Introduction to Stacks and Moduli

Lecture notes for Math 582C, *working draft*

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Abstract

These notes provide the foundations of moduli theory in algebraic geometry using the language of algebraic stacks with the goal of providing a self-contained proof of the following theorem:

Theorem A. *The moduli space $\overline{\mathcal{M}}_g$ of stable curves of genus $g \geq 2$ is a smooth, proper and irreducible Deligne–Mumford stack of dimension $3g-3$ which admits a projective coarse moduli space.*[1]

Along the way we develop the foundations of algebraic spaces and stacks, and we hope to convey that this provides a convenient language to establish geometric properties of moduli spaces. Introducing these foundations requires developing several themes at the same time including:

- using the functorial and groupoid perspective in algebraic geometry: we will introduce the new algebro-geometric structures of algebraic spaces and stacks;
- replacing the Zariski topology on a scheme with the étale topology: we will generalize the concept of a topological space to Grothendieck topologies and systematically use descent theory for étale morphisms; and
- relying on several advanced topics not seen in a first algebraic geometry course: properties of flat, étale and smooth morphisms of schemes, algebraic groups and their actions, deformation theory, Artin approximation, existence of Hilbert schemes, and the birational geometry of surfaces.

Choosing a linear order in presenting the foundations is no easy task. We attempt to mitigate this challenge by relegating much of the background to appendices. We keep the main body of the notes always focused entirely on developing moduli theory with the above goal in mind.

[1] In a future course, I hope to establish an analogous result for the moduli of vector bundles: The moduli space $\mathcal{B}\mathrm{un}^{\mathrm{ss}}_{r,d}(C)$ of semistable vector bundles of rank r and degree d over a smooth, connected and projective curve C of genus g is a smooth, universally closed and irreducible algebraic stack of dimension $r^2(g-1)$ which admits a projective good moduli space.

Contents

Chapter 0

Introduction and motivation

A *moduli space* is a space M (e.g. topological space, complex manifold or algebraic variety) where there is a natural one-to-one correspondence between points of M and isomorphism classes of certain types of algebro-geometric objects (e.g. smooth curves or vector bundles on a fixed curve). While any space M is the moduli space parameterizing points of M, it is much more interesting when alternative descriptions can be provided. For instance, projective space $\mathbb{P}^1$ can be described as the set of points in $\mathbb{P}^1$ (not so interesting) or as the set of lines in the plane passing through the origin (more interesting).

Moduli spaces arise as an attempt to answer one of the most fundamental problems in mathematics, namely the classification problem. In algebraic geometry, we may wish to classify all projective varieties, all vector bundles on a fixed variety or any number of other structures. The moduli space itself is the solution to the classification problem.

Depending on what objects are being parameterized, the moduli space could be discrete or continuous, or a combination of the two. For instance, the moduli space parameterizing line bundles on $\mathbb{P}^1$ is the discrete set $\mathbb{Z}$: every line bundle on $\mathbb{P}^1$ is isomorphic to $\mathcal{O}(n)$ for a unique integer $n \in \mathbb{Z}$. On the other hand, the moduli space parameterizing quadric plane curve $C \subset \mathbb{P}^2$ is the connected space $\mathbb{P}^5$: a plane curve defined by $a_0x^2 + a_1xy + a_2xz + a_3y^2 + a_4yz + a_5z^2$ is uniquely determined by the point $[a_0, \ldots, a_5] \in \mathbb{P}^5$, and as a plane curve varies continuously (i.e. by varying the coefficients a_i), the corresponding point in $\mathbb{P}^5$ does too.

The moduli space parameterizing smooth projective abstract curves has both a discrete and continuous component. While the genus of a smooth curve is a discrete invariant, smooth curves of a fixed genus vary continuously. For instance, varying the coefficients of a homogeneous degree d polynomial in x, y, z describes a continuous family of mostly non-isomorphic curves of genus $(d-1)(d-2)/2$. After fixing the genus g, the moduli space M_g parameterizing genus g curves is a connected (even irreducible) variety of dimension $3g-3$, a deep fact providing the underlying motivation of these notes. Similarly, the moduli space of vector bundles on a fixed curve has a discrete component corresponding to the rank r and degree d of the vector bundle, and it turns out that after fixing these invariants, the moduli space is also irreducible.

An inspiring feature of moduli spaces and one reason they garner so much attention is that their properties inform us about the properties of the objects themselves that are being classified. For instance, knowing that M_g is unirational

(i.e. there is a dominant rational map $\mathbb{P}^N \dashrightarrow M_g$) for a given genus g tells us that a general genus g curve can be written down explicitly in a similar way to how a general genus 3 curve can be expressed as the solution set to a plane quartic whose coefficients are general complex numbers.

Before we can get started discussing the geometry of moduli spaces such as M_g, we need to ask: why do they even exist? We develop the foundations of moduli theory with this single question in mind. Our goal is to establish the truly spectacular result that there is a projective variety whose points are in natural one-to-one correspondence with isomorphism classes of curves (or vector bundles on a fixed curve). In this chapter, we motivate our approach for constructing projective moduli spaces through the language of algebraic stacks.

0.1 Moduli sets

A *moduli set* is a set where elements correspond to isomorphism classes of certain types of algebraic, geometric or topological objects. To be more explicit, defining a moduli set entails specifying two things:

1. a class of certain types of objects, and
2. an equivalence relation on objects.

The word 'moduli' indicates that we are viewing an element of the set of as an equivalence class of certain objects. In the same vein, we will discuss *moduli groupoids*, *moduli varieties/schemes* and *moduli stacks* in the forthcoming sections. Meanwhile, the word 'object' here is intentionally vague as the possibilities are quite broad: one may wish to discuss the moduli of really any type of mathematical structure, e.g. complex structures on a fixed space, flat connections, quiver representations, solutions to PDEs, or instantons. In these notes, we will entirely focus our study on moduli problems appearing in algebraic geometry although many of the ideas we present extend similarly to other branches of mathematics.

The two central examples in these notes are the moduli of curves and the moduli of vector bundles on a fixed curve—two of the most famous and studied moduli spaces in algebraic geometry. While there are simpler examples such as projective space and the Grassmanian that we will study first, the moduli spaces of curves and vector bundles are both complicated enough to reveal many general phenomena of moduli and simple enough that we can provide a self-contained exposition. Certainly, before you hope to study moduli of higher dimensional varieties or moduli of complexes on a surface, you better have mastered these examples.

0.1.1 Moduli of curves

Here's our first attempt at defining M_g:

Example 0.1.1 (Moduli set of smooth curves)**.** The *moduli set of smooth curves*, denoted as M_g, is defined as followed: the objects are smooth, connected and projective curves of genus g over $\mathbb{C}$ and the equivalence relation is given by isomorphism.

There are alternative descriptions. We could take the objects to be complex structures on a fixed oriented compact surface Σ of genus g and the equivalence relation to be biholomorphism. Or we could take the objects to be pairs (X, ϕ)

where X is a hyberbolic surface and $\phi\colon \Sigma \to X$ is a diffeomorphism (the set of such pairs is the Teichmüller space) and the equivalence relation is isotopy (induced from the action of the mapping class group of Σ).

Each description hints at different additional structures that M_g should inherit.

There are many related examples parameterizing curves with additional structures as well as different choices for the equivalences relations.

Example 0.1.2 (Moduli set of plane curves)**.** The objects here are degree d plane curves $C \subset \mathbb{P}^2$ but there are several choices for how we could define two plane curves C and C' to be equivalent:

(1) C and C' are equal as subschemes;

(2) C and C' are projectively equivalent (i.e. there is an automorphism of $\mathbb{P}^2$ taking C to C'); or

(3) C and C' are abstractly isomorphic.

The three equivalence relations define three different moduli sets. The moduli set (1) is naturally bijective to the projectivization $\mathbb{P}(\mathrm{Sym}^d \mathbb{C}^3)$ of the space of degree d homogeneous polynomials in x, y, z while the moduli set (2) is naturally bijective to the quotient set $\mathbb{P}(\mathrm{Sym}^d \mathbb{C}^3)/\operatorname{Aut}(\mathbb{P}^2)$. The moduli set (3) is the subset of the moduli set of (possibly singular) abstract curves which admit planar embeddings.

Example 0.1.3 (Moduli set of curves with level n structure)**.** The objects are smooth, connected and projective curves C of genus g over $\mathbb{C}$ together with a basis $(\alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g)$ of $H_1(C, \mathbb{Z}/n\mathbb{Z})$ such that the intersection pairing is symplectic. We say that $(C, \alpha_i, \beta_i) \sim (C', \alpha_i', \beta_i')$ if there is an isomorphism $C \to C'$ taking α_i and β_i to α_i' and β_i'.

A rational function f/g on a curve C defines a map $C \to \mathbb{P}^1$ given by $x \mapsto [f(x), g(x)]$. Visualizing a curve as a cover of $\mathbb{P}^1$ is extremely instructive providing a handle to its geometry. Likewise it is instructive to consider the moduli of such covers.

Example 0.1.4 (Moduli of branched covers)**.** We define the *Hurwitz moduli set* $\mathrm{Hur}_{d,g}$ where an object is a smooth, connected and projective curve of genus g together with a finite morphisms $f\colon C \to \mathbb{P}^1$ of degree d, and we declare $(C \xrightarrow{f} \mathbb{P}^1) \sim (C' \xrightarrow{f'} \mathbb{P}^1)$ if there is an isomorphism $\alpha\colon C \to C'$ over $\mathbb{P}^1$ (i.e. $f' = f \circ \alpha$). By Riemann–Hurwitz, any such map $C \to \mathbb{P}^1$ has $2d + 2g - 2$ branch points. Conversely, given a general collection of $2d + 2g - 2$ points of $\mathbb{P}^1$, there exists a genus g curve C and a map $C \to \mathbb{P}^1$ branched over precisely these points. In fact there are only finitely many such covers $C \to \mathbb{P}^1$ as any cover is uniquely determined by the ramification type over the branched points and the finite number of permutations specifying how the unramified covering over the complement of the branched locus is obtained by gluing trivial coverings. In other words, the map $\mathrm{Hur}_{d,g} \to \mathrm{Sym}^{2d+2g-2} \mathbb{P}^1$, assigning a cover to its branched points, has dense image and finite fibers.

Likewise, for a fixed curve C, we could consider the moduli set $\mathrm{Hur}_{d,C}$ parameterizing degree d covers $C \to \mathbb{P}^1$ where the equivalence relation is equality. There is a map $\mathrm{Hur}_{d,g} \to M_g$ defined by $(C \to \mathbb{P}^1) \mapsto C$, and the fiber over a curve C is precisely $\mathrm{Hur}_{d,C}$. Equivalently, $\mathrm{Hur}_{d,C}$ can be described as parameterizing line bundles L on C together with linearly independent sections s_1, s_2 where $(L, s_1, s_2) \sim (L', s_1', s_2')$ if there exists an isomorphism $\alpha\colon L \to L'$ such that $s_i' = \alpha(s_i)$.

Application: number of moduli of M_g

Even before we attempt to give M_g the structure of a variety so that in particular its dimension makes sense, for $g \geq 2$ we can use a parameter count to determine the *number of moduli of* M_g or in modern terminology the *dimension of the local deformation spaces*. Historically Riemann computed the number of moduli in the mid 19th century (in fact using several different methods) well before it was known that M_g is a variety. Following [Rie57], the main idea is to compute the number of moduli of $\mathrm{Hur}_{d,g}$ in two different ways using the diagram

$$\begin{array}{ccccc} \mathrm{Hur}_{d,C} & \hookrightarrow & \mathrm{Hur}_{d,g} & & \\ & \searrow \quad \swarrow & & \overset{\text{finite fibers}}{\searrow} & \\ \{C\} & \hookrightarrow & M_g & & \mathrm{Sym}^{2d+2g-2}\,\mathbb{P}^1 \end{array} \tag{0.1.1}$$

We first compute the number of moduli of $\mathrm{Hur}_{d,C}$ and we might as well assume that d is sufficiently large (or explicitly $d > 2g$). For a fixed curve C, a degree d map $f\colon C \to \mathbb{P}^1$ is determined by an effective divisor $D := f^{-1}(0) = \sum_i p_i \in \mathrm{Sym}^d C$ and a section $t \in H^0(C, \mathcal{O}(D))$ (so that $f(p) = [s(p), t(p)]$ where $s \in \Gamma(C, \mathcal{O}(D))$ defines D). Using that $H^1(C, \mathcal{O}(D)) = H^0(C, \mathcal{O}(K_C - D)) = 0$, Riemann–Roch implies that $h^0(\mathcal{O}(D)) = d - g + 1$. Thus the number of moduli of $\mathrm{Hur}_{d,C}$ is the sum of the number of parameters determining D and the section t

$$\#\text{ of moduli of } \mathrm{Hur}_{d,C} = d + (d - g + 1) = 2d - g + 1.$$

Using (0.1.1), we compute that

$$\begin{aligned} \#\text{ of moduli of } M_g &= \#\text{ of moduli of } \mathrm{Hur}_{d,g} - \#\text{ of moduli of } \mathrm{Hur}_{d,C} \\ &= \#\text{ of moduli of } \mathrm{Sym}^{2d+2g-2}\,\mathbb{P}^1 - \#\text{ of moduli of } \mathrm{Hur}_{d,C} \\ &= (2d + 2g - 2) - (2d - g + 1) \\ &= 3g - 3. \end{aligned}$$

One goal of these notes is to put this calculation on a more solid footing. The interested reader may wish to consult [GH78, pg. 255-257] or [Mir95, pg. 211-215] for further discussion on the number of moduli of M_g, or [AJP16] for a historical background of Riemann's computations.

0.1.2 Moduli of vector bundles

The moduli of vector bundles on a fixed curve provides our second primary example of a moduli set:

Example 0.1.5 (Moduli set of vector bundles on a curve)**.** Let C be a fixed smooth, connected and projective curve over $\mathbb{C}$, and fix integers $r \geq 0$ and d. The objects of interest are vector bundles E (i.e. locally free $\mathcal{O}_C$-modules of finite rank) of rank r and degree d, and the equivalence relation is isomorphism.

There are alternative descriptions. If V is a fixed C^∞-vector bundle V on C, we can take the objects to be connections on V and the equivalence relation to be gauge equivalence. Or we can take the objects to be representations $\pi_1(C) \to \mathrm{GL}_n(\mathbb{C})$ of the fundamental group $\pi_1(C)$ and declare two representations to be equivalent if they have the same dimension n and are conjugate under an

element of $\mathrm{GL}_n(\mathbb{C})$. This last description uses the observation that a vector bundle induces a monodromy representation of $\pi_1(C)$ and conversely that a representation V of $\pi_1(C)$ induces a vector bundle $(\widetilde{C} \times V)/\pi_1(C)$ on C, where $\widetilde{C}$ denotes the universal cover of C.

Specializing to the rank one case is a model for the general case: the moduli set $\mathrm{Pic}^d(C)$ of line bundles on C of degree d is identified (non-canonically) with the abelian variety $H^1(C, \mathcal{O}_C)/H^1(C, \mathbb{Z})$ by means of the cohomology of the exponential exact sequence

$$\begin{array}{ccccccccc} H^1(C,\mathbb{Z}) & \longrightarrow & H^1(C,\mathcal{O}_C) & \longrightarrow & \mathrm{Pic}(C) & \longrightarrow & H^2(C,\mathbb{Z}) & \longrightarrow & 0 \\ & & & & L & \mapsto & \deg(L) & & \end{array}$$

There is a group structure on $\mathrm{Pic}^0(C)$ corresponding to the tensor product of line bundles.

Example 0.1.6 (Moduli of vector bundles on $\mathbb{P}^1$)**.** Since all vector bundles on $\mathbb{A}^1$ are trivial, a vector bundle of rank n on $\mathbb{P}^1$ is described by an element of $\mathrm{GL}_n(k[x]_x)$ specifying how trivial vector bundles on $\{x \neq 0\}$ and $\{y \neq 0\}$ are glued. We can thus describe this moduli set by taking the objects to be elements of $\mathrm{GL}_n(k[x]_x)$ where two elements g and g' are declared equivalent if there exists $\alpha \in \mathrm{GL}_n(k[x])$ and $\beta \in \mathrm{GL}_n(k[1/x])$ (i.e. automorphisms of the trivial vector bundles on $\{x \neq 0\}$ and $\{y \neq 0\}$) such that $g' = \alpha g \beta$.

The Birkhoff–Grothendieck theorem asserts that any vector bundle E on $\mathbb{P}^1$ is isomorphic to $\mathcal{O}(a_1) \oplus \cdots \oplus \mathcal{O}(a_r)$ for unique integers $a_1 \leq \cdots \leq a_r$.[1] This implies that the moduli set of degree d vector bundles of rank r on $\mathbb{P}^1$ is bijective to the set of increasing tuples $(a_1, \ldots, a_r) \in \mathbb{Z}^r$ of integers with $\sum_i a_i = d$. One would be mistaken though to think that the moduli space of vector bundles on $\mathbb{P}^1$ with fixed rank and degree is discrete. For instance, if $d = 0$ and $r = 2$, the group of extensions

$$\mathrm{Ext}^1(\mathcal{O}_{\mathbb{P}^1}(1), \mathcal{O}_{\mathbb{P}^1}(-1)) = H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2)) = H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) = \mathbb{C}$$

is one-dimensional and the universal extension (see Example 0.4.21) is a vector bundle $\mathcal{E}$ on $\mathbb{P}^1 \times \mathbb{A}^1$ such that $\mathcal{E}|_{\mathbb{P}^1 \times \{t\}}$ is the non-trivial extension $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}$ for $t \neq 0$ and the trivial extension $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(1)$ for $t = 0$. This shows that $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}$ and $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(1)$ should be in the same connected component of the moduli space.

0.1.3 Wait—why are we just defining sets?

It is indeed a bit silly to define these moduli spaces as sets. After all, any two complex projective varieties are bijective so we should be demanding a lot more structure than a variety whose points are in bijective correspondence with isomorphism classes. However, spelling out what properties we desire of the moduli space is by no means easy. What we would really like is a quasi-projective variety

[1] Birkhoff proved this in 1909 using linear algebra by explicitly showing that an element $\mathrm{GL}_n(k[x]_x)$ can be multiplied on the left and right by elements of $\mathrm{GL}_n(k[x])$ and $\mathrm{GL}_n(k[1/x])$ to be a diagonal matrix $\mathrm{diag}(x^{a_1}, \ldots, x^{a_r})$ [Bir09] while Grothendieck proved this in 1957 via induction and cohomology by exhibiting a line subbundle $\mathcal{O}(a) \subset E$ such that the corresponding short exact sequence splits [Gro57].

M_g with a universal family $U_g \to M_g$ such that the fiber of a point $[C] \in M_g$ is precisely that curve. This is where the difficulty lies—automorphisms of curves obstruct the existence of such a family—and this is the main reason we want to expand our notion of a geometric space from schemes to algebraic stacks. Algebraic stacks provide a nice approach ensuring the existence of a universal family but it is by no means the only approach.

Historically, it was not clear what structure M_g should have. Riemann introduced the word 'Mannigfaltigkeiten' (or 'manifoldness') but did not specify what this means–complex manifolds were only introduced in the 1940s following Teichmüller, Chern and Weil. The first claim that M_g exists as an algebraic variety was perhaps due to Weil in [Wei58]: "As for M_g there is virtually no doubt that it can be provided with the structure of an algebraic variety." Grothendieck, aware that the functor of smooth families of curves was not representable, studied the functor of smooth families of curves with level structure $r \geq 3$ [Gro61]. While he could show representability, he struggled to show quasi-projectivity. It was only later that Mumford proved that M_g is a quasi-projective variety, an accomplishment for which he was awarded the Field Medal in 1974, by introducing and then applying Geometric Invariant Theory (GIT) to construct M_g as a quotient [GIT]. For further historical background, we recommend [JP13], [AJP16] and [Kol18].

In these notes, we take a similar approach to Mumford's original construction and integrate later influential results due to Deligne, Kollár, Mumford and others such as the seminal paper [DM69] which simultaneously introduced stable curves and stacks with the application of irreducibility of M_g in any characteristic. In this chapter, we motivate our approach by gradually building in additional structure: first as a groupoid (Section 0.3), then as a presheaf (i.e. contravariant functor) (Section 0.4), then as a stack (Section 0.7) and then ultimately as a projective variety (Section 0.9).

One of the challenges of learning moduli stacks is that it requires simultaneously extending the theory of schemes in several orthogonal directions including:

(1) the functorial approach: thinking of a scheme X not as topological space with a sheaf of rings but rather in terms of the functor $\mathrm{Sch} \to \mathrm{Sets}$ defined by $T \mapsto \mathrm{Mor}(T, X)$. For moduli problems, this means specifying not just objects but families of objects; and

(2) the groupoid approach: rather than specifying just the points we also specify their symmetries. For moduli problems, this means specifying not just the objects but their automorphism groups.

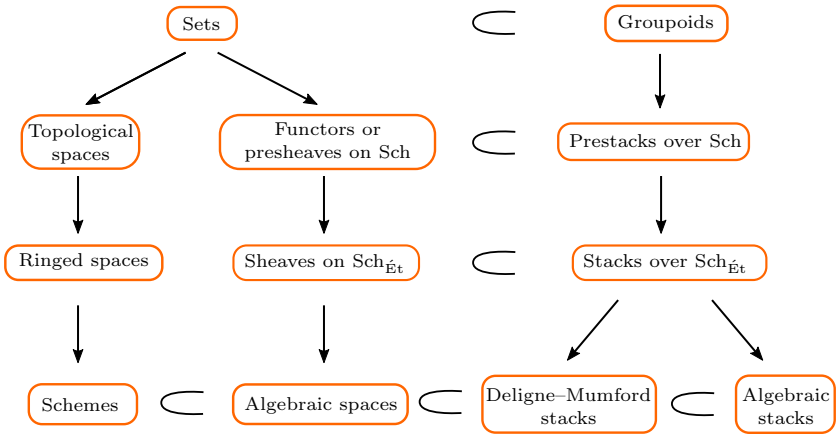


Figure 1: Schematic diagram featuring algebro-geometric enrichments of sets and groupoids where arrows indicate additional geometric conditions.

0.2 Toy example: moduli of triangles

Before we dive deeper into the moduli of curves or vector bundles, we will study the simple yet surprisingly fruitful example of the moduli of triangles which is easy both to visualize and construct. In fact, we present several variants of the moduli of triangles that highlight various concepts in moduli theory. The moduli spaces of labelled triangles and labelled triangles up to similarity have natural functorial descriptions and universal families while the moduli space of *unlabelled* triangles does not admit a universal family due to the presence of symmetries—in exploring this example, we are led to the concept of a moduli groupoid and ultimately to moduli stacks. Michael Artin is attributed to remarking that you can understand most concepts in moduli through the moduli space of triangles.

0.2.1 Labelled triangles

A *labelled triangle* is a triangle in $\mathbb{R}^2$ where the vertices are labelled with '1', '2' and '3', and the distances of the edges are denoted as a, b, and c. We require that triangles have non-zero area or equivalently that their vertices are not colinear.

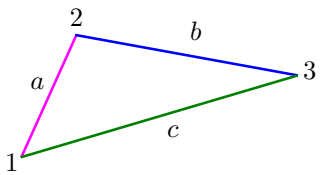


Figure 2: To keep track of the labelling, we color the edges as above.

We define the *moduli set of labelled triangles* M as the set of labelled triangles where two triangles are said to be equivalent if they are the same triangle in $\mathbb{R}^2$ with the same vertices and same labeling. By writing (x_1, y_1), (x_2, y_2) and (x_3, y_3)

as the coordinates of the labelled vertices, we obtain a bijection

$$M \cong \left\{(x_1, y_1, x_2, y_2, x_3, y_3) \mid \det \begin{pmatrix} x_2 - x_1 & x_3 - x_1 \\ y_2 - y_1 & y_3 - y_1 \end{pmatrix} \neq 0\right\} \subset \mathbb{R}^6 \tag{0.2.1}$$

with the open subset of $\mathbb{R}^6$ whose complement is the codimension 1 closed subset defined by the condition that the vectors $(x_2, y_2) - (x_1, y_1)$ and $(x_3, y_3) - (x_1, y_1)$ are linearly dependent.

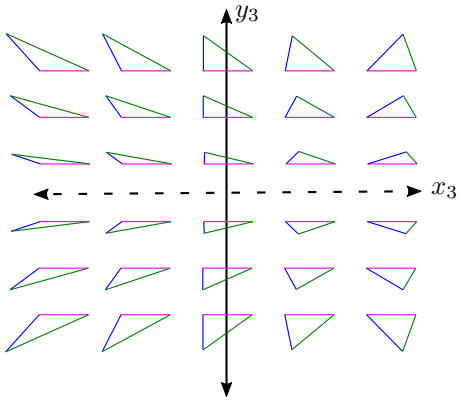


Figure 3: Picture of the slice of the moduli space M where $(x_1, y_1) = (0, 0)$ and $(x_2, y_2) = (1, 0)$. Triangles are described by their third vertex (x_3, y_3) with $y_3 \neq 0$. We've drawn representative triangles for a handful of points in the x_3y_3- plane.

0.2.2 Labelled triangles up to similarity

We define the *moduli set of labelled triangles up to similarity*, denoted by M^{lab}, by taking the same class of objects as in the previous example—labelled triangles—but changing the equivalence relation to label-preserving similarity.

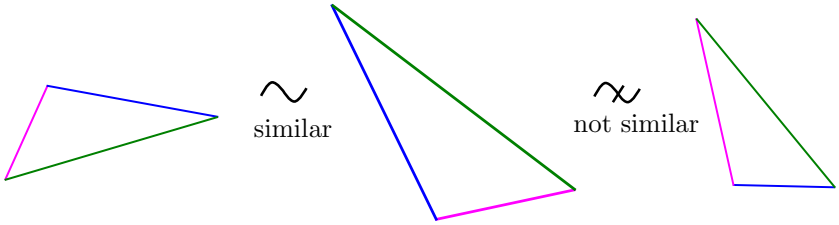


Figure 4: The two triangles on the left are similar, but the third is not.

Every labelled triangle is similar to a unique labelled triangle with perimeter $a + b + c = 2$. We have the description

$$M^{\text{lab}} = \left\{ (a, b, c) \,\middle|\, \begin{array}{c} a + b + c = 2 \\ 0 < a < b + c \\ 0 < b < a + c \\ 0 < c < a + b \end{array} \right\}. \tag{0.2.2}$$

By setting $c = 2 - a - b$, we may visualize M^{lab} as the analytic open subset of $\mathbb{R}^2$ defined by pairs (a, b) satisfying $0 < a, b < 1$ and $a + b > 1$.

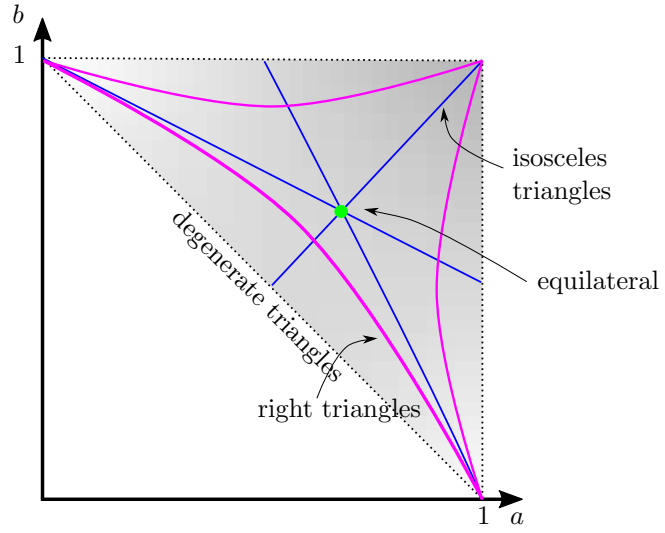


Figure 5: M^{lab} is the shaded area above. The pink lines represent the right triangles defined by $a^2 + b^2 = c^2$, $a^2 + c^2 = b^2$ and $b^2 + c^2 = a^2$, the blue lines represent isosceles triangles defined by $a = b$, $b = c$ and $a = c$, and the green point is the unique equilateral triangle defined by $a = b = c$.

0.2.3 Unlabelled triangles up to similarity

We now turn to the moduli of unlabelled triangles up to similarity, which reveals a new feature not seen in to the two above examples: symmetry!

We define the *moduli set of unlabelled triangles up to similarity*, denoted by M^{unl}, where the objects are unlabelled triangles in $\mathbb{R}^2$ and the equivalence relation is symmetry. We can describe a unlabelled triangle uniquely by the ordered tuple (a, b, c) of increasing side lengths as follows:

$$M^{\text{unl}} = \left\{ (a,b,c) \;\middle|\; \begin{array}{l} 0 < a \le b \le c < a+b \\ a+b+c = 2 \end{array} \right\}. \tag{0.2.3}$$

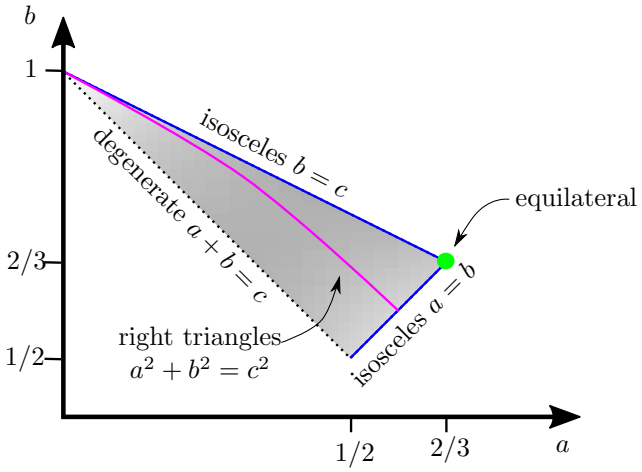


Figure 6: Picture of M^{unl} where $c = 2 - a - b$.

The isosceles triangles with $a = b$ or $b = c$ and the equilateral triangle with $a = b = c$ have symmetry groups of $\mathbb{Z}/2$ and S_3, respectively. This is unfortunately not encoded into our description M^{unl} above. However, we can identify M^{unl} as the quotient M^{lab}/S_3 of the moduli set of labelled triangles up to similarity modulo the natural action of S_3 on the labellings. Under this action, the stabilizers of isosceles and equilateral triangles are precisely their symmetry groups $\mathbb{Z}/2$ and S_3. The action of S_3 on the complement of the set of isosceles and equilateral triangles is free.

0.3 Moduli groupoids

We now change our perspective: rather than specifying *when* two objects are identified, we specify *how*! One of the most desirable properties of a moduli space is the existence of a universal family (see §0.4.5) and the presence of automorphisms obstructs its existence (see §0.4.6). Encoding automorphisms into our descriptions will allow us to get around this problem. A convenient mathematical structure to encode this information is a *groupoid*.

Definition 0.3.1. A *groupoid* is a category $\mathcal{C}$ where every morphism is an isomorphism.

0.3.1 Specifying a moduli groupoid

A *moduli groupoid* is described by

1. a class of certain algebraic, geometric or topological objects; and
2. a set of equivalences between two objects.

where (1) describes the objects and (2) the morphisms of a groupoid. In particular, the moduli groupoid encodes $\mathrm{Aut}(E)$ for every object E.

We say that two groupoids $\mathcal{C}_1$ and $\mathcal{C}_2$ are *equivalent* if there is an equivalence of categories (i.e. a fully faithful and essentially surjective functor) $\mathcal{C}_1 \to \mathcal{C}_2$. Moreover, we say that a groupoid $\mathcal{C}$ *is equivalent to a set* Σ if there is an equivalence of categories $\mathcal{C} \to \mathcal{C}_\Sigma$ (where $\mathcal{C}_\Sigma$ is defined in Example 0.3.2).

0.3.2 Examples

We will return to our two main examples—curves and vector bundles—in a moment but it will be useful first to consider a number of simpler examples.

Example 0.3.2. If Σ is a set, the category $\mathcal{C}_\Sigma$, whose objects are elements of Σ and whose morphisms consist of only the identity morphism, is a groupoid.

Example 0.3.3. If G is a group, the *classifying groupoid* BG of G, defined as the category with one object $\star$ such that $\mathrm{Mor}(\star, \star) = G$, is a groupoid.

Example 0.3.4. The category FB of finite sets where morphisms are bijections is a groupoid. Observe that the isomorphism classes of FB are in bijection with $\mathbb{N}$ but the groupoid FB retains the information of the permutation groups S_n.

Example 0.3.5 (Projective space)**.** Projective space can be defined as a moduli groupoid where the objects are lines $L \subset \mathbb{A}^{n+1}$ through the origin and whose morphisms consist of only the identity, or alternatively where the objects are

non-zero linear maps $x = (x_0, \ldots, x_n) \colon \mathbb{C} \to \mathbb{C}^{n+1}$ such that there is a unique morphism $x \to x'$ if $\mathrm{im}(x) = \mathrm{im}(x') \subset \mathbb{C}^{n+1}$ (i.e. there exists a $\lambda \in \mathbb{C}^*$ such that $x' = \lambda x$) and no morphisms otherwise.

0.3.3 Moduli groupoid of orbits

Example 0.3.6 (Moduli groupoid of orbits)**.** Given an action of a group G on a set X, we define the *moduli groupoid of orbits* $[X/G]$[2] by taking the objects to be all elements $x \in X$ and by declaring $\mathrm{Mor}(x, x') = \{g \in G \,|\, x' = gx\}$.

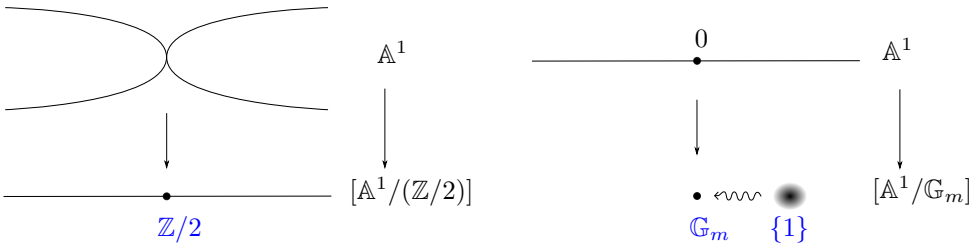


Figure 7: Pictures of the scaling actions of $\mathbb{Z}/2 = \{\pm 1\}$ and $\mathbb{G}_m$ on $\mathbb{A}^1$ over $\mathbb{C}$ with the automorphism groups listed in blue. Note that $[\mathbb{A}^1/\mathbb{G}_m]$ has two isomorphism classes of objects—0 and 1—corresponding to the two orbits—0 and $\mathbb{A}^1 \setminus 0$—such that $0 \in \overline{\{1\}}$ if the set $\mathbb{A}^1/\mathbb{G}_m$ is endowed with the quotient topology.

Exercise 0.3.7.

(1) Show that the moduli groupoid of orbits $[X/G]$ in Example 0.3.6 is equivalent to a set if and only if the action of G on X is free.

(2) Show that a groupoid $\mathcal{C}$ is equivalent to a set if and only if $\mathcal{C} \to \mathcal{C} \times \mathcal{C}$ is fully faithful.

Example 0.3.8. Consider the category $\mathcal{C}$ with two objects x_1 and x_2 such that $\mathrm{Mor}(x_i, x_j) = \{\pm 1\}$ for $i, j = 1, 2$ where composition of morphisms is given by multiplication. Then $\mathcal{C}$ is equivalent $B\mathbb{Z}/2$.

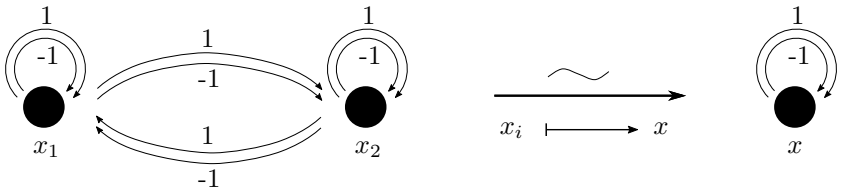


Figure 8: An equivalence of groupoids

Exercise 0.3.9. In Example 0.3.8, show that there is an equivalence of categories inducing a bijection on objects between $\mathcal{C}$ and either $[(\mathbb{Z}/2)/(\mathbb{Z}/4)]$ or $[(\mathbb{Z}/2)/(\mathbb{Z}/2 \times \mathbb{Z}/2)]$ where the action is given by the surjections $\mathbb{Z}/4 \to \mathbb{Z}/2$ or $\mathbb{Z}/2 \times \mathbb{Z}/2 \to \mathbb{Z}/2$.

[2]We use brackets to distinguish the groupoid quotient $[X/G]$ from the set quotient X/G. Later when G and X are enriched with more structure (e.g. an algebraic group acting on a variety), then $[X/G]$ will be correspondingly enriched (e.g. as an algebraic stack).

Example 0.3.10 (Projective space as a quotient)**.** The moduli groupoid of projective space (Example 0.3.5) can also be described as the moduli groupoid of orbits $[(\mathbb{A}^{n+1} \setminus 0)/\mathbb{G}_m]$.

We can also consider the quotient groupoid $[\mathbb{A}^{n+1}/\mathbb{G}_m]$, which is equivalent to the groupoid whose objects are (possibly zero) linear maps $x = (x_0, \dots, x_n) \colon \mathbb{C} \to \mathbb{C}^{n+1}$ such that $\mathrm{Mor}(x, x') = \{t \in \mathbb{C}^* \mid x'_i = tx_i \text{ for all } i\}$. We can thus view $\mathbb{P}^n$ as a subgroupoid of $[\mathbb{A}^{n+1}/\mathbb{G}_m]$.

Exercise 0.3.11. If a group G acts on a set X and $x \in X$ is any point, there exists a fully faithful functor $BG_x \to [X/G]$. If the action is transitive, show that it is an equivalence.

A morphisms of groupoids $\mathcal{C}_1 \to \mathcal{C}_2$ is simply a functor, and we define the category $\mathrm{MOR}(\mathcal{C}_1, \mathcal{C}_2)$ whose objects are functors and whose morphisms are natural transformations.

Exercise 0.3.12. If $\mathcal{C}_1$ and $\mathcal{C}_2$ are groupoids, show that $\mathrm{MOR}(\mathcal{C}_1, \mathcal{C}_2)$ is a groupoid.

Exercise 0.3.13. If H and G are groups, show that there is an equivalence

$$\mathrm{MOR}(BH, BG) = \coprod_{\phi \in \mathrm{Conj}(H,G)} BN_G(\operatorname{im}\phi)$$

where $\mathrm{Conj}(H, G)$ denotes a set of representatives of homomorphisms $H \to G$ up to conjugation by G, and $N_G(\operatorname{im}\phi)$ denotes the normalizer of $\operatorname{im}\phi$ in G.

Exercise 0.3.14. Provide an example of group actions of H and G on sets X and Y and a map $[X/H] \to [Y/G]$ of groupoids that *does not* arise from a group homomorphism $\phi \colon H \to G$ and a ϕ-equivariant map $X \to Y$.

0.3.4 Moduli groupoids of curves and vector bundles

We return to the two main examples in these notes.

Example 0.3.15 (Moduli groupoid of smooth curves)**.** In this case, the objects are smooth, connected and projective curves of genus g over $\mathbb{C}$ and for two curves C, C', the set of morphisms is defined as the set of isomorphisms

$$\mathrm{Mor}(C, C') = \{\text{isomorphisms } \alpha \colon C \xrightarrow{\sim} C'\}.$$

Example 0.3.16 (Moduli groupoid of vector bundles on a curve)**.** Let C be a fixed smooth, connected and projective curve over $\mathbb{C}$, and fix integers $r \geq 0$ and d. The objects are vector bundles E of rank r and degree d, and the morphisms are isomorphisms of vector bundles.

0.3.5 Moduli groupoid of unlabelled triangles up to similarity

We now revisit Section 0.2.3 of the moduli set M^{unl} of unlabelled triangles up to similarity. We will show later that this moduli set does not admit a natural functorial descriptions nor universal family due to presence of symmetries

(Example 0.4.37). Since these are such desirable properties, we will pursue a work around where we encode the symmetries into the definition.

We define the *moduli groupoid of unlabelled triangles up to similarity*, denoted by $\mathcal{M}^{\text{unl}}$ (note the calligraphic font), where the objects are unlabelled triangles in $\mathbb{R}^2$ and where for triangles $T_1, T_2 \subset \mathbb{R}^2$, the set $\text{Mor}(T_1, T_2)$ consists of the symmetries σ (corresponding to the permutations of the vertices) such that T_1 is similar to $\sigma(T_2)$. For example, an isosceles triangle (resp. equilateral triangle) has automorphism group $\mathbb{Z}/2$ (resp. S_3).

We can draw essentially the same picture as Figure 6 except we mark the automorphisms of triangles.

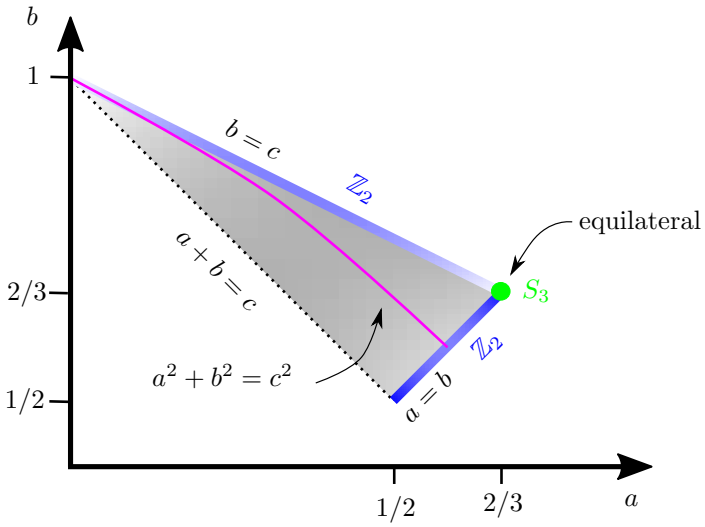


Figure 9: Picture of the moduli groupoid $\mathcal{M}^{\text{unl}}$ with non-trivial automorphism groups labelled.

There is a functor

$$\mathcal{M}^{\text{unl}} \to M^{\text{unl}}$$

which is the identity on objects and collapses all morphisms to the identity. This could be called a *coarse moduli set* where by forgetting some information (i.e. the symmetry groups of isosceles and equilateral triangles), we can study the moduli problem as a more familiar object (i.e. a set rather than groupoid).

Exercise 0.3.17. Recall that the moduli set M^{lab} of labelled triangles up to similarity has the description as the set of tuples (a, b, c) such that $a + b + c = 2$, $0 < a < b + c$, $0 < b < a + c$, and $0 < c < a + b$ (see from (0.2.1)) Show that there is a natural action of S_3 on the moduli set M^{lab} of unlabelled triangles up to similarity and that the functor obtained by forgetting the labelling

$$[M^{\text{lab}}/S_3] \to \mathcal{M}^{\text{unl}}$$

is an equivalence of categories.

Exercise 0.3.18. Define a moduli groupoid of *oriented triangles* and investigate its relation to the moduli sets and groupoids of triangles we've defined above.

0.4 Moduli functors

We now undertake the challenging task of motivating moduli functors, which will be our approach for endowing moduli sets with the enriched structure of a topological space or scheme. This will require a leap in abstraction that is not at all the most intuitive, especially if you are seeing for the first time. The idea due to Grothendieck is to study a scheme X by studying all maps to it!

It may seem that this leap made life more difficult for us: rather than just specifying the points of a moduli space, we need to define all maps to the moduli space. In fact, it is easier than you may expect. Let's take M_g as an example. If S is a scheme and $f\colon S \to M_g$ is a map of sets, then for every point $s \in S$, the image $f(s) \in M_g$ corresponds to an isomorphism class of a curve C_s. But we don't want to consider arbitrary maps of sets. If M_g is enriched as a topological space (resp. scheme), then a continuous (resp. algebraic) map $f\colon S \to M_g$ should mean that the curves C_s are varying continuously (resp. algebraically). A nice way of packaging this is via *families of curves*, i.e. smooth and proper morphisms $\mathcal{C} \to S$ such that every fiber $\mathcal{C}_s$ is a curve.

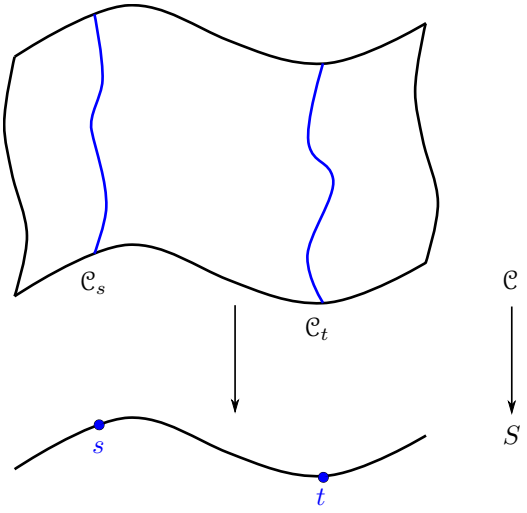


Figure 10: A family of curves over a curve S.

This suggests we define M_g as a functor Sch $\to$ Sets assigning a scheme S to the set of families of curves over S.

0.4.1 Yoneda's lemma

The fact that schemes are determined by maps into it follows from a completely formal argument that holds in any category. If X is an object of a category $\mathcal{C}$, the contravariant functor

$$h_X\colon \mathcal{C} \to \text{Sets}, \qquad S \mapsto \text{Mor}(S, X)$$

recovers the object X itself: this is the content of Yoneda's lemma:

Lemma 0.4.1 (Yoneda's lemma)**.** *Let $\mathcal{C}$ be a category and X be an object. For any contravariant functor $G\colon \mathcal{C} \to \mathrm{Sets}$, the map*

$$\mathrm{Mor}(h_X, G) \to G(X), \qquad \alpha \mapsto \alpha_X(\mathrm{id}_X)$$

is bijective and functorial with respect to both X and G.

Remark 0.4.2. The set $\mathrm{Mor}(h_X, G)$ consists of morphisms or natural transformations $h_X \to G$, and α_X denotes the map $h_X(X) = \mathrm{Mor}(X, X) \to G(X)$.

⚠ **Warning 0.4.3.** We will consistently abuse notation by conflating an element $g \in G(X)$ and the corresponding morphism $h_X \to G$, which we will often write simply as $X \to G$.

Exercise 0.4.4.

1. Spell out precisely what 'functorial with respect to both X and G' means.
2. Prove Yoneda's lemma.

Remark 0.4.5. It is instructive to imagine constructive proofs of Yoneda's lemma. Here we try to explicitly recover a *variety* X over $\mathbb{C}$ from its functor $h_X\colon \mathrm{Sch}/\mathbb{C} \to \mathrm{Sets}$. Clearly, we can recover the closed points of X by simply evaluating $h_X(\operatorname{Spec}\mathbb{C})$. To get all points, we need to allow points whose residue fields are extensions of $\mathbb{C}$. The underlying set of X is

$$\Sigma_X := \bigsqcup_{\mathbb{C}\subset k} h_X(\operatorname{Spec} k)/\sim$$

where we say $x \in h_X(k)$ and $x' \in h_X(k')$ are equivalent if there is a further field extension $\mathbb{C} \subset k''$ containing both k and k' such that the images of x and x' in $h_X(k'')$ are equal under the natural maps $h_X(k) \to h_X(k'')$ and $h_X(k') \to h_X(k'')$. Later, we will follow the same approach when *defining* points of algebraic spaces and stacks (see Definition 2.2.6).

How can we recover the topological space? Here's a tautological way: we say a subset $A \subset \Sigma_X$ is open if there is an open immersion $U \hookrightarrow X$ with image A. Here's a better approach: we say a subset $A \subset \Sigma_X$ is open if for every map $f\colon S \to X$ of schemes, the subset $f^{-1}(A) \subset S$ is open.

What about recovering the sheaf of rings $\mathcal{O}_X$? For an open subset $U \subset \Sigma_X$, we define the functions on U as continuous maps $U \to \mathbb{A}^1$ such that for every morphism $f\colon S \to X$ of schemes, the composition (as a continuous map) $f^{-1}(U) \to U \to \mathbb{A}^1$ is an algebraic function (i.e. corresponds to an element $\Gamma(S, f^{-1}(U))$.

Exercise 0.4.6.

(a) Can the above argument be extended if X is non-reduced?

(b) Is it possible to explicitly recover a scheme X from its *covariant* functor $\mathrm{Sch} \to \mathrm{Sets}, S \mapsto \mathrm{Mor}(X, S)$?

0.4.2 Specifying a moduli functor

Defining a moduli functor requires specifying:

(1) families of objects;

(2) when two families of objects are isomorphic; and

(3) and how families pull back under morphisms.

In defining a moduli functor $F\colon \mathrm{Sch} \to \mathrm{Sets}$, then (1) and (2) specify $F(S)$ for a scheme S and (3) specifies the pull back $F(S) \to F(S')$ for maps $S' \to S$.

Example 0.4.7 (Moduli functor of smooth curves)**.** A *family of smooth curves* (of genus g) is a smooth, proper morphism $\mathcal{C} \to S$ of schemes such that for every $s \in S$, the fiber $\mathcal{C}_s$ is a connected curve (of genus g). The *moduli functor of smooth curves of genus g* is

$$F_{M_g}\colon \mathrm{Sch} \to \mathrm{Sets}, \quad S \mapsto \{\text{families of smooth curves } \mathcal{C} \to S \text{ of genus } g\} / \sim,$$

where two families $\mathcal{C} \to S$ and $\mathcal{C}' \to S$ are equivalent if there is a S-isomorphism $\mathcal{C} \to \mathcal{C}'$. If $S' \to S$ is a map of schemes and $\mathcal{C} \to S$ is a family of curves, the pull back is defined as the family $\mathcal{C} \times_S S' \to S'$.

Example 0.4.8 (Moduli functor of vector bundles on a curve)**.** Let C be a fixed smooth, connected and projective curve over $\mathbb{C}$, and fix integers $r \geq 0$ and d. A *family of vector bundles* (of rank r and degree d) over a scheme S is a vector bundle $\mathcal{E}$ on $C \times S$ (such that for all $s \in S$, the restriction $\mathcal{E}_s := \mathcal{E}|_{C \times \operatorname{Spec} \kappa(s)}$ has rank r and degree d on $C_{\kappa(s)}$). The *moduli functor of vector bundles on C of rank r and degree d* is

$$\mathrm{Sch} \to \mathrm{Sets} \qquad S \mapsto \left\{ \begin{array}{l} \text{families of vector bundles } \mathcal{E} \text{ on } C \times S \\ \text{of rank } r \text{ and degree } d \end{array} \right\} / \sim,$$

where equivalence $\sim$ is given by isomorphism. If $S' \to S$ is a map of schemes and E is a vector bundle on $C \times S$, the pull back is defined as the vector bundle $(\mathrm{id} \times f)^* \mathcal{E}$ on $C \times S'$

Example 0.4.9 (Moduli functor of orbits)**.** Revisiting Example 0.3.6, consider an algebraic group G acting on a scheme X. For every scheme S, the abstract group $G(S)$ acts on the set $X(S)$ (in fact, giving such actions functorial in S uniquely specifies the group action). We can consider the functor

$$\mathrm{Sch} \to \mathrm{Sets} \qquad S \mapsto X(S)/G(S).$$

Elements of the quotient set $X(S)/G(S)$ is our first candidate for a notion of a family of orbits, which we will modify later.

To gain intuition of any moduli functor $F\colon \mathrm{Sch} \to \mathrm{Sets}$, it is always useful to plug in special test schemes. For instance, plugging in a field K should give the K-points of the moduli problem, plugging in $\mathbb{C}[\epsilon]/(\epsilon^2)$ should give pairs of $\mathbb{C}$-points together with tangent vectors, and plugging in a curve (e.g. a DVR) gives families of objects over the curve.

In some cases, even though you may know exactly what objects you want to parameterize, it is not always clear how to define families of objects. In fact, there may be several candidates for families corresponding to different scheme structures on the same topological space. This is the case for instance for the moduli of higher dimensional varieties.

0.4.3 Representable functors

Definition 0.4.10. We say that a functor $F\colon \mathrm{Sch} \to \mathrm{Sets}$ is *representable by a scheme* if there exists a scheme X and an isomorphism of functors $F \xrightarrow{\sim} h_X$.

We would like to know when a given a moduli functor F is representable by a scheme. Unfortunately, each of the functors considered in Examples 0.4.7 to 0.4.9 is *not* representable; see Section 0.4.6. We begin though by considering a few simpler moduli functors which are in fact representable.

Theorem 0.4.11 (Projective space as a functor)**.** *[Har77, Thm. II.7.1] There is a functorial bijection*

$$\mathrm{Mor}(S, \mathbb{P}^n_{\mathbb{Z}}) \cong \left\{ (L, (s_0, \ldots, s_n)) \;\middle|\; \begin{array}{l} L \text{ is a line bundle on } S \text{ globally generated} \\ \text{by sections } s_0, \ldots, s_n \in \Gamma(S, L) \end{array} \right\} / \sim,$$

where $(L, (s_i)) \sim (L', (s_i'))$ *if there exists* $t \in \Gamma(S, \mathcal{O}_S)^*$ *such that* $s_i' = t s_i$ *for all* i.

In other words, the theorem states the functor defined on the right is representable by the scheme $\mathbb{P}^n_{\mathbb{Z}}$. The condition that the sections s_i are globally generated translates to the condition that for every $x \in S$, at least one section $s_i(x) \in L \otimes \kappa(t)$ is non-zero, or equivalently to the surjectivity of $(s_0, \ldots, s_n) \colon \mathcal{O}_S^{n+1} \to L$. This perspective of viewing projective space as parameterizing rank 1 quotients of the trivial bundle will be generalized when we study the Grassmanian in Section 0.5 and even further generalized when we study the Hilbert and Quot schemes. For now, we mention the following mild generalization:

Definition 0.4.12. If S is a scheme and E is a vector bundle on S, we define the contravariant functor

$$\begin{aligned} \mathbb{P}(E) \colon \mathrm{Sch}/S &\to \mathrm{Sets} \\ (T \xrightarrow{f} S) &\mapsto \{\text{quotients } f^*E \overset{q}{\twoheadrightarrow} L \text{ where } L \text{ is a line bundle on } T\}/\sim \end{aligned}$$

where $[f^*E \overset{q}{\twoheadrightarrow} L] \sim [f^*E \overset{q'}{\twoheadrightarrow} L']$ if there is an isomorphism $\alpha \colon L \to L'$ with $q' = \alpha' \circ q'$.

Observe that there is an isomorphism $\mathbb{P}^n_{\mathbb{Z}} \cong \mathbb{P}(\mathcal{O}^{n+1}_{\operatorname{Spec}\mathbb{Z}})$ of functors.

Exercise 0.4.13. Show that $\mathbb{P}(E)$ is representable by the usual projectivization of a vector bundle.

Exercise 0.4.14. Provide functorial descriptions of:

(a) $\mathbb{A}^n \setminus 0$; and

(b) the blowup $\mathrm{Bl}_p \mathbb{P}^n$ of $\mathbb{P}^n$ at a point.

Exercise 0.4.15. Let X be a scheme, and let E and G be $\mathcal{O}_X$-modules. The group $\mathrm{Ext}^1(G, E)$ classifies extensions $0 \to E \to F \to G \to 0$ of $\mathcal{O}_X$-modules where two extensions are identified if there is an isomorphism of short exact sequences inducing the identity map on E and G [Har77, Exer. III.6.1].

Show that the affine scheme $\underline{\mathrm{Ext}}^1_{\mathcal{O}_X}(G, E) := \operatorname{Spec} \operatorname{Sym} \mathrm{Ext}^1(G, E)^\vee$ represents the functor

$$\mathrm{Sch} \to \mathrm{Sets}, \quad T \mapsto \mathrm{Ext}^1_{\mathcal{O}_{X \times T}}(p_1^* G, p_1^* E)$$

where $p_1 \colon X \times T \to X$.

0.4.4 Working with functors

We can form a category $\mathrm{Fun}(\mathrm{Sch}, \mathrm{Sets})$ whose objects are contravariant functors $F\colon \mathrm{Sch} \to \mathrm{Sets}$ and whose morphisms are natural transformations. This category has fiber products: given a morphism $F \xrightarrow{\alpha} G$ and $G' \xrightarrow{\beta} G$, we define

$$\begin{aligned} F \times_G G'\colon \mathrm{Sch} &\to \mathrm{Sets} \\ S &\mapsto \{(a,b) \in F(S) \times G'(S) \mid \alpha_S(a) = \beta_S(b)\} \end{aligned}$$

Exercise 0.4.16. Show that that $F \times_G G'$ satisfies the universal property for fiber products in $\mathrm{Fun}(\mathrm{Sch}, \mathrm{Sets})$.

Definition 0.4.17.

(1) We say that a morphism $F \to G$ of contravariant functors is *representable by schemes* if for any map $S \to G$ from a scheme S, the fiber product $F \times_G S$ is representable by a scheme.

(2) We say that a morphism $F \to G$ is an *open immersion* or that a subfunctor $F \subset G$ is *open* if for any morphism $S \to G$ from a scheme S, $F \times_G S$ is representable by an open subscheme of S.

(3) We say that a set of open subfunctors $\{F_i\}$ is a *Zariski-open cover* of F if for any morphism $S \to F$ from a scheme S, $\{F_i \times_F S\}$ is a Zariski-open cover of S.

Each of these conditions can be checked on *affine* schemes

By appealing to Yoneda's lemma (Lemma 0.4.1), one can define a scheme as a functor $F\colon \mathrm{Sch} \to \mathrm{Sets}$ such that there exists a Zariski-open cover $\{F_i\}$ where each F_i is representable by an affine scheme. Furthermore, this perspective also gives us a recipe for checking that a given functor F is representable by a scheme: simply find a Zariski-open cover $\{F_i\}$ where each F_i is representable.

Exercise 0.4.18. Show that a scheme can be equivalently defined as a contravariant functor $F\colon \mathrm{AffSch} \to \mathrm{Sets}$ on the category of affine schemes (or covariant functor on the category of rings) such that there is Zariski-open cover $\{F_i\}$ where each F_i is representable by an affine scheme.

Replacing Zariski-opens with étale-opens (see Section 0.6) leads to the definition of an algebraic space (Definition 2.1.2).

0.4.5 Universal families

Definition 0.4.19. Let $F\colon \mathrm{Sch} \to \mathrm{Sets}$ be a moduli functor representable by a scheme X via an isomorphism $\alpha\colon F \xrightarrow{\sim} h_X$ of functors. The *universal family* of F is the object $U \in F(X)$ corresponding under α to the identity morphism $\mathrm{id}_X \in h_X(X) = \mathrm{Mor}(X, X)$.

Suspend your skepticism for a moment and suppose that there actually exists a scheme M_g representing the moduli functor of smooth curves of genus g (Example 0.4.7). Then corresponding to the identity map $M_g \to M_g$ is a family of genus g curves $U_g \to M_g$ satisfying the following universal property: for any smooth family of curves $\mathcal{C} \to S$ over a scheme S, there is a unique map $S \to M_g$

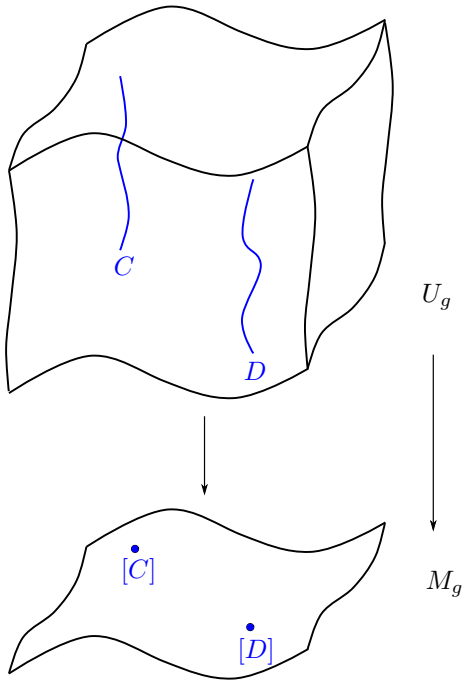


Figure 11: Visualization of a (non-existent) universal family over M_g.

and cartesian diagram

$$\begin{array}{ccc} \mathcal{C} & \longrightarrow & U_g \\ \downarrow & \square & \downarrow \\ S & \longrightarrow & M_g. \end{array}$$

The map $S \to M_g$ sends a point $s \in S$ to the curve $[\mathcal{C}_s] \in M_g$.

Example 0.4.20. The universal family of the moduli functor of projective space (Theorem 0.4.11) is the line bundle $\mathcal{O}(1)$ on $\mathbb{P}^n$ together with the sections $x_0, \ldots, x_n \in \Gamma(\mathbb{P}^n, \mathcal{O}(1))$.

Example 0.4.21 (Universal extensions)**.** If X is a scheme with vector bundles E and G, the universal family for the moduli functor $\underline{\mathrm{Ext}}^1_{\mathcal{O}_X}(G, E)$ of extensions of Exercise 0.4.15 is the extension $0 \to p_1^*G \to \mathcal{F} \to p_1^*E \to 0$ of vector bundle on $X \times \underline{\mathrm{Ext}}^1_{\mathcal{O}_X}(G, E)$. The restriction of this extension to $X \times \{t\}$ is the extension corresponding to $t \in \mathrm{Ext}^1(G, E)$.

Example 0.4.22 (Classifying spaces in algebraic topology)**.** Let G be a topological group and $\mathrm{Top}^{\mathrm{para}}$ be the category of paracompact topological spaces where morphisms are defined up to homotopy. It is a theorem in algebraic topology that the functor

$$\mathrm{Top}^{\mathrm{para}} \to \mathrm{Sets}, \qquad S \mapsto \{\text{principal } G\text{-bundles } P \to S\}/\sim,$$

where $\sim$ denotes isomorphism, is represented by a topological space, which we denote by BG and call the *classifying space*. The universal family is usually denoted by $EG \to BG$.

For example, the classifying space $B\mathbb{C}^*$ is the infinite-dimensional manifold $\mathbb{CP}^\infty$; in algebraic geometry however the classifying stack $B\mathbb{G}_{m,\mathbb{C}}$ is an algebraic stack of dimension -1.

0.4.6 Non-representability of some moduli functors

Suppose $F\colon \mathrm{Sch}/\mathbb{C} \to \mathrm{Sets}$ is a moduli functor parameterizing isomorphism classes of objects, and let's suppose that there is an object E over $\operatorname{Spec}\mathbb{C}$ with a non-trivial automorphism α. This can obstruct the representability of F as the automorphism α can sometimes be used to construct non-trivial families: namely, if $S = S_1 \cup S_2$ is an open cover of a scheme S, we can glue the trivial families $E \times S_1$ and $E \times S_2$ using α to obtain a family $\mathcal{E}$ over S which might be non-trivial.

Proposition 0.4.23. *Let $F\colon \mathrm{Sch}/\mathbb{C} \to \mathrm{Sets}$ be a moduli functor parameterizing isomorphism classes of objects. Suppose there is a family of objects $\mathcal{E} \in F(S)$ over a variety S. For a point $s \in S(\mathbb{C})$, denote by $\mathcal{E}_s$ the pull back of $\mathcal{E}$ along $s\colon \operatorname{Spec}\mathbb{C} \to S$. If*

(a) the fibers $\mathcal{E}_s$ are isomorphic for $s \in S(\mathbb{C})$; and

(b) the family $\mathcal{E}$ is non-trivial, i.e. is not equal to the pull back of an object $E \in F(\mathbb{C})$ along the structure map $S \to \operatorname{Spec}\mathbb{C}$,

then F is not representable.

Proof. Suppose by way of contradiction that F is represented by a scheme X. By condition (a), the restriction $E := \mathcal{E}_s$ is independent of $s \in S(\mathbb{C})$ and defines a unique point $x \in X(\mathbb{C})$. As S is reduced, the map $S \to X$ factors as $S \to \operatorname{Spec}\mathbb{C} \xrightarrow{x} X$. Thus both the family $\mathcal{E}$ and the trivial family correspond to the same constant map $S \to \operatorname{Spec}\mathbb{C} \xrightarrow{x} X$, contradicting condition (b). □

Example 0.4.24 (Moduli of vector bundles over a point)**.** Consider the moduli functor $F\colon \mathrm{Sch}/\mathbb{C} \to \mathrm{Sets}$ assigning a scheme S to the set of isomorphism classes of vector bundles over S. Note that $F(\operatorname{Spec}\mathbb{C}) = \bigsqcup_{r\geq 0}\{\mathcal{O}^r_{\operatorname{Spec}\mathbb{C}}\}$. Since we know there exist non-trivial vector bundles (of any positive rank), we see that F cannot be representable by a scheme.

Exercise 0.4.25. Show that the moduli functor of vector bundles over a curve C is not representable.

Example 0.4.26 (Moduli of elliptic curves)**.** An *elliptic curve* over a field K is a pair (E, P) where E is a smooth, geometrically connected (i.e. $E_{\overline{K}}$ is connected), and projective curve E of genus 1 and $p \in E(K)$. A *family of elliptic curves* over a scheme S is a pair $(\mathcal{E} \to S, \sigma)$ where $\mathcal{E} \to S$ is smooth proper morphism with a section $\sigma\colon S \to \mathcal{E}$ such that for every $s \in S$, the fiber $(\mathcal{E}_s, \sigma(s))$ is an elliptic curve over the residue field $\kappa(s)$. The *moduli functor of elliptic curves* is

$$\begin{aligned} F_{M_{1,1}}\colon \mathrm{Sch} &\to \mathrm{Sets} \\ S &\mapsto \{\text{families } (\mathcal{E} \to S, \sigma) \text{ of elliptic curves }\}/\sim, \end{aligned}$$

where $(\mathcal{E} \to S, \sigma) \sim (\mathcal{E}' \to S, \sigma')$ if there is a S-isomorphism $\alpha\colon \mathcal{E} \to \mathcal{E}'$ compatible with the sections (i.e. $\sigma' = \alpha \circ \sigma$).

Exercise 0.4.27. Consider the family of elliptic curves defined over $\mathbb{A}^1 \setminus 0$ (with coordinate t) by

$$\begin{array}{ccc} \mathcal{E} := V(y^2z - x^3 + tz^3) & \hookrightarrow & (\mathbb{A}^1 \setminus 0) \times \mathbb{P}^2 \\ \downarrow & & \\ \mathbb{A}^1 \setminus 0 & & \end{array}$$

with section $\sigma\colon \mathbb{A}^1 \setminus 0 \to \mathcal{E}$ given by $t \mapsto [0,1,0]$. Show that $(\mathcal{E} \to \mathbb{A}^1 \setminus 0, \sigma)$ satisfies (a) and (b) in Proposition 0.4.23.

Example 0.4.28 (Moduli functor of smooth curves)**.** Let C be a curve with a non-trivial automorphism $\alpha \in \mathrm{Aut}(C)$ and let N be the nodal cubic curve which we can think of as $\mathbb{P}^1$ after glueing 0 and ∞. We can construct a family $\mathcal{C} \to N$ by taking the trivial family $\pi\colon C \times \mathbb{P}^1 \to \mathbb{P}^1$ and gluing the fiber $\pi^{-1}(0)$ with $\pi^{-1}(\infty)$ via the automorphism α.

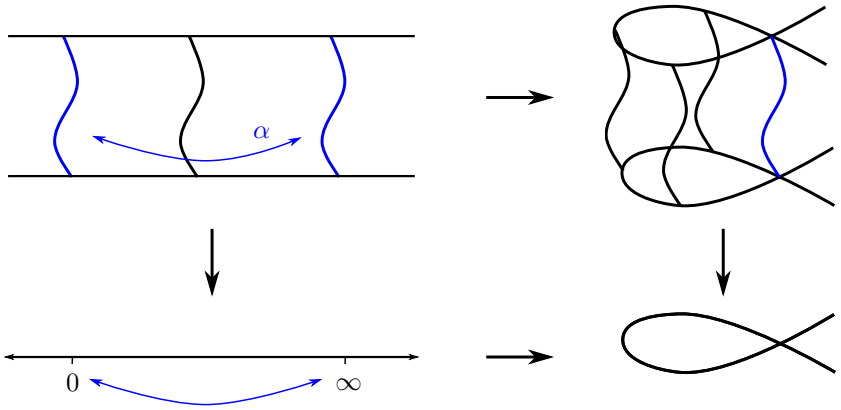


Figure 12: Family of curves over the nodal cubic obtaining by gluing the fibers over 0 and ∞ of the trivial family over $\mathbb{P}^1$ via α. (It would be more illustrative to draw a Mobius band as the family of curves over the nodal cubic.)

To show that the moduli functor of curves is not representable, it suffices to show that $\mathcal{C} \to N$ is non-trivial.

Exercise 0.4.29. Show that $\mathcal{C} \to N$ is a non-trivial family.

0.4.7 Schemes are sheaves

If $F\colon \mathrm{Sch} \to \mathrm{Sets}$ is representable by a scheme X (i.e. $F = \mathrm{Mor}(-, X)$), then F is necessarily a *sheaf in the big Zariski topology*, that is, for any scheme S, the presheaf on the Zariski topology of S defined by assigning to an open subset $U \subset S$ the set $F(U)$ is a sheaf on the Zariski topology of S. This is simply stating that morphisms into the fixed scheme X glue uniquely.

This therefore gives a potential obstruction to the representability of a given moduli functor F: if F is not a sheaf in the big Zariski topology, then F can not be representable.

Example 0.4.30. Consider the functor

$$F\colon \mathrm{Sch} \to \mathrm{Sets}, \quad S \mapsto \{\text{quotients } q\colon \mathcal{O}_S^n \twoheadrightarrow \mathcal{O}_S^k\}/\sim$$

where quotients q and q' are identified if there exists an automorphism Ψ of $\mathcal{O}_S^k$ such that $q' = \Psi \circ q$ or equivalently if $\ker(q) = \ker(q')$.

If F were representable by a scheme, then since morphisms glue in the Zariski topology, sections of F should also glue. But it easy to see that this fails: specializing to $k = 1$ and $S = \mathbb{P}^1$ (with coordinates x and y), consider the cover $S_1 = \{y \neq 0\} = \operatorname{Spec} \mathbb{C}[\frac{x}{y}]$ and $S_2 = \{x \neq 0\} = \operatorname{Spec} \mathbb{C}[\frac{y}{x}]$. The quotients

$$[(1, \frac{x}{y}, 0, \cdots, 0)\colon \mathcal{O}_{S_1}^{\oplus n} \to \mathcal{O}_{S_1}] \in F(S_1) \quad \text{and} \quad [(\frac{y}{x}, 1, 0, \cdots, 0)\colon \mathcal{O}_{S_2}^{\oplus n} \to \mathcal{O}_{S_2}] \in F(S_2)$$

become equivalent in $F(S_1 \cap S_2)$ under the automorphism $\Psi = \frac{y}{x}$ of $\mathcal{O}_{S_1 \cap S_2}$ and *do not* glue to a section of $F(\mathbb{P}^1)$. Of course, the issue is that the structure sheaves on S_1 and S_2 glue to $\mathcal{O}_{\mathbb{P}^1}(1)$—not $\mathcal{O}_{\mathbb{P}^1}$—under Ψ.

The above functor can be modified to define the Grassmanian functor (Definition 0.5.1) where instead of parameterizing *free* rank k quotients of $\mathcal{O}_S^n$, we parameterize *locally free* quotients.

Example 0.4.31. In Example 0.4.9, we introduced the functor $S \mapsto X(S)/G(S)$ associated to an action of an algebraic group G on a scheme X. Even in simple examples of free actions, this functor is not a sheaf; see Exercise 0.4.32

Exercise 0.4.32. Consider $\mathbb{G}_m$ acting on $\mathbb{A}^{n+1} \setminus 0$ with the usual scaling action. Show that the functor $S \mapsto (\mathbb{A}^{n+1} \setminus 0)(S)/\mathbb{G}_m(S)$ is not a sheaf.

Remark 0.4.33. The obstruction of representability due to non-sheafiness is intimately related to the existence of automorphisms. Indeed, the presence of a non-trivial automorphism often implies that a given moduli functor is not a sheaf.

Consider the moduli functor F_{M_g} of smooth curves from Example 0.4.7. Let $\{S_i\}$ be a Zariski-open covering of a scheme S. Suppose we have families of smooth curves $\mathcal{C}_i \to S_i$ and isomorphisms $\alpha_{ij}\colon \mathcal{C}_i|_{S_{ij}} \xrightarrow{\sim} \mathcal{C}_j|_{S_{ij}}$ on the intersection $S_{ij} := S_i \cap S_j$. The requirement that F_{M_g} be a sheaf (when restricted to the Zariski topology on S) implies that the families $\mathcal{C}_i \to S_i$ glue uniquely to a family of curves $\mathcal{C} \to S$. However, we have not required the isomorphisms α_i to be compatible on the triple intersection (i.e. $\alpha_{ij}|_{S_{ijk}} \circ \alpha_{jk}|_{S_{ijk}} = \alpha_{ik}|_{S_{ijk}}$) as is usual with gluing of schemes ([Har77, Exercise II.2.12]). For this reason, F_{M_g} fails to be a sheaf.

Exercise 0.4.34. Show that the moduli functors of smooth curves and elliptic curves are not sheaves by explicitly exhibiting a scheme S, an open cover $\{S_i\}$ and families of curves over S_i that do not glue to a family over S.

0.4.8 Moduli functors of triangles

We will now attempt to define moduli functors of labelled and unlabelled triangles. Since we are primarily interested in constructing these moduli spaces as topological spaces, we will consider the category Top of topological spaces and consider representability as a topological space.

Example 0.4.35 (Labelled triangles)**.** If S is a topological space, then we define a *family of labelled triangles over* S as a tuple $(\mathcal{T}, \sigma_1, \sigma_2, \sigma_3)$ where $\mathcal{T} \subset S \times \mathbb{R}^2$ is a closed subset and $\sigma_i \colon S \to \mathcal{T}$ are continuous sections for $i = 1, 2, 3$ of the projection $\mathcal{T} \to S$ such that for every $s \in S$, the subset $\mathcal{T}_s \subset \mathbb{R}^2$ is a labelled triangle with vertices $\sigma_1(s)$, $\sigma_2(s)$, and $\sigma_3(s)$.

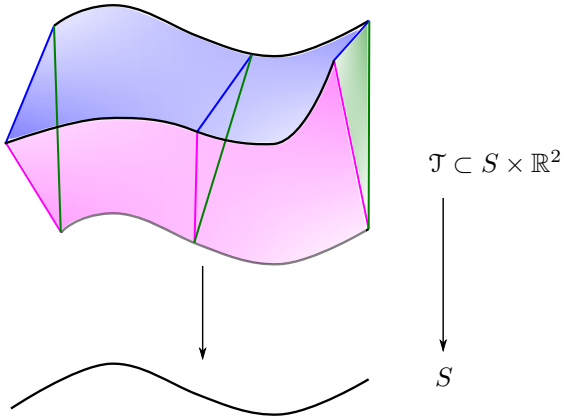


Figure 13: A family of labelled triangles over a curve.

Likewise, we define the *moduli functor of labelled triangles* as

$$F_M \colon \text{Top} \to \text{Sets}, \quad S \mapsto \{\text{families } (\mathcal{T}, \sigma_1, \sigma_2, \sigma_3) \text{ of labelled triangles}\}$$

We claim this functor is represented by the topological space of full rank 2×3 matrices

$$M := \left\{(x_1, y_1, x_2, y_2, x_3, y_3) \mid \det \begin{pmatrix} x_2 - x_1 & x_3 - x_1 \\ y_2 - y_1 & y_3 - y_1 \end{pmatrix} \neq 0\right\} \subset \mathbb{R}^6.$$

There is a bijection of the set $F_M(\text{pt})$ of labelled triangles and M given by taking the coordinates of the vertices. It is easy to see that this bijection can be promoted to an equivalence of functors $F_M \xrightarrow{\sim} h_M$, i.e. to a functorial bijection

$$F_M(S) \xrightarrow{\sim} \text{Mor}(S, M)$$

for each $S \in \text{Top}$, which assigns a family $(\mathcal{T}, \sigma_i)$ of labelled triangles to the map $S \to M$ where $s \mapsto (\sigma_1(s), \sigma_2(s), \sigma_3(s)) \in \mathcal{T}$.

Since F_M is representable by the topological space M, we have a universal family $\mathcal{T}_{\text{univ}} \subset M \times \mathbb{R}^2$ with $\sigma_1, \sigma_2, \sigma_3 \colon M \to \mathcal{T}_{\text{univ}}$. This universal family can be visualized over the locus $(x_1, y_1) = (0, 0)$ and $(x_2, y_2) = (1, 0)$ by taking Figure 3 and drawing the triangles *above* each point rather than *at* each point.

Example 0.4.36 (Labelled triangles up to similarity)**.** We say two families $(\mathcal{T}, (\sigma_i))$ and $(\mathcal{T}', (\sigma_i'))$ of labelled triangles over $S \in \text{Top}$ are *similar* if for each $s \in S$, the labelled triangles $\mathcal{T}_s$ and $\mathcal{T}_s'$ are similar. We define the functor

$$F_{M^{\text{lab}}} \colon \text{Top} \to \text{Sets}, \quad S \mapsto \{\text{families } \mathcal{T} \subset S \times \mathbb{R}^2 \text{ of labelled triangles}\}/\sim$$

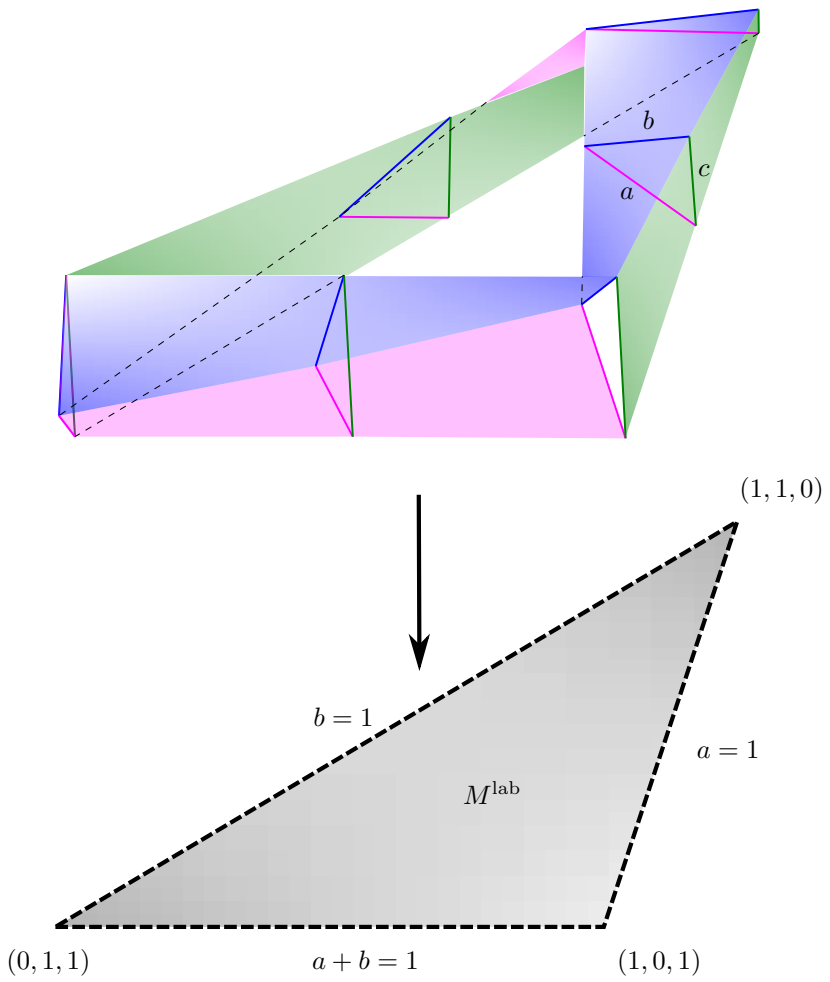


Figure 14: The universal family $U^{\mathrm{lab}} \to M^{\mathrm{lab}}$ of labelled triangles up to similarity.

where $\sim$ denotes similarity. Recall from (0.2.2) that the assignment of a triangle to its side lengths yields a bijection between $F_{M^{\mathrm{lab}}}$ and

$$M^{\mathrm{lab}} = \left\{ (a,b,c) \,\middle|\, \begin{array}{c} a+b+c=2 \\ 0<a<b+c \\ 0<b<a+c \\ 0<c<a+b \end{array} \right\};$$

As in the previous example, this extends to an isomorphism of functors $F_{M^{\mathrm{lab}}} \to \mathrm{Mor}(-, M^{\mathrm{lab}})$, showing that the topological space M^{lab} represents the functor $F_{M^{\mathrm{lab}}}$.

Example 0.4.37 (Unlabelled triangles up to similarity)**.** In Examples 0.4.35 and 0.4.36, we considered the moduli functor of *labelled* triangles up to isomorphism and similarity, respectively. We now consider the unlabelled version.

If S is a topological space, a *family of triangles* is a closed subset $\mathcal{T} \subset S \times \mathbb{R}^2$ such that for all $s \in S$, the fiber $\mathcal{T}_s \subset \mathbb{R}^2$ is a triangle. We say two families $\mathcal{T}, \mathcal{T}'$ over S are similar if the fibers $\mathcal{T}_s$ and $\mathcal{T}'_s$ are similar for all $s \in S$.

We define the functor

$$F\colon \text{Top} \to \text{Sets}, \quad S \mapsto \{\text{families } \mathcal{T} \subset S \times \mathbb{R}^2 \text{ of triangles}\}/\sim$$

where $\sim$ denotes similarity.

This functor is *not* representable as there are non-trivial families of triangles $\mathcal{T}$ such that all fibers are similar triangles (Proposition 0.4.23). For instance, we construct a non-trivial family of triangles over S^1 by gluing two trivial families via a symmetry of an equilateral triangle.

Figure 15: A trivial (left) and non-trivial (right) family of equilateral triangles. Image taken from a video produced by Jonathan Wise: see http://math.colorado.edu/~jonathan.wise/visual/moduli/index.html.

0.5 Illustrating example: Grassmanian

As an illustration of the utility of the functorial approach, we introduce the Grassmanian functor $\text{Gr}(k,n)$ over $\mathbb{Z}$ (Definition 0.5.1) and show that it is representable by a projective scheme (Proposition 0.5.7). Since the Grassmanian parameterizes subspaces V of a fixed vector space, this moduli problem does not have non-trivial symmetries, i.e. automorphisms, and thus we do not need the language of groupoids or stacks. This also provides a warmup to the representability and projectivity of Hilbert and Quot schemes (Chapter D).

0.5.1 Functorial definition

The points of the Grassmanian $\text{Gr}(k,n)$ are k-dimensional *quotients* of n-dimensional space.[3] But what are families of k-dimensional quotients over a scheme S? As motivated by Example 0.4.30, they should be locally free quotients of $\mathcal{O}_S^n$:

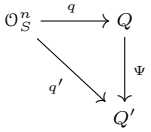
Definition 0.5.1. The *Grassmanian functor* is

$$\begin{aligned}\text{Gr}(k,n)\colon \text{Sch} &\to \text{Sets}\\ S &\mapsto \left\{ \left[\mathcal{O}_S^n \twoheadrightarrow Q\right] \;\middle|\; Q \text{ is a vector bundle of rank } k \right\}/\sim\end{aligned}$$

where $\left[\mathcal{O}_S^n \xrightarrow{q} Q\right] \sim \left[\mathcal{O}_S^n \xrightarrow{q'} Q'\right]$ if there exists an isomorphism $\Psi\colon Q \xrightarrow{\sim} Q'$ such

[3] Alternatively, the points could be considered as k-dimensional *subspaces* but in these notes, we will follow Grothendieck's convention of quotients.

that



commutes (i.e. $q' = \Psi \circ q$) or equivalently $\ker(q) = \ker(q')$.

Pullbacks are defined in the obvious manner. Observe that if $k = 1$, then $\mathrm{Gr}(1, n) \cong \mathbb{P}^{n-1}$.

0.5.2 Representability by a scheme

In this subsection, we show that $\mathrm{Gr}(k, n)$ is representable by a scheme (Proposition 0.5.4). Our strategy will be to find a Zariski-open cover of $\mathrm{Gr}(k, n)$ by representable functors; see Definition 0.4.17. Given a subset $I \subset \{1, \ldots, n\}$ of size k, let $\mathrm{Gr}(k, n)_I \subset \mathrm{Gr}(k, n)$ be the subfunctor where for a scheme S, $\mathrm{Gr}(k, n)_I(S)$ is the subset of $\mathrm{Gr}(k, n)(S)$ consisting of surjections $\mathcal{O}_S^n \overset{q}{\twoheadrightarrow} Q$ such that the composition

$$\mathcal{O}_S^I \xrightarrow{e_I} \mathcal{O}_S^n \overset{q}{\twoheadrightarrow} Q$$

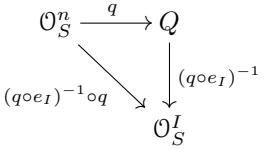
is an isomorphism, where e_I is the canonical inclusion. When there is no possible ambiguity, we set $\mathrm{Gr}_I := \mathrm{Gr}(k, n)_I$.

Lemma 0.5.2. *For each $I \subset \{1, \ldots, n\}$ of size k, the functor Gr_I is representable by affine space $\mathbb{A}_{\mathbb{Z}}^{k\times(n-k)}$*

Proof. We may assume that $I = \{1, \ldots, k\}$. We define a map of functors $\phi\colon \mathbb{A}^{k\times(n-k)} \to \mathrm{Gr}_I$ where over a scheme S, a $k \times (n-k)$ matrix $f = \{f_{i,j}\}_{\substack{1\le i\le n\\ 1\le j\le k}}$ of global functions on S is mapped to the quotient

$$\left(\begin{array}{ccc|ccc} 1 & & & f_{1,1} & \cdots & f_{1,n-k} \\ & 1 & & f_{2,1} & \cdots & f_{2,n-k} \\ & & \ddots & \vdots & & \\ & & 1 & f_{k,1} & \cdots & f_{k,n-k} \end{array}\right) : \mathcal{O}_S^n \to \mathcal{O}_S^k. \tag{0.5.1}$$

The injectivity of $\phi(S)\colon \mathbb{A}^{k\times(n-k)}(S) \to \mathrm{Gr}_I(S)$ is clear. To see surjectivity, let $[\mathcal{O}_S^n \overset{q}{\to} Q] \in \mathrm{Gr}_I(S)$ where by definition $\mathcal{O}_S^I \xrightarrow{e_I} \mathcal{O}_S^n \overset{q}{\twoheadrightarrow} Q$ is an isomorphism. The tautological commutative diagram



shows that $[\mathcal{O}_S^n \overset{q}{\twoheadrightarrow} Q] = [\mathcal{O}_S^n \overset{(q\circ e_I)^{-1}\circ q}{\twoheadrightarrow} \mathcal{O}_S^I] \in \mathrm{Gr}(k, n)(S)$. Since the composition $\mathcal{O}_S^I \xrightarrow{e_I} \mathcal{O}_S^n \overset{(q\circ e_I)^{-1}}{\twoheadrightarrow} \mathcal{O}_S^I$ is the identity, the $k \times n$ matrix corresponding to $(q \circ e_I)^{-1} \circ q$ has the same form as (0.5.1) for functions $f_{i,j} \in \Gamma(S, \mathcal{O}_S)$ and therefore $\phi(S)(\{f_{i,j}\}) = [\mathcal{O}_S^n \overset{q}{\twoheadrightarrow} Q] \in \mathrm{Gr}(k, n)(S)$. □

Lemma 0.5.3. *$\{\mathrm{Gr}_I\}$ is an open cover of $\mathrm{Gr}(k,n)$ where I ranges over all subsets of size k.*

Proof. For a fixed subset I, we first show that $\mathrm{Gr}_I \subset \mathrm{Gr}(k,n)$ is an open subfunctor. To this end, we consider a scheme S and a morphism $S \to \mathrm{Gr}(k,n)$ corresponding to a quotient $q\colon \mathcal{O}_S^n \to Q$. Let C denote the cokernel of the composition $q \circ e_I\colon \mathcal{O}_S^I \to Q$. Notice that if $C = 0$, then q is an isomorphism. The fiber product

$$\begin{array}{ccc} F_I & \longrightarrow & S \\ \downarrow & \square & \downarrow{\scriptstyle [\mathcal{O}_S^n \xrightarrow{q} Q]} \\ \mathrm{Gr}_I & \longrightarrow & \mathrm{Gr}(k,n) \end{array}$$

of functors is representable by the open subscheme $U = S \setminus \mathrm{Supp}(C)$ (the reader is encouraged to verify this claim).

To check the surjectivity of $\bigsqcup_I F_I \to S$, let $s \in S$ be a point. Since $\kappa(s)^n \overset{q\otimes\kappa(s)}{\twoheadrightarrow} Q \otimes \kappa(s)$ is a surjection of vector spaces, there is a non-zero $k \times k$ minor, given by a subset I, of the $k \times n$ matrix $q \otimes \kappa(s)$. This implies that $[\kappa(s)^n \overset{q\otimes\kappa(s)}{\twoheadrightarrow} Q \otimes \kappa(s)] \in F_I(\kappa(s))$. □

Lemmas 0.5.2 and 0.5.3 together imply:

Proposition 0.5.4. *The functor $\mathrm{Gr}(k,n)$ is representable by a scheme.* □

⚠ **Warning 0.5.5.** We will abuse notation by denoting both the functor and the scheme as $\mathrm{Gr}(k,n)$.

Exercise 0.5.6. Use the valuative criterion of properness to show that $\mathrm{Gr}(k,n) \to \operatorname{Spec}\mathbb{Z}$ is proper.

0.5.3 Projectivity of the Grassmanian

We show that the Grassmanian scheme $\mathrm{Gr}(k,n)$ is projective (Proposition 0.5.7) by explicitly providing a projective embedding using the functorial approach. The *Plücker embedding* is the map of functors

$$P\colon \mathrm{Gr}(k,n) \to \mathbb{P}(\bigwedge^k \mathcal{O}^n_{\operatorname{Spec}\mathbb{Z}})$$

defined over a scheme S by mapping a rank k quotient $\mathcal{O}_S^n \overset{q}{\twoheadrightarrow} Q$ to the corresponding rank 1 quotient $\bigwedge^k \mathcal{O}_S^n \to \bigwedge^k Q$. As both sides are representable by schemes, the morphism P corresponds to a morphism of schemes via Yoneda's lemma.

Proposition 0.5.7. *The morphism $P\colon \mathrm{Gr}(k,n) \to \mathbb{P}(\bigwedge^k \mathcal{O}^n_{\operatorname{Spec}\mathbb{Z}})$ of schemes is a closed immersion. In particular, $\mathrm{Gr}(k,n)$ is a projective scheme.*

Proof. Let $I \subset \{1,\dots,n\}$ be a subset which corresponds to a coordinate x_I on $\mathbb{P}(\bigwedge^k \mathcal{O}^n_{\operatorname{Spec}\mathbb{Z}})$. Let $\mathbb{P}(\bigwedge^k \mathcal{O}^n_{\operatorname{Spec}\mathbb{Z}})_I$ be the open locus where $x_I \neq 0$. Viewing

$\mathbb{P}(\bigwedge^k \mathcal{O}^n_{\operatorname{Spec}\mathbb{Z}}) \cong \operatorname{Gr}(1, \binom{n}{k})$, then $\mathbb{P}(\bigwedge^k \mathcal{O}^n_{\operatorname{Spec}\mathbb{Z}})_I \cong \operatorname{Gr}(1, \binom{n}{k})_{\{I\}}$ (viewing $\{I\}$ as the corresponding subset of $\{1, \dots, \binom{n}{k}\}$ of size 1). Since

$$\begin{array}{ccc} \operatorname{Gr}(k,n)_I & \xrightarrow{P_I} & \mathbb{P}(\bigwedge^k \mathcal{O}^n_{\operatorname{Spec}\mathbb{Z}})_I \\ \downarrow & \square & \downarrow \\ \operatorname{Gr}(k,n) & \xrightarrow{P} & \mathbb{P}(\bigwedge^k \mathcal{O}^n_{\operatorname{Spec}\mathbb{Z}}) \end{array}$$

is a cartesian diagram of functors, it suffices to show that P_I is a closed immersion. Under the isomorphisms of Lemma 0.5.2, P_I corresponds to the map

$$\mathbb{A}_{\mathbb{Z}}^{k\times(n-k)} \to \mathbb{A}_{\mathbb{Z}}^{\binom{n}{k}-1}$$

assigning a $k \times (n-k)$ matrix $A = \{a_{i,j}\}$ to the element of $\mathbb{A}_{\mathbb{Z}}^{\binom{n}{k}-1}$ whose Jth coordinate, where $J \subset \{1, \dots, n\}$ is a subset of length k distinct from I, is the $\{1, \dots, k\} \times J$ minor of the $k \times n$ block matrix

$$\left(\begin{array}{ccc|ccc} 1 & & & a_{1,1} & \cdots & a_{1,n-k} \\ & 1 & & a_{2,1} & \cdots & a_{2,n-k} \\ & & \ddots & \vdots & & \\ & & 1 & a_{k,1} & \cdots & a_{k,n-k} \end{array}\right)$$

(of the same form as (0.5.1)). The coordinate $x_{i,j}$ on $\mathbb{A}_{\mathbb{Z}}^{k\times(n-k)}$ is the pull back of the coordinate corresponding to the subset $\{1, \cdots, \widehat{i}, \cdots, k, k+j\}$ (see Figure 16). This shows that the corresponding ring map is surjective thereby establishing that P_I is a closed immersion.

$$x_{i,j} = \det \left(\begin{array}{ccc|ccc} 1 & & & x_{1,1} & \cdots & x_{1,j} & \cdots & x_{1,n-k} \\ & \ddots & & \vdots & & \vdots & & \vdots \\ & & 1 & x_{i,1} & \cdots & x_{i,j} & \cdots & x_{i,n-k} \\ & & \ddots & \vdots & & \vdots & & \vdots \\ & & & 1 & x_{k,1} & \cdots & x_{k,j} & \cdots & x_{k,n-k} \end{array}\right)$$

Figure 16: The minor obtained by removing the ith column and all columns $k+1, \dots, n$ other than $k+j$ is precisely $a_{i,j}$.

