PART IV: Moduli of stable curves

Today

0. Recap of Keel-Mori
1. Refresher on smooth curves
2. Nodal curves

§0. Recap

Definition. A map $\pi\colon \mathcal{X} \to X$ from an algebraic stack to an algebraic space is a *coarse moduli space* if

(1) for all k alg. closed, $\mathcal{X}(k)/\sim \xrightarrow{\sim} X(k)$

(2) π is universal for maps to algebraic spaces.

Theorem (Keel-Mori)**.** *Let $\mathcal{X}$ be a Deligne-Mumford stack separated and of finite type over a noetherian algebraic space S. Then there exists a coarse moduli space $\pi\colon \mathcal{X} \to X$ with $\mathcal{O}_X = \pi_*\mathcal{O}_{\mathcal{X}}$ such that*

(1) X is separated and of finite type over S,

(2) π is a proper universal homeomorphism, and

(3) for any flat map $X' \to X$ of noetherian algebraic spaces, $\mathcal{X} \times_X X' \to X'$ is a coarse moduli space.

Key hypothesis: $\mathcal{X}$ separated

This means that $\mathcal{X} \xrightarrow{\text{proper}} \mathcal{X} \times \mathcal{X}$

Since $\mathcal{X}$ is DM, $\mathcal{X} \xrightarrow{\Delta\ \text{finite}} \mathcal{X} \times \mathcal{X}$

$\mathcal{X}$ sep $\iff \mathcal{X} \xrightarrow{\text{finite}} \mathcal{X} \times \mathcal{X}$

Later: We will show that $\mathcal{M}_g$ & $\overline{\mathcal{M}}_{g,n}$ are separated

Keel-Mori thm $\implies$

$\exists\ \overline{\mathcal{M}}_{g,n} \xrightarrow{\text{cms}} \overline{M}_{g,n}$

alg. space

DM case of Keel-Mori

Assume $\mathcal{X}$ f.type / noeth ring

Thm If $\mathcal{X} \to \mathcal{X}\times\mathcal{X}$ unramified ($\mathcal{X}$ DM) & finite ($\mathcal{X}$ sep), then $\exists\, \mathcal{X} \xrightarrow{cms} X$

Sketch Let $x: \operatorname{Spec} k \to \mathcal{X}$ geom. pt with closed image

① Let $(U,u) \xrightarrow{sm} (\mathcal{X},x)$ & then slice U so that

$$\begin{tikzcd} \operatorname{Spec} k \arrow[r, "u"] \arrow[d] \arrow[dr, phantom, "\square"] & U \arrow[d, "\text{ét}"] \\ BG_x \arrow[r] & \mathcal{X} \end{tikzcd}$$

$d = |G_x|$

② $S_d \curvearrowright W \subset (U/\mathcal{X})^d := U \times_{\mathcal{X}} \cdots \times_{\mathcal{X}} U$

complement of all Δ's

Then $[W/S_d]$ param.

$$\begin{tikzcd} Z \arrow[r, hook] \arrow[dr, "\text{fin ét}", "\deg = d"'] & U_S \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & U \arrow[d] \\ & S \arrow[r] & \mathcal{X} \end{tikzcd}$$

Showed $[W/S_d] \to \mathcal{X}$ ét & repn

$w \longmapsto x$ $\quad \operatorname{Aut}(w) \xrightarrow{\sim} \operatorname{Aut}(x)$

Finally, arrange W affine, replace S_d w/ G_x

General Keel-Mori Thm

Thm If $I_{\mathcal{X}} \xrightarrow{fin} \mathcal{X}$, then $\exists\, \mathcal{X} \xrightarrow{cms} X$

Recall:

$$\begin{tikzcd} I_{\mathcal{X}} \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\ \mathcal{X} \arrow[r] & \mathcal{X}\times\mathcal{X} \end{tikzcd}$$

