LECTURE 16: Gluing and forgetful morphisms

TODAY'S OUTLINE

- Recap
- Explicit stable reduction (compute stable limit of $y^2 = x^5 + t$)
- Uniqueness of the stable limit (i.e. $\overline{M}_{g,n}$ separated)
- Gluing morphisms (Will allow us to define boundary divisors)
  - $\overline{M}_{g_1,n_1} \times \overline{M}_{g_2,n_2} \to \overline{M}_{g_1+g_2,n_1+n_2-2}$
  - $\overline{M}_{g,n} \to \overline{M}_{g+1,n-2}$
- Forgetful morphism
  - $\overline{M}_{g,n+1} \to \overline{M}_{g,n}$ (Will show this is a univ. family)

§0. Recap of stable reduction

Theorem (Stable Reduction). *If* $(\mathcal{C}^* \to \Delta^*, s_1^*, \ldots, s_n^*)$ *is a family of* n*-pointed stable curves of genus* g*, then there exists a finite cover* $\Delta' \to \Delta$ *of spectrums of DVRs and a family* $(\mathcal{C}' \to \Delta', s_1', \ldots, s_n')$ *of stable curves extending* $\mathcal{C}^* \times_{\Delta^*} \Delta'^* \to \Delta'^*$.

(n=0 case) Given $\quad \exists$

$$\begin{tikzcd} \mathcal{C}^\times \arrow[r, dashed, hook] \arrow[d, "\text{stable}"] & \mathcal{C} \arrow[d, "\text{stable family}"] \\ \operatorname{Spec} K = \Delta^\times \arrow[r, hook] & \Delta = \operatorname{Spec} R \end{tikzcd}$$

After possibly an extension $\Delta' \to \Delta$

$t \mapsto t^n$

Stable reduction

$\Longrightarrow$ existence part of the valuative criterion of properness

STEP 1 Reduce to case where generic fiber $\mathcal{C}^\times \to \Delta^\times$ is smooth

STEP 2 Find some extension $\mathcal{C} \xrightarrow{\text{flat}} \Delta$.

STEP 3 Use Embedded Resolutions

$$\begin{tikzcd} \exists \ \widetilde{\mathcal{C}} \arrow[rr, "\text{proj}", "\text{bir'l}"'] \arrow[dr] & & \mathcal{C} \arrow[dl] \\ & \Delta & \end{tikzcd}$$

regular

finite sequence of blowups at closed pts of $\mathcal{C}_0$

s.t. $\widetilde{\mathcal{C}}_0$ set-theoretic normal crossings

(i.e. $(\widetilde{\mathcal{C}}_0)_{red}$ nodal curve)

STEP 4 Take ramified base extension

$\Delta' = \operatorname{Spec} R \longrightarrow \Delta = \operatorname{Spec} R, \quad t \mapsto t^m$

s.t. central fiber of the normalization $\widetilde{\widetilde{\mathcal{C}} \times_\Delta \Delta}$ is reduced & nodal.

STEP 5 Take minimal resolution & Contract rational tail & bridges

(i.e. take stable model)

§1. Explicit stable reduction

- The biggest challenge in computing the stable limit is in Step 4: computing the normalization $\mathcal{C} \times_\Delta \Delta'$ after base changing $\Delta' \to \Delta$, $t \mapsto t^m$.

- In practice, it is useful to factor $\Delta' \to \Delta$ as a composition of prime order base changes $\Delta' \to \Delta$, $t \mapsto t^p$

Proposition.

- *Let $\mathcal{C} \to \Delta$ be a generically smooth family of curves such that $(\mathcal{C}_0)_{\mathrm{red}}$ is nodal.*

- *Define the divisor $\mathcal{C}_0 = \sum a_i D_i$ on $\mathcal{C}$ where a_i is the multiplicity of the irreducible component D_i.*

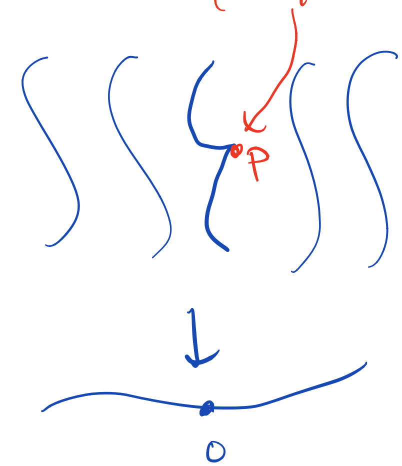
- *Let $\Delta' \to \Delta$ be defined by $t \mapsto t^p$ where p is prime, and set $\mathcal{C}' := \mathcal{C} \times_\Delta \Delta'$ with normalization $\widetilde{\mathcal{C}}'$.*

Then $\widetilde{\mathcal{C}}' \to \mathcal{C}$ is a branched cover ramified over $\sum (a_i \mod p) D_i$.