□

Exercise 0.5.8. For a field K, let $\operatorname{Gr}(k,n)_K$ be the K-scheme $\operatorname{Gr}(k,n) \times_{\mathbb{Z}} K$, and $p = [K^n \overset{q}{\twoheadrightarrow} Q]$ be a quotient with kernel $K = \ker(q)$. Show that there is a natural bijection of the tangent space

$$T_p \operatorname{Gr}(k,n)_K \xrightarrow{\sim} \operatorname{Hom}(K, Q).$$

with the vector space of K-linear maps $K \to Q$.

Exercise 0.5.9.

(1) Show that the functor $P\colon \mathrm{Gr}(k,n) \to \mathbb{P}(\bigwedge^k \mathcal{O}^n_{\operatorname{Spec}\mathbb{Z}})$ is injective on points and tangent spaces.

Hint: You may want to use the identification of the tangent space of $\mathrm{Gr}(k,n)$ *from Exercise 0.5.8. Alternatively you can also show it is a monomorphism.*

(2) Use Exercise 0.5.6, part (1) above and a criterion for a closed immersion (c.f.[Har77, Prop. II.7.3]) to provide an alternative proof that $\mathrm{Gr}(k,n)_K$ is projective.

0.6 Motivation: why the étale topology?

Why is the Zariski topology not sufficient for our purposes? The short answer is that there are not enough Zariski-open subsets and that étale morphisms are an algebro-geometric replacement of analytic open subsets.

0.6.1 What is an étale morphism anyway?

I'm always baffled when a student is intimidated by étale morphisms, especially when the student has already mastered the conceptually more difficult notions of say properness and flatness. One reason may be due to the fact that the definition is buried in [Har77, Exercises III.10.3-6] and its importance is not highlighted there.

The geometric picture of étaleness that you should have in your head is a covering space. The precise definition of an étale morphism is of course more algebraic, and there are in fact many equivalent formulations. This is possibly another point of intimidation for students as it is not at all obvious why the different notions are equivalent, and indeed some of the proofs are quite involved. Nevertheless, if you can take the equivalences on faith, it requires very little effort to not only internalize the concept, but to master its use.

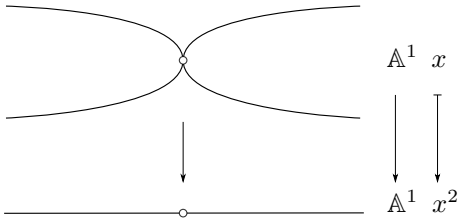


Figure 17: Picture of an étale double cover of $\mathbb{A}^1 \setminus 0$

For a morphism $f\colon X \to Y$ of schemes of finite type over $\mathbb{C}$, the following are equivalent characterizations of étaleness:

- f is smooth of relative dimension 0 (i.e. f is flat and all fibers are smooth of dimension 0);
- f is flat and unramified (i.e. for all $y \in Y(\mathbb{C})$, the scheme-theoretic fiber X_y is isomorphic to a disjoint union $\bigsqcup_i \operatorname{Spec}\mathbb{C}$ of points);
- f is flat and $\Omega_{X/Y} = 0$;

- for all $x \in X(\mathbb{C})$, the induced map $\widehat{\mathcal{O}}_{Y,f(x)} \to \widehat{\mathcal{O}}_{X,x}$ on completions is an isomorphism; and
- (assuming in addition that X and Y are smooth) for all $x \in X(\mathbb{C})$, the induced map $T_{X,x} \to T_{Y,f(x)}$ on tangent spaces is an isomorphism.

We say that f is *étale at* $x \in X$ if there is an open neighborhood U of x such that $f|_U$ is étale.

Exercise 0.6.1. Show that $f\colon \mathbb{A}^1 \to \mathbb{A}^1, x \mapsto x^2$ is étale over $\mathbb{A}^1 \setminus 0$ but is not étale at the origin.

Try to show this for as many of the above definitions as you can.

Étale and smooth morphisms are discussed in much greater detail and generality in Section A.3.

0.6.2 What can you see in the étale topology?

Working with the étale topology is like putting on a better pair of glasses allowing you to see what you couldn't before. Or perhaps more accurately, it is like getting magnifying lenses for your algebraic geometry glasses allowing you to visualize what you already could using your differential geometry glasses.

Example 0.6.2 (Irreducibility of the node)**.** Consider the plane nodal cubic C defined by $y^2 = x^2(x-1)$ in the plane. While there is an analytic open neighborhood of the node $p = (0,0)$ which is reducible, there is no such Zariski-open neighborhood. However, taking a 'square root' of $x-1$ yields a reducible étale neighborhood. More specifically, define $C' = \operatorname{Spec} k[x,y,t]_t/(y^2-x^3+x^2, t^2-x+1)$ and consider

$$C' \to C, \qquad (x,y,t) \mapsto (x,y)$$

Since $y^2 - x^3 + x^2 = (y - xt)(y + xt)$, we see that C' is reducible.

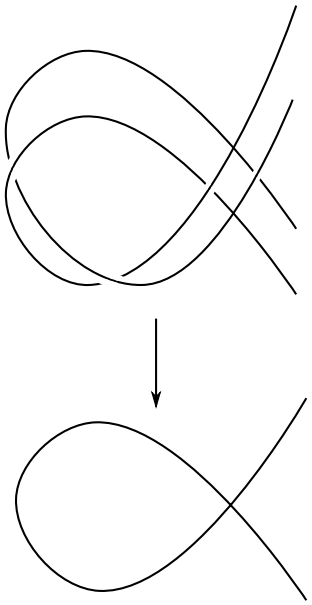


Figure 18: After an étale cover, the nodal cubic becomes reducible.

Example 0.6.3 (Étale cohomology)**.** Sheaf cohomology for the Zarisk-topology can be extended to the étale topology leading to the extremely robust theory of *étale cohomology*. As an example, consider a smooth projective curve C over $\mathbb{C}$ (or equivalently a Riemann surface of genus g), then the étale cohomology $H^1(C_{\text{ét}}, \mathbb{Z}/n)$ of the finite constant sheaf is isomorphic to $(\mathbb{Z}/n)^{2g}$ just like the ordinary cohomology groups, while the sheaf cohomology $H^1(C, \mathbb{Z}/n)$ in the Zariski-topology is 0.

Finally, we would be remiss without mentioning the spectacular application of étale cohomology to prove the Weil conjectures.

Example 0.6.4 (Étale fundamental group)**.** Have you ever thought that there is a similarity between the bijection in Galois theory between intermediate field extensions and subgroups of the Galois group, and the bijection in algebraic topology between covering spaces and subgroups of the fundamental group? Well, you're in good company—Grothendieck also considered this and developed a beautiful theory of the *étale fundamental group* which packages Galois groups and fundamental groups in the same framework.

We only point out here that this connection between étale morphisms and Galois theory is perhaps not so surprising given that a finite field extension L/K is étale (i.e. $\operatorname{Spec} L \to \operatorname{Spec} K$ is étale) if and only if L/K is separable. While we only defined étaleness above for $\mathbb{C}$-varieties, the general notion is not much more complicated; see Étale Equivalences A.3.2.

For the reader interested in reading more about étale cohomology or the étale fundamental group, we recommend [Mil80].

Example 0.6.5 (Quotients by free actions of finite groups)**.** If G is a finite group acting freely on a projective variety X, then there exists a quotient X/G as a projective variety. The essential reason for this is that any G-orbit (or in fact any finite set of points) is contained in an affine variety U, which is the complement of some hypersurface. Then the intersection $V = \bigcap_g gU$ of the G-translates is a G-invariant affine open containing Gx. One can then show that $V/G = \operatorname{Spec} \Gamma(V, \mathcal{O}_V)^G$ and that these local quotients glue to form X/G.

However, if X is not projective, the quotient does not necessarily exist as a scheme. As with most phenomenon for smooth proper varieties that are not-projective, a counterexample is provided by Hironaka's examples of smooth, proper 3-folds; [Har77, App. B, Ex. 3.4.1]. One can construct an example which has a free action by $G = \mathbb{Z}/2$ such that there is an orbit Gx not contained in any G-invariant affine open. This shows that X/G cannot exist as a scheme; indeed, if it did, then the image of x under the finite morphism $X \to X/G$ would be contained in some affine and its inverse would be an affine open containing Gx. See [Knu71, Ex. 1.3] or [Ols16, Ex. 5.3.2] for details.

Nevertheless, for any free action of a finite group G on a scheme X, there does exist a G-invariant étale morphism $U \to X$ from an affine scheme, and the quotients U/G can be glued in the *étale topology* to construct X/G as an *algebraic space*. The upshot is that we can always take quotients of free actions by finite groups, a very desirable feature given the ubiquity of group actions in algebraic geometry; this however comes at the cost of enlarging our category from schemes to algebraic spaces.

Example 0.6.6 (Artin approximation)**.** *Artin approximation* is a powerful and extremely deep result, due to Michael Artin, which implies that most properties

which hold for the completion $\widehat{\mathcal{O}}_{X,x}$ of the local ring is also true in an étale neighborhood of x. More precisely, let $F\colon \mathrm{Sch}\,/X \to \mathrm{Sets}$ be a functor locally of finite presentation (i.e. satisfying the functorial property of Proposition A.1.2), $\widehat{a} \in F(\widehat{\mathcal{O}}_{X,x})$ and N a positive integer. Under the weak hypothesis of excellency on X (which holds if X is locally of finite type over $\mathbb{Z}$ or a field), Artin approximation states that there exists an étale neighborhood $(X', x') \to (X, x)$ with $\kappa(x') = \kappa(x)$ and an element $a' \in F(X')$ agreeing with a on the Nth order neighborhood of x.

For example, in Example 0.6.2, it's not hard to use properties of power series rings to establish that $\widehat{\mathcal{O}}_{C,p} \cong \mathbb{C}[[x,y]]/(y^2 - x^2)$ (e.g. take a power series expansion of $\sqrt{x-1}$), which is reducible. If we consider the functor

$$F\colon \mathrm{Sch}\,/C \to \mathrm{Sets}, \qquad (C' \xrightarrow{\pi} C) \mapsto \{\text{decompositions } C' = C'_1 \cup C'_2\}$$

then applying Artin approximation yields an étale cover $C' \to C$ with C' reducible. Of course, we already knew this from an explicit construction in Example 0.6.2, but hopefully this example shows the potential power of Artin approximation.

0.6.3 Working with the étale topology: descent theory

Another reason why the étale topology is so useful is that many properties of schemes and their morphisms can be checked on étale covers. For instance, you already know that to check if a scheme X is noetherian, finite type over $\mathbb{C}$, reduced or smooth, it suffices to find a Zariski-open cover $\{U_i\}$ such that the property holds for each U_i. *Descent theory* implies the same with respect to a collection $\{U_i \to U\}$ of étale morphisms such that $\bigsqcup_i U_i \to U$ is surjective: X has the property if and only if each U_i does. Descent theory is developed in Chapter B and is used to prove just about everything concerning algebraic spaces and stacks.

0.7 Moduli stacks: moduli with automorphisms

The failure of the representability of the moduli functors of curves and vector bundles is a motivating factor for introducing moduli stacks, which encode the automorphisms groups as part of the data. We will synthesize the approaches from Section 0.3 on moduli groupoids and Section 0.4 on moduli functors.

0.7.1 Specifying a moduli stack

To define a moduli stack, we need to specify

1. families of objects;
2. how two families of objects are isomorphic; and
3. how families pull back under morphisms.

Notice the difference from specifying a moduli functor (Section 0.4.2) is that rather than specifying *when* two families are isomorphic, we specify *how*.

To specify a moduli stack in the algebro-geometric setting, we need to specify for each scheme T a *groupoid* Fam_T of families of objects over T. As a natural generalization of functors to sets, we could consider assignments

$$F\colon \mathrm{Sch} \to \mathrm{Groupoids}, \quad T \mapsto \mathrm{Fam}_T\,.$$

This presents the technical difficulty of considering functors between the category of schemes and the 'category' of groupoids. Morphisms of groupoids are functors but there are also morphisms of functors (i.e. natural transformations) which we call *2-morphisms*. This leads to a '2-category' of groupoids.

What is actually involved in defining such an assignment F? In addition to defining the groupoids Fam_T over each scheme T, we need pullback functors $f^*\colon \mathrm{Fam}_T \to \mathrm{Fam}_S$ for each morphism $f\colon S \to T$. But what should be the compatibility for a composition $S \xrightarrow{f} T \xrightarrow{g} U$ of schemes? Well, there should be an isomorphism of functors (i.e. a 2-morphism) $\mu_{f,g}\colon (f^* \circ g^*) \xrightarrow{\sim} (g \circ f)^*$. Should the isomorphisms $\mu_{f,g}$ satisfy a compatibility condition under triples $S \xrightarrow{f} T \xrightarrow{g} U \xrightarrow{h} V$? Yes, but we won't spell it out here (although we encourage the reader to work it out). Altogether this leads to the concept of a *pseudo-functor* (see [SP, Tag 003N]). We will take another approach however in specifying prestacks that avoids specifying such compatibility data.

0.7.2 Motivating the definition of a prestack

Instead of trying to define an assignment $T \mapsto \mathrm{Fam}_T$, we will build one massive category $\mathcal{X}$ encoding all of the groupoids Fam_T which will live over the category Sch of schemes. Loosely speaking, the objects of $\mathcal{X}$ will be a family a of objects over a scheme S, i.e. $a \in \mathrm{Fam}_S$. If $a \in \mathrm{Fam}_S$ and $b \in \mathrm{Fam}_T$, a morphism $a \to b$ in $\mathcal{X}$ will be a morphism $f\colon S \to T$ together with an isomorphism $a \xrightarrow{\sim} f^*b$.

A *prestack* over Sch is a category $\mathcal{X}$ together with functor $p\colon \mathcal{X} \to \mathrm{Sch}$, which we visualize as

$$\begin{array}{ccc} \mathcal{X} & & a & \xrightarrow{\alpha} & b \\ \downarrow{\scriptstyle p} & & \downarrow & & \downarrow \\ \mathrm{Sch} & & S & \xrightarrow{f} & T \end{array}$$

where the lower case letters a, b are objects in $\mathcal{X}$ and the upper case letters S, T are objects in Sch. We say that a is over S and $\alpha\colon a \to b$ is over $f\colon S \to T$. Moreover, we need to require certain natural axioms to hold for $\mathcal{X} \xrightarrow{p} \mathrm{Sch}$. This will be given in full later but vaguely we need to require the existence and uniqueness of pullbacks: given a map $S \to T$ and object $b \in \mathcal{X}$ over T, there should exist an arrow $a \xrightarrow{\alpha} b$ over f satisfying a suitable universal property. See Definition 1.3.1 for a precise definition.

Given a scheme S, the *fiber category* $\mathcal{X}(S)$ is the category of objects over S whose morphisms are over id_S. If $\mathcal{X}$ is built from the groupoids Fam_S as above, then the fiber category $\mathcal{X}(S) = \mathrm{Fam}_S$.

Example 0.7.1 (Viewing a moduli functor as a moduli prestack)**.** A moduli functor $F\colon \mathrm{Sch} \to \mathrm{Sets}$ can be encoded as a moduli prestack as follows: we define the category $\mathcal{X}_F$ of pairs (S, a) where S is a scheme and $a \in F(S)$. A map $(S', a) \to (S, a)$ is a map $f\colon S' \to S$ such that $a' = f^*a$, where f^* is convenient shorthand for $F(f)\colon F(S) \to F(S')$. Observe that the fiber categories $\mathcal{X}_F(S)$ are equivalent (even equal) to the set $F(S)$.

Example 0.7.2 (Moduli prestack of smooth curves)**.** We define the *moduli prestack of smooth curves* as the category $\mathcal{M}_g$ of families of smooth curves $\mathcal{C} \to S$ together with the functor $p\colon \mathcal{M}_g \to \mathrm{Sch}$ where $(\mathcal{C} \to S) \mapsto S$. A map $(\mathcal{C}' \to S') \to$

$(\mathcal{C} \to S)$ is the data of maps $\alpha\colon \mathcal{C}' \to \mathcal{C}$ and $f\colon S' \to S$ such that the diagram

$$\begin{array}{ccc} \mathcal{C}' & \xrightarrow{\alpha} & \mathcal{C} \\ \downarrow & \square & \downarrow \\ S' & \xrightarrow{f} & S \end{array}$$

is cartesian.

Example 0.7.3 (Moduli prestack of vector bundles)**.** Let C be a fixed smooth, connected and projective curve over $\mathbb{C}$, and fix integers $r \geq 0$ and d. We define the *moduli prestack of vector bundles on* C as the category $\mathcal{B}\mathrm{un}_{r,d}(C)$ of pairs (E, S) where S is a scheme and E is a vector bundle on $C_S = C \times_{\mathbb{C}} S$ together with the functor $p\colon \mathcal{B}\mathrm{un}_{r,d}(C) \to \mathrm{Sch}/\mathbb{C}, (E, S) \mapsto S$. A map $(E', S') \to (E, S)$ consists of a map of schemes $f\colon S' \to S$ together with a map $E \to (\mathrm{id} \times f)_* E'$ of $\mathcal{O}_{C_S}$-modules whose adjoint is an isomorphism (i.e. for any choice of pull back $(\mathrm{id} \times f)^* E$, the adjoint map $(\mathrm{id} \times f)^* E \to E'$ is an isomorphism). Note that a map $(E', S) \to (E, S)$ over the identity map id_S consists simply of an isomorphism $E' \to E$.

Remark 0.7.4. We have formulated morphisms using the adjoint because the pull back is only defined up to isomorphism while the pushforward is canonical. If we were to instead parameterize the total spaces of vector bundles (i.e. $\mathbb{A}(E)$ rather than E), then a morphism $(V', S') \to (V, S)$ would consist of morphisms $\alpha\colon V' \to V$ and $f\colon S' \to S$ such that $V' \to V \times_{C_S} C_{S'}$ is an isomorphism of vector bundles.

0.7.3 Motivating the definition of a stack

A stack is to a prestack as a sheaf is to a presheaf. The concept could not be more intuitive: we require that objects and morphisms glue uniquely.

Example 0.7.5 (Moduli stack of sheaves over a point)**.** Define the category $\mathcal{X}$ over Sch of pairs (E, S) where E is a sheaf of abelian groups on a scheme S, and the functor $p\colon \mathcal{X} \to \mathrm{Sch}$ given by $(E, S) \mapsto S$. A map $(E', S') \to (E, S)$ in $\mathcal{X}$ is a map of schemes $f\colon S' \to S$ together with a map $E \to f_* E'$ of $\mathcal{O}_{S'}$-modules whose adjoint is an isomorphism.

You already know that morphisms of sheaves glue [Har77, Exercise II.1.15]: let E and F be sheaves on schemes S and T, and let $f\colon S \to T$ be a map. If $\{S_i\}$ is a Zariski-open cover of S, then giving a morphism $\alpha\colon (E, S) \to (F, T)$ is the same data as giving morphisms $\alpha_i\colon (E|_{S_i}, S_i) \to (F, T)$ such that $\alpha_i|_{S_{ij}} = \alpha_j|_{S_{ij}}$.

You also know how sheaves themselves glue [Har77, Exercise II.1.22]—it is more complicated than gluing morphisms since sheaves have automorphisms and given two sheaves, we prefer to say that they are isomorphic rather than equal. If $\{S_i\}$ is a Zariski-open cover of a scheme S, then giving a sheaf E on S is equivalent to giving a sheaf E_i on S_i and isomorphisms $\phi_{ij}\colon E_i|_{S_{ij}} \to E_j|_{S_{ij}}$ such that $\phi_{ik} = \phi_{jk} \circ \phi_{ij}$ on the triple intersection S_{ijk}.

In an identical way, we could have considered the moduli stack of $\mathcal{O}$-modules, quasi-coherent sheaves or vector bundles.

The definition of a stack simply axiomitizes these two natural gluing concepts; it is postponed until Definition 1.4.1.

Exercise 0.7.6. Convince yourself that Examples 0.7.2 and 0.7.3 satisfy the same gluing axioms. (See also Propositions 1.4.6 and 1.4.8.)

0.7.4 Motivating the definition of an algebraic stack

There are functors $F\colon \mathrm{Sch} \to \mathrm{Sets}$ that are sheaves when restricted to the Zariski topology on any scheme T but that are not necessarily representable by schemes; see for instance Examples 2.9.1 and 2.9.2. In a similar way, there are prestacks $\mathcal{X}$ that are stacks but that are not sufficiently algebro-geometric. If we wish to bring our algebraic geometry toolkit (e.g. coherent sheaves, commutative algebra, cohomology, ...) to study stacks in a similar way that we study schemes, we must impose an algebraicity condition.

The condition we impose on a stack to be algebraic is very natural. Recall that a functor $F\colon \mathrm{Sch} \to \mathrm{Sets}$ is representable by a scheme if and only if there is a Zariski-open cover $\{U_i \subset F\}$ such that U_i is an affine scheme. Similarly, we will say that a stack $\mathcal{X} \to \mathrm{Sch}$ is *algebraic* if

- there is a smooth cover $\{U_i \to \mathcal{X}\}$ where each U_i is an affine scheme.

To make this precise, we need to define what it means for $\{U_i \to \mathcal{X}\}$ to be a smooth cover. Just like in the definition of Zariski-open cover (Definition 0.4.17(3)), we require that for every morphism $T \to \mathcal{X}$ from a scheme T, the fiber product (fiber products of prestacks will be formally introduced in §1.3.5) $U_i \times_{\mathcal{X}} T$ is representable (by an algebraic space) such that $\bigsqcup_i U_i \times_{\mathcal{X}} T \to T$ is a smooth and surjective morphism. See Definition 2.1.5 for the precise definition of an algebraic stack.

Constructing a smooth cover of a given moduli stack is a geometric problem inherent to the moduli problem. It can often be solved by ridigifying the moduli problem by parameterizing additional information. This concept is best absorbed in examples.

Example 0.7.7 (Moduli stack of elliptic curves)**.** An elliptic curve (E, p) over $\mathbb{C}$ is embedded into $\mathbb{P}^2$ via $\mathcal{O}_E(3p)$ such that E is defined by a Weierstrass equation $y^2z = x(x-z)(x-\lambda z)$ for some $\lambda \neq 0, 1$ [Har77, Prop. 4.6]. Let $U = \mathbb{A}^1 \setminus \{0, 1\}$ with coordinate λ. The family $\mathcal{E} \subset U \times \mathbb{P}^2$ of elliptic curves defined by the Weierstrass equation gives a smooth (even étale) cover $U \to \mathcal{M}_{1,1}$.

Example 0.7.8 (Moduli stack of smooth curves)**.** For any smooth, connected and projective curve C of genus $g \geq 2$, the third tensor power $\omega_C^{\otimes 3}$ is very ample and gives an embedding $C \hookrightarrow \mathbb{P}(H^0(C, \omega_c^{\otimes 3})) \cong \mathbb{P}^{5g-6}$. There is a Hilbert scheme H parameterizing closed subschemes of $\mathbb{P}^{5g-6}$ with the same Hilbert polynomial as $C \subset \mathbb{P}^{5g-6}$, and there is a locally closed subscheme $H' \subset H$ parameterizing *smooth* subschemes such that $\omega_C^{\otimes 3} \cong \mathcal{O}_C(1)$. The universal subscheme over H' yields a smooth cover $H' \to \mathcal{M}_g$.

Example 0.7.9 (Moduli stack of vector bundles)**.** For any vector bundle E of rank r and degree d on a smooth, connected and projective curve C, the twist $E(m)$ is globally generated for sufficiently large m. Taking $N = h^0(C, E(m))$, we can view E as a quotient $\mathcal{O}_C(-m)^N \twoheadrightarrow E$. There is a Quot scheme Q_m parameterizing quotients $\mathcal{O}_C(-m)^N \xrightarrow{\pi} F$ with the same Hilbert polynomial as E and a locally closed subscheme $Q'_m \subset Q$ parameterizing quotients where E is a vector bundle and such that the induced map $H^0(\pi \otimes \mathcal{O}_C(m))\colon \mathbb{C}^N \to H^0(C, E(m))$

is an isomorphism. The universal quotient over Q'_m defines a smooth map $Q'_m \to \mathcal{B}\mathrm{un}_{r,d}(C)$ and the collection $\{Q'_m \to \mathcal{B}\mathrm{un}_{r,d}(C)\}$ over $m \gg 0$ defines a smooth cover.

0.7.5 Deligne–Mumford stacks and algebraic spaces

A *Deligne–Mumford stack* can be defined in two equivalent ways:

- a stack $\mathcal{X}$ such that there exists an étale (rather than smooth) cover $\{U_i \to \mathcal{X}\}$ by schemes; or
- an algebraic stack such that all automorphisms groups of field-valued points are étale, i.e. discrete (e.g. finite) and reduced.

The moduli stacks $\mathcal{M}_g$ and $\overline{\mathcal{M}}_g$ are Deligne–Mumford for $g \geq 2$, but $\mathcal{B}\mathrm{un}_{r,d}(C)$ is not. Similarly, an *algebraic space* can be defined in two equivalent ways:

- a sheaf (i.e. a contravariant functor $F\colon \mathrm{Sch} \to \mathrm{Sets}$ that is a sheaf in the big étale topology) such that there exists an étale cover $\{U_i \to F\}$ by schemes; or
- an algebraic stack such that all automorphisms groups of field-valued points are trivial.

In other words, an algebraic space is an algebraic stack without any stackiness.

Table 1: Schemes, algebraic spaces, Deligne–Mumford stacks, and algebraic stacks are obtained by gluing affine schemes in certain topologies

Algebro-geometric space	Type of object	Obtained by gluing
Schemes	sheaf	affine schemes in the Zariski topology
Algebraic spaces	sheaf	affine schemes in the étale topology
Deligne–Mumford stacks	stack	affine schemes in the étale topology
Algebraic stacks	stack	affine schemes in the smooth topology

Example 0.7.10 (Quotients by finite groups)**.** Quotients by free actions of finite groups exist as algebraic spaces! See Corollary 2.1.9.

0.8 Moduli stacks and quotients

One of the most important examples of a stack is a quotient stack $[X/G]$ arising from an action of a smooth algebraic group G on a scheme X. The geometry of $[X/G]$ couldn't be simpler: it's the G-equivariant geometry of X.

Similar to how toric varieties provide concrete examples of schemes, quotient stacks provide both concrete examples useful to gain geometric intuition of general algebraic stacks and a fertile testing ground for conjectural results. On the other hand, it turns out that many algebraic stacks are quotient stacks (or at least locally quotient stacks) and therefore any (local) property that holds for quotient stacks also holds for many algebraic stacks.

0.8.1 Motivating the definition of the quotient stack

The quotient functor Sch $\to$ Sets defined by $S \mapsto X(S)/G(S)$ is not a sheaf even when the action is free (see Example 0.4.31). We therefore first need to consider a better notion for a family of orbits.

For simplicity, let's assume that G and X are defined over $\mathbb{C}$. For $x \in X(\mathbb{C})$, there is a G-equivariant map $\sigma_x \colon G \to X$ defined by $g \mapsto g \cdot x$. Note that two points x, x' are in the same G-orbit (say $x = hx'$), if and only if there is a G-equivariant morphism $\varphi \colon G \to G$ (say by $g \mapsto gh$) such that $\sigma_x = \sigma_{x'} \circ \varphi$.

We can try the same thing for a T-point $T \xrightarrow{f} X$ by considering

$$\begin{array}{ccc} G \times T & \xrightarrow{f} & X, \\ \downarrow{\scriptstyle p_2} & & \\ T & & \end{array} \qquad (g,t) \longmapsto g \cdot f(t)$$

and noting that $f \colon G \times T \to X$ is a G-equivariant map. If we define a prestack consisting of such families, it fails to be a stack as objects don't glue: given a Zariski-cover $\{T_i\}$ of T, maps $T_i \xrightarrow{f_i} X$ and isomorphisms of the restrictions to T_{ij}, the trivial bundles $G \times T_i \to T_i$ will glue to a G-torsor $P \to T$ but it will *not* necessarily be trivial (i.e. $P \cong G \times T$). It is clear then how to correct this using the language of G-torsors (see Section C.3):

Definition 0.8.1 (Quotient stack)**.** We define $[X/G]$ as the category over Sch whose objects over a scheme S are diagrams

$$\begin{array}{ccc} P & \xrightarrow{f} & X \\ \downarrow & & \\ S & & \end{array}$$

where $P \to S$ is a G-torsor and $f \colon P \to X$ is a G-equivariant morphism. A morphism $(P' \to S', P' \xrightarrow{f'} X) \to (P \to S, P \xrightarrow{f} X)$ consists a maps $g \colon S' \to S$ and $\varphi \colon P' \to P$ of schemes such that the diagram

$$\begin{array}{ccccc} & & f' & & \\ & \overset{\varphi}{\nearrow} & & \overset{f}{\searrow} & \\ P' & \longrightarrow & P & \longrightarrow & X \\ \downarrow & \square & \downarrow & & \\ S' & \xrightarrow{g} & S & & \end{array}$$

commutes with the left square cartesian.

There is an object of $[X/G]$ over X given by the diagram

$$\begin{array}{ccc} G \times X & \xrightarrow{\sigma} & X \\ \downarrow{\scriptstyle p_2} & & \\ X, & & \end{array}$$

where σ denotes the action map. This corresponds to a map $X \to [X/G]$ via a 2-categorical version of Yoneda's lemma.

The map $X \to [X/G]$ is a G-torsor even if the action of G on X is not free. We state that again: **the map $X \to [X/G]$ is a G-torsor even if the action of G on X is not free**. Pause for a moment to appreciate how remarkable that is!

In particular, the map $X \to [X/G]$ is smooth and it follows that $[X/G]$ is algebraic. At the expense of enlarging our category from schemes to algebraic stacks, we are able to (tautologically) construct the quotient $[X/G]$ as a 'geometric space' with desirable geometric properties.

Example 0.8.2. Specializing to the case that $X = \operatorname{Spec} \mathbb{C}$ is a point, we define the *classifying stack of* G as the category $BG := [\operatorname{Spec} \mathbb{C}/G]$ of G-torsors $P \to S$. The projection $\operatorname{Spec} \mathbb{C} \to BG$ is not only a G-torsor; it is the *universal G-torsor*. Given any other G-torsor $P \to S$, there is a unique map $S \to BG$ and a cartesian diagram

$$\begin{array}{ccc} P & \longrightarrow & \operatorname{Spec} \mathbb{C} \\ \downarrow & \square & \downarrow \\ S & \longrightarrow & BG. \end{array}$$

Exercise 0.8.3. What is the universal family over the quotient stack $[X/G]$?

0.8.2 Moduli as quotient stacks

Moduli stacks can often be described as quotient stacks, and these descriptions can be leveraged to establish properties of the moduli stack.

Example 0.8.4 (Moduli stack of smooth curves)**.** In Example 0.7.8, the embedding of a smooth curve C via $C \overset{|\omega_C^{\otimes 3}|}{\hookrightarrow} \mathbb{P}^{5g-6}$ depends on a choice of basis $H^0(C, \omega_C^{\otimes 3}) \cong \mathbb{C}^{5g-5}$ and therefore is only unique up to a projective automorphism, i.e. an element of $\mathrm{PGL}_{5g-5} = \operatorname{Aut}(\mathbb{P}^{5g-6})$. The action of the algebraic group PGL_{5g-5} on the scheme H', parameterizing *smooth* subschemes such that $\omega_C \cong \mathcal{O}_C(3)$, yields an identification $\mathcal{M}_g \cong [H'/\operatorname{PGL}_{5g-6}]$. See Theorem 2.1.11.

Example 0.8.5 (Moduli stack of vector bundles)**.** In Example 0.7.9, the presentation of a vector bundle E as a quotient $\mathcal{O}_C(-m)^N \twoheadrightarrow E$ depends on a choice of basis $H^0(C, E(m)) \cong \mathbb{C}^N$. The algebraic group PGL_{N-1} acts on the scheme Q'_m, parameterizing vector bundle quotients of $\mathcal{O}_C(-m)^N$ such that $\mathbb{C}^N \xrightarrow{\sim} H^0(C, E(m))$, yields an identification $\mathcal{B}\mathrm{un}_{r,d}(C) \cong \bigcup_{m \gg 0} [Q'_m/\operatorname{PGL}_{N-1}]$. See Theorem 2.1.15.

0.8.3 Geometry of $[X/G]$

While the definition of the quotient stack $[X/G]$ may appear abstract, its geometry is very familiar. The table below provides a dictionary between the geometry of a

quotient stack $[X/G]$ and the G-equivariant geometry of X. The stack-theoretic concepts on the left-hand side will be introduced later. For simplicity we work over $\mathbb{C}$.

Table 2: Dictionary

Geometry of $[X/G]$	G-equivariant geometry of X
$\mathbb{C}$-point $x \in [X/G]$	orbit Gx
automorphism group $\mathrm{Aut}(x)$	stabilizer G_x
function $f \in \Gamma([X/G], \mathcal{O}_{[X/G]})$	G-equivariant function $f \in \Gamma(X, \mathcal{O}_X)^G$
map $[X/G] \to Y$ to a scheme Y	G-equivariant map $X \to Y$
line bundle	G-equivariant line bundle (or linearization)
quasi-coherent sheaf	G-equivariant quasi-coherent sheaf
tangent space $T_{[X/G],x}$	normal space $T_{X,x}/T_{Gx,x}$ to the orbit
coarse moduli space $[X/G] \to Y$	geometric quotient $X \to Y$
good moduli space $[X/G] \to Y$	good GIT quotient $X \to Y$

0.9 Constructing moduli spaces as projective varieties

One of the primary reasons for introducing algebraic stacks to begin with is to ensure that a given moduli problem $\mathcal{M}$ is in fact represented by a bona fide algebro-geometric space equipped with a universal family. Many geometric questions can be answered (and arguably should be answered) by studying the moduli stack $\mathcal{M}$ itself. However, even in the presence of automorphisms, there still may exist a scheme—even a projective variety— that closely approximates the moduli problem. If we are willing to sacrifice some desirable properties (e.g. a universal family), we can sometimes construct a more familiar algebro-geometric space—namely a projective variety—where we have the much larger toolkit of projective geometry (e.g., Hodge theory, birational geometry, intersection theory, ...) at our disposal.

In this section, we present a general strategy for a constructing a moduli space specifically as a *projective variety*.

0.9.1 Boundedness

The first potential problem is that our moduli problem may simply have too many objects so that there is no hope of representing it by a *finite type* or *quasi-compact* scheme. We say that a moduli functor or stack $\mathcal{M}$ over $\mathbb{C}$ is *bounded* if there exists a scheme X of finite type over $\mathbb{C}$ and a family of objects $\mathcal{E}$ over X such that every object E of $\mathcal{M}$ is isomorphic to a fiber $E \cong \mathcal{E}_x$ for some (not necessarily unique) $x \in X(\mathbb{C})$.

Example 0.9.1. Let Vect be the algebraic stack over $\mathbb{C}$ where objects over a scheme S consist of vector bundles. Since we have not specified the rank, $\mathrm{Vect}_{\mathbb{C}}$ is not bounded. In fact, if we let $\mathrm{Vect}_r \subset \mathrm{Vect}$ be the substack parameterizing vector bundles of rank r, then $\mathrm{Vect} = \bigsqcup_{r\geq 0} \mathrm{Vect}_r$ While Vect is locally of finite type over $\mathbb{C}$, it is not of finite type (or equivalently quasi-compact).

Exercise 0.9.2. Show that Vect_r is isomorphic to the classifying stack $B\,\mathrm{GL}_r$ (Example 0.8.2).

Example 0.9.3. Let $\mathcal{V}$ be the stack of *all* vector bundles over a smooth, connected and projective curve C. The stack $\mathcal{V}$ is clearly not bounded since we haven't specified the rank and degree. But even the substack $\mathcal{B}\mathrm{un}_{r,d}(C)$ of vector bundles with prescribed rank and degree is not bounded! For example, on $\mathbb{P}^1$, there are vector bundles $\mathcal{O}(-d) \oplus \mathcal{O}(d)$ of rank 2 and degree 0 for every $d \in \mathbb{Z}$, and not all of them can arise as the fibers of a single vector bundle on a finite type $\mathbb{C}$-scheme.

Exercise 0.9.4. Prove that $\mathcal{B}\mathrm{un}_{r,d}(C)$ is not bounded for any curve C.

Although $\mathcal{B}\mathrm{un}_{r,d}(C)$ is not bounded, we will study the substack $\mathcal{B}\mathrm{un}_{r,d}(C)^{\mathrm{ss}}$ of *semistable* vector bundles which is bounded. Semistable vector bundles admit a number of remarkable properties with boundedness being one of the most important.

0.9.2 Compactness

Projective varieties are compact so if we are going to have any hope to construct a projective moduli space, the moduli stack better be compact as well. However, many moduli stacks such as $\mathcal{M}_g$ are not compact as they don't have *enough* objects. This is in contrast to the issue of non-boundedness where there may be *too many* objects.

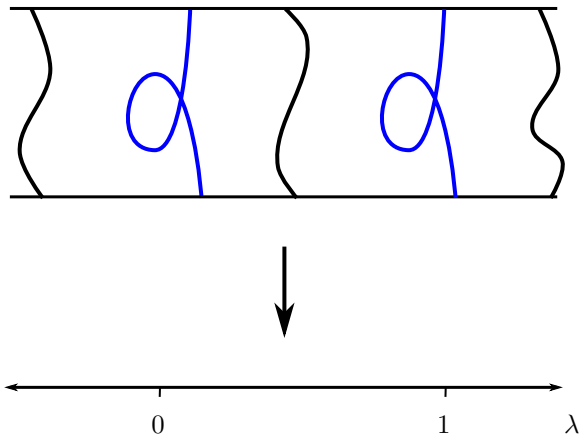


Figure 19: The family of elliptic curves $y^2z = x(x-z)(x-\lambda z)$ degenerates to the nodal cubic over $\lambda = 0, 1$.

The scheme-theoretic notion for compactness is *properness*—universally closed, separated and of finite type. There is a conceptual criterion to test properness called the *valuative criterion* which loosely speaking requires one-dimensional

limits to exist. The usefulness of the valuative criterion is arguably best witnessed through studying moduli problems.

More precisely, a moduli stack $\mathcal{M}$ of finite type over $\mathbb{C}$ is *proper* (resp. *universally closed*, *separated*) if for every DVR R with fraction field K and for any diagram

$$\begin{array}{ccc} \operatorname{Spec} K & \longrightarrow & \mathcal{M} \\ \downarrow & \nearrow & \\ \operatorname{Spec} R, & & \end{array} \tag{0.9.1}$$

after possibly allowing for an extension of R, there exists a unique extension (resp. there exists an extension, resp. there exists at most one extension) of the above diagram.[4] Since $\mathcal{M}$ is a moduli stack, a map $\operatorname{Spec} K \to \mathcal{M}$ corresponds to an object $E^\times$ over $\operatorname{Spec} K$ and a dotted arrow corresponds to a family of objects E over $\operatorname{Spec} R$ and an isomorphism $E|_{\operatorname{Spec} K} \cong E^\times$. In other words, properness of $\mathcal{M}$ means that every object E^* over the punctured disk $\operatorname{Spec} K$ extends uniquely (after possibly allowing for an extension of R) to a family E of objects over the entire disk $\operatorname{Spec} R$.

Example 0.9.5. The moduli stack $\mathcal{M}_g$ of smooth curves is not proper as exhibited in Figure 19. The pioneering insight of Deligne and Mumford is that there is a *moduli-theoretic compactification*! Namely, there is an algebraic stack $\overline{\mathcal{M}_g}$ parameterizing *Deligne–Mumford stable curves*, i.e. proper curves C with at worst nodal singularities such that any smooth rational subcurve $\mathbb{P}^1 \subset C$ intersects the rest of the curve along at least three points. The stack $\overline{\mathcal{M}}_g$ is a proper algebraic stack (due to the stable reduction theorem for curves) and contains $\mathcal{M}_g$ as an open substack.

Example 0.9.6. Let $\mathcal{B}\mathrm{un}_{r,d}(C)^{\mathrm{ss}}$ be the moduli stack parameterizing semistable vector bundles over a curve of prescribed rank and degree. We will later show that $\mathcal{B}\mathrm{un}_{r,d}(C)^{\mathrm{ss}}$ is an algebraic stack of finite type over $\mathbb{C}$. Langton's semistable reduction theorem states that $\mathcal{B}\mathrm{un}_{r,d}(C)^{\mathrm{ss}}$ is universally closed, i.e. satisfies the existence part of the above valuative criterion.

However $\mathcal{B}\mathrm{un}_{r,d}(C)^{\mathrm{ss}}$ is not separated as there may exist several non-isomorphic extensions of a vector bundle on C_K to C_R. Indeed, let E be vector bundle and consider the trivial family E_K on C_K. This extends to trivial family E_R over C_R but the data of an extension

$$\begin{array}{ccc} \operatorname{Spec} K & \xrightarrow{[E_K]} & \mathcal{B}\mathrm{un}_{r,d}(C)^{\mathrm{ss}} \\ \downarrow & \nearrow_{[E_R]} & \\ \operatorname{Spec} R, & & \end{array}$$

also consists of an isomorphism $E_R|_{C_K} = E_K \xrightarrow{\sim} E_K$ or equivalently a K-point of $\mathrm{Aut}(E)$. There are many such isomorphisms and some don't extend to R-points. The automorphism group of a vector bundle is a positive dimensional (affine) algebraic group containing a copy of $\mathbb{G}_m$ corresponding to scaling. For instance,

[4] The valuative criterion can be equivalently formulated by replacing the local curve $\operatorname{Spec} R$ with a smooth curve C and $\operatorname{Spec} K$ with a puncture curve $C \setminus p$.

if $\pi \in K$ is a uniformizing parameter, the automorphism $1/\pi \in \mathbb{G}_m(K)$ does not extend to $\mathbb{G}_m(R)$ so (E_R, id) and $(E_R, 1/\pi)$ give non-isomorphic extensions of E_K. In a similar way, any moduli stack which has an object with a positive dimensional affine automorphism group is not separated.

0.9.3 Enlarging a moduli stack

It is often useful to consider enlargements $\mathcal{X} \subset \mathcal{M}$ of a given moduli stack $\mathcal{X}$ by parameterizing a larger collection of objects. For instance, rather than just considering smooth or Deligne–Mumford stable curve, you could consider all curves, or rather than considering semistable vector bundles, you could consider all vector bundles or even all coherent sheaves.

Let's call an object of $\mathcal{M}$ *semistable* if is isomorphic to an object of $\mathcal{X}$; in this way, we can view $\mathcal{X} = \mathcal{M}^{\mathrm{ss}} \subset \mathcal{M}$ as the substack of semistable objects. Often it is easier to show properties (e.g. algebraicity) for $\mathcal{M}$ and then infer the corresponding property for $\mathcal{M}^{\mathrm{ss}}$.

0.9.4 The six steps toward projective moduli

In the setting of a moduli stack $\mathcal{M}^{\mathrm{ss}}$ of semistable objects and an enlargement $\mathcal{M}^{\mathrm{ss}} \subset \mathcal{M}$, we outline the steps to construct a projective moduli scheme M^{ss} approximating $\mathcal{M}^{\mathrm{ss}}$.[5]

Step 1 (Algebraicity): $\mathcal{M}$ is an algebraic stack locally of finite type over $\mathbb{C}$.

> This requires first defining $\mathcal{M}$ by specifying both (1) families of objects over an arbitrary $\mathbb{C}$-scheme S, (2) how two families are isomorphic, and (3) how families pull back; see Section 0.7.1. One must then check that $\mathcal{M}$ is a stack.
>
> To check that $\mathcal{M}$ is an algebraic stack locally of finite type over $\mathbb{C}$ entails finding a smooth cover of $\{U_i \to \mathcal{M}\}$ by affine schemes (see Section 0.7.4) where each U_i is of finite type over $\mathbb{C}$.
>
> An alternative approach is to verify 'Artin's criteria' for algebraicity which essentially amounts to verifying local properties of the moduli problem and in particular requires an understanding of the deformation and obstruction theory.

Step 2 (Openness of semistability): semistability is an open condition, i.e. $\mathcal{M}^{\mathrm{ss}} \subset \mathcal{M}$ is an open substack.

> If E is an object of $\mathcal{M}$ over T, one must show that the locus of points $t \in T$ such that the restriction E_t is semistable is an open subset of T. Indeed, just like in the definition of an open subfunctor, a substack $\mathcal{M}^{\mathrm{ss}} \subset \mathcal{M}$ is open if and only if for all maps $T \to \mathcal{M}$, the fiber product $\mathcal{M}_{\mathrm{ss}} \times_{\mathcal{M}} T$ is an open subscheme of T. This ensures in particular that $\mathcal{M}^{\mathrm{ss}}$ is also an algebraic stack locally of finite type over $\mathbb{C}$.

Step 3 (Boundedness of semistability): semistability is bounded, i.e. $\mathcal{M}^{\mathrm{ss}}$ is of finite type over $\mathbb{C}$.

[5] The calligraphic font $\mathcal{M}^{\mathrm{ss}}$ denotes an algebraic stack while the Roman font M^{ss} denotes an algebraic space. This notation will be continued throughout the notes.

One must verify the existence of a scheme T of finite type over $\mathbb{C}$ and a family $\mathcal{E}$ of objects over T such that every semistable object $E \in \mathcal{M}^{ss}(\mathbb{C})$ appears as a fiber of $\mathcal{E}$; see Section 0.9.1. In other words, one must exhibit a morphism $U \to \mathcal{M}$ from a scheme U of finite type whose image contains $\mathcal{M}^{ss}$. It is worth noting that since we already know $\mathcal{M}$ is locally of finite type, the finite typeness of $\mathcal{M}$ is equivalent to quasi-compactness; *boundedness* is casual term often used to refer to this property.

Step 4 (Existence of coarse/good moduli space): there exists either a coarse or good moduli space $\mathcal{M}^{ss} \to M^{ss}$ where M^{ss} is a separated algebraic space.

The algebraic space M^{ss} can be viewed as the best possible approximation of $\mathcal{M}^{ss}$ which is an algebraic space. If automorphisms are finite and $\mathcal{M}^{ss}$ is a proper Deligne–Mumford stack, the Keel–Mori theorem ensures that there exists a *coarse moduli space* $\pi\colon \mathcal{M}^{ss} \to M^{ss}$ with M^{ss} proper; this means that (1) π is universal for maps to algebraic spaces and (2) π induces a bijection between the isomorphism classes of $\mathbb{C}$-points of $\mathcal{M}^{ss}$ and the $\mathbb{C}$-points of M^{ss}.

In the case of infinite automorphisms, we often cannot expect the existence of a coarse moduli space (as defined above) and we therefore relax the notion to a *good moduli space* $\pi\colon \mathcal{M}^{ss} \to M^{ss}$ which may identify non-isomorphic objects. In fact, it identifies precisely the $\mathbb{C}$-points whose closures in $\mathcal{M}^{ss}$ intersect in an analogous way to the orbit closure equivalence relation in GIT. A good moduli space is also universal for maps to algebraic spaces even if this property is not obvious from the definitions. We will use an analogue of the Keel–Mori theorem which ensures the existence of a proper good moduli space as long as $\mathcal{M}^{ss}$ can be verified to be both 'S-complete' and 'Θ-reductive'.

Step 5 (Semistable reduction): $\mathcal{M}^{ss}$ is universally closed, i.e. satisfies the existence part of the valuative criterion for properness.

This requires checking that any family of objects $E^{\times}$ over a punctured DVR or smooth curve $C^{\times} = C \setminus p$ has at least one extension to a family of objects over C after possibly taking an extension of C; see Section 0.9.2. For moduli problems with finite automorphisms, the uniqueness of the extension can usually be verified, which implies the properness of $\mathcal{M}$. For moduli problems with infinite affine automorphism groups, the extension is never unique. While $\mathcal{M}$ is therefore not separated, you can often still verify a condition called 'S-completeness', which enjoys properties analogous to separatedness. This property is often referred to as *stable or semistable reduction*.

As a consequence, we conclude that M^{ss} is a proper algebraic space.

Step 6 (Projectivity): a tautological line bundle on $\mathcal{M}^{ss}$ descends to an ample line bundle on M^{ss}.

This is often the most challenging step in this process. It requires a solid understanding of the geometry of the moduli problem and often relies on techniques in higher dimensional geometry.

0.9.5 An alternative approach using Geometric Invariant Theory

The approach outlined above is by no means the only way to construct moduli spaces. One alternative approach is Mumford's Geometric Invariant Theory, which has been wildly successful in both constructing and studying moduli spaces. The main idea is to rigidify the moduli stack $\mathcal{M}^{\mathrm{ss}}$ (e.g. $\overline{\mathcal{M}}_g$) by parameterizing additional data (e.g. a stable curve C *and* an embedding $C \overset{|\omega_C^{\otimes 3}|}{\hookrightarrow} \mathbb{P}^N$) in such way that it represented by a projective scheme X and such that the different choices of additional data correspond to different orbits for the action of an algebraic group G acting on X. This provides an identification of the moduli stack $\mathcal{M}^{\mathrm{ss}}$ as an open substack of the quotient stack $[X/G]$. Given a *choice* of equivariant embedding $X \hookrightarrow \mathbb{P}^n$, GIT constructs the quotient as the projective variety

$$X/\!\!/G := \operatorname{Proj} \bigoplus_{d \geq 0} \Gamma(X, \mathcal{O}(d))^G$$

The rational map $X \dashrightarrow X/\!\!/G$ is defined on an open subscheme X^{ss}, which we call the *GIT semistable locus*. To make this procedure work (and this is the hard part!), one must show that an element $x \in X$ is GIT semistable if and only if the corresponding object of $[X/G]$ is semistable (i.e. is in $\mathcal{M}^{\mathrm{ss}}$).

One of the striking features of GIT is that it handles all six steps at once and in particular constructs the moduli space as a projective variety. Moreover, if we do not know a priori how to compactify a moduli problem, GIT can sometimes tell you how.

Example 0.9.7 (Deligne–Mumford stable curves)**.** Using the quotient presentation $\overline{\mathcal{M}}_g = [H'/\operatorname{PGL}_{5g-6}]$ of Example 0.8.4, the closure $\overline{H}'$ of H' in the Hilbert scheme inherits an action of $\operatorname{PGL}_{5g-6}$ and one must show than an element in H' is GIT semistable if and only if the corresponding curve is Deligne–Mumford stable.

Example 0.9.8 (Semistable vector bundles)**.** Using the quotient presentation $\mathcal{B}\mathrm{un}_{r,d}(C)^{\mathrm{ss}} = [Q'_m/\operatorname{PGL}_{N-1}]$ of Example 0.8.5, the closure $\overline{Q}'_m$ has a PGL_{N-1}-action and one must show that an element in $\overline{Q}'_m$ is GIT semistable if and only if the corresponding quotient is semistable.

0.9.6 Trichotomy of moduli spaces

Table 3: The trichotomy of moduli

	No Auts	Finite Auts	Infinite Auts
Type of space	Algebraic variety / space	Deligne–Mumford stack	algebraic stack
Defining property	Zariski/étale-locally an affine scheme	étale-locally an affine scheme	smooth-locally an affine scheme
Examples	$\mathbb{P}^n$, $\mathrm{Gr}(k,n)$, Hilb, Quot	$\mathcal{M}_g$	$\mathcal{B}\mathrm{un}_{r,d}(C)$
Quotient stacks $[X/G]$	action is free	finite stabilizers	any action
Existence of moduli varieties / spaces	already an algebraic variety/space	coarse moduli space	good moduli space

Notes

For a more detailed exposition of the moduli stack of triangles, we recommend Behrend's notes [Beh14].

Chapter 1

Sites, sheaves and stacks

1.1 Grothendieck topologies and sites

We would like to consider a topology on a scheme where étale morphisms are the open sets. This doesn't make sense using the conventional notion of a topological space so we simply adapt our definitions.

Definition 1.1.1. A *Grothendieck topology* on a category $\mathcal{S}$ consists of the following data: for each object $X \in \mathcal{S}$, there is a set $\mathrm{Cov}(X)$ consisting of *coverings* of X, i.e. collections of morphisms $\{X_i \to X\}_{i\in I}$ in $\mathcal{S}$. We require that:

(1) (identity) If $X' \to X$ is an isomorphism, then $(X' \to X) \in \mathrm{Cov}(X)$.

(2) (restriction) If $\{X_i \to X\}_{i\in I} \in \mathrm{Cov}(X)$ and $Y \to X$ is any morphism, then the fiber products $X_i \times_X Y$ exist in $\mathcal{S}$ and the collection $\{X_i \times_X Y \to Y\}_{i\in I} \in \mathrm{Cov}(Y)$.

(3) (composition) If $\{X_i \to X\}_{i\in I} \in \mathrm{Cov}(X)$ and $\{X_{ij} \to X_i\}_{j\in J_i} \in \mathrm{Cov}(X_i)$ for each $i \in I$, then $\{X_{ij} \to X_i \to X\}_{i\in I, j\in J_i} \in \mathrm{Cov}(X)$.

A *site* is a category $\mathcal{S}$ with a Grothendieck topology.

Example 1.1.2 (Topological spaces)**.** If X is a topological space, let $\mathrm{Op}(X)$ denote the category of open sets $U \subset X$ where there is a unique morphism $U \to V$ if $U \subset V$ and no other morphisms. We say that a covering of U (i.e. an element of $\mathrm{Cov}(U)$) is a collection of open immersions $\{U_i \to U\}_{i\in I}$ such that $U = \bigcup_{i\in I} U_i$. This defines a Grothendieck topology on $\mathrm{Op}(X)$.

In particular, if X is a scheme, the Zariski-topology on X yields a site, which we refer to as the *small Zariski site on* X.

Example 1.1.3 (Small étale site)**.** If X is a scheme, the *small étale site on* X is the category $X_{\text{ét}}$ of étale morphisms $U \to X$ such that a morphism $(U \to X) \to (V \to X)$ is simply an X-morphism $U \to V$ (which is necessarily étale). In other words, $X_{\text{ét}}$ is the full subcategory of Sch/X consisting of schemes étale over X. A covering of an object $(U \to X) \in X_{\text{ét}}$ is a collection of étale morphisms $\{U_i \to U\}$ such that $\bigsqcup_i U_i \to U$ is surjective.

Example 1.1.4 (Big Zariski and étale sites)**.** The *big Zariski site* (resp. *big étale site*) is the category Sch where a covering of a scheme U is a collection of open immersions (resp. étale morphisms) $\{U_i \to U\}$ in Sch such that $\bigsqcup_i U_i \to U$ is surjective. We denote these sites as $\mathrm{Sch}_{\mathrm{Zar}}$ and $\mathrm{Sch}_{\text{Ét}}$.

Example 1.1.5 (Localized categories and sites)**.** If $\mathcal{S}$ is a category and $S \in \mathcal{S}$, define the category $\mathcal{S}/S$ whose objects are maps $T \to S$ in $\mathcal{S}$. A morphism $(T' \to S) \to (T \to S)$ is a map $T' \to T$ over S. If $\mathcal{S}$ is a site, $\mathcal{S}/S$ is also a site where a covering of $T \to S$ in $\mathcal{S}/S$ is a covering $\{T_i \to T\}$ in $\mathcal{S}$.

Applying this construction for a scheme S yields the big Zariski and étale sites $(\mathrm{Sch}\,/S)_{\mathrm{Zar}}$ and $(\mathrm{Sch}\,/S)_{\text{Ét}}$ over a scheme S.

Replacing étale morphisms with other properties of morphisms yields other sites.

1.2 Presheaves and sheaves

Recall that if X is a topological space, a presheaf of sets on X is simply a contravariant functor $F\colon \mathrm{Op}(X) \to \mathrm{Sets}$ on the category $\mathrm{Op}(X)$ of open sets. The sheaf axiom translates succinctly into the condition that for each covering $U = \bigcup_i U_i$, the sequence

$$F(U) \to \prod_i F(U_i) \rightrightarrows \prod_{i,j} F(U_i \cap U_j)$$

is exact (i.e. is an equalizer diagram), where the two maps $F(U_i) \rightrightarrows F(U_i \cap U_j)$ are induced by the two inclusions $U_i \cap U_j \subset U_i$ and $U_i \cap U_j \subset U_j$. Also note that the intersections $U_i \cap U_j$ can also be viewed as fiber products $U_i \times_X U_j$.

1.2.1 Definitions

Definition 1.2.1. A *presheaf* on a category $\mathcal{S}$ is a contravariant functor $\mathcal{S} \to \mathrm{Sets}$.

Remark 1.2.2. If $F\colon \mathcal{S} \to \mathrm{Sets}$ is a presheaf and $S \xrightarrow{f} T$ is a map in $\mathcal{S}$, then the pullback $F(f)(b)$ of an element $b \in F(T)$ is sometimes denoted as f^*b or $b|_S$.

Definition 1.2.3. A *sheaf* on a site $\mathcal{S}$ is a presheaf $F\colon \mathcal{S} \to \mathrm{Sets}$ such that for every object S and covering $\{S_i \to S\} \in \mathrm{Cov}(S)$, the sequence

$$F(S) \to \prod_i F(S_i) \rightrightarrows \prod_{i,j} F(S_i \times_S S_j) \tag{1.2.1}$$

is exact, where the two maps $F(S_i) \rightrightarrows F(S_i \times_S S_j)$ are induced by the two maps $S_i \times_S S_j \to S_i$ and $S_i \times_S S_j \to S_i$.

Remark 1.2.4. The exactness of (1.2.1) means that it is an equalizer diagram: $F(S)$ is precisely the subset of $\prod_i F(S_i)$ consisting of elements whose images under the two maps $F(S_i) \rightrightarrows F(S_i \times_S S_j)$ are equal.

Example 1.2.5 (Schemes are sheaves)**.** If X is a scheme, then $\mathrm{Mor}(-, X)\colon \mathrm{Sch} \to \mathrm{Sets}$ is a sheaf on $\mathrm{Sch}_{\text{Ét}}$ since morphisms glue uniquely in the étale topology. Indeed, Proposition B.2.1 implies that the sheaf axiom holds for a cover given by a single morphism $S' \to S$ which is étale and surjective. The sheaf axiom for an an étale covering $\{S_i \to S\}$ can be easily reduced to this case (see Exercise 1.2.6).

Similarly, if $X \to S$ is a morphism of schemes, then $\mathrm{Mor}_S(-, X)\colon \mathrm{Sch}\,/S \to \mathrm{Sets}$ is a sheaf on $(\mathrm{Sch}\,/S)_{\text{Ét}}$. We will abuse notation by using X and $X \to S$ to denote the sheaves $\mathrm{Mor}(-, X)$ and $\mathrm{Mor}_S(-, X)$.

Exercise 1.2.6. Let F be a presheaf on Sch.

(1) Show that F is a sheaf on $\mathrm{Sch}_{\text{Ét}}$ if and only if for every étale surjective morphism $S' \to S$ of schemes, the sequence $F(S) \to F(S') \rightrightarrows S' \times_S S'$ is exact.

(2) Show that F is a sheaf on $\mathrm{Sch}_{\text{Ét}}$ if and only if

- F is a sheaf in the big Zariski topology $\mathrm{Sch}_{\mathrm{Zar}}$; and
- or every étale surjective morphism $S' \to S$ of *affine* schemes, the sequence $F(S) \to F(S') \rightrightarrows F(S' \times_S S')$ is exact.

Exercise 1.2.7. If $X \to Y$ is a surjective smooth morphism of schemes, show that $X \to Y$ is an epimorphism of sheaves on $\mathrm{Sch}_{\text{Ét}}$.

1.2.2 Morphisms and fiber products

A *morphism* of presheaves or sheaves is by definition a natural transformation. By Yoneda's lemma (Lemma 0.4.1), if X is a scheme and F is a presheaf on Sch, a morphism $\alpha\colon X \to F$ (which we interpret as a morphism of presheaves $\mathrm{Mor}(-, X) \to F$) corresponds to an element in $F(X)$, which by abuse of notation we also denote by α.

Given morphisms $F \xrightarrow{\alpha} G$ and $G' \xrightarrow{\beta} G$ of presheaves on a category $\mathcal{S}$, consider the presheaf

$$\begin{aligned} &\mathcal{S} \to \mathrm{Sets} \\ &S \mapsto F(S) \times_{G(S)} G'(S) = \{(a, b) \in F(S) \times G'(S) \,|\, \alpha_S(a) = \beta_S(b)\}. \end{aligned} \tag{1.2.2}$$

Exercise 1.2.8.

(1) Show that that (1.2.2) is a fiber product $F \times_G G'$ in $\mathrm{Pre}(\mathcal{S})$. (This is a generalization of Exercise 0.4.16 but the same proof should work.)

(2) Show that if F, G and G' are sheaves on a site $\mathcal{S}$, then so is $F \times_G G'$. In particular, (1.2.2) is also a fiber product $F \times_G G'$ in $\mathrm{Sh}(\mathcal{S})$.

1.2.3 Sheafification

Theorem 1.2.9 (Sheafification)**.** *Let $\mathcal{S}$ be a site. The forgetful functor $\mathrm{Sh}(\mathcal{S}) \to \mathrm{Pre}(\mathcal{S})$ admits a left adjoint $F \mapsto F^{\mathrm{sh}}$, called the sheafification.*

Proof. A presheaf F on $\mathcal{S}$ is called *separated* if for every covering $\{S_i \to S\}$ of an object S, the map $F(S) \to \prod_i F(S_i)$ is injective (i.e. if sections glue, they glue uniquely). Let $\mathrm{Pre}^{\mathrm{sep}}(\mathcal{S})$ be the full subcategory of $\mathrm{Pre}(\mathcal{S})$ consisting of separated presheaves. We will construct left adjoints

$$\mathrm{Sh}(\mathcal{S}) \underset{\mathrm{sh}_2}{\hookleftarrow} \mathrm{Pre}^{\mathrm{sep}}(\mathcal{S}) \underset{\mathrm{sh}_1}{\hookleftarrow} \mathrm{Pre}(\mathcal{S}).$$

For $F \in \mathrm{Pre}(\mathcal{S})$, we define $\mathrm{sh}_1(F)$ by $S \mapsto F(S)/\sim$ where $a \sim b$ if there exists a covering $\{S_i \to S\}$ such that $a|_{S_i} = b|_{S_i}$ for all i.

For $F \in \mathrm{Pre}^{\mathrm{sep}}(\mathcal{S})$, we define $\mathrm{sh}_2(F)$ by

$$S \mapsto \left\{ (\{S_i \to S\}, \{a_i\}) \;\middle|\; \begin{array}{l} \text{where } \{S_i \to S\} \in \mathrm{Cov}(S) \text{ and } a_i \in F(S_i) \\ \text{such that } a_i|_{S_{ij}} = a_j|_{S_{ij}} \text{ for all } i, j \end{array} \right\} / \sim$$

where $(\{S_i \to S\}, \{a_i\}) \sim (\{S'_j \to S\}, \{a'_j\})$ if $a_i|_{S_i \times_S S'_j} = a'_j|_{S_i \times_S S'_j}$ for all i, j. The details are left to the reader. □

Remark 1.2.10 (Topos)**.** A *topos* is a category equivalent to the category of sheaves on a site. Two different sites may have equivalent categories of sheaves, and the topos can be viewed as a more fundamental invariant. While topoi are undoubtedly useful in moduli theory, they will not play a role in these notes.

1.3 Prestacks

In Section 0.7.1, we motivated the concept of a prestack on a category $\mathcal{S}$ as a generalization of a presheaf $\mathcal{S} \to \mathrm{Sets}$. By trying to keep track of automorphisms, we were naively led to consider a 'functor' $F\colon \mathcal{S} \to \mathrm{Groupoids}$ but decided instead to package this data into one large category $\mathcal{X}$ over $\mathcal{S}$ parameterizing pairs (a, S) where $S \in \mathcal{S}$ and $a \in F(S)$.

1.3.1 Definition of a prestack

Let $\mathcal{S}$ be a category and $p\colon \mathcal{X} \to \mathcal{S}$ be a functor of categories. We visualize this data as

$$\begin{array}{ccc} \mathcal{X} & & a \xrightarrow{\alpha} b \\ \downarrow{\scriptstyle p} & & \downarrow \quad \downarrow \\ \mathcal{S} & & S \xrightarrow{f} T \end{array}$$

where the lower case letters a, b are objects of $\mathcal{X}$ and the upper case letters S, T are objects of $\mathcal{S}$. We say that a is over S and $\alpha\colon a \to b$ is over $f\colon S \to T$.

Definition 1.3.1. A functor $p\colon \mathcal{X} \to \mathcal{S}$ is a *prestack over a category* $\mathcal{S}$ if

(1) (pullbacks exist) for any diagram

$$\begin{array}{ccc} a & \dashrightarrow & b \\ \downarrow & & \downarrow \\ S & \longrightarrow & T \end{array}$$

of solid arrows, there exist a morphism $a \to b$ over $S \to T$; and

(2) (universal property for pullbacks) for any diagram

$$\begin{array}{ccccc} a & \dashrightarrow & b & \longrightarrow & c \\ \downarrow & & \downarrow & & \downarrow \\ R & \longrightarrow & S & \longrightarrow & T \end{array}$$

of solid arrows, there exists a unique arrow $a \to b$ over $R \to S$ filling in the diagram.

⚠ **Warning 1.3.2.** When defining and discussing prestacks, we often simply write $\mathcal{X}$ instead of $\mathcal{X} \to \mathcal{S}$. In most examples it is clear what the functor $\mathcal{X} \to \mathcal{S}$ is. When necessary, we denote the projection by $p_{\mathcal{X}}\colon \mathcal{X} \to \mathcal{S}$.

Moreover, when defining a prestack $\mathcal{X}$, we often only define the objects and morphisms in $\mathcal{X}$, and we leave the definition of the composition law to the reader.

Remark 1.3.3. Axiom (2) above implies that the pullback in Axiom (1) is unique up to unique isomorphism. We often write f^*b or simply $b|_S$ to indicate a *choice* of a pullback.

Definition 1.3.4. If $\mathcal{X}$ is a prestack over $\mathcal{S}$, the *fiber category* $\mathcal{X}(S)$ over $S \in \mathcal{S}$ is the category of objects in $\mathcal{X}$ over S with morphisms over id_S.

Exercise 1.3.5. Show that the fiber category $\mathcal{X}(S)$ is a groupoid.

⚠ **Warning 1.3.6.** Our terminology is not standard. Prestacks are usually referred to as *categories fibered in groupoids*. In the literature (c.f. [Vis05], [Ols16]) a prestack is sometimes defined as a category fibered in groupoids together with Axiom (1) of a stack (Definition 1.4.1).

It is also standard to call a morphism $b \to c$ in $\mathcal{X}$ *cartesian* if it satisfies the universal property in (2) and $p\colon \mathcal{X} \to S$ a *fibered category* if for any diagram as in (1), there exists a cartesian morphism $a \to b$ over $S \to T$. With this terminology, a prestack (as we've defined it) is a fibered category where every arrow is cartesian or equivalently where every fiber category $\mathcal{X}(S)$ is a groupoid.

1.3.2 Examples

Example 1.3.7 (Presheaves are prestacks)**.** If $F\colon \mathcal{S} \to \mathrm{Sets}$ is a presheaf, we can construct a prestack $\mathcal{X}_F$ as the category of pairs (a, S) where $S \in \mathcal{S}$ and $a \in F(S)$. A map $(a', S') \to (a, S)$ is a map $f\colon S' \to S$ such that $a' = f^*a$, where f^* is convenient shorthand for $F(f)\colon F(S) \to F(S')$. Observe that the fiber categories $\mathcal{X}_F(S)$ are equivalent (even equal) to the set $F(S)$. We will often abuse notation by conflating F and $\mathcal{X}_F$.

Example 1.3.8 (Schemes are prestacks)**.** For a scheme X, applying the previous example to the functor $\mathrm{Mor}(-, X)\colon \mathrm{Sch} \to \mathrm{Sets}$ yields a prestack $\mathcal{X}_X$. This allows us to view a scheme X as a prestack and we will often abuse notation by referring to $\mathcal{X}_X$ as X.

Example 1.3.9 (Prestack of smooth curves)**.** We define the prestack $\mathcal{M}_g$ over Sch as the category of families of smooth curves $\mathcal{C} \to S$ of genus g, i.e. smooth and proper morphisms $\mathcal{C} \to S$ (of finite presentation) of schemes such that every geometric fiber is a connected curve of genus g. A map $(\mathcal{C}' \to S') \to (\mathcal{C} \to S)$ is the data of maps $\alpha\colon \mathcal{C}' \to \mathcal{C}$ and $f\colon S' \to S$ such that the diagram

$$\begin{array}{ccc} \mathcal{C}' & \xrightarrow{\alpha} & \mathcal{C} \\ \downarrow & \square & \downarrow \\ S' & \xrightarrow{f} & S \end{array}$$

is cartesian. Note that the fiber category $\mathcal{M}_g(\mathbb{C})$ over $\mathrm{Spec}\,\mathbb{C}$ is the groupoid of smooth connected projective complex curves C of genus g such that $\mathrm{Mor}_{\mathcal{M}_g(\mathbb{C})}(C, C') = \mathrm{Isom}_{\mathrm{Sch}/\mathbb{C}}(C, C')$.

Example 1.3.10 (Prestack of vector bundles)**.** Let C be a fixed smooth connected projective curve over $\mathbb{C}$, and fix integers $r \geq 0$ and d. We define the prestack $\mathcal{B}\mathrm{un}_{r,d}(C)$ over $\mathrm{Sch}/\mathbb{C}$ where objects are pairs (E, S) where S is a scheme over $\mathbb{C}$ and E is a vector bundle on $C_S = C \times_{\mathbb{C}} S$. A map $(E', S') \to (E, S)$ consists of a

map of schemes $f\colon S' \to S$ together with a map $E \to (\mathrm{id} \times f)_* E'$ of $\mathcal{O}_{C_S}$-modules whose adjoint is an isomorphism (i.e. for any *choice* of pullback $(\mathrm{id} \times f)^* E$, the adjoint map $(\mathrm{id} \times f)^* E \to E'$ is an isomorphism).

Exercise 1.3.11. Verify that $\mathcal{M}_g$ and $\mathcal{B}\mathrm{un}_{r,d}(C)$ are prestacks.

Definition 1.3.12 (Quotient and classifying prestacks)**.** Let $G \to S$ be a group scheme acting on a scheme $X \to S$ via $\sigma\colon G \times_S X \to X$. We define the *quotient prestack* $[X/G]^{\mathrm{pre}}$ as the category over Sch/S where the fiber category over an S-scheme T is quotient groupoid $[X(T)/G(T)]$ of the (abstract) group $G(T)$ acting on the set $X(T)$; see Example 0.3.6. A morphism $(T' \to X) \to (T \to X)$ over $T' \to T$ is an element $\gamma \in G(T')$ such that $(T' \to X) = \gamma \cdot (T' \to T \to X) \in X(T')$

We now define the prestack $[X/G]$ (which we will call the *quotient stack*) as the category over Sch/S whose objects over an S-scheme T are diagrams

$$\begin{array}{ccc} P & \xrightarrow{f} & X \\ \downarrow & & \\ T & & \end{array}$$

where $P \to T$ is a G-torsor (see Definition C.3.12) and $f\colon P \to X$ is a G-equivariant morphism. A morphism $(P' \to T', P' \xrightarrow{f'} X) \to (P \to T, P \xrightarrow{f} X)$ consists a maps $g\colon T' \to T$ and $\varphi\colon P' \to P$ of schemes such that the diagram

$$\begin{array}{ccccc} & & f' & \overset{\curvearrowright}{} & \\ P' & \xrightarrow{\varphi} & P & \xrightarrow{f} & X \\ \downarrow & \square & \downarrow & & \\ T' & \xrightarrow{g} & T & & \end{array}$$

commutes with the left square cartesian. See Section 0.8.1 for motivation of the above definition.

We define the *classifying prestack* as $\mathbf{B}_S G = [S/G]$ arising as the special case when $X = S$. When S is understood, we simply write $\mathbf{B}G$.

Exercise 1.3.13. Verify that $[X/G]^{\mathrm{pre}}$ and $[X/G]$ are prestacks over Sch/S.

1.3.3 Morphisms of prestacks

Definition 1.3.14.

(1) A *morphism of prestacks* $f\colon \mathcal{X} \to \mathcal{Y}$ is a functor $f\colon \mathcal{X} \to \mathcal{Y}$ such that the diagram

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{f} & \mathcal{Y} \\ & {\scriptstyle p_{\mathcal{X}}} \searrow \quad \swarrow {\scriptstyle p_{\mathcal{Y}}} & \\ & \mathcal{S} & \end{array}$$

strictly commutes, i.e. for every object $a \in \mathrm{Ob}(\mathcal{X})$, there is an *equality* $p_{\mathcal{X}}(a) = p_{\mathcal{Y}}(f(a))$ of objects in $\mathcal{S}$.

(2) If $f, g\colon \mathcal{X} \to \mathcal{Y}$ are morphisms of prestacks, a *2-morphism* (or *2-isomorphism*) $\alpha\colon f \to g$ is a natural transformation $\alpha\colon f \to g$ such that for every object $a \in \mathcal{X}$, the morphism $\alpha_a\colon f(a) \to g(a)$ in $\mathcal{Y}$ (which is an isomorphism) is over the identity in $\mathcal{S}$. We often describe the 2-morphism α schematically as

$$\mathcal{X} \overset{f}{\underset{g}{\Downarrow \alpha}} \mathcal{Y}.$$

(3) We define the category $\mathrm{MOR}(\mathcal{X}, \mathcal{Y})$ whose objects are morphisms of prestacks and whose morphisms are 2-morphisms.

(4) We say that a diagram

$$\begin{array}{ccc} \mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}' & \xrightarrow{f'} & \mathcal{Y}' \\ \downarrow{\scriptstyle g'} & \swarrow_{\alpha} & \downarrow{\scriptstyle g} \\ \mathcal{X} & \xrightarrow{f} & \mathcal{Y} \end{array}$$

together with a 2-isomorphism $\alpha\colon g \circ f' \xrightarrow{\sim} f \circ g'$ is *2-commutative*.

(5) A morphism $f\colon \mathcal{X} \to \mathcal{Y}$ of prestacks is an *isomorphism* if there exists a morphism $g\colon \mathcal{Y} \to \mathcal{X}$ and 2-isomorphisms $g \circ f \xrightarrow{\sim} \mathrm{id}_{\mathcal{X}}$ and $f \circ g \xrightarrow{\sim} \mathrm{id}_{\mathcal{Y}}$.

Exercise 1.3.15. Show that any 2-morphism is an isomorphism of functors, or in other words that $\mathrm{MOR}(\mathcal{X}, \mathcal{Y})$ is a groupoid.

Exercise 1.3.16. Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a morphism of prestacks over a category $\mathcal{S}$.

(a) Show that f is fully faithful if and only if $f_S\colon \mathcal{X}(S) \to \mathcal{Y}(S)$ is fully faithful for every $S \in \mathcal{S}$.

(b) Show that f is an isomorphism if and only if $f_S\colon \mathcal{X}(S) \to \mathcal{Y}(S)$ is an equivalence of categories for every $S \in \mathcal{S}$.

A prestack $\mathcal{X}$ is *equivalent to a presheaf* if there is a presheaf F and an isomorphism between $\mathcal{X}$ and the stack $\mathcal{X}_F$ corresponding to F (see Example 1.3.7).

Exercise 1.3.17. Let $G \to S$ be a group scheme acting on a scheme $X \to S$ via $\sigma\colon G \times_S X \to X$. Show that the prestacks $[X/G]^{\mathrm{pre}}$ and $[X/G]$ are equivalent to presheaves if and only if the action is free (i.e. $(\sigma, p_2)\colon G \times_S X \to X \times_S X$ is a monomorphism).

1.3.4 The 2-Yoneda lemma

Recall that Yoneda's lemma (Lemma 0.4.1) implies that for a presheaf $F\colon \mathcal{S} \to \mathrm{Sets}$ on a category $\mathcal{S}$ and an object $X \in \mathcal{S}$, there is a bijection $\mathrm{Mor}(S, F) \xrightarrow{\sim} F(S)$, where we view S as a presheaf via $\mathrm{Mor}(-, S)$. We will need an analogue of Yoneda's lemma for prestacks. First we recall that an object $S \in \mathcal{S}$ defines a prestack over $\mathcal{S}$, which we also denote by S, whose objects over $T \in \mathcal{S}$ are morphisms $T \to S$ and a morphism $(T \to S) \to (T' \to S)$ is an S-morphism $T \to T'$.

Lemma 1.3.18 (The 2-Yoneda Lemma)**.** *Let $\mathcal{X}$ be a prestack over a category $\mathcal{S}$ and $S \in \mathcal{S}$. The functor*

$$\mathrm{MOR}(S, \mathcal{X}) \to \mathcal{X}(S), \qquad f \mapsto f_S(\mathrm{id}_S)$$

is an equivalence of categories.

Proof. We will construct a quasi-inverse $\Psi\colon \mathcal{X}(S) \to \mathrm{MOR}(S, \mathcal{X})$ as follows.

On objects: For $a \in \mathcal{X}(S)$, we define $\Psi(a)\colon S \to \mathcal{X}$ as the morphism of prestacks sending an object $(T \xrightarrow{f} S)$ (of the prestack corresponding to S) over T to a *choice* of pullback $f^*a \in \mathcal{X}(T)$ and a morphism $(T' \xrightarrow{f'} S) \to (T \xrightarrow{f} S)$ given by an S-morphism $g\colon T' \to T$ to the morphism $f'^*a \to f^*a$ uniquely filling in the diagram

$$\begin{array}{ccccc} f'^*a & \dashrightarrow & f^*a & \longrightarrow & a \\ \downarrow & & \downarrow & & \downarrow \\ T' & \xrightarrow{g} & T & \xrightarrow{f} & S, \end{array}$$

using Axiom (2) of a prestack.

On morphisms: If $\alpha\colon a' \to a$ is a morphism in $\mathcal{X}(S)$, then $\Psi(\alpha)\colon \Psi(a') \to \Psi(a)$ is defined as the morphism of functors which maps a morphism $T \xrightarrow{f} S$ (i.e. an object in S over T) to the unique morphism $f^*a' \to f^*a$ filling in the diagram

$$\begin{array}{ccc} f^*a' & \dashrightarrow & f^*a \\ \downarrow & & \downarrow \\ a' & \xrightarrow{\alpha} & a \end{array} \qquad \text{over} \qquad \begin{array}{c} T \\ \downarrow f \\ S \end{array}$$

using again Axiom (2) of a prestack.

We leave the verification that Ψ is a quasi-inverse to the reader. □

We will use the 2-Yoneda lemma, often without mention, throughout these notes in passing between morphisms $S \to \mathcal{X}$ and objects of $\mathcal{X}$ over S.

Example 1.3.19 (Quotient stack presentations)**.** Consider the prestack $[X/G]$ in Definition 1.3.12 arising from a group action $\sigma\colon G \times_S X \to X$. The object of $[X/G]$ over X given by the diagram

$$\begin{array}{ccc} G \times_S X & \xrightarrow{\sigma} & X \\ \downarrow{\scriptstyle p_2} & & \\ X & & \end{array}$$

corresponds via the 2-Yoneda lemma (Lemma 1.3.18) to a morphism $X \to [X/G]$.

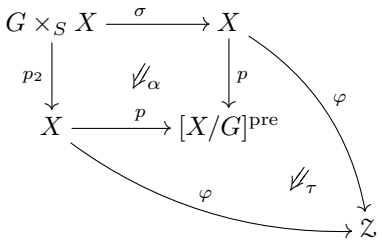
Exercise 1.3.20.

(1) Show that there is a morphism $p\colon X \to [X/G]^{\mathrm{pre}}$ and a 2-commutative diagram

$$\begin{array}{ccc} G \times_S X & \xrightarrow{\sigma} & X \\ \downarrow{\scriptstyle p_2} & \swarrow\!\!\!\swarrow{\scriptstyle \alpha} & \downarrow{\scriptstyle p} \\ X & \xrightarrow{p} & [X/G]^{\mathrm{pre}} \end{array}$$

(2) Show that $X \to [X/G]^{\mathrm{pre}}$ is a categorical quotient among prestacks, i.e. for

any 2-commutative diagram



of prestacks, there exists a morphism $\chi\colon [X/G]^{\mathrm{pre}} \to \mathcal{Z}$ and a 2-isomorphism $\beta\colon \varphi \xrightarrow{\sim} \chi \circ p$ which is compatible with α and τ (i.e. the two natural transformations $\varphi \circ \sigma \xrightarrow{\beta\circ\sigma} \chi \circ p \circ \sigma \xrightarrow{\chi\circ\alpha} \chi \circ p \circ p_2$ and $\varphi \circ \sigma \xrightarrow{\tau} \varphi \circ p_2 \xrightarrow{\beta\circ p_2} \chi \circ p \circ p_2$ agree.

1.3.5 Fiber products

We discuss fiber products for prestacks and in particular prove their existence. Recall that for morphisms $X \to Y$ and $Y' \to Y$ of presheaves on a category $\mathcal{S}$, the fiber product can be constructed as the presheaf mapping an object $S \in \mathcal{S}$ to the fiber product $X(S) \times_{Y(S)} Y'(S)$ of sets. Essentially the same construction works for morphisms $\mathcal{X} \to \mathcal{Y}$ and $\mathcal{Y}' \to \mathcal{Y}$ of prestacks but since we are dealing with groupoids rather than sets, the fiber category over an object $S \in \mathcal{S}$ should be the fiber product $\mathcal{X}(S) \times_{\mathcal{Y}(S)} \mathcal{Y}'(S)$ of groupoids.

The reader may first want to work on Exercises 1.3.24 and 1.3.25 on fiber products of groupoids as they not only provide a warmup to fiber products of prestacks but motivate its construction.

Construction 1.3.21. Let $f\colon \mathcal{X} \to \mathcal{Y}$ and $g\colon \mathcal{Y}' \to \mathcal{Y}$ be morphisms of prestacks over a category $\mathcal{S}$. Define the prestack $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}'$ over $\mathcal{S}$ as the category of triples (x, y', γ) where $x \in \mathcal{X}$ and $y' \in \mathcal{Y}'$ are objects over the *same* object $S := p_{\mathcal{X}}(x) = p_{\mathcal{Y}'}(y') \in \mathcal{S}$, and $\gamma\colon f(x) \xrightarrow{\sim} g(y')$ is an isomorphism in $\mathcal{Y}(S)$. A morphism $(x_1, y_1', \gamma_1) \to (x_2, y_2', \gamma_2)$ consists of a triple (f, χ, γ') where $f\colon p_{\mathcal{X}}(x_1) = p_{\mathcal{Y}'}(y_1') \to p_{\mathcal{Y}'}(y_2') = p_{\mathcal{X}}(x_2)$ is a morphism in $\mathcal{S}$, and $\chi\colon x_1 \xrightarrow{\sim} x_2$ and $\gamma'\colon y_1' \xrightarrow{\sim} y_2'$ are morphisms in $\mathcal{X}$ and $\mathcal{Y}'$ over f such that

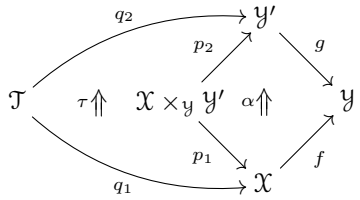
$$\begin{array}{ccc} f(x_1) & \xrightarrow{f(\chi)} & f(x_2) \\ \downarrow{\scriptstyle \gamma_1} & & \downarrow{\scriptstyle \gamma_2} \\ g(y_1') & \xrightarrow{g(\gamma')} & g(y_2') \end{array}$$

commutes.

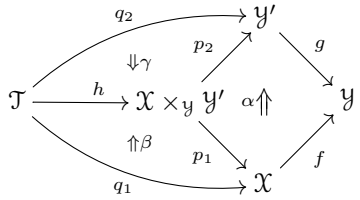
Let $p_1\colon \mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}' \to \mathcal{X}$ and $p_2\colon \mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}' \to \mathcal{X}$ denote the projections $(x, y', \gamma) \mapsto x$ and $(x, y', \gamma) \mapsto y'$. There is a 2-isomorphism $\alpha\colon f \circ p_1 \xrightarrow{\sim} g \circ p_2$ defined on an object $(x, y', \gamma) \in \mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}'$ by $\alpha_{(x,y',\gamma)}\colon f(x) \xrightarrow{\gamma} g(y')$. This yields a 2-commutative diagram

$$\begin{array}{ccc} \mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}' & \xrightarrow{p_2} & \mathcal{Y}' \\ \downarrow{\scriptstyle p_1} & \overset{\alpha}{\swarrow} & \downarrow{\scriptstyle g} \\ \mathcal{X} & \xrightarrow{f} & \mathcal{Y} \end{array} \tag{1.3.1}$$

Theorem 1.3.22. *The prestack $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}'$ together with the morphisms p_1 and p_2 and the 2-isomorphism α as in (1.3.1) satisfy the following universal property: for any 2-commutative diagram*



with 2-isomorphism $\tau\colon f \circ q_1 \xrightarrow{\sim} g \circ q_2$, there exist a morphism $h\colon \mathcal{T} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}'$ and 2-isomorphisms $\beta\colon q_1 \to p_1 \circ h$ and $\gamma\colon q_2 \to p_2 \circ h$ yielding a 2-commutative diagram



such that

$$\begin{array}{ccc} f \circ q_1 & \xrightarrow{f(\beta)} & f \circ p_1 \circ h \\ \downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle \alpha \circ h} \\ g \circ q_2 & \xrightarrow{g(\gamma)} & g \circ p_2 \circ h \end{array}$$

commutes. The data (h, β, γ) is unique up to unique isomorphism.

Proof. We define $h\colon \mathcal{T} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}'$ on objects by $t \mapsto \big(q_1(t), q_2(t), f(q_1(t)) \xrightarrow{\tau_t} g(q_2(t))\big)$ and on morphisms as $(t \xrightarrow{\Psi} t') \mapsto (p_{\mathcal{T}}(\Psi), q_1(\Psi), q_2(\Psi))$. There are *equalities* of functors $q_1 = p_1 \circ h$ and $q_2 = p_2 \circ h$ so we define β and γ as the identity natural transformation. The remaining details are left to the reader. □

Definition 1.3.23. We say that a 2-commutative diagram

$$\begin{array}{ccc} \mathcal{X}' & \longrightarrow & \mathcal{Y}' \\ \downarrow & \swarrow_{\alpha} & \downarrow \\ \mathcal{X} & \longrightarrow & \mathcal{Y} \end{array}$$

is *cartesian* if it satisfies the universal property of Theorem 1.3.22.

1.3.6 Examples of fiber products

Exercise 1.3.24.

(1) If $\mathcal{C} \xrightarrow{f} \mathcal{D}$ and $\mathcal{D}' \xrightarrow{g} \mathcal{D}$ are morphisms of groupoids, define the groupoid $\mathcal{C} \times_{\mathcal{D}} \mathcal{D}'$ whose objects are triples (c, d', δ) where $c \in \mathcal{C}$ and $d' \in \mathcal{D}'$ are objects, and $\delta \colon f(c) \xrightarrow{\sim} g(d')$ is an isomorphism in $\mathcal{D}$. A morphism $(c_1, d_1', \delta_1) \to (c_2, d_2', \delta_2)$ is the data of morphisms $\gamma \colon c_1 \xrightarrow{\sim} c_2$ and $\delta' \colon d_1' \xrightarrow{\sim} d_2'$ such that

$$\begin{array}{ccc} f(c_1) & \xrightarrow{f(\gamma)} & f(c_2) \\ \downarrow{\scriptstyle \delta_1} & & \downarrow{\scriptstyle \delta_2} \\ g(d_1') & \xrightarrow{g(\delta')} & g(d_2') \end{array}$$

commutes. Formulate a university property for fiber products of groupoids and show that $\mathcal{C} \times_{\mathcal{D}} \mathcal{D}'$ satisfies it.

(2) If $f \colon \mathcal{X} \to \mathcal{Y}$ and $g \colon \mathcal{Y}' \to \mathcal{Y}$ are morphisms of prestacks over a category $\mathcal{S}$, show that for every $S \in \mathcal{S}$, the fiber category $(\mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}')(S)$ is a fiber product $\mathcal{X}(S) \times_{\mathcal{Y}(S)} \mathcal{Y}'(S)$ of groupoids.

Exercise 1.3.25. Let G be a group acting on a set X via $\sigma \colon G \times X \to X$. Let $[X/G]$ denote the quotient groupoid (Exercise 0.3.7) with projection $p \colon X \to [X/G]$.

(1) Show that there are cartesian diagrams

$$\begin{array}{ccc} G \times X & \xrightarrow{\sigma} & X \\ \downarrow{\scriptstyle p_2} & \swarrow\!\!\!\swarrow & \downarrow{\scriptstyle p} \\ X & \xrightarrow{p} & [X/G] \end{array} \qquad \text{and} \qquad \begin{array}{ccc} G \times X & \xrightarrow{(\sigma, p_2)} & X \times X \\ \downarrow & \swarrow\!\!\!\swarrow & \downarrow{\scriptstyle p \times p} \\ [X/G] & \xrightarrow{\Delta} & [X/G] \times [X/G]. \end{array}$$

(2) Show that if $P \to T$ is any G-torsor and $P \to X$ is a G-equivariant map, there is a morphism $T \to [X/G]$, unique up to unique isomorphism, and a cartesian diagram

$$\begin{array}{ccc} P & \longrightarrow & X \\ \downarrow & \swarrow\!\!\!\swarrow & \downarrow \\ T & \longrightarrow & [X/G]. \end{array}$$

(If $G \to S$ is a smooth affine group scheme, we will later see that $[X/G]$ is an algebraic stack and that $X \to [X/G]$ is G-torsor (Theorem 2.1.8). Therefore the G-torsor $X \to [X/G]$ and the identity map $X \to X$ is the universal family over $[X/G]$ (corresponding to the identiy map $[X/G] \to [X/G]$).

Exercise 1.3.26.

(1) If $x \in X$, show that there is a morphism $\mathbf{B}G_x \to [X/G]$ of groupoids and a cartesian diagram

$$\begin{array}{ccc} Gx & \longrightarrow & X \\ \downarrow & \swarrow\!\!\!\swarrow & \downarrow{\scriptstyle p} \\ \mathbf{B}G_x & \longrightarrow & [X/G]. \end{array}$$

(2) Let $\phi \colon H \to G$ be a homomorphism of groups. Show that there is an induced morphism $\mathbf{B}H \to \mathbf{B}G$ of groupoids and that $\mathbf{B}H \times_{\mathbf{B}G} \mathrm{pt} \cong [G/H]$. If $G' \to G$ is a homomorphism of groups, can you describe $\mathbf{B}H \times_{\mathbf{B}G} \mathbf{B}G'$?

Exercise 1.3.27 (Magic Square)**.** Let $\mathcal{X}$ be a prestack. Show that for any morphism $a\colon S \to \mathcal{X}$ and $b\colon T \to \mathcal{X}$, there is a cartesian diagram

$$\begin{array}{ccc} S \times_{\mathcal{X}} T & \longrightarrow & S \times T \\ \downarrow & \swarrow\!\!\swarrow & \downarrow{\scriptstyle a\times b} \\ \mathcal{X} & \xrightarrow{\Delta} & \mathcal{X} \times \mathcal{X}. \end{array}$$

Exercise 1.3.28 (Isom presheaf)**.**

(1) Let $\mathcal{X}$ be a prestack over a category $\mathcal{S}$ and let a and b be objects over $S \in \mathcal{S}$. Recall that $\mathcal{S}/S$ denotes the localized category whose objects are morphisms $T \to S$ in $\mathcal{S}$ and whose morphisms are S-morphisms. Show that

$$\begin{aligned} \underline{\mathrm{Isom}}_{\mathcal{X}(S)}(a,b)\colon \mathcal{S}/S &\to \mathrm{Sets} \\ (T \xrightarrow{f} S) &\mapsto \mathrm{Mor}_{\mathcal{X}(T)}(f^*a, f^*b), \end{aligned}$$

where f^*a and f^*b are choices of a pullback, defines a presheaf on $\mathcal{S}/S$.

(2) Show that there is a cartesian diagram

$$\begin{array}{ccc} \underline{\mathrm{Isom}}_{\mathcal{X}(S)}(a,b) & \longrightarrow & S \\ \downarrow & \swarrow\!\!\swarrow & \downarrow{\scriptstyle (a,b)} \\ \mathcal{X} & \xrightarrow{\Delta} & \mathcal{X} \times \mathcal{X}. \end{array}$$

(3) Show that the presheaf $\underline{\mathrm{Aut}}_{\mathcal{X}(T)}(a) = \underline{\mathrm{Isom}}_{\mathcal{X}(T)}(a,a)$ is naturally a presheaf in groups.