$\mathcal{X}$ sep $\Rightarrow I_{\mathcal{X}} \xrightarrow{fin} \mathcal{X}$

① Same argument $\Longrightarrow$

$$\begin{tikzcd} \operatorname{Spec} k \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & U \arrow[d, "\text{q.finite flat}"] \\ BG_x \arrow[r] & \mathcal{X} \end{tikzcd}$$

② Consider rel. Hilbert scheme

$\mathcal{H}$ param

$$\begin{tikzcd} Z \arrow[r, hook] \arrow[dr, "\text{fin. flat}", "\deg = d"'] & U_S \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & U \arrow[d] \\ & S \arrow[r] & \mathcal{X} \end{tikzcd}$$

Univ. family $\rightsquigarrow$

Arrange affine

$$\begin{tikzcd} W \arrow[r, hook] \arrow[dr, "\text{fin flat}"'] & U_{\mathcal{H}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & U \arrow[d] \\ & \mathcal{H} \arrow[r, "\text{ét}", "\text{repn}"'] & \mathcal{X} \end{tikzcd}$$

$w \longmapsto x$ $\quad \operatorname{Aut}(w) \xrightarrow{\sim} \operatorname{Aut}(x)$

THM $\mathcal{X} \to \mathcal{X}\times\mathcal{X}$ unramified & finite $\implies \exists\, \mathcal{X} \xrightarrow{cms} X$

Know: $\exists\, [W/G_x] \xrightarrow{ét,\ rep} \mathcal{X}$, W affine

③ Show $[\operatorname{Spec} A/G] \xrightarrow{cms} \operatorname{Spec} \underbrace{A^G}_{\text{global sections of } [\operatorname{Spec} A/G]}$

④ Use $I_\mathcal{X} \to \mathcal{X}$ finite to arrange that $[\operatorname{Spec} A/G_x] \to \mathcal{X}$ preserve all stabilizers

⑤ Show

$$\begin{tikzcd} \mathcal{X} \arrow[r, "f"] \arrow[d, "cms"] & \mathcal{Y} \arrow[d, "cms"] \\ X \arrow[r, "\bar f"] & Y \end{tikzcd}$$

DM stack

f étale & preserves stab $\implies$ $\bar f$ étale & diagram is cartesian

⑥

$$\begin{tikzcd} R = W\times_\mathcal{X} W \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & W = [\operatorname{Spec} A/G] \arrow[r, "ét"] \arrow[d, "cms"] & \mathcal{X} \Rightarrow X \arrow[d, "cms"] \\ R \arrow[r, shift left] \arrow[r, shift right] & \operatorname{Spec} A^G = W \arrow[r] & W/R \end{tikzcd}$$

THM $I_\mathcal{X} \to \mathcal{X}$ finite $\implies \exists\, \mathcal{X} \xrightarrow{cms} X$

Know:
$$\begin{tikzcd} W \text{ affine} \arrow[d, "\text{fin flat}"] & \\ \exists\, \mathcal{H} \arrow[r, "ét,\ rep"] & \mathcal{X} \end{tikzcd}$$

③ $R = W\times_\mathcal{H} W \rightrightarrows W = \operatorname{Spec} A \to \mathcal{H}$

finite flat groupoid

Show $\mathcal{H} \xrightarrow{cms} \operatorname{Spec} \underbrace{\Gamma(\mathcal{H}, \mathcal{O}_\mathcal{H})}_{A^R}$

④ Same

⑤ Same

⑥ Same

Show this is étale equiv. relation $\implies W/R$ alg. space

§1. Refresher on smooth curves

A *curve* over a field k is a pure 1-dim'l scheme C of fin type$/k$.

- If C is proper, define genus of $C = g = h^1(X, \mathcal{O}_X)$

FACTS

① C proper $\Rightarrow$ C projective

② If C separated alg. space of dim $= 1$ then C is a scheme.