Example Suppose

Local eqn: $y^2 = x^5 + t$

$$\mathcal{C} \xrightarrow{\text{gen smooth family of curves}} \Delta = \operatorname{Spec} R$$

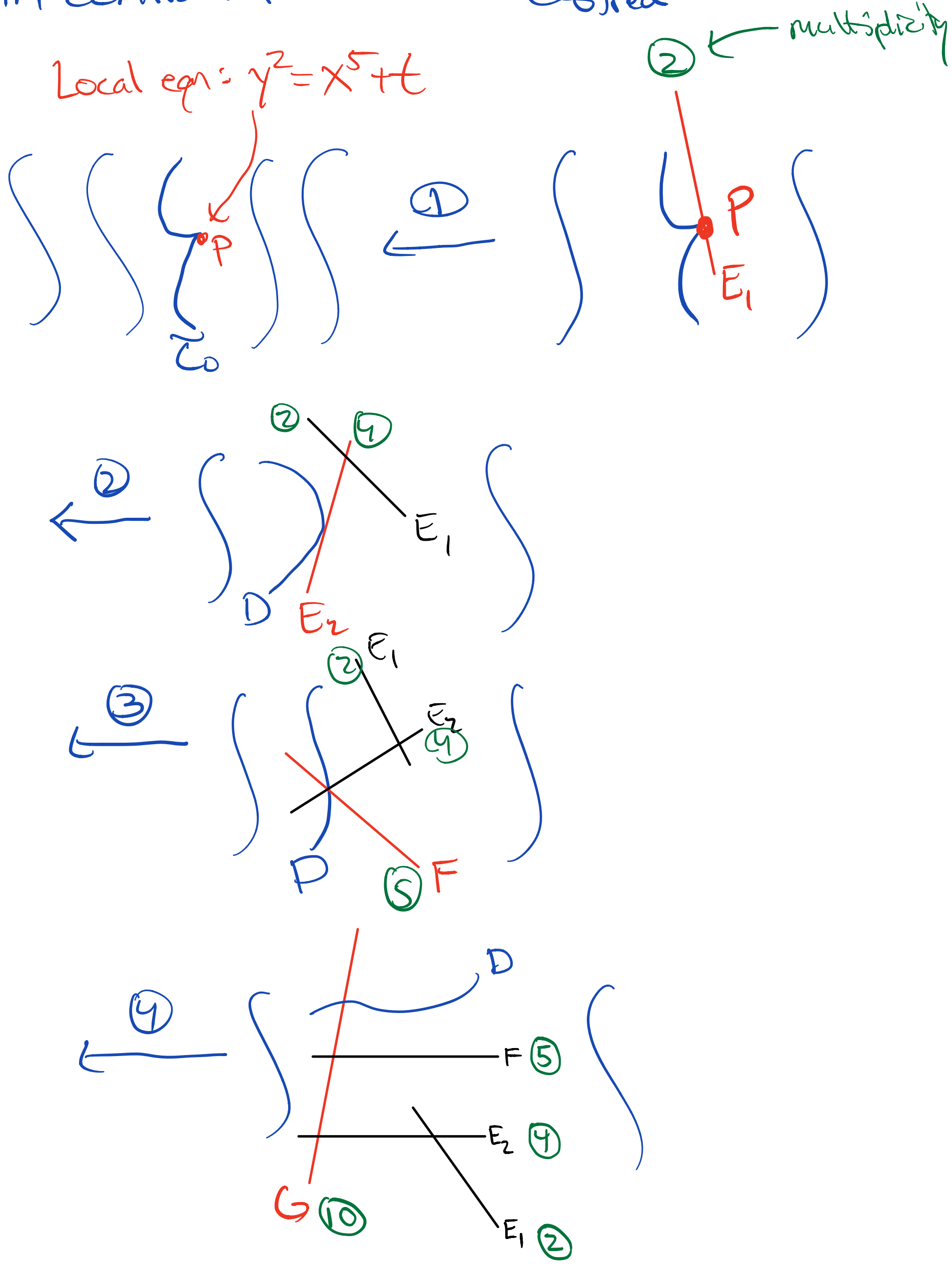


Here $t \in R$ uniformizer

when $t = 0$, central fiber $\mathcal{C}_0$ is not stable & $p \in \mathcal{C}_0$ is rhamphoid cusp $y^2 = x^5$

Ques: What is the stable limit?