Exercise 1.3.29. If $n \geq 2$, show that $[\mathbb{A}^n/\mathbb{G}_m^n] \cong \underbrace{[\mathbb{A}^1/\mathbb{G}_m] \times \cdots \times [\mathbb{A}^1/\mathbb{G}_m]}_{n \text{ times}}$.

Exercise 1.3.30.

(1) Show that if $H \to G$ is a morphism of group schemes over a scheme S, there is an induced morphism of prestacks $\mathbf{B}H \to \mathbf{B}G$ over Sch/S.

(2) Show that $\mathbf{B}H \times_{\mathbf{B}G} S \cong [G/H]$.

1.4 Stacks

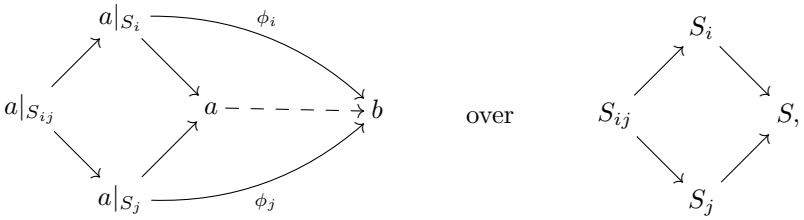
In this subsection, we will define a stack over a site $\mathcal{S}$ as a prestack $\mathcal{X}$ such that objects and morphisms glue uniquely in the Grothendieck topology of $\mathcal{S}$ (Definition 1.4.1). Verifying a given prestack is a stack reduces to a *descent* condition on objects and morphisms with respect to the covers of $\mathcal{S}$. The theory of descent is discussed in Section B.1 and is essential for verifying the stack axioms.

For a motivating example, consider the prestack of sheaves (Example 0.7.5) over the big Zariski site $(\mathrm{Sch})_{\mathrm{Zar}}$ whose objects over a scheme S are sheaves of abelian groups. Since sheaves and morphisms of sheaves glue in the Zariski-topology, this is a stack. It is also a stack in the big étale site $(\mathrm{Sch})_{\text{Ét}}$ and this requires the analogous gluing results in the étale topology (Propositions B.1.3 and B.1.5).

1.4.1 Definition of a stack

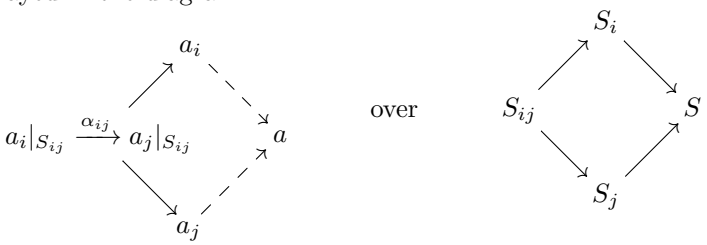
Definition 1.4.1. A prestack $\mathcal{X}$ over a site $\mathcal{S}$ is a *stack* if the following conditions hold for all coverings $\{S_i \to S\}$ of an object $S \in \mathcal{S}$:

(1) (morphisms glue) For objects a and b in $\mathcal{X}$ over S and morphisms $\phi_i \colon a|_{S_i} \to b$ such that $\phi_i|_{S_{ij}} = \phi_j|_{S_{ij}}$ as displayed in the diagram



there exists a unique morphism $\phi \colon a \to b$ with $\phi|_{S_i} = \phi_i$.

(2) (objects glue) For objects a_i over S_i and isomorphisms $\alpha_{ij} \colon a_i|_{S_{ij}} \to a_j|_{S_{ij}}$, as displayed in the diagram



satisfying the cocycle condition $\alpha_{ij}|_{S_{ijk}} \circ \alpha_{jk}|_{S_{ijk}} = \alpha_{ik}|_{S_{ijk}}$ on S_{ijk}, then there exists an object a over S and isomorphisms $\phi_i \colon a|_{S_i} \to a_i$ such that $\alpha_{ij} \circ \phi_i|_{S_{ij}} = \phi_j|_{S_{ij}}$ on S_{ij}.

Remark 1.4.2. There is an alternative description of the stack axioms analogous to the sheaf axiom of a presheaf $F \colon \mathcal{S} \to \mathrm{Sets}$, i.e. that $F(S) \to \prod_i F(S_i) \rightrightarrows \prod_{i,j} F(S_i \times_S S_j)$ is exact for coverings $\{S_i \to S\}$. Namely, we add an additional layer to the diagram corresponding to triple intersections and the stack axiom translates to the 'exactness' of

$$\mathcal{X}(S) \longrightarrow \prod_i \mathcal{X}(S_i) \rightrightarrows \prod_{i,j} \mathcal{X}(S_i \times_S S_j) \mathrel{\substack{\rightarrow\\[-1ex]\rightarrow\\[-1ex]\rightarrow}} \prod_{i,j,k} \mathcal{X}(S_i \times_S S_j \times_S S_k).$$

Exercise 1.4.3. Show that Axiom (1) is equivalent to the condition that for all objects a and b of $\mathcal{X}$ over $S \in \mathcal{S}$, the Isom presheaf $\underline{\mathrm{Isom}}_{\mathcal{X}(S)}(a, b)$ (see Exercise 1.3.28) is a sheaf on $\mathcal{S}/S$.

A *morphism* of stacks is a morphism of prestacks.

Exercise 1.4.4 (Fiber product of stacks)**.** Show that if $\mathcal{X} \to \mathcal{Y}$ and $\mathcal{Y}' \to \mathcal{Y}$ are morphisms of stacks over a site $\mathcal{S}$, then $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}'$ is also a stack over $\mathcal{S}$.

1.4.2 Examples of stacks

Example 1.4.5 (Sheaves and schemes are stacks)**.** Recall that if F is a presheaf on a site $\mathcal{S}$, we can construct a prestack $\mathcal{X}_F$ over $\mathcal{S}$ as the category of pairs (a, S)

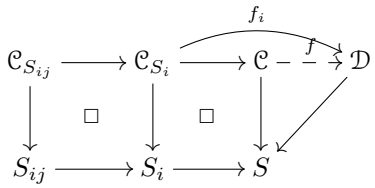
where $S \in \mathcal{S}$ and $a \in F(S)$ (see Example 1.3.7). If F is a sheaf, then $\mathcal{X}_F$ is a stack. We often abuse notation by writing F also as the stack $\mathcal{X}_F$.

Since schemes are sheaves on $\mathrm{Sch}_{\text{Ét}}$ (Example 1.2.5), a scheme X defines a stack over $\mathrm{Sch}_{\text{Ét}}$ (where objects over a scheme S are morphisms $S \to X$), which we also denote as X.

Let $\mathcal{M}_g$ denote the prestack of families of smooth curves $\mathcal{C} \to S$ of genus g; see Example 1.3.9.

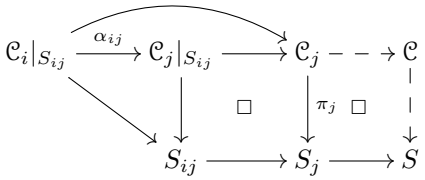
Proposition 1.4.6 (Moduli stack of smooth curves)**.** *If $g \geq 2$, then $\mathcal{M}_g$ is a stack over* $\mathrm{Sch}_{\text{Ét}}$.

Proof. Axiom (1) translates to: for families of smooth curves $\mathcal{C} \to S$ and $\mathcal{D} \to S$ of genus g and commutative diagrams



of solid arrows for all i, j (i.e. morphisms $f_i \colon \mathcal{C}_{S_i} \to \mathcal{D}$ such that $f_i|_{\mathcal{C}_{S_{ij}}} = f_j|_{\mathcal{C}_{S_{ij}}}$), there exists a unique morphism filling in the diagram (i.e. $f_i = f|_{\mathcal{C}_{S_i}}$). The existence and uniqueness of f follows from étale descent for morphisms (Proposition B.2.1). The fact that f is an isomorphism also follows from étale descent (Proposition B.4.1).

Axiom (2) is more difficult: we must show that given diagrams

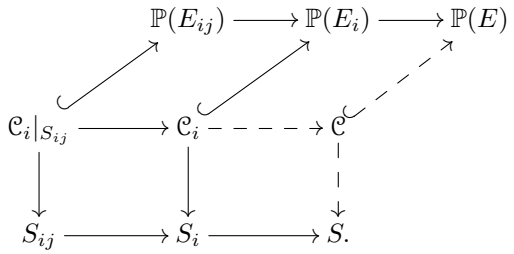


for all i, j where $\pi_i \colon \mathcal{C}_i \to S_i$ are families of smooth curves of genus g and $\alpha_{ij} \colon \mathcal{C}_i|_{S_{ij}} \to \mathcal{C}_j|_{S_{ij}}$ are isomorphisms satisfying the cocycle condition $\alpha_{ij} \circ \alpha_{jk} = \alpha_{ik}$, there is family of smooth curves $\mathcal{C} \to S$ and isomorphisms $\phi_i \colon \mathcal{C}|_{S_i} \to \mathcal{C}_i$ such that $\alpha_{ij} \circ \phi_i|_{\mathcal{C}_{S_{ij}}} = \phi_j|_{\mathcal{C}_{S_{ij}}}$.

We will use the following property of families of smooth curves (see Proposition 4.1.8): for a family of smooth curves $\pi \colon \mathcal{C} \to S$, $\omega_{\mathcal{C}/S}^{\otimes 3}$ is relatively very ample on S (as $g > 2$) and $F := \pi_* \omega_{\mathcal{C}/S}^{\otimes 3}$ is a vector bundle of rank $5(g-1)$. In particular, $\omega_{\mathcal{C}/S}^{\otimes 3}$ yields a closed immersion $\mathcal{C} \hookrightarrow \mathbb{P}(F)$ over S.

Therefore, if we set $E_i = (\pi_i)_*(\omega_{\mathcal{C}_i/S_i})$, there is a closed immersion $\mathcal{C}_i \hookrightarrow \mathbb{P}(E_i)$ over S_i. The isomorphisms α_{ij} induce isomorphisms $\beta_{ij} \colon E_i|_{S_{ij}} \to E_j|_{S_{ij}}$ satisfying the cocycle condition $\beta_{ij} \circ \beta_{jk} = \beta_{ik}$ on S_{ijk}. Descent for quasi-coherent sheaves (Proposition B.1.5) implies there is a quasi-coherent sheaf E on S and isomorphisms $\Psi_i \colon E|_{S_{ij}} \to E_i$ such that $\beta_{ij} \circ \Psi_i|_{S_{ij}} = \Psi_j|_{S_{ij}}$. It follows again from descent that

E is in fact a vector bundle (Proposition B.4.4). Pictorially, we have



Since the preimages of $\mathcal{C}_i \subset \mathbb{P}(E_i)$ and $\mathcal{C}_j \subset \mathbb{P}(E_j)$ in $\mathbb{P}(E_{ij})$ are equal, it follows from descent for closed subschemes (Proposition B.3.1) that there exists $\mathcal{C} \to S$ and isomorphisms ϕ_i such that $\alpha_{ij} \circ \phi_i|_{\mathcal{C}_{S_{ij}}} \to \phi_j|_{\mathcal{C}_{S_{ij}}}$. Since smoothness and properness are étale-local property on the target (Proposition B.4.1), $\mathcal{C} \to S$ is smooth and proper. The geometric fibers of $\mathcal{C} \to S$ are connected genus g curves since the geometric fibers of $\mathcal{C}_i \to S_i$ are. □

Exercise 1.4.7.

(1) Show that the prestack $\mathcal{M}_0$ is a stack on $\mathrm{Sch}_{\text{Ét}}$ isomorphic to $B\,\mathrm{PGL}_2$.

(2) Show that the moduli stack $\mathcal{M}_{1,1}$, whose objects are families of elliptic curves (see Example 0.4.26) is a stack on $\mathrm{Sch}_{\text{Ét}}$.

(3) Can you show that $\mathcal{M}_1$ is a stack on $\mathrm{Sch}_{\text{Ét}}$?

Let C be a smooth connected projective curve over $\mathbb{C}$, and fix integers $r \geq 0$ and d. Recall from Example 1.3.10 that $\mathcal{B}\mathrm{un}_{r,d}(C)$ denotes the prestack over $\mathrm{Sch}_{/\mathbb{C}}$ consisting of pairs (E, S) where S is a scheme over $\mathbb{C}$ and E is a vector bundle on C_S.

Proposition 1.4.8 (Moduli stack of vector bundles over a curve)**.** *For all integers r, d with $r \geq 0$, $\mathcal{B}\mathrm{un}_{r,d}(C)$ is a stack over $(\mathrm{Sch}_{/\mathbb{C}})_{\text{Ét}}$.*

Proof. The prestack $\mathcal{B}\mathrm{un}_{r,d}(C)$ is a stack: Axioms (1) and (2) are precisely descent for morphisms of quasi-coherent sheaves (Propositions B.1.3 and B.1.5) coupled with the fact that the property of a quasi-coherent sheaf being a vector bundle is étale-local (Proposition B.4.4). □

Let $G \to S$ be a smooth affine group scheme acting on a scheme $X \to S$. Let $[X/G]$ be the prestack defined in Definition 1.3.12 whose objects over a scheme S are G-torsors $P \to S$ together with G-equivariant maps $P \to X$. The following proposition justifies calling $[X/G]$ the *quotient stack*.

Proposition 1.4.9 (Quotient stack)**.** *The prestack $[X/G]$ is a stack.*

Proof. Axiom (1) follows from descent for morphisms of schemes (Proposition B.2.1). For Axiom (2), if $\{T_i \to T\}$ is an étale covering and $(\mathcal{P}_i \to T_i, \mathcal{P}_i \to X)$ are objects over T_i with isomorphisms on the restrictions satisfying the cocycle condition, then the existence of a G-torsor $\mathcal{P} \to T$ follows from descent for G-torsors (Proposition C.3.11) and the existence of $\mathcal{P} \to X$ follows from descent for morphisms of schemes (Proposition B.2.1). □

1.4.3 Stackification

To any presheaf F on a site $\mathcal{S}$, there is a sheafification $F \to F^{\mathrm{sh}}$ which is a left adjoint to the inclusion, i.e. $\mathrm{Mor}(F^{\mathrm{sh}}, G) \to \mathrm{Mor}(F, G)$ is bijective for any sheaf G on $\mathcal{S}$ (Theorem 1.2.9). Similarly, there is a stackification $\mathcal{X} \to \mathcal{X}^{\mathrm{st}}$ of any prestack $\mathcal{X}$ over $\mathcal{S}$.

Theorem 1.4.10 (Stackification)**.** *If $\mathcal{X}$ is a prestack over a site $\mathcal{S}$, there exists a stack $\mathcal{X}^{\mathrm{st}}$, which we call the* stackification*, and a morphism $\mathcal{X} \to \mathcal{X}^{\mathrm{st}}$ of prestacks such that for any stack $\mathcal{Y}$ over $\mathcal{S}$, the induced functor*

$$\mathrm{MOR}(\mathcal{X}^{\mathrm{st}}, \mathcal{Y}) \to \mathrm{MOR}(\mathcal{X}, \mathcal{Y}) \tag{1.4.1}$$

is an equivalence of categories.

Proof. As in the construction of the sheafification (see the proof of Theorem 1.2.9), we construct the stackification in stages. Most details are left to the reader.

First, given a prestack $\mathcal{X}$, we can construct a prestack $\mathcal{X}^{\mathrm{st}_1}$ satisfying Axiom (1) and a morphism $\mathcal{X} \to \mathcal{X}^{\mathrm{st}_1}$ of prestacks such that

$$\mathrm{MOR}(\mathcal{X}^{\mathrm{st}_1}, \mathcal{Y}) \to \mathrm{MOR}(\mathcal{X}, \mathcal{Y})$$

is an equivalence for all prestacks $\mathcal{Y}$ satisfying Axiom (1). Specifically, the objects of $\mathcal{X}^{\mathrm{st}_1}$ are the same as $\mathcal{X}$, and for objects $a, b \in \mathcal{X}$ over $S, T \in \mathcal{S}$, the set of morphisms $a \to b$ in $\mathcal{X}^{\mathrm{st}_1}$ over a given morphism $f\colon S \to T$ is the global sections $\Gamma(S, \underline{\mathrm{Isom}}_{\mathcal{X}(S)}(a, f^*b)^{\mathrm{sh}})$ of the sheafification of the Isom presheaf (Exercise 1.3.28).

Second, given a prestack $\mathcal{X}$ satisfying Axiom (1), we construct a stack $\mathcal{X}$ and a morphism $\mathcal{X} \to \mathcal{X}^{\mathrm{st}}$ of prestacks such that (1.4.1) is an equivalence for all stacks $\mathcal{Y}$. An object of $\mathcal{X}^{\mathrm{st}}$ over $S \in \mathcal{S}$ is given by a triple consisting of a covering $\{S_i \to S\}$, objects a_i of $\mathcal{X}$ over S_i, and isomorphisms $\alpha_{ij}\colon a_i|_{S_{ij}} \to a_j|_{S_{ij}}$ satisfying the cocycle condition $\alpha_{ij}|_{S_{ijk}} \circ \alpha_{jk}|_{S_{ijk}} = \alpha_{ik}|_{S_{ijk}}$ on S_{ijk}. Morphisms

$$\big(\{S_i \to S\}, \{a_i\}, \{\alpha_{ij}\}\big) \to \big(\{T_\mu \to T\}, \{b_\mu\}, \{\beta_{\mu\nu}\}\big)$$

in $\mathcal{X}^{\mathrm{st}}$ over $S \to T$ are defined as follows: first consider the induced cover $\{S_i \times_S T_\mu \to S\}_{i,\mu}$ and choose pullbacks $a_i|_{S_i \times_S T_\mu}$ and $b_\mu|_{S_i \times_S T_\mu}$. A morphism is then the data of maps $\Psi_{i\mu}\colon a_i|_{S_i \times_S T_\mu} \to b_\mu|_{S_i \times_S T_\mu}$ for all i, μ which are compatible with α_{ij} and $\beta_{\mu\nu}$ (i.e. $\Psi_{j\nu} \circ \alpha_{ij} = \beta_{\mu\nu} \circ \Psi_{i\mu}$ on $S_{ij} \times_T T_{\mu\nu}$). □

Exercise 1.4.11. Show that stackification commutes with fiber products: if $\mathcal{X} \to \mathcal{Y}$ and $\mathcal{Z} \to \mathcal{Y}$ are morphisms of prestacks, then $(\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z})^{\mathrm{st}} \cong \mathcal{X}^{\mathrm{st}} \times_{\mathcal{Y}^{\mathrm{st}}} \mathcal{Z}^{\mathrm{st}}$.

Exercise 1.4.12. Recall the prestacks $[X/G]^{\mathrm{pre}}$ and $[X/G]$ from Definition 1.3.12.

(1) Show that $[X/G]^{\mathrm{pre}}$ satisfies Axiom (1) of a stack.

(2) Show that the $[X/G]$ is isomorphic to the stackification of $[X/G]^{\mathrm{pre}}$ and that $[X/G]^{\mathrm{pre}} \to [X/G]$ is fully faithful.

Exercise 1.4.13. Extending Exercise 1.3.20, show that $X \to [X/G]$ is is a categorical quotient among stacks.

Notes

Grothendieck topologies and stacks were introduced in [SGA4] and our exposition closely follows [Art62], [Vis05], and [Ols16, Ch. 2].

Chapter 2

Algebraic spaces and stacks

2.1 Definitions of algebraic spaces and stacks

We present a streamlined approach to defining algebraic spaces (Definition 2.1.2), Deligne–Mumford stacks (Definition 2.1.4) and algebraic stacks (Definition 2.1.5), and we verify the algebraicity of quotient stacks (Theorem 2.1.8), the moduli stack of curves (Theorem 2.1.11) and the moduli stack of vector bundles (Theorem 2.1.15).

2.1.1 Algebraic spaces

Definition 2.1.1 (Morphisms representable by schemes)**.** A morphism $\mathcal{X} \to \mathcal{Y}$ of prestacks (or presheaves) over Sch is *representable by schemes* if for every morphism $V \to \mathcal{Y}$ from a scheme, the fiber product $\mathcal{X} \times_{\mathcal{Y}} V$ is a scheme.

If $\mathcal{P}$ is a property of morphisms of schemes (e.g. surjective or étale), a morphism $\mathcal{X} \to \mathcal{Y}$ of prestacks representable by schemes *has property* $\mathcal{P}$ if for every morphism $V \to \mathcal{Y}$ from a scheme, the morphism $\mathcal{X} \times_{\mathcal{Y}} V \to V$ of schemes has property $\mathcal{P}$.

Definition 2.1.2. An *algebraic space* is a sheaf X on $\mathrm{Sch}_{\acute{\mathrm{E}}\mathrm{t}}$ such that there exist a scheme U and a surjective étale morphism $U \to X$ representable by schemes.

The morphism $U \to X$ is called an *étale presentation*. Morphisms of algebraic spaces are by definition morphisms of sheaves. Any scheme is an algebraic space.

2.1.2 Deligne–Mumford stacks

Definition 2.1.3 (Representable morphisms)**.** A morphism $\mathcal{X} \to \mathcal{Y}$ of prestacks (or presheaves) over Sch is *representable* if for every morphism $V \to \mathcal{Y}$ from a scheme V, the fiber product $\mathcal{X} \times_{\mathcal{Y}} V$ is an algebraic space.

If $\mathcal{P}$ is a property of morphisms of schemes which is étale-local on the source (e.g., surjective, étale, or smooth), we say that a representable morphism $\mathcal{X} \to \mathcal{Y}$ of prestacks *has property* $\mathcal{P}$ if for every morphism $V \to \mathcal{Y}$ from a scheme and étale presentation $U \to \mathcal{X} \times_{\mathcal{Y}} V$ by a scheme, the composition $U \to \mathcal{X} \times_{\mathcal{Y}} V \to V$ has property $\mathcal{P}$.

Definition 2.1.4. A *Deligne–Mumford stack* is a stack $\mathcal{X}$ over $\mathrm{Sch}_{\acute{\mathrm{E}}\mathrm{t}}$ such that there exist a scheme U and a surjective, étale and representable morphism $U \to \mathcal{X}$.

The morphism $U \to \mathcal{X}$ is called an *étale presentation*. Morphisms of Deligne–Mumford stacks are by definition morphisms of stacks. Any algebraic space is a Deligne–Mumford stack via Example 1.3.7.

2.1.3 Algebraic stacks

Definition 2.1.5. An *algebraic stack* is a stack $\mathcal{X}$ over $\mathrm{Sch}_{\text{Ét}}$ such that there exist a scheme U and a surjective, smooth and representable morphism $U \to \mathcal{X}$.

The morphism $U \to \mathcal{X}$ is called a *smooth presentation*. For any smooth-local property $\mathcal{P}$ of schemes, we can say that $\mathcal{X}$ has $\mathcal{P}$ if U does. Morphisms of algebraic stacks are by definition morphisms of prestacks. Any scheme, algebraic space or Deligne–Mumford stack is also an algebraic stack.

⚠ **Warning 2.1.6.** The definitions above are not standard as most authors also add a representability condition on the diagonal. They are nevertheless equivalent to the standard definitions: we show in Theorem 2.4.1 that the diagonal of an algebraic space is representable by schemes and that the diagonal of an algebraic stack is representable.

Exercise 2.1.7 (Fiber products)**.** Show that fiber products exist for algebraic spaces, Deligne–Mumford stacks and algebraic stacks

2.1.4 Algebraicity of quotient stacks

We will now show that if G is a smooth affine group scheme acting on an algebraic space U over a base T, the quotient stack $[U/G]$ is algebraic and $U \to [U/G]$ is a G-torsor (Theorem 2.1.8).

Since we want to allow for the case that U is not a scheme, we need to generalize a few definitions. An *action* of a smooth affine group scheme $G \to T$ on an algebraic space U over T is a morphism $\sigma\colon G \times_T U \to U$ satisfying the same axioms as in Definition C.1.6, and we define as in Definition 1.3.12 the quotient stack $[U/G]$ as the stackification of the prestack $[U/G]^{\mathrm{pre}}$, whose fiber category over an T-scheme S is the quotient groupoid $[U(S)/G(S)]$. Objects of $[U/G]$ over an T-scheme S are G-torsors $P \to S$ and G-equivariant morphisms $S \to U$. Since morphisms to algebraic spaces glue uniquely in the étale topology (by definition), the argument of Proposition 1.4.9 shows that $[U/G]$ is a stack. Using Definition 2.1.3, the morphism $U \to [U/G]$ is a *G-torsor* if for every morphism $S \to \mathcal{X}$ from a scheme S, the algebraic space $U \times_{\mathcal{X}} S$ with the induced G-action is a G-torsor over S.

Theorem 2.1.8 (Algebraicity of Quotient Stacks)**.** *If $G \to T$ is a smooth, affine group scheme acting on an algebraic space $U \to T$, the quotient stack $[U/G]$ is an algebraic stack over T such that $U \to [U/G]$ is a G-torsor and in particular surjective, smooth and affine.*

Proof. If $S \to [U/G]$ is a morphism from a scheme corresponding to a G-torsor $\mathcal{P} \to S$ and a G-equivariant map $\mathcal{P} \xrightarrow{f} U$, there is a cartesian diagram

$$\begin{array}{ccc} \mathcal{P} & \xrightarrow{f} & U \\ \downarrow & & \downarrow \\ S & \longrightarrow & [U/G] \end{array}$$

(see Exercise 1.3.25). This shows that $U \to [U/G]$ is a G-torsor. If $U' \to U$ is an étale presentation by a scheme, then $U' \to U \to [U/G]$ provides a smooth presentation. □

Corollary 2.1.9. *If G is a finite group acting freely on an algebraic space U, then the quotient sheaf U/G is an algebraic space.*

Proof. Theorem 2.1.8 implies that U/G is an algebraic stack and that $U \to U/G$ is a G-torsor so in particular finite, étale, surjective and representable by schemes. Taking $U' \to U$ to be any étale presentation by a scheme, the composition $U' \to U \to U/G$ yields an étale presentation of U/G. □

Remark 2.1.10. This resolves the troubling issue from Example 0.6.5 where we saw that the quotient of a finite group acting freely on a scheme need not exist as a scheme. In addition, it shows that the category of algebraic spaces is reasonably well-behaved as it is closed under taking quotients by free actions of finite groups.

2.1.5 Algebraicity of $\mathcal{M}_g$

We now show that $\mathcal{M}_g$ is an algebraic stack. The main idea is quite simple: every smooth connected projective curve C is tri-canonically embedded $C \overset{|\omega_C^{\otimes 3}|}{\hookrightarrow} \mathbb{P}^{5g-6}$ and the locally closed subscheme $H' \subset \mathrm{Hilb}_P(\mathbb{P}^{5g-6})$ parameterizing smooth families of tri-canonically embedded curves provides a smooth presentation $H' \to \mathcal{M}_g$.

Theorem 2.1.11 (Algebraicity of the stack of smooth curves)**.** *If $g \geq 2$, then $\mathcal{M}_g$ is an algebraic stack over* $\operatorname{Spec}\mathbb{Z}$.

Proof. As in the proof that $\mathcal{M}_g$ is a stack (Proposition 1.4.6), we will use Properties of Families of Smooth Curves (see Proposition 4.1.8) which implies that for a family of smooth curves $\pi\colon \mathcal{C} \to S$, $\omega_{\mathcal{C}/S}^{\otimes 3}$ is relatively very ample on S (as $g > 2$) and $\pi_*\omega_{\mathcal{C}/S}^{\otimes 3}$ is a vector bundle of rank $5(g-1)$. In particular, $\omega_{\mathcal{C}/S}^{\otimes 3}$ yields a closed immersion $\mathcal{C} \hookrightarrow \mathbb{P}(\pi_*\omega_{\mathcal{C}/S}^{\otimes 3})$ over S. By Riemann–Roch, the Hilbert polynomial of any fiber $\mathcal{C}_s \hookrightarrow \mathbb{P}^{5g-6}_{\kappa(s)}$ is given by

$$P(n) := \chi(\mathcal{O}_{\mathcal{C}_s}(n)) = \deg(\omega_{\mathcal{C}_s}^{\otimes 3n}) + 1 - g = (6n-1)(g-1).$$

Let

$$H := \mathrm{Hilb}^P(\mathbb{P}^{5g-6}_{\mathbb{Z}})$$

by the Hilbert scheme parameterizing closed subschemes of $\mathbb{P}^{5g-6}$ with Hilbert polynomial P (Theorem D.0.1). Let $\mathcal{C} \hookrightarrow \mathbb{P}^{5g-6} \times H$ be the universal closed subscheme and let $\pi\colon \mathcal{C} \to H$. We claim that there is a unique locally closed subscheme $H' \subset H$ consisting of points $h \in H$ satisfying

(a) $\mathcal{C}_h \to \operatorname{Spec}\kappa(h)$ is smooth and geometrically connected; and
(b) $\mathcal{C}_h \hookrightarrow \mathbb{P}^{5g-6}_{\kappa(h)}$ is embedded by the complete linear series $\omega^{\otimes 3}_{\mathcal{C}_h/\kappa(h)}$.
(c) denoting $\mathcal{C}' = \mathcal{C}|_{H'} \to H'$, the coherent sheaves $\omega^{\otimes 3}_{\mathcal{C}'/H'}$ and $\mathcal{O}_{\mathcal{C}'}(1)$ differ by a pullback of a line bundle from H'.

Since the condition that a fiber of a proper morphism (of finite presentation) is smooth is an open condition on the target (Corollary A.3.8), the condition that $\mathcal{C}_h$ is smooth is open. Consider the Stein factorization [Har77, Cor. 11.5]

$\mathcal{C} \to \widetilde{H} = \operatorname{Spec}_H \pi_* \mathcal{O}_{\mathcal{C}} \to H$ where $\mathcal{C} \to \widetilde{H}$ has geometrically connected fibers and $\widetilde{H} \to H$ is finite. Since the kernel and cokernel of $\mathcal{O}_H \to \pi_* \mathcal{O}_{\mathcal{C}}$ have closed support (as they are coherent), $\widetilde{H} \to H$ is an isomorphism over an open subscheme of H, which is precisely where the fibers of $\mathcal{C} \to H$ are geometrically connected. In summary, the set of $h \in H$ satisfying (a) is an open subscheme of H, which we will denote by H_1.

The relative canonical sheaf $\omega_{\mathcal{C}_1/H_1}$ of the family $\mathcal{C}_1 := \mathcal{C}|_{H_1}$ is a line bundle. As a consequence Theorem 2.1.12, there exists a locally closed subscheme $H_2 \hookrightarrow H_1$ such that a morphism $T \to H_1$ factor through H_2 if and only if $\omega_{\mathcal{C}_1/H_1}|_{\mathcal{C}_T}$ and $\mathcal{O}_{\mathcal{C}}(1)|_{\mathcal{C}_T}$ differ by the pullback of a line bundle on T. In particular, (c) holds and for every $h \in H_2$, there is an isomorphism $\omega_{\mathcal{C}_h/\kappa(h)}^{\otimes 3} \cong \mathcal{O}_{\mathcal{C}_h}(1)$. To arrange (b), consider the restriction of the universal curve $\pi_2 \colon \mathcal{C}_2 \to H_2$. There is a canonical map $\alpha \colon H^0(\mathbb{P}^{5g-6}, \mathcal{O}(1)) \otimes \mathcal{O}_{H_2} \to (\pi_2)_* \omega_{\mathcal{C}_2/H_2}$ of vector bundles of rank $5g-5$ on H_2 whose fiber over a point $h \in H_2$ is the map $\alpha_h \colon H^0(\mathbb{P}^{5g-6}_{\kappa(h)}, \mathcal{O}(1)) \to H^0(\mathcal{C}_h, \omega_{\mathcal{C}_h/\kappa(h)}^{\otimes 3})$. The closed locus defined by the support of $\operatorname{coker}(\alpha)$ is precisely the locus where α_h is not an isomorphism (as the vector bundles have the same rank). The closed subscheme $H' = H_2 \setminus \operatorname{Supp}(\operatorname{coker}(\alpha))$ satisfies (a)-(c).

The group scheme $\mathrm{PGL}_{5g-5} = \underline{\mathrm{Aut}}(\mathbb{P}^{5g-6}_{\mathbb{Z}})$ over $\mathbb{Z}$ acts naturally on H: if $g \in \mathrm{Aut}(\mathbb{P}^{5g-6}_S)$ and $[\mathcal{D} \subset \mathbb{P}^{5g-6}_S] \in H(S)$, then $g \cdot [\mathcal{D} \subset \mathbb{P}^{5g-6}_S] = [g(\mathcal{D}) \subset \mathbb{P}^{5g-6}_S]$. The closed subscheme $H' \subset H$ is PGL_{5g-5}-invariant and we claim that $\mathcal{M}_g \cong [H'/\mathrm{PGL}_{5g-5}]$. This establishes the theorem since $[H'/\mathrm{PGL}_{5g-5}]$ is algebraic (Theorem 2.1.8).

Consider the morphism $H' \to \mathcal{M}_g$ which forgets the embedding, i.e. assigns a closed subscheme $\mathcal{C} \subset \mathbb{P}^{5g-6}_S$ to the family $\mathcal{C} \to S$. This morphism descends to a morphism $\Psi^{\mathrm{pre}} \colon [H'/\mathrm{PGL}_{5g-5}]^{\mathrm{pre}} \to \mathcal{M}_g$ of prestacks. The map Ψ^{pre} is fully faithful since for a family $\mathcal{C} \subset \mathbb{P}^{5g-6}_S$ of closed subschemes in H', any automorphism of $\mathcal{C} \to S$ induces an automorphism of $\omega_{\mathcal{C}/S}^{\otimes 3}$ and therefore an automorphism of $\mathbb{P}^{5g-6}_S$ preserving $\mathcal{C}$.

Since $\mathcal{M}_g$ is a stack (Theorem 2.1.8), the universal property of stackification yields a morphism $\Psi \colon [H'/\mathrm{PGL}_{5g-5}] \to \mathcal{M}_g$. Since $[H'/\mathrm{PGL}_{5g-5}]^{\mathrm{pre}} \to [H'/\mathrm{PGL}_{5g-5}]$ is fully faithful (Exercise 1.4.12), so is Ψ. It remains to check that Ψ is essentially surjective. For this, it suffices to check that if $\pi \colon \mathcal{C} \to S$ is a family of smooth curves, then there exists an étale cover $\{S_i \to S\}$ such that each $\mathcal{C}|_{S_i}$ is in the image of $H' \to \mathcal{M}_g$. Since $\pi_* \omega_{\mathcal{C}/S}$ is locally free of rank $5g-5$ and there is a closed immersion $\mathcal{C} \hookrightarrow \mathbb{P}(\pi_* \omega_{\mathcal{C}/S}^{\otimes 3})$ over S, we may simply take $\{S_i\}$ to be any Zariski-open cover (and thus étale cover) where $\pi_* \omega_{\mathcal{C}/S}^{\otimes 3}$ is free. □

The above proof used the following statement which provides conditions on a morphism $X \to S$ and a line bundle L on X ensuring that the locus in S consisting of points $s \in S$ such that $L|_{X_s}$ is trivial is closed. See [SP, Tag 0BEZ,Tag 0BF0] (and [Mum70, Cor. II.5.6, Thm. III.10] for the case when X is a product over S) .

Theorem 2.1.12. *Let $f \colon X \to S$ be a flat, proper morphism of finite presentation with geometrically integral fibers. Let L be a line bundle on X. Assume that for any morphism $T \to S$, the base change $f_T \colon X_T \to T$ satisfies $\mathcal{O}_T \xrightarrow{\sim} (f_T)_* \mathcal{O}_{X_T}$. Let L be a line bundle on X. Then there exists a closed subscheme $Z \hookrightarrow S$ of finite presentation such that a morphism $T \to S$ factors through Z if and only if $L|_{X_T}$ is the pullback of a line bundle on T.*

Exercise 2.1.13. Let $f\colon X \to S$ be a morphism as in Theorem 2.1.12. Define the *Picard functor of* $f\colon X \to S$ as

$$\mathrm{Pic}_{X/S}\colon \mathrm{Sch}\,/S \to \mathrm{Sets}, \quad T \mapsto \mathrm{Pic}(X_T)/f_T^*\,\mathrm{Pic}(T).$$

Show that the above theorem is equivalent to the diagonal morphism $\mathrm{Pic}_{X/S} \to \mathrm{Pic}_{X/S} \times_S \mathrm{Pic}_{X/S}$ of presheaves over $\mathrm{Sch}\,/S$ being representable by closed immersions, i.e. $\mathrm{Pic}_{X/S}$ is separated over S.

Exercise 2.1.14. Show that $\mathcal{M}_{1,1}$ is an algebraic stack.

2.1.6 Algebraicity of $\mathcal{B}\mathrm{un}_{r,d}(C)$

We now show that the stack of vector bundles over a fixed curve is algebraic.

Theorem 2.1.15 (Algebraicity of the stack of vector bundles)**.** *Let C be a smooth, projective and connected curve over a field $\Bbbk$, and let r and d be integers with $r \geq 0$. The stack $\mathcal{B}\mathrm{un}_{r,d}(C)$ is an algebraic stack over* $\mathrm{Spec}\,\Bbbk$.

Proof. For any vector bundle E on C of rank r and degree d, by Serre vanishing $E(m)$ is globally generated and $H^1(C, E(m)) = 0$ for $m \gg 0$. In particular,

$$\Gamma(C, E(m)) \otimes \mathcal{O}_C \twoheadrightarrow E(m)$$

is surjective which by construction induces an isomorphism on global sections. By Riemann–Roch, the Hilbert polynomial of E is

$$P(n) = \chi(E(n)) = \deg(E(n)) + \mathrm{rk}(E(n))(1-g) = d + rn + r(1-g).$$

For any scheme S, we have the diagram

$$\begin{array}{ccccc} & & C \times S & & \\ & \swarrow^{p_1} & & \searrow^{p_2} & \\ C & & & & S. \end{array}$$

For each integer m, consider the substack $\mathcal{B}\mathrm{un}_{r,d}(C)^m$ parameterizing families $\mathcal{E}$ of vector bundles on $C \times S$ over S such that $p_1^* p_{2,*}\mathcal{E}(m) \to \mathcal{E}(m)$ is surjective and $R^1 p_{2,*}\mathcal{E}(m) = 0$. It follows from Cohomology and Base Change [Har77, Thm III.12.11] that $\mathcal{B}\mathrm{un}_{r,d}(C)^m \subset \mathcal{B}\mathrm{un}_{r,d}(C)$ is an open substack.

For each m, let $N_m = P(m)$ and consider the Quot scheme

$$Q_m := \mathrm{Quot}^P(C, \mathcal{O}_C(-m)^{N_m})$$

parameterizing quotients $\mathcal{O}_C(-m)^{N_m} \twoheadrightarrow F$ with Hilbert polynomial P (Theorem D.0.2). Let $\mathcal{O}_{C\times Q_m}(-m)^{N_m} \to \mathcal{E}_m$ be the universal quotient on $C \times Q_m$ and consider the induced map

$$\Psi\colon \mathcal{O}_{Q_m}^{N_m} \xrightarrow{\sim} p_{2,*}\mathcal{O}_{C\times Q_m}^{N_m} \to p_{2,*}(\mathcal{E}_m(m))$$

The cokernel of Ψ has closed support in Q_m and its complement $Q'_m \subset Q_m$ is precisely the locus over which Ψ is an isomorphism.

The Quot scheme Q_m inherits a natural action from GL such that Q'_m is invariant. The morphism $Q'_m \to \mathcal{B}\mathrm{un}_{r,d}(C)^m$, defined by $[\mathcal{O}_C(-m)^{N_m} \twoheadrightarrow F] \mapsto F$, factors to a yield a morphism $\Psi^{\mathrm{pre}}\colon [Q'_m/\,\mathrm{GL}_{N_m}]^{\mathrm{pre}} \to \mathcal{B}\mathrm{un}_{r,d}(C)^m$ of prestacks. The

map Ψ^{pre} is fully faithful since any automorphism of a family $\mathcal{F} \in \mathcal{B}\mathrm{un}_{r,d}(C)^m(S)$ of vector bundles on $C \times S$ induces an automorphism of $p_{2,*}\mathcal{F}(m) = \mathcal{O}_S^{N_m}$ which is an element of $\mathrm{GL}_{N_m}(S)$, and this element acts on $\mathcal{O}_C(-m)^{N_m}$ preserving the quotient F.

Since $\mathcal{B}\mathrm{un}_{r,d}(C)$ is a stack (Proposition 1.4.8), there is an induced morphism $\Psi\colon [Q'_m/\operatorname{GL}_{N_m}] \to \mathcal{B}\mathrm{un}_{r,d}(C)^m$ of stacks which is also fully faithful (Exercise 1.4.12) and by construction essentially surjective. We conclude that

$$\mathcal{B}\mathrm{un}_{r,d}(C) = \bigcup_m \left[Q'_m/\operatorname{GL}_{N_m}\right]$$

and the result follows from the algebraicity of quotient stacks (Theorem 2.1.8. □

Remark 2.1.16. Note that while $\mathcal{B}\mathrm{un}_{r,d}(C)$ itself is not quasi-compact (Definition 2.2.7), the proof establishes that any quasi-compact open substack of $\mathcal{B}\mathrm{un}_{r,d}(C)$ is a quotient stack.

2.1.7 Survey of important results

We will develop the foundations of algebraic spaces and stacks in the forthcoming chapters but it is worth first highlighting some of the most important results.

The importance of the diagonal

When overhearing others discussing algebraic stacks, you may have wondered what's all the fuss about the diagonal? Well, I'll tell you—the diagonal encodes the stackiness!

First and foremost, the diagonal $\mathcal{X} \to \mathcal{X}\times\mathcal{X}$ of an algebraic stack is representable and the diagonal $X \to X \times X$ of an algebraic space is representable by schemes. Many authors in fact include this condition in the definition of algebraicity.

Recall that if $\mathcal{X}$ is a prestack over Sch and x, y are objects over a scheme T, then there is a cartesian diagram

$$\begin{array}{ccc} \underline{\mathrm{Isom}}_{\mathcal{X}(T)}(x,y) & \longrightarrow & T \\ \downarrow & & \downarrow{\scriptstyle (x,y)} \\ \mathcal{X} & \xrightarrow{\Delta} & \mathcal{X}\times\mathcal{X}; \end{array}$$

see Exercise 1.3.28. Axiom (1) of a stack is the condition that $\underline{\mathrm{Isom}}_{\mathcal{X}(T)}(x,y)$ is a sheaf on $(\mathrm{Sch}/T)_{\text{Ét}}$ and Representability of the Diagonal (Theorem 2.4.1) shows that $\underline{\mathrm{Isom}}_{\mathcal{X}(T)}(x,y)$ is an algebraic space. Moreover, $\underline{\mathrm{Aut}}_{\mathcal{X}(T)}(x) = \underline{\mathrm{Isom}}_{\mathcal{X}(T)}(x,x)$ is naturally a sheaf in groups and thus a group algebraic space over T. Taking T to be the spectrum of a field K, we define the *stabilizer of* x: $\operatorname{Spec} K \to \mathcal{X}$ as

$$G_x := \underline{\mathrm{Aut}}_{\mathcal{X}(K)}(x).$$

For schemes (resp. separated schemes), the diagonal is an immersion (resp. closed immersion). For algebraic stacks, the diagonal is not necessarily a monomorphism as the fiber over $(x,x)\colon \operatorname{Spec} K \to \mathcal{X}\times\mathcal{X}$, or in other words the stabilizer G_x, may be non-trivial. Properties of the diagonal in fact characterize algebraic spaces and Deligne–Mumford stacks: an algebraic stack is an algebraic space

(resp. Deligne–Mumford stack) if and only if $\mathcal{X} \to \mathcal{X} \to \mathcal{X}$ is a monomorphism (resp. unramified)—see Theorems 2.6.3 and 2.6.5. Properties of the stabilizer also provide characterizations as in the table below:

Table 2.1: Characterization of algebraic spaces and Deligne–Mumford stacks

Type of space	Property of the diagonal	Property of stabilizers
algebraic space	monomorphism	trivial
Deligne–Mumford stack	unramified	discrete and reduced groups
algebraic stack	arbitrary	arbitrary

As a consequence of these characterizations, we will generalize Corollary 2.1.9: the quotient of a free action of a smooth algebraic group on an algebraic space exists as an algebraic space. We will also be able to establish that $\mathcal{M}_g$ is Deligne–Mumford rather than just algebraic (Theorem 2.1.11).

We now summarize additional important properties of algebraic spaces, Deligne–Mumford stacks and algebraic stacks. The reader may also wish to consult Table 3 for a brief recap of the trichotomy of moduli spaces.

Properties of algebraic spaces

- If $R \rightrightarrows U$ is an étale equivalence relation of schemes, the quotient sheaf U/R is an algebraic space.
- If X is a quasi-separated algebraic space, there exists a dense open subspace $U \subset X$ which is a scheme.
- If $X \to Y$ is a separated and quasi-finite morphism of noetherian algebraic spaces, then there exists a factorization $X \hookrightarrow \widetilde{X} \to Y$ where $X \hookrightarrow \widetilde{X}$ is an open immersion and $\widetilde{X} \to Y$ is finite (Zariski's Main Theorem). In particular, $X \to Y$ is quasi-affine.

Properties of Deligne–Mumford stacks

- If $R \rightrightarrows U$ is an étale groupoid of schemes, the quotient stack $[U/R]$ is a Deligne–Mumford stack.
- If $\mathcal{X}$ is a Deligne–Mumford stack (e.g. algebraic space), there exists a scheme U and a finite morphism $U \to \mathcal{X}$.
- If $\mathcal{X}$ is a Deligne–Mumford stack and $x \in \mathcal{X}(k)$ is any field-valued point, there exists an étale neighborhood $[\mathrm{Spec}(A)/G] \to \mathcal{X}$ of x where G is a finite group, which can be arranged to be the stabilizer of x (Local Structure of Deligne–Mumford Stacks).
- If $\mathcal{X}$ is a separated Deligne–Mumford stack, there exists a coarse moduli space $\mathcal{X} \to X$ where X is a separated algebraic space (Keel-Mori theorem).

Properties of algebraic stacks

- If $R \rightrightarrows U$ is a smooth groupoid of schemes, the quotient stack $[U/R]$ is an algebraic stack.
- If $\mathcal{X}$ is an algebraic stack of finite type over an algebraically closed field $\Bbbk$ with affine diagonal, any point $x \in \mathcal{X}(k)$ with linearly reductive stabilizer has an affine étale neighborhood $[\mathrm{Spec}(A)/G_x] \to \mathcal{X}$ of x where G is a finite group (Local Structure of Algebraic Stacks).
- Let $\mathcal{X}$ be an algebraic stack of finite type over an algebraically closed field $\Bbbk$ of characteristic 0 with affine diagonal. If $\mathcal{X}$ is S-complete and Θ-reductive, there exists a good moduli space $\mathcal{X} \to X$ where X is a separated algebraic space of finite type over $\Bbbk$.

Notes

Deligne–Mumford and algebraic stacks were first introduced in [DM69] and [Art74]—and in both cases referred to as *algebraic stacks*—with conventions slightlly different than ours. Namely, [DM69, Def. 4.6] assumed in addition to the existence of an étale presentation that the diagonal is representable by schemes (which is automatic if the diagonal is separated and quasi-compact). On the other hand, [Art74, Def. 5.1] assumed in addition to the existence of a smooth presentation that the stack is locally of finite type over an excellent Dedekind domain. We will not use the term *Artin stack* which is often used to refer to algebraic stacks that satisfy Artin's axioms (e.g. algebraic stacks locally of finite type over an excellent scheme with quasi-compact and separated diagonal) as *Artin stacks*.

We follow the conventions of [Ols16] and [SP] (with the exception that we work over the site $\mathrm{Sch}_{\text{Ét}}$ while [SP] works over $\mathrm{Sch}_{\text{fppf}}$). We begin this section by discussing properties of algebraic spaces and stacks and their morphisms. We discuss equivalence relations and groupoids (Definition 2.3.1) and establish that their quotient sheaves or stacks are algebraic (Theorem 2.3.8, Corollary 2.4.5). We verify that the diagonal of an algebraic space (resp. algebraic stack) is representable by schemes (resp. representable) (Theorem 2.4.1), and provide equivalent characterizations of algebraic spaces and Deligne–Mumford stacks in terms of the diagonal (Theorem 2.6.3). We discuss the Formal Lifting Criteria (Proposition 2.7.1) for smooth, étale and unramified morphisms, and the Valuative Criteria (Proposition 2.8.5) for separated, universally closed and proper morphisms. Finally, §2.9 provides examples.

2.2 First properties

2.2.1 Properties of morphisms

Recall that a morphism of stacks $\mathcal{X} \to \mathcal{Y}$ over $(\mathrm{Sch})_{\text{Ét}}$ is representable by schemes (resp. representable) if for every morphism $T \to \mathcal{Y}$ from a scheme, the base change $\mathcal{X} \times_{\mathcal{Y}} T$ is a scheme (resp. algebraic space).

Definition 2.2.1. Let $\mathcal{P}$ be a property of morphisms of schemes.

(1) If $\mathcal{P}$ is stable under composition and base change and is étale-local (resp. smooth-local) on the source and target, a morphism $\mathcal{X} \to \mathcal{Y}$ of Deligne–Mumford stacks (resp. algebraic stacks) *has property* $\mathcal{P}$ if there for all étale (resp. smooth) presentations (equivalently there exists a presentations) $V \to \mathcal{Y}$ and $U \to \mathcal{X} \times_{\mathcal{Y}} V$, in the diagram

$$\begin{array}{ccccc} U & \longrightarrow & \mathcal{X} \times_{\mathcal{Y}} V & \longrightarrow & V \\ & & \downarrow & \square & \downarrow \\ & & \mathcal{X} & \longrightarrow & \mathcal{Y} \end{array}$$

the composition $U \to V$ has $\mathcal{P}$.

(2) A morphism $\mathcal{X} \to \mathcal{Y}$ of algebraic stacks representable by schemes *has property* $\mathcal{P}$ if for every morphism $T \to Y$ from a scheme, the base change $\mathcal{X} \times_{\mathcal{Y}} T \to T$ has $\mathcal{P}$.

(3) A morphism $\mathcal{X} \to \mathcal{Y}$ of algebraic stacks is an *open immersion*, *closed immersion*, *locally closed immersion*, *affine*, or *quasi-affine* if it is representable by schemes and has the corresponding property in the sense of (2).

The properties of flatness, smoothness, surjectivity, locally of finite presentation, and locally of finite type are smooth-local and étaleness is étale-local

and thus using (1), we have the corresponding notions for morphisms of algebraic stacks and Deligne–Mumford stacks, respectively. Such properties are clearly stable under composition and base change.

Example 2.2.2. If $G \to S$ is a group scheme acting on an algebraic space $U \to S$, then $[U/G] \to S$ is locally of finite type if and only if $U \to S$ is. In particular, using Theorems 2.1.11 and 2.1.15, we conclude that $\mathcal{M}_g$ is locally of finite type over $\mathbb{Z}$ and $\mathcal{B}\mathrm{un}_{r,d}(C)$ is locally of finite type over $\Bbbk$.

The properties of being an open immersion, closed immersion, locally closed immersion, affine, or quasi-affine are smooth-local on the target (but not the source) and therefore it suffices to show that for a smooth presentation $V \to \mathcal{Y}$, the base change $\mathcal{X} \times_{\mathcal{Y}} V \to V$ has $\mathcal{P}$. Again these properties are clearly stable under composition and base change. In fact, we will show now that they also descend.

Proposition 2.2.3. *Let $\mathcal{P}$ be one of the following properties of morphisms of algebraic spaces: open immersion, closed immersion, locally closed immersion, affine, or quasi-affine. Consider a cartesian diagram*

$$\begin{array}{ccc} X' & \longrightarrow & Y' \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

of algebraic spaces such that $Y' \to Y$ is surjective, flat and locally of finite presentation. Then $X \to Y$ has $\mathcal{P}$ if and only if $X' \to Y'$ has $\mathcal{P}$.

Proof. We may assume that $X' \to Y'$ is a morphism of schemes. The challenge here is to show that $X \to Y$ is representable by schemes but fortunately this is a consequence of Effective Descent (Corollary B.3.6). □

Remark 2.2.4. Once we know that an algebraic stack $\mathcal{X}$ is isomorphic to an algebraic space if and only if $\mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is a monomorphism (Theorem 2.6.3), we can show that representable morphisms satisfy descent and therefore can extend this result to algebraic stacks.

We can now define unramified and étale morphisms.

Definition 2.2.5.

(1) A morphism of stacks $\mathcal{X} \to \mathcal{Y}$ over $(\mathrm{Sch})_{\text{Ét}}$ is *representable by Deligne–Mumford stacks* if for every morphism $T \to \mathcal{Y}$ from a scheme, the fiber product $\mathcal{X} \times_{\mathcal{Y}} T$ is a Deligne–Mumford stack.

(2) If $\mathcal{P}$ is a property of morphisms of schemes that is étale-local on the source and smooth-local on the target, we say that a morphism $\mathcal{X} \to \mathcal{Y}$ of stacks representable by Deligne–Mumford stacks *has property* $\mathcal{P}$ if for a smooth presentation $V \to \mathcal{Y}$ and an étale presentation $U \to \mathcal{X} \times_{\mathcal{Y}} V$ (equivalently for all such presentations), in the diagram

$$\begin{array}{ccccc} U & \longrightarrow & \mathcal{X} \times_{\mathcal{Y}} V & \longrightarrow & V \\ & & \downarrow & \square & \downarrow \\ & & \mathcal{X} & \longrightarrow & \mathcal{Y} \end{array}$$

the composition $U \to V$ has $\mathcal{P}$.

(3) A morphism $\mathcal{X} \to \mathcal{Y}$ of algebraic stacks is *unramified* or *étale* if $\mathcal{X} \to \mathcal{Y}$ is representable by Deligne–Mumford stacks and has the corresponding property in the sense of (2).

2.2.2 The topological space of a stack

We can associate a topological space $|\mathcal{X}|$ (in the usual sense) to any algebraic stack $\mathcal{X}$.

Definition 2.2.6 (Topological space of an algebraic stack)**.** If $\mathcal{X}$ is an algebraic stack, we define *the topological space of* $\mathcal{X}$ as the set $|\mathcal{X}|$ consisting of field-valued morphisms $x\colon \operatorname{Spec} K \to \mathcal{X}$. Two morphisms $x_1\colon \operatorname{Spec} K_1 \to \mathcal{X}$ and $x_2\colon \operatorname{Spec} K_2 \to \mathcal{X}$ are identified in $|\mathcal{X}|$ if there exists field extensions $K_1 \to K_3$ and $K_2 \to K_3$ such that $x_1|_{\operatorname{Spec} K_3}$ and $x_2|_{\operatorname{Spec} K_3}$ are isomorphic in $\mathcal{X}(K_3)$. A subset $U \subset |\mathcal{X}|$ is open if there exists an open immersion $\mathcal{U} \hookrightarrow \mathcal{X}$ such that U is the image of $|\mathcal{U}| \to |\mathcal{X}|$.

A morphism of stacks $\mathcal{X} \to \mathcal{Y}$ induces a continuous map $|\mathcal{X}| \to |\mathcal{Y}|$. We can now define topological properties of algebraic stacks and their morphisms.

Definition 2.2.7. We say that an algebraic stack $\mathcal{X}$ is *quasi-compact*, *connected*, or *irreducible* if $|\mathcal{X}|$ is.

Exercise 2.2.8. Show that an algebraic stack $\mathcal{X}$ is quasi-compact if and only if there exists a smooth presentation $\operatorname{Spec} A \to \mathcal{X}$.

Definition 2.2.9. A morphism $\mathcal{X} \to \mathcal{Y}$ of algebraic stacks is *quasi-compact* if for every morphism $\operatorname{Spec} B \to \mathcal{Y}$, the fiber product $\mathcal{X} \times_{\mathcal{Y}} \operatorname{Spec} B$ is quasi-compact. We say that $\mathcal{X} \to \mathcal{Y}$ is *of finite type* if $\mathcal{X} \to \mathcal{Y}$ is locally of finite type and quasi-compact.

After discussing properties of the diagonal, we will define $\mathcal{X}$ to be *noetherian* if $\mathcal{X}$ is locally noetherian (Definition 2.2.16), quasi-compact *and* quasi-separated (Definition 2.4.15).

Exercise 2.2.10. Show that an algebraic stack $\mathcal{X}$ is quasi-compact if and only if there exists a smooth presentation $\operatorname{Spec} A \to \mathcal{X}$.

Example 2.2.11. Using the quotient presentation in Theorem 2.1.11, we see that $\mathcal{M}_g$ is quasi-compact and in particular of finite type over $\mathbb{Z}$.

Exercise 2.2.12. If $\mathcal{X}$ is an algebraic stack such that the diagonal $\mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is quasi-compact, show that $|\mathcal{X}|$ is a sober topological space, i.e. every irreducible closed subset has a generic point. (This is a difficult exercise and should perhaps be postponed until we have developed more theory.)

Exercise 2.2.13. Let $x \in |\mathcal{X}|$ be a point of an algebraic stack with two representatives $x_1 \colon \operatorname{Spec} K_1 \to \mathcal{X}$ and $x_2 \colon \operatorname{Spec} K_2 \to \mathcal{X}$. Show that the stabilizer group G_{x_1} is smooth (resp. affine) if and only if G_{x_2} is. (It thus makes sense to say that $x \in |\mathcal{X}|$ has smooth or affine stabilizer.)

Definition 2.2.14. A point $x \in |\mathcal{X}|$ in an algebraic stack is *of finite type* if there exists a representative $\operatorname{Spec} K \to \mathcal{X}$ of finite type.

Remark 2.2.15. If X is a scheme, then a morphism $\operatorname{Spec} K \to X$ with image $x \in X$ is of finite type if and only if the image $x \in X$ is locally closed (i.e. closed in an open neighborhood U) and $\kappa(x)/K$ is a finite extension. In particular, $x \in X$ is of finite type if and only if $x \in X$ is locally closed, and we will later show that the same holds for points of noetherian algebraic stacks (Proposition 2.5.14).

An example of a finite type point that is not closed is the generic point of a DVR. However, if X is a scheme of finite type over $\Bbbk$, then any finite type point is in fact a closed point. The analogous fact is *not* true for algebraic stacks of finite type over $\Bbbk$, e.g. $\operatorname{Spec} \Bbbk \xrightarrow{1} [\mathbb{A}^1/\mathbb{G}_m]$ is an open finite type point.

We can also define étale and smooth-local properties of algebraic spaces and stacks:

Definition 2.2.16 (Properties of algebraic spaces and stacks)**.** Let $\mathcal{P}$ be a property of schemes which is étale (resp. smooth) local. We say that a Deligne–Mumford stack (resp. algebraic stack) $\mathcal{X}$ *has property* $\mathcal{P}$ if for an étale (resp. smooth) presentation (equivalently for all presentations) $U \to \mathcal{X}$, the scheme U has $\mathcal{P}$.

The properties of being locally noetherian, reduced or regular are smooth-local.

Example 2.2.17. Let $G \to S$ be a smooth, affine group scheme acting on a scheme U over S. If $\mathcal{P}$ is smooth-local on the source, then $[U/G]$ has $\mathcal{P}$ if and only if U has $\mathcal{P}$.

2.3 Equivalence relations and groupoids

2.3.1 Definitions

Definition 2.3.1. An *étale (resp. smooth) groupoid of schemes* is a pair of schemes U and R together with étale (resp. smooth) morphisms $s \colon R \to U$ called the *source* and $t \colon R \to U$ called the *target*, and a *composition* morphism $c \colon R \times_{t,U,s} R \to R$ satisfying:

(1) (associativity) the following diagram commutes

$$\begin{array}{ccc} R \times_{t,U,s} R \times_{t,U,s} R & \xrightarrow{c\times \mathrm{id}} & R \times_{t,U,s} R \\ \downarrow{\scriptstyle \mathrm{id}\times c} & & \downarrow{\scriptstyle c} \\ R \times_{t,U,s} R & \xrightarrow{c} & R, \end{array}$$

(2) (identity) there exists a morphism $e\colon U \to R$ (called the *identity*) such that the following diagrams commute

$$\begin{array}{ccccc} & & U & & \\ & \swarrow^{\mathrm{id}} & \downarrow{\scriptstyle e} & \searrow^{\mathrm{id}} & \\ U & \xleftarrow{s} & R & \xrightarrow{t} & U \end{array} \qquad \begin{array}{ccccc} R & \xrightarrow{e\circ s,\mathrm{id}} & R \times_{t,U,s} R & \xleftarrow{e\circ t,\mathrm{id}} & R \\ & \searrow_{\mathrm{id}} & \downarrow{\scriptstyle c} & \swarrow_{\mathrm{id}} & \\ & & R, & & \end{array}$$

(3) (inverse) there exists a morphism $i\colon R \to R$ (called the *inverse*) such that the following diagrams commute

$$\begin{array}{ccccc} R & \xrightarrow{i} & R & \xrightarrow{i} & R \\ & \searrow_{s} & \downarrow{\scriptstyle t} & \swarrow_{s} & \\ & & U & & \end{array} \qquad \begin{array}{ccc} R & \xrightarrow{s} & U \\ \downarrow{\scriptstyle (\mathrm{id},i)} & & \downarrow{\scriptstyle e} \\ R \times_{t,U,s} R & \xrightarrow{c} & R \end{array} \qquad \begin{array}{ccc} R & \xrightarrow{t} & U \\ \downarrow{\scriptstyle (i,\mathrm{id})} & & \downarrow{\scriptstyle e} \\ R \times_{t,U,s} R & \xrightarrow{c} & R. \end{array}$$

We will often denote this data as $s,t\colon R \rightrightarrows U$.

If $(s,t)\colon R \to U \times U$ is a monomorphism, then we say that $s,t\colon R \rightrightarrows U$ is an *étale (resp. smooth) equivalence relation.*

A morphism of groupoids from $R' \rightrightarrows U'$ to $R \rightrightarrows U$ is defined as morphisms $R' \to R$ and $U' \to U$ compatible with the source, target and composition morphisms.

We can view R as a *scheme of relations on* U: a point $r \in R$ specifies a relation on the points $s(r), t(r) \in U$, which we sometimes write as $s(r) \xrightarrow{r} t(r)$. For any scheme T, the morphisms $R(T) \rightrightarrows U(T)$ define a groupoid of sets, i.e. there is composition morphism $R(T) \times_{t,U(T),s} R(T) \to R(T)$ satisfying axioms analogous to (1)-(3). We can think of an element $r \in R(T)$ as specifying a relation $u \xrightarrow{r} v$ between elements $u, v \in U(T)$. The composition morphism composes relations $u \xrightarrow{r} v$ and $v \xrightarrow{r'} w$ to the relation $u \xrightarrow{r'\circ r} w$ while the identity morphism takes $u \in U(T)$ to $u \xrightarrow{\mathrm{id}} u$ and the inverse morphism takes $u \xrightarrow{r} v$ to $v \xrightarrow{r^{-1}} u$. When $R \rightrightarrows U$ is an equivalence relation, the morphism $R(T) \to U(T) \times U(T)$ is injective and there is thus at most one relation between any two elements of $U(T)$.

Exercise 2.3.2. Show that the identity and inverse morphism are uniquely determined.

Example 2.3.3. If $G \to S$ is an étale (resp. smooth) group scheme acting on a scheme U over S via multiplication $\sigma\colon G \times U \to U$, then

$$\sigma, p_2\colon G \times_S U \rightrightarrows U$$

is an étale (resp. smooth) groupoid of schemes. The composition is

$$(G \times_S U) \times_{p_2,U,\sigma} (G \times_S U), \qquad \big((g,u),(g',u')\big) \mapsto (gg',u').$$

Here (g,u) is a T-valued point of $G \times_S U$ and can be viewed as the relation $gu \to u$.

Example 2.3.4. Let $\mathcal{X}$ be a Deligne–Mumford stack (resp. algebraic stack) and $U \to \mathcal{X}$ be an étale (resp. smooth) presentation which we assume is not only representable but representable by schemes. Define the scheme $R := U \times_{\mathcal{X}} U$, the source morphism $s = p_1 \colon R \to U$, the target morphism $t = p_2 \colon R \to U$ and the composition morphism $(s \circ p_1, t \circ p_2) \colon R \times_{t,U,s} R \to R := U \times_{\mathcal{X}} U$. This gives the structure of an étale (resp. smooth) groupoid $R \rightrightarrows U$. If X is an algebraic space, then $R \rightrightarrows U$ is an étale equivalence relation.

Choosing different presentation yields different groupoids which are equivalent under a notion called *Morita equivalence*; we will not use this notion in these notes.

2.3.2 Quotient of a groupoid

Definition 2.3.5 (Quotient stack of a groupoid)**.** Let $R \rightrightarrows U$ be a smooth groupoid. Define $[U/R]^{\mathrm{pre}}$ to be the prestack whose objects are morphisms $T \to U$ from a scheme T. A morphism $(S \xrightarrow{a} U) \to (T \xrightarrow{b} U)$ is the data of a morphism of schemes $f \colon S \to T$ and an element $r \in R(S)$ such that $s(r) = a$ and $t(r) = f \circ b$.

Define $[U/R]$ to be the stackification of $[U/R]^{\mathrm{pre}}$.

If in addition $R \rightrightarrows U$ is an equivalence relation, then $[U/R]$ is isomorphic to a sheaf (Exercise 2.3.6) and we denote it as U/R.

The fiber category $[U/R]^{\mathrm{pre}}(T)$ is the groupoid whose objects are $U(T)$ and morphisms are $R(T)$. The identity morphism $\mathrm{id} \colon U \to U$ defines a map $U \to [U/R]^{\mathrm{pre}}$ and therefore a map $p \colon U \to [U/R]$.

Exercise 2.3.6. Let $R \rightrightarrows U$ be a smooth groupoid of schemes. Show that $[U/R]$ is equivalent to a sheaf if and only if $R \rightrightarrows U$ is an equivalence relation.

Exercise 2.3.7. Extend Exercise 1.3.25 to show that if $s,t \colon R \rightrightarrows U$ is a smooth groupoid, the following diagrams are cartesian:

$$\begin{array}{ccc} R & \xrightarrow{s} & U \\ \downarrow{\scriptstyle t} & \square & \downarrow{\scriptstyle p} \\ U & \xrightarrow{p} & [U/R] \end{array} \qquad \text{and} \qquad \begin{array}{ccc} R & \xrightarrow{(s,t)} & U \times U \\ \downarrow & \square & \downarrow{\scriptstyle p\times p} \\ [U/R] & \xrightarrow{\Delta} & [U/R] \times [U/R]. \end{array}$$

2.3.3 Algebraicity of groupoid quotients

Theorem 2.3.8 (Algebraicity of Groupoid Quotients)**.** *Let $R \rightrightarrows U$ be an étale (resp. smooth) groupoid of schemes. Then $[U/R]$ is a Deligne–Mumford stack (resp. algebraic stack) and $U \to [U/R]$ is an étale (resp. smooth) presentation.*

Proof. We follow a similar argument to Theorem 2.1.8. We need to check that $U \to \mathcal{X} := [U/R]$ is representable. Let $T \to \mathcal{X}$ be a morphism from a scheme T.

It follows from the definition of $[U/R]$ as the stackification of $[U/R]^{\mathrm{pre}}$ that there exists an étale cover $T' \to T$ and a commutative diagram

$$\begin{array}{ccc} T' & \longrightarrow & U \\ \downarrow & & \downarrow \\ T & \longrightarrow & \mathcal{X}. \end{array}$$

In the commutative cube

$$\begin{array}{ccccc} & & U_{T'} & \longrightarrow & T' \\ & \swarrow & & \swarrow & \\ R & & \longrightarrow & & U \\ & & U_T & \longrightarrow & T \\ & \swarrow & & \swarrow & \\ U & & \longrightarrow & & \mathcal{X} \end{array} \tag{2.3.1}$$

the front, back, top and bottom squares are cartesian, and U_T is a sheaf. Since $T' \to T$ is a surjective étale morphism representable by schemes, so is $U_{T'} \to U_T$. This establishes that U_T is an algebraic space. □

Remark 2.3.9. In Corollary 2.4.5, we show that the quotient U/R of an étale equivalence relation is an algebraic space. This requires a little additional work as we need to show that $U \to U/R$ is representable by schemes, rather than algebraic spaces.

However, if we were place additional assumptions on an étale equivalence relation $s, t\colon R \rightrightarrows U$, namely that s and t are separated, then the above argument can be used show that U/R is an algebraic space. Indeed, in (2.3.1), as $R \to U$ is separated and locally quasi-finite, so is $U_{T'} \to T'$. By Effective Descent for Separated and Locally Quasi-finite Morphisms (Corollary B.3.6), U_T is a scheme and $U_T \to T$ is étale and surjective.

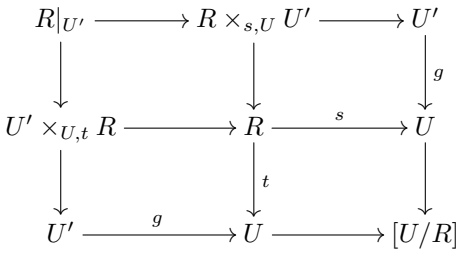
2.3.4 Slicing a groupoid

We end this subsection by introducing a useful technique to pullback a smooth groupoid $s, t\colon R \rightrightarrows U$ along a morphism $g\colon U' \to U$. Namely, define $R|_{U'}$ as the fiber product

$$\begin{array}{ccc} R|_{U'} & \xrightarrow{(t',s')} & U' \times U' \\ \downarrow & & \downarrow \\ R & \xrightarrow{(t,s)} & U \times U \end{array}$$

Exercise 2.3.10.

(1) Show that $R|_{U'}$ fits into a cartesian diagram



Assume in addition that $U' \times_{U,t} R \to R \xrightarrow{s} U$ is étale (resp. smooth).

(2) Show that $R|_{U'} \rightrightarrows U'$ is an étale (resp. smooth) groupoid.

(3) Show that that there is an open immersion $[U'/R|_{U'}] \to [U/R]$.

(4) Show that $[U'/R|_{U'}] \to [U/R]$ is an isomorphism if and only if for every every point $u \in U$, there exists a pont $u' \in U$ and a relation $u \to g(u')$ in R.

2.4 Representability of the diagonal

2.4.1 Representability

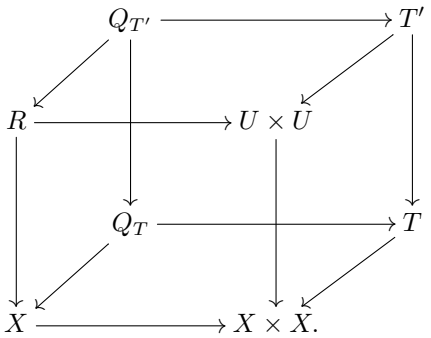
We now show that the diagonal of an algebraic space or stack is representable.

Theorem 2.4.1 (Representability of the Diagonal)**.**

(1) The diagonal of an algebraic space is representable by schemes.

(2) The diagonal of an algebraic stack is representable.

Proof. Let X be an algebraic space and $T \to X \times X$ be any morphism from a scheme. We need to show that the sheaf $Q_T = X \times_{X \times X} T$ is in fact a scheme. Let $U \to X$ be an étale presentation. If $U \to X$ is an étale presentation, then so is $U \times U \to X \times X$. The base change of $T \to X \times X$ by $U \times U \to X \times X$ is a scheme T' which is surjective étale over T. Consider the cartesian cube



(2.4.1)

Since $R \to U \times U$ is a separated and locally quasi-finite morphism of schemes, so is $Q_{T'} \to T'$. (If X had quasi-compact diagonal, then by Zariski's main theorem $R \to U \times U$ is quasi-affine and thus so is $Q_{T'} \to T'$.) Since the sheaf Q_T pulls back to the scheme $Q_{T'}$, we may apply Effective Descent for Separated and Locally Quasi-finite Morphisms (Corollary B.3.6) to conclude that Q_T is a scheme.

If $\mathcal{X}$ is an algebraic stack and $U \to \mathcal{X}$ is a smooth presentation, we may imitate the above argument. The base change of a morphism $T \to \mathcal{X} \times \mathcal{X}$ along $U \times U \to \mathcal{X} \times \mathcal{X}$, yields an algebraic space T_1 which is surjective smooth over T. Choose an étale presentation $T_2 \to T_1$. Then $T_2 \to T$ is a surjective smooth morphism of schemes which has a section after an étale cover $T' \to T$ (Proposition A.3.5). The composition $T' \to T_2 \to T_1 \to U \times U$ provides a lift of $T \to \mathcal{X} \times \mathcal{X}$. We obtain a diagram similar to (2.4.1) but where the left and right squares are not necessarily cartesian. Since $Q_{T'}$ is a scheme and $Q_{T'} \to Q_T$ is étale, surjective and representable by schemes (as $T' \to T$ is), Q_T is an algebraic space. □

Exercise 2.4.2. Extend the above argument to establish:

(1) If $\mathcal{X} \to \mathcal{Y}$ is a representable morphism of algebraic stacks (e.g. a morphism of algebraic spaces), then $\mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ is representable by schemes.

(2) If $\mathcal{X} \to \mathcal{Y}$ is a morphism of algebraic stacks, then $\mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ is representable.

Proof. TO BE ADDED □

Exercise 2.4.3. Show that the diagonal of any morphism $\mathcal{X} \to \mathcal{Y}$ of algebraic stacks is locally of finite type.

Corollary 2.4.4.

(1) Any morphism from a scheme to an algebraic space is representable by schemes.

(2) Any morphism from a scheme to an algebraic stack is representable.

Proof. This follows directly from Representability of the Diagonal (Theorem 2.4.1) and the cartesian diagram

$$\begin{array}{ccc} T_1 \times_{\mathcal{X}} T_2 & \longrightarrow & T_1 \times T_2 \\ \downarrow & & \downarrow \\ \mathcal{X} & \longrightarrow & \mathcal{X} \times \mathcal{X}. \end{array}$$

associated to any two maps $T_1 \to \mathcal{X}$ and $T_2 \to \mathcal{X}$ from schemes to an algebraic stack. □

Corollary 2.4.5. *If $R \rightrightarrows U$ be an étale equivalence relation of schemes, then U/R is an algebraic space and $U \to U/R$ is an étale presentation.*

Proof. It suffices to show that the diagonal of the quotient sheaf $X := U/R$ is representable by schemes. Indeed, this implies that $U \to X$ is representable by schemes via the argument of Corollary 2.4.4 and étale descent implies that $U \to X$ is étale and surjective.

Let $T \to X \times X$ be a morphism from a scheme and consider the commutative cube as in (2.4.1). Since $R \to U \times U$ is separated and locally quasi-finite, so is $Q_{T'} \to T'$. Effective Descent for Separated and Locally Quasi-finite Morphisms (Proposition B.3.5) implies that sheaf Q_T is a scheme. □

2.4.2 Stabilizer groups and the inertia stack

Now that we know that the diagonal is representable, we can discuss its properties. One of the most important features of the diagonal is that it encodes the stabilizer groups.

Definition 2.4.6 (Stabilizers)**.** If $\mathcal{X}$ is an algebraic stack and $x\colon \operatorname{Spec} K \to \mathcal{X}$ is a field-valued point, the *stabilizer of* x is defined as the group scheme $G_x := \underline{\mathrm{Aut}}_{\mathcal{X}(K)}(x)$.

By Exercise 1.3.28, we can identify G_x with the fiber product

$$\begin{array}{ccc} G_x := \underline{\mathrm{Aut}}_{\mathcal{X}(K)}(x) & \longrightarrow & \operatorname{Spec} K \\ \downarrow & & \downarrow{\scriptstyle (x,x)} \\ \mathcal{X} & \xrightarrow{\Delta} & \mathcal{X}\times\mathcal{X} \end{array}$$

By Representability of the Diagonal (Theorem 2.4.1), G_x is a group algebraic space. (In fact, G_x is actually a group scheme.)

Exercise 2.4.7. Let G be a group scheme over a field $\Bbbk$ acting on a $\Bbbk$-scheme X, and let $x \in X(\Bbbk)$. Show that the stabilizer of the image of x in $[X/G]$ is the usual stabilizer group scheme.

Exercise 2.4.8.

(1) Show that the stabilizer of a field-valued point of a fiber product of algebraic stacks is the fiber product of stabilizers, i.e. for $x' \in (\mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}')(\Bbbk)$, then $G_{x'} = G_x \times_{G_y} G_{y'}$ where x, y and y' are the images of x'.

(2) Does you argument for part (1) suggest any generalizations?

Exercise 2.4.9. Let $\mathcal{X}$ be a Deligne–Mumford stack.

(1) For any field-valued point $x \in \mathcal{X}(\Bbbk)$, show that G_x is an étale group scheme over $\Bbbk$.

(2) For a geometric point $x \in \mathcal{X}(\Bbbk)$ (i.e. $\Bbbk$ is algebraically closed), show that G_x is a discrete and reduced group scheme corresponding to an abstract group. In particular, if $\mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is quasi-compact, G_x is a finite (abstract) group.

(3) Show that the diagonal of $\mathcal{X}$ is unramified.

(4) For $x \in |\mathcal{X}|$, show that the discrete group $G_{\overline{x}}$ defined as the stabilizer of a geometric point $\overline{x}\colon \operatorname{Spec}\Bbbk \to \mathcal{X}$ with image x is independent of the choice of representative $\overline{x}$.

We will see later that these properties characterize Deligne–Mumford stacks.

Part (4) shows that the following definition is well-defined.

Definition 2.4.10. If $\mathcal{X}$ is a Deligne–Mumford stack and $x \in |\mathcal{X}|$, we define the *geometric stabilizer of* x as the discrete group $G = G_{\overline{x}}$ where $\overline{x}\colon \operatorname{Spec}\Bbbk \to \mathcal{X}$ is any representative of x with $\Bbbk$ algebraically closed.

Varying the point x of $\mathcal{X}$, the stabilizer group varies and naturally forms a family. In fact, we've already seen this: if $a\colon T \to \mathcal{X}$ is an object, then $\mathrm{Isom}_{\mathcal{X}(T)}(a) \to S$ is a group algebraic space such that the fiber over a point $s \in S$ is the stabilizer of the restriction $a|_{\operatorname{Spec}\kappa(s)}$ of a to $\operatorname{Spec}\kappa(s)$. Applying this to the identity map $\mathrm{id}_{\mathcal{X}}\colon \mathcal{X} \to \mathcal{X}$ yields the construction of the inertia stack.

Definition 2.4.11 (Inertia stack)**.** The *inertia stack* of an algebraic stack $\mathcal{X}$ is the fiber product

$$\begin{array}{ccc} I_{\mathcal{X}} & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ \mathcal{X} & \longrightarrow & \mathcal{X} \times \mathcal{X}. \end{array}$$

The fiber of $I_{\mathcal{X}} \to \mathcal{X}$ over a field-valued point $x\colon \operatorname{Spec} K \to \mathcal{X}$ is precisely the stabilizer G_x. We therefore think of $I_{\mathcal{X}}$ as a group scheme (or really group algebraic space) over $\mathcal{X}$ incorporating all of the stabilizers of $\mathcal{X}$.

Exercise 2.4.12. Let $G \to S$ be a group scheme acting on a scheme $X \to S$, and let $\mathcal{X} = [X/G]$ be the quotient stack. Show that there is a cartesian diagram

$$\begin{array}{ccc} S_X & \longrightarrow & X \\ \downarrow & & \downarrow \\ I_{\mathcal{X}} & \longrightarrow & \mathcal{X} \end{array}$$

where $S_X \to X$ is the stabilizer group scheme, i.e. the fiber product of the action map $G \times X \to X \times X$ and the diagonal $X \to X \times X$.

2.4.3 Properties of the diagonal

Conditions on the diagonal yield conditions on the Isom presheaves and in particular on the stabilizer groups. For instance, if the diagonal is affine, then $\mathrm{Isom}_{\mathcal{X}(T)}(a) \to T$ is an affine group scheme and in particular G_x is an (affine) algebraic group. The condition that the diagonal is affine is satisfied by most moduli problems (except for example $\mathcal{M}_1$).

In Theorem 2.1.8, we showed that if $G \to S$ is a smooth, affine group scheme acting on an algebraic space $U \to S$, the quotient stack $[U/G]$ is an algebraic stack over S.

Corollary 2.4.13. *The diagonal of $[U/G]$ is representable. If U has quasi-affine diagonal (resp. has affine diagonal), then $[U/G]$ has quasi-affine diagonal (resp. affine diagonal).*

Proof. The first statement is a direct consequence of Representability of the Diagonal (Theorem 2.4.1(2)). For the other statements, we use the cartesian diagram

$$\begin{array}{ccc} G \times_S U & \longrightarrow & U \times_S U \\ \downarrow & & \downarrow \\ [U/G] & \longrightarrow & [U/G] \times [U/G]. \end{array}$$

Since G is affine, so is the composition $G \times_S U \to U \times_S U \xrightarrow{p_1} U$. The statement follows from the Cancellation Law and descent. □

Example 2.4.14. In Theorem 2.1.11, we showed that the stack $\mathcal{M}_g$ for $g \geq 2$ is algebraic by expressing it as a quotient stack $[H'/\operatorname{PGL}_N]$ where H' is a locally closed subschemes of the (projective) Hilbert scheme. We therefore can conclude

that $\mathcal{M}_g$ has affine diagonal. We will show later that $\mathcal{M}_g$ is separated or in other words that the diagonal of $\mathcal{M}_g$ is a finite morphism.

Similarly in Theorem 2.1.15, we expressed any quasi-compact open substack of $\mathcal{B}\mathrm{un}_{r,d}(C)$ as a quotient stack $[Q'/\operatorname{PGL}_N]$ where Q' is a locally closed subschemes of the (projective) Quot scheme. To show that $\mathcal{B}\mathrm{un}_{r,d}(C)$ has affine diagonal, it suffices to consider morphisms $T \to \mathcal{B}\mathrm{un}_{r,d}(C) \times \mathcal{B}\mathrm{un}_{r,d}(C)$ from an affine scheme. But such a morphism factors through $\mathcal{U} \times \mathcal{U}$ for some quasi-compact open substack $\mathcal{U} \subset \mathcal{B}\mathrm{un}_{r,d}(C)$ and we know that $\mathcal{U}$ has affine diagonal.

2.4.4 Some separation properties

Definition 2.4.15.

(1) A morphism of algebraic stack $\mathcal{X} \to \mathcal{Y}$ is *quasi-separated* if $\mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ is quasi-compact.

(2) A representable morphism $\mathcal{X} \to \mathcal{Y}$ of algebraic stacks is *separated* if the morphism $\mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$, which is representable by schemes (Exercise 2.4.2), is proper.

(3) An algebraic stack $\mathcal{X}$ is *quasi-separated* if it is quasi-separated over $\operatorname{Spec} \mathbb{Z}$.

(4) An algebraic stack $\mathcal{X}$ is *noetherian* if it is locally noetherian, quasi-compact and quasi-separated.

Remark 2.4.16. A quasi-separated Deligne–Mumford stacks has *finite* and reduced stabilizer groups (see Exercise 2.4.9)

For morphisms of schemes, the definition of properness above agrees with the usual notation since proper monomorphisms of schemes are closed immersions. We postpone the definition of separatedness for non-representable morphisms until Definition 2.8.1.

2.5 Dimension, tangent spaces and residual gerbes

2.5.1 Dimension

Recall that the dimension of a scheme X is the Krull dimension of the underlying topological space while the dimension at a point $x \in X$ is the minimum dimension of open subsets containing x (which is in general distinct from $\dim \mathcal{O}_{X,x}$). We now extend these definitions to algebraic spaces and stacks.

Definition 2.5.1.

(1) Let X be a noetherian algebraic space and $x \in |X|$. We define the *dimension of X at x* to be

$$\dim_x X = \dim_u U \in \mathbb{Z}_{\geq 0} \cup \infty$$

where $U \to X$ is any étale presentation and $u \in U$ is any preimage.

(2) Let $\mathcal{X}$ be an algebraic stack with smooth presentation $U \to \mathcal{X}$ and corresponding smooth groupoid $s, t \colon R \rightrightarrows U$, and let $u \in U$ be a preimage of $x \in |\mathcal{X}|$. We define the *dimension of $\mathcal{X}$ at x* to be

$$\dim_x \mathcal{X} = \dim_u U - \dim_{e(u)} R_u \in \mathbb{Z} \cup \infty$$

where R_u is the fiber of $s \colon R \to U$ over u and $e \colon U \to R$ denotes the identity morphism in the groupoid.

(3) If $\mathcal{X}$ is a noetherian algebraic space or stack, we define the *dimension of* $\mathcal{X}$ to be

$$\dim \mathcal{X} = \sup_{x \in |\mathcal{X}|} \dim_x \mathcal{X} \in \mathbb{Z} \cup \infty.$$

Proposition 2.5.2. *The definition of the dimension* $\dim_x \mathcal{X}$ *of a noetherian algebraic stack* $\mathcal{X}$ *at a point* $x \in |\mathcal{X}|$ *is independent of the presentation* $U \to \mathcal{X}$ *and of the choice of preimage* u *of* x*.*

Proof. The definition of the dimension of an algebraic space at a point is clearly well defined as étale morphisms have relative dimension 0.

If $U \to \mathcal{X}$ is a smooth presentation, then by definition U is a scheme and a preimage $u \in U$ has a residue field $\kappa(u)$. The fiber R_u is is identified with the fiber product

$$\begin{array}{ccccc} R_u & \longrightarrow & R & \xrightarrow{t} & U \\ \downarrow & & \downarrow{\scriptstyle s} & & \downarrow \\ \operatorname{Spec} \kappa(u) & \longrightarrow & U & \longrightarrow & \mathcal{X}, \end{array}$$

and is a smooth algebraic space over $\kappa(u)$.

If $U' \to \mathcal{X}$ is a second presentation and $u' \in U'$ a preimage of x, then define the algebraic space $U'' := U \times_{\mathcal{X}} U'$. Observe that there is a cartesian diagram

$$\begin{array}{ccccc} U''_u & \longrightarrow & U'' & \longrightarrow & U' \\ \downarrow & & \downarrow & & \downarrow \\ \operatorname{Spec} \kappa(u) & \longrightarrow & U & \longrightarrow & \mathcal{X} \end{array} \tag{2.5.1}$$

where the fiber U''_u is identified with $R'_{u'}$. By Exercise 2.5.3 applied to $U'' \to U$, we have the identity

$$\dim_{u''} U'' = \dim_u U + \dim_{u''} U''_u = \dim_u U + \dim_{e'(u')} R'_{u'}. \tag{2.5.2}$$

Choose a representative $\operatorname{Spec} L \to U''$ in $|U''|$ mapping to u and u'. Note that the compositions $\operatorname{Spec} \kappa(u) \to U \to \mathcal{X}$, $\operatorname{Spec} \kappa(u') \to U' \to \mathcal{X}$ and $\operatorname{Spec} L \to U'' \to \mathcal{X}$ all define the same point $x \in |\mathcal{X}|$. Let $R \rightrightarrows U$ and $R' \rightrightarrows U'$ be the corresponding smooth groupoids, and set $R''_{u''} = U'' \times_{\mathcal{X}} \operatorname{Spec} L$.

We need to show that

$$\dim_u U - \dim_{e(u)} R_u = \dim_{u'} U' - \dim_{e'(u')} R'_{u'}$$

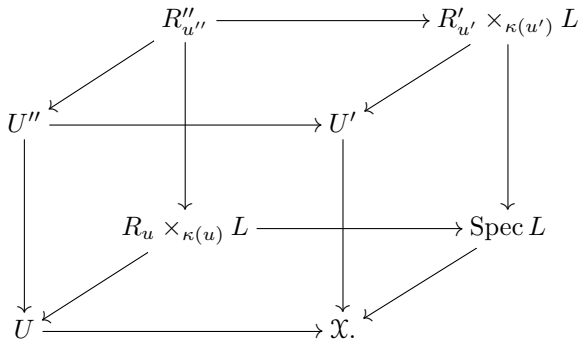
and by symmetry between U and U', it suffices to show that

$$\dim_u U - \dim_{e(u)} R_u = \dim_{u''} U'' - \dim_{e''(u'')} R''_{u''}$$

where $e''(u'') \in |R''_{u''}|$ is the image of the map $\operatorname{Spec} L \to R''_{u''} = U'' \times_{\mathcal{X}} \operatorname{Spec} L$ defined by the identity automorphism of u''. By (2.5.2), this is in turn equivalent to

$$\dim_{e''(u'')} R''_{u''} = \dim_{e(u)} R_u + \dim_{e'(u')} R'_{u'}$$

This last fact follows from the cartesian cube



and properties of dimension (see Exercise 2.5.3). □

Exercise 2.5.3.

1. Show that the analogue of Proposition A.3.9 holds for algebraic spaces; that is, if $X \to Y$ is a smooth morphism of noetherian algebraic spaces, and if $x \in |X|$ is a point with image $y \in |Y|$, then
$$\dim_x(X) = \dim_y(Y) + \dim_x(X_y).$$
2. If X and X' are noetherian algebraic spaces over a field $\Bbbk$ with $\Bbbk$-points x and x', show that
$$\dim_{(x,x')} X \times_{\Bbbk} X' = \dim_x X + \dim_{x'} X'.$$
3. Let $\mathcal{X}$ be a noetherian algebraic space over a field $\Bbbk$ and $\Bbbk \to L$ be a field extension. Set $\mathcal{X}_L = \mathcal{X} \times_{\Bbbk} L$. If $x' \in |\mathcal{X}_L|$ is a point with image $x \in |\mathcal{X}|$, show that $\dim_{x'} \mathcal{X} \times_{\Bbbk} L = \dim_x \mathcal{X}$.

Example 2.5.4. If X is a scheme of pure dimension with an action of an algebraic group G (which is necessarily of pure dimension) over a field $\Bbbk$, then
$$\dim[X/G] = \dim X - \dim G.$$
In particular, the classifying stack has dimension $\dim \mathbf{B}G = -\dim G$ and we see that the dimension may be negative!

2.5.2 Tangent spaces

For a field $\Bbbk$, we will will abuse notation by writing $\Bbbk[\epsilon]$ as $\Bbbk[\epsilon]/\epsilon^2$. We call $\Bbbk[\epsilon]$ the ring of *dual numbers*.

Definition 2.5.5. If $\mathcal{X}$ is an algebraic stack and $x\colon \operatorname{Spec}\Bbbk \to \mathcal{X}$, we define the *Zariski tangent space* or simply the *tangent space* of $\mathcal{X}$ at x as the set
$$T_{\mathcal{X},x} := \left\{ \text{2-commutative diagrams } \begin{array}{ccc} \operatorname{Spec}\Bbbk & & \\ \downarrow & \overset{\alpha}{\swarrow} & \searrow^{x} \\ \operatorname{Spec}\Bbbk[\epsilon] & \xrightarrow{\tau} & \mathcal{X} \end{array} \right\} \Big/ \sim$$

or in other words the set of pairs (τ, α) where $\tau\colon \operatorname{Spec} \Bbbk[\epsilon] \to \mathcal{X}$ and $\alpha\colon x \xrightarrow{\sim} \tau|_{\Bbbk}$. Two pairs are equivalent $(\tau, \alpha) \sim (\tau', \alpha')$ if there is an isomorphism $\beta\colon \tau \xrightarrow{\sim} \tau'$ in $\mathcal{X}(k[\epsilon])$ compatible with α and α', i.e. $\alpha' = \beta|_{\operatorname{Spec} \Bbbk} \circ \alpha$

Proposition 2.5.6. *If $\mathcal{X}$ is an algebraic stack with affine diagonal and $x \in \mathcal{X}(\Bbbk)$, then $T_{\mathcal{X},x}$ is naturally a $\Bbbk$-vector space.*

Proof. Scalar multiplication of $c \in \Bbbk$ on $(\tau, \alpha) \in T_{\mathcal{X},x}$ is defined as the composition $\operatorname{Spec} k[\epsilon] \to \operatorname{Spec} k[\epsilon] \xrightarrow{\tau} \mathcal{X}$ where the first map is defined by $\epsilon \mapsto c\epsilon$ and with the same 2-isomorphism α.