Theorem (Easy Riemann–Roch). *Let C be an integral projective curve of genus g. If L is a line bundle on C, then*

$$\chi(C, L) = \deg L + 1 - g.$$

$h^0(L) - h^0(\omega_C \otimes L^{-1})$

line bundle $\omega_C = \Omega_C$

Theorem (Serre–Duality). *If C is a smooth projective curve over k, then Ω_C is a dualizing sheaf, i.e. there is a linear map $\mathrm{tr}\colon H^1(C, \Omega_C) \to k$ such that for any coherent sheaf $\mathcal{F}$, the natural pairing*

$$\mathrm{Hom}(\mathcal{F}, \Omega_C) \times H^1(C, \mathcal{F}) \to H^1(C, \Omega_C) \xrightarrow{\mathrm{tr}} k$$

is perfect.

$\Rightarrow h^1(C, L) = h^0(C, \omega_C \otimes L^{-1})$

Positivity of divisors

Corollary. *Let C be a smooth projective curve over k and L be a line bundle on C.*

1. *if $\deg L < 0$, then $h^0(C, L) = 0$.*
2. *if $\deg L > 0$, then L is ample;*
3. *if $\deg L \geq 2g$, then L is basepoint free; and*
4. *if $\deg L \geq 2g + 1$, then L is very ample.*

Assume C is geom conn$/k$

If genus $g \geq 2$,

(a) $h^0(C, \omega_C) \overset{SD}{=} h^1(C, \mathcal{O}_C) = g$

(b) $h^1(C, \omega_C) = h^0(C, \mathcal{O}_C) = 1$

(c) RR applied to ω_C

$\Rightarrow \deg \omega_C = 2g - 2$

$\Rightarrow$ ample

For $k > 1$,

(a) $h^0(C, \omega_C^{\otimes k}) = (2k-1)(g-1)$

(b) $h^1(\quad) = 0$

(c) $\deg \omega_C^{\otimes k} = k(2g-2)$

$\Rightarrow \omega_C^{\otimes k}$ very ample if $k \geq 3$

Families of smooth curves

Definition. A *family of smooth curves of genus g* over a scheme S is a smooth and proper morphism $\mathcal{C} \to S$ of schemes such that every geometric fiber is a connected curve of genus g.

$\implies \Omega_{\mathcal{C}/S}$ line bdl

- For $s \in S$, $\Omega_{\mathcal{C}/S}\big|_{\mathcal{C}_s} = \Omega_{\mathcal{C}_s/k(s)}$

Proposition (Properties of Families of Smooth Curves)**.** *Let $\mathcal{C} \to S$ be a family of smooth curves of genus $g \geq 2$. Then for $k \geq 3$, $\Omega_{\mathcal{C}/S}^{\otimes k}$ is relatively very ample and $\pi_*(\Omega_{\mathcal{C}/S}^{\otimes k})$ is a vector bundle of rank $(2k-1)(g-1)$.*

§2. Nodal curves

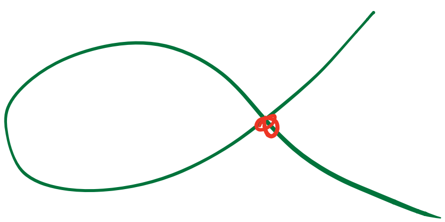
DEF Let C be a curve over k.

(1) If $k = \bar{k}$, $p \in C$ is a <u>node</u> if

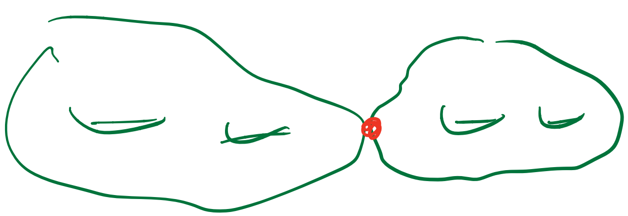
$$\hat{\mathcal{O}}_{C,p} \cong k[[x,y]]/(xy)$$

$$\| \qquad \varprojlim \mathcal{O}_{C,p}/\mathfrak{m}_p^n$$

(2) In general, $p \in C$ is a <u>node</u> if $\exists$ node $p' \in C_{\bar{k}}$ over p.