Example: Since we already have some limit, can begin with Step 3: blowup points in central fiber until $(C_0)_{red}$ is nodal



Computations

$$\begin{tikzcd} E \arrow[d] & \subset & \tilde{C}_0 = Bl_p C_0 \arrow[d] \\ p & \in & C_0 \end{tikzcd}$$

Chart 1

$$\begin{tikzcd} E|_{U_1} = V(\bar{x}) \subset U_1 \subset \tilde{C}_0 \arrow[d] & (\bar{x},\bar{y}) \arrow[d, mapsto] \\ C_0 & (\bar{x}, \bar{x}\bar{y}) \end{tikzcd}$$

Chart 2

$$\begin{tikzcd} E|_{U_2} = V(\bar{y}) \subset U_2 \subset \tilde{C}_0 \arrow[d] & (\bar{x},\bar{y}) \arrow[d, mapsto] \\ C_0 & (\bar{x}\bar{y}, \bar{y}) \end{tikzcd}$$

- **Blowup 1**

1^{st} chart: $y^2 - x^5 = (\tilde{x}\tilde{y})^2 - \tilde{x}^5 = \underbrace{\tilde{x}^2}_{E_1}\underbrace{(\tilde{y}^2 - \tilde{x}^3)}_{\text{cuspidal}}$

- **Blow up 2**

1^{st} chart: $x^2(y^2-x^3) = \tilde{x}^2((\tilde{x}\tilde{y})^2 - \tilde{x}^3) = \underbrace{\tilde{x}^4}_{E_2}\underbrace{(\tilde{y}^2-\tilde{x})}_{D}$

2^{nd} chart: $x^2(y^2-x^3) = \underbrace{\tilde{x}^2}_{E_1}\underbrace{\tilde{y}^4}_{E_2}\underbrace{(1-\tilde{x}^3\tilde{y})}_{D}$

- **Blow up 3**

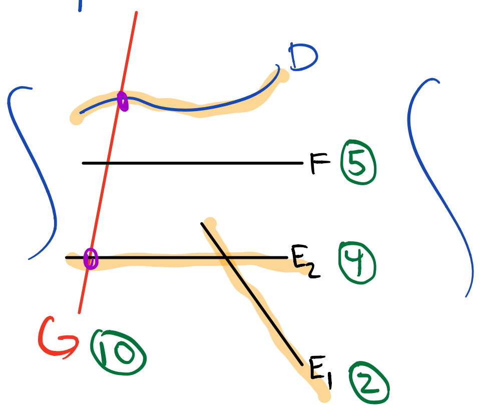
2^{nd} chart: $x^4(y^2-x) = \tilde{x}^4\tilde{y}^4(\tilde{y}^2 - \tilde{x}\tilde{y})$

$= \underbrace{\tilde{x}^4}_{E_2}\underbrace{\tilde{y}^5}_{F}\underbrace{(\tilde{y}-\tilde{x})}_{D}$ ~~partial~~ normalization of original central fiber

- **Blow up 4**

1^{st} chart: $x^4y^5(y-x) = \underbrace{\tilde{x}^{10}}_{G}\underbrace{\tilde{y}^5}_{F}\underbrace{(\tilde{y}-1)}_{D}$

Have a family $\mathcal{C} \to \Delta$ with $(\mathcal{C}_0)_{red}$ nodal



$$\boxed{\mathcal{C}_0 = D + 10G + 5F + 4E_2 + 2E_1}$$

① Base change by $t \mapsto t^5$ & normalize

$\widetilde{\mathcal{C}}' \xrightarrow{\pi} \mathcal{C}$ ramified over $D + E_2 + E_1$

Preimage

$G' := \pi^{-1}(G) \xrightarrow{5:1} G = \mathbb{P}^1$ branched over 2 pts each w/ ram index 4

Riemann-Hurwitz $\Rightarrow$

$2g(G')-2 = 5(-2)+R = -10+8$

$\Rightarrow g(G') = 0 \Rightarrow G' = \mathbb{P}^1$

Preimage

$F' := \pi^{-1}(F) \xrightarrow{5:1} F = \mathbb{P}^1$ unramified

$\Rightarrow F' = F_1 \cup \cdots \cup F_5 \quad F_i = \mathbb{P}^1$

Proposition.

- *Let $\mathcal{C} \to \Delta$ be a generically smooth family of curves such that $(\mathcal{C}_0)_{\mathrm{red}}$ is nodal.*
- *Define the divisor $\mathcal{C}_0 = \sum a_i D_i$ on $\mathcal{C}$ where a_i is the multiplicity of the irreducible component D_i.*
- *Let $\Delta' \to \Delta$ be defined by $t \mapsto t^p$ where p is prime, and set $\mathcal{C}' := \mathcal{C} \times_\Delta \Delta'$ with normalization $\widetilde{\mathcal{C}}'$.*

Then $\widetilde{\mathcal{C}}' \to \mathcal{C}$ is a branched cover ramified over $\sum (a_i \mod p) D_i$.