To define addition, we will show that there is an equivalence of categories

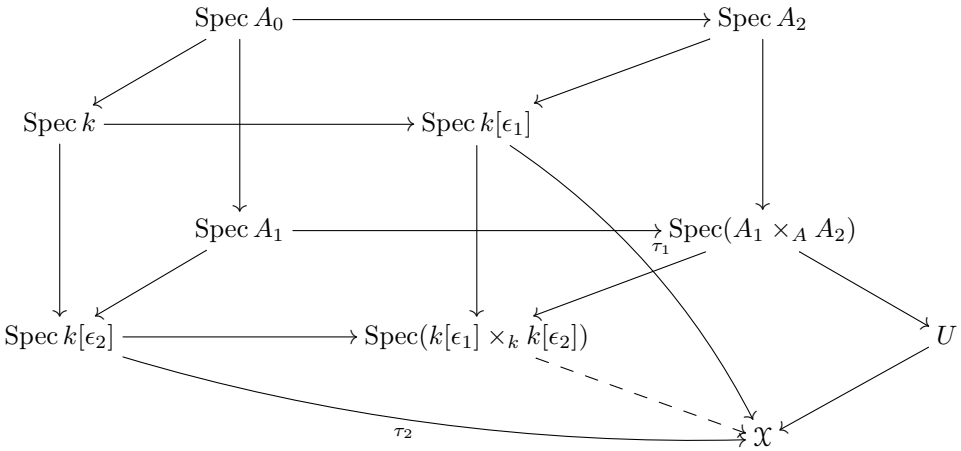
$$\mathcal{X}(k[\epsilon_1] \times_k k[\epsilon_2]) \to \mathcal{X}(k[\epsilon_1]) \times_{\mathcal{X}(k)} \mathcal{X}(k[\epsilon_2]) \tag{2.5.3}$$

or in other words that

$$\begin{array}{ccc} \operatorname{Spec} k & \hookrightarrow & \operatorname{Spec} k[\epsilon_1] \\ \downarrow & & \downarrow \\ \operatorname{Spec} k[\epsilon_2] & \longrightarrow & \operatorname{Spec}(k[\epsilon_1] \times_k k[\epsilon_2]) \end{array}$$

is a pushout in algebraic stacks with affine diagonal. Once this is established, we define addition of (τ_1, α_1) and (τ_2, α_2) by the composition $\operatorname{Spec} k[\epsilon] \to \operatorname{Spec}(k[\epsilon_1] \times_k k[\epsilon_2]) \to \mathcal{X}$ where the first map is defined sending both $(\epsilon_1, 0)$ and $(0, \epsilon_2)$ to ϵ.

Choose a smooth morphism $(U, u) \to (\mathcal{X}, x)$ from an affine scheme U. Since $\mathcal{X}$ has affine diagonal $U \to \mathcal{X}$ is an affine morphism. Let $\operatorname{Spec} A_0 = \operatorname{Spec} k \times_{\mathcal{X}} U$, $\operatorname{Spec} A_1 = \operatorname{Spec} k[\epsilon_1] \times_{\mathcal{X}} U$ and $\operatorname{Spec} A_2 = \operatorname{Spec} k[\epsilon_2] \times_{\mathcal{X}} U$. Since $\operatorname{Spec}(A_1 \times_A A_2)$ is clearly the pushout of $\operatorname{Spec} A_0 \hookrightarrow \operatorname{Spec} A_1$ and $\operatorname{Spec} A_0 \hookrightarrow \operatorname{Spec} A_2$ in the category of affine schemes, there are unique morphisms $\operatorname{Spec}(A_1 \times_A A_2) \to \operatorname{Spec}(k[\epsilon_1] \times_k k[\epsilon_2])$ and $\operatorname{Spec}(A_1 \times_A A_2) \to U$ completing the diagram



By the flatness criterion over artinian algebras (i.e. a module is flat if and only if its free), we see that the map $\operatorname{Spec}(A_1 \times_A A_2) \to \operatorname{Spec}(k[\epsilon_1] \times_k k[\epsilon_2])$ is faithfully flat. By repeating this argument on $U \times_{\mathcal{X}} U$, one argues that the $\operatorname{Spec}(A_1 \times_A A_2) \to U$ descends uniquely providing the desired dotted arrow. □

Exercise 2.5.7. Show that Proposition 2.5.6 remains true without the affine diagonal condition.

Remark 2.5.8. Suppose that $A_0 \to A_1$ and $A_0 \to A_2$ of local artinian rings such that $A_0 \twoheadrightarrow A_1$ is surjective. If $\mathcal{X}$ is an algebraic stack, the same argument shows that

$$\mathcal{X}(A_1 \times_{A_0} A_2) \to \mathcal{X}(A_1) \times_{\mathcal{X}(A_0)} \mathcal{X}(A_2)$$

is an equivalence of categories.

This condition is usually referred as *homogeneity* and this exercise states that homogeneity is a necessary condition for algebraicity. Conversely, in Schlessinger's criteria (resp. Artin's criteria), a variant of the above homogeneity condition is in fact one of the conditions ensuring the existence of a formal miniversal deformation space (resp. algebraicity of $\mathcal{X}$).

Exercise 2.5.9. Show that $T_{\mathcal{X},x}$ is naturally a representation of G_x which is given set-theoretically by: $g \cdot (\tau, \alpha) = (\tau, g \circ \alpha)$ for $g \in G_x$ and $(\tau, \alpha) \in T_{\mathcal{X},x}$.

2.5.3 Residual gerbes

If X is a scheme, any point $x \in X$ has a residue field $\kappa(x)$ and there is a unique monomorphism $\operatorname{Spec} \kappa(x) \to X$ with image x. We would like an analogous fact for algebraic stacks but the existence of non-trivial stabilizers prevents field-valued points from being monomorphism, e.g. $\mathbf{B}_{\Bbbk} G$ for a finite group G admits no monomorphisms from fields. Residual gerbes however provide an analogous notion.

Definition 2.5.10. Let $\mathcal{X}$ be an algebraic stack and $x \in |\mathcal{X}|$ be a point. Choose a smooth presentation $(U, u) \to (\mathcal{X}, x)$. The *residual gerbe of* x is the substack $\mathcal{G}_x \subset \mathcal{X}$ defined as the stackification of the full subcategory $\mathcal{G}_x^{\mathrm{pre}}$ consisting of objects $a \in \mathcal{X}$ over S which factor as $a \colon S \to \operatorname{Spec} \kappa(u) \to \mathcal{X}$.

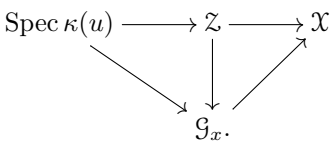
The residual gerbe $\mathcal{G}_x$ is in fact a gerbe over the spectrum of a field $\kappa(x)$ which we call the *residual field of* x (see Exercise 2.9.17). If in addition $\mathcal{X}$ is of finite type over $\Bbbk$ and $x \in \mathcal{X}(k)$, then $\mathcal{G}_x = \mathbf{B}G_x$ is the trivial gerbe (Exercise 2.5.13).

Exercise 2.5.11. Show that the definition of the residual gerbe is independent of the presentation.

Exercise 2.5.12. Let $\mathcal{X}$ be an algebraic stack and $x \in |\mathcal{X}|$ be a point. Choose a smooth presentation $(U, u) \to (\mathcal{X}, x)$.

(1) Show that there is a factorization $\operatorname{Spec} \kappa(u) \to \mathcal{G}_x \to \mathcal{X}$ where $\operatorname{Spec} \kappa(u) \to \mathcal{G}_x$ is an epimorphism and $\mathcal{G}_x \to \mathcal{X}$ is a monomorphism.

(2) Show that $\mathcal{G}_x$ satisfies the following universal property: given any other factorization $\operatorname{Spec} \kappa(u) \to \mathcal{Z} \to \mathcal{X}$ where $\operatorname{Spec} \kappa(u) \to \mathcal{Z}$ is an epimorphism and $\mathcal{Z} \to \mathcal{X}$ is a monomorphism, there is a morphism $\mathcal{Z} \to \mathcal{G}_x$ unique up to unique isomorphism and a 2-commutative diagram



Exercise 2.5.13. Let $\mathcal{X}$ be a noetherian algebraic stack and $x \colon \operatorname{Spec} \Bbbk \to \mathcal{X}$. Suppose that the stabilizer G_x is smooth and affine. The classifying stack $\mathbf{B}G_x$ is the trivial gerbe banded by G_x (Exercise 2.9.16).

(1) Show that there is a canonical morphism $\mathbf{B}G_x \to \mathcal{X}$.

(2) If $\mathcal{X}$ is of finite type over $\Bbbk$, show that the map $\mathbf{B}G_x \hookrightarrow \mathcal{X}$ is a monomorphism and identified with the residual gerbe $\mathcal{G}_x \hookrightarrow \mathcal{X}$ of $x \in |\mathcal{X}|$.

Recall that we say that a point $x \in |\mathcal{X}|$ of an algebraic stack has *smooth* or *affine* stabilizer if any representative (or equivalently all) does (Exercise 2.2.13).

Proposition 2.5.14 (Residual Gerbes are Algebraic)**.** *Let $\mathcal{X}$ be a noetherian algebraic stack. Let $x \in |\mathcal{X}|$ be a finite type point with smooth stabilizer. Then $\mathcal{G}_x$ is an algebraic stack and $\mathcal{G}_x \hookrightarrow \mathcal{X}$ is a locally closed immersion. Moreover, if $(U, u) \to (\mathcal{X}, x)$ is a smooth morphism from a scheme U, then there is a cartesian diagram*

$$\begin{array}{ccc} O(u) & \hookrightarrow & U \\ \downarrow & & \downarrow \\ \mathcal{G}_x & \hookrightarrow & \mathcal{X} \end{array} \tag{2.5.4}$$

where $O(u)$ is the orbit $s(t^{-1}(u))$ of the induced groupoid $s, t\colon R := U \times_{\mathcal{X}} U \rightrightarrows U$.

If in addition $\mathcal{X}$ is of finite type over $\Bbbk$ and $x \in \mathcal{X}(k)$, then $\mathcal{G}_x = \mathbf{B}G_x$.

Remark 2.5.15. Residual gerbes are algebraic for arbitrary quasi-separated algebraic stacks and arbitrary points $x \in |\mathcal{X}|$. The known proofs of this however rely on knowing the quotient stack of an fppf equivalence relation is algebraic.

The locally closedness of the orbit $O(u) \hookrightarrow U$ should be compared to the familiar fact that orbits of algebraic group actions are locally closed; indeed, this is the implication when $\mathcal{X}$ is a quotient stack $[U/G]$.

Proof. Let $\mathcal{G} = \mathcal{G}_x$ be the residual gerbe. After replacing $\mathcal{X}$ with the smallest closed substack containing x, we may assume that $x \in |\mathcal{X}|$ is dense. Let $(U, u) \to (\mathcal{X}, x)$ be a smooth presentation. Then the morphism $\operatorname{Spec} \kappa(u) \to U \to \mathcal{X}$ is of finite type and is a representative of x. Consider the cartesian diagram

$$\begin{array}{ccccc} U_k & \longrightarrow & U_{\mathcal{G}} & \longrightarrow & U \\ \downarrow & & \downarrow & & \downarrow \\ \operatorname{Spec} k & \longrightarrow & \mathcal{G} & \longrightarrow & \mathcal{X}. \end{array}$$

By Generic Flatness (Proposition A.2.6) and descent, $\operatorname{Spec} k \to \mathcal{X}$ is flat over an open substack of $\mathcal{X}$. But since $x \in |\mathcal{X}|$ is dense, $\operatorname{Spec} k \to \mathcal{X}$ is flat. As $\operatorname{Spec} k \to \mathcal{X}$ is also of finite type, we may apply descent and Openness of Fppf Morphisms (Proposition A.2.2) to conclude that the image of $\operatorname{Spec} k \to \mathcal{X}$ is open. Thus, after replacing $\mathcal{X}$ with an open substack, we may assume that $\operatorname{Spec} k \to \mathcal{X}$ is fppf.

The stabilizer of $\operatorname{Spec} k \to \mathcal{X}$ is smooth by assumption and identified with $\operatorname{Spec} k \times_{\mathcal{X}} \operatorname{Spec} k$. By applying descent again, we see that $\operatorname{Spec} k \to \mathcal{X}$ is smooth and surjective.

Since the diagonal of $\mathcal{X}$ is representable by schemes, U_k is a scheme. Moreover, $U_{\mathcal{G}}$ is identified as the sheaf-theoretic image of the morphism $U_k \to U$ of sheaves on $\mathrm{Sch}_{\text{ét}}$, and thus can be identified with the quotient sheaf $U_k/(U_k \times_U U_k)$ of the smooth equivalence relations $U_k \times_U U_k \rightrightarrows U_k$. By Theorem 2.3.8 (generalized to allow smooth equivalence relations of algebraic spaces), $U_{\mathcal{G}}$ is an algebraic

stack and $U_k \to U_{\mathcal{G}}$ is a smooth presentation. (While $U_{\mathcal{G}}$ is also a sheaf, we don't know yet that $U_{\mathcal{G}}$ is an algebraic space.) Since $U \to \mathcal{X}$ is surjective, smooth and representable, so is $U_{\mathcal{G}} \to \mathcal{G}$, and it follows that $U_k \to U_{\mathcal{G}} \to \mathcal{G}$ is a smooth presentation. This shows that $\mathcal{G}$ is an algebraic stack. Since $U_k \to U_{\mathcal{G}}$ is smooth and surjective, it follows from descent that $\operatorname{Spec} k \to \mathcal{G}$ is a smooth presentation of $\mathcal{G}$. Since smoothness is smooth-local, we have that $\mathcal{G} \to \mathcal{X}$ is also smooth.

To summarize, we know now that $\mathcal{G} \to \mathcal{X}$ is a surjective and smooth monomorphism. We wish to show that $\mathcal{G} \to \mathcal{X}$ is an isomorphism. It suffices to show that $U_{\mathcal{G}} \to U$ is an isomorphism of sheaves in $\mathrm{Sch}_{\text{ét}}$ and for this, it suffices to show that any morphism $T \to U$ from a scheme lifts to a morphism $T' \to U_{\mathcal{G}}$ after an étale cover $T' \to T$. Considering the cartesian diagram

$$\begin{array}{ccccc} T' & \longrightarrow & V & \longrightarrow & T \\ \downarrow & & \downarrow & & \downarrow \\ U_k & \longrightarrow & U_{\mathcal{G}} & \longrightarrow & U, \end{array}$$

we have a smooth cover $T' \to T$ and a lift $T' \to U_{\mathcal{G}}$ of $T \to U$. Since smooth morphisms étale locally have sections (Proposition A.3.5), we have established the desire claim. □

Exercise 2.5.16. Let $\mathcal{X}$ be a noetherian algebraic stack with affine diagonal and $x \in |\mathcal{X}|$ be a finite type point with smooth stabilizer. Let $\overline{x}\colon \operatorname{Spec} \Bbbk \to \mathcal{X}$ be a representative of x. Show that $\dim \mathcal{G}_x = -\dim G_{\overline{x}}$.

2.6 Characterization of Deligne–Mumford stacks

2.6.1 Slicing presentations

Theorem 2.6.1 (Existence of Miniversal Presentations)**.** *Let $\mathcal{X}$ be a noetherian algebraic stack and $x \in |\mathcal{X}|$ a finite type point with smooth stabilizer G_x. Then there exists a smooth morphism $(U, u) \to (\mathcal{X}, x)$ of relative dimension $\dim G_x$ from a scheme U such that the diagram*

$$\begin{array}{ccc} \operatorname{Spec} \kappa(u) & \hookrightarrow & U \\ \downarrow & & \downarrow \\ \mathcal{G}_x & \hookrightarrow & \mathcal{X} \end{array}$$

is cartesian.

In particular, if G_x is finite and reduced, there is an étale morphism $(U, u) \to (\mathcal{X}, x)$ from a scheme.

Remark 2.6.2. While the stabilizer group $G_{\overline{x}}$ depends on the choice of representative $\overline{x}\colon \operatorname{Spec} \Bbbk \to \mathcal{X}$ of $x \in |\mathcal{X}|$, its dimension—which we denote by $\dim G_x$—is independent of this choice. Similarly, the properties of being smooth, affine, finite and reduced are also independent of this choice.

A smooth presentation $p\colon U \to \mathcal{X}$ is called a *miniversal at $u \in U(\Bbbk)$* if $T_{U,u} \to T_{\mathcal{X},p(u)}$ is an isomorphism of $\Bbbk$-vector spaces. We will see that the above presentations are miniversal in Proposition 2.7.3 as a consequence of the Formal Lifting Criteria for smoothness; see also Section 2.7.4.

Proof. Let $(U,u) \to (\mathcal{X},x)$ be any smooth morphism of relative dimension n from a scheme and consider

$$\begin{array}{ccc} O(u) & \hookrightarrow & U \\ \downarrow & & \downarrow \\ \mathcal{G}_x & \hookrightarrow & \mathcal{X} \end{array}$$

The residual gerbe $\mathcal{G}_x$ is smooth of dimension $-\dim G_x$ and thus $O(u)$ is smooth at u of dimension $c := n - \dim G_x$. Let $f_1,\dots,f_c \in \mathcal{O}_{O(u),u}$ be a regular sequence generating the maximal ideal at u. After replacing U with an open affine neighborhood of u, we may assume that each f_i is a global function on U. We can consider the closed subscheme $W := V(f_1,\dots,f_c)$ which by design intersects $O(u)$ transversely at U, i.e. $W \cap O(u) = \operatorname{Spec}\kappa(u)$ scheme-theoretically.

We will inductively apply a version of the local criterion of flatness: if $(A,\mathfrak{m}_A) \to (B,\mathfrak{m}_B)$ is a flat local ring homomorphism of local noetherian rings and $f \in \mathfrak{m}_B$ is a non-zero divisor in $B \otimes_A A/\mathfrak{m}_A$, then $A \to B \to B/f$ is flat. By working on the groupoid $R \rightrightarrows U$ and using descent, this local criterion implies that the composition $W \hookrightarrow U \to \mathcal{X}$ is flat at u. Since G_x is smooth, so is $\operatorname{Spec}\kappa(u) \to \mathcal{G}_x$. For flat morphisms, smoothness is a property that can be checked on fibers and thus (again arguing on $R \rightrightarrows U$ and using descent) $W \to \mathcal{X}$ is smooth at u. We win after replacing W with an open neighborhood of u. □

2.6.2 Equivalent characterizations

Theorem 2.6.3 (Characterization of Deligne–Mumford Stacks)**.** *Let $\mathcal{X}$ be a noetherian algebraic stack. The following are equivalent:*

(1) the stack $\mathcal{X}$ is a Deligne–Mumford;

(2) the diagonal $\mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is unramified; and

(3) every point of $\mathcal{X}$ has a finite and reduced stabilizer group.

Remark 2.6.4. The equivalence (2) $\iff$ (3) is essentially the definition of unramified. Indeed, since the diagonal $\mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is always locally of finite type (Exercise 2.4.3), it is unramified if and only if every geometric fiber (which is either empty or isomorphic to a stabilizer) is discrete and reduced.

The result remains true without the noetherian hypothesis as long as one replaces "finite" with "discrete" (as the diagonal may not be quasi-compact).

Proof. A Deligne–Mumford stack has unramified diagonal or equivalently discrete and reduced stabilizer groups (Exercise 2.4.9). For the converse, Existence of Miniversal Presentations (Theorem 2.6.1) provides an étale cover of $\mathcal{X}$ which is necessarily representable (Corollary 2.4.4).

□

Theorem 2.6.5 (Characterization of Algebraic Spaces)**.** *Let $\mathcal{X}$ be a noetherian algebraic stack whose diagonal is representable by schemes. The following are equivalent:*

(1) the stack $\mathcal{X}$ is an algebraic space;

(2) the diagonal $\mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is a monomorphism; and

(3) every point of $\mathcal{X}$ has a trivial stabilizer.

Remark 2.6.6. The result is true without the ugly hypothesis that $\Delta_{\mathcal{X}}$ is representable by schemes. However, establishing this generalization requires some work.

Proof. Condition (2) is equivalent to the condition that $\mathcal{X}$ is a sheaf. The implication (1) $\implies$ (2) follows from the definition of an algebraic space. For the converse, if $\mathcal{X}$ is a sheaf, then Theorem 2.6.1 implies that there exists a representable, étale and surjective morphism $U \to \mathcal{X}$ from a scheme. Since $\Delta_{\mathcal{X}}$ is representable by schemes, so is $U \to \mathcal{X}$.

The equivalence (2) $\iff$ (3) follows from the fact that a separated group scheme of finite type is trivial if and only if every fiber is trivial (Exercise C.2.2). □

Corollary 2.6.7. *Let $\mathcal{X} \to \mathcal{Y}$ be a morphism of noetherian algebraic stacks whose diagonal is representable by schemes. Then $\mathcal{X} \to \mathcal{Y}$ is representable if and only if for every geometric point $x \in \mathcal{X}(K)$, the map $G_x \to G_{f(x)}$ on automorphism groups is injective.* □

2.6.3 Applications

Corollary 2.6.8. *If $g \geq 2$, $\mathcal{M}_g$ is a Deligne–Mumford of finite type over $\mathbb{Z}$ with affine diagonal.*

Proof. It only remains to show that $\mathcal{M}_g$ is Deligne–Mumford and by Theorem 2.6.3 it suffices to show that for any smooth, connected and proper curve C over $\Bbbk$ that $G = \mathrm{Aut}(C)$ is discrete and reduced, or in other words that the dimension of the Lie algebra $\dim T_{G,e} = 0$. The vector space $T_{G,e}$ is identified with the automorphism group of the trivial first order deformation of C. Infinitesimal Deformation theory (Theorem E.1.1) implies that $\dim T_{G,e} = H^0(C, T_C)$, but this vanishes since the degree of $T_C = \Omega_C^\vee$ is $2 - 2g < 0$. □

2.7 Smoothness and the Formal Lifting Criterion

We state and prove the Formal Lifting Criteria (Proposition 2.7.1) which provides an extremely useful functorial criteria to check that moduli stacks are smooth. We apply this criteria to establish that the moduli stacks $\mathcal{M}_g$ of smooth curves and $\mathcal{B}\mathrm{un}_{r,d}(C)$ of vector bundles are smooth (Propositions 2.7.4 and 2.7.5)

2.7.1 Formal Lifting Criteria

Since flatness and smoothness are smooth-local properties on the source and target, we have the notions of smoothness and flatness for arbitrary morphisms of algebraic stacks (Definition 2.2.1). Since étaleness and unramifiedness are étale-local on the source and smooth-local on the target, we can make sense of étale or unramified morphisms of algebraic stacks as a relatively Deligne–Mumford morphism (Definition 2.2.5) with the corresponding property.

Proposition 2.7.1 (Formal Lifting Criteria for Unramifed/Étale/Smooth Morphisms)**.** *Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks and consider a 2-*

commutative diagram

$$\begin{array}{ccc} \operatorname{Spec} A_0 & \longrightarrow & \mathcal{X} \\ \downarrow & \nearrow & \downarrow f \\ \operatorname{Spec} A & \longrightarrow & \mathcal{Y}, \end{array} \tag{2.7.1}$$

of solid arrows where $A \to A_0$ is a surjection of rings with nilpotent kernel. Then

(1) f is unramified if and only if f is locally of finite type and for every 2-commutative diagram (2.7.1), any two liftings are isomorphic.

(2) f is étale if and only if f is locally of finite presentation and for every 2-commutative diagram (2.7.1), there exists a lifting which is unique up to unique isomorphism.

(3) f is smooth if and only if f is locally of finite presentation and for every 2-commutative diagram (2.7.1), there exists a lifting.

Moreover, to verify that f is unramified, étale, or smooth, it suffices to restrict to diagrams (2.7.1) where A and A_0 are local artinian rings with residue field K and $\ker(A \to A_0) \cong K$.

Remark 2.7.2. To be explicit, a *lifting* of a 2-commutative diagram

$$\begin{array}{ccc} S & \xrightarrow{x} & \mathcal{X} \\ {\scriptstyle g}\downarrow & \swarrow_{\alpha} & \downarrow{\scriptstyle f} \\ T & \xrightarrow[y]{} & \mathcal{Y}, \end{array} \tag{2.7.2}$$

is the data of a morphism $\widetilde{x}\colon T \to \mathcal{X}$ as pictured

$$\begin{array}{ccc} S & \xrightarrow{x} & \mathcal{X} \\ {\scriptstyle g}\downarrow & \overset{\beta}{\Uparrow}\ \nearrow & \downarrow{\scriptstyle f} \\ T & \xrightarrow[y]{\ \Downarrow\gamma\ } & \mathcal{Y}, \end{array}$$

together with 2-morphisms $\beta\colon \widetilde{x} \circ g \xrightarrow{\sim} x$ and $\gamma\colon f \circ \widetilde{x} \xrightarrow{\sim} y$ such that

$$\begin{array}{ccc} & f \circ x & \\ {\scriptstyle\alpha}\swarrow & & \nwarrow{\scriptstyle f(\beta)} \\ y \circ g & \xleftarrow[g^*\gamma]{} & f \circ \widetilde{x} \circ g \end{array}$$

commutes. A morphism $(\widetilde{x}, \beta, \gamma) \xrightarrow{\sim} (\widetilde{x}', \beta', \gamma')$ of liftings is a 2-morphism $\Theta\colon \widetilde{x} \to \widetilde{x}'$ such that $\beta = \beta' \circ (\Theta \circ g)$ and $\gamma = \gamma' \circ f(\Theta)$.

We can also interpret liftings using the map $\Psi\colon \mathcal{X}(T) \to \mathcal{X}(S) \times_{\mathcal{Y}(S)} \mathcal{Y}(T)$ of groupoids. The 2-commutativity of (2.7.2) defines an object $(x, y, \alpha) \in \mathcal{X}(S) \times_{\mathcal{Y}(S)} \mathcal{Y}(T)$ and the category of liftings is the fiber category over this object, e.g. a lifting is an object $\widetilde{x} \in \mathcal{X}(T)$ together with an isomorphism $\Psi(\widetilde{x}) \to (x, y, \alpha)$. For instance, the existence of a lifting translates to the essential surjectivity of $\mathcal{X}(T) \to \mathcal{X}(S) \times_{\mathcal{Y}(S)} \mathcal{Y}(T)$.

Proof. TO ADD

□

2.7.2 Applications

As a first application, we see that the presentations produced by Existence of Miniversal Presentations (Theorem 2.6.1) are in fact miniversal, and that the dimension of a smooth algebraic stack can be computed in terms of its tangent space and stabilizer.

Proposition 2.7.3. *Let $\mathcal{X}$ be a noetherian algebraic stack and $x \in |\mathcal{X}|$ be a finite type point with smooth stabilizer. Let $f\colon (U,u) \to (\mathcal{X},x)$ be a smooth morphism from a scheme such that $\mathcal{G}_x \times_{\mathcal{X}} U \cong \operatorname{Spec}\kappa(u)$. Then $U \to \mathcal{X}$ is miniversal at u, i.e. $T_{U,u} \to T_{\mathcal{X},f(u)}$ is an isomorphism of $\kappa(u)$-vector spaces.*

In particular, if $\mathcal{X}$ is a smooth over a field $\Bbbk$ and $x \in \mathcal{X}(\Bbbk)$ is a point with smooth stabilizer. Then

$$\dim_x \mathcal{X} = \dim T_{\mathcal{X},x} - \dim G_x.$$

Proof. Surjectivity of $T_{U,u} \to T_{\mathcal{X},f(u)}$ follows from the Formal Lifting Criterion (Proposition 2.7.1). Let $\Bbbk = \kappa(u)$. Injectivity follows from the fact that

$$\begin{array}{ccc} \operatorname{Spec}\Bbbk & \hookrightarrow & U \\ \downarrow & & \downarrow \\ \mathcal{G}_x & \hookrightarrow & \mathcal{X} \end{array}$$

is cartesian. Indeed, if $\tau\colon \operatorname{Spec}\Bbbk[\epsilon] \to U$ is an element of $T_{U,u}$ mapping to $0 \in T_{\mathcal{X},f(u)}$, then by the definition of the residual gerbe, the composition $\operatorname{Spec}\Bbbk[\epsilon] \to U \to \mathcal{X}$ factors through $\mathcal{G}_x$ and therefore also factors through the fiber product $\operatorname{Spec}\Bbbk$. We conclude that $\tau = 0$.

For the last statement, Existence of Miniversal Presentations (Theorem 2.6.1) produces a smooth morphism $(U,u) \to (\mathcal{X},x)$ miniversal at u and whose relative dimension is equal to $\dim G_x$. Therefore $\dim_x \mathcal{X} = \dim_u U - \dim G_x$ but since U is smooth at u, we have $\dim_u U = \dim T_{U,u} = \dim T_{\mathcal{X},x}$. □

2.7.3 Smoothness of moduli problems

The Formal Lifting Criterion for Smoothness and infinitesimal deformation theory provide a useful technique to verify smoothness of a moduli problem and to compute its dimension.

Proposition 2.7.4. *For $g \geq 2$, the Deligne–Mumford stack $\mathcal{M}_g$ is smooth over $\operatorname{Spec}\mathbb{Z}$ of relative dimension $3g-3$.*

Proof. Let $\operatorname{Spec}\Bbbk \to \mathcal{M}_g$ be a morphism from a field $\Bbbk$ corresponding to smooth projective and connected curve $C \to \operatorname{Spec}\Bbbk$. Consider a diagram

$$\begin{array}{ccccc} \operatorname{Spec}\Bbbk & \longrightarrow & \operatorname{Spec}A_0 & \xrightarrow{[C]} & \mathcal{M}_g \\ & & \downarrow & \swarrow_{\alpha} & \downarrow{\scriptstyle f} \\ & & \operatorname{Spec}A & \longrightarrow & \operatorname{Spec}\mathbb{Z}, \end{array} \tag{2.7.3}$$

where $A \to A_0$ is surjection of local artinian rings with residue field $\Bbbk$ such that $\Bbbk = \ker(A \to A_0)$. The map $\operatorname{Spec} A_0 \to \mathcal{M}_g$ corresponds to a family of curves $\mathcal{C}_0 \to \operatorname{Spec} A_0$ and a cartesian diagram

$$\begin{array}{ccccc} C & \hookrightarrow & \mathcal{C}_0 & \dashrightarrow & \mathcal{C} \\ \downarrow & & \downarrow & & \vdots \\ \operatorname{Spec} \Bbbk & \hookrightarrow & \operatorname{Spec} A_0 & \hookrightarrow & \operatorname{Spec} A \end{array}$$

of solid arrows: a lifting of the diagram (2.7.3) corresponds to a family $\mathcal{C} \to \operatorname{Spec} A$ extending $\mathcal{C}_0 \to \operatorname{Spec} A_0$. By Theorem E.1.1, there is cohomology class $\mathrm{ob} \in H^2(C, T_C)$ such that $\mathrm{ob} = 0$ if and only if there exists a lifting. Since C is a curve, $H^2(C, T_C) = 0$.

Finally, Theorem E.1.1 also provides an identification of the tangent space of $\mathcal{M}_{g,\Bbbk} := \mathcal{M}_g \times_{\mathbb{Z}} \Bbbk$ at C (i.e. the set of extensions of C to families of curves over $\operatorname{Spec} \Bbbk[\epsilon]$) with $H^1(C, T_C)$. Since $\deg T_C < 0$, $H^0(C, T_C) = 0$ and Riemann–Roch implies

$$\dim H^1(C, T_C) = -\chi(T_C) = -(\deg T_C + (1-g)) = 3g - 3.$$

□

Proposition 2.7.5. *The algebraic stack* $\mathcal{B}\mathrm{un}_{r,d}(C)$ *is smooth over* $\operatorname{Spec} \Bbbk$ *of dimension* $r^2(g-1)$.

Proof. Let $[F] \in \mathcal{B}\mathrm{un}_{r,d}(C)(\Bbbk)$ be a vector bundle on C of rank r and degree d. Let $A \to A_0$ be a surjection of local artinian rings with residue field $\Bbbk$ such that $\Bbbk = \ker(A \to A_0)$. We need to check that any vector bundle $\mathcal{F}_0$ on C_{A_0} that restricts to F extends to a vector bundle $\mathcal{F}$ on C_A. By Infinitesimal Deformation Theory (missing reference) there is an element $\mathrm{ob} \in \operatorname{Ext}^2(F, F)$ such that $\mathrm{ob} = 0$ if and only if there exists an extension. Since C is a smooth curve, $\operatorname{Ext}^2(F, F) = H^2(C_{\Bbbk}, F \otimes F^\vee) = 0$.

Infinitesimal Deformation Theory also provides an identification of the tangent space of $\mathcal{B}\mathrm{un}_{r,d}(C)(\Bbbk)$ (i.e. the set of extensions of F to vector bundles on $C_{\Bbbk[\epsilon]}$) with $\operatorname{Ext}^1(F, F)$. Since $\dim \operatorname{Aut}(F) = \dim Ext^0(F, F) = \dim H^0(C, F \otimes F^\vee)$, we compute by Riemann–Roch that

$$\begin{aligned} \dim \operatorname{Ext}^1(F, F) &= \dim H^1(C, F \otimes F^\vee) \\ &= \dim H^1(C, F \otimes F^\vee) - \big(\deg(F \otimes F^\vee) + \mathrm{rk}(F \otimes F^\vee)(1-g)\big) \\ &= \dim \operatorname{Aut}(F) + r^2(1-g). \end{aligned}$$

and therefore $\dim_{[F]} \mathcal{B}\mathrm{un}_{r,d}(C) = \dim \operatorname{Ext}^1(F, F) - \dim \operatorname{Aut}(F) = r^2(g-1)$. □

2.7.4 Aside: Criteria of Schlessinger–Rim and Artin

The criteria of Schlessinger–Rim and Artin provide an alternative approach to constructing miniversal presentations for a stack $\mathcal{X}$ and at the same for verifying the algebraicity of $\mathcal{X}$. Restricting to a stack $\mathcal{X}$ (not assumed algebraic) defined over an algebraically closed field $\Bbbk$, the Schlessinger–Rim's conditions require that

- for maps of local artinian $\Bbbk$-algebras $A_1 \to A_0$ and $A_2 \to A_0$, the functor

$$\mathcal{X}(A_1 \times_{A_0} A_2) \to \mathcal{X}(A_1) \times_{\mathcal{X}(A_0)} \mathcal{X}(A_2)$$

  is essentially surjective when $A_2 \twoheadrightarrow A_0$ is a small extension (i.e. $\ker(A_2 \to A_0) \cong k$) and an equivalence when $A_0 = \Bbbk$ and $A_2 = \Bbbk[\epsilon]$ (this is a variant of the *homogeneity* condition—see Remark 2.5.8), and

- $T_{\mathcal{X},x}$ is finite dimensional as a $\Bbbk$-vector space.

These conditions imply that if $x_0 \in \mathcal{X}(k)$, there exists a complete local noetherian ring $(A, \mathfrak{m})$ and a compatible sequence of objects $x_n \in \mathcal{X}(A/\mathfrak{m}^{n+1})$ which are formally smooth, i.e. satisfies a Formal Lifting Criteria for local artinian $\Bbbk$-algebras (we encourage the reader to spell out precisely what this means), and miniversal, i.e. $(\mathfrak{m}/\mathfrak{m}^2)^\vee \to T_{\mathcal{X},x}$ is an isomorphism. One sometimes writes $(x_n)\colon \operatorname{Spf} A \to \mathcal{X}$ which we call the 'formal miniversal deformation.'

Grothendieck's Existence Theorem often implies an equivalence of categories $\mathcal{X}(A) \to \varprojlim \mathcal{X}(A/\mathfrak{m}^{n+1})$ and allows us to 'effective' the formal deformation (x_n) to an object $\widehat{x}\colon \operatorname{Spec} A \to \mathcal{X}$, which we can all an 'effective formal versal deformation.' Finally, by verifying additional properties of the local deformation and obstruction theory of $\mathcal{X}$, Artin's criteria can be used to 'smooth out' (x_n) to obtain a *finite type* scheme U and a smooth morphism $(U, u) \to (\mathcal{X}, x)$ miniversal at u such that $A \cong \widehat{\mathcal{O}}_{U,u}$ and such that for each $n \geq 0$, x_n is isomorphic to the restriction $\operatorname{Spec} \widehat{\mathcal{O}}_{U,u}/\mathfrak{m}_u^{n+1} \to U \to \mathcal{X}$ under $\widehat{\mathcal{O}}_{U,u}/\mathfrak{m}_u^{n+1} \cong A/\mathfrak{m}_A^{n+1}$.

2.8 Properness and the Valuative Criterion

2.8.1 Definitions

With some care, we define separatedness and properness for morphisms of algebraic stacks. Recall from Definition 2.4.15 that we say a representable morphism $\mathcal{X} \to \mathcal{Y}$ of algebraic stacks is *separated* if the diagonal $\mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ (which is representable by schemes) is proper.

Definition 2.8.1.

(1) A morphism $\mathcal{X} \to \mathcal{Y}$ of algebraic stacks is *universally closed* if for every morphism $\mathcal{Y}' \to \mathcal{Y}$ of algebraic stacks, the morphism $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}' \to \mathcal{Y}'$ induces a closed map $|\mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}'| \to |\mathcal{Y}|$.

(2) A representable morphism $\mathcal{X} \to \mathcal{Y}$ of algebraic stacks is *proper* if it is universally closed, separated and of finite type.

(3) A morphism $\mathcal{X} \to \mathcal{Y}$ of algebraic stacks is *separated* if the representable morphism $\mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ is proper.

(4) A morphism $\mathcal{X} \to \mathcal{Y}$ of algebraic stacks is *proper* if it is universally closed, separated and of finite type.

Remark 2.8.2. Notice that we have not defined properness by requiring the diagonal is a closed immersion as with schemes. Indeed, the diagonal of a morphism of algebraic stacks is not a monomorphism. For schemes or algebraic spaces, the diagonal is proper if and only if it is a closed immersion; this follows from the fact that proper monomorphisms of schemes are closed immersions.

Remark 2.8.3. The property of being universally closed is smooth-local on the target so to verify that a morphism $\mathcal{X} \to \mathcal{Y}$ is universally closed it suffices to check the condition on a smooth presentation $V \to \mathcal{Y}$.

Remark 2.8.4. Recall that the stabilizer G_x of a field-valued point $x\colon \operatorname{Spec} \Bbbk \to \mathcal{X}$ is given by the cartesian diagram

$$\begin{array}{ccc} G_x & \longrightarrow & \operatorname{Spec} \Bbbk \\ \downarrow & & \downarrow{\scriptstyle (x,x)} \\ \mathcal{X} & \longrightarrow & \mathcal{X} \times \mathcal{X}. \end{array}$$

If $\mathcal{X}$ is a separated algebraic stack over $\operatorname{Spec} \mathbb{Z}$ (or in fact over any scheme S), then G_x is a proper group algebraic space over $\Bbbk$. If an addition $\mathcal{X}$ has affine diagonal, then the stabilizer group G_x is proper and affine, thus finite. Since $\mathcal{B}\mathrm{un}_{r,d}(C)$ has affine diagonal (Example 2.4.14) and infinite automorphism groups, we see that $\mathcal{B}\mathrm{un}_{r,d}(C)$ is *not* separated.

2.8.2 Valuative criteria

Proposition 2.8.5 (Valuative Criteria for Universally Closed/Separated/Proper Morphisms)**.** *Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a finite type morphism of algebraic stacks. Consider a 2-commutative diagram*

$$\begin{array}{ccc} \operatorname{Spec} K & \longrightarrow & \mathcal{X} \\ \downarrow & \swarrow_{\alpha} & \downarrow{\scriptstyle f} \\ \operatorname{Spec} R & \longrightarrow & \mathcal{Y} \end{array} \tag{2.8.1}$$

where R is a valuation ring with fraction field K. Then

(1) f is universally closed if and only if for every diagram (2.8.1), there exists an extension $R \to R'$ of valuation rings and fraction fields $K \to K'$ and a lifting

$$\begin{array}{ccccc} \operatorname{Spec} K' & \longrightarrow & \operatorname{Spec} K & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow & \nearrow & \downarrow{\scriptstyle f} \\ \operatorname{Spec} R' & \longrightarrow & \operatorname{Spec} R & \longrightarrow & \mathcal{Y}. \end{array}$$

(2) f is separated if and only if any two liftings of a diagram (2.8.1) are isomorphic.

(3) f is proper if and only if every diagram (2.8.1) has a lifting after an extension $R \to R'$ and any two liftings of a diagram (2.8.1) are isomorphic.

Moreover, if $f\colon \mathcal{X} \to \mathcal{Y}$ is a finite type morphism of noetherian algebraic stacks, than it suffices to consider DVRs R and extensions such that $K \to K'$ is of finite transcendence degree.

Proof. TO ADD □

2.9 Examples

In addition to providing examples of algebraic spaces (Section 2.9.1), Deligne–Mumford stacks (Section 2.9.2) and algebraic stacks (Section 2.9.3), we provide several "counterexamples" of algebraic spaces and stacks (Section 2.9.4).

2.9.1 Examples of algebraic spaces

Example 2.9.1. As discussed in Example 0.6.5, there exists a smooth proper complex 3-fold U with a free of $\mathbb{Z}/2$-action such that there is an orbit not contained in any affine. The quotient sheaf $U/(\mathbb{Z}/2)$ is an algebraic space (Corollary 2.1.9) which is not a scheme.

Example 2.9.2 (The bug-eyed cover)**.** Let $\Bbbk$ be field with $\operatorname{char}(k) \neq 2$. Let $\mathbb{Z}/2 = \{\pm 1\}$ act on the non-separated affine line $U = \mathbb{A}^1_\Bbbk \bigcup_{\mathbb{A}^1_\Bbbk \setminus 0} \mathbb{A}^1_\Bbbk$ by swapping the origins and by $(-1) \cdot x = -x$ for $x \neq 0$. Since the orbit of an origin is not contained in any affine, the quotient sheaf $U/(\mathbb{Z}/2)$ is not representable by a scheme; it is however an algebraic space (Corollary 2.1.9).

We now provide another description of the same algebraic space. If we let $\mathbb{Z}/2 = \{\pm 1\}$ act on $\mathbb{A}^1_\Bbbk$ via $-1 \cdot x = -x$, and if we remove the non-identity element of the stabilizer of the origin, we obtain a scheme $R = (\mathbb{Z}/2 \times \mathbb{A}^1) \setminus \{(-1, 0)\}$ and an equivalence relation $\sigma, p_2 \colon R \rightrightarrows \mathbb{A}^1_\Bbbk$. The algebraic space quotient $\mathbb{A}^1_\Bbbk/R$ is isomorphic to $U/(\mathbb{Z}/2)$ (Exercise 2.9.3(1))

Another way to see that $X = \mathbb{A}^1_\Bbbk/R$ is not a scheme is the observation that the diagonal $X \to X \times X$ is not a locally closed immersion. This can be seen from the cartesian diagram

$$\begin{array}{ccc} (\mathbb{A}^1_\Bbbk \setminus 0) \sqcup \{0\} & \longrightarrow & \mathbb{A}^1_\Bbbk \\ \downarrow & & \downarrow \\ R & \xrightarrow{(\sigma, p_2)} & \mathbb{A}^1_\Bbbk \times \mathbb{A}^1_\Bbbk \\ \downarrow & & \downarrow \\ X & \longrightarrow & X \times X. \end{array} \qquad \begin{array}{c} x \\ \downarrow \\ (x, -x) \end{array}$$

Exercise 2.9.3.

(1) Show that $X = \mathbb{A}^1_\Bbbk/R$ is isomorphic to $U/(\mathbb{Z}/2)$.

(2) Show that there is a universal homeomorphism $X \to \mathbb{A}^1_\Bbbk$ which is ramifed over the origin.

(3) Show that any map to a scheme $X \to Z$ factors through $X \to \mathbb{A}^1_\Bbbk$. (In other words, while $\mathbb{A}^1_\Bbbk$ may be the categorical quotient of U by $\mathbb{Z}/2$ (or equivalently $\mathbb{A}^1_\Bbbk$ by R) in the category of schemes, it is distinct from the algebraic space quotient.

(4) Consider the SL_2 action on $V_d = \operatorname{Sym}^d \Bbbk^2$, the space of homogeneous polynomials in x and y of degree d. Let $W \subset V_1 \times V_4$ be the reduced locally closed subscheme defined as the set (L, F) such that $L \neq 0$ and F is the square of a homogeneous quadratic with discriminant 1. Show that the induced SL_2-action on W is free (i.e. $\mathrm{SL}_2 \times W \to W \times W$ is a monomorphism) and that quotient sheaf W/SL_2 is an algebraic space isomorphic to $\mathbb{A}^1_\Bbbk/R$ and

$U/(\mathbb{Z}/2)$.

While the descriptions of X as $\mathbb{A}^1_{\Bbbk}/R$ and $U/(\mathbb{Z}/2)$ may seem pathological, this exercise shows that in fact this algebraic space arises also as a quotient of a quasi-affine variety by SL_2.

Example 2.9.4. Let $\mathbb{Z}/2 = \{\pm 1\}$ act on $\mathbb{A}^1_{\mathbb{C}}$ via conjugation over $\operatorname{Spec}\mathbb{R}$. Note that the action defined over $\mathbb{R}$ of $\mathbb{Z}/2$ on $\operatorname{Spec}\mathbb{C}$ is free, and therefore the product action of $\mathbb{Z}/2$ on $\mathbb{A}^1_{\mathbb{C}} = \mathbb{A}^1_{\mathbb{R}} \times_{\mathbb{R}} \mathbb{C}$ (which is trivial on the first factor) is also free. Letting $R = (\mathbb{Z}/2 \times \mathbb{A}^1_{\mathbb{C}}) \setminus \{(-1,0)\}$ and an equivalence relation $\sigma, p_2 \colon R \rightrightarrows U$. The algebraic space quotient $X = \mathbb{A}^1_{\mathbb{C}}/R$ is not a scheme by the same argument as in Example 2.9.2. The quotient X looks like $\mathbb{A}^1_{\mathbb{R}}$ except that the origin has residue field $\mathbb{C}$.

2.9.2 Examples of stacks with finite stabilizers

In characteristic 0, each of the following examples are Deligne–Mumford stacks.

Example 2.9.5 (Classifying stacks)**.** If G is a finite group scheme over a field $\Bbbk$, then $\mathbf{B}_{\Bbbk}G$ is the stack defined as the category of pairs (T, P) where T is a scheme and $P \to T$ is a G-torsor (Definition 1.3.12). This is called the *classifying stack of G*). The diagonal $\mathbf{B}_{\Bbbk}G \to \mathbf{B}_{\Bbbk}G \times \mathbf{B}_{\Bbbk}G$ is finite since its base change by $\operatorname{Spec}\Bbbk \to \mathbf{B}_{\Bbbk}G \times \mathbf{B}_{\Bbbk}G$ is isomorphic to G. In particular, $\mathbf{B}_{\Bbbk}G$ is separated.

If G is étale (which is guaranteed if $\operatorname{char}(\Bbbk) = 0$), then $\mathbf{B}_{\Bbbk}G$ is a smooth and separated Deligne–Mumford of dimension 0.

Exercise 2.9.6.

(1) Show that $\mathbf{B}_{\Bbbk}\boldsymbol{\mu}_n$ is the stack parameterizing the data of a triple (T, L, α) where T is a scheme, L is a line bundle on T and $\alpha \colon \mathcal{O}_T \to L^{\otimes n}$ is a trivialization.

(2) Show that $\mathbf{B}_{\Bbbk}\boldsymbol{\mu}_n$ is a smooth algebraic stack in any characteristic by identifying it with the quotient of $\mathbb{G}_m$ acting on $\mathbb{G}_m$ via $t \cdot x = t^n x$.

(3) Show that $\mathbf{B}_{\Bbbk}\boldsymbol{\mu}_n$ is a Deligne–Mumford stack if and only if n is prime to the characteristic.

(4) In $\operatorname{char}(\Bbbk) = p$, compute the dimension of both the stack $\mathbf{B}\boldsymbol{\mu}_n$ and its tangent space at the unique point.

Example 2.9.7 (Weighted projective stacks)**.** For a tuple of positive integers $(d_0, \ldots, d_n)$, let $\mathbb{G}_m$ act on $\mathbb{A}^{n+1}$ via $t \cdot (x_0, \ldots, x_n) = (t^{d_0}x_0, \ldots, t^{d_n}x_n)$. We define the *weighted projective stack* as

$$\mathcal{P}(d_0, \ldots, d_n) = [(\mathbb{A}^{n+1} \setminus 0)/\mathbb{G}_m].$$

If the d_i are all 1, then we recover projective space $\mathbb{P}^n$; otherwise, $\mathcal{P}(d_0, \ldots, d_n)$ is not an algebraic space.

More generally, if R is any finitely generated positively graded $\Bbbk$-algebra, we can define *stacky proj* as $\mathcal{P}\operatorname{roj} R = [(\operatorname{Spec}(R) \setminus 0)/\mathbb{G}_m]$, where $\mathbb{G}_m$ acts such that the weight of x_i is the same as its degree.

Exercise 2.9.8.

1. If $\Bbbk$ is a field of characteristic p, show that $\mathcal{P}(d_0, \ldots, d_n)$ is a Deligne–Mumford stack if and only if p doesn't divide each d_i.

2. Classify all the points of $\mathcal{P}(3,3,4,6)$ that have non-trivial stabilizers.
3. We say that an algebraic stack $\mathcal{X}$ has *generically trivial stabilizer* if there exists a dense open substack $U \subset \mathcal{X}$ which is an algebraic space. Provide conditions for when $\mathcal{P}(d_0,\dots,d_n)$ has generically trivial stabilizer.
4. Show that there is a bijective morphism $\mathcal{P}(d_0,\dots,d_n)$ to weighted projective space $\operatorname{Proj}\Bbbk[x_0,\dots,x_n]$, where x_i has degree d_i. (We will later call this a coarse moduli space.)
5. Show that $\overline{\mathcal{M}}_{1,1} \cong \mathbb{P}(4,6)$ over $\operatorname{Spec}\mathbb{Z}[1/6]$.

Example 2.9.9. Suppose $\operatorname{char}(k) \neq 2$. Let $\mathbb{Z}/2$ act on $\mathbb{A}^2_{\Bbbk}$ via $-1\cdot(x,y) = (-x,-y)$. Show that $[\mathbb{A}^2_{\Bbbk}/(\mathbb{Z}/2)]$ is a smooth algebraic stack over a field $\Bbbk$ and that there is a proper and bijective morphism $[\mathbb{A}^2_{\Bbbk}/(\mathbb{Z}/2)] \to Y$ where Y is the singular variety $\operatorname{Spec}\Bbbk[x^2,xy,y^2]$ defined by the $\mathbb{Z}/2$-invariants of $\Gamma(\mathbb{A}^2_{\Bbbk},\mathcal{O}_{\mathbb{A}^2_{\Bbbk}})$.

Exercise 2.9.10. Let G be a finite abelian group acting on a scheme X, and let $\mathcal{X} = [X/G]$. Show that the inertia stack $I_{\mathcal{X}}$ is isomorphic to

$$I_{\mathcal{X}} = \coprod_{g\in G} [X^g/G]$$

where $X^g = \{x \in X \mid gx = x\}$ or more precisely the fiber product of the diagonal $X \to X\times X$ and the map $X \to X\times X$ defined by $x \mapsto (x,gx)$.

Example 2.9.11 (Stacky curves)**.** A *stacky curve* is a smooth proper irreducible 1-dimensional Deligne-Mumford stack over a field $\Bbbk$ with generically trivial stabilizer.

Exercise 2.9.12. If d_1 and d_2 are relatively prime positive integers, then $\mathbb{P}(d_1,d_2)$ is a stacky curve.

2.9.3 Examples of algebraic stacks

Example 2.9.13. The classifying stack $\mathbf{B}\operatorname{GL}_n$ parameterizes vector bundles of rank n. When $n=1$, $\mathbf{B}\mathbb{G}_m = \mathbf{B}\operatorname{GL}_1$ parameterizes line bundles. The stack $\mathbf{B}\operatorname{GL}_n$ is an algebraic stack (but not Deligne–Mumford) smooth over $\operatorname{Spec}\mathbb{Z}$ of relative dimension $-n^2$ with affine diagonal.

Example 2.9.14. If $\mathbb{G}_m$ acts on $\mathbb{A}^1$ via scaling, the quotient stack $[\mathbb{A}^1/\mathbb{G}_m]$ is an algebraic stack (but not Deligne–Mumford) which is smooth of relative dimension 0 over $\operatorname{Spec}\mathbb{Z}$ (but not étale!) with affine diagonal. An object of $[\mathbb{A}^1/\mathbb{G}_m]$ is a triple (T,L,s) where T is a scheme, L is a line bundle on T and $s \in \Gamma(T,L)$.

If $\Bbbk$ is a field, $[\mathbb{A}^1_{\Bbbk}/\mathbb{G}_{m,\Bbbk}]$ has two points—one open and one closed—corresponding to the two $\mathbb{G}_m$-orbits (see Figure 7). There is an open immersion and closed immersion

$$\operatorname{Spec}\Bbbk \hookrightarrow [\mathbb{A}^1_{\Bbbk}/\mathbb{G}_{m,\Bbbk}] \hookleftarrow \mathbf{B}_{\Bbbk}\mathbb{G}_m.$$

The morphism $[\mathbb{A}^1_{\Bbbk}/\mathbb{G}_{m,\Bbbk}] \to \operatorname{Spec}\Bbbk$ identifies the two orbits and is an example of a good moduli space.

Let $\mathbb{G}_{m,\Bbbk}$ act on $\mathbb{A}^2_{\Bbbk}$ via $t\cdot(x,y) = (tx,t^{-1}y)$. The quotient stack $\mathcal{X} = [\mathbb{A}^2_{\Bbbk}/\mathbb{G}_{m,\Bbbk}]$ is a smooth algebraic stack. An object of $\mathcal{X}$ is a tuple (T,L,s,t) where T is a scheme, L is a line bundle on T, $s \in \Gamma(T,L)$ and $t \in \Gamma(T,L^{-1})$. The complement $\mathcal{X}\setminus 0$ of the origin is isomorphic to the non-separated affine line. There is a morphism $\mathcal{X} \to \mathbb{A}^1_{\Bbbk}$ defined by $(x,y) \mapsto xy$, which is an isomorphism over $\mathbb{A}^1_{\Bbbk}\setminus 0$ and identifies the three orbits defined by $xy=0$.

Example 2.9.15 (Gerbes)**.** A stack $\mathcal{X}$ over a site $\mathcal{S}$ is called a *gerbe* if

(1) for every object $S \in \mathcal{S}$, there exists a covering $(S_i \to S)$ in $\mathcal{S}$ such that each $\mathcal{X}(S_i)$ is non-empty; and

(2) for objects $x, y \in \mathcal{X}$ over $S \in \mathcal{S}$, there exists a covering $(S_i \to S)$ in $\mathcal{S}$ such that $x|_{S_i} \xrightarrow{\sim} y|_{S_i}$ for each i.

If G is a sheaf of groups on $\mathcal{S}$, we say that a gerbe $\mathcal{X}$ is a *G-gerbe* if for each object $x \in \mathcal{X}$ over $S \in \mathcal{S}$, the sheaves $\underline{\mathrm{Aut}}_S(x)$ and $G|_S$ on the localized site $\mathcal{S}/S$ (Example 1.1.5) are isomorphic.

We define a *band* of a gerbe $\mathcal{X}$ as the data of an isomorphism $\iota_x \colon G|_S \to \underline{\mathrm{Aut}}(x)$ of sheaves for each object $x \in \mathcal{X}$ over $S \in \mathcal{S}$. We require that:

(3) for every object $S \in \mathcal{S}$ and isomorphism $\alpha \colon x \xrightarrow{\sim} y$ over S, the diagram

$$\begin{array}{ccc} & G|_S & \\ {\scriptstyle \iota_x}\swarrow & & \searrow{\scriptstyle \iota_y} \\ \underline{\mathrm{Aut}}_S(x) & \xrightarrow{\alpha} & \underline{\mathrm{Aut}}_S(y). \end{array}$$

We say that the gerbe $\mathcal{X}$ is *G-banded*.

If $\mathcal{X}$ is a stack over $\mathrm{Sch}_{\text{Ét}}$, we say that a morphism $\mathcal{X} \to X$ to a scheme is a *gerbe*, *G-gerbe* or *banded G-gerbe* if $\mathcal{X}$ has the corresponding property as a stack over the site $(\mathrm{Sch}\,/X)_{\text{Ét}}$. Finally, if X is defined over a scheme S and G is a sheaf of groups on $(\mathrm{Sch}\,/S)_{\text{Ét}}$, we say that $\mathcal{X} \to X$ is a *G-gerbe* or *banded G-gerbe* if it is a G_X-gerbe or banded G_X-gerbe with respect to the pullback $G_X = G \times_S X$.

Exercise 2.9.16. Let $G \to S$ be a group scheme.

(1) Show that $\mathbf{B}_S G \to S$ is a banded G-gerbe.

(2) A G-gerbe $\mathcal{X}$ over a scheme X is said to be *trivial* if $\mathcal{X} \cong \mathbf{B}_S G \times_S X$. If $\mathcal{X} \to X$ is a gerbe and $x \in \mathcal{X}(X)$, then $\mathcal{X}$ is a trivial $\underline{\mathrm{Aut}}_X(x)$-gerbe.

(3) Show that a stack $\mathcal{X}$ over X is a G-gerbe if and only if there exists an étale covering $(X_i \to X)$ such that the pullback

$$\begin{array}{ccc} \mathbf{B}_S G \times_S X_i & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ X_i & \longrightarrow & X \end{array}$$

is trivial.

Exercise 2.9.17 (Residual gerbes and residue fields)**.** Let $\mathcal{X}$ be a noetherian algebraic stack and $x \in |\mathcal{X}|$ be a locally closed point. Recall the residual gerbe $\mathcal{G}_x$ of Definition 2.5.10 and that is an algebraic stack locally closed in $\mathcal{X}$ (Proposition 2.5.14).

(1) Show that the residual gerbe $\mathcal{G}_x$ is a gerbe.

(2) Show that the sheafification of the functor $\mathrm{Sch} \to \mathrm{Sets}$, taking a scheme S to the set of isomorphism classes of $\mathcal{G}_x(S)$, is representable by the spectrum of a field. (We denote this field as $\kappa(x)$ and call it the *residue field*.)

Exercise 2.9.18. Assume that $\mathrm{char}(k) \neq 2$. Show that the j-line $\pi \colon \mathcal{M}_{1,1} \to \mathbb{A}^1_{\Bbbk}$ is a banded $\mathbb{Z}/2$-gerbe over $\mathbb{A}^1_{\Bbbk} \setminus \{0, 1728\}$. Is it trivial?

Remark 2.9.19. We have not developed a cohomology theory of sheaves on sites but it is worth pointing out that if G is an sheaf of *abelian* groups on a site $\mathcal{S}$, then the cohomology group $H^2(\mathcal{S}, G)$ classifies isomorphism classes of G-gerbes. This is analogous to the fact that $H^1(\mathcal{S}, G)$ classifies isomorphism classes of G-torsors. When G is not abelian, then one take this approach to define *non-abelian cohomology groups*; see [Gir71].

Example 2.9.20 (Root stacks I)**.** Let X be a scheme and L be a line bundle. For a positive integer r, define the *rth root stack of X and L* as the fiber product

$$\begin{array}{ccc} \sqrt[r]{L/X} & \longrightarrow & B\mathbb{G}_m \\ \downarrow & & \downarrow{\scriptstyle r} \\ X & \xrightarrow{[L]} & B\mathbb{G}_m \end{array}$$

where $[L]\colon X \to B\mathbb{G}_m$ denotes the morphism corresponding to L and $B\mathbb{G}_m \xrightarrow{r} B\mathbb{G}_m$ is induced from $\mathbb{G}_m \xrightarrow{r} \mathbb{G}_m$ (or defined functorially by the assignment $L \mapsto L^{\otimes r}$).

Example 2.9.21 (Root stacks II)**.** Let X be a scheme, L be a line bundle, and $s \in \Gamma(X, L)$ be a section. For a positive integer r, define the *rth root stack of X and L along s* as the fiber product

$$\begin{array}{ccc} \sqrt[r]{(L,s)/X} & \longrightarrow & [\mathbb{A}^1/\mathbb{G}_m] \\ \downarrow & & \downarrow{\scriptstyle r} \\ \mathcal{X} & \xrightarrow{[L,s]} & [\mathbb{A}^1/\mathbb{G}_m] \end{array}$$

where $[\mathbb{A}^1/\mathbb{G}_m] \xrightarrow{r} [\mathbb{A}^1/\mathbb{G}_m]$ is induced from the map $\mathbb{A}^1 \to \mathbb{A}^1$ sending $x \mapsto x^r$ which is equivariant under $\mathbb{G}_m \xrightarrow{r} \mathbb{G}_m$ (or defined functorially by the assignment $(L, s) \mapsto (L^{\otimes r}, s^r)$).

Exercise 2.9.22.

(1) If $X = \operatorname{Spec} A$ is affine and $L = \mathcal{O}_X$ is trivial, show that

$$\sqrt[r]{L/X} \cong [X/\boldsymbol{\mu}_r] \qquad \sqrt[r]{(L,s)/X} \cong [\operatorname{Spec}(A[x]/(x^r - s))/\boldsymbol{\mu}_r]$$

where $\boldsymbol{\mu}_r$ acts trivially on X and acts on $\operatorname{Spec}\big(A[x]/(x^r - s)\big)$ via $t \cdot x = tx$.

(2) Show that if r is invertible in $\Gamma(X, \mathcal{O}_X)$, then $\sqrt[r]{L/X}$ and $\sqrt[r]{(L,s)/X}$ are Deligne–Mumford stacks.

(3) Show that $\sqrt[r]{L/X}$ has the equivalent description as the category of triples $(T \xrightarrow{t} X, L, \alpha)$ where $T \xrightarrow{t} X$ is a morphism from a scheme, L is a line bundle on X and $\alpha\colon \mathcal{O}_T \to t^*L^{\otimes r}$ is a trivialization.

(4) Provide an equivalent description of $\sqrt[r]{(L,s)/X}$ analogous to (3).

(5) Show that the fiber of $\sqrt[r]{L/X} \to X$ at a point $x \in X$ is isomorphic to $B_{\kappa(x)}\boldsymbol{\mu}_r$. More generally, show that $\sqrt[r]{L/X} \to X$ is a banded $\boldsymbol{\mu}_r$-gerbe.

(6) Show that $\sqrt[r]{(L,s)/X} \to X$ is an isomorphism over the complement X_s of the section s and a banded $\boldsymbol{\mu}_r$-gerbe over the vanishing $V(s) \subset X$ of s.

Example 2.9.23 (Toric stacks)**.** A fan Σ on a lattice $L = \mathbb{Z}^n$ defines a toric variety $X(\Sigma)$, a normal separated variety with an action of $\mathbb{G}_m^n$ such that there is a dense orbit with trivial stabilizer; see [Ful93].

Meanwhile a *stacky fan* is a pair (Σ, β) where Σ is a fan on a lattice L and $\beta\colon L \to N$ is a homomorphism of lattices. As L and N are lattices (i.e. finitely generated free abelian groups), the $\mathbb{Z}$-linear duals define tori $T_L := D(L^\vee)$ and $T_N := D(N^\vee)$ (Example C.1.9) where T_L is a torus for the toric variety $X(\Sigma)$. The map β induces a homomorphism $T_\beta\colon T_L \to T_N$, naturally identifying β with the induced map on lattices of 1-parameter subgroups. We can then define $G_\beta = \ker(T_\beta)$ and the *toric stack*

$$X(\Sigma, \beta) := [X(\Sigma)/G_\beta].$$

Example 2.9.24 (Picard schemes and stacks)**.** If X is a scheme over a field $\Bbbk$, the *Picard functor of* X and *Picard stack of* X are defined as the sheaf $\underline{\mathrm{Pic}}(X)$ and stack $\underline{\mathcal{P}\mathrm{ic}}\,(X)$ on $\mathrm{Sch}_{\text{É}\mathrm{t}}$ by

$$\begin{aligned}\underline{\mathrm{Pic}}(X) &= \text{sheafification of } T \mapsto \mathrm{Pic}(X_T)\\ \underline{\mathcal{P}\mathrm{ic}}\,(X)(T) &= \{\text{groupoid of line bundles } L \text{ on } X_T\}\end{aligned}$$

A morphism $(T, L) \to (T', L')$ in $\underline{\mathcal{P}\mathrm{ic}}\,(X)$ is the data of a morphism $f\colon T \to T'$ of schemes and an isomorphism $\alpha\colon L \to f^*L'$ (or more precisely a morphism $f_*L \to L'$ whose adjoint is an isomorphism).

If X is proper over a field $\Bbbk$, then $\underline{\mathrm{Pic}}(X)$ is a proper scheme and the tensor product of line bundles provides it with the structure of a group scheme, hence an abelian variety. Moreover, $\underline{\mathcal{P}\mathrm{ic}}\,(X)$ is a smooth algebraic stack over $\Bbbk$ and there morphism $\underline{\mathcal{P}\mathrm{ic}}\,(X) \to \underline{\mathrm{Pic}}(X)$ such that the fiber over a line bundle L is isomorphic to $\mathbf{B}_\Bbbk\mathbb{G}_m$. The tensor product of line bundles provides $\underline{\mathcal{P}\mathrm{ic}}\,(X)$ with the structure of a group stack, a notion which we will not spell out precisely.

Exercise 2.9.25. Show that $\underline{\mathcal{P}\mathrm{ic}}\,(X)$ is a banded $\mathbb{G}_m$-gerbe over $\underline{\mathrm{Pic}}(X)$.

2.9.4 Counterexamples

Example 2.9.26 (Example of a non-quasi-separated algebraic space that is not a scheme)**.** Let $\Bbbk$ be a characteristic 0 field. Let $\mathbb{Z}$ act on $\mathbb{A}^1_\Bbbk$ via $n \cdot x = x + n$ for $x \in \mathbb{A}^1_\Bbbk$ and $n \in \mathbb{Z}$. Then $X = \mathbb{A}^1_\Bbbk/\mathbb{Z}$ is an algebraic space (Corollary 2.1.9) which is not quasi-separated (as the action map $\mathbb{Z} \times \mathbb{A}^1_\Bbbk \to \mathbb{A}^1_\Bbbk \times \mathbb{A}^1_\Bbbk$ is not quasi-compact).

If X were a scheme, then there would exist a non-empty open affine subscheme $U = \operatorname{Spec} A \subset X$. Since $p\colon \mathbb{A}^1_\Bbbk \to X$ is an étale presentation, we can compute A as the subring of $\mathbb{Z}$-invariants $\Gamma(p^{-1}(U), \mathcal{O}_{\mathbb{A}^1_\Bbbk})^\mathbb{Z}$, which the reader can check consists of only the constant functions, i.e. $A = k$. As X is obtained by gluing such affine schemes, it follows that $X = \operatorname{Spec}\Bbbk$, a contradiction.

The algebraic space $X = \mathbb{A}^1_\Bbbk/\mathbb{Z}$ provides a counterexample to many facts that hold for all schemes and quasi-separated algebraic spaces but fail for all algebraic spaces (see Exercise 2.9.27).

Similarly, one can consider the algebraic space quotient $\mathbb{A}^1_\mathbb{C}/\mathbb{Z}^2$ where $(a, b)\cdot x = x + a + ib$. While the analytic quotient $\mathbb{C}/\mathbb{Z}^2$ of this action is an elliptic curve over $\mathbb{C}$, the algebraic space quotient is a non-quasi-separated algebraic space that is not a scheme.

Exercise 2.9.27. Let $X = \mathbb{A}^1_\Bbbk/\mathbb{Z}$ be the algebraic space defined above.

1. Show that X is locally noetherian and quasi-compact but not noetherian.
2. Show that the generic point $\operatorname{Spec} \Bbbk(x) \to \mathbb{A}^1_{\Bbbk} \to X$ is fixed under the $\mathbb{Z}$-action.
3. Show that $\operatorname{Spec} \Bbbk(x) \to X$ does not factor through a monomorphism $\operatorname{Spec} L \to X$ for a field L. (In other words, the generic point of X does not have a residue field.)

Example 2.9.28 (Deligne–Mumford stacks with non-separated diagonal)**.** Let $G \to S$ be a finite group scheme. If $H \subset G$ is a subgroup scheme over S, then G/H is separated if and only if $H \subset G$ is closed. For instance, taking $G = \mathbb{Z}/2 \times \mathbb{A}^1 \to \mathbb{A}^1$ and the subgroup $H = G \setminus \{-1, 0\}$, the quotient $Q = G/H$ is the non-separated affine line and is a group scheme over $\mathbb{A}^1$ which is trivial away from the origin and where the fiber over 0 is $\mathbb{Z}/2$. In this case, $\mathbf{B}_{\mathbb{A}^1} Q$ is a Deligne–Mumford stack with non-separated diagonal; however, the diagonal is quasi-compact and representable by schemes.

Chapter 3

Geometry of Deligne–Mumford stacks

3.1 Quasi-coherent sheaves

3.1.1 Sheaves

The small étale site of a Deligne–Mumford stack can be defined analogously to the small étale site of a scheme (Example 1.1.3).

Definition 3.1.1. If $\mathcal{X}$ is a Deligne–Mumford stack, the *small étale site of* $\mathcal{X}$ is the category $\mathcal{X}_{\text{ét}}$ of schemes étale over $\mathcal{X}$. A covering of $U \to \mathcal{X}$ is a collection of étale morphisms $\{U_i \to U\}$ over $\mathcal{X}$ such that $\bigsqcup_i U_i \to U$ is surjective.

We can therefore discuss sheaves on $\mathcal{X}_{\text{ét}}$, and we denote $\mathrm{Sh}(\mathcal{X}_{\text{ét}})$ as the category of sheaves on $\mathcal{X}_{\text{ét}}$. While a sheaf $\mathcal{F}$ on $\mathcal{X}_{\text{ét}}$ by definition only has sections defined on étale morphisms $U \to \mathcal{X}$ from schemes, one can define sections on an étale morphism $\mathcal{U} \to \mathcal{X}$ from a Deligne–Mumford stack by choosing étale presentations $U \to \mathcal{U}$ and $R \to U \times_{\mathcal{U}} U$ by schemes and setting

$$\mathcal{F}(\mathcal{U} \to \mathcal{X}) := \mathrm{Eq}(\mathcal{F}(U \to \mathcal{X}) \rightrightarrows \mathcal{F}(R \to \mathcal{X})).$$

One checks that this is independent of the choice of presentation. In particular, it makes sense to discuss global sections $\Gamma(\mathcal{X}, \mathcal{F}) := \mathcal{F}(\mathcal{X} \xrightarrow{\mathrm{id}} \mathcal{X})$.

Given a morphism $f\colon \mathcal{X} \to \mathcal{Y}$ of Deligne–Mumford stacks, there are functors

$$\mathrm{Sh}(\mathcal{X}_{\text{ét}}) \underset{f^{-1}}{\overset{f_*}{\rightleftarrows}} \mathrm{Sh}(\mathcal{Y}_{\text{ét}})$$

defined by $f_*\mathcal{F}(V \to \mathcal{Y}) := \mathcal{F}(V \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{X})$ and $f^{-1}\mathcal{G}$ as the sheafification of presheaf whose sections on $U \to \mathcal{X}$ is $\lim_{V \to \mathcal{Y}} \mathcal{G}(V \to \mathcal{Y})$ where the limit is taken over the category of pairs of étale morphisms $V \to \mathcal{Y}$ and $U \to V \times_{\mathcal{Y}} \mathcal{X}$ (i.e. étale morphisms $V \to \mathcal{Y}$ and a choice of factorization of $U \to \mathcal{X} \to \mathcal{Y}$ through $V \to \mathcal{Y}$).

Exercise 3.1.2. Show that f^{-1} is left adjoint to f_*.

Note that if X is a scheme, a sheaf $\mathcal{F}$ on $X_{\text{ét}}$ restricts to a sheaf on the ordinary Zariski topology of X.

3.1.2 $\mathcal{O}_{\mathcal{X}}$-modules

The *structure sheaf* of a Deligne–Mumford stack $\mathcal{X}$ is the sheaf $\mathcal{O}_{\mathcal{X}}$ where

$$\mathcal{O}_{\mathcal{X}}(U \to \mathcal{X}) := \Gamma(U, \mathcal{O}_U)$$

for any étale morphism $U \to \mathcal{X}$ from a scheme. As $\mathcal{O}_{\mathcal{X}}$ is a ring object in the category of sheaves on $\mathcal{X}_{\text{ét}}$, we can define:

Definition 3.1.3. If $\mathcal{X}$ is a Deligne–Mumford stack, an $\mathcal{O}_{\mathcal{X}}$-module is a sheaf $\mathcal{F}$ on $\mathcal{X}_{\text{ét}}$ which is a module object for $\mathcal{O}_{\mathcal{X}}$ in the category of sheaves, i.e. for any étale morphism $U \to \mathcal{X}$ from a scheme, $\mathcal{F}(U \to \mathcal{X})$ is an $\mathcal{O}_{\mathcal{X}}(U \to \mathcal{X})$-module and the module structure is compatible with respect to restriction along étale morphisms $V \to U$ over $\mathcal{X}$.

We denote $\text{Mod}_{\mathcal{O}_{\mathcal{X}}}$ for the category of $\mathcal{O}_{\mathcal{X}}$-modules. Given two $\mathcal{O}_{\mathcal{X}}$-modules $\mathcal{F}$ and $\mathcal{G}$, there is a tensor product $\mathcal{F} \otimes \mathcal{G} := \mathcal{F} \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{G}$ which is the $\mathcal{O}_{\mathcal{X}}$-module defined as the sheafification of $(U \to \mathcal{X}) \mapsto \mathcal{F}(U \to \mathcal{X}) \otimes_{\mathcal{O}_{\mathcal{X}}(U \to \mathcal{X})} \mathcal{G}(U \to \mathcal{X})$.

Given a morphism $f \colon \mathcal{X} \to \mathcal{Y}$ of Deligne–Mumford stacks, there are functors

$$\text{Mod}(\mathcal{O}_{\mathcal{X}}) \underset{f^*}{\overset{f_*}{\rightleftarrows}} \text{Mod}(\mathcal{O}_{\mathcal{Y}})$$

where for an $\mathcal{O}_{\mathcal{X}}$-module $\mathcal{F}$, $f_*\mathcal{F}$ is the pushforward as sheaves and is naturally an $\mathcal{O}_{\mathcal{Y}}$-module. For an $\mathcal{O}_{\mathcal{Y}}$-module $\mathcal{G}$, since there is a morphism $f^{-1}\mathcal{O}_{\mathcal{Y}} \to \mathcal{O}_{\mathcal{X}}$ of sheaves of rings in $\mathcal{X}_{\text{ét}}$ and $f^{-1}\mathcal{G}$ is a $f^{-1}\mathcal{O}_{\mathcal{Y}}$-module, it makes sense to define the pullback $\mathcal{O}_{\mathcal{X}}$-module

$$f^*\mathcal{G} := f^{-1}\mathcal{G} \otimes_{f^{-1}\mathcal{O}_{\mathcal{Y}}} \mathcal{O}_{\mathcal{X}},$$

i.e. the sheafification of $(U \to \mathcal{X}) \mapsto f^{-1}\mathcal{G}(U \to \mathcal{X}) \otimes_{f^{-1}\mathcal{O}_{\mathcal{Y}}(U \to \mathcal{X})} \mathcal{O}_{\mathcal{X}}(U \to \mathcal{X})$.

Exercise 3.1.4. Show that f^* is left adjoint to f_*.

If $f \colon U \to \mathcal{X}$ is an étale morphism from a scheme to a Deligne–Mumford stack and $\mathcal{F}$ is a sheaf on $\mathcal{X}_{\text{ét}}$, then $f^{-1}\mathcal{F}$ is the sheaf on $U_{\text{ét}}$ satisfying $f^{-1}\mathcal{F}(V \to U) = \mathcal{F}(V \to U \to \mathcal{X})$. We can also restrict $f^{-1}\mathcal{F}$ to the Zariski topology of U and we denote this sheaf as $\mathcal{F}|_U$. Note also that if $\mathcal{F}$ is an $\mathcal{O}_{\mathcal{X}}$-module, there is an identification $f^*\mathcal{F} = f^{-1}\mathcal{F}$.

3.1.3 Quasi-coherent sheaves

Definition 3.1.5. Let $\mathcal{X}$ be a Deligne–Mumford stack. An $\mathcal{O}_{\mathcal{X}}$-module $\mathcal{F}$ is *quasi-coherent* if for any étale morphism $U \to \mathcal{X}$ from a scheme, the restriction $\mathcal{F}|_U$ of $\mathcal{F}$ to the Zariski topology of U is a quasi-coherent $\mathcal{O}_U$-module.

Exercise 3.1.6. Show that for a scheme X, this definition agrees with the usual definition of quasi-coherence.

We denote the category of quasi-coherent $\mathcal{O}_{\mathcal{X}}$-modules as $\text{QCoh}(\mathcal{X})$.

Exercise 3.1.7. Let $f \colon \mathcal{X} \to \mathcal{Y}$ be a morphism of Deligne–Mumford stacks.

(1) Show that if $\mathcal{G}$ is a quasi-coherent $\mathcal{O}_{\mathcal{Y}}$-module, then $f^*\mathcal{G}$ is quasi-coherent.

Assume in addition that f is quasi-compact and quasi-separated.

(2) Show that if $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_\mathcal{X}$-module, then $f_*\mathcal{F}$ is quasi-coherent.

(3) Show that the functors

$$\mathrm{QCoh}(\mathcal{X}) \underset{f^*}{\overset{f_*}{\rightleftarrows}} \mathrm{QCoh}(\mathcal{Y})$$

are adjoints (with f_* the right adjoint).

Exercise 3.1.8. Let G be a finite group.

(1) Show that a quasi-coherent sheaf on $\mathbf{B}_\Bbbk G$ corresponds to a representation V of G.

(2) Show that the pullback of V along $\operatorname{Spec}\Bbbk \to \mathbf{B}_\Bbbk G$ is V as a vector space (forgetting the G-action) and that the pushforward of V along $\mathbf{B}_\Bbbk G \to \operatorname{Spec}\Bbbk$ is the G-invariants V^G.

Exercise 3.1.9. Let G be a finite group acting on a $\Bbbk$-scheme $\operatorname{Spec} A$.

(1) Show that a quasi-coherent sheaf on $[\operatorname{Spec} A/G]$ corresponds to an A-module M which as $\Bbbk$-module has the additional structure of a G-representation.

Consider the diagram

$$\begin{array}{ccccc} \operatorname{Spec} A & \longrightarrow & [\operatorname{Spec} A/G] & \longrightarrow & \mathbf{B}_\Bbbk G \\ & & \downarrow & & \\ & & \operatorname{Spec} A^G & & \end{array}$$

(2) Show that the pullback of M along $\operatorname{Spec} A \to [\operatorname{Spec} A/G]$ is M as an A-module (forgetting the G-action).

(3) Show that the pushforward of M along $[\operatorname{Spec} A/G] \to \mathbf{B}_\Bbbk G$ is M as a $\Bbbk$-vector space (forgetting the A-module structure).

(4) Show that the pushforward of M along $[\operatorname{Spec} A/G] \to \operatorname{Spec} A^G$ is the G-invariants M^G of the representation which is naturally an A^G-module.

The condition of being a vector bundle, line bundle or coherent (in the noetherian setting) are étale local (Proposition B.4.4).

Definition 3.1.10. Let $\mathcal{X}$ be a Deligne–Mumford stack and $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_\mathcal{X}$-module.

(1) We say that $\mathcal{F}$ is a *vector bundle* (resp. *line bundle*) if for every (equivalently, there exists) étale presentation $U \to \mathcal{X}$ from a scheme, the pullback $f^*\mathcal{F}$ is a vector bundle (resp. line bundle).

(2) If in addition $\mathcal{X}$ is noetherian, we say $\mathcal{F}$ is *coherent* if for every (equivalently there exists) étale presentation $U \to \mathcal{X}$ from a scheme, the pullback $f^*\mathcal{F}$ is a vector bundle.

Definition 3.1.11 (Quasi-coherent algebras)**.** Let $\mathcal{X}$ be a Deligne–Mumford stack. A *quasi-coherent $\mathcal{O}_\mathcal{X}$-algebra* is a quasi-coherent $\mathcal{O}_\mathcal{X}$-module $\mathcal{A}$ which is a ring object in $\mathrm{Sh}(\mathcal{X}_{\text{ét}})$.

Definition 3.1.12 (Relative spectrum)**.** If $\mathcal{A}$ is a quasi-coherent $\mathcal{O}_{\mathcal{X}}$-algebra on a Deligne–Mumford stack $\mathcal{A}$, let $\mathcal{S}\mathrm{pec}_{\mathcal{X}}\mathcal{A}$ be the stack whose objects over a scheme S consists of a morphism $f\colon T \to \mathcal{X}$ and a morphism $f^*\mathcal{A} \to \mathcal{O}_T$ of $\mathcal{O}_T$-algebras.

Exercise 3.1.13. Show that $\mathcal{S}\mathrm{pec}_{\mathcal{X}}\mathcal{A}$ is an algebraic stack affine over $\mathcal{X}$.

Exercise 3.1.14. Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a quasi-compact and quasi-separated morphism of Deligne–Mumford stacks.

(1) Show that there is factorization $f\colon \mathcal{X} \to \mathcal{S}\mathrm{pec}\, f_*\mathcal{O}_{\mathcal{X}} \to \mathcal{Y}$.

(2) Show that f is affine if and only if $\mathcal{X} \to \mathcal{S}\mathrm{pec}\, f_*\mathcal{O}_{\mathcal{X}}$ is an isomorphism.

(3) Show that f is quasi-affine if and only if $\mathcal{X} \to \mathcal{S}\mathrm{pec}\, f_*\mathcal{O}_{\mathcal{X}}$ is an open immersion.

3.1.4 Line bundles

Here we collect facts about line bundles that will come in handy later. We first extend Proposition F.2.2 from schemes to algebraic spaces.

Proposition 3.1.15. *Let $f\colon X \to Y$ be a finite morphism of algebraic spaces and L be a line bundle on Y. If L is ample, then so is f^*L. If f is surjective, then the converse is true.*

Proof. To be added. □

3.2 Local quotient structure of Deligne–Mumford stacks

In this section, we show that any Deligne–Mumford stack $\mathcal{X}$ is étale-locally near a point x isomorphic to a quotient stack $[\mathrm{Spec}\, A/G_x]$ of an affine scheme by the stabilizer group scheme. Conceptually, this tells us that just as schemes (resp. algebraic spaces) are obtained by gluing affine schemes in the Zariski-topology (resp. étale-topology), Deligne–Mumford stacks are obtained by gluing quotient stacks $[\mathrm{Spec}\, A/G]$ in the étale topology.[1] Practically, this allows one to reduce many properties of Deligne–Mumford stacks to quotient stacks $[\mathrm{Spec}\, A/G]$. We will take advantage of this local structure in order to construct a coarse moduli space (Theorem 3.3.17).

Theorem 3.2.1 (Local Structure Theorem of Deligne–Mumford Stacks)**.** *Let $\mathcal{X}$ be a separated Deligne–Mumford stack and $x \in \mathcal{X}(\Bbbk)$ be a geometric point with stabilizer G_x. There exists an affine and étale morphism*

$$f\colon ([\mathrm{Spec}\, A/G_x], w) \to (\mathcal{X}, x)$$

where $w \in (\mathrm{Spec}\, A)(\Bbbk)$ such that f induces an isomorphism of stabilizer groups at w. Moreover, it can be arranged that $f^{-1}(BG_x) \cong BG_w$.

[1] Of course, Deligne–Mumford stacks are also étale locally schemes but the étale neighborhoods $([\mathrm{Spec}\, A/G_x], w) \to (\mathcal{X}, x)$ produced by Theorem 3.2.1 preserve the stabilizer group at w.

Proof. Let $(U,u) \to (\mathcal{X},x)$ be an étale representable morphism from an affine scheme, and let d be the degree over x, i.e. the cardinality of $\operatorname{Spec}\Bbbk \times_{\mathcal{X}} U$. Since $\mathcal{X}$ is separated, $U \to \mathcal{X}$ is affine. Define the scheme

$$(U/\mathcal{X})^d := \underbrace{U \times_{\mathcal{X}} \cdots \times_{\mathcal{X}} U}_{d \text{ times}}.$$

For a scheme S, a morphism $S \to (U/\mathcal{X})^d$ corresponds to a morphism $S \to \mathcal{X}$ and d sections $s_1,\dots,s_d$ of $U_S := U \times_{\mathcal{X}} S \to S$.

Let $(U/\mathcal{X})^d_0$ be the quasi-affine subscheme $(U/\mathcal{X})^d$ which is the complement of all pairwise diagonals, i.e. a map $S \to (U/\mathcal{X})^d_0$ corresponds to $S \to \mathcal{X}$ and n *distinct* sections $s_1,\dots,s_n \colon S \to U_S$ (distinct meaning the intersection of s_i and s_j is empty for $i \neq j$). There is an action of S_d on $(U/\mathcal{X})^d$ by permuting the sections and $(U/\mathcal{X})^d_0 \subset (U/\mathcal{X})^d$ is equivariant. An object of the quotient stack $[(U/\mathcal{X})^d_0/S_d]$ over a scheme S corresponds to a diagram

$$\begin{array}{ccccc} Z & \hookrightarrow & U_S & \longrightarrow & U \\ & \searrow & \downarrow & & \downarrow \\ & & S & \longrightarrow & \mathcal{X} \end{array}$$

where $Z \hookrightarrow U_S$ is a closed subscheme such that $Z \to S$ is finite étale of degree d. Note that we have a point $w \in [(U/\mathcal{X})^d_0/S_d](\Bbbk)$ corresponding to $Z \xrightarrow{\sim} \operatorname{Spec}\Bbbk \times_{\mathcal{X}} U$. There is an induced representable morphism $[(U/\mathcal{X})^d_0/S_d] \to \mathcal{X}$ and a commutative diagram

$$\begin{array}{ccc} (U/\mathcal{X})^d_0 & \hookrightarrow & (U/\mathcal{X})^d \\ \downarrow & & \downarrow \\ [(U/\mathcal{X})^d_0/S_d] & & U \\ & \searrow & \downarrow \\ & & \mathcal{X} \end{array}$$

Set $W := (U/\mathcal{X})^d_0$. The morphism $[W/S_d] \to \mathcal{X}$ is étale at w and induces an isomorphism of stabilizer groups at w. By quotienting out by $G_x \subset S_d$ instead, we also have a morphism $[W/G_x] \to \mathcal{X}$ which is étale and stabilizer preserving at w. By removing all other preimages of x and we may replace W with a G_x-invariant affine open subscheme of W containing w. The induced morphism $f \colon [W/G_x] \to \mathcal{X}$ is affine and étale such that $f^{-1}(BG_x) \cong BG_w$. □

Remark 3.2.2. A variant of Theorem 3.2.1 holds for non-separated Deligne–Mumford stacks. If $\mathcal{X}$ is a Deligne–Mumford stack with separated and quasi-compact diagonal, there exists a representable (but not necessarily affine) and étale morphism $f \colon ([\operatorname{Spec} A/G_x], w) \to (\mathcal{X},x)$. In fact, the same proof works as long as we rely on the fact that such Deligne–Mumford stacks have quasi-affine diagonal.

Exercise 3.2.3. Use a similar technique to the above proof to establish that if $\mathcal{X}$ is any algebraic stack with separated and quasi-compact diagonal and $x \in \mathcal{X}(\Bbbk)$ is a field-valued point, then there exists a smooth presentation $U \to \mathcal{X}$ from a scheme and a point $u \in U(\Bbbk)$ over x.

Exercise 3.2.4. Use Exercise 3.2.3 to show that Theorem 3.2.1 remains true if $x \in \mathcal{X}(\Bbbk)$ is an arbitrary field-valued point.

Exercise 3.2.5. Let $\mathcal{X}$ be a Deligne–Mumford stack. Show that $\mathcal{X}$ is isomorphic to a quotient stack $[U/G]$ where U is an affine scheme (resp. scheme, algebraic space) and G is a finite group if and only if there exists a finite étale morphism $V \to \mathcal{X}$ from an affine scheme (resp. scheme, algebraic space).

3.3 Existence of coarse moduli spaces: The Keel-Mori Theorem

The goal of this section is to establish the Keel–Mori Theorem: any separated Deligne–Mumford stack $\mathcal{X}$ of finite type over a noetherian scheme admits a separated coarse moduli space $\pi\colon \mathcal{X} \to X$ (see Theorem 3.3.17). One can view this theorem as a way to remove the stackiness of a Deligne–Mumford stack where at the expense of sacrificing universal properties of $\mathcal{X}$ (e.g. existence of a universal family), one can replace $\mathcal{X}$ with an algebraic space without changing the underlying topological space.

We will later apply this theorem to show that the Deligne–Mumford stack $\overline{\mathcal{M}}_g$ parameterizing stable curves admits a coarse moduli space $\pi\colon \overline{\mathcal{M}}_g \to \overline{M}_g$ where $\overline{M}_g$ is a separated algebraic space, which we later show to be proper and then finally projective.

To prove Theorem 3.3.17, we first show that if $\mathcal{X}$ is a quotient stack $[\operatorname{Spec} A/G]$ by a finite group, then $[\operatorname{Spec} A/G] \to \operatorname{Spec} A^G$ is a coarse moduli space (Theorem 3.3.8). We then reduce to this case by applying the Local Structure Theorem of Deligne–Mumford Stacks (Theorem 3.2.1) we construct étale neighboorhods $[\operatorname{Spec}(A_i)/G] \to \mathcal{X}$ and show that their coarse moduli spaces $\operatorname{Spec}(A_i^G)$ glue in the étale topology to a coarse moduli space of $\mathcal{X}$.

3.3.1 Coarse moduli spaces

We begin with the definition:

Definition 3.3.1. A morphism $\pi\colon \mathcal{X} \to X$ from an algebraic stack to an algebraic space is a *coarse moduli space* if

(1) for any algebraically closed field $\Bbbk$, the induced map $\mathcal{X}(\Bbbk)/\sim \to X(\Bbbk)$, from the set of isomorphism classes of objects of $\mathcal{X}$ over $\Bbbk$, is bijective, and

(2) π is universal for maps to algebraic spaces, i.e. any other map from $\mathcal{X} \to Y$ factors uniquely as

$$\begin{array}{ccc} \mathcal{X} & & \\ \downarrow{\scriptstyle \pi} & \searrow & \\ X & \dashrightarrow & Y. \end{array}$$

If in addition $\mathcal{X} = [U/G]$ is a quotient stack, we often write the coarse moduli space as U/G and call it the *geometric quotient of U by G*.

Remark 3.3.2. In practice, we desire coarse moduli spaces with additional properties of $\pi\colon \mathcal{X} \to X$ as otherwise it is difficult to work with this notion. For instance, it is not true that this notion is stable under étale base change (or even

open immersions) or that $\pi_*\mathcal{O}_\mathcal{X} = \mathcal{O}_X$. However, we emphasize that the Keel–Mori Theorem produces a coarse moduli space $\pi\colon \mathcal{X} \to X$ with the additional properties: (a) it is stable under flat base change, (b) $\pi_*\mathcal{O}_\mathcal{X} = \mathcal{O}_X$, (c) π is proper (and in particular separated!) and (d) π is a universal homeomorphism.

Lemma 3.3.3. *Let $\pi\colon \mathcal{X} \to X$ be a coarse moduli space such that for any étale morphism $X' \to X$ from an affine scheme, the base change $\mathcal{X} \times_X X' \to X'$ is a coarse moduli space. Then the natural map $\mathcal{O}_X \to \pi_*\mathcal{O}_\mathcal{X}$ is an isomorphism.*

Proof. As π is universal for maps to algebraic spaces, we have that $\mathrm{Map}(X, \mathbb{A}^1) \to \mathrm{Map}(\mathcal{X}, \mathbb{A}^1)$ is bijective or in other words $\Gamma(X, \mathcal{O}_X) \cong \Gamma(\mathcal{X}, \mathcal{O}_\mathcal{X})$. For any étale map $X' \to X$, the base change $\mathcal{X}' = \mathcal{X} \times_X X' \to X'$ is also a coarse moduli space and thus $\Gamma(X', \mathcal{O}_{X'}) \cong \Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})$. This shows that $\mathcal{O}_X \to \pi_*\mathcal{O}_\mathcal{X}$ is isomorphism. □

The property that a given map is a coarse moduli space can be checked étale locally

Lemma 3.3.4. *Let $\pi\colon \mathcal{X} \to X$ be a morphism to an algebraic space. Suppose that there is an fppf covering $\{X_i \to X\}$ such that $\mathcal{X} \times_X X_i \to X_i$ is a coarse moduli space for each i. Then $\pi\colon \mathcal{X} \to X$ is a coarse moduli space.* □

Proof. Axiom (1) of a coarse moduli space is a condition on geometric fibers and can thus be checked étale-locally while Axiom (2) follows from descent of morphisms of algebraic spaces in the étale topology. □

3.3.2 Quotients by finite groups

In this section, we establish a special case of the Keel–Mori Theorem for quotient stacks $[\operatorname{Spec} A/G]$ by finite groups (Theorem 3.3.8).

If a finite group G acts on an affine scheme $\operatorname{Spec} A$, then G also acts on the ring A and we define the *invariant ring* as

$$A^G = \{f \in A \mid g \cdot f = f \text{ for all } g \in G\}.$$

Lemma 3.3.5. *Let R be a noetherian ring. Let G be a finite group acting on an affine scheme $\operatorname{Spec} A$ of finite type over R. Then $A^G \to A$ is finite and A^G is finitely generated over R.*

Proof. We first show that $A^G \to A$ is integral. If $a \in A$, then $\prod_{g\in G}(x - ga) \in A^G[x]$ is polynomial with invariant coefficients such that a as a root. Since $A^G \to A$ is of finite type (as $R \to A$ is of finite type) and integral, it is finite (c.f [AM69, Cor. 5.2]). Since R is noetherian, we may conclude that $R \to A^G$ is of finite type (c.f. [AM69, Prop. 7.8]. □

The invariant ring is compatible with flat base change.

