ = cat. of all schemes

$Cov(X) = \{ \{X_i \xrightarrow{ét} X\} \mid \coprod X_i \twoheadrightarrow X \}$

$\rightsquigarrow$ big étale site $\boxed{Sch_{Ét}}$ site

Exer: These are sites.

Rmks Could also define
- big Zariski site
- big étale site relative to S $(Sch/S)_{Ét}$
- smooth, fppf, fpqc sites

§3. Sheaves

Definition. A *presheaf* on a category $\mathcal{S}$ is a contravariant functor $\mathcal{S} \to$ Sets.

Definition. A *sheaf* on a site $\mathcal{S}$ is a presheaf $F\colon \mathcal{S} \to$ Sets such that for any covering $\{S_i \to S\} \in \mathrm{Cov}(S)$ of an object $S \in \mathcal{S}$, the sequence

section glue

$$F(S) \to \prod_i F(S_i) \rightrightarrows \prod_{i,j} F(S_i \times_S S_j)$$

inj

Two maps are induced by $S_i \times_S S_j \to S_i$, $\to S_j$

is exact.

Ex 1 If X top. space and $Op(X)$ site of open sets, then for $S_i, S_j \subset S$ $S_i \times_S S_j = S_i \cap S_j$

$\rightsquigarrow$ Obtain usual notion of sheaf

Ex 2 Let X be a scheme

Then $h_X : Sch \to Sets$

$S \mapsto Mor(S, X)$

is a sheaf on Sch_{Et} ⟵ big étale top

Reason: If $\{S_i \to S\}$ étale covering

$$\begin{tikzcd} S_i \times_S S_j \arrow[d, shift left] \arrow[d, shift right] & \\ S_i \arrow[d] \arrow[dr, "g_i"] & \\ S \arrow[r, dashed] & X \end{tikzcd}$$

Descent theory

$(g_i)_i$

- If each $S_i \hookrightarrow S$ is open, this is usual fact that morphisms glue uniquely.
- True more generally!

⚠ Abuse of notation

Write X for sheaf h_X

Fiber products Consider

$$\begin{tikzcd} & F \arrow[d] \\ G' \arrow[r] & G \end{tikzcd}$$

maps of presheaves on a cat $\mathcal{S}$

Exer: Show $F \times_G G'$ defined as

$$S \mapsto F(S) \times_{G(S)} G'(S)$$

is a fiber product in $Pre(\mathcal{S})$ — cat of presheaves

Exer: F, G, G' sheaves $\Rightarrow$ $F \times_G G'$ sheaf

We have fiber product in $Sh(\mathcal{S})$ — cat. sheaves on a site

Theorem (Sheafification). *Let $\mathcal{S}$ be a site. The forgetful functor* $\mathrm{Sh}(\mathcal{S}) \to \mathrm{Pre}(\mathcal{S})$ *admits a left adjoint* $F \mapsto F^{\mathrm{sh}}$*, called the sheafification.*

Proof.

- Call a presheaf F *separated* if for every covering $\{S_i \to S\}$, the map $F(S) \to \coprod_i F(S_i)$ is injective.

Let $\mathrm{Pre}^{\mathrm{sep}}(\mathcal{S}) \subset \mathrm{Pre}(\mathcal{S})$ be full subcat of sep presheaves.

- We will construct left adjoints

$$\mathrm{Sh}(\mathcal{S}) \underset{\mathrm{sh}_2}{\overset{}{\leftrightarrows}} \mathrm{Pre}^{\mathrm{sep}}(\mathcal{S}) \underset{\mathrm{sh}_1}{\leftrightarrows} \mathrm{Pre}(\mathcal{S}). \ni F$$

- Define $\mathrm{sh}_1(F)$ by $S \mapsto F(S)/\sim$ where $a \sim b$ if there exists a covering $\{S_i \to S\}$ such that $a|_{S_i} = b|_{S_i}$ $\forall i$.

- Define $\mathrm{sh}_2(F)$ by

$$S \mapsto \left\{ (\{S_i \to S\}, \{a_i\}) \;\middle|\; \begin{array}{l} \{S_i \to S\} \in \mathrm{Cov}(S),\ a_i \in F(S_i) \\ \text{s.t. } a_i|_{S_{ij}} = a_j|_{S_{ij}}\ \forall i,j \end{array} \right\} / \sim$$

where $(\{S_i \to S\}, \{a_i\}) \sim (\{S'_j \to S\}, \{a'_j\})$ if $a_i|_{S_i \times_S S'_j} = a'_j|_{S_i \times_S S'_j}$ for all i, j.

- Details left to you.

$$F \longrightarrow F^{sh}$$
$$\exists!$$
$$G \leftarrow \text{sheaf}$$

$$\mathrm{Hom}_{\mathrm{Pre}}(F, G) = \mathrm{Hom}_{\mathrm{Sh}}(F^{sh}, G)$$

sheaf

if sections glue they glue uniquely

Exercises in course notes