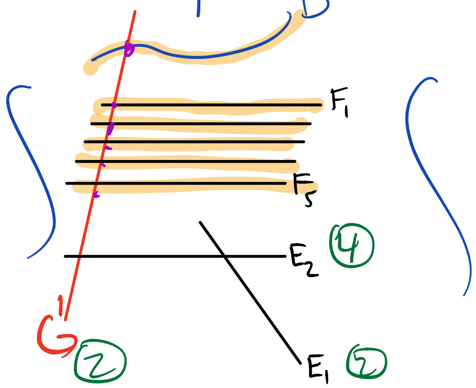
Over Δ, new central fiber

$$\mathcal{C}'_0 = 5D + 10G' + 5(F_1 + \cdots + F_5) + 20E_2 + 10E_1$$

Over Δ',

$$\mathcal{C}'_0 = D + 2G' + (F_1 + \cdots + F_5) + 4E_2 + 2E_1$$

We now have a family $\mathcal{C} \to \Delta$ with $(\mathcal{C}_0)_{red}$



$$\boxed{\mathcal{C}_0' = D + 2G' + F_1 + \cdots + F_5 + 4E_2 + 2E_1}$$

② Base change by $t \mapsto t^2$ & normalize

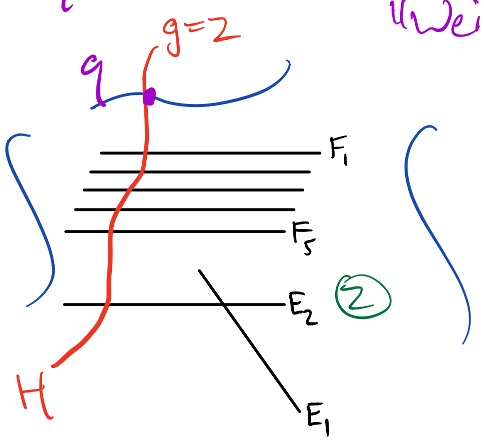
$\widetilde{\mathcal{C}}' \xrightarrow[\pi]{2:1} \mathcal{C}$ ramified over $D + F_1 + \cdots + F_5$

$H := \pi^{-1}(G') \xrightarrow{2:1} G' = \mathbb{P}^1$ ramified over 6 points

$RH \Rightarrow H$ genus 2

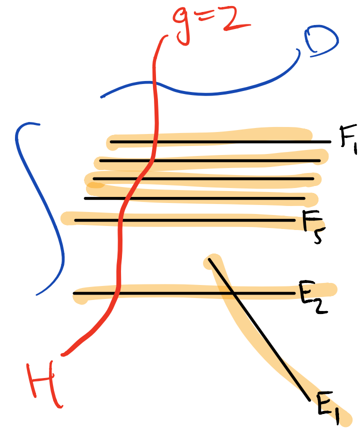
$q = H \cap D$ ram. pt of $H \to \mathbb{P}^1$

"Weierstrass pt"

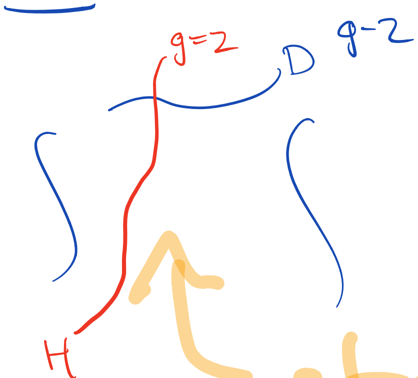


③ Repeat $t \mapsto t^2$ one more time to get reduced central fiber

no multiplicity



STEP 5 Contract rational tails



stable limit

Algorithm: Tim Dokchitser

Ques:

(1) Which genus 2 curve is it?