Lemma 3.3.6. *Let G be a finite group acting on an affine scheme $\operatorname{Spec} A$. If $A^G \to B$ is a flat ring homomorphism, then G acts on the affine scheme $\operatorname{Spec}(B \otimes_{A^G} A)$ and $B = (B \otimes_{A^G} A)^G$.*

Proof. By definition, the invariant ring is the equalizer

$$0 \to A^G \to A \overset{p_1}{\underset{p_2}{\rightrightarrows}} \prod_{g\in G} A$$

where $p_1(f) = (f)_{g\in G}$ and $p_2(f) = (gf)_{g\in G}$. Since $A^G \to B$ is flat, we have that

$$0 \to B \to A \otimes_{A^G} B \overset{p_1}{\underset{p_2}{\rightrightarrows}} \prod_{g\in G} A \otimes_{A^G} B$$

is also exact and we conclude that $B = (B \otimes_{A^G} A)^G$. □

Exercise 3.3.7. Let $A^G \to B$ be an arbitrary ring homomorphism and consider the commutative diagram

$$\begin{array}{ccccc} & & \operatorname{Spec} B \otimes_{A^G} A & \longrightarrow & \operatorname{Spec} A \\ & \swarrow & \downarrow & & \downarrow \\ \operatorname{Spec}(B \otimes_{A^G} A)^G & \longrightarrow & \operatorname{Spec} B & \longrightarrow & \operatorname{Spec} A^G. \end{array}$$

(1) Show that $\operatorname{Spec}(B \otimes_{A^G} A)^G \to \operatorname{Spec} B$ is an integral homeomorphism.

(2) If $|G|$ is invertible in A, show that $B \to (B \otimes_{A^G} A)^G$ is an isomorphism.

Theorem 3.3.8. *Let R be a noetherian ring. Let G be a finite group acting on an affine scheme $\operatorname{Spec} A$ of finite type over R. Then $\pi\colon [\operatorname{Spec} A/G] \to \operatorname{Spec} A^G$ is a coarse moduli space such that*

(1) A^G is finitely generated over R,

(2) π is a proper universal homeomorphism, and

(3) the base change of π along any flat morphism $X' \to \operatorname{Spec} A^G$ of noetherian algebraic spaces is a coarse moduli space.

Proof. We've seen in Lemma 3.3.5 that $A^G \to A$ is finite and A^G is finitely generated over R. We divide the remainder of the proof into three steps:

Step 1: π is a proper universal homeomorphism and in particular bijective on geometric points: Since $\operatorname{Spec} A \to \operatorname{Spec} A^G$ and $\operatorname{Spec} A \to [\operatorname{Spec} A/G]$ are both proper, so is π. As $\operatorname{Spec} A \to \operatorname{Spec} A^G$ is finite and dominant, it is surjective and therefore so is π. To see that π is injective on geometric points, suppose that x and x' are two $\Bbbk$-points of $\operatorname{Spec} A$ with distinct orbits in $\operatorname{Spec}(A \otimes_R \Bbbk)$. Let $f \in A \otimes_R \Bbbk$ be a function with $f|_{Gx} = 0$ and $f|_{Gx'} = 1$. Then $\widetilde{f} = \prod_{g\in G} gf$ is a G-invariant function with $f'(\pi(x)) = 0$ and $f'(\pi(x')) = 1$. Thus $\pi(x) \neq \pi(x')$. Finally, as π is bijective and universally closed, its set-theoretic inverse is continuous, and thus π is a homeomorphism. The base change of π along a morphism $\operatorname{Spec} B \to \operatorname{Spec} A^G$ factors as $[\operatorname{Spec}(B \otimes_{A^G} A)/G] \to \operatorname{Spec}(B \otimes_{A^G} A)^G \to \operatorname{Spec} B$ where the first map is a homeomorphism by the above argument and the second is by Exercise 3.3.7, and we conclude that π is a universal homeomorphism.

Step 2: π is universal for maps to algebraic spaces: Set $\mathcal{X} = [\operatorname{Spec} A/G]$ and $X = \operatorname{Spec} A^G$. We need to show that if Y is any algebraic space, then the natural map

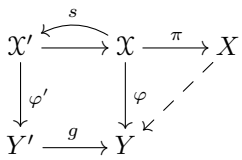
$$\operatorname{Map}(X, Y) \to \operatorname{Map}(\mathcal{X}, Y) \tag{3.3.1}$$

is bijective. If $h_1, h_2\colon X \to Y$ are two maps such that $h_1 \circ \pi = h_2 \circ \pi$, let $E \to X$ be the equalizer of h_1 and h_2, i.e. the pullback of the diagonal $Y \to Y \times Y$ along $(h_1, h_2)\colon X \to Y \times Y$. The equalizer $E \to X$ is a monomorphism and locally of finite type. By construction $\pi\colon \mathcal{X} \to X$ factors through $E \to X$ and since π is

universally closed and schematically dominant (i.e. $\mathcal{O}_X \to \pi_*\mathcal{O}_\mathcal{X}$ is injective), so is $E \to X$. As any universally closed and locally of finite type monomorphism is a closed immersion, we conclude that $E \to X$ is an isomorphism. (Note that this argument only used that $\mathcal{X} \to X$ is universally closed and schematically dominant.)

For the surjectivity of (3.3.1), let $\varphi\colon \mathcal{X} \to Y$ be a map. We claim that the question is étale-local on X. Indeed, if $X' \to X$ is an étale cover and $\chi'\colon X' \to Y$ is a morphism such that the two compositions $X' \times_X \mathcal{X} \to X' \xrightarrow{\chi'} Y$ and $X' \times_X \mathcal{X} \to \mathcal{X} \xrightarrow{\varphi} Y$ agree, then by the injectivity of (3.3.1), the two compositions $X' \times_X X' \rightrightarrows X' \xrightarrow{\chi} Y$ agree and $\chi'\colon X' \to Y$ descends to a morphism $\chi\colon X \to Y$. Étale descent also implies the commutativity of $\varphi = \chi \circ \pi$.

We may therefore assume that A^G is strictly henselian. Since $\mathcal{X}$ is quasi-compact, we may assume that Y is quasi-compact as $\varphi\colon \mathcal{X} \to Y$ factors through a quasi-compact open algebraic subspace of Y. Let $Y' \to Y$ be an étale presentation from an affine scheme and let $\mathcal{X}' := \mathcal{X} \times_Y Y'$. As $A^G \to A$ is finite, A is also strictly henselian and $f\colon Y' \times_Y \operatorname{Spec} A \to \operatorname{Spec} A$ has a section $s\colon \operatorname{Spec} A \to Y' \times_Y \operatorname{Spec} A$, which is a G-invariant open and closed immersion, and descends to a section $s\colon \mathcal{X} \to \mathcal{X}'$ of $\mathcal{X}' \to \mathcal{X}$. To summarize, we have a commutative diagram



where $\mathcal{X} \xrightarrow{\varphi} Y$ factors as $\mathcal{X} \xrightarrow{s} \mathcal{X}' \xrightarrow{\varphi'} Y' \xrightarrow{g} Y$. Since X and Y' are affine, the equality $\Gamma(X, \mathcal{O}_X) = \Gamma(\mathcal{X}, \mathcal{O}_\mathcal{X})$ implies that $\mathcal{X} \xrightarrow{s} \mathcal{X}' \xrightarrow{\varphi'} Y'$ factors through $\pi\colon \mathcal{X} \to X$ via a morphism $X \to Y'$. The composition $X \to Y' \to Y$ yields the desired dotted arrow above.

Step 3: the coarse moduli space property is preserved under flat base change: By Lemma 3.3.4, it suffices to consider morphisms $Y' \to Y$ from an affine scheme. But in this case, Lemma 3.3.6 implies that the base change $\mathcal{X} \times_Y Y' \cong [\operatorname{Spec} B/G]$ with $Y' \cong \operatorname{Spec} B^G$ and the above argument shows that $\mathcal{X} \times_Y Y' \xrightarrow{g} Y'$ is also a coarse moduli space. □

Example 3.3.9. If G is a finite group, the map $BG = [\operatorname{Spec} \Bbbk/G] \to \operatorname{Spec} \Bbbk$ is a coarse moduli space.

Example 3.3.10. Assume $\operatorname{char}(\Bbbk) \neq 2$. If $G = \mathbb{Z}/2$ acts on $\mathbb{A}^1 = \operatorname{Spec} \Bbbk[x]$ via $-1 \cdot x = -x$, then $[\mathbb{A}^1/G] \to \operatorname{Spec} \Bbbk[x^2]$ is a coarse moduli space.

Similarly, if $G = \mathbb{Z}/2$ acts on $\mathbb{A}^2 = \operatorname{Spec} k[x, y]$ via $-1 \cdot (x, y) = (-x, -y)$, then

$$[\mathbb{A}^2/G] \to \mathbb{A}^2/G = \operatorname{Spec} \Bbbk[x^2, xy, y^2]$$

is a coarse moduli space. By setting $A = x^2, B = xy$ and $C = y^2$, the invariant ring can be identified with $\Bbbk[A, B, C]/(B^2 - AC)$ so that the quotient $\mathbb{A}^2/G$ is a cone over a conic, and in particular singular.

Exercise 3.3.11. Suppose that G is a finite group acting on an affine scheme $\operatorname{Spec} A$ of finite type over a noetherian ring R. If $x \in \operatorname{Spec} A$ is a closed point, show that there is an isomorphism

$$\widehat{A}^{G_x} \cong \widehat{A^G}$$

between the G_x-invariants of the completion at $\operatorname{Spec} A$ at x and the completion of $\operatorname{Spec} A^G$ at the image of x.

The following exercise generalizes Theorem 3.3.8 from quotients of finite groups to quotients of finite flat groupoids.

Exercise 3.3.12. Let $\mathcal{X}$ be an algebraic stack separated and of finite type over a noetherian ring B. Suppose that there exists a finite and flat morphism $U = \operatorname{Spec} A \to \mathcal{X}$ from an affine scheme. Set $R := U \times_{\mathcal{X}} U$ so that $R \rightrightarrows U$ is a finite flat groupoid of affine schemes, and define $A^R \subset A$ as the subring of R-invariants, i.e. the subring of elements $a \in A$ such that $s^*a = t^*a \in \Gamma(R, \mathcal{O}_R)$. Note that A^R is identified with $\Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$. Show that $\pi\colon \mathcal{X} \to \operatorname{Spec} A^R$ is a coarse moduli space satisfying properties (1)-(3) of Theorem 3.3.8.

3.3.3 Descending étale morphisms to quotients

Proposition 3.3.13. *Let G be a finite group and $f\colon \operatorname{Spec} A \to \operatorname{Spec} B$ be a G-equivariant morphism of affine schemes of finite type over a noetherian ring R. Let $x \in \operatorname{Spec} A$ be a closed point. Assume that*

(a) f is étale at x and

(b) the induced map $G_x \to G_{f(x)}$ of stabilizer groups is bijective.

Then there is an open affine neighborhood $W \subset \operatorname{Spec} A^G$ of the image of x such that $W \to \operatorname{Spec} A^G \to \operatorname{Spec} B^G$ is étale and $\pi_A^{-1}(W) \cong W \times_{\operatorname{Spec} B^G} [\operatorname{Spec} B/G]$, where $\pi_A\colon [\operatorname{Spec} A/G] \to \operatorname{Spec} A^G$.

Remark 3.3.14. In other words, after replacing $\operatorname{Spec} A^G$ with an affine neighborhood W of $\pi_A(x)$ and $\operatorname{Spec} A$ with $\pi_A^{-1}(W)$, it can be arranged that the diagram

$$\begin{array}{ccc} [\operatorname{Spec} A/G] & \xrightarrow{f} & [\operatorname{Spec} B/G] \\ \downarrow{\scriptstyle \pi_A} & & \downarrow{\scriptstyle \pi_B} \\ \operatorname{Spec} A^G & \longrightarrow & \operatorname{Spec} B^G \end{array} \tag{3.3.2}$$

is cartesian where both horizontal maps are étale.

In the proof of the Keel–Mori Theorem (Theorem 3.3.17), this proposition will be applied in the following form.

Corollary 3.3.15. *Let G be a finite group and $f\colon \operatorname{Spec} A \to \operatorname{Spec} B$ be a G-equivariant morphism of affine schemes of finite type over a noetherian ring R. Assume that for every closed point $x \in \operatorname{Spec} A$,*

(a) f is étale at x and

(b) the induced map $G_x \to G_{f(x)}$ of stabilizer groups is bijective.

Then $\operatorname{Spec} A^G \to \operatorname{Spec} B^G$ *is étale and* (3.3.2) *is cartesian.* □

Proof of Proposition 3.3.13. Set $y = f(x)$. We first claim that the question is étale local around $\pi_B(y) \in \operatorname{Spec} B^G$. Indeed, if $Y' \to \operatorname{Spec} B^G$ is an affine étale neighborhood of $\pi_B(y)$, we let $X', \mathcal{X}'$ and $\mathcal{Y}'$ denote the base changes of $\operatorname{Spec} A^G, [\operatorname{Spec} A/G]$, and $[\operatorname{Spec} B/G]$. By Lemma 3.3.6, we know that $\mathcal{Y}' \cong [\operatorname{Spec} B'/G]$ with $Y' \cong \operatorname{Spec} B'^G$ and similarly for $\mathcal{X}'$ and X'. If the result holds after this base change, there is an open neighborhood $W' \subset X'$ containing a preimage of $\pi_A(x)$ such that $W' \hookrightarrow X' \to Y'$ is étale and such that the preimage of W' in $\mathcal{X}'$ is isomorphic to $W' \times_{Y'} \mathcal{Y}'$. Taking W as the image of W' under $X' \to \operatorname{Spec} A^G$ and applying étale descent yields the desired claim.

We may therefore assume that B^G is strictly henselian. As $B^G \to B$ is finite (Lemma 3.3.5), B is also strictly henselian. Thus $\operatorname{Spec} A \to \operatorname{Spec} B$ has a section $s\colon \operatorname{Spec} B \to \operatorname{Spec} A$ taking y to x, and the morphism s is necessarily a G-invariant open and closed immersion. The result is clear in this case. □

Remark 3.3.16. Here's a conceptual reason for why we should expect the induced map of quotients to be étale. For simplicity, assume that $R = \Bbbk$ is an algebraically closed field. Let $\widehat{A}$ and $\widehat{B}$ be the completions of the local rings at x and $f(x)$. The stabilizers G_x and $G_{f(x)}$ act on $\operatorname{Spec} \widehat{A}$ and $\operatorname{Spec} \widehat{B}$, respectively, and the map $\operatorname{Spec} \widehat{A} \to \operatorname{Spec} \widehat{B}$ is equivariant with respect to the map $G_x \to G_{f(x)}$. The completion $\widehat{A^G}$ of A^G at the image of x is isomorphic to $\widehat{A}^{G_x}$ (Exercise 3.3.11) and similarly $\widehat{B^G} = \widehat{B}^{G_{f(x)}}$. Since f is étale at x, $\widehat{B} \to \widehat{A}$ is an isomorphism and since $G_x \to G_{f(x)}$ is bijective, the induced map $\widehat{B^G} \to \widehat{A^G}$ is an isomorphism and $\operatorname{Spec} A^G \to \operatorname{Spec} B^G$ is étale at the image of x.

3.3.4 The Keel–Mori Theorem

We now state and prove the Keel–Mori Theorem.

Theorem 3.3.17. *Let* $\mathcal{X}$ *be a Deligne-Mumford stack separated and of finite type over a noetherian algebraic space* S*. Then there exists a coarse moduli space* $\pi\colon \mathcal{X} \to X$ *with* $\mathcal{O}_X = \pi_* \mathcal{O}_{\mathcal{X}}$ *such that*

(1) X *is separated and of finite type over* S*,*

(2) π *is a proper universal homeomorphism, and*

(3) for any flat morphism $X' \to X$ *of noetherian algebraic spaces, the base change* $\mathcal{X} \times_X X' \to X'$ *is a coarse moduli space.*

Remark 3.3.18. More generally for any algebraic stack $\mathcal{X}$ (without any noetherian or finiteness conditions) such that the inertia stack $I_{\mathcal{X}}$ is finite over $\mathcal{X}$, there exists a coarse moduli space $\pi\colon \mathcal{X} \to X$ with π separated [KM97, Con05, Ryd13]. In particular, it holds for algebraic stacks with finite but *non-reduced* automorphism groups.

Proof. We first handle the case when $S = \operatorname{Spec} R$ is affine. The question is Zariski-local on $\mathcal{X}$: if $\{\mathcal{X}_i\}$ is a Zariski-open covering of $\mathcal{X}$ with coarse moduli spaces $\mathcal{X}_i \to X_i$, then since coarse moduli spaces are unique (Definition 3.3.1(2)), the X_i's glue to form an algebraic space X and a map $\mathcal{X} \to X$, which is a coarse moduli space by Lemma 3.3.4. It thus suffices to show that any closed point $x \in |\mathcal{X}|$ has an open neighborhood which admits a coarse moduli space.

By the Local Structure Theorem of Deligne–Mumford Stacks (Theorem 3.2.1), there exists an affine and étale morphism

$$f\colon \big(\mathcal{W} = [\operatorname{Spec} A/G_x], w\big) \to (\mathcal{X}, x)$$

such that f induces an isomorphism of geometric stabilizer groups at w. (Recall that the geometric stabilizer of w is simply the stabilizer group of any geometric point $\operatorname{Spec} \Bbbk \to \mathcal{W}$ with image w; see Definition 2.4.10).

We claim that since $\mathcal{X}$ is separated, the locus $\mathcal{U}$ consisting of points $z \in |\mathcal{W}|$, such that f induces an isomorphism of geometric stabilizer groups at z, is open. To establish this, we will analyze the natural morphism $I_{\mathcal{W}} \to I_{\mathcal{X}} \times_{\mathcal{X}} \mathcal{W}$ of relative group schemes over $\mathcal{W}$ as the fiber of this morphism over $z \in \mathcal{W}(\Bbbk)$ is precisely the morphism $G_z \to G_{f(z)}$ of stabilizers. We will exploit the cartesian diagram

$$\begin{array}{ccc} I_{\mathcal{W}} & \xrightarrow{\Psi} & I_{\mathcal{X}} \times_{\mathcal{X}} \mathcal{W} \\ \downarrow & & \downarrow \\ \mathcal{W} & \longrightarrow & \mathcal{W} \times_{\mathcal{X}} \mathcal{W}. \end{array}$$

Since $\mathcal{W} \to \mathcal{X}$ is representable, étale and separated, the diagonal $\mathcal{W} \to \mathcal{W} \times_{\mathcal{X}} \mathcal{W}$ is an open and closed immersion and thus so is Ψ. Since $I_{\mathcal{X}} \to \mathcal{X}$ is finite, so is $p_2\colon I_{\mathcal{X}} \times_{\mathcal{X}} \mathcal{W} \to \mathcal{W}$. Thus $p_2(|I_{\mathcal{X}} \times_{\mathcal{X}} \mathcal{W}| \setminus |I_{\mathcal{W}}|) \subset |\mathcal{W}|$ is closed and its complement, which is identified with the locus $\mathcal{U}$, is open.

Let $\pi_{\mathcal{W}}\colon \mathcal{W} \to W = \operatorname{Spec}(A^{G_x})$ be the coarse moduli space (Theorem 3.3.8). Choose an affine open subscheme $X_1 \subset W$ containing $\pi_{\mathcal{W}}(w)$. Then $\mathcal{X}_1 = \pi_{\mathcal{W}}^{-1}(X_1)$ is isomorphic to a quotient stack $[\operatorname{Spec}(A_1)/G_x]$ such that $X_1 = \operatorname{Spec}(A_1^{G_x})$. This provides an étale and affine morphism

$$g\colon (\mathcal{X}_1 = [\operatorname{Spec}(A_1)/G_x], w) \to (\mathcal{X}, x)$$

which induces a bijection on all stabilizer groups.

We now show that the open substack $\mathcal{X}_0 := \operatorname{im}(f)$ admits a coarse moduli space. Define $\mathcal{X}_2 := \mathcal{X}_1 \times_{\mathcal{X}} \mathcal{X}_1$ and $\mathcal{X}_3 := \mathcal{X}_1 \times_{\mathcal{X}} \mathcal{X}_1 \times_{\mathcal{X}} \mathcal{X}_1$. Since g is affine, each $\mathcal{X}_i$ is of the form $[\operatorname{Spec}(A_i)/G_x]$ and there is a coarse moduli space $\pi_i\colon \mathcal{X}_i \to X_i = \operatorname{Spec}(A_i^{G_x})$. By universality of coarse moduli spaces, there is a diagram

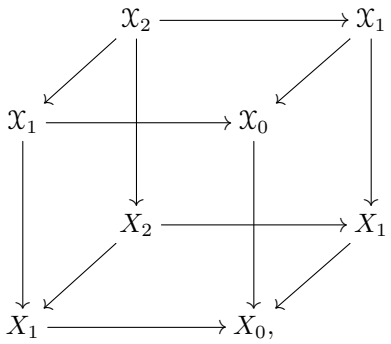
$$\begin{array}{ccccccc} \mathcal{X}_3 & \rightrightarrows & \mathcal{X}_2 & \rightrightarrows & \mathcal{X}_1 & \xrightarrow{g} & \mathcal{X}_0 = \operatorname{im}(f) \\ \downarrow{\scriptstyle \pi_3} & & \downarrow{\scriptstyle \pi_2} & & \downarrow{\scriptstyle \pi_1} & & \vdots{\scriptstyle \pi_0} \\ X_3 & \rightrightarrows & X_2 & \rightrightarrows & X_1 & \dashrightarrow & X_0 \end{array} \tag{3.3.3}$$

where the natural squares commute. Since g induces bijection of stabilizer groups at all points, the same is true for each projection $\mathcal{X}_2 \to \mathcal{X}_1$ and $\mathcal{X}_3 \to \mathcal{X}_2$. Corollary 3.3.15 implies that each map $X_2 \to X_1$ and $X_3 \to X_2$ is étale, and the natural squares of solid arrows in (3.3.3) are cartesian.

The universality of coarse moduli spaces induces an étale groupoid structure $X_2 \rightrightarrows X_1$. To check that this is an étale equivalence relation, it suffices to check that $X_2 \to X_1 \times X_1$ is injective on k-points but this follows easily from the observation the $|\mathcal{X}_2| \to |\mathcal{X}_1| \times |\mathcal{X}_1|$ is injective on closed points. Therefore there is an algebraic space quotient $X_0 := X_1/X_2$ and a map $X_1 \to X_0$. By étale descent

along $\mathcal{X}_1 \to \mathcal{X}_0$, there is a map $\pi_0 \colon \mathcal{X}_0 \to X_0$ making the right square in (3.3.3) commute.

To argue that $\pi \colon \mathcal{X}_0 \to X_0$ is a coarse moduli space, we will use the commutative cube



where the top, left, and bottom faces are cartesian. It follows from étale descent along $\mathcal{X}_1 \to \mathcal{X}_0$ that the right face is also cartesian and since being a coarse moduli space is étale local on X_0 (Lemma 3.3.4), we conclude that $\mathcal{X}_0 \to X_0$ is a coarse moduli space. Except for the separatedness, the additional properties in the statement are étale local on X_0 so they follow from the analogous properties of the coarse moduli space $[\mathrm{Spec}(A_1)/G_x] \to \mathrm{Spec}(A_1^{G_x})$ from Theorem 3.3.8. As $\mathcal{X}_0 \to X_0$ is proper, the separatedness of $\mathcal{X}_0$ is equivalent to the separatedness of X_0.

Finally, the case when S is a noetherian algebraic space can be reduced to the affine case by imitating the above argument to étale-locally construct the coarse moduli space of $\mathcal{X}$. □

Remark 3.3.19. The more general case when $\mathcal{X}$ is an algebraic stack with finite inertia $I_{\mathcal{X}} \to \mathcal{X}$ (see Remark 3.3.18) is proven in an analogous but more technical manner. Namely, the use of the Local Structure Theorem for Deligne–Mumford stacks (Theorem 3.2.1) is replaced by the existence of an étale neighborhood $\mathcal{W} \to \mathcal{X}$ around any closed point such that $\mathcal{W}$ admits a finite flat presentation $V \to \mathcal{W}$ from an affine scheme and the corresponding groupoid $R := V \times_{\mathcal{W}} V \rightrightarrows V$ is a finite flat groupoid of affine schemes. This in turn is proven in an analogous way to Theorem 3.2.1 where one chooses a quasi-finite and flat surjection $U \to \mathcal{X}$ and one replaces the use of $[(U/\mathcal{X})_0^d/S_d]$) with a Hilbert stack $\mathcal{H}$ whose objects over a scheme S consists of a morphism $S \to \mathcal{X}$ and a closed subscheme $Z \hookrightarrow U_S$ finite and flat (rather than finite and étale) over S. Finally, the existence of a coarse moduli space for quotients $[V/R]$ is proven analogously to Theorem 3.3.8 (see Exercise 3.3.12).

3.3.5 Refinements and examples

The Local Structure Theorem of Deligne–Mumford Stacks (Theorem 3.2.1) can also be formulated étale locally on a coarse moduli space:

Corollary 3.3.20 (Local Structure of Coarse Moduli Spaces)**.** *Let $\mathcal{X}$ be a Deligne–Mumford stack of finite type and separated over a noetherian algebraic space S, and let $\pi \colon \mathcal{X} \to X$ be its coarse moduli space. For any closed point $x \in |\mathcal{X}|$ with*

geometric stabilizer group G_x, there exists a cartesian diagram

$$\begin{array}{ccc} [\operatorname{Spec} A/G_x] & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow{\scriptstyle \pi} \\ \operatorname{Spec} A^{G_x} & \longrightarrow & X \end{array}$$

such that $\operatorname{Spec} A^{G_x} \to X$ *is an étale neighborhood of* $\pi(x) \in |X|$.

Proof. This follows from the uniqueness of coarse moduli spaces and the construction of the coarse moduli space in the proof of Theorem 3.3.17. Alternatively, it follows from the Local Structure Theorem of Deligne–Mumford stacks (Theorem 3.2.1) and Exercise 3.3.21 □

Exercise 3.3.21. Establish the following generalization of Proposition 3.3.13: Let S be a noetherian algebraic space. Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a morphism of Deligne–Mumford stacks separated and of finite type over S and

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{f} & \mathcal{Y} \\ \downarrow{\scriptstyle \pi_{\mathcal{X}}} & & \downarrow{\scriptstyle \pi_{\mathcal{Y}}} \\ X & \longrightarrow & Y \end{array}$$

be a commutative diagram where $\pi_{\mathcal{X}}\colon \mathcal{X} \to X$ and $\pi_{\mathcal{Y}}\colon \mathcal{Y} \to Y$ are coarse moduli spaces. Let $x \in |\mathcal{X}|$ be a closed point such that

(1) f is étale at x and

(2) the induced map $G_x \to G_{f(x)}$ of geometric stabilizer groups is bijective.

Then there exists an open neighborhood $U \subset X$ of $\pi_{\mathcal{X}}(x)$ such that $U \to X \to Y$ is étale and $\pi_{\mathcal{X}}(U) \cong U \times_Y \mathcal{Y}$.

Exercise 3.3.22. With the hypotheses of Theorem 3.3.17, if in addition the order of every stabilizer group is invertible in S, show that $\mathcal{X} \to X$ is a universal coarse moduli space, i.e. for every $X' \to X$, the base change $\mathcal{X} \times_X X' \to X'$ is a coarse moduli space.

Exercise 3.3.23. Let $\operatorname{char}(\Bbbk) \neq 2$ and $G = \mathbb{Z}/2$.

(1) Let G act on the non-separated union $X = \mathbb{A}^1 \bigcup_{x \neq 0} \mathbb{A}^1$ simultaneously exchanging the copies of $\mathbb{A}^1$. The quotient $[X/G]$ is a Deligne–Mumford stack with quasi-finite but not finite inertia, and in particular non-separated. Show nevertheless that there is a coarse moduli space $[X/G] \to \mathbb{A}^1$.

(2) Let X be the non-separated union $\mathbb{A}^2 \bigcup_{x \neq 0} \mathbb{A}^2$. Let $G = \mathbb{Z}/2$ act on X by simultaneously exchanging the copies of $\mathbb{A}^2$ and by acting via the involution $y \mapsto -y$ on each copy. Show that $[X/G]$ does not admit a coarse moduli space.

3.3.6 Descending line bundles to the coarse moduli space

Definition 3.3.24. A noetherian Deligne–Mumford stack $\mathcal{X}$ is *tame* if for every geometric point $x \in \mathcal{X}(K)$, the order of $\operatorname{Aut}_{\mathcal{X}(K)}(x)$ is invertible in $\Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$.

Remark 3.3.25. If $\mathcal{X}$ is defined over a field $\Bbbk$, then this means that the order of $\mathrm{Aut}_{\mathcal{X}(K)}(x)$ is prime to the characteristic of $\Bbbk$.

We say that a vector bundle $\mathcal{E}$ on $\mathcal{X}$ *descends to its coarse moduli space* $\pi\colon \mathcal{X} \to X$ if there exists a vector bundle E on X and an isomorphism $\mathcal{E} \cong \pi^* E$.

Proposition 3.3.26. *Let $\mathcal{X}$ be a tame Deligne-Mumford stack separated and of finite type over a noetherian algebraic space S, and let $\pi\colon \mathcal{X} \to X$ be its coarse moduli space. A vector bundle $\mathcal{E}$ on $\mathcal{X}$ descends to a vector bundle on X if and only if for every closed point $x \in |\mathcal{X}|$, the pullback of $\mathcal{E}$ to the residual gerbe $\mathcal{G}_x$ is trivial.*

Remark 3.3.27. The condition that $\mathcal{E}|_{\mathcal{G}_x}$ is trivial is equivalent to the condition that automorphism group $\mathrm{Aut}_{\mathcal{X}(K)}(\overline{x})$ of a representative $\overline{x}\colon \operatorname{Spec} K \to \mathcal{X}$ of x acts trivially on the fiber $\mathcal{E} \otimes K = \overline{x}^* \mathcal{E}$. Note that this latter condition is insensitive to field extensions.

Proof. To be added. □

When $\mathcal{X}$ is not tame, we have the following variant of descending line bundles.

Proposition 3.3.28. *Let $\mathcal{X}$ be a Deligne-Mumford stack separated and of finite type over a noetherian algebraic space S, and let $\pi\colon \mathcal{X} \to X$ be its coarse moduli space. If $\mathcal{L}$ is a line bundle on $\mathcal{X}$, then for N sufficiently divisible $\mathcal{L}^{\otimes N}$ descends to X.*

Proof. To be added. □

3.4 Global structure of algebraic spaces and Deligne–Mumford stacks

Theorem 3.4.1. *Any noetherian algebraic space has a dense open algebraic subspace which is a scheme.*

Proof. Let $f\colon V \to X$ be an étale presentation of a noetherian algebraic space X. There exists an open algebraic subspace $U \subset X$ such that $f^{-1}(U) \to U$ is finite and étale. By Exercise 3.2.5, U is isomorphic to a quotient stack $[V/G]$ for the free action of a finite group G on a scheme V. If $V_1 \subset V$ is a dense open subscheme which is affine, then $V_2 = \bigcap_g gV_1$ is a G-invariant quasi-affine open. Repeating this argument, we can choose a dense open affine subscheme $V_3 \subset V_1$ and now $V_4 = \bigcap_g gV_3$ is a G-invariant affine open. Theorem 3.3.8 implies that $V_4/G \cong \operatorname{Spec} A^G$ is a dense open algebraic subspace of U. □

Remark 3.4.2. The above result is not necessarily true if X is not quasi-separated, e.g. $\mathbb{A}^1/\mathbb{Z}$ (Example 2.9.26).

Theorem 3.4.3 (Le Lemme de Gabber)**.** *Let $\mathcal{X}$ be a Deligne-Mumford stack separated and of finite type over a noetherian algebraic space S. Then there exists a finite, generically étale, and surjective morphism $U \to \mathcal{X}$ from a scheme U.*

Proof. To be added. □

By applying Chow's lemma to U, we obtain:

Corollary 3.4.4. *There exists a projective, generically étale, and surjective morphism $V \to \mathcal{X}$ from a scheme V quasi-projective over S.*

Chapter 4

Moduli of stable curves

4.1 Nodal curves

A *curve* over a field $\Bbbk$ is a pure one-dimensional scheme C of finite type over $\Bbbk$. If C is a proper curve, we define the *arithmetic genus of* C or simply the *genus of* C as

$$g(C) = 1 - \chi(C, \mathcal{O}_C)$$

which is equal to $h^1(C, \mathcal{O}_C)$ if C is geometrically connected and reduced.

The following easy version of Riemann-Roch holds for singular curves (c.f. [Har77, Exer. IV.1.7]):

Theorem 4.1.1 (Easy Riemann–Roch)**.** *Let C be an integral projective curve of genus g. If L is a line bundle on C, then*

$$\chi(C, L) = \deg L + 1 - g.$$

Proper curves are projective: the case of smooth curves is handled in [Har77, Prop. II.6.7] and in general properties of ampleness reduce the projectivity of a curve C to the projectivity of the normalization $\widetilde{C}_{\mathrm{red}}$ of the reduction. It is also true that any pure one-dimensional separated algebraic space is a scheme (see [SP, Tag 0ADD]).

4.1.1 Review of smooth curves

We review some basic properties of smooth curves which we would later like to generalize for nodal curves.

If C is a smooth curve, then the sheaf of differentials Ω_C is a line bundle. Serre Duality is a deep result, indispensable in the study of curves, that states that Ω_C is in fact a dualizing sheaf on C.

Theorem 4.1.2 (Serre–Duality for Smooth Curves)**.** *[Har77, Cor. III.7.12] If C is a smooth projective curve over $\Bbbk$, then Ω_C is a dualizing sheaf, i.e. there is a linear map $\mathrm{tr}\colon H^1(C, \Omega_C) \to \Bbbk$ such that for any coherent sheaf $\mathcal{F}$, the natural pairing*

$$\mathrm{Hom}(\mathcal{F}, \Omega_C) \times H^1(C, \mathcal{F}) \to H^1(C, \Omega_C) \xrightarrow{\mathrm{tr}} \Bbbk$$

is perfect.

Remark 4.1.3. The pairing being perfect means that the $\mathrm{Hom}(\mathcal{F}, \Omega_C)$ is identified with the dual $H^1(C, \mathcal{F})^\vee$. If $\mathcal{F}$ is a vector bundle, there is a functorial isomorphism

$$H^0(C, \mathcal{F}^\vee \otimes \omega_C) \cong H^1(C, \mathcal{F})^\vee.$$

Taking $\mathcal{F} = \omega_C$, we see that $H^1(C, \omega_C) \cong H^0(C, \mathcal{O}_C)^\vee$ and in particular that the trace map $\mathrm{tr}\colon H^1(C, \omega_C) \to \Bbbk$ is an isomorphism if C is connected and $H^0(C, \mathcal{O}_C) = \Bbbk$ (e.g. $\Bbbk$ is algebraically closed).

While Easy Riemann–Roch (Theorem 4.1.1) is essentially a trivial consequence of long exact sequences in cohomology, Serre–Duality is substantially more difficult, even for smooth projective curves. Combining Easy Riemann–Roch and Serre Duality leads to the more powerful version of Riemann–Roch.

Theorem 4.1.4 (Riemann–Roch)**.** *[Har77, Thm IV.1.3] Let C be a smooth projective curve of genus g. If L is a line bundle on C, then*

$$h^0(C, L) - h^0(C, \Omega_C \otimes L^\vee) = \deg L + 1 - g \tag{4.1.1}$$

Remark 4.1.5. This is often written in divisor form as $h^0(C, L) - h^0(C, K - L) = \deg L + 1 - g$ where K denotes the canonical divisor, i.e. $\Omega_C = \mathcal{O}_C(K)$.

Like Riemann–Roch, Riemann–Hurwitz plays an essential role in the study of smooth curves. Riemann–Hurwitz informs us on how the sheaf of differentials behaves under finite morphisms of smooth curves; the statement is postponed until our discussion of branched covers; see Theorem 4.6.3.

Positivity of divisors on curves

The following consequence of Riemann–Roch provides a useful criteria to determine whether a given line bundle is basepoint-free, ample or very ample.

Corollary 4.1.6. *[Har77, Cor. IV.3.2] Let C be a smooth projective curve over $\Bbbk$ and L be a line bundle on C.*

(1) if $\deg L < 0$, then $h^0(C, L) = 0$;

(2) if $\deg L > 0$, then L is ample;

(3) if $\deg L \geq 2g$, then L is basepoint-free; and

(4) if $\deg L \geq 2g + 1$, then L is very ample.

For a smooth projective curve C of genus $g > 1$, we can use Riemann–Roch and Serre–Duality to compute that: (a) $h^0(C, \omega_C) = h^1(C, \mathcal{O}_C) = g$, (b) $h^1(C, \omega_C) = h^0(C, \mathcal{O}_C) = \Bbbk$ and (c) Ω_C has degree $2g - 2$ and is thus ample on C. Similarly, if $k > 1$, we have: (a) $h^0(C, \omega_C^{\otimes k}) = (2k - 1)(g - 1)$, (b) $h^1(C, \omega_C^{\otimes k}) = 0$ and (c) $\Omega_C^{\otimes k}$ has degree $2k(g - 1)$ and is very ample if $k \geq 3$. Note that Ω_C is not very ample precisely when C is *hyperelliptic*.

On the other hand, if $g = 1$ then $\Omega_C \cong \mathcal{O}_C$, and if $g = 0$ then $C = \mathbb{P}^1$ and $\omega_C = \mathcal{O}(-2)$.

Families of smooth curves

Definition 4.1.7. A *family of smooth curves (of genus g)* over a scheme S is a smooth and proper morphism $\mathcal{C} \to S$ of schemes such that every geometric fiber is a connected curve (of genus g).

The relative sheaf of differentials $\Omega_{\mathcal{C}/S}$ is a line bundle on $\mathcal{C}$ such that for any geometric point $s\colon \operatorname{Spec} K \to S$, the restriction $\Omega_{\mathcal{C}/S}|_{\mathcal{C}_s}$ is identified with $\Omega_{\mathcal{C}_s}$. More generally, for any morphism $T \to S$ of schemes, the pullback of $\Omega_{\mathcal{C}/S}$ to $\mathcal{C} \times_S T$ is canonically isomorphic to $\Omega_{\mathcal{C}\times_S T/T}$.

We now will apply Cohomology and Base Change [Har77, Thm III.12.11] to show that for $k \geq 3$, the kth relative pluricanonical sheaf $\Omega_{\mathcal{C}/S}^{\otimes k}$ is relatively very ample and that its pushforward is a vector bundle on S.

Proposition 4.1.8 (Properties of Families of Smooth Curves)**.** *Let $\mathcal{C} \to S$ be a family of smooth curves of genus $g \geq 2$. Then for $k \geq 3$, $\Omega_{\mathcal{C}/S}^{\otimes k}$ is relatively very ample and $\pi_*(\Omega_{\mathcal{C}/S}^{\otimes k})$ is a vector bundle of rank $(2k-1)(g-1)$.*

Proof. For any geometric point $s\colon \operatorname{Spec} K \to S$, the fiber $\Omega_{\mathcal{C}/S}^{\otimes k} \otimes K = \Omega_{\mathcal{C}_s}^{\otimes k}$ is very ample (Corollary 4.1.6) and therefore $\Omega_{\mathcal{C}/S}^{\otimes k}$ is relatively very ample as this property can be checked on geometric fibers for a proper morphisms locally of finite presentation. Since $H^1(\mathcal{C}_s, \Omega_{\mathcal{C}_s/\kappa(s)}^{\otimes k}) = 0$ for all $s \in S$, Cohomology and Base Change implies that $\pi_*(\Omega_{\mathcal{C}/S}^{\otimes k})$ is a vector bundle whose fiber at s is identified with $H^0(\mathcal{C}_s, \Omega_{\mathcal{C}_s/\kappa(s)}^{\otimes k})$, which has dimension $(2k-1)(g-1)$ by Riemann–Roch. □

It is also true that the relative sheaf of differentials $\Omega_{\mathcal{C}/S}$ is also a relative dualizing sheaf, i.e. satisfies a relative version of Serre–Duality.

4.1.2 Nodes

Definition 4.1.9 (Nodes)**.** Let C be a curve over a field $\Bbbk$.

- If $\Bbbk$ is algebraically closed, we say that $p \in C$ is a *node* if there is an isomorphism $\widehat{\mathcal{O}}_{C,p} \cong \Bbbk[[x,y]]/(xy)$.
- If $\Bbbk$ is an arbitrary field, we say that $p \in C$ is a *node* if there exists a node $p' \in C_{\overline{\Bbbk}}$ over p.

We say that C is a *nodal curve* if every closed point is either smooth or nodal.

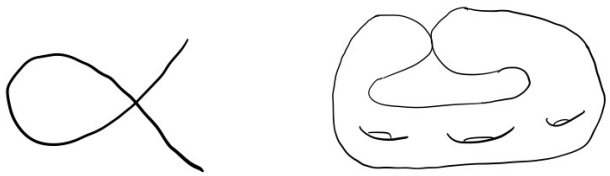


Figure 4.1: A node of a curve over $\mathbb{C}$ viewed algebraically (left hand side) or analytically (right hand side).

If $\Bbbk$ is not algebraically closed, the completion of the local ring of a node may not be isomorphic to $\Bbbk[[x,y]]/(xy)$. However, it becomes isomorphic after a separable field extension of $\Bbbk$.

Lemma 4.1.10. *If C is a curve over a field $\Bbbk$ and $p \in C$ is a node, there exists a finite separable field extension $\Bbbk \to \Bbbk'$, a point $p' \in C_{\Bbbk'}$ over p and an isomorphism $\widehat{\mathcal{O}}_{C_{\Bbbk'},p'} \cong \Bbbk'[[x,y]]/(xy)$.*

Proof. TO BE ADDED □

If C is a curve over a field $\Bbbk$ and $p \in C$ is a node, then there exists a finite separable field extension $\Bbbk \to \Bbbk'$ and a common étale neighborhood

$$\begin{array}{ccc} & (U,u) & \\ \swarrow & & \searrow \\ (C,p) & & (\operatorname{Spec} \Bbbk'[x,y]/(xy),0). \end{array} \tag{4.1.2}$$

Indeed, this follows directly from Lemma 4.1.10 and an application of Artin Approximation (Corollary A.4.17). We will prove a more general and relative statement in Theorem 4.1.21.

Exercise 4.1.11. Provide a proof of the existence of the common étale neighborhood in (4.1.2) without appealing to Artin Approximation.

4.1.3 Genus formula

Let C be a connected, nodal and projective curve over an algebraically closed field $\Bbbk$ with δ nodes $z_1, \dots, z_\delta \in C$ and ν irreducible components $C_1, \dots, C_\nu$. Let $g(\widetilde{C}_i)$ be the genus of the normalization $\widetilde{C}_i$ of C_i, i.e. the geometric genus of C_i. The normalization $\pi\colon \widetilde{C} \to C$ induces a short exact sequence

$$0 \to \mathcal{O}_C \to \pi_* \mathcal{O}_{\widetilde{C}} \to \bigoplus_i \kappa(p_i) \to 0.$$

The long exact sequence of cohomology is

$$0 \to \underbrace{H^0(C,\mathcal{O}_C)}_{1} \to \underbrace{H^0(\widetilde{C},\mathcal{O}_{\widetilde{C}})}_{\nu} \to \underbrace{\oplus_i \kappa(p_i)}_{\delta} \to \underbrace{H^1(C,\mathcal{O}_C)}_{g} \to \underbrace{H^1(\widetilde{C},\mathcal{O}_{\widetilde{C}})}_{\sum_i g(\widetilde{C}_i)} \to 0$$

with the labels underneath indicating the dimension. We therefore obtain:

Proposition 4.1.12 (Genus Formula)**.** *The genus g of C satisfies $g = \sum_{i=1}^{\nu} g(\widetilde{C}_i) + \delta - \nu + 1$.* □

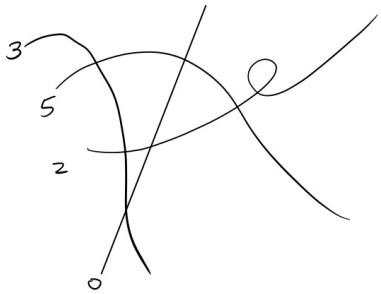


Figure 4.2: An example of a nodal curve of genus 14.

Remark 4.1.13. Notice that $\delta - \nu + 1$ is precisely the number of connected regions bounded by the one-dimensional picture of C. Thus, the genus of a nodal curve can be easily computed from the picture by summing the geometric genera of the irreducible components and adding the number of such regions.

4.1.4 The dualizing sheaf

If C is a projective nodal curve, then C is locally a complete intersection and therefore it has a *dualizing sheaf* ω_C and trace map $\mathrm{tr}_C\colon H^1(C,\omega_C)\to k$ [Har77, III.7.11], i.e. for any coherent sheaf $\mathcal{F}$, the natural pairing

$$\mathrm{Hom}(\mathcal{F},\omega_C)\times H^1(C,\mathcal{F})\to H^1(C,\omega_C)\xrightarrow{\mathrm{tr}}\Bbbk$$

is perfect. The most important properties of the dualizing sheaf ω_C is that (1) it exists and is a line bundle and (2) that its powers $\omega_C^{\otimes k}$ are very ample for $k\geq 3$ and thereby provide pluricanonical embeddings into projective space.

Explicit description of ω_C

Given the importance of dualizing sheaf ω_C, it is natural to seek a more explicit description. Let $\Sigma := C^{\mathrm{sing}}$ be the singular locus and $U = C\setminus\sigma$. Let $\pi\colon \widetilde{C}\to C$ be the normalization of C, and let $\widetilde{\Sigma}$ and $\widetilde{U}$ be the preimages of Σ and U as in the diagram

$$\begin{array}{ccccc}
\widetilde{U} & \xhookrightarrow{\tilde{j}} & \widetilde{C} & \hookleftarrow & \widetilde{\Sigma} \\
\downarrow & & \downarrow{\scriptstyle \pi} & & \downarrow \\
U & \xhookrightarrow{j} & C & \hookleftarrow & \Sigma
\end{array}$$

Let $\Sigma=\{z_1,\dots,z_n\}$ be an ordering of the points and $\pi^{-1}(z_i)=\{p_i,q_i\}$. Since $\widetilde{C}$ is smooth, the sheaf of differentials $\Omega_{\widetilde{C}}$ is a dualizing sheaf and is a line bundle. There is a short exact sequence

$$0\to\Omega_{\widetilde{C}}\to\Omega_{\widetilde{C}}(\widetilde{\Sigma})\to\mathcal{O}_{\widetilde{\Sigma}}\to 0 \tag{4.1.3}$$

induced by a choice of non-zero section $t\in\Gamma(\widetilde{C},\mathcal{O}_{\widetilde{C}}(\widetilde{\Sigma}))$. As $\Omega_{\widetilde{C}}(\widetilde{\Sigma})|_{\widetilde{U}}=\Omega_{\widetilde{U}}$, we can interpret sections of $\Omega_{\widetilde{C}}(\widetilde{\Sigma})$ as rational sections of $\Omega_{\widetilde{C}}$ with at worst simple poles along $\widetilde{\Sigma}$. Evaluating (4.1.3) on an open $\widetilde{V}\subset\widetilde{C}$ yields

$$\begin{array}{ccccc}
0 & \longrightarrow & \Gamma(\widetilde{V},\Omega_{\widetilde{C}}) & \longrightarrow & \Gamma(\widetilde{V},\Omega_{\widetilde{C}}(\widetilde{\Sigma})) \longrightarrow \bigoplus_{y\in\widetilde{V}\cap\widetilde{\Sigma}}\kappa(y) \\
 & & & & s \longmapsto (\mathrm{res}_y(s))
\end{array}$$

where the last map takes a rational section $s\in\Gamma(\widetilde{V}\cap\widetilde{U},\Omega_{\widetilde{C}})$ to the tuple whose coordinate at $y\in\widetilde{V}\cap\widetilde{\Sigma}$ is the *residue* $\mathrm{res}_y(s)$ *of* s *at* y.

Definition 4.1.14. We define the subsheaf $\omega_C\subset\pi_*\Omega_{\widetilde{C}}(\widetilde{\Sigma})$ by declaring that sections along $V\subset C$ consist of rational sections s of $\Omega_{\widetilde{C}}$ along $\pi^{-1}(V)$ with at worst simple poles along $\widetilde{\Sigma}$ such that for all $z_i\in V\cap\Sigma$, $\mathrm{res}_{p_i}(s)+\mathrm{res}_{q_i}(s)=0$.

The definition implies that ω_C sits in the following two exact sequences:

$$\begin{array}{ccccccc}
0 & \longrightarrow & \omega_C & \longrightarrow & \pi_*\Omega_{\widetilde{C}}(\widetilde{\Sigma}) & \longrightarrow & \bigoplus_{z_i\in\Sigma}\Bbbk \longrightarrow 0 \\
 & & & & s & \mapsto & (\mathrm{res}_{p_i}(s)-\mathrm{res}_{q_i}(s))
\end{array} \tag{4.1.4}$$

$$0 \longrightarrow \pi_*\Omega_{\widetilde{C}} \longrightarrow \omega_C \longrightarrow \bigoplus_{z_i \in \Sigma} \Bbbk \longrightarrow 0 \qquad (4.1.5)$$
$$s \mapsto (\mathrm{res}_{p_i}(s))$$

Example 4.1.15 (Local calculation)**.** Let $C = \operatorname{Spec} \Bbbk[x,y]/(xy)$. Then $\widetilde{C} = \mathbb{A}^1 \sqcup \mathbb{A}^1$ with coordinates x and y respectively. The singular locus of C is $\Sigma = \{0\}$ with preimage $\widetilde{\Sigma} = \{p, q\}$ consisting of the two origins. Then $\Gamma(\widetilde{C}, \Omega_{\widetilde{C}}) = \Gamma(\mathbb{A}^1, \omega_{\mathbb{A}^1}) \times \Gamma(\mathbb{A}^1, \omega_{\mathbb{A}^1})$ and $(\frac{dx}{x}, -\frac{dy}{y})$ is a rational section with opposite residues at p and q. In fact, any section of $\Gamma(C, \omega_C)$ is of the form

$$\left(f(x)\frac{dx}{x}, g(y)\frac{-dy}{y}\right) = (f(0) + f(x) + g(y)) \cdot (\frac{dx}{x}, \frac{-dy}{y})$$

for polynomials $f(x)$ and $g(y)$ such that $f(0) = g(0)$, which is precisely the condition for $(f, g) \in \Gamma(\widetilde{C}, \mathcal{O}_{\widetilde{C}})$ to descend to a global function on C. In other words, $\omega_C \cong \mathcal{O}_C$ with generator $(\frac{dx}{x}, -\frac{dy}{y})$.

Example 4.1.16. Let C be the nodal projective plane cubic and $\mathbb{P}^1 \to C$ be the normalization with coordinates $[x : y]$ such that 0 and ∞ are the fibers of the node. Observe that the rational differential $\eta := \frac{dx}{x} = -\frac{dy}{y}$ on $\mathbb{P}^1$ satisfies $\mathrm{res}_0 \eta + \mathrm{res}_\infty \eta = 0$. It is easy to see that any local section of ω_C is a multiple of η or in other words that $\eta \colon \mathcal{O}_C \to \omega_C$ is an isomorphism.

Exercise 4.1.17. Let C be a nodal curve over $\Bbbk$.

1. Show that if $\pi \colon C' \to C$ is an étale morphism, then $\pi^*\omega_C \cong \omega_{C'}$. (Hint: Use the fact that normalization commutes with étale base change.)
2. Conclude that ω_C is a line bundle.

Despite the explicit description of ω_C in Definition 4.1.14, it is difficult to directly establish Serre Duality for C. However, this can be shown using the normalization $\widetilde{C} \to C$ and Serre Duality for Smooth Curves (Theorem 4.1.2) which itself is a difficult result.

Exercise 4.1.18. If C is a nodal proper curve over $\Bbbk$, show that ω_C is a dualizing sheaf.

Combining the two exercises above, we have:

Proposition 4.1.19. *If C is a nodal proper curve over $\Bbbk$, then ω_C is a dualizing line bundle.* □

Exercise 4.1.20. If C is a nodal curve and $T \subset C$ is a subcurve with complement $T^c := \overline{C \setminus T}$, show that

$$\omega_C|_T = \omega_T(T \cap T^c)$$

4.1.5 Local structure of nodes

Recall that if $\mathcal{C} \to S$ is a smooth family of curves, then any point $p \in \mathcal{C}$ over $s \in S$ is étale-locally isomorphic to relative affine space of dimension 1. More precisely,

there are étale neighborhoods

$$\begin{array}{ccccc} (\mathcal{C},p) & \xleftarrow{\text{ét}} & (\mathcal{U},u) & \xrightarrow{\text{ét}} & (S'\times_{\mathbb{Z}}\mathbb{A}^1_{\mathbb{Z}},(s',0)) \\ \downarrow & & \downarrow & \swarrow & \\ (S,s) & \xleftarrow{\text{ét}} & (S',s') & & \end{array}$$

where $S'\times_{\mathbb{Z}}\mathbb{A}^1_{\mathbb{Z}}=\mathbb{A}^1_{S'}\to S'$ is the base change of $\mathbb{A}^1_{\mathbb{Z}}\to\operatorname{Spec}\mathbb{Z}$.

We apply the powerful result of Artin Approximation to obtain a similar structure result around a node.

Theorem 4.1.21 (Local Structure of Nodes)**.** *Let $\pi\colon\mathcal{C}\to S$ be a flat and finitely presented morphism such that every geometric fiber is a curve (i.e. pure one-dimensional). Let $p\in\mathcal{C}$ be a node in a fiber $\mathcal{C}_s$. There is a commutative diagram*

$$\begin{array}{ccccc} (\mathcal{C},p) & \xleftarrow{\text{ét}} & (\mathcal{U},u) & \xrightarrow{\text{ét}} & (\operatorname{Spec}A[x,y]/(xy-f),(s',0)) \\ \downarrow & & \downarrow & \swarrow & \\ (S,s) & \xleftarrow{\text{ét}} & (\operatorname{Spec}A,s') & & \end{array} \tag{4.1.6}$$

where each horizontal arrow is a residually-trivial pointed étale morphism and $f\in A$ is a function vanishing at s'.

Remark 4.1.22. In other words, any such morphism is étale-locally on the source and target the base change of the morphism

$$\operatorname{Spec}\mathbb{Z}[x,y,t]/(xy-t)\to\operatorname{Spec}\mathbb{Z}[t]$$

by the map $\operatorname{Spec}A\to\operatorname{Spec}\mathbb{Z}[t]$ induced by $t\mapsto f$.

Sketch. See [SP, Tag 0CBY] for more details.

Step 1: Reduce to S of finite type over $\mathbb{Z}$. Using Absolute Noetherian Approximation, we may assume that S is of finite type over $\mathbb{Z}$.

Step 2: Reduce to the case where $\widehat{\mathcal{O}}_{\mathcal{C}_s,p}\cong\kappa(s)[[x,y]]/(xy)$. By Lemma 4.1.10, there exists a finite separable field extension $\kappa(s)\to\Bbbk'$ and a point $p'\in\mathcal{C}_s\times_{\kappa(s)}\Bbbk'$ such that the completions of its local ring is isomorphic to $\Bbbk'[[x,y]]/(xy)$. Let $(S',s')\to(S,s)$ be an étale morphism such that $\kappa(s')\cong\Bbbk'$ are isomorphic over $\kappa(s)$. After replacing S with S', we may assume that $\widehat{\mathcal{O}}_{\mathcal{C}_s,p}\cong\kappa(s)[[x,y]]/(xy)$.

Step 3: Show that $\widehat{\mathcal{O}}_{\mathcal{C},p}\cong\widehat{\mathcal{O}}_{S,s}[[x,y]]/(xy-f)$ where $f\in\widehat{\mathfrak{m}}_s$. We will use Schlessinger's theorem in formal deformation theory. If $\widehat{\mathcal{O}}_{S,s}$ is characteristic 0, set $\Lambda=\kappa(s)$ with maximal ideal $\mathfrak{m}_\Lambda=0$; otherwise let $(\Lambda,\mathfrak{m}_\Lambda)$ be a complete discrete valuation ring of characteristic zero with residue field $\Lambda/\mathfrak{m}_\Lambda\cong\kappa(s)$ (this is unique by Cohen's structure theorem). Schlessinger's theorem applied to the local deformation functor of the node $p\in\mathcal{C}_s$ implies that if $\mathcal{D}\to\operatorname{Spec}B$ is a flat family over a complete local ring $(B,\mathfrak{m}_B)$ such that the central fiber $\mathcal{D}\times_B B/\mathfrak{m}_B$ is isomorphic to $\mathcal{C}_s$, then there exists dotted arrows completing a cartesian diagram

$$\begin{array}{ccccc} \mathcal{C}_s & \hookrightarrow & \mathcal{D} & \dashrightarrow & \operatorname{Spec}\Lambda[[t,x,y]]/(xy-t) \\ \downarrow & & \downarrow & & \downarrow \\ \operatorname{Spec}\kappa(s) & \hookrightarrow & \operatorname{Spec}B & \dashrightarrow & \operatorname{Spec}\Lambda[[t]] \end{array}$$

Setting $\widehat{\mathcal{C}} := \mathcal{C} \times_S \operatorname{Spec} \widehat{\mathcal{O}}_{S,s}$ and $\widehat{p} = (p, s) \in \widehat{\mathcal{C}}$, we apply the above result to $\operatorname{Spec} \widehat{\mathcal{O}}_{\widehat{\mathcal{C}},\widehat{p}} \to \operatorname{Spec} \widehat{\mathcal{O}}_{S,s}$ yields a cartesian diagram

$$\begin{array}{ccc} \operatorname{Spec} \widehat{\mathcal{O}}_{\widehat{\mathcal{C}},\widehat{p}} & \longrightarrow & \operatorname{Spec} \Lambda[[t,x,y]]/(xy-t) \\ \downarrow & & \downarrow \\ \operatorname{Spec} \widehat{\mathcal{O}}_{S,s} & \longrightarrow & \operatorname{Spec} \Lambda[[t]] \end{array} \tag{4.1.7}$$

The completion of the local ring of $\mathcal{C}$ at p is identified with the completion of $\mathcal{C} \times_S \operatorname{Spec} \widehat{\mathcal{O}}_{S,s}$ at (p, s), and by the diagram above is further identified with $\widehat{\mathcal{O}}_{S,s}[[x, y]]/(xy - f)$.

Step 4: Apply Artin Approximation. We will apply Artin Approximation to the following functor:

$$F\colon (\mathrm{Sch}\,/S) \to \mathrm{Sets}$$

$$(T \to S) \mapsto \left\{ \text{commutative diagrams } \begin{array}{ccccc} \mathcal{C} & \longleftarrow & \mathcal{C}_T & \longrightarrow & \operatorname{Spec} \mathbb{Z}[t,x,y]/(xy-t) \\ \downarrow & & \downarrow & & \downarrow \\ S & \longleftarrow & T & \longrightarrow & \operatorname{Spec} \mathbb{Z}[t] \end{array} \right\}$$

The diagram (4.1.7), after post-composing with $\operatorname{Spec} \kappa(s)[[t] \to \operatorname{Spec} \mathbb{Z}[t]$ and $\operatorname{Spec} \kappa(s)[[t, x, y]](xy-t) \to \operatorname{Spec} \mathbb{Z}[t, x, y]/(xy-t)$ yields an element $\widehat{\xi} \in F(\operatorname{Spec} \widehat{\mathcal{O}}_{S,s})$. (Missing details here: we only obtain a map $\operatorname{Spec} \widehat{\mathcal{O}}_{\widehat{\mathcal{C}},\widehat{p}} \to \operatorname{Spec} \Lambda[[t, x, y]]/(xy - t))$ but we need a map $\widehat{\mathcal{C}} \to \operatorname{Spec} \mathbb{Z}[t, x, y]/(xy - t)$). Applying Artin Approximation (Corollary A.4.17) with $N = 2$ produces a diagram as in (4.1.6) where $\operatorname{Spec} A \to S$ and $\mathcal{U} \to \mathcal{C}$ are étale. To check that $\mathcal{U} \to \operatorname{Spec} A[x, y]/(xy - f)$ is étale at u, we will argue that $\mathcal{U} \to \operatorname{Spec} A[x, y]/(xy - f)$ induces an isomorphism on completions at u. Step 3 implies that $\widehat{\mathcal{O}}_{\mathcal{C},p} \cong \widehat{\mathcal{O}}_{\mathcal{U},u}$ is isomorphic to the completion of $\operatorname{Spec} A[x, y]/(xy - f)$ at $(s', 0)$. Thus, $\mathcal{U} \to \operatorname{Spec} A[x, y]/(xy - f)$ induces an endomorphism $R \to R$ of a complete local noetherian ring $R \cong \widehat{\mathcal{O}}_{\mathcal{U},u}$ which is surjective modulo $\mathfrak{m}_R^2$. Lemma A.4.18 implies that $R \to R$ is surjective, and applying the general fact that surjective endomorphisms of noetherian rings are isomorphism, we conclude that $R \to R$ is an isomorphism. □

As a result, for a morphism $\mathcal{C} \to S$ as in Theorem 4.1.21, the locus $\mathcal{C}^{\leq \mathrm{nod}} \subset \mathcal{C}$ of points which are either smooth or nodal is open. And if we add a properness condition on $\mathcal{C} \to S$, then $\pi(\mathcal{C} \setminus \mathcal{C}^{\leq \mathrm{nod}}) \subset S$ is closed and therefore the locus of points $s \in S$ such that $\mathcal{C}_s$ is a nodal curve is the *open* subscheme $S \setminus \pi(\mathcal{C} \setminus \mathcal{C}^{\leq \mathrm{nod}})$.

Corollary 4.1.23. *If $\mathcal{C} \to S$ is a flat, proper and finitely presented morphism of schemes such that every geometric fiber is a curve, then the locus of points $s \in S$ such that $\mathcal{C}_s$ is nodal is open.* □

We will apply the above corollary later to conclude that the stack parameterizing families of nodal curves is an open substack of the stack of all curves.

The following exercise establishes a similar structure statement for a family of possibly non-reduced curves over a DVR.

Exercise 4.1.24. Let R be a DVR with uniforming parameter t. Let $\mathcal{C} \to \Delta = \operatorname{Spec} R$ be a flat, proper and finitely presented morphisms such that each geometric fiber is a curve. Assume that $\mathcal{C}$ is *regular*. Let $p \in \mathcal{C}_0$.

(1) If p is a smooth point in the reduced fiber $(\mathcal{C}_0)_{\mathrm{red}}$. Show that after possibly an extension of DVRS, there exists an étale neighborhood of p (defined over R)

$$\operatorname{Spec} R[x,y]/(x^a - t) \to \mathcal{C}$$

(2) If p is a node in the reduced fiber $(\mathcal{C}_0)_{\mathrm{red}}$. Show that there exists an étale neighborhood of p (defined over R)

$$\operatorname{Spec} R[x,y]/(x^a y^b - t) \to \mathcal{C}.$$

4.2 Stable curves

Stable curves were introduced in joint work by Mayer and Mumford [Mum64].

4.2.1 Definition and equivalences

An *n-pointed curve* is a curve C over a field $\Bbbk$ together with an ordered collection of k-points $p_1, \ldots, p_n \in C$ which we call the *marked points*. A point $q \in C$ of an n-pointed curve is called *special* if q is a node or a marked point.

Definition 4.2.1 (Stable curves)**.** An n-pointed curve $(C, p_1, \ldots, p_n)$ over $\Bbbk$ is *stable* if C is a connected, nodal and projective curve, and $p_1, \ldots, p_n \in C$ are distinct smooth points such that

(1) every smooth rational subcurve $\mathbb{P}^1 \subset C$ contains at least 3 special points, and

(2) C is not of genus 1 without marked points.

We define $(C, p_1, \ldots, p_n)$ to be *semistable* by replacing (1) with the condition that every smooth rational subcurve $\mathbb{P}^1 \subset C$ contains at least 2 (rather than 3) special points, we obtain the notion of a *semistable* curve. We define $(C, p_1, \ldots, p_n)$ to be *prestable* by dropping both condition (1) and (2), i.e. C is a connected, nodal and projective curve and the points p_i are distinct smooth points of C. Note that in the unpointed case for connected projective curves, there is no distinction between a nodal curve and prestable curve.

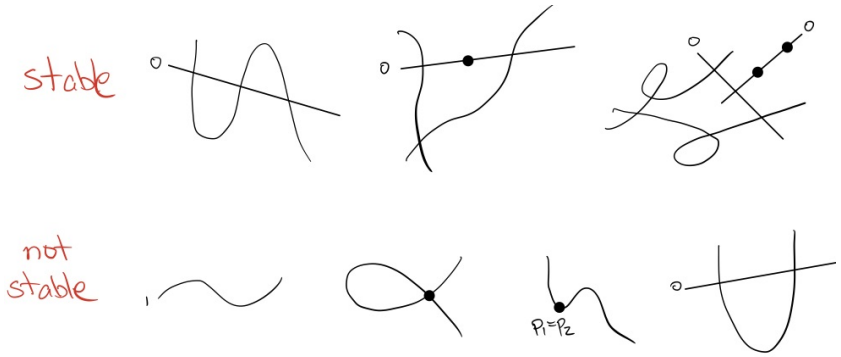


Figure 4.3: The curves in the top row are stable while those in the second row are not.

Remark 4.2.2. Note that there are no n-pointed stable curve of genus g if $(g,n) \in \{(0,0),(0,1),(0,2),(1,0)\}$ or equivalently $2g-2+n \leq 0$. We will often impose the condition that $2g-2+n>0$ in order to exclude these special cases.

An automorphism of a stable curve $(C,p_1,\ldots,p_n)$ is an automorphism $\alpha\colon C \xrightarrow{\sim} C$ such that $\alpha(p_i)=p_i$. We denote by $\mathrm{Aut}(C,p_1,\ldots,p_n)$ the (abstract) group of automorphisms.

Proposition 4.2.3. *Let $(C,p_1,\ldots,p_n)$ be an n-pointed prestable curve. The following are equivalent:*

(1) $(C,p_1,\ldots,p_n)$ is stable;

(2) $\mathrm{Aut}(C,p_1,\ldots,p_n)$ is finite; and

(3) $\omega_C(p_1+\cdots+p_n)$ is ample.

Proof. The equivalence (1) $\iff$ (2) follows Exercise 4.2.4 and the fact that the only smooth, connected and projective n-pointed curves $(C,\{p_i\})$ with positive dimensional automorphism groups are when either $C=\mathbb{P}^1$ with $n\leq 2$ or C is a genus 1 curve with $n=0$.

To see the equivalence with (3), we will use the fact that for a subcurve $T\subset C$, we have $\omega_C|_T = \omega_T(T\cap T^c)$ (Exercise 4.1.20). If $\pi\colon \widetilde{C}\to C$ is the normalization, then: $\omega_C(p_1+\cdots+p_n)$ is ample $\iff$ $\pi^*\big(\omega_C(p_1+\cdots+p_n)\big)$ is ample $\iff$ for each irreducible component $T\subset C$, $\omega_C(p_1+\cdots+p_n)|_T = \omega_T(\sum_{p_i\in T} p_i + (T\cap T^c))$ is ample. This latter condition holds precisely if each $\mathbb{P}^1\subset\widetilde{C}$ contains at least three points that lie over nodes or marked points. □

Exercise 4.2.4. Let $(C,p_1,\ldots,p_n)$ be an n-pointed nodal projective curve such that the points p_i are distinct and smooth. Let $\pi\colon\widetilde{C}\to C$ be the normalization of C, $\widetilde{p}_i\in\widetilde{C}$ be the unique preimage of p_i and $\widetilde{q}_1,\ldots,\widetilde{q}_m\in\widetilde{C}$ be an ordering of the preimages of nodes.

(1) Show that $(C,\{p_i\})$ is stable if and only if every connected component of $(\widetilde{C},\{\widetilde{p}_i\},\{\widetilde{q}_i\})$ is stable.

(2) Show that the automorphism group scheme $\underline{\mathrm{Aut}}(C,\{p_i\})$ is an algebraic group.

(3) Show that $\underline{\mathrm{Aut}}(C, \{p_i\})$ is naturally a closed subgroup of $\underline{\mathrm{Aut}}(\widetilde{C}, \{\widetilde{p}_i\}, \{\widetilde{q}_i\})$ with the same connected component of the identity (i.e. $\underline{\mathrm{Aut}}(C, \{p_i\})^\circ = \underline{\mathrm{Aut}}(\widetilde{C}, \{\widetilde{p}_i\}, \{\widetilde{q}_i\})^\circ$).

(4) Provide an example where $\underline{\mathrm{Aut}}(C, \{p_i\}) \neq \underline{\mathrm{Aut}}(\widetilde{C}, \{\widetilde{p}_i\}, \{\widetilde{q}_i\})$.

4.2.2 Positivity of ω_C

Exercise 4.2.5. If C is a stable curve, show that $\omega_C^{\otimes k}$ is very ample for $k \geq 3$ by showing that its sections separates points and tangent vectors. In other words, for distinct points $x, y \in C(\Bbbk)$, you must show that the maps

$$H^0(C, \omega_C^{\otimes k}) \to \left(\omega_C^{\otimes k} \otimes \kappa(x)\right) \oplus \left(\omega_C^{\otimes k} \otimes \kappa(y)\right) \qquad H^0(C, \omega_C^{\otimes k}) \to \omega_C^{\otimes k} \otimes \mathcal{O}_{C,x}/\mathfrak{m}_x^2$$

are surjective.

Hint: It suffices to prove that $H^1(C, I_x I_y \cdot \omega_C^{\otimes k}) = 0$ where I_x and I_y are the ideal sheaves of (possibly equal) points $x, y \in C(k)$. By applying Serre Duality, it suffices to show that $\mathrm{Hom}(I_x I_y, \omega_C^{\otimes(1-k)})$. Establish this by a case analysis on whether x, y are smooth or nodal.

Exercise 4.2.6. Prove that for an n-pointed stable curve $(C, p_1, \ldots, p_n)$ that $\left(\omega_C(\sum_i p_i)\right)^{\otimes k}$ is very ample for $k \geq 3$.

Exercise 4.2.7.

(1) If C is the nodal union $C_1 \cup C_2$ of genus i and $g - i$ curves along a single node $p = C_1 \cap C_2 \in C$, show that ω_C has a basepoint at p.

(2) If C the nodal union $C_1 \cup E \cup C_2$ of genus i, 1 and $g - i - 1$ curves along nodes at $C_1 \cap C_2 = p_1$ and $C_1 \cap C_2 = p_2$, show that $\omega_C^{\otimes 2}$ is not ample.

4.2.3 Families of stable curves

Definition 4.2.8.

(1) A *family of n-pointed nodal curves* is a flat, proper and finitely presented morphism $\mathcal{C} \to S$ of schemes with n sections $\sigma_1, \ldots, \sigma_n \colon S \to \mathcal{C}$ such that every geometric fiber $\mathcal{C}_s$ is a (reduced) connected nodal curve.

(2) A *family of n-pointed stable curves* (resp. *semistable curves, prestable curves*) is a family $\mathcal{C} \to S$ of n-pointed nodal curves such that every geometric fiber $(\mathcal{C}_s, \sigma_1(s), \ldots, \sigma_n(s))$ is stable (resp. semistable, prestable).

If $\mathcal{C} \to S$ is a family of prestable curves, then $\mathcal{C} \to S$ is locally a complete intersection morphism and thus there is a relative dualizing line bundle $\omega_{\mathcal{C}/S}$ that is compatible with base change $T \to S$ and in particular restricts to the dualizing line bundle $\omega_{\mathcal{C}_s}$ on any fiber of $\mathcal{C} \to S$; see [Har66b] or [Liu02, §6.4]. Note also that since the geometric fibers are stable curves, the image of each σ_i is a divisor contained in the smooth locus and we can form the line bundle $\omega_{\mathcal{C}/S}(\sigma_1 + \cdots \sigma_n)$.

We have the following generalization of Proposition 4.1.8 which is proven in the same way but using the very ampleness of third tensor power of $\omega_C(p_1 + \cdots + p_n)$ in Exercise 4.2.6.

Proposition 4.2.9 (Properties of Families of Stable Curves)**.** *Let $(\mathcal{C} \to S, \{\sigma_i\})$ be a family of n-pointed stable curves of genus g, and set $L := \omega_{\mathcal{C}/S}(\sum_i \sigma_i)$. If*

$k \geq 3$, then $L^{\otimes k}$ is relatively very ample and $\pi_* L^{\otimes k}$ is a vector bundle of rank $(2k-1)(g-1)+kn$. □

Proposition 4.2.10 (Openness of Stability)**.** *Let $(\mathcal{C} \to S, \{\sigma_i\})$ be a family of n-pointed nodal curves. The locus of points $s \in S$ such that $(\mathcal{C}_s, \{\sigma_i(s)\})$ is stable is open.*

Proof. The locus in S where $\sigma_1(s), \ldots, \sigma_n(s)$ are distinct and smooth is open. We may thus assume that $(\mathcal{C} \to S, \{\sigma_i\})$ is a family of prestable n-pointed curves.

Argument 1: since $\mathrm{Aut}(\mathcal{C}/S, \sigma_1, \ldots, \sigma_n) \to S$ is a group scheme of finite type, upper semicontinuity implies that the locus of points $s \in S$ such that $\mathrm{Aut}(\mathcal{C}_s, \sigma_1(s), \ldots, \sigma_n(s))$ is finite is open. By the equivalence Proposition 4.2.3(2), this open subset is identified with the stable locus.

Argument 2: there is a relative dualizing sheaf $\omega_{\mathcal{C}/S}$ and the locus of points $s \in S$ such that $\omega_{\mathcal{C}/S}|_{\mathcal{C}_s} \cong \omega_{\mathcal{C}_s}$ is ample is open. By the equivalence Proposition 4.2.3(3), this is open subset is identified with the stable locus. □

4.2.4 Automorphisms, deformations and obstructions

Automorphisms, deformations and obstructions of a stable curve C are governed by $\mathrm{Ext}^0(\Omega_C, \mathcal{O}_C)$, $\mathrm{Ext}^1(\Omega_C, \mathcal{O}_C)$, and $\mathrm{Ext}^2(\Omega_C, \mathcal{O}_C)$, respectively.

Proposition 4.2.11. *Let $(C, p_1, \ldots, p_n)$ be an n-pointed stable curve of genus g over $\Bbbk$. Then*

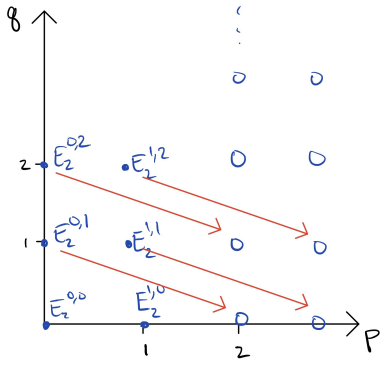
$$\dim_{\Bbbk} \mathrm{Ext}^i(\Omega_C(\sum_i p_i), \mathcal{O}_C) = \begin{cases} 0 & \text{if } i = 0 \\ 3g-3+n & \text{if } i = 1 \\ 0 & \text{if } i = 2 \end{cases}$$

Proof. We may assume $\Bbbk = \overline{\Bbbk}$ and for simplicity we handle the case that there are no marked points, i.e. $n = 0$. Let $\pi\colon \widetilde{C} \to C$ be the normalization, $\Sigma \subset C$ be the nodes of C and $\widetilde{\Sigma} = \pi^{-1}(\Sigma)$.

In the proof, we will use the local-to-global spectral sequence

$$E_2^{p,q} = H^p(C, \mathscr{E}xt^q_{\mathcal{O}_C}(\Omega_C, \mathcal{O}_C)) \Rightarrow \mathrm{Ext}^{p+q}_{\mathcal{O}_C}(\Omega_C, \mathcal{O}_C).$$

Since $\dim C = 1$, we have that $E_2^{p,q} = 0$ if $p \geq 2$.



To compute Ext^0, we claim that there is an isomorphism $\mathrm{Hom}_{\mathcal{O}_{\widetilde{C}}}(\Omega_{\widetilde{C}}(\widetilde{\Sigma}), \mathcal{O}_{\widetilde{C}}) \to \mathrm{Hom}_{\mathcal{O}_C}(\Omega_C, \mathcal{O}_C)$ (details omitted), or in other words that regular vector fields on C correspond to regular vector fields on $\widetilde{C}$ vanishing at the preimages of nodes. Since the pointed normalization $(\widetilde{C}, \widetilde{\Sigma})$ is smooth and each connected component is stable (Exercise 4.2.4), the degree of the restriction of $T_{\widetilde{C}}(-\widetilde{\Sigma})$ to each connected component of $\widetilde{C}$ is strictly negative. Thus, $\mathrm{Hom}_{\mathcal{O}_{\widetilde{C}}}(\Omega_{\widetilde{C}}(\widetilde{\Sigma}), \mathcal{O}_{\widetilde{C}}) = H^0(\widetilde{C}, T_{\widetilde{C}}(-\widetilde{\Sigma})) = 0$.

To compute that $\mathrm{Ext}^2 = 0$, we will show that $E_2^{1,1} = E_2^{0,2} = 0$. For $q > 0$, $\mathscr{E}xt^q_{\mathcal{O}_C}(\Omega_C, \mathcal{O}_C)_z = \mathrm{Ext}^1_{\mathcal{O}_{C,z}}(\Omega_{C,z}, \mathcal{O}_{C,z})$ vanishes at smooth points $z \in C$ as $\Omega_{C,z}$ is locally free. Thus $\mathscr{E}xt^1_{\mathcal{O}_C}(\Omega_C, \mathcal{O}_C)$ is a zero-dimensional sheaf supported only at the nodes of C and $E_2^{1,1} = H^1(C, \mathscr{E}xt^1_{\mathcal{O}_C}(\Omega_C, \mathcal{O}_C)) = 0$. Similarly, $\mathscr{E}xt^2_{\mathcal{O}_C}(\Omega_C, \mathcal{O}_C)$ is supported at the nodes, and for a node $z \in C$, we have an identification $\mathscr{E}xt^2_{\mathcal{O}_C}(\Omega_C, \mathcal{O}_C)_z = \mathrm{Ext}^2_{\widehat{\mathcal{O}}_{C,z}}(\Omega_{\widehat{\mathcal{O}}_{C,z}/\Bbbk}, \widehat{\mathcal{O}}_{C,z})$. At a node $z \in C$, we can write $\widehat{\mathcal{O}}_{C,z} = \Bbbk[[x,y]]/(xy)$ and we have a locally free resolution

$$0 \to \widehat{\mathcal{O}}_{C,z} \xrightarrow{\binom{y}{x}} \widehat{\mathcal{O}}_{C,z}^{\oplus 2} \xrightarrow{(dx,dy)} \Omega_{\widehat{\mathcal{O}}_{C,z}/\Bbbk} \to 0 \tag{4.2.1}$$

from which we conclude that $\mathscr{E}xt^2_{\mathcal{O}_C}(\Omega_C, \mathcal{O}_C)_z = 0$ and thus $E_2^{0,2} = 0$. (Alternatively, we can find a Zariski-open neighborhood $U \subset C$ of z and an embedding $U \hookrightarrow \mathbb{A}^n$ defined by an ideal sheaf $\mathcal{I}$, and use the locally free resolution $0 \to \mathcal{I}/\mathcal{I}^2 \to \Omega_{\mathbb{A}^n} \otimes \mathcal{O}_C \to \Omega_C \to 0$.) As $E_2^{0,2} = E_2^{1,1} = E_2^{2,0} = 0$, we have $\mathrm{Ext}^2_{\mathcal{O}_C}(\Omega_C, \mathcal{O}_C) = 0$.

To compute Ext^1, we analyze the low degree exact sequence associated to the the above spectral sequence:

$$0 \to E_2^{1,0} \to \mathrm{Ext}^1_{\mathcal{O}_C}(\Omega_C, \mathcal{O}_C) \to E_2^{0,1} \to E_2^{2,0} = 0.$$

Since $\mathscr{E}xt(\omega_C, \mathcal{O}_C)$ is supported only at the nodes,

$$\begin{aligned} E_2^{0,1} &= H^0(C, \mathscr{E}xt^1_{\mathcal{O}_C}(\omega_C, \mathcal{O}_C)) \\ &= \prod_{z \in \Sigma} \mathrm{Ext}^1_{\mathcal{O}_{C,z}}(\Omega_{C,z}, \mathcal{O}_{C,z}) \\ &= \prod_{z \in \Sigma} \mathrm{Ext}^1_{\widehat{\mathcal{O}}_{C,z}}(\Omega_{\widehat{\mathcal{O}}_{C,z}}, \widehat{\mathcal{O}}_{C,z}) \qquad \text{using } \widehat{\Omega}_{C,z} = \Omega_{\widehat{\mathcal{O}}_{C,z}} \end{aligned} \tag{4.2.2}$$

By Infinitesimal Deformation Theory, the group $\mathrm{Ext}^1_{\mathcal{O}_C}(\Omega_C, \mathcal{O}_C)$ classifies first order deformations of C and likewise $\mathrm{Ext}^1_{\widehat{\mathcal{O}}_{C,z}}(\Omega_{\widehat{\mathcal{O}}_{C,z}}, \widehat{\mathcal{O}}_{C,z})$ classifies first order deformations of the singularity $\widehat{\mathcal{O}}_{C,z}$. The natural map

$$\mathrm{Ext}^1_{\mathcal{O}_C}(\Omega_C, \mathcal{O}_C) \to \mathrm{Ext}^1_{\widehat{\mathcal{O}}_{C,z}}(\Omega_{\widehat{\mathcal{O}}_{C,z}}, \widehat{\mathcal{O}}_{C,z})$$

is given by restricting first order deformations

$$\left[\begin{array}{ccc} C & \hookrightarrow & \mathcal{C} \\ \downarrow & \square & \downarrow \\ \operatorname{Spec} \Bbbk & \hookrightarrow & \operatorname{Spec} \Bbbk[\epsilon]. \end{array}\right] \mapsto \left[\begin{array}{ccc} \operatorname{Spec} \widehat{\mathcal{O}}_{C,z} & \hookrightarrow & \operatorname{Spec} \widehat{\mathcal{O}}_{\mathcal{C},z} \\ \downarrow & \square & \downarrow \\ \operatorname{Spec} \Bbbk & \hookrightarrow & \operatorname{Spec} \Bbbk[\epsilon]. \end{array}\right]$$

and corresponds to the map $\operatorname{Ext}^1_{\mathcal{O}_C}(\Omega_C, \mathcal{O}_C) \to E_2^{0,1}$ under the identification in (4.2.2). The kernel of this map is identified with first order deformations of C that preserve all the nodes, which in turn is identified with first order deformations of the pointed normalization $(\widetilde{C}, \widetilde{\Sigma})$, which are classified by $\operatorname{Ext}^1_{\mathcal{O}_{\widetilde{C}}}(\Omega_{\widetilde{C}}(\widetilde{\Sigma}), \mathcal{O}_{\widetilde{C}})$. This yields an identification

$$E_2^{1,0} = H^1(C, \mathscr{H}om(\Omega_C), \mathcal{O}_C) = \operatorname{Ext}^1_{\mathcal{O}_{\widetilde{C}}}(\Omega_{\widetilde{C}}(\widetilde{\Sigma}), \mathcal{O}_{\widetilde{C}}). \tag{4.2.3}$$

We are finally prepared to calculate $\dim_{\Bbbk} \operatorname{Ext}^1_{\mathcal{O}_C}(\Omega_C, \mathcal{O}_C) = \dim_k E_2^{1,0} + \dim_k E_2^{0,1}$. First, we may use the locally free resolution in (4.2.1) to compute that $\dim_{\Bbbk} \operatorname{Ext}^1_{\widehat{\mathcal{O}}_{C,z}}(\Omega_{\widehat{\mathcal{O}}_{C,z}}, \widehat{\mathcal{O}}_{C,z}) = 1$ and the identification (4.2.2) implies that

$$\dim_k E_2^{0,1} = (\#\text{ nodes}). \tag{4.2.4}$$

On the other hand, writing $\widetilde{C} = \bigsqcup_i \widetilde{C}_i$ and $\widetilde{\Sigma}_i = \widetilde{C}_i \cap \widetilde{\Sigma}$ and using that $\Omega_{\widetilde{C}_i}$ is a line bundle, we can compute using the identification (4.2.3) that

$$\begin{aligned}
\dim_k E_2^{0,1} &= \dim_{\Bbbk} \operatorname{Ext}^1_{\mathcal{O}_{\widetilde{C}}}(\Omega_{\widetilde{C}}(\widetilde{\Sigma}), \mathcal{O}_{\widetilde{C}}) \\
&= \sum_i \dim_{\Bbbk} \operatorname{Ext}^1_{\mathcal{O}_{\widetilde{C}_i}}(\Omega_{\widetilde{C}_i}(\widetilde{\Sigma}_i), \mathcal{O}_{\widetilde{C}}) \\
&= \sum_i h^1(\widetilde{C}_i, T_{\widetilde{C}_i}(-\widetilde{\Sigma}_i)) \\
&= \sum_i h^0(\widetilde{C}_i, \Omega^{\otimes 2}_{\widetilde{C}_i}(\widetilde{\Sigma}_i)) \qquad \text{(Serre Duality)} \\
&= \sum_i \big(\deg \Omega^{\otimes 2}_{\widetilde{C}_i}(\widetilde{\Sigma}_i) + 1 - g(\widetilde{C}_i)\big) \qquad \text{(Riemann–Roch)} \\
&= \sum_i \big(3g(\widetilde{C}_i) - 3 + |\widetilde{\Sigma}_i|\big) \\
&= 3\sum_i g(\widetilde{C}_i) - 3(\#\text{ comp}) + 2(\#\text{ nodes}).
\end{aligned} \tag{4.2.5}$$

Adding (4.2.4) and (4.2.5) together with the Genus Formula (Proposition 4.1.12) that $g = \sum_i g(\widetilde{C}_i) - (\#\text{ comp}) + (\#\text{ nodes}) + 1$, we can wrap up the calculation:

$$\dim_k E_2^{1,0} + \dim_k E_2^{0,1} = 3\sum_i g(\widetilde{C}_i) - 3(\#\text{ comp}) + 3(\#\text{ nodes}) = 3g - 3.$$

□

We have already seen that the $\Bbbk$-points of the automorphism group scheme $\underline{\operatorname{Aut}}(C, p_1, \ldots, p_n)$ is a finite (abstract) group (Proposition 4.2.3). The vanishing of Ext^0 implies that an n-pointed stable curve $(C, p_1, \ldots, p_n)$ has no infinitesimal automorphisms, i.e. that the Lie algebra $T_e\underline{\operatorname{Aut}}(C, p_1, \ldots, p_n)$ is trivial. Since the automorphism group scheme $\underline{\operatorname{Aut}}(C, p_1, \ldots, p_n)$ is of finite type, this implies that $\underline{\operatorname{Aut}}(C, p_1, \ldots, p_n)$ is finite and discrete. Once we know that the algebraicity of the stack $\overline{\mathcal{M}}_{g,n}$, we can conclude by Theorem 2.6.3 that $\overline{\mathcal{M}}_{g,n}$ is Deligne–Mumford.

Meanwhile, the vanishing of Ext^2 implies that there are no obstructions to deforming C. Assuming the algebraicity of $\overline{\mathcal{M}}_{g,n}$, this will allow us to invoke the

Formal Lifting Criterion (Proposition 2.7.1) to establish that $\overline{\mathcal{M}}_{g,n}$ is smooth over $\operatorname{Spec}\mathbb{Z}$.

Since Ext^1 parametrizes isomorphism classes of deformations of $(C, p_1, \ldots, p_n)$, it is identified with the Zariski tangent space of $\overline{\mathcal{M}}_{g,n}$ at the point corresponding to $(C, p_1, \ldots, p_n)$. This will allow us to conclude that $\overline{\mathcal{M}}_{g,n}$ has relative dimension $3g - 3 + n$ over $\operatorname{Spec}\mathbb{Z}$.

4.2.5 Rational tails and bridges

Definition 4.2.12 (Rational tails and bridges)**.** Let $(C, p_1, \ldots, p_n)$ be an n-pointed prestable curve. We say that a smooth rational subcurve $E \cong \mathbb{P}^1 \subset C$ is

- a *rational tail* if $E \cap E^c = 1$ and E contains no marked points;
- a *rational bridge* if either $E \cap E^c = 2$ and E contains no marked points, or $E \cap E^c = 1$ and E contains one marked point.

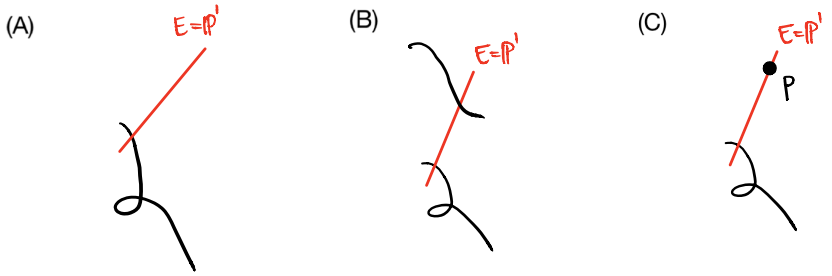


Figure 4.4: (A) features a rational tail while (B) and (C) feature rational bridges.

From the definition of stability (Definition 4.2.1), we see that if $(C, p_1, \ldots, p_n)$ is not stable and $(g, n) \neq (1, 0)$, then C necessarily contains a rational tail or bridge. Note that C can also contain a chain of rational tails or bridges of arbitrary length.

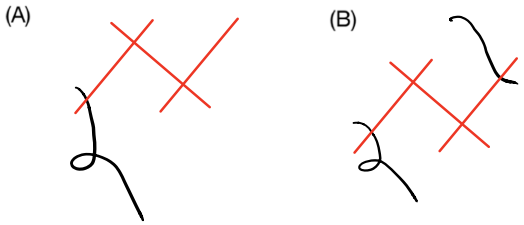


Figure 4.5: Examples of chains of rational tails and bridges

Suppose that $\mathcal{C} \to \Delta = \operatorname{Spec} R$ is a family of nodal curves over a DVR R such that the generic fiber $\mathcal{C}^*$ is smooth. If $E \cong \mathbb{P}^1 \subset \mathcal{C}'_0$ is a smooth rational

subcurve in the central fiber, then $E^2 = -E \cdot E^c$; indeed this follows from $0 = E \cdot \mathcal{C}_0 = E \cdot E + E \cdot E^c$. Thus if E is rational tail (resp. rational bridge without a marked point), then $E^2 = -1$ (resp. $E^2 = -2$).

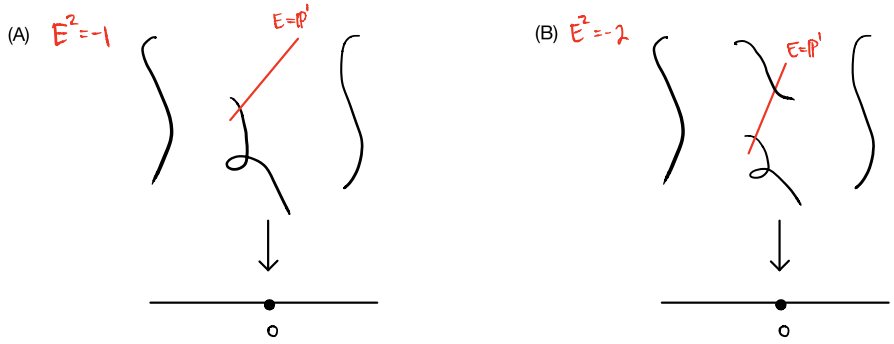


Figure 4.6: In (A) (resp. (B)), the exceptional component E meets the rest of the curve at one point (resp. two points) and $E^2 = -1$ (resp. $E^2 = -2$).

4.2.6 The stable model

Let $(C, p_1, \ldots, p_n)$ be an n-pointed prestable curve. Let C^{st} be the curve obtained by removing all rational bridges and tails E_i, i.e. $C^{\mathrm{st}} = \overline{C \setminus \cup_i E_i}$, and let $\pi\colon C \to C^{\mathrm{st}}$ be the induced morphism. If we set $p_i' = \pi(p_i)$, then $(C^{\mathrm{st}}, p_1', \ldots, p_n')$ is a stable curve, which we call the *stable model of* $(C, \{p_i\})$ and $\pi\colon C \to C^{\mathrm{st}}$ the *stabilization morphism*.

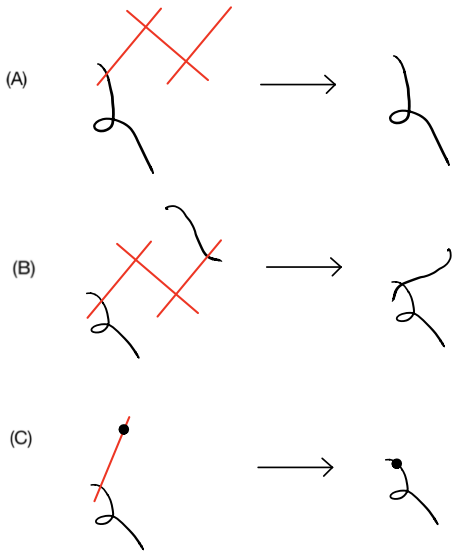


Figure 4.7: Examples of stable models of curves with rational tails and bridges.