algebraic 1-dim'l



complex

Exer: If $p \in C$ is node, $\exists\, k \to k'$ fin & sep & $p' \in C_{k'}$ over p s.t. $\hat{\mathcal{O}}_{C_{k'},p'} = k'[[x,y]]/xy$

Exer: If $p \in C$ is a node,

$$\begin{tikzcd} & u \in \mathcal{U} \arrow[dl, "\text{ét}"'] \arrow[dr, "\text{ét}"] & \\ p \in C & & 0 \in \operatorname{Spec} k[x,y]/xy \subset \mathbb{A}^2 \end{tikzcd}$$

Later: relative version

Genus

- Let C be a conn, nodal and proj curve$/k=\bar{k}$
- Let $p_1,\dots,p_\delta \in C$ nodes $\delta = \#$ nodes
- Let $C_1,\dots,C_\nu$ irred. comp $\nu = \#$ irr. comp
- Let $g_i = $ genus $\underbrace{\tilde{C}_i}_{\text{normalization}}$

$$\begin{tikzcd} \tilde{C} = \coprod \tilde{C}_i \arrow[d, "\pi"] \\ C \end{tikzcd} \qquad 0 \to \mathcal{O}_C \to \pi_* \mathcal{O}_{\tilde{C}} \to \bigoplus_i k(p_i) \to 0$$

$\Longrightarrow$

$$0 \to \underbrace{H^0(C,\mathcal{O}_C)}_{1} \to \underbrace{H^0(\tilde{C},\mathcal{O}_{\tilde{C}})}_{\nu} \to \underbrace{\bigoplus_i k(p_i)}_{\delta} \to$$

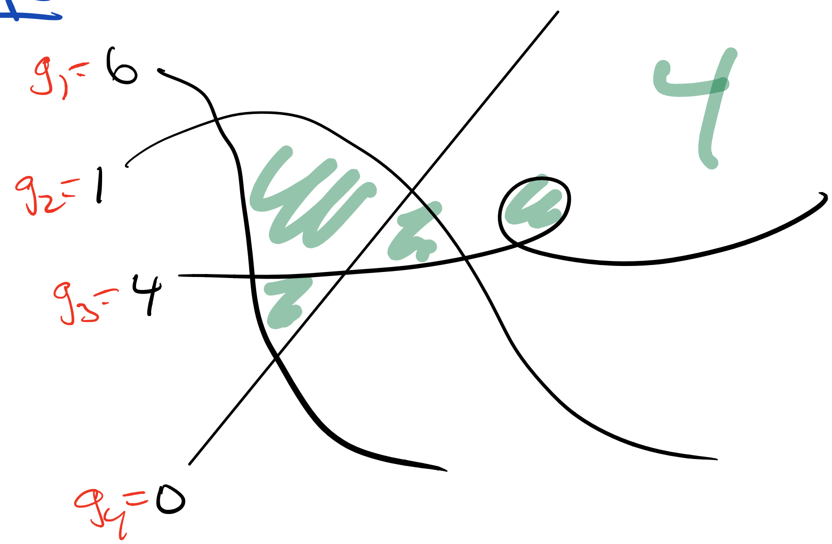
$$\to \underbrace{H^1(C,\mathcal{O}_C)}_{g} \to \underbrace{H^1(\tilde{C},\mathcal{O}_{\tilde{C}})}_{\sum g_i} \to 0$$

$$\Longrightarrow g = \sum g_i + \delta - \nu + 1$$

Example

$\delta = 7$

$\nu = 4$



$$g = (6+1+4+0)+7-4+1$$
$$= 11+4$$
$$= 15$$

§3. Dualizing sheaf

A nodal curve C is a local complete int $\Longrightarrow \exists$ dualizing sheaf ω_C