(2) More generally, what happens for

$$y^2 = x^5 + a_3(t)x^3 + \cdots + a_6(t)$$

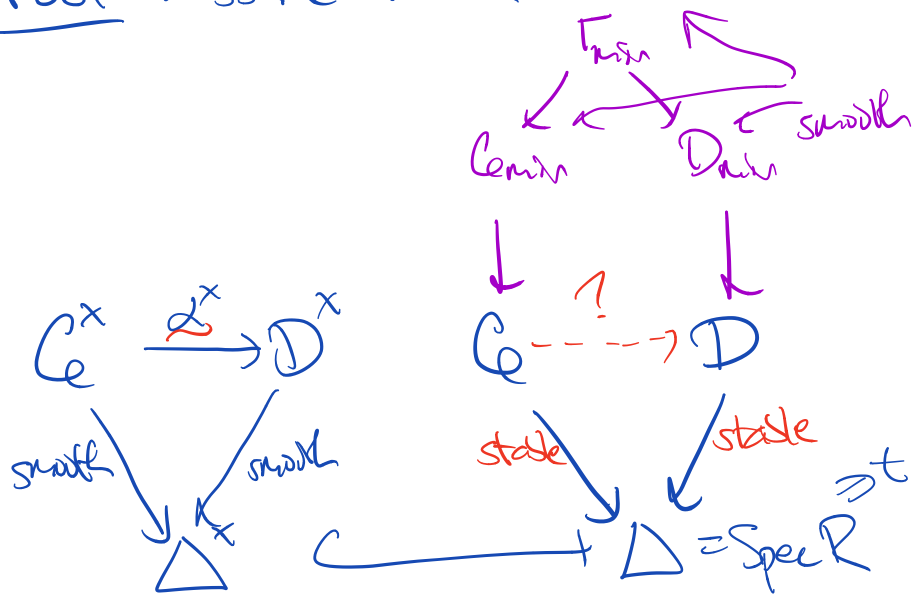
x^3

see Harris-Morrison

§2. Uniqueness of stable limit

Proposition (Separatedness). *If $(\mathcal{C} \to \Delta, \sigma_1, \ldots, \sigma_n)$ and $(\mathcal{D} \to \Delta, \tau_1, \ldots, \tau_n)$ are families of n-pointed stable curves, then any isomorphism $\alpha^* \colon \mathcal{C}^* \to \mathcal{D}^*$ over Δ^* with $\tau_i^* = \alpha^* \circ \sigma_i^*$ of the generic fibers extends to a unique isomorphism $\alpha \colon \mathcal{C} \to \mathcal{D}$ over Δ with $\tau_i = \alpha \circ \sigma_i$.*

Proof Assume $n=0$ & $\mathcal{C}^\times \cong \mathcal{D}^\times$ smooth



Local structure of $z \in \mathcal{C}$ if $z \in \mathcal{C}_0$ node

$xy = t^{n+1}$ ⟵ A_n-singularity

Step 1 Take minimal resolutions of $\mathcal{C}$ & $\mathcal{D}$

$\pi_{\mathcal{C}}^{-1}(z) = E_1 \cup \cdots \cup E_n \qquad E_i^2 = -2$

↑ node in $\mathcal{C}_0$

Step 2 Take minimal resolution $\Gamma_{min} \to \Gamma$ of $\Gamma = \mathrm{im}(\mathcal{C}^\times \to \mathcal{C}_{min} \times_\Delta \mathcal{D}_{min})$.

Step 3 As $\Gamma_{min} \to \mathcal{C}_{min}$ & $\Gamma_{min} \to \mathcal{D}_{min}$ are birational maps of smooth proj. surfaces,

$$\Gamma(\omega^{\otimes k}_{\mathcal{C}_{min}/\Delta}) = \Gamma(\omega^{\otimes k}_{\Gamma_{min}/\Delta}) = \Gamma(\omega^{\otimes k}_{\mathcal{D}_{min}/\Delta})$$

Step 4 $\mathcal{C}$ & $\mathcal{D}$ are rel. stable model of $\mathcal{C}_{min} \cong \mathcal{D}_{min}$. By uniqueness of stable models, $\mathcal{C} \cong \mathcal{D}$.

Know

$$\mathcal{C} = \mathrm{Proj} \bigoplus_{k \geq 0} \Gamma(\omega^{\otimes k}_{\mathcal{C}_{min}/\Delta})$$

$$\cong$$

$$\mathcal{D} = \mathrm{Proj} \bigoplus_{k \geq 0} \Gamma(\omega^{\otimes k}_{\mathcal{D}_{min}/\Delta})$$

Recap of argument

$$\supset E \subset \pi^{-1}(E_D) = E+F \qquad \supset E_D \not\cong \mathbb{P}^1$$

$$\begin{tikzcd}
 & & & & \Gamma_{min} \arrow[dl, "p"'] \arrow[dr, "\pi"] & \\
 & & & C_{min} \arrow[d] & & D_{min} \arrow[d, "\pi_D"] \\
C^\times \arrow[rr, "\alpha^\times"] \arrow[dr, "smooth"'] & & D^\times \arrow[dl, "smooth"] & C \arrow[rr, dashed, "?"] \arrow[dr] & & D \arrow[dl] \\
 & \Delta^\times \arrow[rrr, hook] & & & \Delta = \operatorname{Spec} R &
\end{tikzcd}$$

More explicit argument that $\Gamma_{min} \xrightarrow{\sim} C_{min}$ & $\Gamma_{min} \xrightarrow{\sim} D_{min}$

- If not, $\exists\, E = \mathbb{P}^1 \subset \Gamma_{min}$ that is contracted under $\Gamma_{min} \to C_{min}$ but not $\Gamma_{min} \to D_{min}$
- Let $E_D = \pi_D(E)$ & $\pi^{-1}(E_D) = E \cup F$ (π_D-exceptional)
- Blowing up decreases self-intersection: By proj formula $\Longrightarrow E_D^2 = E\cdot(E+F) \geq E^2 = -1$
- Hodge Index for exceptional curves $\Longrightarrow E_D^2 < 0$
- Since $E_D^2 = -1$, $E_D \subset D_{min}$ is singular. As $\Gamma_{min} \to \Gamma$ resolves this singularity, $E\cdot F \geq 1 \Longrightarrow E_D^2 \geq 0$, contradiction!