Note that if $(C, \{p_i\})$ is semistable, then $\omega_C(\sum_i p_i)$ is trivial on rational bridges and is the pullback of the ample line bundle $\omega_{C^{\mathrm{st}}}(\sum_i p_i')$ under the stabilization morphism $\pi\colon C \to C^{\mathrm{st}}$.

The stabilization construction extends to families of nodal curves.

Proposition 4.2.13. *Let $(\mathcal{C} \to S, \sigma_1, \dots, \sigma_n)$ be a family of n-pointed prestable curves. There is a unique morphism $\pi\colon \mathcal{C} \to \mathcal{C}^{\mathrm{st}}$ over S such that*

(1) $(\mathcal{C}^{\mathrm{st}} \to S, \{\sigma_i'\})$ is an n-pointed family of stable curves where $\sigma_i' = \pi \circ \sigma_i$;

(2) for each $s \in S$, $(\mathcal{C}_s, \{\sigma_i(s)\}) \to (\mathcal{C}_s^{\mathrm{st}}, \{\sigma_i'(s)\})$ is the stable model; and

(3) $\mathcal{O}_{\mathcal{C}^{\mathrm{st}}} = \pi_* \mathcal{O}_{\mathcal{C}}$ and $R^1\pi_* \mathcal{O}_{\mathcal{C}} = 0$ and this remains true after base change by any morphism $S' \to S$ of schemes;

(4) If $\mathcal{C} \to S$ is a family of semistable curves, then $\omega_{C/S}(\sum_i \sigma_i)$ is the pullback of the relatively ample line bundle $\omega_{\mathcal{C}^{\mathrm{st}}/S}(\sum_i \sigma_i')$.

Proof. TO ADD. See [SP, Tag 0E7B] or [ACG11, Prop. 10.6.7]. □

4.3 The stack of all curves

4.3.1 Families of arbitrary curves

In this subsection, we redefine a *curve over a field k* to mean a scheme C of finite type over k of dimension 1 (rather than pure dimension 1). The genus of C is defined as $g(C) = 1 - \chi(C, \mathcal{O}_C)$.

Remark 4.3.1. The reason we allow for non-pure dimensional and non-connected curves is that they may arise as deformations of connected pure one-dimensional curves; without this relaxation, the stack of all curves would fail to be algebraic. For instance, consider a rational normal curve $\mathbb{P}^1 \hookrightarrow \mathbb{P}^3$ embedded via $[x, y] \mapsto [x^3, x^2y, xy^2, ty^3]$ for any $t \neq 0$. As $t \to 0$, these curves degenerate in a flat family to a non-reduced curve C_0 which is supported along a plane nodal cubic and has an embedded point at the node; see [Har77, Ex. 9.8.4]. On the other hand, the curve C_0 deforms to the disjoint union of a plane nodal cubic and a point in $\mathbb{P}^3$.

A *family of curves* over a scheme S is a flat, proper and finitely presented morphism $\mathcal{C} \to S$ of algebraic spaces such that every fiber is a curve.

A *family of n-pointed curves* is a family of curves $\mathcal{C} \to S$ together with n sections $\sigma_1, \dots, \sigma_n\colon S \to \mathcal{C}$ (with no condition on whether they are distinct or land in the relative smooth locus of $\mathcal{C}$ over S).

Remark 4.3.2. While any pure one-dimensional separated algebraic space over a field is in fact a scheme, in the relative setting the total family $\mathcal{C}$ may not be a scheme. There are examples of a family of prestable genus 0 curves [Ful10, Ex. 2.3] and a family of smooth genus 1 curves [Ray70, XIII 3.2] where the total family is not a scheme .

Therefore, if we wish define a *stack* of all curves, then in order to satisfy the decent condition, we better allow for the case that the total family is not a scheme. In the stable case however there is no difference: if $\mathcal{C} \to S$ is a family of curves (with $\mathcal{C}$ an algebraic space) such that every geometric fiber is stable, then $\omega_{\mathcal{C}/S}$ is relatively ample (Proposition 4.2.9) and $\mathcal{C} \to S$ is projective; in particular, $\mathcal{C}$ is a scheme.

Proposition 4.3.3. *If $\mathcal{C} \to S$ is a family of curves over a scheme S, there exists an étale cover $S' \to S$ such that $\mathcal{C}_{S'} \to S'$ is projective.*

Vague sketch. Approach 1: Local to global

For a point $s \in S$, define $S_n = \operatorname{Spec} \mathcal{O}_{S,s}/\mathfrak{m}_s^{n+1}$ and $\widehat{S} = \operatorname{Spec} \widehat{\mathcal{O}}_{S,s}$. Consider the cartesian diagram

$$\begin{array}{ccccc} \mathcal{C}_s = \mathcal{C}_0 & \hookrightarrow & \mathcal{C}_1 & \hookrightarrow & \cdots \\ \downarrow & & \downarrow & & \\ \operatorname{Spec} \kappa(s) = S_0 & \hookrightarrow & S_1 & \hookrightarrow & \cdots \end{array} \qquad \begin{array}{ccc} \widehat{\mathcal{C}} & \longrightarrow & \mathcal{C} \\ \downarrow & & \downarrow \\ \widehat{S} & \longrightarrow & S \end{array}$$

Case 1: $\mathcal{C}_s \to \operatorname{Spec} \kappa(s)$. Since separated one-dimensional algebraic spaces are schemes and that proper one-dimensional schemes are projective, there exists an ample line bundle L_0 on $\mathcal{C}_0$.

Case 2: $\mathcal{C}_n \to S_n$. The obstruction to deforming a line bundle L_n on $\mathcal{C}_n$ to L_{n+1} on $\mathcal{C}_{n+1}$ lives in $H^2(\mathcal{C}_0, \mathcal{O}_{\mathcal{C}_0})$ and thus vanishes as $\dim \mathcal{C}_0 = 1$. Thus there exists a compatible sequence of line bundles L_n on $\mathcal{C}_n$. Since ampleness is an open condition in families and L_0 is ample, L_n is also ample.

Case 3: $\widehat{\mathcal{C}} \to \widehat{S}$ *with* $\widehat{S}$ *noetherian.* Use Grothendieck's Existence Theorem: $\operatorname{Coh}(\widehat{cC}) \to \varprojlim \operatorname{Coh}(\mathcal{C}_n)$ is an equivalence of categories. The classical case is when $\widehat{\mathcal{C}} \to \widehat{S}$ is a proper morphism of schemes. Chow's Lemma for Algebraic Spaces implies that there exists a projective birational morphism $\mathcal{C}' \to \widehat{\mathcal{C}}$ of algebraic spaces such that $\mathcal{C}' \to \widehat{S}$ is projective. This allows one to reduce Grothendieck's Existence Theorem for $\widehat{\mathcal{C}} \to \widehat{S}$ to $\mathcal{C}' \to \widehat{S}$ using devissage similar to how the proper case of schemes is reduced to the projective case.

As a result, we can extend the sequence of line bundle L_n to a line bundle $\widehat{L}$ on $\widehat{\mathcal{C}}$ which is ample (using again that ampleness is an open condition in families).

Case 4: S *is of finite type over* $\mathbb{Z}$. For any closed point $s \in S$, apply Artin Approximation to the functor

$$\operatorname{Sch}/S \to \operatorname{Sets}, \qquad (T \to S) \mapsto \operatorname{Pic}(\mathcal{C}_T)$$

to obtain an étale neighborhood $(S', s') \to (S, s)$ of s and a line bundle L' on $\mathcal{C}_{S'}$ extending L_0. By openness of ampleness, we can replace S' with an open neighborhood of s' such that L' is relatively ample over S'.

Case 5: S *is an arbitrary scheme.* Apply Noetherian Approximation.

Approach 2: Explicitly extend an ample line bundle

The idea here is to use geometric methods to extend a line bundle L_s on $\mathcal{C}_s$ to a line bundle on $\mathcal{C}$. If we assume in addition that every fiber of $\mathcal{C} \to S$ is generically reduced (and thus also generically smooth), then we may follow the argument of [Ols16, Cor. 13.2.5]. Choose smooth points $p_1, \ldots, p_n \in \mathcal{C}_s$ such that every irreducible one-dimensional component of $\mathcal{C}_s$ contains at least one of the p_i's. Our hypothesis imply that the relative smooth locus $\mathcal{C}^0$ of $\mathcal{C} \to S$ surjects onto S. As smooth morphisms étale locally have sections, there is an étale neighborhood $S' \to S$ of s and sections $\sigma_i \colon S' \to \mathcal{C}^0$ extending p_i. The line bundle $L' := \mathcal{O}_{\mathcal{C}_{S'}}(\sigma_1 + \cdots + \sigma_n)$ extends the ample line bundle $L_s := \mathcal{O}_{\mathcal{C}_s}(p_1 + \cdots + p_n)$. By openness of ampleness in families, L' is relatively ample over S' in an open neighborhood of s'.

(An alternative argument that works without any restrictions is presented in [Hal13, Lem. 1.2] (based on ideas in [SGA4$\frac{1}{2}$, IV.4.1]) where one first uses

Noetherian approximation and étale localization to reduce to $S = \operatorname{Spec} R$ where R is an excellent strictly henselian local ring. One can then reduce to the case where $\mathcal{C}$ is a scheme by appealing to the fact that there exists a finite surjection $\mathcal{C}' \to \mathcal{C}$ from a scheme and the fact that $\mathcal{C}$ satisfies the Chevalley-Kleiman property (i.e. every finite set of points is contained in an open affine) if and only if $\mathcal{C}'$ does. Using deformation theory as above, one can further reduce to the case where $\mathcal{C}$ is reduced. Finally, one attempts to explicitly extend an ample line bundle on $\mathcal{C}_s$ by extending a function $f \in \Gamma(U, \mathcal{O}_{\mathcal{C}_s})$ to a function defined on an open neighborhood of $s \in \mathcal{C}$ so that it defines an effective Cartier divisor.) □

Remark 4.3.4. Raynaud gives an example of a family of smooth $g = 1$ curves over an affine curve which is Zariski-locally projective but not projective [Ray70, XIII 3.1]. The examples in Remark 4.3.2 are not even Zariski-locally projective.

4.3.2 Algebraicity of the stack of all curves

Definition 4.3.5. Let $\mathcal{M}_{g,n}^{\mathrm{all}}$ denote the category over $\mathrm{Sch}_{\mathrm{\acute{e}t}}$ whose objects over a scheme S consists of families of curves $\mathcal{C} \to S$ and n sections $\sigma_1, \ldots, \sigma_n \colon S \to \mathcal{C}$. A morphism $(\mathcal{C}' \to S', \sigma_1', \ldots, \sigma_n') \to (\mathcal{C} \to S, \sigma_1, \ldots, \sigma_n)$ is the data of a cartesian diagram

$$\begin{array}{ccc} \mathcal{C}' & \xrightarrow{g} & \mathcal{C} \\ {\scriptstyle \sigma_i'}\uparrow \downarrow & & {\scriptstyle \sigma_i}\uparrow \downarrow \\ S' & \xrightarrow{f} & S \end{array}$$

such that $g \circ \sigma_i' = \sigma_i \circ f$.

As a stepping stone to the algebraicity of $\mathcal{M}_{g,n}^{\mathrm{all}}$, we first show that the diagonal is representable.

Lemma 4.3.6. *The diagonal $\mathcal{M}_{g,n}^{\mathrm{all}} \to \mathcal{M}_{g,n}^{\mathrm{all}} \times \mathcal{M}_{g,n}^{\mathrm{all}}$ is representable.*

Proof. For simplicity, we handle the case when $n = 0$. Let S be a scheme and $S \to \mathcal{M}_g^{\mathrm{all}} \times \mathcal{M}_g^{\mathrm{all}}$ be a morphism corresponding to families of curves $\mathcal{C}_1 \to S$ and $\mathcal{C}_2 \to S$. Considering the cartesian diagram

$$\begin{array}{ccc} \underline{\mathrm{Isom}}_S(\mathcal{C}_1, \mathcal{C}_2) & \longrightarrow & S \\ \downarrow & & \downarrow \\ \mathcal{M}_g^{\mathrm{all}} & \longrightarrow & \mathcal{M}_g^{\mathrm{all}} \times \mathcal{M}_g^{\mathrm{all}}, \end{array}$$

we need to show that $\underline{\mathrm{Isom}}_S(\mathcal{C}_1, \mathcal{C}_2)$ is an algebraic space. By Proposition 4.3.3, there exists an étale cover $S' \to S$ such that $\mathcal{C}_{S'} \to S'$ is projective. Since $\underline{\mathrm{Isom}}_S(\mathcal{C}_1, \mathcal{C}_2) \times_S S' = \underline{\mathrm{Isom}}_{S'}(\mathcal{C}_{1,S'}, \mathcal{C}_{2,S'})$, the morphism $\underline{\mathrm{Isom}}_{S'}(\mathcal{C}_{1,S'}, \mathcal{C}_{2,S'}) \to \underline{\mathrm{Isom}}_S(\mathcal{C}_1, \mathcal{C}_2)$ is representable, surjective and étale. We may thus assume that $\mathcal{C}_1$ and $\mathcal{C}_2$ are projective over S.

We will use the following fact from scheme theory: if $X \to Y$ is a morphism of schemes each proper over S, there exists an open subscheme $S^0 \subset S$ such that for any map $T \to S$ of schemes $X_T \xrightarrow{\sim} Y_T$ is an isomorphism if and only if $T \to S$ factors through $S_0 \subset S$.

Consider the inclusion of functors:

$$\underline{\mathrm{Isom}}_S(\mathcal{C}_1, \mathcal{C}_2) \subset \underline{\mathrm{Mor}}_S(\mathcal{C}_1, \mathcal{C}_2) \subset \mathrm{Hilb}_S(\mathcal{C}_1 \times_S \mathcal{C}_2)$$

where the second inclusion assigns to a morphism $\mathcal{C}_1 \xrightarrow{\alpha} \mathcal{C}_2$ the graph $\mathcal{C}_1 \xhookrightarrow{\Gamma_\alpha} \mathcal{C}_1 \times_S \mathcal{C}_2$ (and is similarly defined on T-valued points). The first inclusion is a representable open immersion by the above fact. Analyzing the second inclusion, we see that a subscheme $[\mathcal{Z} \subset \mathcal{C}_1 \times_S \mathcal{C}_2] \in \mathrm{Hilb}_S(\mathcal{C}_1 \times_S \mathcal{C}_2)(S)$ is in the image of element of $\underline{\mathrm{Mor}}(\mathcal{C}_1, \mathcal{C}_2)(S)$ if and only if the composition $Z \hookrightarrow \mathcal{C}_1 \times_S \mathcal{C}_2 \xrightarrow{p_1} \mathcal{C}_1$ is an isomorphism (and similarly for T-valued points). Therefore, the above fact also establishes that the second inclusion is a representable open immersion. □

Theorem 4.3.7. *$\mathcal{M}_{g,n}^{\mathrm{all}}$ is an algebraic stack locally of finite type over* $\operatorname{Spec}\mathbb{Z}$.

Sketch.

- Suffices to show the $n = 0$ case: $\mathcal{M}_{g,1}^{\mathrm{all}}$ is the universal family over $\mathcal{M}_g^{\mathrm{all}}$ and more generally $\mathcal{M}_{g,n+1}^{\mathrm{all}}$ is the universal family over $\mathcal{M}_{g,n}^{\mathrm{all}}$. (We will see that the same holds for $\overline{\mathcal{M}}_g$ but this is a more remarkable fact since an n-pointed stable curve can become unstable if a marked point is forgotten.)
- $\mathcal{M}_g^{\mathrm{all}}$ is a stack over $\mathrm{Sch}_{\text{Ét}}$: Suppose $S' \to S$ is an étale cover of schemes, $\mathcal{C}' \to S'$ is a family of curves, and $\alpha\colon p_1^*\mathcal{C}' \to p_2^*\mathcal{C}'$ is an isomorphism over $S' \times_S S'$ satisfying the cocycle condition. The quotient of the étale equivalence relation

  $$R' := p_1^*\mathcal{C}' \underset{p_2 \circ \alpha}{\overset{p_1}{\rightrightarrows}} \mathcal{C}'$$

  is an algebraic space $\mathcal{C} := \mathcal{C}'/R$ and $\mathcal{C} \to S$ is a family of curves such that $\mathcal{C}_{S'} \cong \mathcal{C}'$.
- It to suffices show that for all projective curves C_0 over a field $\Bbbk$, there exists a representable, smooth morphism $U \to \mathcal{M}_g^{\mathrm{all}}$ from a scheme with $[C_0]$ in the image. Choose an embedding $C_0 \hookrightarrow \mathbb{P}^N$ such that $h^1(C_0, \mathcal{O}_{C_0}(1)) = 0$, and let $P(t)$ be its Hilbert polynomial.
- Let $H := \mathrm{Hilb}^P(\mathbb{P}^N_{\mathbb{Z}}/\mathbb{Z})$ be the Hilbert scheme, which is projective over $\mathbb{Z}$ by Theorem D.0.1. Considering the universal family

  $$\begin{array}{ccc} \mathcal{C} & \hookrightarrow & \mathbb{P}^N_H \\ \downarrow & \swarrow & \\ H, & & \end{array}$$

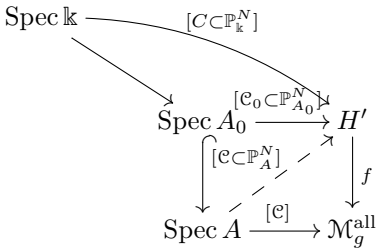
  there is a point $h_0 \in H(\Bbbk)$ such that $\mathcal{C}_{h_0} = C_0$ as closed subschemes of $\mathbb{P}^N_{\Bbbk}$. Cohomology and Base Change implies that there exists an open neighborhood $H' \subset H$ of h_0 such that for all $s \in H'$, $h^1(\mathcal{C}_s, \mathcal{O}_{\mathcal{C}_s}(1)) = 0$.
- Consider the morphism

  $$H' \to \mathcal{M}_g^{\mathrm{all}}, \qquad [C \hookrightarrow \mathbb{P}^n] \mapsto [C],$$

  which is representable by Lemma 4.3.6 and the fact that representability of the diagonal implies that any morphism from a scheme is representable (see the argument of Corollary 2.4.4).

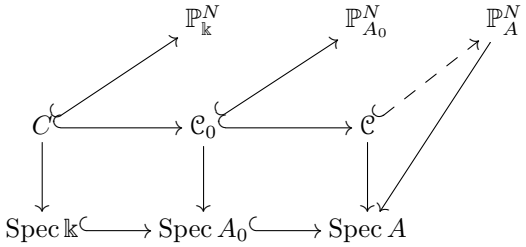
- Claim: $H' \to \mathcal{M}_g^{\text{all}}$ is smooth.

  We will use the Formal Lifting Criterion (Proposition 2.7.1)—even though we don't yet know $\mathcal{M}_g^{\text{all}}$ is algebraic, we may still use this criterion as we know that $H' \to \mathcal{M}_g^{\text{all}}$ is representable so it suffices to show for all maps $S \to \mathcal{M}_g^{\text{all}}$ from a scheme, the induced morphism $H'_S \to S$ is a smooth morphism of algebraic spaces. We need to check that for all surjections $A \to A_0$ of local artinian rings with residue field $\Bbbk$ such that $\Bbbk = \ker(A \to A_0)$ and for all diagrams



(4.3.1)

  of solid arrows, there exists a dotted arrow. The existence of a dotted arrow in the above diagram is equivalent to the existence of a dotted arrow in the below diagram



  of solid arrows: a lifting of the diagram (2.7.3) corresponds to a family $\mathcal{C} \to \operatorname{Spec} A$ extending $\mathcal{C}_0 \to \operatorname{Spec} A_0$. By Theorem E.1.1, there is cohomology class $\text{ob} \in H^2(C, T_C)$ such that $\text{ob} = 0$ if and only if there exists a lifting. Since C is a curve, $H^2(C, T_C) = 0$.

- Use deformation theory to extend $\mathcal{C}_0 \hookrightarrow \mathbb{P}^N_{A_0}$ to $\mathcal{C} \hookrightarrow \mathbb{P}^N_A$. We will use the simplifying assumption that C is a local complete intersection; the general case is handled by more advanced deformation theory (see [Hal13, Prop. 4.2]). This implies that the ideal sheaf $\mathcal{I}$ defining $C \hookrightarrow \mathbb{P}^N_{\Bbbk}$ is cut out locally by a regular sequence and that $\mathcal{I}/\mathcal{I}^2$ is a vector bundle on C fitting into an exact sequence

$$0 \to \mathcal{I}/\mathcal{I}^2 \to \Omega_{\mathbb{P}^N_{\Bbbk}}|_C \to \Omega_C \to 0.$$

  Applying $\operatorname{Hom}(-, \mathcal{O}_C)$ gives a long exact sequence where the relevant terms for us are

$$\operatorname{Hom}(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_C) \to \operatorname{Ext}^1(\Omega_C, \mathcal{O}_C) \to \operatorname{Ext}^1(\Omega_{\mathbb{P}^N_{\Bbbk}}|_C, \mathcal{O}_C) = H^1(C, T_{\mathbb{P}^N_{\Bbbk}}|_C).$$

  The first term classifies embedded deformations of $\mathcal{C}_0 \hookrightarrow \mathbb{P}^N_{A_0}$ over A_0 to $\mathcal{C}' \hookrightarrow \mathbb{P}^N_A$ over A while the second term classifies deformations of $\mathcal{C}_0$ over A_0 to $\mathcal{C}'$ over A. The boundary map $\operatorname{Hom}(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_C) \to \operatorname{Ext}^1(\Omega_C, \mathcal{O}_C)$ assigns an embedded deformation $[\mathcal{C}' \hookrightarrow \mathbb{P}^N_A]$ to $[\mathcal{C}']$.

Finally, we have the restriction of the Euler sequence to C

$$0 \to \mathcal{O}_C \to \mathcal{O}_C(1)^{\oplus(N+1)} \to T_{\mathbb{P}^N}|C \to 0.$$

Since $H^2(C, \mathcal{O}_C) = 0$ (as $\dim C = 1$) and $H^1(C, \mathcal{O}_C(1)) = 0$ (as $[C] \in H'$), we conclude that $H^1(C, T_{\mathbb{P}^N}|C) = 0$. Thus, our given deformation $[\mathcal{C}] \in \mathrm{Ext}^1(\Omega_C, \mathcal{O}_C)$ maps to 0 in $H^1(C, \mathcal{O}_C(1))$, and thus is the image of an embedded deformation $[\mathcal{C} \hookrightarrow \mathbb{P}^N_A] \in \mathrm{Hom}(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_C)$.

□

4.3.3 Algebraicity of $\overline{\mathcal{M}}_{g,n}$: openness and boundedness of stable curves

Consider the inclusions of prestacks

$$\mathcal{M}_{g,n} \subset \overline{\mathcal{M}}_{g,n} \subset \mathcal{M}^{\mathrm{ss}}_{g,n} \subset \mathcal{M}^{\mathrm{pre}}_{g,n} \subset \mathcal{M}^{\leq \mathrm{nodal}}_{g,n} \subset \mathcal{M}^{\mathrm{all}}_{g,n} \tag{4.3.2}$$

where $\overline{\mathcal{M}}_{g,n}$ (resp. $\mathcal{M}^{\mathrm{ss}}_{g,n}, \mathcal{M}^{\mathrm{pre}}_{g,n}$, $\mathcal{M}^{\leq \mathrm{nodal}}_{g,n}$) denotes the full subcategory of $\mathcal{M}^{\mathrm{all}}_{g,n}$ consisting of n-pointed families $(\mathcal{C} \to S, \sigma_1, \ldots, \sigma_n)$ of stable curves (resp. semistable, prestable, and nodal curves).

- By Theorem 4.3.7, $\mathcal{M}^{\mathrm{all}}_{g,n}$ is an algebraic stack locally of finite type over $\operatorname{Spec} \mathbb{Z}$.
- $\mathcal{M}^{\leq \mathrm{nodal}}_{g,n} \subset \mathcal{M}^{\mathrm{all}}_{g,n}$ is an open substack: this is equivalent to showing that $\mathcal{C} \xrightarrow{\pi} S$ is a family of curves (with $\mathcal{C}$ possibly an algebraic space) then the locus $\{s \in S \mid \mathcal{C}_s \text{ is nodal}\} \subset S$ is open. This is established in Corollary 4.1.23 when $\mathcal{C}$ is a scheme by relying on the Local Structure of Nodes (Theorem 4.1.21) and can be established in general by choosing an étale cover $\mathcal{C}' \xrightarrow{g} \mathcal{C}$ by a scheme and using the observation that a point $p \in \mathcal{C}'$ is node in its fiber $\mathcal{C}'_{\pi(p)}$ if and only if $g(p) \in \mathcal{C}_{\pi(p)}$ is a node.
- $\mathcal{M}^{\mathrm{pre}}_{g,n} \subset \mathcal{M}^{\leq \mathrm{nodal}}_{g,n}$ is an open substack: for a family $(\mathcal{C} \to S, \{\sigma_i\})$ of nodal curves, the locus $\{s \in S \mid \sigma_i(s) \text{ are disjont and smooth}\}$ is open.
- $\mathcal{M}^{\mathrm{ss}}_{g,n} \subset \mathcal{M}^{\mathrm{pre}}_{g,n}$ is an open substack: the condition that a prestable curve $(\mathcal{C} \to S, \{\sigma_i\})$ is semistable is equivalent to the nefness of $\omega_{\mathcal{C}/S}(\sigma_1 + \cdots \sigma_n)$ and nefness is an open condition in flat families.
- $\mathcal{M}_{g,n} \subset \mathcal{M}^{\mathrm{ss}}_{g,n}$ is an open substack: the condition that a semistable curve $(\mathcal{C} \to S, \{\sigma_i\})$ is stable is equivalent to the ampleness of $\omega_{\mathcal{C}/S}(\sigma_1 + \cdots \sigma_n)$ and ampleness is an open condition in families. See also Proposition 4.2.10.

It follows that each prestack featured in (4.3.2) is an algebraic stack locally of finite type over $\operatorname{Spec} \mathbb{Z}$.

To show the boundedness (i.e. finite typeness or equivalently quasi-compactness) of $\overline{\mathcal{M}}_{g,n}$, we will appeal to the fact that if $(C, p_1, \ldots, p_n)$ is an n-pointed stable curve over a field $\Bbbk$, then the third power of the twist of the dualizing sheaf $\left(\omega_{C/\Bbbk}(p_1 + \cdots p_n)\right)^{\otimes 3}$ is very ample (Exercise 4.2.5). Let $P(t)$ be the Hilbert polynomial of $C \hookrightarrow \mathbb{P}^N_{base}$ embedded via $\left(\omega_{C/\Bbbk}(p_1 + \cdots p_n)\right)^{\otimes 3}$; this is independent of $[C, \{p_i\}] \in \overline{\mathcal{M}}_{g,n}$. Consider the closed subscheme

$$H \subset \mathrm{Hilb}^P(\mathbb{P}^N_{\mathbb{Z}}/\mathbb{Z}) \times (\mathbb{P}^N)^n$$

of an embedded curve and n points $(C \hookrightarrow \mathbb{P}^N, p_1, \dots, p_n)$ such that $p_i \in C$. There is a forgetful functor

$$H \to \mathcal{M}_{g,n}^{\text{all}} \qquad [C \hookrightarrow \mathbb{P}^N, p_1, \dots, p_n] \mapsto (C, p_1, \dots, p_n).$$

Since $\text{Hilb}^P(\mathbb{P}^N_{\mathbb{Z}}/\mathbb{Z})$ is a projective scheme (Theorem D.0.1) and in particular quasi-compact and the image of $|H| \to |\mathcal{M}_{g,n}^{\text{all}}|$ contains $\overline{\mathcal{M}}_{g,n}$, we conclude that $\overline{\mathcal{M}}_{g,n}$ is quasi-compact.

At this point, we've shown that $\overline{\mathcal{M}}_{g,n}$ is an algebraic stack of finite type over $\operatorname{Spec}\mathbb{Z}$. We now invoke each part of Proposition 4.2.11 characterizing automorphisms, deformations and obstructions of stable curve exactly as in the proof of the analogous fact for $\overline{\mathcal{M}}_g$ (Proposition 2.7.4). Indeed, $\operatorname{Ext}^0(\Omega_C(\sum_i p_i), \mathcal{O}_C) = 0$ implies that the Lie algebra of $\operatorname{Aut}(C, \{p_i\})$ is trivial and thus that $\operatorname{Aut}(C, \{p_i\})$ is a finite and reduced group scheme. By the Characterization of Deligne–Mumford stacks (Theorem 2.6.3), we conclude that $\overline{\mathcal{M}}_{g,n}$ is Deligne–Mumford. Since $\operatorname{Ext}^2(\Omega_C(\sum_i p_i), \mathcal{O}_C) = 0$, there are no obstructions to deforming stable curves and the Formal Lifting Criterion (Proposition 2.7.1) implies that $\overline{\mathcal{M}}_{g,n}$ is smooth over $\operatorname{Spec}\mathbb{Z}$. Finally, since $\dim_{\Bbbk} \operatorname{Ext}^1(\Omega_C(\sum_i p_i), \mathcal{O}_C) = 3g - 3 + n$ and there is a bijection of this Ext group with the Zariski tangent space of $[C, \{p_i\}] \in \overline{\mathcal{M}}_{g,n} \times_{\mathbb{Z}} K$, we see that $\overline{\mathcal{M}}_{g,n} \to \operatorname{Spec}\mathbb{Z}$ has relative dimension $3g - 3 + n$.

Putting everything together, we've proved:

Theorem 4.3.8. *If $2g-2+n > 0$, then $\overline{\mathcal{M}}_{g,n}$ is a quasi-compact Deligne–Mumford stack smooth over $\operatorname{Spec}\mathbb{Z}$ of relative dimension $3g - 3 + n$.* □

Exercise 4.3.9. Show that $\overline{\mathcal{M}}_{g,n}$ is algebraic by following the proof of Theorem 2.1.11.

4.4 Stable reduction: properness of $\overline{\mathcal{M}}_{g,n}$

In the section, we discuss stable reduction of curves. Following the exposition of [HM98, §3.C], we give a complete proof in characteristic 0 relying on the birational geometry of surfaces and specifically the existence of embedded resolutions for curves on surfaces (see Section F.1).

Theorem 4.4.1 (Stable Reduction)**.** *Let R be a DVR with fraction field K, and set $\Delta = \operatorname{Spec} R$ and $\Delta^* = \operatorname{Spec} K$. If $(\mathcal{C}^* \to \Delta^*, s_1^*, \dots, s_n^*)$ is a family of n-pointed stable curves of genus g, then there exists a finite cover $\Delta' \to \Delta$ of spectrums of DVRs and a family $(\mathcal{C}' \to \Delta', s_1', \dots, s_n')$ of stable curves extending $\mathcal{C}^* \times_{\Delta^*} \Delta'^* \to \Delta'^*$.*

Remark 4.4.2. This theorem was first established in [DM69] by embedding the generic fiber into its Jacobian and reducing the statement to semistable reduction for abelian varieties, which had been established in [SGA7-I, SGA7-II]. Interestingly, Gieseker also established this theorem by using GIT rather than the geometry of families of curves over a DVR [Gie82]. Later arguments due to Artin–Winters [AW71] and Saito [Sai87] follow essentially the strategy outlined below. See [SP, Tag 0C2Q] or Remark 4.4.7 for more background.

After introducing the basic strategy to establish Stable Reduction in Section 4.4.1, we prove Stable Reduction (Theorem 4.4.1) in characteristic 0 in

Section 4.4.2. We also illustrate in Sections 4.4.3 and 4.4.4 how one can explicitly compute the stable limit of a given family $\mathcal{C}^* \to \Delta^*$ of stable curves: while the proof of Stable Reduction offers a strategy, additional care and techniques are needed to get an explicit handle on the stable limit. Finally, in Section 4.4.5, we prove the uniqueness of the stable limit (Proposition 4.4.14) in arbitrary (possibly mixed) characteristic. This implies the properness of $\overline{\mathcal{M}}_{g,n}$ via the Valuative Criterion for Properness (Proposition 2.8.5).

Theorem 4.4.3. *If $2g-2+n>0$, the Deligne–Mumford stack $\overline{\mathcal{M}}_{g,n}$ is proper over* $\operatorname{Spec}\mathbb{Z}$. □

By applying the Keel–Mori Theorem (Theorem 3.3.17), we obtain:

Corollary 4.4.4. *If $2g-2+n>0$, there exists a coarse moduli space $\overline{\mathcal{M}}_{g,n} \to \overline{M}_{g,n}$ where $\overline{M}_{g,n}$ is an algebraic space proper over* $\operatorname{Spec}\mathbb{Z}$. □

4.4.1 Basic strategy

We provide the basic strategy to exhibit the existence of stable reduction for a given family $\mathcal{C}^* \to \Delta^*$ of stable curves. For simplicity of notation, we assume that there are no marked points, i.e. $n = 0$.

Throughout, we use the notation: $\Delta = \operatorname{Spec} R$ for a DVR R, $\Delta^* = \operatorname{Spec} K$ with K the fraction field of R, $t \in R$ uniformizer, and $0 = (t) \in \operatorname{Spec} R$ the unique closed point.

Step 0: Reduce to the case where $\mathcal{C}^* \to \Delta^*$ is smooth. If $\mathcal{C}^*$ has k nodes, then possibly after a finite extension of K we can arrange that each node is given by a K-point $p_i \in \mathcal{C}^*(K)$. Let $(\widetilde{\mathcal{C}}^*, \widetilde{p}_1, \dots, \widetilde{p}_{2k})$ be the pointed normalization. By induction on the genus g (relying on stable reduction for *$2k$-pointed curves* of genus $< g$), we perform stable reduction on each connected component and then take the nodal union along sections. After possibly an extension of K (and R), this produces a family of curves $\mathcal{C} \to \Delta$ extending $\mathcal{C}^* \to \Delta^*$.

Step 1: Find some flat extension $\mathcal{C} \to \Delta$.

Using that $\omega_{\mathcal{C}^*/\Delta^*}^{\otimes 3}$ is very ample (Proposition 4.2.9), we may embed $\mathcal{C}^*$ as a closed subscheme of $\mathbb{P}^{5g-6} \times \Delta^*$. The scheme-theoretic image $\mathcal{C}$ of $\mathcal{C}^* \hookrightarrow \mathbb{P}^{5g-6} \times \Delta$ is flat over Δ using the Flatness Criterion over Smooth Curves (Proposition A.2.4) and the fact the closure doesn't introduce any embedded points in the central fiber. Thus we have a proper flat family of curves $\mathcal{C} \to \Delta$ extending $\mathcal{C}^* \to \Delta^*$. (This is the same argument that establishes the valuative criterion for properness of the Hilbert scheme.)

Step 2: Use Embedded Resolutions to find a resolution of singularities $\widetilde{\mathcal{C}} \to \mathcal{C}$ so that the reduced central fiber $(\widetilde{\mathcal{C}}_0)_{\mathrm{red}}$ is nodal.

Applying Embedded Resolutions (Theorem F.1.2), there is a finite sequence of blow-ups at closed points of $\mathcal{C}_0$ yielding a projective birational morphism

$$\begin{array}{ccc} \widetilde{\mathcal{C}} & \longrightarrow & \mathcal{C} \\ & \searrow & \downarrow \\ & & \Delta \end{array}$$

such that $\widetilde{\mathcal{C}}$ is regular, $\widetilde{\mathcal{C}} \to \Delta$ is a (flat) family of curves and such that the preimage $\widetilde{\mathcal{C}}_0$ of $\mathcal{C}_0$ has set-theoretic normal crossings, i.e. $(\widetilde{\mathcal{C}}_0)_{\mathrm{red}}$ is nodal. Replace $\mathcal{C}$ with $\widetilde{\mathcal{C}}$.

Step 3: Take a ramified base extension $\Delta' = \operatorname{Spec} R \to \operatorname{Spec} R = \Delta$ by $t \mapsto t^m$ such that the central fiber of the normalization of $\mathcal{C} \times_\Delta \Delta'$ becomes reduced and nodal.

We will explain the details of this step in Section 4.4.2. This step is where we will use the characteristic 0 assumption. Replacing $\mathcal{C}$ with the normalization $\widetilde{\mathcal{C}}'$ of $\widetilde{\mathcal{C}}' = \mathcal{C} \times_\Delta \Delta'$, we may assume that $\mathcal{C} \to \Delta$ is a prestable family (i.e. nodal family) of curves with $\mathcal{C}$ regular.

Step 4: After taking the minimal model $\widetilde{\mathcal{C}}_{\min} \to \mathcal{C}$, contract all rational tails and bridges in the central fiber.

In other words, we take the stable model of the family $\widetilde{\mathcal{C}}_{\min} \to \Delta$ as in Proposition 4.2.13. Alternatively as we argue in Section 4.4.2, one can explicitly contract the rational tails (smooth rational -1 curves) and rational bridges (smooth rational -2) curves.

Semistable reduction

In Step 4 above, if we stop after contracting only rational tails (and not the rational bridges), i.e. the smooth rational -1 curves, then we obtain a family $\mathcal{C} \to \Delta$ of semistable curves such that $\mathcal{C}$ is regular (by Theorem F.1.5). This is called Semistable Reduction, an important variant of Stable Reduction.

Theorem 4.4.5 (Semistable Reduction)**.** *Let R be a DVR with fraction field K, and set $\Delta = \operatorname{Spec} R$ and $\Delta^* = \operatorname{Spec} K$. If $\mathcal{C}^*$ is a smooth projective curve over Δ^*, there exists a cover $\Delta' \to \Delta$ of spectrums of DVRs and a family $\mathcal{C}' \to \Delta'$ of semistable curves extending $\mathcal{C}^* \times_{\Delta^*} \Delta'^* \to \Delta'^*$ such that $\mathcal{C}'$ is regular.*

4.4.2 Proof of stable reduction in characteristic 0

Proof of Theorem 4.4.1 in characteristic 0. Following Steps 0-2 in the basic strategy discussed in Section 4.4.1, we may assume that $\mathcal{C} \to \Delta$ is a generically smooth family of stable curves such that the reduced central fiber $(\mathcal{C}_0)_{\mathrm{red}}$ is nodal and $\mathcal{C}$ is regular.

Step 3: Perform a base change $\Delta' \to \Delta$ such that the normalization of the total family $\mathcal{C} \times_\Delta \Delta'$ has a reduced and nodal central fiber. Around any point $p \in \mathcal{C}_0$, we can choose local coordinates x, y (either étale-locally or formally locally at p) such that the morphism $\mathcal{C} \to \Delta$ can be described explicitly as follows (Exercise 4.1.24):

- If $p \in (\mathcal{C}_0)_{\mathrm{red}}$ is a smooth point, then $(x, y) \mapsto x^a$ and the multiplicity of the irreducible component of $\mathcal{C}_0$ containing p is a.
- If $p \in (\mathcal{C}_0)_{\mathrm{red}}$ is a separating node (i.e. $\mathcal{C}_0 \setminus p$ is disconnected), then $(x, y) \mapsto x^a y^b$ and the two components of $\mathcal{C}_0$ containing p have multiplicities a and b.
- If $p \in (\mathcal{C}_0)_{\mathrm{red}}$ is a non-separating node, then $(x, y) \mapsto x^a y^a$ and the components of $\mathcal{C}_0$ containing p has multiplicity a.

Let m be the least common multiple of the multiplicities of the irreducible components of $\mathcal{C}_0$. Let $\Delta' = \operatorname{Spec} R \to \operatorname{Spec} R = \Delta$ be defined by $t \mapsto t^m$ where t denotes a uniformizing parameter. Let $\mathcal{C}' := \mathcal{C} \times_\Delta \Delta'$ and $\widetilde{\mathcal{C}}'$ be its normalization Let ρ be a primitive mth root of unity. If $p \in (\mathcal{C}_0)_{\mathrm{red}}$ is a smooth point, then $\mathcal{C}'$ locally around the unique preimage of p is defined by $x^a = t^m$ which factors as $\prod_{i=0}^{a-1}(x - \rho^i t^{m/a})$. Thus $p \in \mathcal{C}$ has a preimages in $\mathcal{C}'$ and each preimage is locally defined by $x = \rho^i t^{m/a}$ and is thus a smooth point in the central fiber $\widetilde{\mathcal{C}}'_0$. If $p \in (\mathcal{C}_0)_{\mathrm{red}}$ is a node defined by $x^a y^b$, then one computes that each preimage of p is locally defined by $t^k = xy$ (see Exercise 4.4.6) and thus is a reduced and nodal point in $\widetilde{\mathcal{C}}'_0$. Note that if $k > 1$, then $\widetilde{\mathcal{C}}'$ has an A_{k-1}-singularity at the preimage.

We now replace $\mathcal{C}$ with $\widetilde{\mathcal{C}}'$. At the expense of introducing singularities into the total family, we have arranged the central fiber to be reduced and nodal.

Step 4: Take a minimal resolution of $\mathcal{C}$ and contract curves with negative self-intersection. Let $\mathcal{C}' \to \mathcal{C}$ be a Minimal Resolution (Theorem F.1.1) which replaces each A_k-singular with a chain of $\lfloor \frac{k}{2} \rfloor$ rational curves. At this stage $\mathcal{C}' \to \Delta$ is a prestable family of curves, i.e. a proper flat family of reduced nodal curves, such that the total family $\mathcal{C}'$ is regular. The central fiber $\mathcal{C}'_0$ however may not be stable.

If $\mathcal{C}'_0$ is not stable, then it contains either a rational tail or bridge as in Figure 4.6. Each rational tail E has self-intersection -1 can be blown down by Castelnuovo's Contraction Theorem (Theorem F.1.5). Contracting all rational tails yields a projective birational morphism $\mathcal{C}' \to \mathcal{C}'_{\mathrm{min}}$, which is the Relative Minimal Model (Corollary F.1.7). Replacing $\mathcal{C}$ with $\mathcal{C}'_{\mathrm{min}}$, we obtain a semistable family $\mathcal{C} \to \Delta$ of curves such that the total family $\mathcal{C}$ is regular.

Finally, we apply the stabilization construction (Proposition 4.2.13) to obtain a morphism $\mathcal{C} \to \mathcal{C}^{\mathrm{st}}$ contracting each rational bridge and where $\mathcal{C}^{\mathrm{st}} \to \Delta$ is a stable family of curves. We note that $\mathcal{C}^{\mathrm{st}}$ is precisely the relative canonical model of $\mathcal{C}$ (Proposition 4.2.13(4)). Alternatively, one can realize this final step by iteratively contracting each rational bridge E since each such subcurve satisfies $E^2 = -2$. Indeed, a version of Castelnuovo's Contraction Theorem is valid even if $E^2 < -1$ (the only difference is that the contracted surface may not be regular) and the contraction yields a family of stable curves. □

Exercise 4.4.6. Let a, b, m be positive integers such that both a and b divide m. Let $X = \operatorname{Spec} \Bbbk[x, y, t]/(t^m - x^a y^b)$ and $\widetilde{X} \to X$ be its normalization. Show that each preimage of the origin is locally defined by $t^k = xy$, and in particular is a reduced and nodal point in the fiber over $t = 0$.

Remark 4.4.7. The above argument fails if the residue field of R has positive characteristic p. Indeed, in Step 3, if any of the multiplicities of the components of the central fiber are divisible by p, then the extension $\operatorname{Spec} R \to \operatorname{Spec} R$ given by $t \mapsto t^m$ is not tamely ramified and the base change $\mathcal{C} \times_\Delta \Delta'$ may remain non-reduced.

A different approach is therefore needed in positive characteristic. The approach of [AW71] starts as above by taking a resolution of singularities $\mathcal{C}$ of some family of curves over R extending $\mathcal{C}^*$. One then chooses an extension $K \to K'$ (and a corresponding extension $R \to R'$ of DVRs) such that $\mathcal{C}^*$ has a K'-point and such that the l-torsion $\operatorname{Pic}(\mathcal{C}^*_{K'})[l] \cong (\mathbb{Z}/l\mathbb{Z})^{2g}$ for a sufficiently large prime $l \neq p$. This magically forces the central fiber of $\mathcal{C} \times_R R'$ to be reduced and nodal! See [AW71] or [SP, Tag 0E8C].

4.4.3 First examples

In these examples $\Delta = \operatorname{Spec} R$ where R is a DVR with uniformizing parameter t.

Example 4.4.8 (Nodal elliptic curves)**.** Consider the family of elliptic curves $(\mathcal{C}^* \to \Delta^*, \sigma)$ defined by the equation $y^2z = x(x-z)(x-tz)$ in $\mathbb{P}^2 \times \Delta$ and the section $\sigma(t) = [0,1,0]$. The stable limit in $\overline{\mathcal{M}}_{1,1}$ as $t \to 0$ is the nodal cubic $y^2z = x^2(x-z)$; see Figure 19.

Example 4.4.9 (Colliding marked points)**.** Let C be a smooth curve and consider the constant family $\mathcal{C} = C \times \Delta$. Let $p \in C$ be a $\Bbbk$-point and $\sigma_1 \colon \Delta \to \mathcal{C}$ be the constant section $t \mapsto p$. Suppose that $\sigma_2 \colon \Delta \to \mathcal{C}$ is another section meeting σ_1 transversely at $(p,0) \in \mathcal{C}$ as shown below:

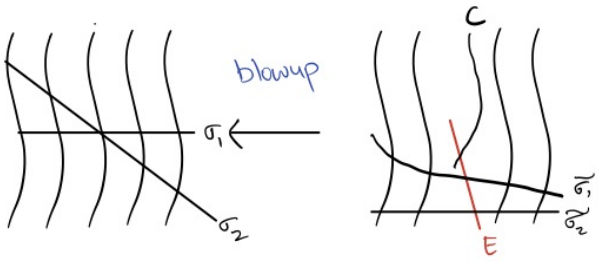


Figure 4.8:

To obtain the stable limit, we simply blowing up the surface at $(p,0)$. The stable limit is the nodal union of C and $\mathbb{P}^1$ at p.

For a more involved example of colliding points, consider again the constant family $C \times \Delta$ with sections locally defined by $(\sigma_1, \sigma_2, \sigma_3) = (t^2, -t^2, 4t)$.

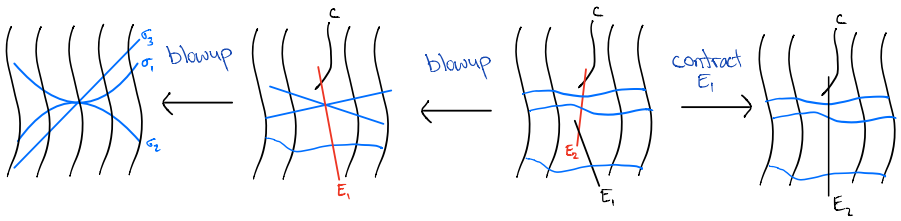


Figure 4.9:

After blowing up twice, the sections become disjoint but the central fiber is unstable as the exceptional component $E_1 \cong \mathbb{P}^1$ only has one node and one marked point. The stable limit is obtained by contracting E_1.

Example 4.4.10 (A node degenerating to a cusp)**.** Consider a smooth curve C with two points $p, q \in C$. Gluing p and q yields a nodal curve. Now if we fix p and slide q toward p, we have a family of nodal curves $\mathcal{C}^* \to C \setminus p$ as in Figure 4.10. For instance, this family could be defined locally by $y^2 = x^3 + tx^2$ in which case we have an extension $\mathcal{C} \to C$ where the central fiber $\mathcal{C}_p$ (given by $t = 0$) has a cusp. We would like to compute the stable limit.

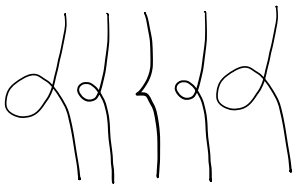


Figure 4.10: What is the stable limit of the above nodal degeneration?

In this case, the base curve is C itself but it would be no different to work over $\operatorname{Spec} \mathcal{O}_{C,p}$. The pointed normalization of the family $\mathcal{C}^*$ extends to a family $C \times C \to C$ with the diagonal section Δ and the constant section $\Gamma_p = \{p\} \times C$.

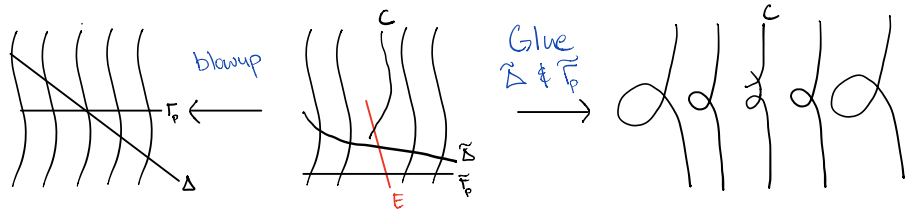


Figure 4.11: Recipe for computing the stable reduction

We first find the stable limit of the pointed normalization exactly as in Example 4.4.9: we blowup so that the strict transforms $\widetilde{\Delta}$ and $\widetilde{\Gamma}_p$ become disjoint. We then glue the sections $\widetilde{\Delta}$ and $\widetilde{\Gamma}_p$ to obtain a family $\mathcal{C} \to C$ of nodal curves where the central fiber is the nodal union of C and a rational nodal curve at the point $p \in C$.

The above examples are too simple to reveal the general stable reduction procedure as no base changes were needed.

4.4.4 Explicit stable reduction

The biggest challenge in explicitly computing the stable limit of a family $\mathcal{C}^* \to \Delta^*$ following the basic strategy of Section 4.4.1 is in Step 3: computing the normalization $\widetilde{\mathcal{C}}'$ of the family $\mathcal{C}' = \mathcal{C} \times_\Delta \Delta'$ obtained by base changing $\mathcal{C} \to \Delta$ along a ramified cover $\Delta' \to \Delta$ defined by $t \mapsto t^m$. It is often simpler to factor $\Delta' \to \Delta$ as a composition of prime order base changes and use the following observation.

Proposition 4.4.11. *Let $\mathcal{C} \to \Delta$ be a generically smooth, proper and flat family such that $(\mathcal{C}_0)_{\mathrm{red}}$ is nodal. As a divisor on $\mathcal{C}$, we may write $\mathcal{C}_0 = \sum a_i D_i$ where a_i is the multiplicity of the irreducible component D_i. Let $\Delta' \to \Delta$ be defined by $t \mapsto t^p$ where p is prime, and set $\mathcal{C}' := \mathcal{C} \times_\Delta \Delta'$ with normalization $\widetilde{\mathcal{C}}'$. Then $\widetilde{\mathcal{C}}' \to \mathcal{C}$ is a branched cover ramified over $\sum (a_i \mod p) D_i$.*

Example 4.4.12 (Stable Reduction of an A_{2k+1}-singularity)**.** Suppose $\mathcal{C} \to \Delta$ is a generically smooth family degenerating to a A_{2k+1}-singularity in the central fiber such that the local equation around the singular point is $y^2 = x^{2k+1} + t$. In particular, the total family $\mathcal{C}$ is smooth. Figure 4.12 provides a pictorial representation of the stable reduction procedure.

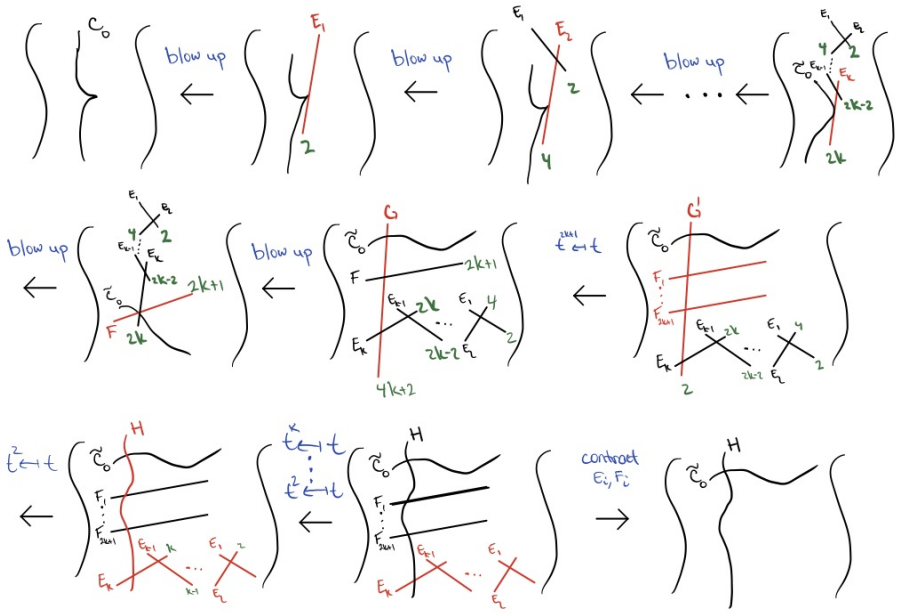


Figure 4.12: Recipe for computing the stable limit of a A_{2k+1}-singularity. The altered components in each step are colored in red while the green numbers indicate the multiplicity of the component.

We are already given a flat limit $\mathcal{C} \to \Delta$ so may begin with Step 2.

Step 2: Repeatedly blow-up to find a resolution of singularities $\widetilde{\mathcal{C}} \to \mathcal{C}$ so that the reduced central fiber $(\widetilde{\mathcal{C}}_0)_{\text{red}}$ is nodal.

We repeatedly blow up the (reduced) singular point in the central fiber. To keep track of the local equations, we will always use local coordinates x, y on the original surface and $\widetilde{x}, \widetilde{y}$ on the new surface. In one chart of the blowup, $\widetilde{x} = x, \widetilde{y} = y/x$ with exceptional divisor $\widetilde{x} = 0$ while in the other chart, $\widetilde{x} = x/y, \widetilde{y} = y$ with exceptional divisor $\widetilde{y} = 0$.

For the first blow up, the preimage of $y^2 - x^{2k+1}$ in the chart $\widetilde{x} = x, \widetilde{y} = y/x$ is given by $\widetilde{x}^2(\widetilde{y}^2 - \widetilde{x}^{2k-1})$ and in the other chart by $\widetilde{y}^2(1 - \widetilde{x}^{2k+1}\widetilde{y}^{2k-1})$. The exceptional divisor E_1 has multiplicity 2.

For the second blow up, the preimage of $x^2(y^2 - x^{2k-1})$ in the chart $\widetilde{x} = x, \widetilde{y} = y/x$ is given by $\widetilde{x}^4(\widetilde{y}^2 - \widetilde{x}^{2k-3})$ and in the other chart by $\widetilde{x}^2\widetilde{y}^4(1 - \widetilde{x}^{2k-1}\widetilde{y}^{2k-3})$ (where $\widetilde{x}$ defines E_1 and $\widetilde{y}$ defines E_2). The new exceptional divisor E_2 has multiplicity 4.

After k blow ups, one obtains a surface with local equation $x^{2k}(y^2 - x)$ at the singular point in the central fiber. The equation $y^2 - x$ defines the normalization $\widetilde{\mathcal{C}}_0$ of the original central fiber and x defines the exceptional divisor E_k which has multiplicity $2k$. There is a chain of nodally attached exceptional divisors $E_k, \ldots, E_1$ such that the multiplicity of E_i is $2i$.

Blowing up again, the strict transform of $x^{2k}(y^2 - x)$ in the chart $\widetilde{x} = x/y, \widetilde{y} = y$ becomes $\widetilde{x}^{2k}\widetilde{y}^{2k+1}(\widetilde{y} - \widetilde{x})$ where $\widetilde{x}$ defines E_k, $\widetilde{y}$ defines the new exceptional divisor F which has multiplicity $2k + 1$, and $\widetilde{y} - \widetilde{x}$ defines $\widetilde{\mathcal{C}}_0$.

Blowing up one final time, the strict transform of $x^{2k}y^{2k+1}(y - x)$ in the chart $\widetilde{x} = x, \widetilde{y} = y/x$ becomes $\widetilde{x}^{4k+2}\widetilde{y}^{2k+1}(\widetilde{y} - 1)$ where $\widetilde{x}$ defines the new exceptional

divisor G which has multiplicity $4k+2$, $\widetilde{y}$ defines F and $\widetilde{y}-1$ defines $\widetilde{\mathcal{C}}_0$. In particular, the (non-reduced) central fiber is set-theoretically nodal.

Step 3: Perform a base change $\Delta' \to \Delta$ such that the normalization of the total family $\mathcal{C} \times_\Delta \Delta'$ has a reduced and nodal central fiber.

We begin by base changing by $\Delta' \to \Delta, t \mapsto t^{2k+1}$ and normalizing. For this analysis, we assume that $2k+1$ is prime but one can inductively apply the same process to a prime factorization of $2k+1$ and obtain the same result in the end; the only difference is in the numerics of the multiplicities of the exceptional components E_i but these can be resolved in the last step in the same way.

By applying Proposition 4.4.11, the new surface is a degree $2k+1$ cover ramified over $\widetilde{\mathcal{C}}_0 + \sum_i E_i$ as the other components of the central fiber have multiplicities divisible by $2k+1$. The preimage G' of G is $2k+1$ degree cover of $\mathbb{P}^1$ ramified over two points, each with ramification index $2k$. By Riemann–Hurwitz, the genus $g(G')$ of G' satisfies $2g(G')-2 = (2k+1)(g(\mathbb{P}^1)-2)+R$ and since the ramification divisor R has degree $2(2k)$, we see that $g(G') = 0$. Meanwhile, the preimage of F is the disjoint union of $2k+1$ smooth rational curves $F_1, \ldots, F_{2k+1}$. Over Δ, the new special fiber is

$$(2k+1)\widetilde{\mathcal{C}}_0 + (4k+2)G' + (2k+1)\sum_i F_i + (2k+1)\sum_i 2iE_i$$

which over Δ' becomes $\widetilde{\mathcal{C}}_0 + 2G' + \sum_i F_i + \sum_i 2iE_i$

We now base change by $\Delta' \to \Delta, t \mapsto t^2$ and normalize. By Proposition 4.4.11, the new surface is a $2:1$ cover ramified over $\widetilde{\mathcal{C}}_0 + \sum_i F_i$. The preimage H of $G' \cong \mathbb{P}^1$ is a $2:1$ cover ramified over $2k+2$ points, with one of those points being the node $H \cap \widetilde{\mathcal{C}}_0$. Thus G' is a hyperelliptic curve of genus g attached to $\widetilde{\mathcal{C}}_0$ at a ramification point (otherwise known as a Weierstrass point). The new central fiber over Δ' becomes reduced except for the components E_i which have multiplicity i.

Finally, we inductively base change and normalize by the ramified covers defined by $t \mapsto t^k, \ldots, t \mapsto t^2$ so that the central fiber becomes reduced and nodal.

Step 4: Contract rational tails in the central fiber.

The exceptional components F_i are smooth rational -1 curves which we can contract. We then inductive contract $E_1, E_2, \ldots, E_k$ (note that while E_1 is a -1 curve, E_2 is a -2 curve but becomes a -1 curve once E_1 is contracted). In the end, we obtain a reduced central fiber which is the nodal union of the normalization $\widetilde{\mathcal{C}}_0$ of the original central fiber and a hyperelliptic genus k curve H. The node in H is a ramification point of the $2:1$ cover $H \to \mathbb{P}^1$ while the node in $\widetilde{\mathcal{C}}_0$ is the preimage of the singular point of $\mathcal{C}_0$.

The above example begs the following questions:

- Precisely which hyperelliptic curve H appears in the stable limit?
- How does the stable limit depend on the choice of degeneration? By calculating the deformation space of a A_{2k+1}-singularity, one sees that any degeneration can be written locally as $y^2 = x^{2k+1} + a_{2k-1}(t)x^{2k-1} + \cdots + a_0(t)$ for polynomials $a_{2k-1}, \ldots, a_0$. In other words, we are asking how does the stable limit depend on $a_i(t)$. In particular, what happens when the total family of the surface is singular (e.g. $y^2 = x^{2k+1} + t^2$)?

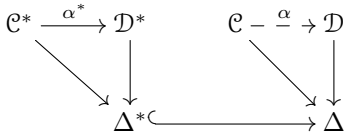
These questions are addressed in detail in [HM98, §3.C] in the case of a cusp $y^2 = x^3$ (i.e. $k = 1$). The reader is also encouraged to refer to *loc. cit.* for additional examples of stable reduction and other aspects of this story.

Exercise 4.4.13. Work out the stable reduction of a smooth family of curves degenerating to an A_{2k+2}-singularity with local equation $y^2 = x^{2k+2} + t$.

4.4.5 Separatedness of $\overline{\mathcal{M}}_{g,n}$

We now show that the stable limit is unique. The following proposition establishes via the Valuative Criterion for Separatedness (Proposition 2.8.5) that $\overline{\mathcal{M}}_{g,n}$ is separated.

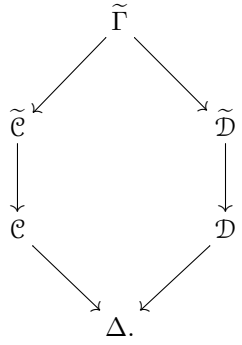
Proposition 4.4.14. *Let R be a DVR with fraction field K, and set $\Delta = \operatorname{Spec} R$ and $\Delta^* = \operatorname{Spec} K$. If $(\mathcal{C} \to \Delta, \sigma_1^*, \dots, \sigma_n^*)$ and $(\mathcal{D} \to \Delta, \tau_1^*, \dots, \tau_n^*)$ are families of n-pointed stable curves, then any isomorphism $\alpha^*\colon \mathcal{C}^* \to \mathcal{D}^*$ over Δ^* with $\tau_i^* = \alpha^* \circ \sigma_i^*$ of the generic fibers as pictured*



extends to a unique isomorphism $\alpha\colon \mathcal{C} \to \mathcal{D}$ over Δ with $\tau_i = \alpha \circ \sigma_i$.

Proof. We will prove the case when there are no marked points ($n = 0$) and the generic fiber $\mathcal{C}^* \cong \mathcal{D}^*$ is smooth over Δ^*. We leave the general case to the reader.

Let $\widetilde{\mathcal{C}} \to \mathcal{C}$ and $\widetilde{\mathcal{D}} \to \mathcal{D}$ be the minimal resolutions (Corollary F.1.7). Let $\Gamma \subset \widetilde{\mathcal{C}} \times_\Delta \widetilde{\mathcal{D}}$ be the closure of the graph $\mathcal{C}^* \xrightarrow{(\mathrm{id},\alpha^*)} \mathcal{C}^* \times_{\Delta^*} \mathcal{D}^*$ of α^* and let $\widetilde{\Gamma} \to \Gamma$ be the minimal resolution. We have a commutative diagram



(4.4.1)

Since $\widetilde{\Gamma} \to \widetilde{\mathcal{C}}$ and $\widetilde{\Gamma} \to \widetilde{\mathcal{C}}$ are birational morphisms of smooth projective surfaces over Δ and the relative dualizing sheaves are line bundles, we have identifications of the pluricanonical sections

$$\Gamma(\widetilde{\mathcal{C}}, \omega_{\widetilde{\mathcal{C}}/\Delta}^{\otimes k}) \cong \Gamma(\widetilde{\Gamma}, \omega_{\widetilde{\Gamma}/\Delta}^{\otimes k}) \cong \Gamma(\widetilde{\mathcal{D}}, \omega_{\widetilde{\mathcal{D}}/\Delta}^{\otimes k})$$

for each non-negative integer k; see [Har77, Thm. II.8.19]. Furthermore, we know that $\mathcal{C}$ and $\mathcal{D}$ are the stable models of $\widetilde{\mathcal{C}}$ and $\widetilde{\mathcal{C}}$ obtained by contracting rational

tails and bridges (Proposition 4.2.13). Thus we have an isomorphism

$$\mathcal{C} \cong \operatorname{Proj} \bigoplus_k \Gamma(\widetilde{\mathcal{C}}, \omega_{\widetilde{\mathcal{C}}/\Delta}^{\otimes k}) \cong \operatorname{Proj} \bigoplus_k \Gamma(\widetilde{\mathcal{D}}, \omega_{\widetilde{\mathcal{D}}/\Delta}^{\otimes k}) \cong \mathcal{D}$$

extending $\alpha^*\colon \mathcal{C}^* \to \mathcal{D}^*$. □

Remark 4.4.15. We can also argue more explicitly using our understanding of the birational geometry of surfaces. First, notice that the local structure of the surface $\mathcal{C}$ or $\mathcal{D}$ around a node z in the central fiber is of the form $xy = t^{n+1}$, where $t \in R$ is a uniformizer (Theorem 4.1.21). This is an A_n-surface singularity and in particular normal, and its preimage under the resolution $\widetilde{\mathcal{C}} \to \mathcal{C}$ is a chain $E_1 \cup \cdots \cup E_n$ of rational bridges with $E_i^2 = -2$. By construction, there are no smooth rational -1 curves in the fibers of $\widetilde{\mathcal{C}} \to \mathcal{C}$ and $\widetilde{\mathcal{D}} \to \mathcal{D}$, and since $\mathcal{C}$ and $\mathcal{D}$ are families of stable curves, they have no rational tails and thus no smooth rational -1 curves. We conclude that $\widetilde{\mathcal{C}}$ and $\widetilde{\mathcal{D}}$ are birational smooth surfaces over Δ with no smooth rational -1 curves whose generic fibers $\mathcal{C}^*$ and $\mathcal{D}^*$ are isomorphic.

By the Structure Theorem of Birational Morphisms of Surfaces (Theorem F.1.3), both $\widetilde{\Gamma} \to \widetilde{\mathcal{C}}$ and $\widetilde{\Gamma} \to \widetilde{\mathcal{D}}$ are the compositions of finite sequences of blow-ups at closed points. Since $\widetilde{\Gamma}$ is minimal over Γ, there are no smooth rational -1 curves in $\widetilde{\Gamma}$ that get contracted under both $\widetilde{\Gamma} \to \widetilde{\mathcal{C}}$ and $\widetilde{\Gamma} \to \widetilde{\mathcal{D}}$.

We now claim that $\widetilde{\Gamma} \to \widetilde{\mathcal{C}}$ and $\widetilde{\Gamma} \to \widetilde{\mathcal{D}}$ are isomorphism. Suppose for instance that $\widetilde{\Gamma} \to \widetilde{\mathcal{C}}$ is not an isomorphism. Then there is a smooth rational -1 curve $E \subset \widetilde{\Gamma}$ not contracted under $\widetilde{\Gamma} \to \widetilde{\mathcal{D}}$ and let $E_{\widetilde{\mathcal{D}}} \subset \widetilde{\mathcal{D}}$ be its image. On the one hand, since blowing up only decreases the self-intersection number (indeed, if we write the pre-image of $E_{\widetilde{\mathcal{D}}}$ in $\widetilde{\Gamma}$ as $E + F$, then the projection formula implies that $E_{\widetilde{\mathcal{D}}}^2 = E \cdot (E + F) = E^2 + E \cdot F$), we have that $E_{\widetilde{\mathcal{D}}}^2 \geq E^2 = -1$. The Hodge Index Theorem for Exceptional Curves (Theorem F.1.4) implies however that the self-intersection of $E_{\widetilde{\mathcal{D}}}$ must be negative, and we conclude that $E_{\widetilde{\mathcal{D}}}^2 = -1$. On the other hand, since $E_{\widetilde{\mathcal{D}}}$ is not a smooth rational -1 curve, $E_{\widetilde{\mathcal{D}}}$ must be a singular curve and one of the blow-ups in the composition $\widetilde{\Gamma} \to \widetilde{\mathcal{D}}$ must be along a singular point of $E_{\widetilde{\mathcal{D}}}$. But this implies that exceptional locus F of $\widetilde{\Gamma} \to \widetilde{\mathcal{D}}$ intersects E with multiplicity at least 2 so that $E_{\widetilde{\mathcal{D}}}^2 \geq E^2 + 2$, a contradiction.

We finish the proof as before by observing that both $\mathcal{C}$ and $\mathcal{D}$ are the stable models of $\widetilde{\mathcal{C}} \cong \widetilde{\mathcal{D}}$. Since the stable model is unique (Proposition 4.2.13), there is an isomorphism $\mathcal{C} \xrightarrow{\sim} \mathcal{D}$ extending $\mathcal{C}^* \xrightarrow{\sim} \mathcal{D}^*$.

4.5 Gluing and forgetful morphisms

4.5.1 Gluing morphisms

Proposition 4.5.1. *There are finite morphisms of algebraic stacks*

$$\begin{aligned} \overline{\mathcal{M}}_{i,n} \times \overline{\mathcal{M}}_{g-i,m} &\to \overline{\mathcal{M}}_{g,n+m-2} \\ \big((C, p_1, \ldots, p_n), (C', p_1', \ldots, p_m')\big) &\mapsto (C \cup C', p_1, \ldots, p_{n-1}, p_1', \ldots, p_m'). \end{aligned} \tag{4.5.1}$$

and

$$\begin{aligned} \overline{\mathcal{M}}_{g-1,n} &\to \overline{\mathcal{M}}_{g,n-2} \\ (C, p_1, \ldots, p_n) &\mapsto (C/_{p_{n-1} \sim p_n}, p_1, \ldots, p_{n-2}). \end{aligned} \tag{4.5.2}$$

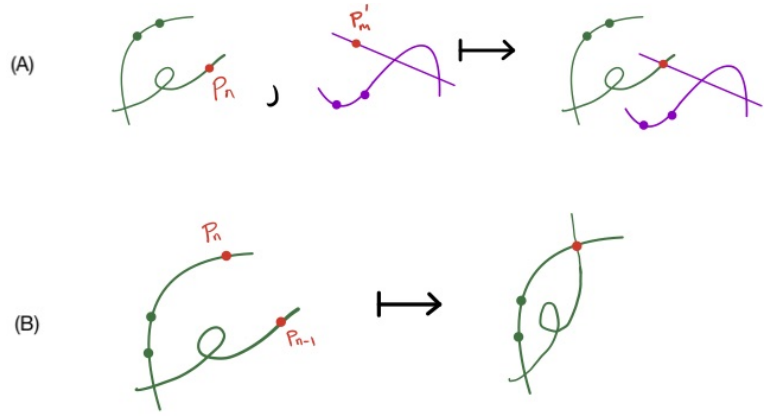


Figure 4.13: (A) is an example of (4.5.1) while (B) is an example of (4.5.2)

Remark 4.5.2. To simplify the notation, we chose to write only the case of gluing the nth marked point p_n and the mth marked point p'_m curve in (4.5.1), and likewise only case of gluing the p_{n-1} and p_n in (4.5.2). Clearly the same holds for the gluing of any two points.

Sketch. To simplify the notation, we will establish the proposition in the following two cases:

(a) In (4.5.1), we assume $n = m = 1$.

(b) In (4.5.2), we assume $n = 2$.

Note that once we establish the existence of the morphisms of algebraic stacks, it follows from Stable Reduction (Theorem 4.4.1) that the morphisms are proper. By inspection, they are clearly representable and have finite fibers and thus follows that the morphisms are finite.

Case (a): Let $(\mathcal{C} \xrightarrow{\pi} S, \sigma)$ and $(\mathcal{C}' \xrightarrow{\pi'} S, \sigma')$ be two families of 1-pointed stable curves over a scheme S.

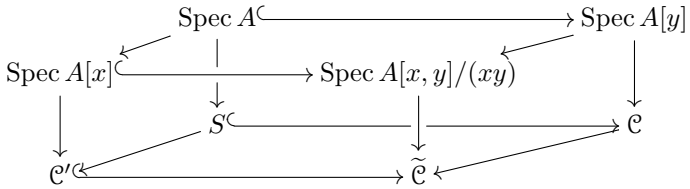
Argument 1 (pushout construction): Consider the pushout diagram

$$\begin{array}{ccc} S & \xrightarrow{\sigma} & \mathcal{C} \\ \downarrow{\scriptstyle \sigma'} & & \downarrow \\ \mathcal{C}' & \longrightarrow & \widetilde{\mathcal{C}} \end{array}$$

which exists by Ferrand's Theorem on the Existence of Pushouts (Theorem 4.5.5). We claim that $\widetilde{\mathcal{C}} \to S$ is a family of stable curves. First, note that $\widetilde{\mathcal{C}} \to S$ is proper as there is a finite cover $\mathcal{C} \sqcup \mathcal{C}' \to \mathcal{C}$ with $\mathcal{C} \sqcup \mathcal{C}'$ proper over S. One can use properties of pushouts to show that $\widetilde{\mathcal{C}} \to S$ is flat (missing details). It remains to chose that the geometric fibers of $\widetilde{\mathcal{C}} \to S$ are stable curves and in particular nodal.

For any point $s \in S$, since $\sigma(s)$ is a smooth point of $\mathcal{C}$, there is an étale neighborhood $\operatorname{Spec} A[x] \to \mathcal{C}$ of $\sigma(s)$ which pulls back to an étale neighborhood $\operatorname{Spec} A \to S$ of s. Since an étale morphism from an affine scheme extend over closed immersions (missing reference), there is an étale neighborhood $\operatorname{Spec} A[y] \to \mathcal{C}'$ of

$\sigma'(s)$ which also pulls back to $\operatorname{Spec} A \to S$. The geometric pushout of $[\operatorname{Spec} A[x] \leftarrow \operatorname{Spec} A \to \operatorname{Spec} A[y]]$ is $\operatorname{Spec} A[x,y]/(xy)$, and we have a commutative cube



We see by Proposition 4.5.6 that $\operatorname{Spec} A[x,y]/(xy) \to \widetilde{\mathcal{C}}$ is an étale neighborhood of the image of s. This shows that $\widetilde{\mathcal{C}} \to S$ is nodal along S and since $\widetilde{\mathcal{C}}$ is either isomorphic to $\mathcal{C}$ or $\mathcal{C}'$ outside S, we see that $\widetilde{\mathcal{C}} \to S$ is a nodal family of curves. Finally one checks (missing details) that $\widetilde{\mathcal{C}}_s$ is identified with the nodal union $\mathcal{C}_s$ and $\mathcal{C}_{s'}$, which is stable.

*Argument 2 (*Proj *construction):* We know that $\omega_{\mathcal{C}}(\sigma)$ is ample. There is a surjection $\omega_{\mathcal{C}}(\sigma) \to \mathcal{O}_{\sigma_1}$ and for each $k \geq 0$, the pushforward of the surjection $(\omega_{\mathcal{C}}(\sigma))^{\otimes k} \to \mathcal{O}_{\sigma_1}$ under $\pi\colon \mathcal{C} \to S$ is $\pi_*(\omega_{\mathcal{C}}(\sigma)^{\otimes k}) \to \mathcal{O}_S$. We have a similar construction for $\pi'\colon \mathcal{C}' \to S$, and we can consider the fiber product of quasi-coherent $\mathcal{O}_S$-modules

$$\begin{array}{ccc} \mathcal{A}_k & \longrightarrow & \pi_*(\omega_{\mathcal{C}}(\sigma)^{\otimes k}) \\ \downarrow & & \downarrow \\ \pi_*(\omega_{\mathcal{C}'}(\sigma')^{\otimes k}) & \longrightarrow & \mathcal{O}_S \end{array}$$

One checks that $\mathcal{A} := \bigoplus_{k \geq 0} \mathcal{A}_k$ is a finitely generated quasi-coherent $\mathcal{O}_S$-algebra and that $\widetilde{\mathcal{C}} := \operatorname{Proj}_S \mathcal{A}$ is a family of stable curves over S such that $\widetilde{\mathcal{C}}_s$ is the nodal union $\mathcal{C}_s$ of $\mathcal{C}_{s'}$.

Case (b): Let $(\mathcal{C} \to S, \sigma_1, \sigma_2)$ be a 2-pointed family of stable curves over a scheme S.

Argument 1 (pushout construction): We use the pushout diagram

$$\begin{array}{ccc} S \sqcup S & \xrightarrow{\sigma_1 \sqcup \sigma_2} & \mathcal{C} \\ \downarrow & & \downarrow \\ S & \longrightarrow & \widetilde{\mathcal{C}} \end{array}$$

By the étale local properties of pushouts (Proposition 4.5.6), the local structure of $\widetilde{\mathcal{C}}$ is determined by the pushout diagram

$$\begin{array}{ccc} \operatorname{Spec} A \times A & \xrightarrow{(0,1)} & \operatorname{Spec} A[t] \\ \downarrow & & \downarrow \\ \operatorname{Spec} A & \longrightarrow & \operatorname{Spec} A \times_{A \times A} A[t]. \end{array}$$

The subalgebra $A \times_{A \times A} A[t] \subset A[t]$ consists of functions $f \in A[t]$ such that $f(0) = f(1) \in A$. The elements $x := t^2 - 1$ and $y := t^3 - t$ generate $A \times_{A \times A} A[t]$ as

an A-algebra and since x and y satisfy $y^2 = x^2(x+1)$, we see that $A \times_{A\times A} A[t] \cong A[x,y]/(y^2 - x^2(x+1))$.

Argument 2 (Proj construction): One defines $\widetilde{\mathcal{C}} := \mathcal{P}\mathrm{roj}_S \bigoplus_{k\geq 0} \mathcal{A}_k$ where $\mathcal{A}_k$ is defined as the fiber product

$$\begin{array}{ccc} \mathcal{A}_k & \longrightarrow & \mathcal{O}_S \\ \downarrow & & \downarrow \Delta \\ \pi_*(\omega_{\mathcal{C}}(\sigma_1)^{\otimes k}) \sqcup \pi_*(\omega_{\mathcal{C}}(\sigma_2)^{\otimes k}) & \longrightarrow & \mathcal{O}_S \sqcup \mathcal{O}_S. \end{array}$$

□

Aside: pushouts

Definition 4.5.3. Consider a commutative diagram of schemes

$$\begin{array}{ccc} X_0 & \xhookrightarrow{i} & X \\ {\scriptstyle f_0}\downarrow & & \downarrow{\scriptstyle f} \\ Y_0 & \xhookrightarrow{j} & Y, \end{array} \tag{4.5.3}$$

where i and j are closed immersions and f_0 and f are affine. If the induced map

$$\mathcal{O}_Y \to j_*\mathcal{O}_{Y_0} \times_{(j\circ f_0)_*\mathcal{O}_{X_0}} f_*\mathcal{O}_X$$

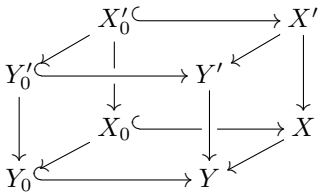
is an isomorphism of sheaves, then we say that the diagram is a *geometric pushout*, and that Y is a *geometric pushout* of the diagram $[Y_0 \xleftarrow{f_0} X_0 \xrightarrow{i} X]$.

Example 4.5.4. In the affine case where $X = \operatorname{Spec} A$, $X_0 = \operatorname{Spec} A_0$, $Y_0 = \operatorname{Spec} B_0$, then $\operatorname{Spec}(A \times_{A_0} B_0)$ is a geometric pushout.

Theorem 4.5.5 (Ferrand's Theorem on the Existence of Pushouts)**.** *[Fer03, Thm. 5.4] Let $f_0 \colon X_0 \to Y_0$ be a finite morphism of schemes and $X_0 \hookrightarrow X$ be a closed immersion of schemes. Assume that X and Y_0 satisfy the following property: every finite set of points is contained in an affine open subscheme. Then there exists a geometric pushout Y such that the corresponding diagram (4.5.3) is a cartesian (i.e. fiber product) and cocartesian (i.e. pushout). Moreover f restricts to an isomorphism $X \setminus X_0 \xrightarrow{f} Y \setminus Y_0$.*

We will also need étale-local properties of the pushout.

Proposition 4.5.6. *Consider a diagram of schemes as in (4.5.3) where f is affine and i is a closed immersion. The question of whether the diagram is a pushout is étale-local on Y. Moreover, suppose*



is a commutative cube of schemes where the back and left faces are cartesian and the top and bottom faces are geometric pushouts. If $Y_0' \to Y_0$ and $X' \to X$ are étale, then so is $X' \to X$.

4.5.2 Boundary divisors of $\overline{\mathcal{M}}_g$

Define the closed substacks

$$\delta_0 = \mathrm{im}(\overline{\mathcal{M}}_{g-1,2} \to \overline{\mathcal{M}}_g)$$
$$\delta_i = \mathrm{im}(\overline{\mathcal{M}}_{i,1} \times \overline{\mathcal{M}}_{g-i,1} \to \overline{\mathcal{M}}_g)$$

where $i = 1, \ldots, \lfloor g/2 \rfloor$.

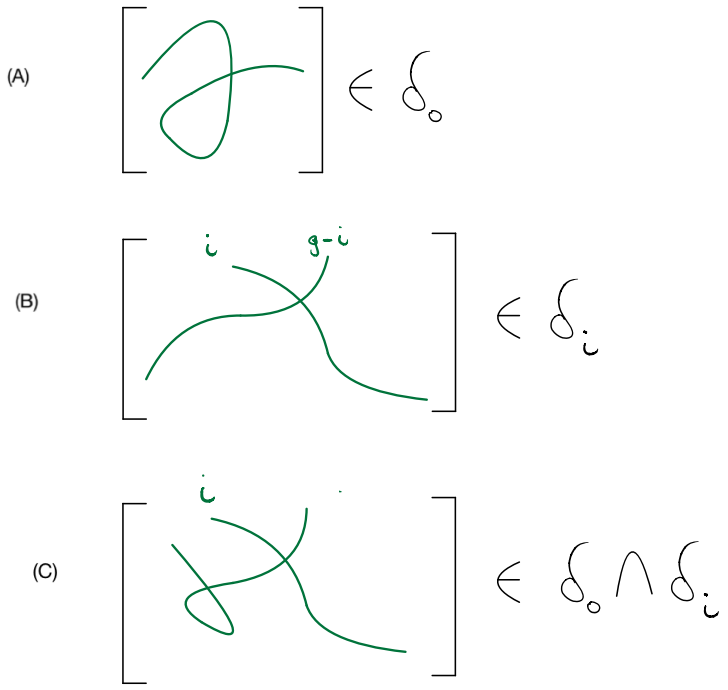


Figure 4.14: Examples of stable curves in the boundary.

Once we show that $\mathcal{M}_g$ is dense in $\overline{\mathcal{M}}_g$, it will follow that δ_0 and δ_i are the closure of the locus of curves with a single node as featured in (A) and (B) of Figure 4.14.

To see that δ_0 and δ_i are divisors in $\overline{\mathcal{M}}_g$, we can do a simple dimension count. As $\overline{\mathcal{M}}_{g-1,2} \to \overline{\mathcal{M}}_g$ and $\overline{\mathcal{M}}_{i,1} \times \overline{\mathcal{M}}_{g-i,1} \to \overline{\mathcal{M}}_g$ are finite morphisms, we compute that $\dim \delta_0 = \dim \overline{\mathcal{M}}_{g-1,2} = 3(g-1) - 3 + 2 = 3g - 4$ and that $\dim \delta_i = \dim \overline{\mathcal{M}}_{i,1} + \dim \overline{\mathcal{M}}_{g-i,1} = (3i - 3 + 1) + (3(g-i) - 3 + 1) = 3g - 4$.

By analyzing the formal deformation space of a stable curve, one can show that more is true: $\delta = \delta_0 \cup \cdots \cup \delta_{\lfloor \frac{g}{2} \rfloor}$ is a normal crossings divisor.

4.5.3 The forgetful morphism

Proposition 4.5.7. *There is a morphism of algebraic stacks*

$$\overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_{g,n-1}$$
$$(C, p_1, \ldots, p_n)) \mapsto (C^{\mathrm{st}}, p_1, \ldots, p_{n-1}).$$

where $(C^{\mathrm{st}}, p_1, \ldots, p_{n-1})$ *is the stable model of* $(C, p_1, \ldots, p_{n-1})$.

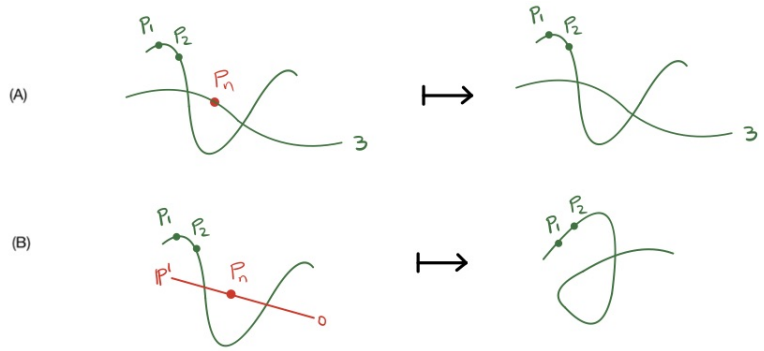


Figure 4.15: In (A), the nth point is simply forgotten. In (B), if p_n is forgotten, then the curve is no longer stable and we must contract the rational bridge.

Proof. If $(\mathcal{C} \to S, \sigma_1, \ldots, \sigma_n)$ is an n-pointed family of stable curves, then if we forget the nth section, the $(n-1)$-pointed family $(\mathcal{C} \to S, \sigma_1, \ldots, \sigma_{n-1})$ may not be stable. However, we have already constructed the stable model $(\mathcal{C}^{\mathrm{st}} \to S, \sigma_1, \ldots, \sigma_{n-1})$ in Proposition 4.2.13. □

4.5.4 The universal family $\overline{\mathcal{M}}_{g,1} \to \overline{\mathcal{M}}_g$

Let $\mathcal{U}_g \to \overline{\mathcal{M}}_g$ be the universal family: this is a proper and flat morphism of algebraic stacks whose geometric fibers are genus g curves. (The existence of the universal family follows from applying descent and the 2-Yoneda Lemma (Lemma 1.3.18) to the identity morphism $\mathrm{id}\colon \overline{\mathcal{M}}_g \to \overline{\mathcal{M}}_g$.) Objects of $\mathcal{U}_g$ over a scheme S correspond to a family of stable curves $\mathcal{C} \to S$ and a section $\sigma\colon S \to \mathcal{C}$ (that may land in the relative singular locus).

There is a morphism of algebraic stacks

$$\overline{\mathcal{M}}_{g,1} \to \mathcal{U}_g$$

sending $(\mathcal{C} \to S, \sigma)$ to $(\mathcal{C}^{\mathrm{st}} \to S, \sigma^{\mathrm{st}})$ where $\mathcal{C} \to \mathcal{C}^{\mathrm{st}}$ is the stabilization of $\mathcal{C} \to S$ (see Proposition 4.2.13) and $\sigma^{\mathrm{st}} = \pi \circ \sigma$. In other words, there is a 2-commutative diagram

$$\begin{array}{ccc} \overline{\mathcal{M}}_{g,1} & \longrightarrow & \mathcal{U}_g \\ & \searrow & \downarrow \\ & & \overline{\mathcal{M}}_g \end{array}$$

where $\overline{\mathcal{M}}_{g,1} \to \overline{\mathcal{M}}_g$ is the forgetful morphism of Proposition 4.5.7.

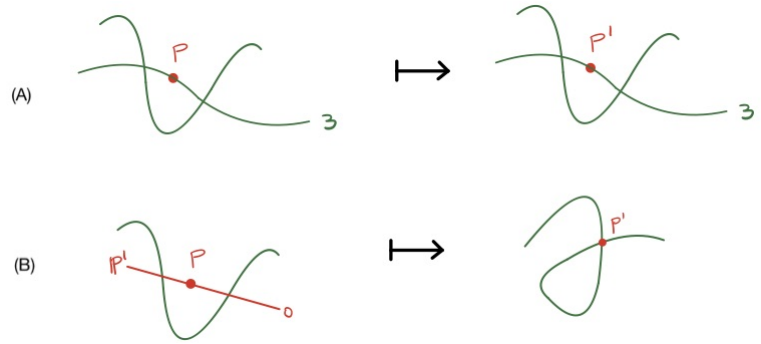


Figure 4.16: In Example (A), $\overline{\mathcal{M}}_{g,1} \to \mathcal{U}_g$ sends (C, p) to itself while in Example (B), the morphism sends (C, p) to the curve (C', p') obtained by contracting the rational bridge.

Proposition 4.5.8. *The morphism $\overline{\mathcal{M}}_{g,1} \to \mathcal{U}_g$ is an isomorphism over $\overline{\mathcal{M}}_g$. In other words, the morphism $\overline{\mathcal{M}}_{g,1} \to \overline{\mathcal{M}}_g$, which forgets the marked point and stabilizes the the curve, is the universal family.*

Sketch.

Strategy 1: Construct an inverse [Knu83, Thm. 2.4].

We construct an inverse morphism $\overline{\mathcal{U}}_g \to \overline{\mathcal{M}}_{g,1}$ defined as follows. Let $\mathcal{C} \to S$ be a family of stable curves and $\sigma\colon S \to \mathcal{C}$ be an arbitrary section defined by an ideal sheaf $\mathcal{I}_\sigma$. Define the coherent $\mathcal{O}_\mathcal{C}$-module $\mathcal{K}$ by

$$0 \to \mathcal{O}_\mathcal{C} \xrightarrow{\delta} \mathcal{I}_\sigma^\vee \oplus \mathcal{O}_\mathcal{C} \to \mathcal{K} \to 0$$

where $\delta = (\iota^\vee, \mathrm{id})$ with $\iota\colon \mathcal{I} \hookrightarrow \mathcal{O}_\mathcal{C}$ denoting the inclusion, and define

$$\mathcal{C}' = \mathcal{P}\mathrm{roj}\ \mathrm{Sym}\,\mathcal{K} \to S$$

One then needs to show that $\sigma^*(\mathcal{I}_\sigma^\vee/\mathcal{O}_\mathcal{C})$ is a line bundle [Knu83, Lem. 2.2]. This requires some work: Knudsen introduces the notion of a *stably reflexive* $\mathcal{O}_\mathcal{C}$-module, shows that $\mathcal{I}_\sigma$ is stable reflexive, and uses this to deduce that that $\sigma^*(\mathcal{I}_\sigma^\vee/\mathcal{O}_\mathcal{C})$ is a line bundle. The surjection $\sigma^*\mathcal{K} \to \sigma^*(\mathcal{K}/\mathcal{O}_\mathcal{C}) \cong \sigma^*(\mathcal{I}_\sigma^\vee/\mathcal{O}_\mathcal{C})$ defines a section $\sigma'\colon S \to \mathcal{C}$ and one checks that $(\mathcal{C}' \to S, \sigma')$ is a 1-pointed family of stable curves.

Strategy 1: Show $\overline{\mathcal{M}}_{g,1} \to \mathcal{U}_g$ is an isomorphism by showing it separates points and tangent vectors.

We will use the following exercise (which can be established via descent by reducing to the case of schemes):

Exercise 4.5.9. Suppose $f\colon \mathcal{X} \to \mathcal{Y}$ is a proper morphism of algebraic stacks such that

(a) For every geometric point $x \in \mathcal{X}(\Bbbk)$, f induces a fully faithful functor of groupoids $\mathcal{X}(\Bbbk) \to \mathcal{Y}(\Bbbk)$. (This encodes both that f is injective on geometric points and that f induces an isomorphism of automorphism groups at any geometric point.)

(b) f induces an injection on the tangent spaces at geometric points.

Then f is a closed immersion.

By Stable Reduction (Theorem 4.4.1), both $\overline{\mathcal{M}}_{g,1}$ and $\mathcal{U}_g$ are proper; thus $\overline{\mathcal{M}}_{g,1} \to \mathcal{U}_g$ is also proper. It is easy to directly check that $\overline{\mathcal{M}}_{g,1} \to \mathcal{U}_g$ induces an equivalence of groupoids at geometric points. It remains to check (c). Once we do show (c), then Exercise 4.5.9 implies that $\overline{\mathcal{M}}_{g,1} \to \mathcal{U}_g$ is a closed immersion. But since this map is also surjective and $\mathcal{U}_g$ is reduced, we conclude that it is an isomorphism.

Let $(C', p') \in \overline{\mathcal{M}}_{g,1}$ be a one-pointed stable curve over $\Bbbk$, and (C, p) be its image in $\mathcal{U}_g$. Let I_p and $I_{p'}$ be the ideal sheaf defining p and p' respectively. We claim that the map

$$\operatorname{Ext}^1(\Omega_{C'}, I_{p'}) = T_{\overline{\mathcal{M}}_{g,1},(C',p')} \to T_{\mathcal{U}_g,(C,p)} = \operatorname{Ext}^1(\Omega_C, I_p)$$

is injective.

If C' is a stable curve, the statement is obvious. Otherwise, denote $\pi\colon C' \to C$ as the stabilization. We have the following properties:

(1) $\pi_* \mathcal{O}_{C'} = \mathcal{O}_C$ and $R^1\pi_* \mathcal{O}_{C'} = 0$.

(2) $\pi_* I_{p'} = I_p$ and $R^1\pi_* I_{p'} = 0$.

(3) $0 \to \kappa(p) \to \Omega_C \to \pi_* \Omega_{C'} \to 0$ is exact.