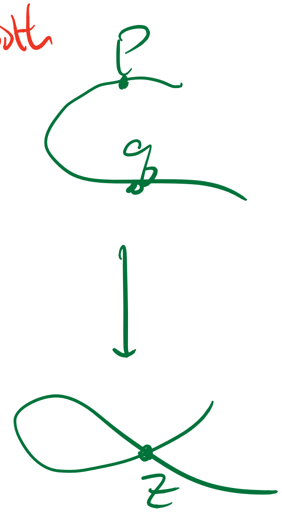
Goal: Explicit description of ω_C

Consider

divisor   smooth

$$\begin{array}{ccccc} p_i, q_i & \in & \tilde{\Sigma} = \pi^{-1}(\Sigma) & \subset & \tilde{C} \\ \downarrow & & \downarrow & & \downarrow \pi \\ z_i & \in & \Sigma = C^{sing} & \subset & C \end{array}$$

set of nodes



Consider

sheaf of meromorphic/rational section of $\Omega_{\tilde{C}}$ with a pole of order ≤ 1

$$0 \to \Omega_{\tilde{C}} \to \Omega_{\tilde{C}}(\tilde{\Sigma}) \to \mathcal{O}_{\tilde{\Sigma}} \to 0$$

$$\mathcal{O}_{\tilde{\Sigma}} = \bigoplus_{y \in \tilde{\Sigma}} k(y)$$

$$s \longmapsto (\mathrm{res}_y(s))$$

defn of residue

DEF The subsheaf $\omega_C \subset \pi_* \Omega_{\tilde{C}}(\tilde{\Sigma})$ is defined on $V \subset C$

$$\Gamma(V, \omega_C) = \{ s \in \Gamma(\pi^{-1}(V), \Omega_{\tilde{C}}(\tilde{\Sigma})) \mid \forall \text{ nodes } z_i \in \Sigma,\ \mathrm{res}_{p_i}(s) + \mathrm{res}_{q_i}(s) = 0 \}$$

Two exact sequences

① $$0 \to \omega_C \to \pi_* \Omega_{\tilde{C}}(\tilde{\Sigma}) \to \bigoplus_{z_i \in \Sigma} k(z_i) \to 0$$

$$s \longmapsto (\mathrm{res}_{p_i}(s) - \mathrm{res}_{q_i}(s))$$

$$0 \to \pi_* \Omega_{\tilde{C}} \to \omega_C \to \bigoplus_{z_i \in C} k(z_i) \to 0$$

$$s \longmapsto (\mathrm{res}_{p_i}(s))$$

DEF The subsheaf $\omega_C \subset \pi_* \Omega_{\tilde{C}}(\Sigma)$ is defined on $V \subset C$

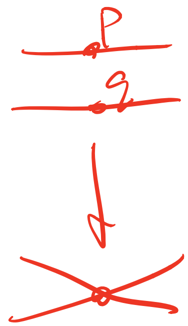
$$\Gamma(V,\omega_C) = \{ s \in \Gamma(\pi^{-1}(V), \Omega_{\tilde{C}}(\Sigma)) \mid \forall \text{ nodes } z_i \in \Sigma,\ res_{p_i}(s) + res_{q_i}(s) = 0 \}$$

Basic example

$$\{p,q\} \subseteq \tilde{C} = \operatorname{Spec} k[x]\times k[y]$$

$$\downarrow \pi$$

$$\Sigma = \{0\} \in C = \operatorname{Spec} k[x,y]/xy \subset \mathbb{A}^2$$



$k[x,y]/xy \hookrightarrow k[x]\times k[y]$ subring of $(f(x),g(y))$ s.t. $f(0)=g(0)$

Let $\nu = \left(\frac{dx}{x}, -\frac{dy}{y}\right) \in \Gamma(C,\omega_C)$

If $\left(f(x)\frac{dx}{x}, g(y)\left(-\frac{dy}{y}\right)\right) \in \Gamma(C,\omega_C)$

then $f(0)=g(0)$

$$(\underbrace{f(0)+f(x)+g(y)})\cdot \nu$$

$\Rightarrow \omega_C \cong \mathcal{O}_C$ with generator ν

Exer: $C' \xrightarrow{et} C \longrightarrow$

$f^*\omega_C \cong \omega_{C'}$

$\Rightarrow \omega_C$ line bundle

Exer Use Serre-Duality for smooth curves to show ω_C is a dualizing sheaf.