Upshot: $\overline{\mathcal{M}}_g$ proper

While $\overline{\mathcal{M}}_g \to \operatorname{Spec}\mathbb{Z}$ is proper, we only have proved this in char $=0$

By Keel-Mori thm, $\exists$ coarse mod space

$$\exists\ \overline{\mathcal{M}}_g \xrightarrow{cms} \overline{M}_g$$

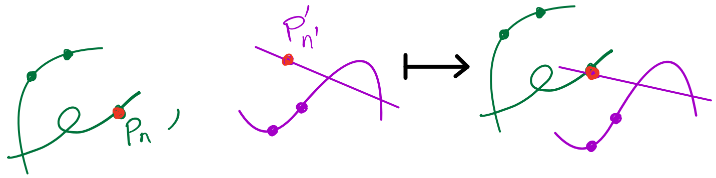
$\overline{M}_g$: proper alg. space

§3. Gluing morphisms

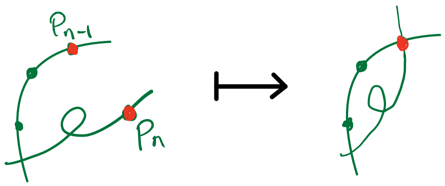
Reference: Knudsen, Projectivity II (1983)

Proposition. *There are morphisms of algebraic stacks*

(a) $$\overline{\mathcal{M}}_{g,n} \times \overline{\mathcal{M}}_{g',n'} \to \overline{\mathcal{M}}_{g+g',n+n'-2}$$



(b) $$\overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_{\cancel{g,n-2}\, g+1,n-2}$$



While it is hopefully conceptuall clear what these maps do on points (i.e curves over a field) we need the construction in families.

Sketch of (a) Assume $n=n'=1$.

$$\begin{tikzcd} \mathcal{C} \arrow[dr] & & \mathcal{C}' \arrow[dl] \\ & S \arrow[ul, bend left, "\sigma"] \arrow[ur, bend left, "\sigma'"] & \end{tikzcd}$$

family of genus g stable

genus g'

Approach 1: Use pushout

$$\begin{tikzcd} S \arrow[r, "\sigma'", "\text{closed imm}"'] \arrow[d, "\sigma"', "\text{closed im}"] & \mathcal{C}' \arrow[d] \\ \mathcal{C} \arrow[r] & \widetilde{\mathcal{C}} \end{tikzcd}$$

Exists by Ferrand

Well behaved when one map is a closed imm & the other is finite.

Need to show $\widetilde{\mathcal{C}} \to S$ desired family of stable curves. We will use

(1) Pushout is étale local on $S, \mathcal{C}, \mathcal{C}'$

(2) Local structure of smooth maps

($\mathcal{C}$ & $\mathcal{C}'$ are smooth along σ & σ')

Reduce pushout computation

$$\begin{tikzcd} \operatorname{Spec} A \arrow[r, "\sigma'=0"] \arrow[d, "0=\sigma"'] & \operatorname{Spec} A[y] \arrow[d] \\ \operatorname{Spec} A[x] \arrow[r] & \operatorname{Spec} A[x] \times_A A[y] \end{tikzcd}$$

$$A[x] \times_A A[y] = \{ (f(x), g(y)) \mid f(0) = g(0) \}$$
$$= A[x,y]/xy$$

$\Rightarrow$ $\widetilde{\mathcal{C}} \to S$ nodal family

stability can be checked on fibers

Approach 2 Proj construction

$$\begin{tikzcd} C \arrow[dr, "\pi"] & & C' \arrow[dl, "\pi'"'] \\ & S \arrow[ul, bend left, "\sigma"] \arrow[ur, bend right, "\sigma'"'] & \end{tikzcd}$$

Know $\omega_{C/S}(\sigma)$, $\omega_{C'/S}(\sigma')$ ample

$$0 \to \omega_C \to \underline{\underline{\omega_C(\sigma)}} \to \mathcal{O}_\sigma \to 0$$

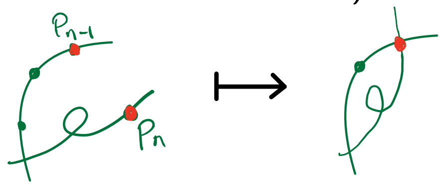
$$\omega_C(\sigma)^{\otimes k} \to \mathcal{O}_\sigma \quad \forall k$$

$$\exists \quad \begin{tikzcd} A_k \arrow[r] \arrow[d] & \pi'_*(\omega_{C'/S}(\sigma')^{\otimes k}) \arrow[d] \\ \pi_*(\omega_{C/S}(\sigma)^{\otimes k}) \arrow[r] & \mathcal{O}_S \end{tikzcd}$$

Check $\mathcal{C} := \operatorname{Proj} \bigoplus_{k \geq 0} A_k \to S$ desired stable family

Sketch of (b)

$\overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_{g+1,n-2}$



Assume $n=2$ $\quad (C_e \longrightarrow S, \sigma_1, \sigma_2)$

Pushout

$$\begin{tikzcd} S \sqcup S \arrow[r, "\sigma_1 \sqcup \sigma_2", "\text{closed}"'] \arrow[d, "\text{finite}"] & C_e \arrow[d] \\ S \arrow[r] & \widetilde{C}_e \end{tikzcd}$$