The map on Ext's takes an extension

$$[0 \to I_{p'} \to E' \to \Omega_{C'} \to 0] \in \operatorname{Ext}^1(\Omega_{C'}, I_{p'})$$

to the extension $[E] \in \operatorname{Ext}^1(\Omega_C, I_p)$ defined by the diagram

$$\begin{array}{ccccccccc}
 & & & & \kappa(p) & = & \kappa(p) & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & E & \xrightarrow{s} & \Omega_C & & \\
 & & & \swarrow & & & & & \\
0 & \longrightarrow & I_p & \longrightarrow & E & \longrightarrow & \Omega_C & \longrightarrow & 0 \\
 & & \| & & \downarrow & \swarrow_{\overline{s}} & \downarrow & & \\
0 & \longrightarrow & \pi_* I_{p'} & \longrightarrow & \pi_* E' & \longrightarrow & \pi_* \Omega_{C'} & \longrightarrow & 0.
\end{array}$$

We need to show that if E is the trivial extension, then so is E'. If $[E] = 0 \in \operatorname{Ext}^1(\Omega_C, I_p)$, let $s\colon \Omega_C \to E$ be a section of $E \to \Omega_C$. One checks that s is the identity on $\kappa(p)$ and therefore descends to a morphism $\overline{s}\colon \pi_* \Omega_{C'} \to \pi_* E'$. Adjunction gives an natural equivalence

$$\operatorname{Hom}_{\mathcal{O}_C}(\pi_* \Omega_{C'}, \pi_* E') = \operatorname{Hom}_{\mathcal{O}_C}(\pi^* \pi_* \Omega_{C'}, E')$$

By abuse of notation, denote by $\overline{s}$ the corresponding homomorphism $\pi^* \pi_* \Omega_{C'} \to E'$, and one checks that it descends to a map $s'\colon \Omega_{C'} \to E'$ as pictured:

$$\begin{array}{ccccccccc}
 & & & & \pi^* \pi_* \Omega_{C'} & & & & \\
 & & & \swarrow_{\overline{s}} & \downarrow & & & & \\
0 & \longrightarrow & I_{p'} & \longrightarrow & E' & \longrightarrow & \Omega_{C'} & \longrightarrow & 0 \\
 & & & & & \nwarrow_{s'} & & &
\end{array}$$

Thus the original extension E' was trivial. □

Exercise 4.5.10. Show that the above arguments can be modified to show that $\overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ is a universal family.

4.6 Irreducibility

In this section, we show that the algebraic stack $\overline{\mathcal{M}}_{g,n}$ is irreducible over any algebraically closed field $\Bbbk$. After reviewing properties of branched coverings in §4.6.1, we provide the classical topological argument due to Clebsch and Hurwitz in the late 19th century establishing irreducibility of M_g in characteristic 0 (Theorem 4.6.14). We then provide a purely algebraic argument for the irreducibility of $\overline{\mathcal{M}}_{g,n}$ (Theorem 4.6.17) by using admissible covers to show that every smooth curves degenerates to a singular stable curve and induction on the genus to show that the boundary $\delta = \overline{\mathcal{M}}_g \setminus \mathcal{M}_g$ is connected. Finally, in §4.6.4, we provide the arguments from the seminal papers from 1969 of Deligne and Mumford [DM69] and Fulton [Ful69] which establish the irreducibility of $\overline{M}_{g,n}$ in positive characteristic (where Fulton's argument has the restriction $p > g + 1$) by reduction to characteristic 0.

Before beginning, we make a few simple remarks regarding the relation between connectedness and irreducibility of $\overline{\mathcal{M}}_{g,n}$, between connectedness/irreducibility of $\overline{\mathcal{M}}_g$ versus $\mathcal{M}_g$, and between the connectedness/irreducibility of the stack versus coarse moduli space. Since $\overline{\mathcal{M}}_{g,n}$ is a smooth algebraic stack, its irreducibility is equivalent to its connectedness. Moreover, since $\overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ is the universal family (Proposition 4.5.8) and in particular has connected fibers, it suffices to verify the connectedness of $\overline{\mathcal{M}}_g$. We thus have equivalences

$$\begin{aligned} \overline{\mathcal{M}}_{g,n} \text{ irreducible} &\iff \overline{\mathcal{M}}_{g,n} \text{ connected} \\ &\iff \overline{\mathcal{M}}_g \text{ connected} \\ &\iff \mathcal{M}_g \text{ connected and dense in } \overline{\mathcal{M}}_g \end{aligned}$$

Finally, we note that since the coarse moduli space $\overline{\mathcal{M}}_{g,n} \to \overline{M}_{g,n}$ induces a homeomorphism $|\overline{\mathcal{M}}_{g,n}| \xrightarrow{\sim} |\overline{M}_{g,n}|$ on topological spaces, each statement above can be equivalently stated in terms of the coarse moduli space.

4.6.1 Branched coverings

Recall that a finite morphism $f\colon C \to \mathbb{P}^1$ is said to be *ramified at* $p \in \mathbb{P}^1$ *of index* $e(p)$ if the $\mathcal{O}_{C,p}$-module $(\Omega_{C/\mathbb{P}^1})_p$ is non-zero of length $e(p) - 1$. Note that by definition f is unramified at p precisely when $(\Omega_{C/\mathbb{P}^1})_p = 0$. If in addition f is flat or equivalently all associated points of f map to the generic point of $\mathbb{P}^1$ (which is guaranteed for instance if C is connected and reduced), then this is equivalent to f being étale at p.

Definition 4.6.1. A *branched covering of* $\mathbb{P}^1$ is a finite morphism $f\colon C \to \mathbb{P}^1$ from a smooth connected curve C such that the extension $K(\mathbb{P}^1) \to K(C)$ on functions fields is separable. We say that $f\colon C \to \mathbb{P}^1$ is *simply branched* if for each branched point $x \in \mathbb{P}^1$, there is at most one ramification point in the fiber $f^{-1}(x)$ and such a point has index 2.

Remark 4.6.2. Note that in the definition, we require C to be connected.

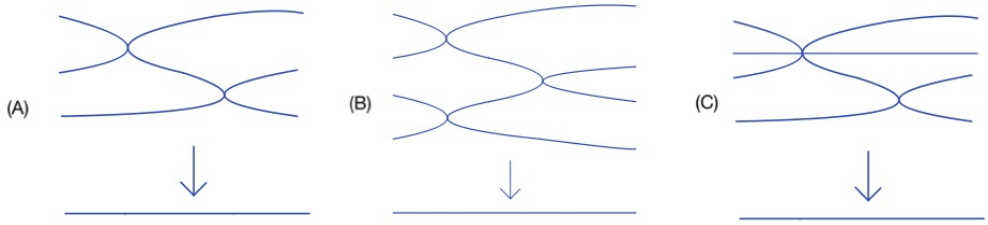


Figure 4.17: Examples of branched coverings: (A) is simply branched while (B) and (C) are not. While the picture may suggest that the source curve C is not smooth, C is in fact smooth over the base field $\Bbbk$. However, the map $C \to \mathbb{P}^1$ is not smooth and the pictures above are designed to reflect the singularities of C over $\mathbb{P}^1$.

Theorem 4.6.3 (Riemann–Hurwtiz)**.** *[Har77, Prop. IV.2.3] If $f\colon C \to \mathbb{P}^1$ is a branched covering and $R = \sum_{P\in C} \operatorname{length}(\Omega_{C/\mathbb{P}^1})_P \cdot P$ is the ramification divisor on C, then $\Omega_C \cong f^*\Omega_{\mathbb{P}^1} \otimes \mathcal{O}(R)$. In particular,*

$$2g(C) - 2 = \deg(f)(-2) + \deg R.$$

Riemann–Hurwitz implies that a simply branched covering $f\colon C \to \mathbb{P}^1$ is ramified over $b = 2g + 2d - 2$ distinct points.

Remark 4.6.4. More generally, if $C \to D$ is a finite morphism of smooth, connected and projective curves such that $K(D) \to K(C)$ is separable, then Riemann–Hurwitz states that $\Omega_C \cong f^*\Omega_D \otimes \mathcal{O}(R)$ and $2g(C)-2 = \deg(f)(2g(D)-2) + \deg R$.

Example 4.6.5. For a local model of a branched cover, consider the map $f\colon \mathbb{A}^1_{\Bbbk} \to \mathbb{A}^1_{\Bbbk}$ defined by $x \mapsto x^n$. The relative sheaf of differentials is $\Omega_{\mathbb{A}^1_{\Bbbk}/\mathbb{A}^1_{\Bbbk}} = k[x]\langle dx\rangle/(nx^{n-1}dx)$ and thus if $\operatorname{char}(\Bbbk)$ does not divide n, then $f\colon \mathbb{A}^1_{\Bbbk} \to \mathbb{A}^1_{\Bbbk}$ is étale over $\mathbb{A}^1_{\Bbbk}$ and ramified at 0 with index $n-1$.

Exercise 4.6.6. Show that any branched covering is étale-locally isomorphic to $\mathbb{A}^1_{\Bbbk} \to \mathbb{A}^1_{\Bbbk}, x \mapsto x^n$ around a branched point of index $n-1$.

Lemma 4.6.7. *Let C be a smooth, projective and connected curve of genus g over an algebraically closed field $\Bbbk$ of characteristic 0. If L is a line bundle of degree $d \geq g+1$, then for a general linear series $V \subset H^0(C, L)$ of dimension 2, $C \xrightarrow{V} \mathbb{P}^1$ is simply branched.*

Proof. We proceed with a dimension count. Since $h^0(C, L) = d + 1 - g$, the dimension of the Grassmanian $\operatorname{Gr}(2, H^0(L))$ of 2-dimensional subspaces is $2(d - g - 1)$. Since $\operatorname{char}(\Bbbk) = 0$, any finite morphism $C \to \mathbb{P}^1$ is automatically separable. Thus, if $C \xrightarrow{V} \mathbb{P}^1$ is not simply branched, then one of the following three conditions must hold:

(a) V has a base point;

(b) there exists a ramification point with index > 2; or

(c) there exists 2 ramification points in the same fiber.

We handle only case (b) and leave the other cases to the reader. There must exist a section $s \in V$ vanishing to order 3 at a point $p \in C$, i.e. $s \in H^0(C, L(-3p))$.

The dimension of $V \in \mathrm{Gr}(2, H^0(L))$ having a branched point at $p \in C$ with index at least 3 can be calculated as

$$\dim \mathbb{P}H^0(L(-3p)) + \dim \mathbb{P}(H^0(L)/\langle s \rangle) = 2d - 2g - 4.$$

Varying $p \in C$, the locus of all $V \in \mathrm{Gr}(2, H^0(L))$ failing condition (b) is thus $2d - 2g - 3 = \dim \mathrm{Gr}(2, H^0(L)) - 1$. □

For a branched cover $C \to \mathbb{P}^1$, we denote by $\mathrm{Aut}(C/\mathbb{P}^1) = 1$ the group of automorphisms $C \to C$ over $\mathbb{P}^1$.

Lemma 4.6.8. *If $C \to \mathbb{P}^1$ is a simply branched cover of degree $d > 2$ in characteristic 0, then $\mathrm{Aut}(C/\mathbb{P}^1)$ is trivial.*

Proof. Any automorphism $C \to C$ over $\mathbb{P}^1$ must fix the $2g+2d-2$ branched points but this contradicts the classical result of Mayer which asserts that there are no non-trivial automorphisms of a smooth curve fixing more than $2g + 2$ points. □

The above lemma shows that there are no stacky issues that arises when defining moduli spaces of simply branched covering. We define

$$H_{d,b} := \{C \to \mathbb{P}^1 \text{ simply branched covering of degree } d \text{ over } b \text{ points}\}$$

as the moduli space simply branched coverings where $b = 2g + 2d - 2$. The moduli space $H_{d,b}$ can be defined either as a topological space (if $\Bbbk = \mathbb{C}$) or as an algebraic space; we leave the details to the reader. Denoting $\mathrm{Sym}^b \mathbb{P}^1 \setminus \Delta$ as the variety of b unordered distinct points in $\mathbb{P}^1$ (which can also be written as the complement $\mathbb{P}^b \setminus \Delta$ of the discriminant hypersurface), we have a diagram

$$\begin{array}{ccc} & H_{d,b} & \\ \swarrow & & \searrow \\ \mathcal{M}_g & & \mathrm{Sym}^b \mathbb{P}^1 \setminus \Delta \end{array} \tag{4.6.1}$$

where a simple branched covering $[C \to \mathbb{P}^1]$ gets mapped to $[C]$ under $H_{d,b} \to \mathcal{M}_g$ and the b branched points under $H_{d,b} \to \mathrm{Sym}^b \mathbb{P}^1 \setminus \Delta$.

Lemma 4.6.9. *In characteristic 0, the morphism $H_{d,b} \to \mathrm{Sym}^b \mathbb{P}^1 \setminus \Delta$ is finite and étale.*

Proof. We only establish étaleness. It is straightforward to see that $H_{d,b} \to \mathrm{Sym}^b \mathbb{P}^1 \setminus \Delta$ is a topological covering space. Consulting Figure 4.18, given a branched covering $f\colon C \to \mathbb{P}^1$ and a branched point $p \in C$, we can choose an analytic open neighborhood $U \subset \mathbb{P}^1$ around $f(p)$ such that $f^{-1}(U) \to U$ is isomorphic to an open neighborhood of $\mathbb{C} \to \mathbb{C}, x \mapsto x^n$. For any other point $q' \in U$, we can construct a branched cover $C' \to \mathbb{P}^1$ which outside U is the same as $C \to \mathbb{P}^1$ and over U is locally isomorphic to $x \mapsto x^n$ but centered over q' (rather than $f(p)$).

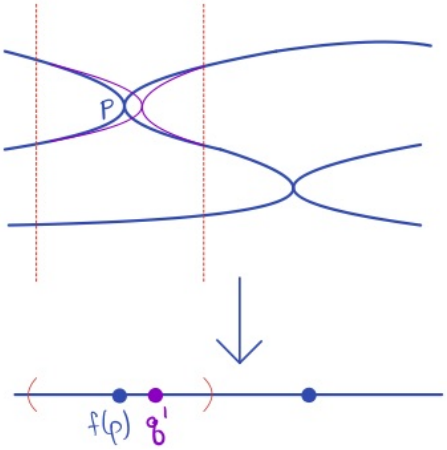


Figure 4.18:

For an algebraic argument, it suffices to show that for a covering $f\colon C \to \mathbb{P}^1$ simply branched over $p_1, \ldots, p_b$, the map

$$\mathrm{Def}(C \xrightarrow{f} \mathbb{P}^1) \to \mathrm{Def}(\{p_i\}_{i=1}^b \subset \mathbb{P}^1)$$

on first order deformation spaces is bijective. There is an identification $\mathrm{Def}(C \xrightarrow{f} \mathbb{P}^1) = H^0(C, N_f)$ where N_f sits in a short exact sequence

$$0 \to T_C \to f^* T_{\mathbb{P}^1} \to N_f \to 0.$$

On cohomology, this induces a short exact sequence

$$0 \to H^0(C, f^* T_{\mathbb{P}^1}) \to H^0(C, N_f) \to H^1(C, T_C) \to 0.$$

Riemann–Roch allows us to compute $h^0(C, f^* T_{\mathbb{P}^1}) = 2d+1-g$ and $h^1(T_C) = 3g-3$, and thus $\dim \mathrm{Def}(C \xrightarrow{f} \mathbb{P}^1) = h^0(C, N_f) = 2d + 2g - 2 = b$ is the same as the dimension of $\mathrm{Def}(\{p_i\}_{i=1}^b \subset \mathbb{P}^1)$. We leave the remaining details to the reader. □

Relation between algebraic and topological branched coverings

The Clebsch–Hurwitz argument below relies on the following correspondence between topological, analytic, and algebraic branched coverings. (Topological and analytic coverings can be defined analogously to algebraic coverings—to be added.) This can be viewed as a version of the Riemann Existence Theorem.

Proposition 4.6.10. *Over $\mathbb{C}$, there are natural bijections*

$$\{C \to \mathbb{P}^1 \textit{ algebraic branched coverings}\} \longleftrightarrow \{C \to \mathbb{P}^1 \textit{ topological branched coverings}\} \longleftrightarrow \{C \to \mathbb{P}^1 \textit{ analytic branched coverings}\}$$

Proof. An algebraic branched covering is clearly topological and if $C \to \mathbb{P}^1$ is a topological covering, then the holomorphic structure on $\mathbb{P}^1$ induces naturally a holomorphic structure on C such that $C \to \mathbb{P}^1$ is analytic. The Riemann Existence Theorem implies than any holomorphic branched covering is in fact algebraic. □

Monodromy actions

Let $C \to \mathbb{P}^1$ be a (topological) branched covering over $\mathbb{C}$ and $B \subset \mathbb{P}^1$ its ramification locus. Choose a base point $p \in \mathbb{P}^1 \setminus B$. The *monodromy action of* $\pi_1(\mathbb{P}^1 \setminus B, p)$ *on the fiber* $\pi^{-1}(p)$ is defined as follows: for $\gamma \in \pi_1(\mathbb{P}^1 \setminus B, p)$ and $q \in \pi^{-1}(p)$, then the path $\gamma\colon [0,1] \to \mathbb{P}^1$ lifts uniquely to a path $\widetilde{\gamma}\colon [0,1] \to C$ such that $\widetilde{\gamma}(0) = q$ and the action is defined by $\gamma \cdot q = \widetilde{\gamma}(1)$.

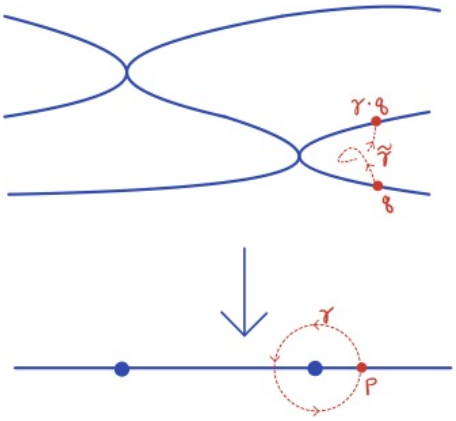


Figure 4.19:

We now summarize some of the key properties of the monodromy action.

Proposition 4.6.11. *Let $B \subset \mathbb{P}^1$ be a finite subset, $p \in \mathbb{P}^1 \setminus B$ be a point, and $d > 0$ a positive integer. There is a natural bijection between topological branched coverings $C \to \mathbb{P}^1$ of degree d and group homomorphisms $\rho\colon \pi_1(X \setminus B, x) \to S_d$ such that* $\mathrm{im}(\rho) \subset S_d$ *is a transitive subgroup. Here two branched covers $C \to \mathbb{P}^1$ and $C' \to \mathbb{P}^1$ are equivalent if there is an isomorphism $C \to C'$ over $\mathbb{P}^1$, and two homomorphisms $\rho, \rho'\colon \pi_1(X \setminus B, x) \to S_d$ are equivalent if they differ by an inner automorphism of S_d, i.e. $\exists h \in S_d$ such that $\rho' = h^{-1}\rho h$.*

Moreover, if we let $\sigma_1, \ldots, \sigma_b$ be simple loops around the b distinct points of B, then $\pi_1(\mathbb{P}^1 \setminus B, x) = \langle \sigma_i | \sigma_1 \cdots \sigma_b = 1 \rangle$, and under this correspondence a simply branched cover corresponds to a homomorphism $\pi_1(X \setminus B, p) \to S_d$ such that each σ_i maps to a transposition.

Remark 4.6.12. Recall that by definition in a branched covering $C \to \mathbb{P}^1$, the curve C is necessarily connected. This is the reason for the condition above that $\mathrm{im}(\rho) \subset S_d$ is transitive: any group homomorphism $\rho\colon \pi_1(X \setminus B, x) \to S_d$ corresponds to a possibly non-connected branched covering $C \to \mathbb{P}^1$, and C is connected if and only if $\mathrm{im}(\rho) \subset S_d$ is transitive.

Remark 4.6.13. Like with Riemann–Hurwitz, the fact that the base is $\mathbb{P}^1$ plays no role: the above proposition holds for arbitrary branched covers of smooth curves (except for the explicit description of π_1).

4.6.2 The Clebsch–Hurwitz argument

We now provide the classical argument due to Clebsch [Cle73] and Hurwitz [Hur91] that $\mathcal{M}_g$ is connected over $\mathbb{C}$. For a modern treatment, see [Ful69, §1].

This argument uses a single non-algebraic input, namely Riemann's Existence Theorem in the form of Proposition 4.6.10. There are of course other non-algebraic approaches, e.g. using Teichmüller theory.

By taking $d \geq g+1$, we know that every smooth, projective and connected complex curve C of genus g admits a map $C \to \mathbb{P}^1$ which is a covering simply branched over $b = 2d+2g-2$ points (Lemma 4.6.7). This shows that the map

$$H_{d,b} \to \mathcal{M}_g, \qquad [C \to \mathbb{P}^1] \mapsto [C]$$

is surjective, where $H_{d,b}$ is the moduli space of coverings $C \to \mathbb{P}^1$ simply branched over b points. The connectedness of $H_{d,b}$ thus implies the connectedness of $\mathcal{M}_g$.

Theorem 4.6.14 (Clebsch, Hurwitz)**.** *$H_{d,b}$ is connected.*

Proof. We will use the diagram

$$\begin{array}{ccc} & H_{d,b} & \\ {\scriptstyle}\swarrow & & \searrow{\scriptstyle \beta} \\ \mathcal{M}_g & & \operatorname{Sym}^b \mathbb{P}^1 \setminus \Delta \end{array}$$

where $H_{d,b} \to \mathcal{M}_g$ is surjective (Lemma 4.6.7) and $\beta\colon H_{d,b} \to \operatorname{Sym}^b \mathbb{P}^1 \setminus \Delta$ is finite and étale (Lemma 4.6.9).

For any finite set $B = \{p_1, \dots, p_b\} \subset \mathbb{P}^1$ of $b = 2d+2g-2$ points and $p \in \mathbb{P}^1 \setminus B$, the fundamental group $\pi_1(\mathbb{P}^1 \setminus B, p) = \langle \sigma_i | \sigma_1 \cdots \sigma_b = 1 \rangle$ acts on the fiber $\pi^{-1}(p)$ of a simply branched covering $\pi\colon C \to \mathbb{P}^1$. Similarly, $\pi_1(\operatorname{Sym}^b \mathbb{P}^1 \setminus \Delta, B)$ acts on the fiber $H_{d,B} := \beta^{-1}(B)$ of $\beta\colon H_{d,b} \to \operatorname{Sym}^b \mathbb{P}^1 \setminus \Delta$. Using Proposition 4.6.11, we have bijections

$$\begin{aligned} H_{d,b} = \beta^{-1}(B) &= \{\text{coverings } C \to \mathbb{P}^1 \text{ simply branched over } B\} \\ &= \{\text{group homomorphisms } \pi_1(\mathbb{P}^1 \setminus B, p) \xrightarrow{\rho} S_d \text{ such that} \\ &\qquad \operatorname{im}(\rho) \subset S_d \text{ is transitive and each } \rho(\sigma_i) \text{ is a transposition}\} \\ &= \{(\tau_1, \dots, \tau_b) \in (S_d)^b \mid \text{each } \tau_i \text{ is a transposition and } \tau_1 \cdots \tau_b = 1\}. \end{aligned}$$

The connectedness of $H_{d,b}$ is equivalent to the transitivity of the action of $\pi_1(\operatorname{Sym}^b \mathbb{P}^1 \setminus \Delta, B)$ on the fiber $H_{d,B}$. The strategy of proof is to find loops in $\operatorname{Sym}^b \mathbb{P}^1 \setminus \Delta$ that act on $(\tau_1, \dots, \tau_b) \in H_{d,B}$ in a prescribed way and to find enough loops so that we can show that each orbit contains the element

$$\boldsymbol{\tau}^* := \big(\underbrace{(12), (12), (13), (13), \dots, (1\,d-1), (1\,d-1)}_{2(d-2)}, \underbrace{(1d), (1d), \cdots, (1d)}_{2g+2} \big).$$

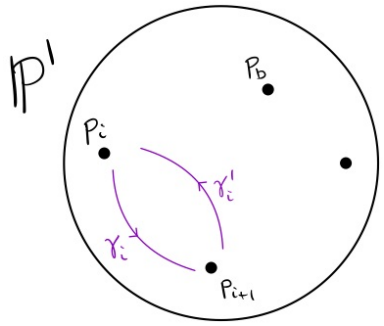


Figure 4.20:

Referring to Figure 4.20, we define the loop

$$\begin{aligned}\Gamma_i \colon [0,1] &\to \operatorname{Sym}^b \mathbb{P}^1 \setminus \Delta \\ t &\mapsto (p_1, \ldots, p_{i-1}, \gamma_i(t), \gamma_i'(t), p_{i+2}, \ldots, p_b).\end{aligned}$$

One checks that

$$\Gamma_i \cdot (\tau_1, \ldots, \tau_b) = (\tau_1, \ldots, \tau_{i-1}, \tau_i^{-1}\tau_{i+1}\tau_i, \tau_i, \tau_{i+2}, \ldots, \tau_b)$$

and that for any element $(\tau_1, \ldots, \tau_b) \in H_{d,B}$, there exists a sequence $\Gamma_{i_1}, \ldots, \Gamma_{i_k}$ of loops such that $\boldsymbol{\tau}^* = \Gamma_{i_1}\Gamma_{i_2}\cdots\Gamma_{i_k} \cdot (\tau_1, \ldots, \tau_b)$. We leave the details of this combinatorial problem to the reader.

□

4.6.3 Irreducibility using admissible covers

We now give a completely algebraic argument of the irreducibility of $\overline{\mathcal{M}}_g$ in characteristic 0. The main idea is to show that every smooth curve of genus g degenerates in a one-dimensional family to a singular stable curve (Proposition 4.6.15) and to show the connectedness of $\delta = \overline{\mathcal{M}}_g \setminus \mathcal{M}_g$ using the inductive structure of the boundary and explicitly the gluing maps of Proposition 4.5.1. The most challenging aspect of this argument is in degenerating a smooth curve to a singular stable curve. To achieve this, we will use the theory of admissible covers. We follow the treatment in Fulton's appendix of the paper [HM82] by Harris and Mumford that introduced admissible covers as a means to compute the Kodaira dimension of $\overline{M}_g$.

Proposition 4.6.15. *Let C be a smooth, projective and connected curve of genus g over an algebraically closed field $\Bbbk$ of characteristic 0. There exists a connected curve T with points $t_1, t_2 \in T$ and a family $\mathcal{C} \to T$ of stable curves such that $\mathcal{C}_{t_1} \cong C$ and $\mathcal{C}_{t_2}$ is a single stable curve.*

Proof. By Lemma 4.6.7, for $d \gg 0$ there exists a finite covering $C \to \mathbb{P}^1$ of degree d simply branched over $b = 2g + 2d - 2$ distinct points $p_1, \ldots, p_b \in \mathbb{P}^1$. This defines a b-pointed stable curve $G = [\mathbb{P}^1, \{p_i\}] \in M_{0,n}$. By Lemma 4.6.9, we may assume that $G \in M_{0,n}$ is general. Since $\overline{M}_{0,n}$ is connected, G degenerates to the b-pointed rational curve $(D_0, q_1, \ldots, q_b)$ which is the nodal union of a chain of $b-2$ $\mathbb{P}^1$'s where q_1, q_2 lie on the first $\mathbb{P}^1$, q_3 on the second $\mathbb{P}^1$, and so on with q_{b-1}, q_b lying on the last $\mathbb{P}^1$; see Figure 4.21.

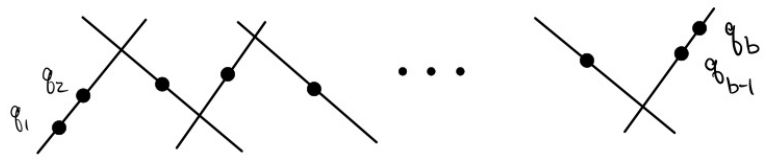


Figure 4.21:

In other words, there is a DVR R with fraction field K and a map $\Delta = \operatorname{Spec} R \to \overline{M}_{0,n}$ corresponding to a b-pointed stable family $(\mathcal{D} \to \Delta, \sigma_i)$ such that the generic fiber $(\mathcal{D}^*, \sigma_i^*)$ is isomorphic to $G = (\mathbb{P}^1, \{g_i\})$ and the special fiber to $(D_0, \{q_i\})$. We have a simply branched covering $\mathcal{C}^* \to \Delta^*$ which fits into a diagram

$$\begin{array}{ccc} \mathcal{C}^* & \dashrightarrow & \mathcal{C} \\ \downarrow & & \vdots \\ \mathcal{D}^* & \hookrightarrow & \mathcal{D} \\ \downarrow & & \downarrow \\ \Delta^* & \hookrightarrow & \Delta \end{array}$$

and extends to a finite morphism $\mathcal{C} \to \mathcal{D}$ by taking $\mathcal{C}$ as the integral closure of $\mathcal{O}_{\mathcal{D}}$ in $K(\mathcal{C}^*)$.

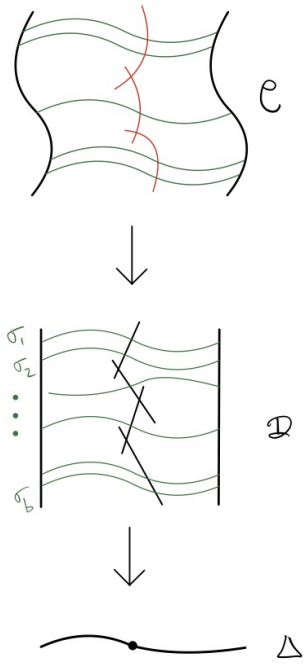


Figure 4.22:

Purity of the branch locus implies that the ramification of $\mathcal{C} \to \mathcal{D}$ is a divisor when restricted to the relative smooth locus of $\mathcal{C} \to \mathcal{D}$. Therefore, the central fiber

$\mathcal{C}_0 \to \mathcal{D}_0$ is ramified over $\sigma_1(0), \dots, \sigma_b(0)$ and possibly over irreducible components of $\mathcal{D}_0$ (where $\mathcal{C}_0$ may be non-reduced). As in the proof of stable reduction, after a suitable base change $\Delta \to \Delta, t \mapsto t^m$ and replacing $\mathcal{C}$ with the normalization $\mathcal{C} \times_\Delta \Delta$, we can arrange that $\mathcal{C}_0 \to \mathcal{D}_0$ is ramified only over $\sigma_i(0)$ and possibly over nodes of $\mathcal{D}_0$. By an analysis of possible extensions $\mathcal{C} \to \mathcal{D}$, one can show that $\mathcal{C}_0$ is a nodal curve (missing details). Therefore $\mathcal{C} \to \Delta$ is a family of nodal curves.

Since $\mathcal{C}_0$ necessarily has nodes, we are done if $\mathcal{C}_0$ is a stable curve! Otherwise, we can contract rational tails and bridges to obtain the stable model $\mathcal{C}^{\mathrm{st}} \to \Delta$ (Proposition 4.2.13). We must check that $\mathcal{C}_0^{\mathrm{st}}$ is not smooth. Let $T \subset \mathcal{C}_0^{\mathrm{st}}$ be any smooth irreducible component. Applying Riemann–Hurwitz to the induced morphism $T \to \mathbb{P}^1 \subset \mathcal{D}_0$ shows that $2g(T) - 2 = -2d + R$ where R is the degree of the ramification divisor on T. If the component $\mathbb{P}^1 \subset \mathcal{D}_0$ is a rational tail (i.e. is either the first or last $\mathbb{P}^1$ in the chain), then $R \leq 2 + (d-1)$ as $T \to \mathbb{P}^1$ is simply ramified over the two marked points and has index at worst $d-1$ over the node. On the other hand, if $\mathbb{P}^1 \subset \mathcal{D}_0$ is a rational bridge, then $R \leq 1 + 2(d-1)$. In either case, we have $R \leq 2d - 1$ and $2g(T) - 2 \leq -2 + (2d-1) = 1$ which establishes that $g(T) = 0$. We've shown every smooth irreducible component of $\mathcal{C}_0^{\mathrm{st}}$ is rational which immediately implies that $\mathcal{C}_0^{\mathrm{st}}$ is singular. □

Proposition 4.6.16. *If we assume that $\overline{\mathcal{M}}_{g',n'}$ is irreducible for all $g' < g$, then the boundary $\delta = \overline{\mathcal{M}}_g \setminus \mathcal{M}_g$ is connected.*

Proof. We write $\delta = \delta_0 \cup \cdots \cup \delta_{\lfloor g/2 \rfloor}$ where $\delta_0 = \mathrm{im}(\overline{\mathcal{M}}_{g-1,2} \to \overline{\mathcal{M}}_g)$ and $\delta_i = \mathrm{im}(\overline{\mathcal{M}}_{i,1} \times \overline{\mathcal{M}}_{g-i,1} \to \overline{\mathcal{M}}_g)$ as defined in §4.5.2 using the gluing maps from Proposition 4.5.1. The hypotheses imply that δ_0 and δ_i are connected (and even irreducible). But on the other hand, the boundary divisors δ_i intersect! Namely, for any $i, j = 0, \dots, \lfloor g/2 \rfloor$, the intersection $\delta_i \cap \delta_j$ contains curves as in Figure 4.23.

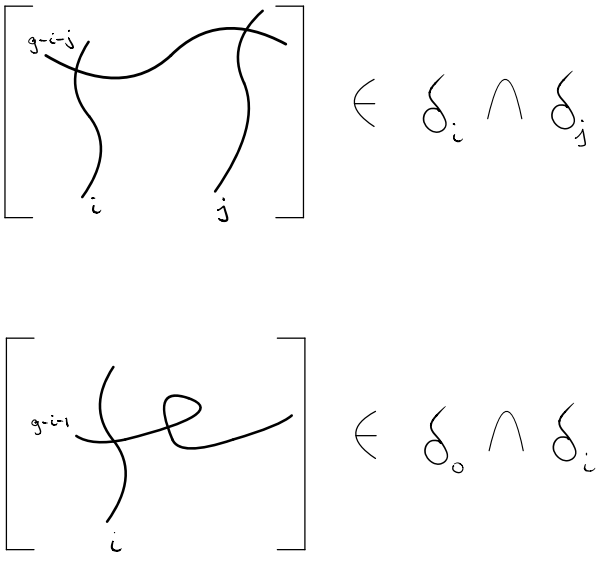


Figure 4.23:

□

Theorem 4.6.17. *$\overline{\mathcal{M}}_{g,n}$ is irreducible.*

Proof. Since $\overline{\mathcal{M}}_{g,n}$ is smooth (Theorem 4.3.8), the irreducibility of $\overline{\mathcal{M}}_{g,n}$ is equivalent to its connectedness. Since $\overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ is the universal family (Proposition 4.5.8) and in particular has connected fibers, it suffices to verify the connectedness of $\overline{\mathcal{M}}_g$. Since every smooth curve degenerates to a stable singular curve in the boundary $\delta = \overline{\mathcal{M}}_g \setminus \mathcal{M}_g$ (Proposition 4.6.15) and the boundary δ itself is connected (Proposition 4.6.16) by induction on g, we obtain that $\overline{\mathcal{M}}_g$ is connected. □

Remark 4.6.18 (Admissible Covers)**.** The above argument was motivated by the theory of admissible covers as introduced by Harris and Mumford [HM82]. Admissible covers are a generalization of simply branched covers $C \to \mathbb{P}^1$ where the source *and* target curve are allowed to have nodal singularities. The main goal is to extend the map $\mathcal{H}_{d,b} \to \mathcal{M}_g$ taking $[C \to \mathbb{P}^1] \to [C]$ to a map $\overline{\mathcal{H}}_{d,b} \to \overline{\mathcal{M}}_g$ over the boundary where $\overline{\mathcal{H}}_{d,b}$ also has a moduli interpretation.

An *admissible cover of degree d* over a stable b-pointed genus 0 curve $(B, p_1, \dots, p_b)$ is a morphism $f\colon C \to B$ such that

(a) $f^{-1}(B^{\mathrm{sm}}) = C^{\mathrm{sm}}$ and $C^{\mathrm{sm}} \to B^{\mathrm{sm}}$ is simply branched of degree d over the points p_i, i.e. each ramification index is 2 and there is at most one ramification point in every fiber; and

(b) for every node $q \in B$ and every node $r \in C$ over q, the local structure (either formally or étale) of $C \to B$ at r is of the form $\Bbbk[x,y]/(xy) \to \Bbbk[x,y]/(xy)$ defined by $(x,y) \mapsto (x^m, y^m)$ for some m.

This definition extends to families of admissible covers and the stack $\overline{\mathcal{H}}_{d,b}$ parameterizing admissible covers of degree d branched over b points is a proper Deligne–Mumford stack.

The total space C of an admissible cover need not be stable. Nevertheless, using the contraction morphism (Proposition 4.2.13), there is a morphism $\overline{\mathcal{H}}_{d,b} \to \overline{\mathcal{M}}_g$ sending an admissible cover $[C \to B]$ to the stable model C^{st} of C. There is also a *finite* morphism $\overline{\mathcal{H}}_{d,b} \to \overline{M}_{0,n}$ sending $[C \to B]$ to $(B, \{p_i\})$ where $p_i \in B$ are the branched points. To summarize, there is a diagram

$$\begin{array}{ccc} & \overline{\mathcal{H}}_{d,b} & \\ \swarrow & & \searrow \\ \overline{\mathcal{M}}_g & & \overline{M}_{0,n} \end{array}$$

extending the uncompactified diagram (4.6.1).

The argument of Proposition 4.6.15 can be rewritten in this language. For $d \gg 0$, given a smooth curve $[C] \in \mathcal{M}_g$, we choose a preimage $[C \to \mathbb{P}^1] \in \overline{\mathcal{H}}_{d,b}$ (Lemma 4.6.7). By Lemma 4.6.9, we can assume that the branched points $g_1, \dots, g_b \in \mathbb{P}^1$ are general. Since $\overline{M}_{0,n}$ is connected, there is a map $\Delta = \operatorname{Spec} R \to \overline{M}_{0,n}$ (where R is a DVR) such that the generic point maps to $(\mathbb{P}^1, \{g_i\})$ and the closed points maps to the b-pointed stable curve $(D_0, q_1, \dots, q_b)$ of Figure 4.21. Since $\overline{\mathcal{H}}_{d,b} \to \overline{M}_{0,n}$ is finite, we may use the valuative criterion to lift $\Delta \to \overline{M}_{0,n}$ to $\Delta \to \overline{\mathcal{H}}_{d,b}$ such that the image of the generic point is $[C \to \mathbb{P}^1]$. The composition $\Delta \to \overline{\mathcal{H}}_{d,b} \to \overline{\mathcal{M}}_g$ gives the desired degeneration.

4.6.4 Irreducibility in positive characteristic: Deligne–Mumford and Fulton's arguments

The year 1969 was a remarkable year for mathematics in part due to the seminal contributions of Deligne and Mumford's paper [DM69] and Fulton's paper [Ful69]. The papers provided independent arguments for the irreducibility of $\overline{M}_g$ in positive characteristic (where Fulton's argument has the restriction that $p > g+1$). Both papers relied on the connectedness of M_g over $\mathbb{C}$ and the time, there was no purely algebraic argument; the algebraic argument establishing Theorem 4.6.17 used admissible covers and became available only in 1982. The connectedness of M_g over $\mathbb{C}$ is a classical result. Clebsch and Hurwitz's arguments in the 19th century (featured in Theorem 4.6.14) used the Hurwitz space of branched covers and used on a single non-algebraic input, namely the Riemann's Existence Theorem. There are of course other non-algebraic arguments, e.g. using the Teichmüller space.

Deligne–Mumford's first argument

The first argument appearing [DM69] is very similar in spirit to the argument in §4.6.3. As with most results, there are many approaches to construct a proof and the first approach in [DM69, §3] reflects the state of technology at the time.

For a field $\Bbbk$ of characteristic p, the argument for irreducibility of $M_g \times_{\mathbb{Z}} \Bbbk$ proceeds along three steps:

Step 1: There is no proper connected component of $M_g \times_{\mathbb{Z}} \Bbbk$.

Let $W(\Bbbk)$ be the Witt vectors for $\Bbbk$; $W(\Bbbk)$ is a complete local noetherian ring whose generic point η has characteristic 0 and whose closed point 0 has residue field is $\Bbbk$. (For example, $W(\mathbb{F}_p) = \mathbb{Z}_p$ is the ring of p-adics.) We now use the existence of a quasi-projective coarse moduli space $\mathcal{M}_g \to M_g$ over $W(\Bbbk)$ as established in [GIT]. (Although appearing in the definitive book on GIT, this would not be viewed as a "GIT construction" today as it relies on some ad hoc techniques and doesn't use the Hilbert–Mumford criterion. Indeed, the standard GIT toolkit only became available in positive characteristic in 1975 after Haboush resolved Mumford's conjecture [Hab75] and in the relative setting in 1977 after Seshadri's paper [Ses77].)

Choosing a projective compactification $M_g \subset X$ over $W(\Bbbk)$, the connectedness of the generic fiber of $M_g \to \operatorname{Spec} W(\Bbbk)$ ensures that the generic fiber of X_η is also connected. The scheme M_g is normal as GIT quotients (or alternatively coarse moduli spaces) preserve normality. By taking the normalization of X, we can assume that X is also normal. Zariski's connectedness theorem implies that the number of connected components in a fiber X_w is independent of $w \in W(\Bbbk)$. Thus, X_0 is also connected.

Suppose $Y \subset M_g \times_{W(\Bbbk)} \Bbbk$ is a proper connected component. Then $Y \subset M_g \times_{W(\Bbbk)} \Bbbk \subset X_0$ is an open subscheme; but it's also a closed subscheme since Y is proper. Since X_0 is connected, we conclude that $Y = M_g \times_{W(\Bbbk)} \Bbbk$ is proper and irreducible. To obtain a contradiction, denote by $A_{g,\Bbbk}$ the moduli of principally polarized g-dimensional abelian varieties over $\Bbbk$ and consider the morphism

$$\Theta\colon M_g \times_{W(\Bbbk)} \Bbbk \to A_{g,\Bbbk}, \qquad C \mapsto \operatorname{Jac}(C)$$

assigning to a smooth curve C its Jacobian $\operatorname{Jac}(C)$. The properness of $M_g \times_{W(\Bbbk)} \Bbbk$ implies that the image would be a closed but there are explicit examples where the closure of the image of Θ contains products of lower dimensional Jacobians.

Step 2: There is no connected component of $\overline{M}_g \times_{\mathbb{Z}} \Bbbk$ consisting entirely of smooth curves.

Let $\overline{M}_{g,1}, \dots, \overline{M}_{g,r}$ be the connected components of $\overline{M}_g$. For each i, Step 1 implies that $\overline{M}_{g,i}$ is not proper. Let $\Delta = \operatorname{Spec} \Bbbk[[t]]$ and $\Delta^* = \operatorname{Spec} \Bbbk((t)) \to M_{g,i} := M_g \cap \overline{M}_{g,i}$ be a morphism that does not extend to Δ. By Stable reduction, after possibly replacing Δ with a finite extension, $\Delta^* \to M_{g,i}$ extends to a morphism $\Delta \to \overline{M}_g$. This shows that $\overline{M}_{g,i} \setminus M_{g,i}$ is non-empty.

Step 3: The boundary $\delta = \overline{M}_g \setminus M_g$ is connected.

Note that Steps 1 and 2 show that every smooth curve degenerates to a singular stable curve (Proposition 4.6.15). This step proceeds precisely as in Proposition 4.6.16 but without using the formalism of the moduli $\overline{\mathcal{M}}_{g,n}$ of n-pointed stable curves and the gluing morphisms.

Deligne–Mumford's second argument

The stack $\overline{\mathcal{M}}_g$ of stable curves is smooth and proper over $\operatorname{Spec} \mathbb{Z}$. Zariski's connectedness theorem implies that for any smooth and proper morphism $X \to Y$ of schemes, the number of connected components of a geometric fiber is a locally constant function on Y. (In fact, for a flat and proper morphism $X \to Y$, this function is lower semi-continuous and it is enough for the fibers of $X \to Y$ to be geometrically normal in order to show constancy.) This fact extends to morphisms of algebraic stacks. Applying this fact to the morphism $\overline{\mathcal{M}}_g \to \operatorname{Spec} \mathbb{Z}$, we see that the connectedness of any geometric fiber follows from the connectedness of $\overline{\mathcal{M}}_g \times_{\mathbb{Z}} \mathbb{C}$. In [DM69, §5], the connectedness of $\overline{\mathcal{M}}_g \times_{\mathbb{Z}} \mathbb{C}$ is argued by relating it to the moduli of Teichmüller structures of level n and the connectedness of the Teichmüller space [Man39].

Fulton's argument

In [Ful69], Fulton defines the Hurwitz scheme $H_{d,b}$ of simply branched covers over $\mathbb{Z}$ and shows that there is a diagram

$$\begin{array}{ccc} & H_{d,b} & \\ \swarrow & & \searrow \\ \mathcal{M}_g & & \operatorname{Sym}^b \mathbb{P}^1 \setminus \Delta \end{array}$$

defined over $\mathbb{Z}$. He shows that the map $H_{d,b} \to \operatorname{Sym}^d \mathbb{P}^1 \setminus \Delta$, taking a simply branched cover to its branch locus, is étale. Moreover, if all primes $p \leq g+1$ are inverted, then $H_{d,b} \to \operatorname{Sym}^d \mathbb{P}^1 \setminus \mathbb{P}^1$ is finite; examples are given where is not finite over primes $p \leq g+1$. Fulton then establishes a "reduction theorem" allowing him to deduce the connectedness of $H_{d,b} \times_{\mathbb{Z}} \overline{\mathbb{F}}_p$ from $H_{d,b} \times_{\mathbb{Z}} \mathbb{C}$ for primes $p > g+1$.

4.7 Projectivity

In this section, we prove that the coarse moduli space $\overline{M}_{g,n}$ is projective (Theorem 4.7.14). We follow the approach introduced by Kollár in [Kol90] partially building on ideas of Viehweg (see [Vie95]). We will primarily focus on the unpointed coarse moduli space $\overline{M}_g$ as this will be enough to deduce the projectivity of $\overline{M}_{g,n}$.

To introduce the general strategy to establish projectivity, we need to introduce some terminology. Let $\pi\colon \mathcal{U}_g \to \overline{\mathcal{M}}_g$ be the universal family and for each integer $k \geq 1$ define the *kth pluri-canonical bundle* as the vector bundle

$$\pi_*(\omega_{\mathcal{U}_g/\overline{\mathcal{M}}_g}^{\otimes k}) \tag{4.7.1}$$

on $\overline{\mathcal{M}}_g$. Its rank $r(k)$ can be computed via Rieman–Roch:

$$r(k) := \begin{cases} g & \text{if } k = 1 \\ (2k-1)(g-1) & \text{if } k > 1. \end{cases} \tag{4.7.2}$$

We obtain line bundles on $\overline{\mathcal{M}}_g$ by taking the determinant

$$\lambda_k := \det \pi_*(\omega_{\mathcal{U}_g/\overline{\mathcal{M}}_g}^{\otimes k}).$$

These provide natural candidates of line bundles on $\overline{\mathcal{M}}_g$ that descend to ample line bundles on $\overline{M}_g$.

Strategy for projectivity: Show that for $k \gg 0$, a positive power of λ_k descends to an *ample* line bundle on the coarse moduli space $\overline{M}_g$.

Outline of this section: In §4.7.1, we prove Kollar's Criterion for ampleness (Theorem 4.7.5). In §4.7.2, we setup the application of Kollár's Criterion to $\overline{M}_g$ by establishing Proposition 4.7.13: projectivity of $\overline{M}_g$ follows from (a) Stable Reduction (Theorem 4.4.1) and (b) the nefness of $\pi_*(\omega_{\mathcal{C}/T}^{\otimes k})$ for a family of stable curves $\mathcal{C} \to T$ over a smooth projective curve and for $k \gg 0$ (Theorem 4.7.17). In §4.7.3, we prove this nefness statement which finishes the proof of projectivity. Finally, in §4.7.4, we compare this argument to the GIT construction of $\overline{M}_g$.

4.7.1 Kollár's criteria

In this section, we prove Kollár's Criterion for projectivity (Theorem 4.7.5), which we will apply to show that λ_k is ample on $\overline{M}_g$ for $k \gg 0$. We first extend ampleness criteria of §F.2.2 to proper algebraic spaces and in particular establish that the Nakai–Moishezon criterion still holds (Theorem 4.7.4).

Lemma 4.7.1. *Let $\mathcal{X}$ be a proper Deligne-Mumford stack with coarse moduli space $\mathcal{X} \to X$. Suppose that L is a line bundle on $\mathcal{X}$ satisfying*

(a) L is semiample (i.e. L^N is basepoint-free for some $N > 0$); and

(b) for every map $f\colon T \to \mathcal{X}$ from a proper integral curve such that $f(T) \subset |\mathcal{X}|$ is not a single point, $\deg L|_T > 0$.

Then for some $N > 0$, $L^{\otimes N}$ descends to an ample line bundle. In particular, X is projective.

Remark 4.7.2. Lemma F.2.16 handles the case when $\mathcal{X}$ is a scheme. Even though we won't actually quote this lemma, it provides a basic technique which underlies many ampleness arguments, e.g. the Nakai-Moishezon criterion.

Proof. For N sufficiently divisible, consider the diagram

$$\begin{array}{ccc} \mathcal{X} & & \\ \downarrow & \searrow & \\ X & \longrightarrow & \mathbb{P}(H^0(\mathcal{X}, L^N)). \end{array}$$

Property (a) implies that $\mathcal{X} \to \mathbb{P}(H^0(\mathcal{X}, L^N))$ is well-defined and (b) implies that it doesn't contract curves. The universal property for coarse moduli spaces gives the existence of the factorization $X \to \mathbb{P}(H^0(\mathcal{X}, L^N))$, which also doesn't contract curves. Thus $X \to \mathbb{P}(H^0(\mathcal{X}, L^N))$ is quasi-finite and proper (as both X and projective space are proper), and thus finite by Zariski's Main Theorem. It follows that the pullback M of $\mathcal{O}(1)$ under $X \to \mathbb{P}(H^0(\mathcal{X}, L^N))$ is ample; moreover, the pullback of M under $\mathcal{X} \to X$ is $L^{\otimes N}$. □

Remark 4.7.3. The semiampleness condition in (a) can be very challenging to verify in practice. Keep in mind that in the GIT approach, semiampleness is hard-coded into the definition of semistability (see Remark 4.7.22). If G is a reductive group acting linearly on projective space $\mathbb{P}(V)$ and $L = \mathcal{O}(1)$ is the corresponding G-line bundle on $[\mathbb{P}(V)/G]$, then a nonzero vector $v \in V$ is in the stable base locus of L (i.e. $s(v) = 0$ for all $s \in \Gamma([\mathbb{P}(V)/G], L^{\otimes d})$ and $d > 0$) if and only if $0 \in \overline{Gv} \subset V$ if and only if there exists a one-parameter subgroup $\lambda\colon \mathbb{G}_m \to G$ such that $\lim_{t\to 0} \lambda(t) \cdot v = 0$. This latter equivalence is the Hilbert–Mumford criterion and can sometimes be verified combinatorially.

On the other hand, in Kollár's Criterion, the existence of sufficient sections of the line bundle follows from the bigness of suitable vector bundles.

Theorem 4.7.4 (Nakai–Moishezon Criterion)**.** *If X is a proper algebraic space, a line bundle L is ample if and only if for all irreducible closed subvarieties $Z \subset X$, $c_1(L)^{\dim Z} \cdot Z > 0$*

Proof. By Le Lemme de Gabber (Theorem 3.4.3), there exists a finite surjection $f\colon X' \to X$ from a scheme X, and L is ample if and only if f^*L is ample (Proposition 3.1.15). The statement then follows for the Nakai–Moisezon Criterion for schemes (Theorem F.2.18). □

Let X be a proper algebraic space over $\Bbbk$. Let $W \to Q$ be a surjection of vector bundles of rank w and q. Suppose that W has structure group $G \to \mathrm{GL}_w$. There is a *classifying map*

$$\begin{array}{rcl} X & \to & [\mathrm{Gr}(q, \Bbbk^w)/G] \\ x & \mapsto & [W \otimes \kappa(x) \twoheadrightarrow Q \otimes \kappa(x)] \end{array}$$

which is well-defined because a choice of isomorphism $W \otimes \kappa(x) \cong \kappa(x)^w$ of the fiber of W over x is well-defined up to the structure group G. Thus, the image of x is identified with the quotient $[\kappa(x)^w \cong W \otimes \kappa(x) \twoheadrightarrow Q \otimes \kappa(x)] \in \mathrm{Gr}(q, \Bbbk^w)$.

For simplicity, we state the following criteria in characteristic 0. The criteria first appears in [Kol90, Lem. 3.9] with improvements from [KP17, Thm. 4.1].

Theorem 4.7.5 (Kollár's Criterion)**.** *Let X be a proper algebraic space over a field $\Bbbk$ of characteristic 0. Let $W \twoheadrightarrow Q$ be a surjection of vector bundles of rank w and q, where W has structure group $G \to \mathrm{GL}_w$. Suppose that*

(a) The classifying map $X(\Bbbk) \to \mathrm{Gr}(q, \Bbbk^w)(\Bbbk)/G(\Bbbk)$ has finite fibers; and

(b) W is nef.

Then $\det Q$ is ample.

Remark 4.7.6. Condition (a) is equivalent to the map $|X| \to |[\mathrm{Gr}(q, \Bbbk^w)/G]|$ on topological spaces having finite fibers. This set-theoretic condition is weaker

than the quasi-finiteness[1] of $X \to [\mathrm{Gr}(q, \Bbbk^w)/G]$, as the latter condition also requires that for every $x \in X(\Bbbk)$ only finitely many elements of $G(\Bbbk)$ leave $\ker(W \otimes \kappa(x) \to Q \otimes \kappa(x))$ invariant (or equivalivalently that the image of x in $[\mathrm{Gr}(q, \Bbbk^w)/G]$ has finite stabilizer.) In fact, Condition (a) is equivalent to quasi-finiteness of the projection morphism $\mathrm{im}(X \to X \times [\mathrm{Gr}(q, \Bbbk^w)/G]) \to [\mathrm{Gr}(q, \Bbbk^w)/G]$ from the scheme-theoretic image of the graph of the classifying map; it is this property that we will use in the proof.

Remark 4.7.7. An easy case of this theorem is when W is the trivial vector bundle so that there is a reduction of structure group to the trivial group $G = \{1\}$. In this case, the classifying map $X \to \mathrm{Gr}(q, \Bbbk^w)$ is quasi-finite by condition (a) and proper since both X and $\mathrm{Gr}(q, \Bbbk^w)$ are proper. Thus $X \to \mathrm{Gr}(q, \Bbbk^w)$ is finite and $\det(Q)$ is ample as its the pullback of the ample line bundle on $\mathrm{Gr}(q, \Bbbk^w)$ defining the Plücker embedding.

Note that in the above theorem, we do not require that the image of X lands in the G-stable locus of $\mathrm{Gr}(q, \Bbbk^w)$. However, if this is true, then we have a commutative diagram

$$\begin{array}{ccc} X & \longrightarrow & [\mathrm{Gr}(q, \Bbbk^w)^{\mathrm{ss}}/G] \\ & \searrow & \downarrow \\ & & \mathrm{Gr}(q, \Bbbk^w)/\!\!/G \end{array}$$

where $\mathrm{Gr}(q, \Bbbk^w)/\!\!/G$ denotes the projective GIT quotient. Since the image of X lands in the *stable* locus, $X \to \mathrm{Gr}(q, \Bbbk^w)/\!\!/G$ is quasi-finite; as it's also proper, we conclude that it's finite. Moreover, we obtain ampleness of $(\det Q)^w \otimes (\det W)^{-q}$, the pullback of the ample line bundle $\mathrm{Gr}(q, \Bbbk^w)/\!\!/G$ coming from GIT. This is a stronger ampleness statement than merely the ampleness of $\det Q$.

Remark 4.7.8. The nefness of $\det(Q)$ is an immediate consequence of the nefness of W as $\det(Q) = \bigwedge^q Q$ is a quotient of $\bigwedge^q W$, which is nef by Proposition F.2.27. The proof will proceed by reducing the ampleness of $\det Q$ to its bigness, which in turn is established by using the quasi-finiteness and nefness to express $\det Q$ as the sum of an effective line bundle and a big and globally generated line bundle.

Proof of Theorem 4.7.5. We will verify the Nakai–Moishezon criterion: for each irreducible subvariety $Z \subset X$, we verify that $\det(Q)|_Z$ is big. Since both conditions (a) and (b) also hold for Z and the restrictions $W|_Z \twoheadrightarrow Q|_Z$, it suffices to verify that if X is an integral scheme with $W \twoheadrightarrow Q$ satisfying (a) and (b), then $\det(Q)$ is big.

The property of bigness (unlike ampleness) is conveniently invariant under birational maps (and we desire this flexibility because in the proof of Proposition 4.7.9 below, we will make a series of reductions where we perform blowups to resolve the indeterminacy locus of certain rational maps). In fact, for a generically quasi-finite and proper morphism $f\colon Y \to X$ of integral schemes, the projection formula implies that $\det(f^*Q)^{\dim Y} = \deg(f) \det(Q)^{\dim X} > 0$ and thus $\det(Q)$ is big if and only if $f^*(\det Q)$ is big. By Le Lemme de Gabber (Corollary 3.4.4), there exists a projective, generically quasi-finite and surjective morphism $f\colon Y \to X$ from a projective integral scheme. By taking the normalization, we can assume

[1] Recall that a morphism $f\colon \mathcal{X} \to \mathcal{Y}$ of algebraic stacks is *quasi-finite* if $|\mathcal{X}| \to |\mathcal{Y}|$ has finite fibers and the relative inertia $I_{\mathcal{X}/\mathcal{Y}}$ is quasi-finite (or equivalently for every field-valued point $x \in \mathcal{X}(K)$ the morphism $\mathrm{Aut}_{\mathcal{X}(K)}(x) \to \mathrm{Aut}_{\mathcal{Y}(K)}(f(x))$ has finite cokernel).

that Y is normal. The theorem therefore follows from the bigness of $f^*(\det Q)$, which is the conclusion of the following proposition. □

Proposition 4.7.9. *Let Y be a normal projective integral scheme over a field $\Bbbk$ of characteristic 0. Let $W \twoheadrightarrow Q$ be a surjection of vector bundles of rank w and q, where W has structure group $G \to \mathrm{GL}_w$. Suppose that*

(a′) The classifying map $Y(\Bbbk) \to \mathrm{Gr}(q, \Bbbk^w)(\Bbbk)/G(\Bbbk)$ generically has finite fibers;

(b) W is nef.

Then $\det Q$ is big.

Remark 4.7.10. Condition (a′) means that there is a non-empty open subscheme $U \subset Y$ such that $U(\Bbbk) \to \mathrm{Gr}(q, \Bbbk^w)(\Bbbk)/G(\Bbbk)$ has finite fibers.

Note that the difference in the hypotheses between Theorem 4.7.5 and Proposition 4.7.9 is that we relaxed the condition on the classifying map from having finite fibers to generically having finite fibers but now we assume that Y is already projective (in addition to being normal and integral). Also the conclusion is weaker in that it asserts the bigness of $\det(Q)$ rather than the ampleness.

Proof.

Step 1: Use the universal basis map to lift the classifying map to a morphism $\mathbb{P} \setminus \Delta \to \mathrm{Gr}(q, \Bbbk^w)$ where $\mathbb{P} \subset \mathbb{P}_Y((W^\vee)^{\oplus w})$ is a closed subscheme and $\Delta \subset \mathbb{P}$ is a divisor.

Define $\widetilde{\mathbb{P}} := \mathbb{P}_Y((W^\vee)^{\oplus w})$ as the projective space of matrices whose columns belong to W, and let $\widetilde{\pi}\colon \widetilde{\mathbb{P}} \to Y$ denote the projection. There is a *universal basis map*

$$\mathcal{O}_{\widetilde{\mathbb{P}}}^{\oplus w} \to \widetilde{\pi}^* W \otimes \mathcal{O}_{\widetilde{\mathbb{P}}}(1) \tag{4.7.3}$$

defined by the isomorphisms

$$H^0(\widetilde{\mathbb{P}}, \widetilde{\pi}^* W \otimes \mathcal{O}_{\widetilde{\mathbb{P}}}(1)) \cong H^0(Y, \widetilde{\pi}_*(\widetilde{\pi}^* W \otimes \mathcal{O}_{\widetilde{\mathbb{P}}}(1))) \cong H^0(Y, W \otimes (W^\vee)^{\oplus w}).$$

The universal basis map (4.7.3) restricts to an isomorphism on the complement $\widetilde{\mathbb{P}} \setminus \Delta$ where $\Delta \subset \widetilde{\mathbb{P}}$ is the divisor of matrices with determinant 0, and thus provides a trivialization of $(\widetilde{\pi}^* W \otimes \mathcal{O}_{\widetilde{\mathbb{P}}}(1))|_{\widetilde{\mathbb{P}} \setminus \Delta}$. Note also that there is a natural PGL_w action on $\widetilde{\mathbb{P}}$ which is free on $\widetilde{\mathbb{P}} \setminus \Delta$ and such that $\widetilde{\pi}\colon \widetilde{\mathbb{P}} \setminus \Delta \to Y$ is a PGL_w-torsor and fits into the cartesian diagram

$$\begin{array}{ccccc} \widetilde{\mathbb{P}} \setminus \Delta & \longrightarrow & \mathrm{Gr}(q, \Bbbk^w) & \longrightarrow & \operatorname{Spec} \Bbbk \\ \downarrow & & \downarrow & & \downarrow \\ Y & \longrightarrow & [\mathrm{Gr}(q, \Bbbk^w)/\mathrm{PGL}_w] & \longrightarrow & B\,\mathrm{PGL}_w\,. \end{array}$$

We can also consider the fiber product with respect to the G-action

$$\mathbb{P} \setminus \Delta := Y \times_{[\mathrm{Gr}(q, \Bbbk^w)/G]} \mathrm{Gr}(q, \Bbbk^w).$$

The inclusion $\mathbb{P} \setminus \Delta \hookrightarrow \widetilde{\mathbb{P}} \setminus \Delta$ is a closed immersion and we define $\mathbb{P} \subset \widetilde{\mathbb{P}}$ to be the closure of $\mathbb{P} \setminus \Delta$, where we abuse notation by using the same symbol Δ for the divisor in $\widetilde{\mathbb{P}}$ and its intersection in $\mathbb{P}$. One way to see that $\mathbb{P} \setminus \Delta = Y \times_{BG} \operatorname{Spec} \Bbbk \hookrightarrow Y \times_{B\,\mathrm{PGL}_w} \operatorname{Spec} \Bbbk = \widetilde{\mathbb{P}} \setminus \Delta$ is a closed immersion is to realize it as the base change

of the diagonal $BG \to BG \times_{B\,\mathrm{PGL}_w} BG$; here we use that $BG \to B\,\mathrm{PGL}_w$ is separated (it is in fact even affine since G is reductive). Alternatively, one can view $\mathbb{P} \subset \widetilde{\mathbb{P}}$ as the closure of a generic G-orbit in $\widetilde{\mathbb{P}}$.

In summary, we have a cartesian diagram

$$\begin{array}{ccccc} \mathbb{P} \setminus \Delta & \longrightarrow & \mathrm{Gr}(q, \Bbbk^w) & \hookrightarrow & \mathbb{P}(\bigwedge^q \Bbbk^w) \\ \downarrow{\scriptstyle \pi} & & \downarrow & & \downarrow \\ Y & \longrightarrow & [\mathrm{Gr}(q, \Bbbk^w)/G] & \hookrightarrow & [\mathbb{P}(\bigwedge^q \Bbbk^w)/G] \end{array}$$

where the right-hand square is given by the Plücker embedding. The map $\mathbb{P}\setminus\Delta \to Y$ extends to a map $\pi\colon \mathbb{P} \to Y$ (i.e. the composition $\mathbb{P} \hookrightarrow \widetilde{\mathbb{P}} \xrightarrow{\widetilde{\pi}} Y$). The map $\mathbb{P} \setminus \Delta \to \mathrm{Gr}(q, \Bbbk^w)$ is defined by the restriction of the composition

$$\mathcal{O}_{\mathbb{P}}^{\oplus w} \to \pi^* W \otimes \mathcal{O}_{\mathbb{P}}(1) \to \pi^* Q \otimes \mathcal{O}_{\mathbb{P}}(1) \tag{4.7.4}$$

of the universal basis map (4.7.3) with the quotient $\pi^* W \to \pi^* Q$. The image of (4.7.4) may not be locally free and thus the rational map $\mathbb{P} \dashrightarrow \mathrm{Gr}(q, \Bbbk^w)$ may not be defined everywhere.

Step 2: Blowup $\mathbb{P}$ in order to extend the map $\mathbb{P} \setminus \Delta \to \mathrm{Gr}(q, \Bbbk^w)$.

(Note that if (4.7.4) is surjective, then $\mathbb{P} \setminus \Delta \to \mathrm{Gr}(q, \Bbbk^w)$ extends to a morphism $\mathbb{P} \to \mathrm{Gr}(q, \Bbbk^w)$ such that the pullback of the Plücker line bundle is $\pi^*(\det Q) \otimes \mathcal{O}_{\mathbb{P}}(q)$.)

We blowup the image ideal sheaf of (4.7.4) (more precisely, if $I \subset \pi^*(\det Q) \otimes \mathcal{O}_{\mathbb{P}}(q)$ denotes the image subsheaf of (4.7.4), we blowup $I \otimes (\pi^*(\det Q) \otimes \mathcal{O}_{\mathbb{P}}(q))^\vee \subset \mathcal{O}_{\mathbb{P}}$). This yields a map $g\colon \mathbb{P}' \to \mathbb{P}$ which is an isomorphism over $\mathbb{P} \setminus \Delta$ and such that $\mathbb{P} \setminus \Delta \to \mathrm{Gr}(q, \Bbbk^w)$ extends to a morphism $\gamma\colon \mathbb{P}' \to \mathrm{Gr}(q, \Bbbk^w)$. This yields a commutative diagram

$$\begin{array}{ccc} \mathbb{P}' & & \\ \downarrow{\scriptstyle g} \searrow{\scriptstyle \gamma} & & \\ \mathbb{P} & \dashrightarrow & \mathrm{Gr}(q, \Bbbk^w) \\ \downarrow{\scriptstyle \pi} & & \downarrow \\ Y & \longrightarrow & [\mathrm{Gr}(q, \Bbbk^w)/G]. \end{array} \qquad \pi' \colon \mathbb{P}' \to Y$$

The effective divisor $E \subset \mathbb{P}'$ satisfies

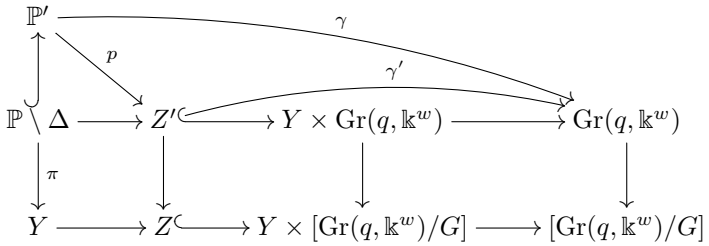
$$g^*(\pi^*(\det Q) \otimes \mathcal{O}_{\mathbb{P}}(q)) \cong \gamma^* \mathcal{O}_{\mathrm{Gr}(q,\Bbbk^w)}(1) \otimes \mathcal{O}_{\mathbb{P}'}(E). \tag{4.7.5}$$

where $\mathcal{O}_{\mathrm{Gr}(q,\Bbbk^w)}(1)$ denotes the Plücker line bundle.

Step 3: Use the generic quasi-finiteness to show that $\gamma^*(\mathcal{O}_{\mathrm{Gr}(q,\Bbbk^w)}(m)) \otimes \pi'^* H^\vee$ is effective for some $m > 0$, where H is a ample line bundle on Y.

(Note that under the stronger assumption that the classifying map $Y \to [\mathrm{Gr}(q, \Bbbk^w)/G]$ is generically quasi-finite, then $\gamma\colon \mathbb{P}' \to \mathrm{Gr}(q, \Bbbk^w)$ is also generically quasi-finite. Thus $\gamma^* \mathcal{O}_{\mathrm{Gr}(q,\Bbbk^w)}(1)$ is big and Kodaira's Lemma (Proposition F.2.8) immediately gives the desired statement.)

Let Z be the scheme-theoretic image of the graph $Y \to Y \times [\mathrm{Gr}(q, \Bbbk^w)/G]$ of the classifying map. The hypothesis that $Y(\Bbbk) \to \mathrm{Gr}(q, \Bbbk^w)(\Bbbk)/G(\Bbbk)$ is generically quasi-finite implies that $Z \to [\mathrm{Gr}(q, \Bbbk^w)/G]$ is generically quasi-finite. Consider the commutative diagram



where the squares are cartesian and where Z' is the scheme-theoretic image of $\mathbb{P} \setminus \Delta \to Y \times \mathrm{Gr}(q, \Bbbk^w)$ (and also of $\mathbb{P}' \to Y \times \mathrm{Gr}(q, \Bbbk^w)$). We see that $\gamma' \colon Z' \to \mathrm{Gr}(q, \Bbbk^w)$ is also generically quasi-finite and it follows that $\gamma'^*(\mathcal{O}_{\mathrm{Gr}(q,\Bbbk^w)}(1))$ is big. If we denote by H' the pullback of H to Z', then by Kodaira's Lemma (Proposition F.2.8), $\gamma'^*(\mathcal{O}_{\mathrm{Gr}(q,\Bbbk^w)}(m)) \otimes H'^\vee$ is effective on Z for some $m > 0$. Its pullback $p^*(\gamma'^*(\mathcal{O}_{\mathrm{Gr}(q,\Bbbk^w)}(m)) \otimes H'^\vee) \cong \gamma^*(\mathcal{O}_{\mathrm{Gr}(q,\Bbbk^w)}(m)) \otimes \pi'^* H^\vee$ is also effective.

Step 4: Pushforward a section to construct a map $\pi_* \mathcal{O}_{\mathbb{P}}(mq)^\vee \to (\det Q)^{\otimes m} \otimes H^\vee$.

Using (4.7.5), we see that

$$\begin{aligned} \gamma^*(\mathcal{O}_{\mathrm{Gr}(q,\Bbbk^w)}(m)) \otimes \pi'^* H^\vee &\cong \pi'^*((\det Q)^{\otimes m} \otimes H^\vee) \otimes g^* \mathcal{O}_{\mathbb{P}}(mq) \otimes \mathcal{O}_{\mathbb{P}'}(-mE) \\ &\subset \pi'^*((\det Q)^{\otimes m} \otimes H^\vee) \otimes g^* \mathcal{O}_{\mathbb{P}}(mq)) \\ &\cong g^*\big(\pi^*((\det Q)^{\otimes m} \otimes H^\vee) \otimes \mathcal{O}_{\mathbb{P}}(mq)\big) \end{aligned}$$

and therefore we may choose a non-zero section

$$\mathcal{O}_{\mathbb{P}'} \to g^*\big(\pi^*((\det Q)^{\otimes m} \otimes H^\vee) \otimes \mathcal{O}_{\mathbb{P}}(mq)\big).$$

Pushing forward under $g \colon \mathbb{P}' \to \mathbb{P}$ and using the projection formula gives a non-zero section

$$\mathcal{O}_{\mathbb{P}} \to \pi^*((\det Q)^{\otimes m} \otimes H^\vee) \otimes \mathcal{O}_{\mathbb{P}}(mq)$$

and pushing forward again under $\pi \colon \mathbb{P} \to Y$ gives a non-zero section

$$\mathcal{O}_Y \to (\det Q)^{\otimes m} \otimes H^\vee \otimes \pi_* \mathcal{O}_{\mathbb{P}}(mq)$$

which we rearrange as

$$\pi_* \mathcal{O}_{\mathbb{P}}(mq)^\vee \to (\det Q)^{\otimes m} \otimes H^\vee. \tag{4.7.6}$$

Step 5: Show that the nefness of W implies the nefness of $\pi_* \mathcal{O}_{\mathbb{P}}(mq)^\vee$.

We compare $\pi_* \mathcal{O}_{\mathbb{P}}(mq)$ to $\pi_* \mathcal{O}_{\widetilde{\mathbb{P}}}(mq) \cong \mathrm{Sym}^{mq}((W^\vee)^{\oplus w})$ (and their duals) under the closed immersion $\mathbb{P} \hookrightarrow \widetilde{\mathbb{P}}$ (where we are using π to denote both projections $\mathbb{P} \to Y$ and $\widetilde{\mathbb{P}} \to Y$). For $m \gg 0$, the map $\pi_* \mathcal{O}_{\widetilde{\mathbb{P}}}(mq) \to \pi_* \mathcal{O}_{\mathbb{P}}(mq)$ is surjective and dualizes to an inclusion

$$(\pi_* \mathcal{O}_{\mathbb{P}}(mq))^\vee \hookrightarrow (\pi_* \mathcal{O}_{\widetilde{\mathbb{P}}}(mq))^\vee \cong \mathrm{Sym}^{mq}((W)^{\oplus w})$$

of vector bundles on Y. Since W is nef, so is $\operatorname{Sym}^{mq}((W)^{\oplus w})$ (Proposition F.2.27) and therefore so is $(\pi_*\mathcal{O}_{\mathbb{P}}(mq))^\vee$ (Proposition 4.7.11).

Step 6: Conclude that $\det Q$ *is big.*

(Note that if (4.7.6) is surjective, then the line bundle quotient $N := (\det Q)^{\otimes m} \otimes H^\vee$ is nef. Thus $(\det Q)^{\otimes m} \cong H \otimes N$ is written as the sum of an ample and nef divisor, which is necessarily big.)

Blowing up the image ideal sheaf of (4.7.6), we obtain a birational morphism $s\colon Y' \to Y$ and a quotient line bundle $s^*\pi_*\mathcal{O}_{\mathbb{P}}(mq)^\vee \twoheadrightarrow N \subset s^*(\det Q)^{\otimes m} \otimes s^*H^\vee$ which is nef. As N is nef and s^*H is big and globally generated, the sub-line bundle $s^*H \otimes N \subset s^*(\det Q)^{\otimes m}$ is big (Proposition F.2.14). The difference of $s^*(\det Q)^{\otimes m}$ and $s^*H \otimes N$ is effective. Since the sum of a big and globally generated line bundle is big, we can conclude that $s^*(\det Q)^{\otimes m}$ is big, which in turn implies that $\det Q$ is big. □

The proof above used the following property of nefness of vector bundles complementing the basic results from Section F.2.3.

Proposition 4.7.11. *Let X be a scheme of finite type over an algebraically closed field $\Bbbk$ of characteristic 0 and W be a vector bundle of rank w. Let G be a reductive group and suppose that W admits a reduction of the structure group $G \to \mathrm{GL}_w$. Let $V \subset W$ be a G-subbundle corresponding a G-invariant subspace $\Bbbk^v \subset \Bbbk^w$. If W is nef, then so is V.*

Proof. In characteristic 0, representations of reductive groups are completely reducible. Therefore $\Bbbk^v \subset \Bbbk^w$ has a G-invariant complement $\Bbbk^{w-v} \subset \Bbbk^w$. Since this expresses V as a quotient of W, we see that V is nef. □

4.7.2 Application to $\overline{M}_g$

To apply Kollár's Criterion to $\overline{M}_g$, we will make use of multiplication maps between pluri-canonical bundles and their symmetric products. Given a morphism $S \to \overline{\mathcal{M}}_g$ corresponding to a family of stable curves $\pi\colon \mathcal{C} \to S$ and an integer $d \geq 0$, we will consider the *multiplication map*

$$\operatorname{Sym}^d \pi_*(\omega_{\mathcal{C}/S}^{\otimes k}) \to \pi_*(\omega_{\mathcal{C}/S}^{\otimes dk}). \tag{4.7.7}$$

For a stable curve C defined over a field $\Bbbk$, this multiplication map is

$$\operatorname{Sym}^d H^0(C, \omega_C^{\otimes k}) \to H^0(C, \omega_C^{\otimes dk})$$

and its kernel consists of degree d equations cutting out the image of $C \xrightarrow{|\omega_C^{\otimes k}|} \mathbb{P}^{r(k)-1}$. If $k \geq 3$, then $\omega_{\mathcal{C}/S}^{\otimes k}$ is relatively very ample and thus $\mathcal{C} \to S$ can be recovered from the kernel of the multiplication map.

Remark 4.7.12. We emphasize here that this construction depends on k and d, the same two integers which the GIT construction depends on (see Section 4.7.4).

Proposition 4.7.13. *Let $g \geq 2$. Assume that*

(a) $\overline{\mathcal{M}}_g$ is a proper Deligne–Mumford stack; and

(b) There exists a $k_0 > 0$ such that for any family of stable curves $\mathcal{C} \to T$ over a smooth projective curve T, $\pi_*(\omega_{\mathcal{C}/T}^{\otimes k})$ is nef for $k \geq k_0$.

Then for $k \gg 0$ and N sufficiently divisible, the line bundle $\lambda_k^{\otimes N}$ on $\overline{\mathcal{M}}_g$ descends to an ample line bundle on the coarse moduli space $\overline{M}_g$. In particular, $\overline{M}_g$ is projective.

Proof. Consider the universal curve $\mathcal{C} = \mathcal{U}_g$ over $S = \overline{\mathcal{M}}_g$. Choose integers k and d such that

- $\omega_{\mathcal{C}/S}^{\otimes k}$ is relatively very ample and $R^1\pi_*\omega_{\mathcal{C}/S}^{\otimes k} = 0$;
- Every stable curve $C \overset{|\omega_C^{\otimes k}|}{\hookrightarrow} \mathbb{P}^{r(k)-1}$ is cut out by equations of degree d; and
- $\pi_*(\omega_{\mathcal{C}/S}^{\otimes k})$ is nef.

The conditions imply that the multiplication map

$$W := \operatorname{Sym}^d \pi_*(\omega_{\mathcal{C}/S}^{\otimes k}) \twoheadrightarrow \pi_*(\omega_{\mathcal{C}/S}^{\otimes dk}) =: Q$$

is surjective. Let $w = \binom{r(k)+d-1}{d}$ and $q = r(dk)$ be the ranks of W and Q, respectively. Note that W has a reduction of the structure group to $G := \mathrm{SL}_{r(k)}$. The classifying map

$$\begin{aligned} \overline{\mathcal{M}}_g &\to [\mathrm{Gr}(q, \Bbbk^w)/G] \\ [C] &\mapsto \big[\underbrace{\operatorname{Sym}^d H^0(C, \omega_C^{\otimes k})}_{\Gamma(\mathbb{P}^{r(k)-1}, \mathcal{O}(d))} \twoheadrightarrow \underbrace{H^0(C, \omega_C^{\otimes dk})}_{\Gamma(C, \mathcal{O}(d))}\big] \end{aligned}$$

is injective as the conditions on d and k imply that the kernel of the multiplication map uniquely determines C.

Let $X \to \overline{\mathcal{M}}_g$ be a finite cover where X is a proper algebraic space (Theorem 3.4.3). By Kollár's Criterion (Theorem 4.7.5), the pullback of λ_k to X is ample for $k \gg 0$. By Proposition 3.3.28, for N sufficiently divisible, $\lambda_k^{\otimes N}$ descends to a line bundle L on $\overline{M}_g$. Since the pullback of L under the finite morphism $X \to \overline{\mathcal{M}}_g \to \overline{M}_g$ is ample, we conclude by Proposition 3.1.15 that L is ample. □

In the next section, we will establish condition (b), the nefness of the pluricanonical bundles. This will allow us to conclude:

Theorem 4.7.14. *If $2g - 2 + n > 0$, then $\overline{M}_{g,n}$ is projective.*

Proof. It suffices to handle the $n = 0$ case as $\overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ is the universal family (Proposition 4.5.8) and is a projective morphism (Proposition 4.2.9). The fact that λ_k descends to an ample line bundle on $\overline{M}_g$ follows from Proposition 4.7.13 as Condition (a) is a consequence of Stable Reduction (see Theorem 4.4.3) while (b) is Theorem 4.7.17. □

Remark 4.7.15. It also possible to show projectivity of $\overline{M}_{g,n}$ directly using Kollár's Criterion applied to the determinant of $\pi_*(L^k)$ where $L := \omega_{\mathcal{U}_{g,n}/\overline{\mathcal{M}}_{g,n}}(\sigma_1 + \cdots + \sigma_n)$ and $\mathcal{U}_{g,n} \to \overline{\mathcal{M}}_{g,n}$ is the universal family with sections $\sigma_1, \ldots, \sigma_n$.

Remark 4.7.16. The criteria of Proposition 4.7.13 for ampleness generalizes to any moduli of polarized varieties (see [Kol90, Thm. 2.6]); this was one of the original motivations of Kollár's paper. In recent years, Kollár's Criterion has been applied in more and more general settings to establish projectivity, e.g. Hassett's moduli space of weighted pointed curves [Has03], the moduli of stable varieties of any dimension [KP17], and the moduli of K-polystable Fano varieties [CP21, XZ20, LXZ21].

4.7.3 Nefness of pluri-canonical bundles

In this section, we establish that $\pi_*(\omega_{\mathcal{C}/T}^{\otimes k})$ is nef for any $k \geq 2$.

Theorem 4.7.17. *For family of stable curves $\mathcal{C} \to T$ over a smooth projective curve T, $\pi_*(\omega_{\mathcal{C}/T}^{\otimes k})$ is nef for $k \geq 2$.*

Proof sketch: Let $\Bbbk$ be the base field.

Step 1: Reduction to characteristic p. Assume that $\operatorname{char}(\Bbbk) = 0$. Since $\mathcal{C}$ and T are finite type over $\Bbbk$, their defining equations only involve finitely many coefficients of $\Bbbk$. Thus there exists a finitely generated $\mathbb{Z}$-subalgebra $A \subset \Bbbk$ and a cartesian diagram

$$\begin{array}{ccc} \mathcal{C} & \longrightarrow & \widetilde{\mathcal{C}} \\ \downarrow & & \downarrow \\ T & \longrightarrow & \widetilde{T} \\ \downarrow & & \downarrow \\ \operatorname{Spec} \Bbbk & \longrightarrow & \operatorname{Spec} A \end{array}$$

where $\widetilde{C}$ and $\widetilde{T}$ are schemes of finite type over A. By possibly enlarging A, we can arrange that $\widetilde{T} \to \operatorname{Spec} A$ is a smooth and projective family of curves and that $\widetilde{C} \to \widetilde{T}$ is a family of stable curves. Finally, by restricting along any morphism $\operatorname{Spec} R \to \operatorname{Spec} A$ from a DVR such that the images of the closed and generic points have characteristic p and 0, respectively, we may assume that A is a DVR. Since nefness is an open condition for such proper flat families (Proposition F.2.28), it suffices to prove the theorem when $\operatorname{char}(\Bbbk) = p > 0$.

Step 2: Second reductions. We reduce to the case where

(a) $\mathcal{C}$ is a smooth and minimal surface;

(b) $\mathcal{C} \to T$ is generically smooth; and

(c) the genus of T is at least 2

(details to be added). These conditions imply that $\mathcal{C}$ is of general type.

Step 3: Positive characteristic case. Let $p = \operatorname{char}(\Bbbk)$. If $\pi_*(\omega_{\mathcal{C}/T}^{\otimes k})$ is not nef, then there exists a quotient line bundle $\pi_*(\omega_{\mathcal{C}/T}^{\otimes k}) \twoheadrightarrow M^\vee$ where $d = \deg M > 0$. Consider the absolute Frobenius morphisms $F\colon \mathcal{C} \to \mathcal{C}$ and $F\colon T \to T$ which fit into a commutative diagram

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathcal{C} \\ \downarrow & & \downarrow \\ T & \xrightarrow{F} & T. \end{array}$$

By properties of the dualizing sheaf, we have $F^*\pi_*(\omega_{\mathcal{C}/T}^{\otimes k}) = \pi_*(\omega_{\mathcal{C}/T}^{\otimes k})$. Since $\deg F^*M = pd$, we can apply the Frobenius repeatedly to arrange that d, the degree of M, is as large as we want. Specifically, we can arrange that $M \cong \omega_T^{\otimes k} \otimes L$ where L is a very ample line bundle on T. (This was the entire point of reducing to characteristic p: to repeatedly apply the Frobenius to jack-up the degree.)

The surjection $\pi_*(\omega_{\mathcal{C}/T}^{\otimes k}) \twoheadrightarrow M^\vee \cong (\omega_T^{\otimes k} \otimes L)^\vee$ yields a surjection

$$\pi_*(\omega_{\mathcal{C}/T}^{\otimes k}) \otimes \omega_T^{\otimes k} \otimes L \twoheadrightarrow \mathcal{O}_T$$

Since $h^1(T, \mathcal{O}_T) \geq 2$, we have $h^1(T, \pi_*(\omega_{\mathcal{C}/T}^{\otimes k}) \otimes \omega_T^{\otimes k} \otimes L) \geq 2$. Using the Leray spectral sequence to relate $H^1(\pi_*(\omega_{\mathcal{C}/T}^{\otimes k}) \otimes \omega_T^{\otimes k} \otimes L)$ to $H^1(\mathcal{C}, \omega_{\mathcal{C}}^{\otimes k} \otimes \pi^* L)$, one can show that $h^1(\mathcal{C}, \omega_{\mathcal{C}}^{\otimes k} \otimes \pi^* L) \geq 2$ (details omitted). This however contradicts Bombieri–Ekedahl vanishing in the form of Lemma 4.7.19 with $D = \pi^* L$. □

Remark 4.7.18. For families of *smooth* curves, $\pi_*(\omega_{\mathcal{C}/T})$ is nef; this fact is somewhat easier and was known earlier. If $\mathcal{C} \to S$ has no hyperelliptic fibers, then Max Noether's theorem on projective normality implies that $\operatorname{Sym}^d \pi_*(\omega_{\mathcal{C}/T}) \to \pi_*(\omega_{\mathcal{C}/T}^{\otimes d})$ is surjective. Therefore, the nefness of $\pi_*(\omega_{\mathcal{C}/T})$ implies the nefness of both $\operatorname{Sym}^d \pi_*(\omega_{\mathcal{C}/T})$ and the quotient $\pi_*(\omega_{\mathcal{C}/T}^{\otimes d})$ (Proposition F.2.27).

Lemma 4.7.19. *Let S be a smooth projective surface over an algebraically closed field $\Bbbk$ which is minimal and of general type. Let D be an effective divisor with $D^2 = 0$. If $\operatorname{char}(\Bbbk) \neq 2$, then $H^1(S, \omega_S^{\otimes n}(D)) = 0$ for all $n \geq 2$. If $\operatorname{char}(\Bbbk) = 2$, then $h^1(S, \omega_S^{\otimes n}(D)) \leq 1$ for all $n \geq 2$.*

Proof. Bombieri–Ekedahl vanishing (Theorem F.3.1) implies that $H^1(S, K_S^{\otimes -n}) = 0$ for all $n \geq 1$. The Serre dual of this statement is that $H^1(S, K_S^{\otimes n}) = 0$ for all $n \geq 2$. The statement follows from using the short exact sequence

$$0 \to \omega_S^{\otimes n} \to \omega_S^{\otimes n}(D) \to \omega_S^{\otimes n}|_D \to 0$$

and adjunction (details omitted). □

4.7.4 Projectivity via Geometric Invariant Theory

The Geometric Invariant Theory (GIT) construction depends on two integers:

- k, the multiple of the dualizing sheaf used to obtain an embedding $C \xrightarrow{|\omega_C^{\otimes k}|} \mathbb{P}^{r(k)-1}$. We need $k \geq 3$ for $\omega_C^{\otimes k}$ to be very ample for a stable curve C but we need $k \geq 5$ for the GIT construction to yield $\overline{M}_g$.
- d, the degree of the equations that we use to embed the Hilbert scheme of k-canonically embedded curves into a Grassmanian. We need $d \gg 0$ to obtain an embedding of the Hilbert scheme.

Assuming that $k \geq 3$, a stable curve C of genus g is pluricanonically embedded via

$$C \xrightarrow{|\omega_C^{\otimes k}|} \mathbb{P}^{r(k)-1}$$

where $r(k) = (2k-1)(g-1)$. Let $P(t) = \chi(C, \omega^{\otimes kt}) = (2kt-1)(g-1)$ be the Hilbert polynomial of C in $\mathbb{P}^{r(k)-1}$. Let $H' \subset \operatorname{Hilb}^P(\mathbb{P}^{r(k)-1})$ be the locally closed subscheme of the Hilbert scheme parameterizing stable curves $[C \hookrightarrow \mathbb{P}^{r(k)-1}]$ embedded via $\omega_C^{\otimes k}$. Note that $\mathrm{PGL}_{r(k)}$ acts naturally on $\operatorname{Hilb}^P(\mathbb{P}^{r(k)-1})$ and that the subscheme H' is $\mathrm{PGL}_{r(k)}$-invariant.

Exercise 4.7.20. Extend Theorem 2.1.11 by establishing that:

(a) $H' \subset \mathrm{Hilb}^P(\mathbb{P}^{r(k)-1}$ is a locally closed $\mathrm{PGL}_{r(k)}$-invariant subscheme, and

(b) $\overline{\mathcal{M}}_g \cong [H'/\mathrm{PGL}_{r(k)}]$.

Let $H = \overline{H'} \subset \mathrm{Hilb}^P(\mathbb{P}^{r(k)-1}$ be the closure of H'. For $d \gg 0$, we have an embedding into the Grassmanian of $P(d)$-dimensional quotients of $\Gamma(\mathbb{P}^{r(k)-1}, \mathcal{O}(d))$

$$\begin{aligned} H &\hookrightarrow \mathrm{Gr}(P(d), \Gamma(\mathbb{P}^{r(k)-1}, \mathcal{O}(d)) \\ [C \hookrightarrow \mathbb{P}^{r(k-1)}] &\mapsto \left[\Gamma(\mathbb{P}^{r(k)-1}, \mathcal{O}(d)) \twoheadrightarrow \Gamma(C, \mathcal{O}(d))\right] \end{aligned}$$

Note that there is a natural identification of this quotient with the multiplication map

$$\begin{array}{ccc} \Gamma(\mathbb{P}^{r(k)-1}, \mathcal{O}(d)) & \twoheadrightarrow & \Gamma(C, \mathcal{O}(d)) \\ \| & & \| \\ \mathrm{Sym}^d H^0(C, \omega_C^{\otimes k}) & \twoheadrightarrow & H^0(C, \omega_C^{\otimes dk}). \end{array}$$

Let $\mathcal{O}_{\mathrm{Gr}}(1)$ be the very ample line bundle on $\mathrm{Gr}(P(d), \Gamma(\mathbb{P}^{r(k)-1}, \mathcal{O}(d))$ obtained via the Plücker embedding

$$\begin{aligned} \mathrm{Gr}(P(d), \Gamma(\mathbb{P}^{r(k)-1}, \mathcal{O}(d)) &\hookrightarrow \mathbb{P}(\wedge^{P(d)}\Gamma(\mathbb{P}^{r(k)-1}, \mathcal{O}(d)) \\ [\Gamma(\mathbb{P}^{r(k)-1}, \mathcal{O}(d)) \twoheadrightarrow \Gamma(C, \mathcal{O}(d))] &\mapsto [\wedge^{P(d)}\Gamma(\mathbb{P}^{r(k)-1}, \mathcal{O}(d)) \twoheadrightarrow \wedge^{P(d)}\Gamma(C, \mathcal{O}(d))]; \end{aligned}$$

see Section 0.5. Finally, let $L_d = \mathcal{O}_{\mathrm{Gr}}(1)|_H$ be the very ample line bundle on H obtained by restricting $\mathcal{O}(1)$ under the composition

$$H \hookrightarrow \mathrm{Gr}(P(d), \Gamma(\mathbb{P}^{r(k)-1}, \mathcal{O}(d)) \hookrightarrow \mathbb{P}(\wedge^{P(d)}\Gamma(\mathbb{P}^{r(k)-1}, \mathcal{O}(d)) \tag{4.7.8}$$

As each morphism in (4.7.8) is $\mathrm{PGL}_{r(k)}$-equivariant, the line bundle L_d inherits a $\mathrm{PGL}_{r(k)}$-linearization.

Definition 4.7.21. A point $h \in H$ is said to be *GIT semistable with respect to* L_d if there exists an equivariant section $s \in \Gamma(H, L_d^{\otimes N})^{\mathrm{PGL}_{r(k)}}$ with $N > 0$ such that $s(h) \neq 0$. The *semistable locus* H^{ss} consisting of GIT semistable points is an open $\mathrm{PGL}_{r(k)}$-invariant subscheme.

Remark 4.7.22. Stack-theoretically, the $\mathrm{PGL}_{r(k)}$-linearization L_d defines a line bundle, which we will also denote by L_d, on the quotient stack $[H/\mathrm{PGL}_{r(k)}]$ and the open substack $[H^{\mathrm{ss}}/\mathrm{PGL}_{r(k)}]$ is the largest open substack such the restriction of L_d is semiample. In other words, $h \in H$ is GIT semistable if and only h does not lie in the stable base locus of L_d on $[H/\mathrm{PGL}_{r(k)}]$.