§4. Local structure of nodes

Local structure of smooth points

If $\mathcal{C} \to S$ is a smooth family of curves, then for any $p \in \mathcal{C}$

$$\exists \quad \begin{tikzcd} \mathcal{C} \arrow[d] & \mathcal{C}' \arrow[l, "\text{ét}"'] \arrow[r, "\text{ét}"] \arrow[d] & \mathbb{A}^1_{S'} \arrow[dl] \\ S & S' \arrow[l, "\text{ét}"] & \end{tikzcd}$$

$p \leftarrow\!\shortmid p' \mapsto 0$

comm not cart

← rel. affine space of dim = 1, pullback of $\mathbb{A}^1_{\mathbb{Z}} \to \operatorname{Spec} \mathbb{Z}$

Theorem (Local Structure of Nodes). *Let $\pi : \mathcal{C} \to S$ be a flat and fin. pres. map s.t. every fiber is a curve. If $p \in \mathcal{C}$ be a node in a fiber $\mathcal{C}_s$, there exists*

$$\begin{tikzcd} (\mathcal{C}, p) \arrow[d] & (\mathcal{U}, u) \arrow[l, "\text{ét}"'] \arrow[r, "\text{ét}"] \arrow[d] & (\operatorname{Spec} A[x,y]/(xy - f), (s', 0)) \arrow[dl] \\ (S, s) & (\operatorname{Spec} A, s') \arrow[l, "\text{ét}"] & \end{tikzcd}$$

where $f \in A$ is a function vanishing at s'.

base change of

$$\begin{tikzcd} & \operatorname{Spec} A \arrow[d] \\ \operatorname{Spec} \mathbb{Z}[x,y,t]/(xy-t) \arrow[r] & \operatorname{Spec} \mathbb{Z}[t] \end{tikzcd}$$

$f \leftarrow\!\shortmid t$

Theorem (Local Structure of Nodes). *Let $\pi\colon \mathcal{C} \to S$ be a flat and fin. pres. map s.t. every fiber is a curve. If $p \in \mathcal{C}$ be a node in a fiber $\mathcal{C}_s$, there exists*

$$\begin{tikzcd}
(\mathcal{C}, p) \arrow[d] & (\mathcal{U}, u) \arrow[l, "\acute{e}t"'] \arrow[r, "\acute{e}t"] \arrow[d] & (\operatorname{Spec} A[x,y]/(xy-f), (s',0)) \arrow[dl] \\
(S, s) & (\operatorname{Spec} A, s') \arrow[l, "\acute{e}t"'] &
\end{tikzcd}$$

where $f \in A$ is a function vanishing at s'.

Sketch

Step 1 Reduce to S f.type over $\mathbb{Z}$ using Absolute Noetherian Approximation

very useful to reduce to noeth. case.

SKIP

Step 2 Reduce to case where $\hat{\mathcal{O}}_{\mathcal{C}_s,p} \cong k(s)[[x,y]]/(xy)$

Apply exercise $\Longrightarrow \exists\, k(s) \to k'$ sep. & $p' \in \mathcal{C}_{k'}$ s.t. ----

Choose

$$\begin{tikzcd}
(S', s') \arrow[r, "\acute{e}t"] & (S,s) & \text{s.t.} & k(s) \arrow[r] \arrow[dr] & k(s') \arrow[d, phantom, "\cong"] \\
 & & & & k'
\end{tikzcd}$$

Step 3 Show $\hat{\mathcal{O}}_{\mathcal{C},p} \cong \hat{\mathcal{O}}_{S,s}[[x,y]]/(xy-f)$ for $f \in \hat{m}_s$ using formal deformation theory

Schlessinger's thm applied to local def. functor of a node says:

Given

$$\begin{tikzcd}
\mathcal{C}_s \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{D} \arrow[r, dashed] \arrow[d, "\text{flat}"] & \operatorname{Spec} k[[t,x,y]]/(xy-t) \arrow[d] \\
\operatorname{Spec} k(s) \arrow[r, hook] & \operatorname{Spec} B \arrow[r, dashed] & \operatorname{Spec} k[[t]]
\end{tikzcd}$$

$k(s) = k$

local artin / complete

Apply this

$$\begin{tikzcd}
\mathcal{C}_s \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{C} \times_S \hat{\mathcal{O}}_{S,s} \arrow[r] \arrow[d] & \operatorname{Spec} k[[t,x,y]]/(xy-t) \arrow[d] \\
\operatorname{Spec} k \arrow[r] & \operatorname{Spec} \hat{\mathcal{O}}_{S,s} \arrow[r] & \operatorname{Spec} k[[t]]
\end{tikzcd}$$

$f \mapsfrom t$

Theorem (Local Structure of Nodes). *Let $\pi\colon \mathcal{C} \to S$ be a flat and fin. pres. map s.t. every fiber is a curve. If $p \in \mathcal{C}$ be a node in a fiber $\mathcal{C}_s$, there exists*

$$\begin{tikzcd}
(\mathcal{C}, p) \arrow[d] & (\mathcal{U}, u) \arrow[l, "\text{ét}"'] \arrow[r, "\text{ét}"] \arrow[d] & (\operatorname{Spec} A[x,y]/(xy - f), (s', 0)) \arrow[dl] \\
(S, s) & (\operatorname{Spec} A, s') \arrow[l, "\text{ét}"] &
\end{tikzcd} \quad (*)$$

where $f \in A$ is a function vanishing at s'.

At this stage, we know

$$\hat{\mathcal{O}}_{\mathcal{C},p} \cong \hat{\mathcal{O}}_{S,s}[[x,y]]/(xy-f) \text{ for } f \in \hat{m}_s$$

Step 4 Apply Artin Approx.

Theorem (Artin Approximation).

- *Let S be an exc. scheme (e.g. finite type / k or $\mathbb{Z}$).*
- *Let $F\colon \mathrm{Sch}/S \to \mathrm{Sets}$ be a limit preserving functor.*
- *Let $\hat{\xi} \in F(\operatorname{Spec} \hat{\mathcal{O}}_{S,s})$ where $s \in S$ is a point.*

For any integer $N \geq 0$, there exist a residually-trivial étale morphism

$$(S', s') \to (S, s) \qquad \text{and} \qquad \xi' \in F(S')$$

such that the restrictions of $\hat{\xi}$ and ξ' to $\operatorname{Spec}(\mathcal{O}_{S,s}/\mathfrak{m}_s^{N+1})$ are equal.

Define

$F: \mathrm{Sch}/S \to \mathrm{Sets}$

$$(T \to S) \longmapsto \left\{ \begin{tikzcd} \mathcal{C} \arrow[d] & \mathcal{C}_T \arrow[l] \arrow[r] \arrow[d] & \operatorname{Spec} \mathbb{Z}[t,x,y]/(xy-t) \arrow[d] \\ S & T \arrow[l] \arrow[r] & \operatorname{Spec} \mathbb{Z}[t] \end{tikzcd} \right\}$$

Over $\operatorname{Spec} \hat{\mathcal{O}}_{S,s} \to S$, we have

$$\begin{tikzcd}
\mathcal{C}_s \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{C} \times_S \hat{\mathcal{O}}_{S,s} \arrow[r] \arrow[d] & \operatorname{Spec} k[t,x,y]/(xy-t) \arrow[d] \\
\operatorname{Spec} k \arrow[r] & \operatorname{Spec} \hat{\mathcal{O}}_{S,s} \arrow[r] & \operatorname{Spec} k[t]
\end{tikzcd}$$

$\to$ define $\hat{\xi} \in F(\hat{\mathcal{O}}_{S,s})$

Artin approx $\Rightarrow$ diagram $(*)$

$N = 2$