Ferrand pushout

Local calc

$$\begin{tikzcd} \operatorname{Spec}(A \times A) \arrow[r, "(0,1)"] \arrow[d] & \operatorname{Spec} A[t] \arrow[d] \\ \operatorname{Spec} A \arrow[r] & \operatorname{Spec} B \end{tikzcd}$$

$A[t]$

$B = A \times_{A \times A} A[t]$

where $B = \{ f \in A[t] \mid f(0) = f(1) \}$

$= A\langle \underbrace{t^2-1}_{x}, \underbrace{t^3-t}_{y} \rangle \subset A[t]$

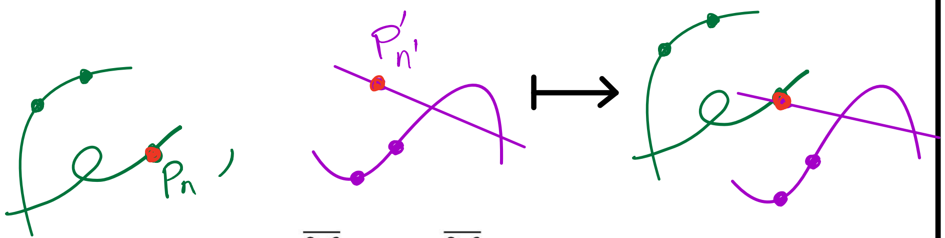
$= A[x,y]/(y^2 - x^2(x+1))$

Proj construction Similar

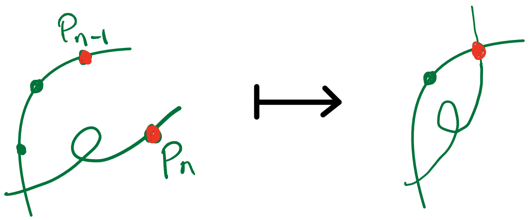
Summary

Proposition. *There are morphisms of algebraic stacks*

$$\overline{\mathcal{M}}_{g,n} \times \overline{\mathcal{M}}_{g',n'} \to \overline{\mathcal{M}}_{g+g',n+n'-2}$$



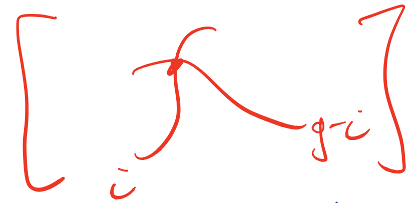
$$\overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_{g+1,n-2}$$



Application (boundary divisors)

- Let $\delta_i = \text{image}\left(\overline{\mathcal{M}}_{i,1} \times \overline{\mathcal{M}}_{g-i,1} \to \overline{\mathcal{M}}_g\right)$

  $i = 1, \dots, \lfloor g/2 \rfloor$

Ex:  $\in \delta_i$

$\delta_0 = \text{image}\left(\overline{\mathcal{M}}_{g-1,2} \to \overline{\mathcal{M}}_g\right)$

 $\in \delta_0$

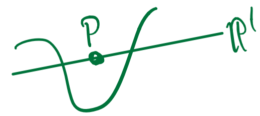
*  $\in \delta_i \cap \delta_0$

$\boxed{\text{IMPORTANT FACT}}$ $\delta = \delta_0 + \delta_1 + \dots + \delta_{\lfloor g/2 \rfloor}$

is a simple normal crossing (snc) divisor.

§4. The universal family $\overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$

Forgetting a marked point can make the curve unstable!



Let $\mathcal{U}_g \xrightarrow[\text{flat}]{\text{proper}} \overline{\mathcal{M}}_g$ be universal family. (alg. stack)

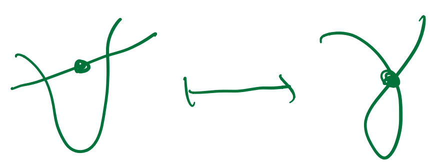
As a stack, $\mathcal{U}_g$ parameterizes

$$\begin{tikzcd} \mathcal{C} \arrow[d, "\text{stable 0-pointed family}"] \\ S \arrow[u, bend left, "\sigma"] \end{tikzcd}$$

& $\sigma : S \to \mathcal{C}$ arbitrary (could land in sing. locus)

We have

$$\overline{\mathcal{M}}_{g,1} \longrightarrow \mathcal{U}_g$$



$$\begin{tikzcd} \mathcal{C} \arrow[d, "\text{stable 1-pted family}"] & & \mathcal{C}^{st} \arrow[d] \\ S \arrow[u, bend left, "\sigma"] \arrow[rr, mapsto] & & S \arrow[u, bend left, "\sigma^{st}"] \end{tikzcd}$$

stable model obtained by contracting rational bridges

Prop $\overline{\mathcal{M}}_{g,1} \xrightarrow{\psi} \mathcal{U}_g$ is isomorphism

Sketch

① Proj construction (following Knudsen)

Explicitly construct an inverse $\mathcal{U}_g \to \overline{\mathcal{M}}_{g,1}$

Let $\mathcal{C} \to S$ stable curve & $\sigma : S \to \mathcal{C}$

Let $I_\sigma \subset \mathcal{O}_{\mathcal{C}}$ be ideal sheaf of σ

Define K as

$$0 \to \mathcal{O}_{\mathcal{C}} \to I_\sigma^\vee \oplus \mathcal{O}_{\mathcal{C}} \to K \to 0$$

Define

$\widetilde{\mathcal{C}} = \operatorname{Proj} \operatorname{Sym} K$ & uses surjection

$$\sigma^* K \twoheadrightarrow \sigma^*(K/\mathcal{O}_{\mathcal{C}}) = \sigma^*(\underbrace{I_\sigma^\vee/\mathcal{O}_{\mathcal{C}}}_{\text{line bdl}})$$

$\to$ gives $\tilde{\sigma} : S \to \widetilde{\mathcal{C}}$

Check: $(\widetilde{\mathcal{C}} \to S, \tilde{\sigma})$ stable

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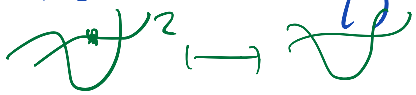
$\mathcal{U}_g \to \overline{\mathcal{M}}_g$ univ family $\Leftrightarrow$

$$\begin{tikzcd} \mathcal{C} \arrow[r, dashed] \arrow[d] \arrow[dr, phantom, "\square\ \exists!"] & \mathcal{U}_g \arrow[d] \\ S \arrow[r, dashed] & \overline{\mathcal{M}}_g \end{tikzcd}$$

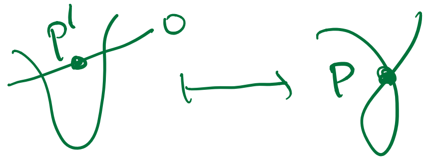
$\leftarrow$ comes from 2-Yoneda's Lemma

Approach 2 (deformation theory)



We have

$$\overline{\mathcal{M}}_{g,1} \xrightarrow{\psi} \mathcal{U}_g$$



$$(C', p') \longmapsto (C, p)$$

stable model of C

$C' \longrightarrow (C')^{st} = C$

$p' \longmapsto p$

- ✓ • ψ is proper & ~~representable~~ stabs pres.
- ✓ • ψ separates points
- ? • ψ separates tangent vectors

$\Rightarrow \psi$ iso

Assume only one rational bridge is contracted and consider

$$T_{\overline{\mathcal{M}}_{g,1},[C',p']} \longrightarrow T_{\mathcal{U}_g,[C,p]}$$

$$\qquad \| \qquad\qquad\qquad\qquad \|$$

def theory

$$\mathrm{Ext}^1(\Omega_{C'}, I_{p'}) \longrightarrow \mathrm{Ext}^1(\Omega_C, I_p)$$

line bdl (at $I_{p'}$) — not line bdl (at I_p)

$k[x,y]/xy$

Properties of contraction map $C' \xrightarrow{\pi} C$

① $\pi_*\mathcal{O}_{C'} = \mathcal{O}_C$ & $R^1\pi_*\mathcal{O}_{C'} = 0$

② $\pi_* I_{p'} = I_p$ & $R^1\pi_* I_{p'} = 0$

③ $0 \to k(p) \to \Omega_C \to \pi_*\Omega_{C'} \to 0$

Map on Ext's is:

$\pi^*\pi_*\Omega_{C'}$

$[0 \to I_{p'} \to E' \to \Omega_{C'} \to 0] \in \mathrm{Ext}^1(\Omega_{C'}, p')$

$[0 \to I_p \to E \to \Omega_C \to 0] \in \mathrm{Ext}^1(\Omega_C, I_p)$

$$\begin{tikzcd} 0 \arrow[r] & I_{p'} \arrow[r] & E' \arrow[r] & \Omega_{C'} \arrow[r] \arrow[l, bend right, "s'"'] & 0 \\ 0 \arrow[r] & I_p \arrow[r] \arrow[d, equal] & E \arrow[r] \arrow[d] & \Omega_C \arrow[r] \arrow[d] \arrow[l, bend right, "s"'] & 0 \\ 0 \arrow[r] & \pi_* I_{p'} \arrow[r] & \pi_* E' \arrow[r] \arrow[ur, phantom, "\square"] & \pi_*\Omega_{C'} \arrow[r] \arrow[l, bend left, "\tilde{s}"] & 0 \end{tikzcd}$$

Suppose E trivial extension

$\Rightarrow \exists$ section s

Check: s descends to $\tilde{s}$

Adjunction

$$\mathrm{Hom}(\pi_*\Omega_{C'}, \pi_* E') = \mathrm{Hom}(\pi^*\pi_*\Omega_{C'}, E')$$

Check descends to s' $\Rightarrow$ E' trivial