Remark 4.7.23. These definitions clearly extend to the action of any algebraic group G on a projective scheme X embedded G-equivariantly $X \hookrightarrow \mathbb{P}^N$ by a G-linearization L. One of the main results of GIT is that if G is reductive, then the graded ring $\bigoplus_{N \geq 0} \Gamma(H, L_d^{\otimes N})^{\mathrm{PGL}_{r(k)}}$ is finitely generated and that the morphism

$$X^{\mathrm{ss}} \to X^{\mathrm{ss}} /\!\!/ G := \mathrm{Proj} \bigoplus_{N \geq 0} \Gamma(H, L_d^{\otimes N})^{\mathrm{PGL}_{r(k)}}$$

is a good quotient. Note that X^{ss} is precisely the maximal locus where the rational map $X \dashrightarrow X^{\mathrm{ss}} /\!\!/ G$ is defined.

The GIT construction of $\overline{M}_g$ rests on the following difficult theorem:

Theorem 4.7.24. *Let $k \geq 5$ and $d \gg 0$. For $h = [C \hookrightarrow \mathbb{P}^{r(k)-1}] \in H$, the curve C is a stable if and only if $h \in H$ is GIT semistable with respect to L_d.*

Remark 4.7.25. This theorem can be established using the Hilbert–Mumford criteria. It is rather difficult to explicitly exhibit sections of $\Gamma(H, L_d^{\otimes N})^{\mathrm{PGL}_{r(k)}}$ and the Hilbert–Mumford criteria allows us to verify that a given point $h \in H$ is semistable by checking that for each one-parameter subgroup $\lambda\colon \mathbb{G}_m \to \mathrm{PGL}_{r(k)}$, the Hilbert–Mumford index $\mu(h, L_d)$, defined as the weight of $\mathbb{G}_m$ on the line in the affine cone $\mathbb{A}^{r(k)}$ over $\lim_{t\to 0} \lambda(t) \cdot h \in H \subset \mathbb{P}^{r(k)-1}$, is negative. The beauty of the Hilbert–Mumford criterion is that it magically guarantees the existence of sections for you! Nevertheless, verifying the Hilbert–Mumford criterion even for a smooth pluricanonical embedded curve is no easy task.

Given Theorem 4.7.24, we obtain $\overline{M}_g$ as the projective variety

$$\overline{M}_g = \operatorname{Proj} \Gamma(H, L_d^{\otimes N})^{\mathrm{PGL}_{r(k)}}.$$

Remark 4.7.26. As a spectacular corollary of Theorem 4.7.24, one obtains an alternative proof of Stable Reduction (Theorem 4.4.1) in arbitrary characteristic. This is perhaps surprising as the GIT argument uses rather little about the geometry of stable curves and their families.

Remark 4.7.27 (The ample cone)**.** For each $k \geq 5$ and $d \gg 0$, GIT constructs a line bundle on on $\overline{\mathcal{M}}_g$ which descends to an ample line bundle on $\overline{M}_g$. This class can be expressed as

$$r(k)\lambda_{dk} - r(dk)\lambda_k.$$

Grothendieck–Riemann–Roch can be used to express each of the line bundles λ_k as a linear combination of λ_1 and δ, the boundary divisor. The asymptotic limit of this class as d goes to infinity is proportional to

$$(12 - \frac{4}{k})\lambda_1 - \delta.$$

Taking $k = 5$, shows that $11.2\lambda - \delta$ is ample.

However, even more is true! By bootstrapping the positivity deduced from GIT, Cornalba and Harris showed that $a\lambda - \delta$ is ample if and only if $a > 11$, thus determining the ample cone of $\overline{M}_g$ in the $\lambda_1\delta$-plane of $\mathrm{NS}^1(\overline{\mathrm{M}}_{\mathrm{g}})$ [CH88].

Appendix A

Properties of morphisms

In this appendix, we recall definitions and summarize properties for certain types morphisms of schemes—locally of finite presentation, flat, smooth, étale, and unramified.

We pay particular attention to properties that can be described *functorially*, i.e. properties of schemes and their morphisms that can be characterized in terms of their functors. The following properties of morphisms can be characterized functorially:

- separated, universally closed and proper;
- locally of finite presentation; and
- smooth, étale and unramified.

Such descriptions are particularly advantageous for us since we systematically study moduli problems via functors and stacks. For example, the valuative criterion for properness for $\overline{\mathcal{M}}_g$ amounts to checking that every family of curves over a punctured curve (i.e. over the generic point of a DVR) can be extended uniquely (after possibly a finite extension of the curve) to the entire curve (i.e. DVR). Similarly, the smoothness of $\overline{\mathcal{M}}_g$ can be shown by using the functorial formal lifting criterion for smoothness.

A.1 Morphisms locally of finite presentation

A morphism of schemes $f\colon X \to Y$ is *locally of finite type* (resp. *locally of finite presentation*) if for all affine open $\operatorname{Spec} B \subset Y$ and $\operatorname{Spec} A \subset f^{-1}(\operatorname{Spec} B)$, there is surjection $A[x_1, \ldots, x_n] \to B$ of A-algebras (resp. a surjection $\phi\colon A[x_1, \ldots, x_n] \to B$ such that the ideal $\ker(\phi) \subset A[x_1, \ldots, x_n]$) is finitely generated). If in addition f is quasi-compact (resp. quasi-compact and quasi-separated), we say that f is *of finite type* (resp. *of finite presentation*).

Remark A.1.1. When Y is locally noetherian, these two notions coincide. However, in the non-noetherian setting even closed immersions may not be locally of finite presentation; e.g. $\operatorname{Spec} \mathbb{C} \hookrightarrow \operatorname{Spec} \mathbb{C}[x_1, x_2, \ldots]$. Since functors and stacks are defined in these notes on the entire category of schemes, it is often necessary to work with non-noetherian schemes. In particular, when defining a moduli

functor or stack, we need to specify what families of objects are over possibly non-noetherian schemes. Morphisms of finite presentation are better behaved than morphisms of finite type and so we often use the former condition. For example, when defining a family of smooth curves $\pi\colon \mathcal{C} \to S$, we require not only that π is proper and smooth, but also of finite presentation.

The following is a very useful functorial criterion for a morphism to be locally of finite presentation. First recall that an *inverse system* (or *projective system*) in a category $\mathcal{C}$ is a partially ordered set $(I, \geq)$ which is filtered (i.e. for every $i, j \in I$ there exists $k \in I$ such that $k \geq i$ and $k \geq j$) together with a functor $I \to \mathcal{C}$.

Proposition A.1.2. *A morphism $f\colon X \to Y$ of schemes is locally of finite presentation if and only if for every inverse system $\{\operatorname{Spec} A_\lambda\}_{\lambda \in I}$ of schemes over Y, the natural map*

$$\varinjlim_\lambda \operatorname{Mor}_Y(\operatorname{Spec} A_\lambda, X) \to \operatorname{Mor}_Y(\operatorname{Spec}(\varinjlim_\lambda A_\lambda), X) \tag{A.1.1}$$

is bijective.

We won't include a proof here but we will mention a conceptual reason for why you might expect this to be true: any ring A (e.g. $\mathbb{C}[x_1, x_2, \ldots]$) is the union (or colimit) of its finitely generated subalgebras A_λ. The requirement that any map $\operatorname{Spec} A \to X$ factors through $\operatorname{Spec} A_\lambda \to X$ for some λ can be viewed as the condition that specifying $\operatorname{Spec} A \to X$ over Y depends on only a finite amount of data and therefore can is a type of finiteness condition on X over Y. We encourage the reader to convince themselves the above proposition holds in the case of a morphism of affine schemes.

Remark A.1.3. As we desire to define and study moduli stacks $\mathcal{X}$ that are of finite type over a field $\Bbbk$, the following analogous condition to (A.1.1) better hold: for all inverse system $\{\operatorname{Spec} A_\lambda\}_{\lambda \in I}$ of $\Bbbk$-schemes, the natural functor

$$\varinjlim_\lambda \operatorname{MOR}_\Bbbk(\operatorname{Spec} A_\lambda, \mathcal{X}) \to \operatorname{MOR}_\Bbbk(\operatorname{Spec}(\varinjlim_\lambda A_\lambda), \mathcal{X})$$

is an equivalence. It turns out for many moduli stacks, this condition can be checked directly even before knowing algebraicity. In fact, this locally of finite presentation condition (often also referred to as *limit-preserving*) is the first axiom in Artin's criteria for algebraicity.

A.2 Flatness

You can't get very far in moduli theory without internalizing the concept of flatness. While its definition is seemingly abstract and algebraic, it is a magical geometric property of a morphism $X \to Y$ that ensures that fibers X_y 'vary nicely' as $y \in Y$ varies. This principle is nicely illustrated by the fact that a subscheme $X \subset \mathbb{P}^n_Y$ is flat over an integral scheme Y if and only if the function assigning a point y to the Hilbert polynomial of the fiber $X_y \subset \mathbb{P}^n_{\kappa(y)}$ is constant (Proposition A.2.5).

A.2.1 Definition and equivalences

A morphism $f\colon X \to Y$ of schemes is *flat* if for all affine opens $\operatorname{Spec} B \subset Y$ and $\operatorname{Spec} A \subset f^{-1}(\operatorname{Spec} B)$, the ring map $B \to A$ is flat, i.e. the functor

$$- \otimes_B A\colon \operatorname{Mod}(B) \to \operatorname{Mod}(A)$$

is exact. More generally, a quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ is *flat over* Y if for all affine opens as above, $\Gamma(\operatorname{Spec} A, \mathcal{F})$ is a flat B-module, i.e. the functor $- \otimes_B \Gamma(\operatorname{Spec} A, \mathcal{F})$ is exact.

Flat Equivalences A.2.1. Let $f\colon X \to Y$ be a morphism of schemes and $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. The following are equivalent:

(1) $\mathcal{F}$ is flat over Y;

(2) There exists a Zariski-cover $\{\operatorname{Spec} B_i\}$ of Y and $\{\operatorname{Spec} A_{ij}\}$ of $f^{-1}(\operatorname{Spec} B_i)$ such that $\Gamma(\operatorname{Spec} A_{ij}, \mathcal{F})$ is flat as an B_i-module under the ring map $B_i \to A_{ij}$;

(3) For all $x \in X$, the $\mathcal{O}_{X,x}$-module $\mathcal{F}_x$ is flat as an $\mathcal{O}_{Y,y}$-module.

(4) The functor

$$\operatorname{QCoh}(Y) \to \operatorname{QCoh}(X), \qquad \mathcal{G} \mapsto f^*\mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{F}$$

is exact.

If $x \in X$, we say that a morphism $f\colon X \to Y$ of schemes is *flat at* x (resp. a quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ is *flat at* x) if there exists a Zariski-open neighborhood $U \subset X$ containing x such that $f|_U$ (resp. $\mathcal{F}|_U$) is flat over Y. This is equivalent to the flatness of $\mathcal{O}_{X,x}$ (resp. $\mathcal{F}_x$) as an $\mathcal{O}_{Y,y}$-module.

A.2.2 Useful geometric properties

Proposition A.2.2 (Openness of Fppf Morphisms)**.** *Let $f\colon X \to Y$ be a morphism of schemes. If f is flat and locally of finite presentation, then $f(U) \subset Y$ is open for every open $U \subset X$.*

The following simple corollary will be used to reduce certain properties of flat and locally of finite presentation morphisms to the affine case.

Corollary A.2.3. *If $f\colon X \to Y$ is a faithfully flat and locally of finite presentation morphism of schemes and $\{V_i\}$ is an affine open cover of Y, then there exist an open cover $\{U_{ij}\}_{j\in J}$ of $f^{-1}(V_i)$ for each i such that U_{ij} is quasi-compact and $f(U_{ij}) = V_i$.*

Proposition A.2.4 (Flatness Criterion over Smooth Curves)**.** *Let C be an integral and regular scheme of dimension 1 (e.g. the spectrum of a DVR or a smooth connected curve over a field) and $X \to C$ a quasi-compact and quasi-separated morphism of schemes. A quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ is flat over C if and only if every associated point of $\mathcal{F}$ maps to the generic point of C.*

Recall that if $X \subset \mathbb{P}^n_K$ is a subscheme and $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_X$-module, the *Hilbert polynomial* of $\mathcal{F}$ is $P_{\mathcal{F}}(n) = \chi(X, \mathcal{F}(n)) \in \mathbb{Q}[n]$.

Proposition A.2.5 (Flatness vs the Hilbert Polynomial)**.** *Let Y be an integral scheme and $X \subset \mathbb{P}^n_Y$ a closed subscheme. A quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ is flat over Y if and only if the function*

$$Y \to \mathbb{Q}[n], \qquad y \mapsto P_{\mathcal{F}|_{X_y}}$$

assigning a point $y \in Y$ to the Hilbert polynomial of the restriction $\mathcal{F}|_{X_y}$ to the fiber $X_y \subset \mathbb{P}^n_{\kappa(y)}$ is constant.

Proposition A.2.6 (Generic flatness)**.** *Let $f\colon X \to S$ be a finite type morphism of schemes and $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_X$-module. If S is reduced, there exists an open dense subscheme $U \subset S$ such that $X_U \to U$ is flat and of presentation and such that $\mathcal{F}|_{X_U}$ is flat over U and of finite presentation as on $\mathcal{O}_{X_U}$-module.*

A.2.3 Faithful flatness

For a ring A, an A-module M is *faithfully flat* if for all non-zero map $\phi\colon N \to N'$ of A-modules, the induced map $\phi \otimes_A M\colon N \otimes_A M \to N' \otimes_A M$ is also non-zero.

Faithfully Flat Equivalences A.2.7. Let R be a ring and M be an A-module. The following are equivalent:

(1) M is faithfully flat;

(2) for any A-module N and non-zero element $n \in N$, the map $M \to N \otimes M$ given by $m \mapsto m \otimes n$ is non-zero;

(3) for any non-zero A-module N, we have $N \otimes_A M$ is non-zero;

(4) the functor $- \otimes_R M\colon \operatorname{Mod}(R) \to \operatorname{Mod}(R)$ is faithfully exact, i.e. a sequence $N' \to N \to N''$ of A-modules is exact if and only if $N' \otimes_A M \to N \otimes_A M \to N'' \otimes_A M$ is exact; and

(5) M is flat and for all maximal ideals $\mathfrak{m} \subset A$, the quotient $M/\mathfrak{m}M$ is non-zero.

If in addition $M = B$ is an A-algebra, then the above are also equivalent to:

(6) $\operatorname{Spec} B \to \operatorname{Spec} A$ is flat and surjective.

A morphism $f\colon X \to Y$ of schemes is *faithfully flat* if f is flat and surjective. This is equivalent to the condition that $f^*\colon \operatorname{QCoh}(Y) \to \operatorname{QCoh}(X)$ is faithfully exact. It is also equivalent to the condition that a quasi-coherent $\mathcal{O}_Y$-module (resp. a morphism of quasi-coherent $\mathcal{O}_Y$-modules) is zero if and only if its pullback is.

A.3 Étale, smooth and unramified morphisms

A.3.1 Smooth morphisms

A morphism $f\colon X \to Y$ of schemes is *smooth* if f is locally of finite presentation and flat, and the geometric fiber $X_{\overline{\kappa(y)}} = X \times_Y \operatorname{Spec} \overline{\kappa(y)}$ of any point $y \in Y$ is regular.

Smooth Equivalences A.3.1. Let $f\colon X \to Y$ be morphism of schemes locally of finite presentation. The following are equivalent:

(1) f is smooth;

(2) f is formally smooth, i.e. for any surjection $A \to A_0$ of rings with nilpotent kernel and any commutative diagram

$$\begin{array}{ccc} \operatorname{Spec} A_0 & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow f \\ \operatorname{Spec} A & \longrightarrow & Y \end{array}$$

of solid arrows, there exists a dotted arrow filling in the diagram;

(This is often referred to as the *Formal Lifting Criterion for Smoothness.*)

(3) for every point $x \in X$, there exist affine open neighborhoods $\operatorname{Spec} B$ of $f(x)$ and $\operatorname{Spec} A \subset f^{-1}(\operatorname{Spec} B)$ of x and an A-algebra isomorphism

$$B \cong \big(A[x_1, \ldots, x_n]/(f_1, \ldots, f_r)\big)_g$$

for some $f_1, \ldots, f_r, g \in A[x_1, \ldots, x_n]$ with $r \leq n$ such that the determinant $\det(\frac{\delta f_j}{\delta x_i})_{1 \leq i,j \leq r} \in B$ of the Jacobi matrix, defined by the partial derivatives with respect *first* r x_i's, is a unit.

(This is often referred to as the *Jacobi Criterion for Smoothness.*)

If in addition X and Y are locally of finite type over an algebraically closed field K, then the above are equivalent to:

(4) for all $x \in X(K)$, there is an isomorphism $\widehat{\mathcal{O}}_{X,x} \cong \widehat{\mathcal{O}}_{Y,y}[[x_1, \ldots, x_r]]$ of $\widehat{\mathcal{O}}_{Y,y}$-algebras.

If $f\colon X \to Y$ is a smooth morphism of schemes, then $\Omega_{X/Y}$ is a locally free $\mathcal{O}_X$-module of finite rank. If Y is connected, the rank of $\Omega_{X/Y}$ is the dimension of any fiber.

A.3.2 Étale morphisms

A morphism $f\colon X \to Y$ of schemes is *étale* if f is smooth of relative dimension 0 (i.e. f is smooth and $\dim X_y = 0$ for all $y \in Y$).

Étale Equivalences A.3.2. Let $f\colon X \to Y$ be morphism of schemes locally of finite presentation. The following are equivalent:

(1) f is étale;

(2) f is smooth and $\Omega_{X/Y} = 0$;

(3) f is flat and for all $y \in Y$, the fiber X_y is isomorphic to a disjoint union $\bigsqcup_i \operatorname{Spec} K_i$ where each K_i is separable field extension of $\kappa(y)$; (This is exactly the condition that f is flat and unramified; see Section A.3.3.)

(4) f is formally étale, i.e. for any surjection $A \to A_0$ of rings with nilpotent kernel and any commutative diagram

$$\begin{array}{ccc} \operatorname{Spec} A_0 & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow f \\ \operatorname{Spec} A & \longrightarrow & Y \end{array}$$

of solid arrows, there exists a unique dotted arrow filling in the diagram;

(This is often referred to as the *Formal Lifting Criterion for Étaleness.*)

(5) for every point $x \in X$, there exist affine open neighborhoods $\operatorname{Spec} B$ of $f(x)$ and $\operatorname{Spec} A \subset f^{-1}(\operatorname{Spec} B)$ of x and an A-algebra isomorphism

$$B \cong \big(A[x_1, \ldots, x_n]/(f_1, \ldots, f_n)\big)_g$$

for some $f_1, \ldots, f_n, g \in A[x_1, \ldots, x_n]$ such that the determinant $\det(\frac{\delta f_j}{\delta x_i})_{1 \leq i,j \leq n} \in B$ is a unit.

(This is often referred to as the *Jacobi criterion for étaleness.*)

If in addition X and Y are locally of finite type over an algebraically closed field K, then the above are equivalent to:

(6) for all $x \in X(K)$, the induced map $\widehat{\mathcal{O}}_{Y,y} \to \widehat{\mathcal{O}}_{X,x}$ on completions is an isomorphism

If in addition X and Y are smooth over K, then the above are equivalent to:

(7) for all $x \in X(K)$, the induced map $T_{X,x} \to T_{Y,y}$ on tangent spaces is an isomorphism.

A.3.3 Unramified morphisms

A morphism $f\colon X \to Y$ of schemes is *unramified* if f is locally of finite type and every geometric fiber is discrete and reduced. Note that this second condition is equivalent to requiring that for all $y \in Y$, the fiber X_y is isomorphic to a disjoint union $\bigsqcup_i \operatorname{Spec} K_i$ where each K_i is separable field extension of $\kappa(y)$.

⚠ **Warning A.3.3.** We are following the conventions of [RG71] and [SP] rather than [EGA] as we only require that f is locally of finite type rather than locally of finite presentation.

Unramified Equivalences A.3.4. Let $f\colon X \to Y$ be morphism of schemes locally of finite type. The following are equivalent:

(1) f is unramified;

(2) $\Omega_{X/Y} = 0$;

(3) f is formally unramifed, i.e. for any surjection $A \to A_0$ of rings with nilpotent kernel and any commutative diagram

$$\begin{array}{ccc} \operatorname{Spec} A_0 & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow f \\ \operatorname{Spec} A & \longrightarrow & Y \end{array}$$

of solid arrows, there exists at most one dotted arrow filling in the diagram.

(This is often referred to as the *Formal Lifting Criterion for Unramifiedness.*)

If in addition X and Y are locally of finite type over an algebraically closed field K, then the above are equivalent to:

(4) for all $x \in X(K)$, the induced map $\widehat{\mathcal{O}}_{Y,y} \to \widehat{\mathcal{O}}_{X,x}$ on completions is surjective.

A.3.4 Further properties

The following proposition states that any smooth morphism $X \to Y$ is étale locally (on the source and target) of the form $\mathbb{A}^n_R \to \operatorname{Spec} R$ and in particular has sections étale locally on the target.

Proposition A.3.5. *Let $X \to Y$ be a morphism of schemes which is smooth at a point $x \in X$. There exists affine open subschemes $\operatorname{Spec} A \subset X$ and $\operatorname{Spec} B \subset Y$ with $x \in \operatorname{Spec} A$, and a commutative diagram*

$$\begin{array}{ccccc} X & \longleftarrow & \operatorname{Spec} A & \longrightarrow & \mathbb{A}^n_B \\ \downarrow & & \downarrow & \swarrow & \\ Y & \longleftarrow & \operatorname{Spec} B & & \end{array}$$

where $U \to \mathbb{A}^n_B$ is étale.

Proposition A.3.6 (Fiberwise criteria for étaleness/smoothness/unramifiedness)**.** *Consider a diagram*

$$\begin{array}{ccc} X & \longrightarrow & Y \\ & \searrow \quad \swarrow & \\ & S & \end{array}$$

of schemes where $X \to S$ and $Y \to S$ are locally of finite presentation. Assume that $X \to S$ is flat in the étale/smooth case. Then $X \to Y$ is étale (resp. smooth, unramified) if and only if $X_s \to Y_s$ is for all $s \in S$.

Remark A.3.7. With the same hypotheses, let $x \in X$ be a point with image $s \in S$. Then $X \to Y$ is étale (resp. smooth, unramified) at $x \in X$ if and only if $X_s \to Y_s$ is at x.

Corollary A.3.8. *If $f\colon X \to Y$ is a proper morphism of finite presentation, then the set $y \in Y$ such that $X_y \to \operatorname{Spec} \kappa(y)$ is smooth defines an open subset.*

Proof. By Remark A.3.7, if $y \in Y$ is a point such that $X_y \to \operatorname{Spec} \kappa(y)$ is smooth, then $f\colon X \to Y$ is smooth in an open neighborhood of X_y. If $Z \subset X$ is the closed locus where $f\colon X \to Y$ is not smooth, then $f(Z) \subset Y$ is precisely the locus where the fibers of f are not smooth. Since f is proper, $f(Z)$ is closed. □

Proposition A.3.9. *Let $X \to Y$ be a smooth morphism of noetherian schemes. For any point $x \in X$ with image $y \in Y$,*

$$\dim_x(X) = \dim_y(Y) + dim_x(X_y).$$

A.4 Artin approximation

In this section, we discuss the deep result of Artin Approximation (Theorem A.4.10) which can be vaguely expressed as the following principle:

> **Principle.** Algebraic properties that hold for the completion $\widehat{\mathcal{O}}_{S,s}$ of the local ring of a scheme S at a point s also hold in an étale neighborhood $(S', s') \to (S, s)$.

Artin approximation is related to another equally deep and powerful result known as Néron–Popescu Desingularization (Theorem A.4.4). Both Artin Approximation and Néron–Popescu are difficult theorems which we will not attempt to prove here. However, we will show at least how Artin Approximation easily follows from Néron–Popescu Desingularization.

A.4.1 Néron–Popescu Desingularization

Definition A.4.1. A ring homomorphism $A \to B$ of noetherian rings is called *geometrically regular* if $A \to B$ is flat and for every prime ideal $\mathfrak{p} \subset A$ and every finite field extension $k(\mathfrak{p}) \to k'$ (where $k(\mathfrak{p}) = A_{\mathfrak{p}}/\mathfrak{p}$), the fiber $B \otimes_A k'$ is regular.

Remark A.4.2. It is important to note that $A \to B$ is *not* assumed to be of finite type. In the case that $A \to B$ is a ring homomorphism (of noetherian rings) of finite type, then $A \to B$ is geometrically regular if and only if $A \to B$ is smooth (i.e. $\operatorname{Spec} B \to \operatorname{Spec} A$ is smooth).

Remark A.4.3. It can be shown that it is equivalent to require the fibers $B \otimes_A k'$ to be regular only for *inseparable* field extensions $k(\mathfrak{p}) \to k'$. In particular, in characteristic 0, $A \to B$ is geometrically regular if it is flat and for every prime ideal $\mathfrak{p} \subset A$, the fiber $B \otimes_A k(\mathfrak{p})$ is regular.

Theorem A.4.4 (Néron–Popescu Desingularization)**.** *Let $A \to B$ be a ring homomorphism of noetherian rings. Then $A \to B$ is geometrically regular if and only if $B = \varinjlim B_\lambda$ is a direct limit of smooth A-algebras.*

Remark A.4.5. This was result was proved by Néron in [Nér64] in the case that A and B are DVRs and in general by Popescu in [Pop85], [Pop86], [Pop90]. We recommend [Swa98] and [SP, Tag 07GC] for an exposition on this result.

Example A.4.6. If l is a field and l^s denotes its separable closure, then $l \to l^s$ is geometrically regular. Clearly, l^s is the direct limit of separable field extensions $l \to l'$ (i.e. étale and thus smooth l-algebras). If l is a perfect field, then any field extension $l \to l'$ is geometrically regular—but if $l \to l'$ is not algebraic, it is not possible to write l' is a direct limit of étale l-algebras. On the other hand, if l is a non-perfect field, then $l \to \bar{l}$ is not geometrically regular as the geometric fiber is non-reduced and thus not regular.

In order to apply Néron–Popescu Desingularization, we will need the following result, which we will also accept as a black box. The proof is substantially easier than Néron–Popescu's result but nevertheless requires some effort.

Theorem A.4.7. *If S is a scheme of finite type over a field $\Bbbk$ or $\mathbb{Z}$ and $s \in S$ is a point, then $\mathcal{O}_{S,s} \to \widehat{\mathcal{O}}_{S,s}$ is geometrically regular.*

Remark A.4.8. See [EGA, IV.7.4.4] or [SP, Tag 07PX] for a proof.

Remark A.4.9. A local ring A is called a *G-ring* if the homomorphism $A \to \widehat{A}$ is geometrically regular. We remark that one of the conditions for a scheme S to be *excellent* is that every local ring is a G-ring. Any scheme that is finite type over a field or $\mathbb{Z}$ is excellent.

A.4.2 Artin Approximation

Let S be a scheme and consider a contravariant functor

$$F\colon \operatorname{Sch}/S \to \operatorname{Sets}$$

where $\mathrm{Sch}\,/S$ denotes the category of schemes over S. An important example of a contravariant functor is the functor representing a scheme: if X is a scheme over S, then the *functor representing* X is:

$$h_X\colon \mathrm{Sch}\,/S \to \mathrm{Sets}, \qquad (T \to S) \mapsto \mathrm{Mor}_S(T, X). \tag{A.4.1}$$

We say that F is *locally of finite presentation* or *limit preserving* if for every direct limit $\varinjlim B_\lambda$ of $\mathcal{O}_S$-algebras B_λ (i.e. a direct limit of commutative rings B_λ together with morphisms $\operatorname{Spec} B_\lambda \to S$), the natural map

$$\varinjlim F(\operatorname{Spec} B_\lambda) \to F(\operatorname{Spec} \varinjlim B_\lambda)$$

is bijective. This should be viewed as a finiteness condition on the functor F. Indeed, a scheme X is locally of finite presentation over S if and only if its function $\mathrm{Mor}_S(-, X)$ is (Proposition A.1.2).

Theorem A.4.10 (Artin Approximation)**.** *Let S be an excellent scheme (e.g. a scheme of finite type over a field or $\mathbb{Z}$) and let*

$$F\colon \mathrm{Sch}\,/S \to \mathrm{Sets}$$

be a limit preserving contravariant functor. Let $s \in S$ be a point and $\widehat{\xi} \in F(\operatorname{Spec} \widehat{\mathcal{O}}_{S,s})$. For any integer $N \geq 0$, there exist a residually-trivial étale morphism

$$(S', s') \to (S, s) \qquad \text{and} \qquad \xi' \in F(S')$$

such that the restrictions of $\widehat{\xi}$ and ξ' to $\operatorname{Spec}(\mathcal{O}_{S,s}/\mathfrak{m}_s^{N+1})$ are equal.

Remark A.4.11. The following theorem was originally proven in [Art69, Cor. 2.2] in the case that S is of finite type over a field or an excellent dedekind domain. We also recommend [BLR90, §3.6] for an accessible account of the case of excellent and henselian DVRs.

Remark A.4.12. The condition that $(S', s') \to (S, s)$ is residually trivial means that the extension of residue fields $\kappa(s) \to \kappa(s')$ is an isomorphism. To make sense of the restriction ξ' to $\operatorname{Spec}(\mathcal{O}_{S,s}/\mathfrak{m}_s^{N+1})$, note that since $(S', s') \to (S, s)$ is a residually-trivial étale morphism, there are compatible identifications $\mathcal{O}_{S,s}/\mathfrak{m}_s^{N+1} \cong \mathcal{O}_{S',s'}/\mathfrak{m}_{s'}^{N+1})$.

Remark A.4.13. It is not possible in general to find $\xi' \in F(S')$ restricting to $\widehat{\xi}$ or even such that the restrictions of ξ' and $\widehat{\xi}$ to $\operatorname{Spec} \mathcal{O}_{S,s}/\mathfrak{m}_s^{n+1}$ agree for all $n \geq 0$. For instance, F could be the functor $\mathrm{Mor}(-, \mathbb{A}^1)$ representing the affine line $\mathbb{A}^1$ and $\widehat{\xi} \in \widehat{\mathcal{O}}_{S,s}$ could be a non-algebraic power series.

A.4.3 Alternative formulation of Artin Approximation

Consider the functor $F\colon \mathrm{Sch}\,/S \to \mathrm{Sets}$ representing an affine scheme $X = \operatorname{Spec} A[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$ of finite type over an excellent affine scheme $S = \operatorname{Spec} A$. Restricted to the category of affine schemes over S (or equivalently A-algebras), the functor is:

$$\begin{aligned} F\colon \mathrm{AffSch}\,/S &\to \mathrm{Sets} \\ \operatorname{Spec} B &\mapsto \{a = (a_1, \ldots, a_n) \in B^{\oplus n} \mid f_i(a) = 0 \text{ for all } i\} \end{aligned}$$

Applying Artin Approximation to the functor F, we obtain:

Corollary A.4.14. *Let R be an excellent ring and A be a finitely generated R-algebra. Let $\mathfrak{m} \subset A$ be a maximal ideal. Let $f_1, \dots, f_m \in A[x_1, \dots, x_n]$ be polynomials. Let $\widehat{a} = (\widehat{a}_1, \dots, \widehat{a}_n) \in \widehat{A}_{\mathfrak{m}}$ be a solution to the equations $f_1(x) = \cdots = f_m(x) = 0$. Then for every $N \geq 0$, there exist a residually-trivial étale ring homomorphism $(A, \mathfrak{m}) \to (A', \mathfrak{m}')$ and a solution $a' = (a'_1, \dots, a'_n) \in A'^{\oplus n}$ to the equations $f_1(x) = \cdots = f_m(x) = 0$ such that $a' \cong \widehat{a} \mod \mathfrak{m}^{N+1}$.* □

Remark A.4.15. Although this corollary may seem weaker than Artin Approximation, it is not hard to see that it in fact directly implies Artin Approximation. Indeed, writing $S = \operatorname{Spec} A$, we may write $\widehat{\mathcal{O}}_{S,s}$ as a direct limit of finite type A-algebras and since F is limit preserving, we can find a commutative diagram

$$\begin{array}{ccc} \operatorname{Spec} \widehat{\mathcal{O}}_{S,s} & & \\ \downarrow & \overset{\widehat{\xi}}{\searrow} & \\ \operatorname{Spec} A[x_1, \dots, x_n]/(f_1, \dots, f_m) & \xrightarrow[\xi]{} & F. \end{array}$$

The vertical morphism corresponds to a solution $\widehat{a} = (\widehat{a}_1, \dots, \widehat{a}_n) \in \widehat{\mathcal{O}}_{S,s}^{\oplus n}$ to the equations $f_1(x) = \cdots = f_m(x) = 0$. Applying Corollary A.4.14 yields the desired étale morphism $(\operatorname{Spec} A', s') \to (\operatorname{Spec} A, s)$ and a solution $a' = (a'_1, \dots, a'_n) \in A'^{\oplus n}$ to the equations $f_1(x) = \cdots = f_m(x) = 0$ agreeing with $\widehat{a}$ up to order N (i.e. congruent modulo $\mathfrak{m}^{N+1}$). This induces a morphism

$$\xi' \colon \operatorname{Spec} A' \to \operatorname{Spec} A[x_1, \dots, x_n]/(f_1, \dots, f_m) \to F$$

which agrees with $\widehat{\xi} \colon \operatorname{Spec} \widehat{\mathcal{O}}_{S,s} \to F$ to order N.

Alternatively, we can state Corollary A.4.14 using henselian rings. Recall that a local ring $(A, \mathfrak{m})$ is called *henselian* if the following analogue of the implicit function theorem holds: if $f_1, \dots, f_n \in A[x_1, \dots, x_n]$ and $\overline{a} = (\overline{a}_1, \dots, \overline{a}_n) \in (A/\mathfrak{m})^{\oplus n}$ is a solution to the equations $f_1(x) = \cdots = f_n(x) = 0$ modulo $\mathfrak{m}$ and $\det \left(\frac{\partial f_i}{\partial x_j}(\overline{a}) \right)_{i,j=1,\dots,n} \neq 0$, then there exists a solution $a = (a_1, \dots, a_n) \in A^{\oplus n}$ to the equations $f_1(x) = \cdots = f_n(x) = 0$. Equivalently, if $(A, \mathfrak{m})$ is a local $\Bbbk$-algebra with $A/\mathfrak{m} \cong k$, then $(A, \mathfrak{m})$ is henselian if every étale homomorphism $(A, \mathfrak{m}) \to (A', \mathfrak{m}')$ of local rings with $A/\mathfrak{m} \cong A'/\mathfrak{m}'$ is an isomorphism. Also, if S is a scheme and $s \in S$ is a point, one defines the *henselization* $\mathcal{O}_{S,s}^h$ *of* S *at* s to be

$$\mathcal{O}_{S,s}^h = \varinjlim_{(S',s') \to (S,s)} \Gamma(S', \mathcal{O}_{S'})$$

where the direct limit is over all étale morphisms $(S', s') \to (S, s)$. In other words, $\mathcal{O}_{S,s}^h$ is the local ring of S at s in the étale topology.

Corollary A.4.16. *Let $(A, \mathfrak{m})$ be an excellent local henselian ring (e.g. the henselization of the local ring of a scheme of finite type over a field or $\mathbb{Z}$). Let $f_1, \dots, f_m \in A[x_1, \dots, x_n]$. Suppose that $\widehat{a} = (\widehat{a}_1, \dots, \widehat{a}_n) \in \widehat{A}^{\oplus n}$ is a solution to the equations $f_1(x) = \cdots = f_m(x) = 0$. For any integer $N \geq 0$, there exists a solution $a = (a_1, \dots, a_n) \in A^{\oplus n}$ to the equations $f_1(x) = \cdots = f_m(x) = 0$ such that $\widehat{a} \cong a \mod \mathfrak{m}^{N+1}$.*

A.4.4 A first application of Artin Approximation

The next corollary states an important fact which you may have taken for granted: if two schemes are formally isomorphic at two points, then they are isomorphic in the étale topology.

Corollary A.4.17. *Let X_1, X_2 be schemes of finite type over an excellent scheme S. Suppose $x_1 \in X_1, x_2 \in X_2$ are points such that $\widehat{\mathcal{O}}_{X_1,x_1}$ and $\widehat{\mathcal{O}}_{X_2,x_2}$ are isomorphic as $\mathcal{O}_S$-algebras. Then there exists a common residually-trivial étale neighborhood*

$$\begin{array}{ccc} & (X_3, x_3) & \\ \swarrow & & \searrow \\ (X_1, x_1) & & (X_2, x_2)\,. \end{array} \tag{A.4.2}$$

Proof. The functor

$$F\colon \operatorname{Sch}/X_1 \to \operatorname{Sets}, \quad (T \to X_1) \mapsto \operatorname{Mor}(T, X_2)$$

is limit preserving as it can be identified with the representable functor $\operatorname{Mor}_{X_1}(-, X_2 \times X_1)$ corresponding to the finite type morphism $X_2 \times X_1 \to X_1$. The isomorphism $\widehat{\mathcal{O}}_{X_1,x_1} \cong \widehat{\mathcal{O}}_{X_2,x_2}$ provides an element of $F(\operatorname{Spec} \widehat{\mathcal{O}}_{X_1,x_1})$. By applying Artin Approximation with $N = 1$, we obtain a diagram as in (A.4.2) with $X_3 \to X_1$ étale at x_3 with $\kappa(x_2) \xrightarrow{\sim} \kappa(x_3)$ and such that $\mathcal{O}_{X_2,x_2}/\mathfrak{m}_{x_2}^2 \to \mathcal{O}_{X_3,x_3}/\mathfrak{m}_{x_3}^2$ is an isomorphism. By Lemma A.4.18, $\widehat{\mathcal{O}}_{X_2,x_2} \to \widehat{\mathcal{O}}_{X_3,x_3}$ is surjective. But we also know that $\widehat{\mathcal{O}}_{X_3,x_3}$ is abstractly isomorphic to $\widehat{\mathcal{O}}_{X_2,x_2}$ and since any surjective endomorphism of a noetherian ring is an isomorphism, we conclude that $\widehat{\mathcal{O}}_{X_2,x_2} \to \widehat{\mathcal{O}}_{X_3,x_3}$ is an isomorphism and therefore that $(X_3, x_3) \to (X_2, x_2)$ is étale. □

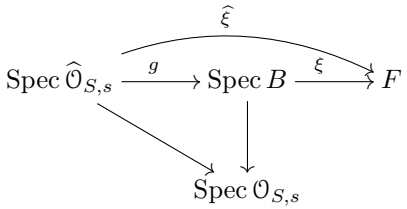
Lemma A.4.18. *Let $(A, \mathfrak{m}_A) \to (B, \mathfrak{m}_B)$ be a local homomorphism of noetherian complete local rings. If $A/\mathfrak{m}_A^2 \to B/\mathfrak{m}_B^2$ is surjective, so is $A \to B$.*

Proof. This follows from the following version of Nakayama's lemma for noetherian complete local rings $(A, \mathfrak{m})$: if M is a (not-necessarily finitely generated) A-module such that $\bigcap_k \mathfrak{m}^k M = 0$ and $m_1, \ldots, m_n \in F$ generate $M/\mathfrak{m}M$, then $m_1, \ldots, m_n$ also generate M (see [Eis95, Exercise 7.2]). □

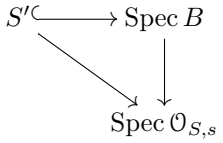
A.4.5 Néron–Pescue Desingularization $\Longrightarrow$ Artin Approximation

By Theorem A.4.7, the morphism $\mathcal{O}_{S,s} \to \widehat{\mathcal{O}}_{S,s}$ is geometrically regular. By Néron–Popescu Desingularization (Theorem A.4.4), $\widehat{\mathcal{O}}_{S,s} = \varinjlim B_\lambda$ is a direct limit of smooth $\mathcal{O}_{S,s}$-algebras. Since F is limit preserving, there exist λ, a factorization $\mathcal{O}_{S,s} \to B_\lambda \to \widehat{\mathcal{O}}_{S,s}$ and an element $\xi_\lambda \in F(\operatorname{Spec} B_\lambda)$ whose restriction to $F(\operatorname{Spec} \widehat{\mathcal{O}}_{S,s})$ is $\widehat{\xi}$.

Let $B = B_\lambda$ and $\xi = \xi_\lambda$. Geometrically, we have a commutative diagram



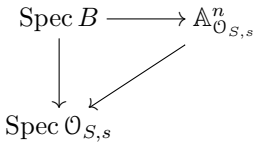
where $\operatorname{Spec} B \to \operatorname{Spec} \mathcal{O}_{S,s}$ is smooth. We claim that we can find a commutative diagram



(A.4.3)

where $S' \hookrightarrow \operatorname{Spec} B$ is a closed immersion, $(S', s') \to (\operatorname{Spec} \mathcal{O}_{S,s}, s)$ is étale, and the composition $\operatorname{Spec} \mathcal{O}_{S,s}/\mathfrak{m}_s^{N+1} \to S' \to \operatorname{Spec} B$ agrees with the restriction of $g\colon \operatorname{Spec} \widehat{\mathcal{O}}_{S,s} \to \operatorname{Spec} B$.[1]

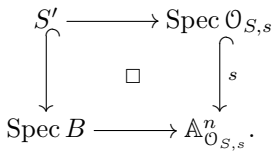
To see this, observe that the B-module of relative differentials $\Omega_{B/\mathcal{O}_{S,s}}$ is locally free. After shrinking $\operatorname{Spec} B$ around the image of the closed point under $\operatorname{Spec} \widehat{\mathcal{O}}_{S,s} \to \operatorname{Spec} B$, we may assume $\Omega_{B/\mathcal{O}_{S,s}}$ is free with basis $db_1, \ldots, db_n$. This induces a homomorphism $\mathcal{O}_{S,s}[x_1, \ldots, x_n] \to B$ defined by $x_i \mapsto b_i$ and provides a factorization



where $\operatorname{Spec} B \to \mathbb{A}^n_{\mathcal{O}_{S,s}}$ is étale. We may choose a lift of the composition

$$\mathcal{O}_{S,s}[x_1, \ldots, x_n] \to B \to \widehat{\mathcal{O}}_{S,s} \to \mathcal{O}_{S,s}/\mathfrak{m}_s^{N+1}$$

to a morphism $\mathcal{O}_{S,s}[x_1, \ldots, x_n] \to \mathcal{O}_{S,s}$. This gives a section $s\colon \operatorname{Spec} \mathcal{O}_{S,s} \to \mathbb{A}^n_{\mathcal{O}_{S,s}}$ and we define S' as the fibered product



This gives the desired Diagram A.4.3. The composition $\xi'\colon S' \to \operatorname{Spec} B \xrightarrow{\xi} F$ is an element which agrees with $\widehat{\xi}$ up to order N.

By "standard direct limit" methods, we may "smear out" the étale morphism $(S', s') \to (\operatorname{Spec} \mathcal{O}_{S,s}, s)$ and the element $\xi'\colon S' \to F$ to find an étale morphism

[1] This is where the approximation occurs. It is not possible to find a morphism $S' \to \operatorname{Spec} B \to \operatorname{Spec} \mathcal{O}_{S,s}$ which is étale at a point s' over s such that the composition $\operatorname{Spec} \widehat{\mathcal{O}}_{S,s} \to S' \to \operatorname{Spec} B$ is equal to g.

$(S'', s'') \to (S, s)$ and an element $\xi'' \colon S'' \to F$ agreeing with $\widehat{\xi}$ up to order N. Since this may not be standard for everyone, we spell out the details. Let $\operatorname{Spec} A \subset S$ be an open affine containing s. We may write $S' = \operatorname{Spec} A'$ and $A' = \mathcal{O}_{S,s}[y_1, \ldots, y_n]/(f'_1, \ldots, f'_m)$. As $\mathcal{O}_{S,s} = \varinjlim_{g \notin \mathfrak{m}_s} A_g$, we can find an element $g \notin \mathfrak{m}_s$ and elements $f''_1, \ldots, f''_m \in A_g[y_1, \ldots, y_n]$ restricting to $f'_1, \ldots, f'_m$. Let $S'' = \operatorname{Spec} A_g[y_1, \ldots, y_n]/(f''_1, \ldots, f''_m)$ and $s'' \in S''$ be the image of s' under $S' \to S''$. Then $S'' \to S$ is étale at s''. As $A' = \varinjlim_{h \notin \mathfrak{m}_s} A_{hg}[y_1, \ldots, y_n]/(f'_1, \ldots, f'_m)$ and F is limit preserving, we can, after replacing g with hg, find an element $\xi'' \in F(S'')$ restricting to ξ' and, in particular, agreeing with $\widehat{\xi}$ up to order N. Finally, we shrink S'' around s'' so that $S'' \to S$ is étale everywhere.

Appendix B

Descent

It is hard to overstate the importance of descent in moduli theory. The central idea of descent is as simple as it is powerful. You already know that many properties of schemes and their morphisms can be checked on a Zariski-cover, and descent theory states that they can also be checked on étale covers or even faithfully flat covers. For example, if $Y' \to Y$ is étale and surjective, then a morphism $X \to Y$ is proper if and only if $X \times_Y Y' \to Y'$ is.

The applications of descent reach far beyond moduli theory. For instance, it can be used to reduce statements about schemes over a field k to the case when k is algebraically closed since $k \to \overline{k}$ is faithfully flat, or reduce statements over a local noetherian ring A to its completion $\widehat{A}$ since $A \to \widehat{A}$ is faithfully flat.

References: [BLR90, Ch.6], [Vis05], [Ols16, Ch. 4], [SP, Tag 0238], [EGA, §IV.2], and [SGA1, §VIII.7] (other descent results are scattered throughout EGA and SGA).

B.1 Descent for quasi-coherent sheaves

Descent theory rests on the following algebraic fact.

Proposition B.1.1. *If $\phi\colon A \to B$ is a faithfully flat ring map, then the sequence*

$$A \xrightarrow{\phi} B \underset{b\mapsto 1\otimes b}{\overset{b\mapsto b\otimes 1}{\rightrightarrows}} B \otimes_A B$$

is exact. More generally, if M is an A-module, the sequence

$$M \xrightarrow{m\mapsto m\otimes 1} M \otimes_A B \underset{m\otimes b\mapsto m\otimes 1\otimes b}{\overset{m\otimes b\mapsto m\otimes b\otimes 1}{\rightrightarrows}} M \otimes_A B \otimes_A B \tag{B.1.1}$$

is exact.

Remark B.1.2. By Faithfully Flat Equivalences A.2.7, $A \to B$ and $M \to M \otimes_A B$ are necessarily injective.

Proof. Since $A \to B$ is faithfully flat, the sequence (B.1.1) is exact if and only if the sequence

$$M \otimes_A B \xrightarrow{m\otimes b'\mapsto m\otimes 1\otimes b'} M \otimes_A B \otimes_A B \underset{m\otimes b\otimes b'\mapsto m\otimes 1\otimes b\otimes b'}{\overset{m\otimes b\otimes b'\mapsto m\otimes b\otimes 1\otimes b'}{\rightrightarrows}} M \otimes_A B \otimes_A B \otimes_A B$$

is exact. The above sequence can be rewritten as

$$M \otimes_A B \xrightarrow{x \mapsto x \otimes 1} (M \otimes_A B) \otimes_B (B \otimes_A B) \underset{x \otimes y \mapsto x \otimes 1 \otimes y}{\overset{x \otimes y \mapsto x \otimes y \otimes 1}{\rightrightarrows}} (M \otimes_A B) \otimes_B (B \otimes_A B) \otimes_B (B \otimes_A B)$$

which is precisely sequence (B.1.1) applied to ring $B \to B \otimes_A B$ given by $b \mapsto 1 \otimes b$ and the B-module $M \otimes_A B$. Since this ring map has a section $B \otimes_A B \to B$ given by $b \otimes b' \mapsto bb'$, we can assume that in the statement $\phi\colon A \to B$ has a section $s\colon B \to A$ with $s \circ \phi = \mathrm{id}_A$. Let $x \in M \otimes_A B$ such that $x \otimes 1 = 1 \otimes x \in M \otimes_A B \otimes_A B$. Applying $\mathrm{id}_M \otimes \mathrm{id}_B \otimes s\colon M \otimes_A B \otimes_A B \to M \otimes_A B \otimes_A A \cong M \otimes_A B$ to the identity $x \otimes 1 = 1 \otimes x$ yields that $x = (\mathrm{id}_M \otimes s)(x) \in M$ where $\mathrm{id}_M \otimes s$ denotes the composition $M \otimes_A B \to M \otimes_A A \xrightarrow{\sim} M$. □

Proposition B.1.3. *Let $f\colon X \to Y$ be a faithfully flat morphism of schemes that is either quasi-compact or locally of finite presentation. Let $\mathcal{F}$ and $\mathcal{G}$ be quasi-coherent $\mathcal{O}_Y$-modules. Let p_1, p_2 denote the two projections $X \times_Y X \to X$ and q denote the composition $X \times_Y X \xrightarrow{p_i} X \xrightarrow{f} Y$. Then the sequence*

$$\mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{F}, \mathcal{G}) \xrightarrow{f^*} \mathrm{Hom}_{\mathcal{O}_X}(f^*\mathcal{F}, f^*\mathcal{G}) \underset{p_2^*}{\overset{p_1^*}{\rightrightarrows}} \mathrm{Hom}_{\mathcal{O}_{X \times_Y X}}(q^*\mathcal{F}, q^*\mathcal{G})$$

is exact.

Remark B.1.4. The special case that $\mathcal{F} = \mathcal{O}_Y$ implies that $0 \to \Gamma(Y, \mathcal{G}) \xrightarrow{f^*} \Gamma(X, f^*\mathcal{G}) \xrightarrow{p_1^* - p_2^*} \Gamma(X \times_Y X, q^*\mathcal{G})$ is exact. When X and Y are affine, this is precisely Proposition B.1.1.

Proof. This can be reduced to Proposition B.1.1 by first reducing to the case that Y is affine. If f is quasi-compact, we reduce to the case that X is affine by choosing a *finite* affine cover $\{U_i\}$ and replacing X with the affine scheme $\bigsqcup_i U_i$. If f is locally of finite presentation, we apply Corollary A.2.3 to reduce to the quasi-compact case. We leave the details to the reader. □

Proposition B.1.5. *Let $f\colon X \to Y$ be a faithfully flat morphism of schemes that is either quasi-compact or locally of finite presentation. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module and $\alpha\colon p_1^*\mathcal{F} \to p_2^*\mathcal{F}$ an isomorphism of $\mathcal{O}_{X \times_Y X}$-modules satisfying the cocycle condition $p_{12}^*\alpha \circ p_{23}^*\alpha = p_{13}^*\alpha$ on $X \times_Y X \times_X Y$. Then there exists a quasi-coherent $\mathcal{O}_Y$-module $\mathcal{G}$ and an isomorphism $\phi\colon \mathcal{F} \to f^*\mathcal{G}$ such that $p_1^*\phi = p_2^*\phi \circ \alpha$ on $X \times_Y X$. The data $(\mathcal{F}, \phi)$ is unique up to unique isomorphism.*

Remark B.1.6. The following diagram may be useful to internalize the above statement:

$$p_{12}^*\alpha \circ p_{23}^*\alpha = p_{13}^*\alpha \qquad p_1^*\mathcal{F} \xrightarrow{\alpha} p_2^*\mathcal{F} \qquad \mathcal{F} \qquad \exists \mathcal{G}$$

$$X \times_Y X \times_Y X \underset{p_{23}}{\overset{p_{12}}{\overset{\longrightarrow}{\overset{p_{13}}{\rightrightarrows}}}} X \times_Y X \underset{p_2}{\overset{p_1}{\rightrightarrows}} X \xrightarrow{f} Y$$

Keep in mind the special case that $X = \bigsqcup_i Y_i$ where $\{Y_i\}$ is an open covering of Y in which case the above fiber products correspond to intersections.

The cocycle condition $p_{12}^*\alpha \circ p_{23}^*\alpha = p_{13}^*\alpha$ should be understood as the commutativity of

$$\begin{array}{ccccc} p_{12}^*p_1^*\mathcal{F} & \xrightarrow{p_{12}^*\alpha} & p_{12}^*p_2^*\mathcal{F} & = & p_{23}^*p_1^*\mathcal{F} \\ \| & & & & \downarrow{\scriptstyle p_{23}^*\alpha} \\ p_{13}^*p_1^*\mathcal{F} & \xrightarrow{p_{13}^*\alpha} & p_{13}^*p_2^*\mathcal{F} & = & p_{23}^*p_2^*\mathcal{F} \end{array}$$

and the condition that $p_1^*\phi = p_2^*\phi \circ \alpha$ should be understood as the commutativity of

$$\begin{array}{ccc} p_1^*\mathcal{F} & \xrightarrow{p_1^*\phi} & p_1^*f^*\mathcal{G} \\ \downarrow{\scriptstyle \alpha} & & \| \\ p_2^*\mathcal{F} & \xrightarrow{p_2^*\phi} & p_2^*f^*\mathcal{G}. \end{array}$$

Remark B.1.7. Propositions B.1.3 and B.1.5 together can be reformulated as the statement that the category $\mathrm{QCoh}(Y)$ is equivalent to the *category of descent datum for* $X \to Y$, denoted by $\mathrm{QCoh}(X \to Y)$. Here the objects of $\mathrm{QCoh}(X \to Y)$ are pairs $(\mathcal{F}, \alpha)$ consisting of a quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ and an isomorphism $\alpha\colon p_1^*\mathcal{F} \to p_2^*\mathcal{F}$ satisfying the cocycle condition. A morphism $(\mathcal{F}', \alpha') \to (\mathcal{F}, \alpha)$ is a morphism $\beta\colon \mathcal{F}' \to \mathcal{F}$ such that

$$\begin{array}{ccc} p_1^*\mathcal{F}' & \xrightarrow{\alpha'} & p_2^*\mathcal{F}' \\ \downarrow{\scriptstyle p_1^*\beta} & & \downarrow{\scriptstyle p_2^*\beta} \\ p_1^*\mathcal{F} & \xrightarrow{\alpha} & p_2^*\mathcal{F} \end{array}$$

commutes.

B.2 Descent for morphisms

The following result implies that if Z is a scheme, the functor $\mathrm{Mor}(-, Z)\colon \mathrm{Sch} \to \mathrm{Sets}$ is a sheaf in the fppf topology.

Proposition B.2.1. *Let $f\colon X \to Y$ be a faithfully flat morphism of schemes that is either quasi-compact or locally of finite presentation. If $g\colon X \to Z$ is any morphism to a scheme such that $p_1 \circ g = p_2 \circ g$ on $X \times_Y X$, then there exists a unique morphism $h\colon Y \to Z$ filling in the commutative diagram*

$$\begin{array}{ccccc} X \times_Y X & \underset{p_2}{\overset{p_1}{\rightrightarrows}} & X & \xrightarrow{f} & Y \\ & & & \searrow^{g} & \downarrow{\scriptstyle h} \\ & & & & Z \end{array}$$

of solid arrows.

B.3 Descending schemes

Proposition B.3.1 (Effective Descent for Open and Closed Immersions)**.** *Let $f\colon X \to Y$ be a faithfully flat morphism of schemes that is either quasi-compact*

or locally of finite presentation. If $Z \subset X$ is a closed (resp. open) subscheme such that ${p_1}^{-1}(Z) = p_2^{-1}(Z)$ as closed (resp. open) subschemes of $X \times_Y X$, then there exists a closed (resp. open) subscheme $W \subset Y$ such that $Z = f^{-1}(W)$.

To formulate effective descent for morphisms that are not monomorphisms, we need to specify an isomorphism of pullbacks satisfying a cocycle condition. We will use the following notation: if $f\colon X \to Y$ and $W \to Y$ are morphisms of schemes, we denote f^*W as the fiber product $X \times_Y W$.

Proposition B.3.2 (Effective Descent for Affine Immersions)**.** *Let $f\colon X \to Y$ be a faithfully flat morphism of schemes that is either quasi-compact or locally of finite presentation. If $Z \to X$ is an affine morphism and $\alpha\colon {p_1}^*(Z) \xrightarrow{\sim} p_2^*(Z)$ is an isomorphism over $X \times_Y X$ satisfying $p_{12}^*\alpha \circ p_{23}^*\alpha = p_{13}^*\alpha$, then there exists an affine morphism $W \to Y$ and an isomorphism $\phi\colon Z \to f^*(W)$ such that $p_1^*\phi = p_2^*\phi \circ \alpha$.*

Remark B.3.3. It is helpful to interpret the above statement using the diagram

$$\begin{array}{ccccccc} p_{12}^*\alpha \circ p_{23}^*\alpha = p_{13}^*\alpha & & p_1^*Z & \xrightarrow{\alpha} & p_2^*Z & & Z & - - \to & W \\ & & \downarrow & & \downarrow & & \downarrow & & \vdots \\ X \times_Y X \times_Y X & \overset{p_{12}}{\underset{p_{23}}{\overset{p_{13}}{\rightrightarrows}}} & X \times_Y X & & \overset{p_1}{\underset{p_2}{\rightrightarrows}} & & X & \xrightarrow{f} & Y. \end{array}$$

Proposition B.3.4 (Effective Descent for Quasi-affine Immersions)**.** *Let $f\colon X \to Y$ be a faithfully flat morphism of schemes that is either quasi-compact or locally of finite presentation. If $Z \to X$ is a quasi-affine morphism and $\alpha\colon {p_1}^*(Z) \xrightarrow{\sim} p_2^*(Z)$ is an isomorphism over $X \times_Y X$ satisfying $p_{12}^*\alpha \circ p_{23}^*\alpha = p_{13}^*\alpha$, then there exists an quasi-affine morphism $W \to Y$ and an isomorphism $\phi\colon Z \to f^*(W)$ such that $p_1^*\phi = p_2^*\phi \circ \alpha$.*

Proposition B.3.5 (Effective Descent for Separated and Locally Quasi-finite Morphisms)**.** *Let $f\colon X \to Y$ be a faithfully flat morphism of schemes that is either quasi-compact or locally of finite presentation. If $Z \to X$ is a separated and locally quasi-finite morphism of schemes and $\alpha\colon {p_1}^*(Z) \xrightarrow{\sim} p_2^*(Z)$ is an isomorphism over $X \times_Y X$ satisfying $p_{12}^*\alpha \circ p_{23}^*\alpha = p_{13}^*\alpha$, then there exists an quasi-affine morphism $W \to Y$ and an isomorphism $\phi\colon Z \to f^*(W)$ such that $p_1^*\phi = p_2^*\phi \circ \alpha$.*

Corollary B.3.6. *Let $\mathcal{P}$ be one of the following properties of morphisms of schemes: open immersion, closed immersion, locally closed immersion, affine, quasi-affine or separated and locally quasi-finite. Let $f\colon X \to Y$ be a faithfully flat morphism of schemes that is either quasi-compact or locally of finite presentation. Let $Q \to Y$ be a map of sheaves and consider the fiber product*

$$\begin{array}{ccc} Q_X & \longrightarrow & Q \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y. \end{array}$$

If Q_X is a scheme and $Q_X \to X$ has $\mathcal{P}$, then Q is a scheme and $Q \to Y$ has $\mathcal{P}$.

Proof. As Q_X is the pullback of Q, there is a canonical isomorphism $\alpha\colon p_1^*Q_X \to p_2^*Q_X$ satisfying the cocycle condition. By Propositions B.3.1, B.3.2, B.3.4 and B.3.5, there exists a quasi-affine morphism $W \to Y$ that pulls back to $Q_X \to X$. The reader to left to check that the natural map $Q \to W$ is an isomorphism. □

B.4 Descending properties of schemes and their morphisms

B.4.1 Descending properties of morphisms

Proposition B.4.1 (Properties flat local on the target)**.** *Let $Y' \to Y$ be a faithfully flat morphism of schemes that is either quasi-compact or locally of finite presentation. Let $\mathcal{P}$ be one of the following properties of a morphism of schemes:*

(i) isomorphism;

(ii) surjective;

(iii) proper;

(iv) flat;

(v) smooth;

(vi) étale;

(vii) unramified.

Then $X \to Y$ has $\mathcal{P}$ if and only if $X \times_Y Y' \to Y'$ does.

Proposition B.4.2 (Properties smooth local on the source)**.** *Let $X' \to X$ be a smooth and surjective morphism of schemes. Let $\mathcal{P}$ be one of the following properties of a morphism of schemes:*

(i) surjective;

(ii) smooth;

Then $X \to Y$ has $\mathcal{P}$ if and only if $X' \to X \to Y$ does.

Proposition B.4.3 (Properties étale local on the source)**.** *Let $X' \to X$ be an étale and surjective morphism of schemes. Let $\mathcal{P}$ be one of the following properties of a morphism of schemes:*

(i) surjective;

(ii) étale;

(iii) smooth.

Then $X \to Y$ has $\mathcal{P}$ if and only if $X' \to X \to Y$ does.

MORE PROPERTIES TO BE ADDED

B.4.2 Descent for properties of quasi-coherent sheaves

Proposition B.4.4. *Let $f\colon X \to Y$ be a faithfully flat morphism of schemes that is either quasi-compact or locally of finite presentation. Let $\mathcal{P} \in \{$finite type, finite presentation, vector bundle$\}$ be a property of quasi-coherent sheaves. If $\mathcal{G}$ is a quasi-coherent $\mathcal{O}_Y$-module, then $\mathcal{G}$ has $\mathcal{P}$ if and only if $f^*\mathcal{G}$ does. If X and Y are noetherian, then the same holds for the property of coherence.*

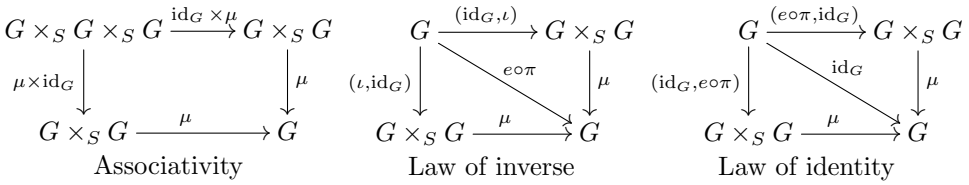
Appendix C

Algebraic groups and actions

C.1 Algebraic groups

C.1.1 Group schemes

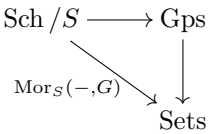
Definition C.1.1. A *group scheme over a scheme* S is a morphism $\pi\colon G \to S$ of schemes together with a multiplication morphism $\mu\colon G \times_S G \to G$, an inverse morphism $\iota\colon G \to G$ and an identity morphism $e\colon S \to G$ (with each morphism over S) such that the following diagrams commute:



A morphism $\phi\colon H \to G$ of schemes over S is a *morphism of group schemes* if $\mu_G \circ (\phi \times \phi) = \phi \circ \mu_H$. A *closed subgroup* of G is a closed subscheme $H \subset G$ such that $H \to G \xrightarrow{\mu_G} G \times G$ factors through $H \times H$.

Remark C.1.2. If G and S are affine, then by reversing the arrows above gives $\Gamma(G, \mathcal{O}_G)$ the structure of a *Hopf algebra* over $\Gamma(S, \mathcal{O}_S)$.

Exercise C.1.3. Show that a group scheme over S is equivalently defined as a scheme G over S together with a factorization



where Gps $\to$ Sets is the forgetful functor.

(*We are not requiring that there exists a factorization; the factorization is part of the data. Indeed, the same scheme can have multiple structures as a group scheme, e.g.* $\mathbb{Z}/4$ *and* $\mathbb{Z}/2 \times \mathbb{Z}/2$ *over* $\mathbb{C}$.)

Example C.1.4. The following examples of group schemes are the most relevant for us. Let $S = \operatorname{Spec} R$ and V be a free R-module of finite rank:

1. The *multiplicative group scheme over* R is $\mathbb{G}_{m,R} = \operatorname{Spec} R[t]_t$ with comultiplication $\mu^*\colon R[t]_t \to R[t]_t \otimes_R R[t']_{t'}$ given by $t \mapsto tt'$.
2. The *additive group scheme over* R is $\mathbb{G}_{a,R} = \operatorname{Spec} R[t]$ with comultiplication $\mu^*\colon R[t] \to R[t] \otimes_R R[t']$ given by $t \mapsto t + t'$.
3. The *general linear group on* V is

$$\mathrm{GL}(V) = \operatorname{Spec}(\operatorname{Sym}^*(\operatorname{End}(V))_{\det})$$

   with the comultiplication $\mu^*\colon \operatorname{Sym}^*(\operatorname{End}(V)) \to \operatorname{Sym}^*(\operatorname{End}(V)) \otimes_R \operatorname{Sym}^*(\operatorname{End}(V))$ which can be defined as following: choose a basis $v_1, \ldots, v_n$ of V and let $x_{ij}\colon V \to V$ where $v_i \mapsto v_j$ and $v_k \mapsto 0$ if $k \neq 0$, and then define $\mu^*(x_{ij}) = x_{i1}x'_{1j} + \cdots + x_{in}x'_{nj}$.
4. The *special linear group on* V is $\mathrm{SL}(V)$ is the closed subgroup of $\mathrm{GL}(V)$ defined by $\det = 1$.
5. The *projective linear group* PGL_n is the affine group scheme

$$\operatorname{Proj}(\operatorname{Sym}^*(\operatorname{End}(V)))_{\det}$$

   with the comultiplication defined similarly to $\mathrm{GL}(V)$.

We write $\mathrm{GL}_{n,R} = \mathrm{GL}(R^n)$, $\mathrm{SL}_{n,R} = \mathrm{GL}(R^n)$ and $\mathrm{PGL}_{n,R} = \mathrm{PGL}(R^n)$. We often simply write $\mathbb{G}_m$, GL_n, SL_n and PGL_n when there is no possible confusion on what the base is.

Exercise C.1.5. (1) Provide functorial descriptions of each of the group schemes above.

(2) Show that any abstract group G can be given the structure of a group scheme $\bigsqcup_{g \in G} S$ over any base S. Provide both explicit and functorial descriptions.

C.1.2 Group actions

Definition C.1.6. An *action* of a group scheme $G \xrightarrow{\pi} S$ on a scheme $X \xrightarrow{p} S$ is a morphism $\sigma\colon G \times_S X \to X$ over S such that the following diagrams commute:

$$\begin{array}{ccc} G \times_S G \times_S X & \xrightarrow{\mathrm{id}_G \times \sigma} & G \times_S X \\ \downarrow{\scriptstyle \sigma \times \mathrm{id}_G} & & \downarrow{\scriptstyle \sigma} \\ G \times_S X & \xrightarrow{\sigma} & G \end{array} \qquad \text{Compatibility}$$

$$\begin{array}{ccc} X & \xrightarrow{e \circ p, \mathrm{id}_X} & G \times_S X \\ & {\scriptstyle \mathrm{id}_X} \searrow & \downarrow{\scriptstyle \sigma} \\ & & X \end{array} \qquad \text{Law of identity}$$

If $X \to S$ and $Y \to S$ are schemes with actions of $G \to S$, a morphism $f\colon X \to Y$ of schemes over S is *G-equivariant* if $\sigma_Y \circ (\mathrm{id} \times f) = f \circ \sigma_X$, and is *$G$-invariant* if G-equivariant and Y has the trivial G-action.

Exercise C.1.7. Show that giving a group action of $G \to S$ on $X \to S$ is the same as giving an action of the functor $\operatorname{Mor}_S(-, G)\colon \operatorname{Sch}/S \to \mathrm{Gps}$ on the functor $\operatorname{Mor}_S(-, X)\colon \operatorname{Sch}/S \to \mathrm{Sets}$.

(*This requires first spelling out what it means for a functor to groups to act on a functor to sets.*)

C.1.3 Representations

To define a representation, for simplicity we specialize to the case when $S = \operatorname{Spec} R$ and G are affine. The case that most interests us of course is when R is a field. A *representation* (or *comodule*) of a group scheme $G \to \operatorname{Spec} R$ is an R-module V together with a homomorphism $\widehat{\sigma}\colon V \to \Gamma(G, \mathcal{O}_G) \otimes_R V$ (often referred to as a *coaction*).

A representation V of G induces an action of G on $\mathbb{A}(V) = \operatorname{Spec} \operatorname{Sym}^* V$, which we refer to as a *linear action*. Morphisms of representations and subrepresentations are defined in the obvious way.

Exercise C.1.8. If $R = k$ is a field and V is a finite dimensional vector space, show that giving V the structure as a representation is the same as giving a homomorphism $G \to \mathrm{GL}(V)$ of group schemes.

A representation V of G is *irreducible* if for every subrepresentation $W \subset V$ is either 0 or V.

Example C.1.9 (Diagonaliable group schemes)**.** If A is a finitely generated abelian group, we let $R[A]$ be the free R-module generated by elements of A. The R-module $R[A]$ has the structure of an R-algebra with multiplication on generators induced from multiplication in A. The comultiplication $R[A] \to R[A] \otimes_R \to R[A']$ defined by $a \mapsto a \otimes a'$ defines a group scheme $D(A) = \operatorname{Spec} R[A]$ over $\operatorname{Spec} R$. A group scheme G over $\operatorname{Spec} R$ is *diagonalizable* if $G \cong D(A)$ for some A.

If $A = \mathbb{Z}^r$, then $D(A) = \mathbb{G}^r_{m,A}$ is the r-dimensional torus. If $A = \mathbb{Z}/n$, then $D(A) = \mu_n = \operatorname{Spec} k[t]/(t^n - 1)$. The classification of finitely generated abelian groups implies that any diagonalizable group scheme is a product of $\mathbb{G}^r_m \times \mu_{n_1} \times \cdots \mu_{n_k}$.

Exercise C.1.10. Describe $D(A)$ as a functor $\mathrm{Sch}\,/R \to \mathrm{Gps}$.

Each element $a \in A$ defines a one-dimensional representation $W_a = A$ of $D(A)$ defined by the coaction $W_a \to R[A] \otimes_R W_a$ defined by $1 \mapsto a \otimes 1$.

Proposition C.1.11. *Any free representation of a diagonalizable group scheme is a direct sum of one-dimensional representations.*

Proof. Let $G = D(A)$ and let $V = A^r$ be a free representation of G with coaction $\widehat{\sigma}\colon V \to R[A] \otimes_R V$. Then for each $a \in A$,

$$V_a := \{v \in V \mid \widehat{\sigma}(v) = a \otimes v\}$$

is isomorphic to $W_a^{\dim V_a}$ as G-representations. Then $V \cong \oplus_{a \in A} V_a$ as G-representations. The details are left to the reader. □

If V is a representation of an affine group scheme G over $\operatorname{Spec} R$ with coaction $\widehat{\sigma}$, the *invariant subrepresentation* is defined as $V^G = \{v \in V \mid \widehat{\sigma}(v) = 1 \otimes v\}$. Observe that $V^G = V_0$ using the notation in the proof above.

C.2 Properties of algebraic groups

An *algebraic group over a field* k if a group scheme G of finite type over k. While we are not assuming that G is affine nor smooth. We are primarily interested in affine algebraic groups.

Algebraic Group Facts C.2.1. Let G be an affine algebraic group over a field k.

(1) Every representation V of G is a union of its finite dimensional subrepresentations.

(2) There exists a finite dimensional representation V and a closed immersion $G \hookrightarrow \mathrm{GL}(V)$ of group schemes.

(3) If G acts on an affine scheme X of finite type over k, there exist a finite dimensional representation V of G and a G-invariant closed immersion $X \hookrightarrow \mathbb{A}(V)$.

(4) If $\mathrm{char}(k) = 0$, then G is smooth.

Exercise C.2.2. Show that a separated group scheme $G \to S$ of finite type is trivial if and only if the fiber G_s is trivial for each $s \in S$.

C.3 Principal G-bundles

The following definition of a principal G-bundle is an algebraic formulation of the topological notion of a fiber bundle $P \to X$ with fiber G where G acts freely on P and $P \to X$ is G-invariant (i.e. equivariant with respect to the trivial action of G on X) with fibers isomorphic to G.

C.3.1 Definition and equivalences

Definition C.3.1. Let $G \to S$ be a flat group scheme locally of finite presentation. A *principal G-bundle over an S-scheme X* is flat morphism $P \to X$ locally of finite presentation with an action of G via $\sigma\colon G \times_S P \to P$ such that $P \to X$ is G-invariant and

$$(\sigma, p_2)\colon G \times_S P \to P \times_X P, \qquad (g, p) \mapsto (gp, p)$$

is an isomorphism.

A principal G-bundle is also often referred to as a *G-torsor* (see Definition C.3.12 and Exercise C.3.13).

Morphisms of principal G-bundles are G-equivariant morphisms.

Exercise C.3.2. Show that $P \to X$ is principal G-bundle over the S-scheme X if and only if $P \to X$ is a principal $G \times_S X$-bundle over the X-scheme X.

Exercise C.3.3. Show that a morphism of principal G-bundles is necessarily an isomorphism.

We call a principal G-bundle $P \to X$ *trivial* if there is a G-equivariant isomorphism $P \cong G \times X$ where G acts on $G \times X$ via multiplication on the first factor. The following proposition characterizes principal G-bundles as morphisms $P \to X$ which are locally trivial.

Proposition C.3.4. *Let $G \to S$ be a flat group scheme locally of finite presentation and $P \to X$ be a G-equivariant morphism of S-schemes where X has the trivial action. Then $P \to X$ is a principal G-bundle if and only if there exists a faithfully flat and locally of finite presentation morphism $X' \to X$, and an isomorphism $P \times_X X' \to G \times_S X'$ of principal G-bundles over X'. Moreover, if $G \to S$ is smooth, then $X' \to X$ can be arranged to be étale.*

Proof. The $\Rightarrow$ direction follows from the definition by taking $X' = P \to X$. For $\Leftarrow$, after base changing $G \to S$ by $X \to S$, we assume that G is defined over X (see Exercise C.3.2). Let $G_{X'}$ and $P_{X'}$ be the base changes of G and P along $X' \to X$. The base change of the action map $(\sigma, p_2)\colon G \times_X P \to P \times_X P$ along $X' \to X$ is the action map $G_{X'} \times_{X'} P_{X'} \to P_{X'} \times_{X'} P_{X'}$ of $G_{X'}$ acting on $P_{X'}$ over X'. Since $P_{X'}$ is trivial, this latter action map is an isomorphism. Since the property of being an isomorphism descends along faithfully flat and locally of finite presentation morphisms (Proposition B.4.1), we conclude that $(\sigma, p_2)\colon G \times_X P \to P \times_X P$ is an isomorphism.

The final statement follows from the fact that smooth morphisms have sections étale-locally (Proposition A.3.5). □

Exercise C.3.5. Let L/K be a finite Galois extension and $G = \mathrm{Gal}(L/K)$ be the finite group scheme over $\operatorname{Spec} K$. Show that $\operatorname{Spec} L \to \operatorname{Spec} K$ is a principal G-bundle.

Exercise C.3.6. If X is a scheme, show that there there is an equivalence of categories

$$\{\text{line bundles on } X\} \to \{\text{principal } \mathbb{G}_m\text{-bundle on } X\}$$
$$L \mapsto \mathbb{A}(L) \setminus 0$$

between the *groupoids* of line bundles on X and $\mathbb{G}_m$-torsors on X and (where the only morphisms allowed are isomorphisms). If L is a line bundle (i.e. invertible $\mathcal{O}_X$-module), then $\mathbb{A}(L)$ denotes the total space $\operatorname{Spec} \operatorname{Sym}^* L^\vee$ and 0 denotes the zero section $X \to \mathbb{A}(L)$.

Exercise C.3.7.

(1) Show that the standard projection $\mathbb{A}^{n+1} \setminus 0 \to \mathbb{P}^n$ is a principal $\mathbb{G}_m$-bundle.

(2) For each line bundle $\mathcal{O}(d)$ on $\mathbb{P}^n$, explicitly determine the corresponding principal $\mathbb{G}_m$-bundle. In particular, for which d does $\mathcal{O}(d)$ correspond to the principal $\mathbb{G}_m$-bundle of (1).

Exercise C.3.8. Let X be a scheme

(1) If E is a vector bundle on X of rank n, define the *frame bundle* is the functor

$$\mathrm{Frame}_X(E)\colon \mathrm{Sch}/X \to \mathrm{Sets}, \quad (T \to X) \mapsto \{\text{trivializations } \alpha\colon f^*E \xrightarrow{\sim} \mathcal{O}_T^n\}.$$

Show that $\mathrm{Frame}_X(E)$ is representable by scheme and that $\mathrm{Frame}_X(E) \to X$ is a principal GL_n-bundle.

(2) If $P \to X$ is a principal GL_n-bundle, then define $P \times^{\mathrm{GL}_n} \mathbb{A}^n := (P \times \mathbb{A}^n)/\mathrm{GL}_n$ where GL_n acts diagonally via its given action on P and the standard action on $\mathbb{A}^n$. (The action is free and the quotient $(P \times \mathbb{A}^n)/\mathrm{GL}_n$ can be interpreted as the sheafification of the quotient presheaf $\mathrm{Sch}/X \to \mathrm{Sets}$ taking $T \mapsto (P \times \mathbb{A}^n)(T)/\mathrm{GL}_n(T)$ in the big Zariski (or big étale) topology or equivalently as the algebraic space quotient (???). Show that $(P \times \mathbb{A}^n)/\mathrm{GL}_n$ is representable by scheme and is the total space of a vector bundle over X.

(3) show that there there is an equivalence of categories

$$\{\text{vector bundles on } X\} \to \{\text{principal } \mathrm{GL}_n\text{-bundles on } X\}$$
$$E \mapsto \mathrm{Frame}_X(E)$$

locally free sheaf associated to $(P \times \mathbb{A}^n)/\mathrm{GL}_n \leftarrow\!\shortmid (P \to X)$

between the *groupoids* of vector bundles on X and principal GL_n-bundles on X.

Exercise C.3.9. What is the GL_2-torsor on $\mathbb{P}^1 \times \mathbb{P}^1$ corresponding to $\mathcal{O}(1) \boxtimes \mathcal{O}(1)$?

Exercise C.3.10. Let $G \to S$ be a smooth, affine group scheme. Let $P \to X$ and $Q \to X$ be principal G-bundles. Show that the functor

$$\begin{aligned} \underline{\mathrm{Isom}}_X(P, Q) \colon \mathrm{Sch}/X &\to \mathrm{Sets} \\ (T \xrightarrow{f} X) &\mapsto \mathrm{Isom}_{\text{principal } G\text{-bundles}/\mathrm{T}}(f^*P, f^*Q) \end{aligned}$$

is representable by a scheme which is a principal G-bundle over X.

C.3.2 Descent for principal G-bundles

Proposition C.3.11 (Effective Descent for Principal G-bundles)**.** *Let $G \to S$ be a flat and affine group scheme of finite presentation Let $f \colon X \to Y$ be a faithfully flat morphism of schemes over S that is either quasi-compact or locally of finite presentation. If $P \to X$ is a principal G-bundle and $\alpha \colon {p_1}^*(P) \xrightarrow{\sim} p_2^*(P)$ is an isomorphism of principal G-bundles over $X \times_Y X$ satisfying $p_{12}^*\alpha \circ p_{23}^*\alpha = p_{13}^*\alpha$, then there exists a principal G-bundle $Q \to Y$ and an isomorphism $\phi \colon P \to f^*(Q)$ of principal G-bundles such that $p_1^*\phi = p_2^*\phi \circ \alpha$.*

C.3.3 G-torsors

A G-torsor is a categorical generalization of a principal G-bundle which makes sense with respect to any sheaf of groups on a site.

Definition C.3.12. Let $\mathcal{S}$ be a site and G a sheaf of groups on $\mathcal{S}$. A *G-torsor on* $\mathcal{S}$ is a sheaf P of sets on $\mathcal{S}$ with a left action $\sigma \colon G \times P \to P$ of G such that

(a) For every object $X \in \mathcal{S}$, there exists a covering $\{X_i \to X\}$ such that $P(X_i) \neq 0$, and

(b) The action map $(\sigma, p_2) \colon G \times P \to P \times P$ is an isomorphism.

Exercise C.3.13. If $G \to S$ is a flat and affine group scheme of finite presentation, show that any G-torsor on the big étale topology $(\mathrm{Sch}/S)_{\text{Ét}}$ is representable by a principal G-bundle.

Appendix D

Hilbert and Quot schemes

In this section, we state that the Hilbert and Quot functors are representable by a projective scheme. Let $X \to S$ be a projective morphism of noetherian schemes and $\mathcal{O}_X(1)$ be a relatively ample line bundle on X. Let $P \in \mathbb{Q}[z]$ be a polynomial.

Theorem D.0.1. *The functor*

$$\mathrm{Hilb}^P(X/S)\colon \mathrm{Sch}/S \to \mathrm{Sets}$$
$$(T \to S) \mapsto \left\{ \begin{array}{l} \text{subschemes } Z \subset X \times_S T \text{ flat and finitely presented over } T \\ \text{such that } Z_t \subset X \times_S \kappa(t) \text{ has Hilbert polynomial } P \text{ for all } t \in T \end{array} \right\}$$

is represented by a scheme projective over S.

Theorem D.0.2. *If F is a coherent sheaf on X, the functor*

$$\mathrm{Quot}^P(F/X/S)\colon \mathrm{Sch}/S \to \mathrm{Sets}$$
$$(T \xrightarrow{f} S) \mapsto \left\{ \begin{array}{l} \text{quotients } f^*F \to Q \text{ of finite presentation such that} \\ Q_t \text{ on } X \times_S \kappa(t) \text{ has Hilbert polynomial } P \text{ for all } t \in T \end{array} \right\}$$

is represented by a scheme projective over S.

Remark D.0.3.

(1) Theorem D.0.1 is a special case of Theorem D.0.2 by taking $F = \mathcal{O}_X$.

(2) A morphism of noetherian schemes $X \to S$ is *projective* if there is a coherent sheaf E on S such that there is a closed immersion $X \hookrightarrow \mathbb{P}(E)$ over S [EGA, §II.5], [SP, Tag 01W8]. The definition of projectivity in [Har77, II.4] is stronger as it requires $X \hookrightarrow \mathbb{P}^n_S$. There is an intermediate notion of *strongly projective* morphisms requiring $X \hookrightarrow \mathbb{P}(E)$ where E is a vector bundle over S. In this case if $X \to S$ is strongly projective, one can show that $\mathrm{Hilb}^P(X/S) \to S$ and $\mathrm{Quot}^P(F/X/S) \to S$ are also strongly projective; [AK80].

(3) When T is noetherian, the conditions that Z be finitely presented and Q be of finite presentation in the definitions of $\mathrm{Hilb}^P(X/S)$ and $\mathrm{Quot}^P(F/X/S)$ are superfluous.

These theorems are the backbone of many results in moduli theory and in particular are essential for establishing properties about the moduli stacks $\overline{\mathcal{M}}_g$

of stable curves and $\mathcal{V}^{\mathrm{ss}}_{r,d}$ of vector bundles over a curve. While the reader could safely treat these results as black boxes (and we encourage some readers to do this), it is also worthwhile to dive into the details. The proof follows the same strategy as the construction of the Grassmanian (Proposition 0.5.7) but it involves several important new ingredients: Castelnuovo–Mumford regularity and flattening stratifications.

Appendix E

Deformation Theory

E.1 Deformations of varieties

E.1.1 Smooth varieties

A *first order deformation* of a scheme X over a field $\Bbbk$ is a flat morphism $\mathcal{X} \to \operatorname{Spec} \Bbbk[\epsilon]$ (where where $\epsilon^2 = 0$) and a cartesian diagram

$$\begin{array}{ccc} X & \hookrightarrow & \mathcal{X} \\ \downarrow & \square & \downarrow \\ \operatorname{Spec} \Bbbk & \hookrightarrow & \operatorname{Spec} \Bbbk[\epsilon]. \end{array} \tag{E.1.1}$$

An *automorphism* of a first order deformation (E.1.1) is an automorphism of $\mathcal{X}$ over $\operatorname{Spec} k[\epsilon]$ that restricts to the identity on X.

We say that a surjection $A \twoheadrightarrow A_0$ of local rings with residue field $\Bbbk$ is a *small extension* if $\ker(A \to A_0) = \Bbbk$. Let $\mathcal{X}_0 \to \operatorname{Spec} A_0$ be a flat morphism, which we may view as a deformation of its central fiber $X = \mathcal{X}_0 \times_{\operatorname{Spec} A_0} \operatorname{Spec} k$. A *small deformation* of $\mathcal{X}_0 \to \operatorname{Spec} A_0$ over $\operatorname{Spec} A_0 \hookrightarrow \operatorname{Spec} A$ is a flat morphism $\mathcal{X} \to \operatorname{Spec} A$ and a cartesian diagram

$$\begin{array}{ccccc} X & \hookrightarrow & \mathcal{X}_0 & \hookrightarrow & \mathcal{X} \\ \downarrow & \square & \downarrow & \square & \downarrow \\ \operatorname{Spec} \Bbbk & \hookrightarrow & \operatorname{Spec} A_0 & \hookrightarrow & \operatorname{Spec} A. \end{array} \tag{E.1.2}$$

An *automorphism* of a small deformation is an automorphism of $\mathcal{X}$ over $\operatorname{Spec} A$ that restricts to the identity on $\mathcal{X}_0$.

Theorem E.1.1. *Let $A \twoheadrightarrow A_0$ be a small extension of local rings with residue field $\Bbbk$. Let $\mathcal{X}_0 \to \operatorname{Spec} A_0$ be a smooth morphism with central fiber $X = \mathcal{X}_0 \times_{\operatorname{Spec} A_0} \operatorname{Spec} k$.*

(1) The group of automorphisms of a small deformation of $\mathcal{X}_0$ over $\operatorname{Spec} A_0 \hookrightarrow \operatorname{Spec} A$ is bijective to $H^0(X, T_X)$.

(2) If there exists a small deformation of $\mathcal{X}_0$ over $\operatorname{Spec} A_0 \hookrightarrow \operatorname{Spec} A$, then the set of isomorphism classes of all such small deformations is bijective to $H^1(X, T_X)$.

(3) There is an element $\mathrm{ob} \in H^2(X, T_X)$ *with the property that there exists a small deformation of* $\mathcal{X}_0$ *over* $\operatorname{Spec} A_0 \hookrightarrow \operatorname{Spec} A$ *if and only if* $\mathrm{ob} = 0$.

Appendix F

Birational Geometry

F.1 Resolution of singularities for surfaces

By a *surface*, we mean an integral scheme of finite type over an algebraically closed field $\Bbbk$ of pure dimension 2.

Theorem F.1.1 (Minimal Resolutions)**.** *Let X be a surface. There exists a unique projective birational morphism $\pi\colon \widetilde{X} \to X$ from a smooth surface such that every other resolution $Y \to X$ factors as $Y \to \widetilde{X} \to X$ (or equivalently such that $K_{\widetilde{X}} \cdot E \geq 0$ for every π-exceptional curve E).*

Proof. See [Kol07, Thm. 2.16]. □

Theorem F.1.2 (Embedded Resolutions of Curves in Surfaces)**.** *Let X be a surface and $X_0 \subset X$ be a curve. There is a finite sequence of blow-ups at reduced points of X_0 yielding a projective birational morphism $\widetilde{X} \to X$ such that $\widetilde{X}$ is smooth and such that the preimage $\widetilde{X}_0$ of X_0 has set-theoretic normal crossings, i.e. $(\widetilde{X}_0)_{\mathrm{red}}$ is nodal.*

Proof. See [Har77, Thm V.3.9] and [Kol07, Thm. 1.47]. □

Theorem F.1.3 (Structure Theorem of Birational Morphisms of Surfaces)**.** *Any projective birational morphism $f\colon X \to Y$ of smooth surfaces is the composition of blowing up smooth points.*

Proof. See [Har77, Thm V.5.5] and [Kol07, Thm 2.13]. □

Theorem F.1.4 (Hodge Index Theorem for Exceptional Curves)**.** *Let $f\colon X \to Y$ be a projective and generically finite morphism of surfaces with X smooth and Y quasi-projective. Let $E_1, \ldots, E_n$ be the exceptional curves. Then the intersection form matrix $(E_i \cdot E_j)$ is negative-definite. In particular, $E_i^2 < 0$ for each i.*

Proof. See [Kol07, Thm 2.12]. □

Theorem F.1.5 (Castelnuovo's Contraction Theorem)**.** *Let X be a smooth projective surface and E a smooth rational curve with $E^2 = -1$. Then there is a projective morphism $X \to Y$ to a smooth surface and a point $y \in Y$ such that $f^{-1}(y) = E$ and $X \setminus E \to Y \setminus \{y\}$ is an isomorphism.*

Proof. See [Har77, Thm. V.5.7] and [Kol07, Thm 2.14]. □

Remark F.1.6. If $E^2 < 0$, then E can still be contracted to a point but the surface may be singular.

One can show that the process of repeatedly contracting smooth rational -1 curves in a smooth projective surface terminates (see [Har77, Thm 5.8]). Thus by applying Castelnuovo's Contractibility Criterion a finite number of times, one obtains:

Corollary F.1.7 (Existence of Relative Minimal Models)**.** *A smooth surface X admits a projective birational morphism $X \to X_{\min}$ to a smooth surface such that every projective birational morphism $X_{\min} \to Y$ to a smooth surface is an isomorphism. In particular $X_{\min}$ has no smooth rational -1 curves.*

F.2 Positivity

The standard reference for this material is [Laz04a, Laz04b].

F.2.1 Positivity of line bundles: basic properties

Ample line bundles

Let X be a proper scheme over an algebraically closed field $\Bbbk$. A line bundle L on X is *ample* if for some $m > 0$, $L^{\otimes m}$ is very ample, i.e. defines an embedding $X \overset{|L^{\otimes m}|}{\hookrightarrow} \mathbb{P}^N$ into projective space. Ampleness can be equivalently characterized by the condition that for every $x \in X$, there exists a section $s \in \Gamma(X, L)$ such that $X_s = \{s \neq 0\}$ is affine and contains x, or cohomologically by the condition that for every coherent sheaf F on X, the cohomology groups $H^i(X, F \otimes L^m) = 0$ vanish for $i > 0$ and $m \gg 0$.

Proposition F.2.1 (Openness of Ampleness)**.** *[Laz04a, Thm. 1.2.17] Let $X \to S$ be a proper morphism of schemes and L be a line bundle on X. If for some $s \in S$, the restriction L_s of s to the fiber X_s is ample, then there exists an open neighborhood $U \subset S$ of s such that L_t is ample on X_t for all $t \in U$.*

We also recall that ampleness can be checked on finite covers

Proposition F.2.2. *[Har77, Exer III.5.7] Let $f\colon X \to Y$ be a finite morphism of schemes and L be a line bundle on Y. If L is ample, then so is f^*L. If f is surjective, then the converse is true.*

Remark F.2.3. As an immediate consequence, we see that a line bundle L on X is ample if and only if its restriction $L_{(X_i)_{\mathrm{red}}}$ to the reduced subscheme of each irreducible component X_i is ample.

Nef line bundles

A line bundle L on a proper scheme X is *nef* if for every irreducible curve

$$\int_C c_1(L) \geq 0.$$

Proposition F.2.4 (Openness of Nefness)**.** *Let X be a proper and flat scheme over a DVR R and L be a line bundle on X. Let $0, \eta \in \operatorname{Spec} R$ be the closed and generic points. If $L|_{X_0}$ is nef, then so is $L|_{X_\eta}$.*

Proof. To be added. □

Remark F.2.5. For proper, flat and surjective morphisms $X \to S$, it is shown in [Laz04a, Prop 1.4.14] that if $L|_{X_s}$ is ample for a point $s \in S$, then there exists a countable union $B \subset S$ of proper subschemes not containing s such that $L|_{X_t}$ is nef for every $t \in S \setminus B$. It is not known whether there exists an open subset $s \in U \subset S$ with $L|_{X_t}$ nef for $t \in S$.

Proposition F.2.6. *Let $f\colon X \to Y$ be a proper morphism and L be a line bundle on Y. If L is nef, then so is f^*L. If f is surjective, then the converse is true.*

Theorem F.2.7 (Kleiman's Theorem)**.** *If L is a line bundle on a proper scheme X, then L is nef if and only if for every irreducible subvariety $Z \subset X$ of dimension k,*

$$\int_Z c_1(L)^k \geq 0.$$

Proof. See [Laz04a, Thm. 1.4.9], [Kol96, Thm. 2.17], or the original source [Kle66]. □

It's also worthwhile to keep in mind that ample and nef line bundles generate cones $\operatorname{Amp}(X), \operatorname{Nef}(X) \subset N^1(X)_{\mathbb{R}}$, called the *ample cone* and *nef cone*. As a consequence of Kleiman's theorem, one can show that for a projective variety, the nef cone is the closure of the ample cone and the ample cone is the interior of the nef cone; see [Laz04a, Thm. 1.4.23].

Effective, globally generated, and semiample line bundles

We have the following notions for a line bundle L on X:

- L is *effective* if $\Gamma(X, L) \neq 0$;
- L is *globally generated* (or *basepoint free*) if for every $x \in X$, there exists $s \in \Gamma(X, L)$ with $s(x) \neq 0$, or equivalent the linear series $|L|$ defines a morphism $X \xrightarrow{|L|} \mathbb{P}^{h^0(X,L)-1}$; and
- L is *semiample* if for some $m > 0$, $L^{\otimes m}$ is globally generated.

A semiample line bundle L is necessarily nef; indeed if for some $m > 0$, $L^{\otimes m}$ defines a morphism $f\colon X \to \mathbb{P}^N$ with $f^*\mathcal{O}(1) \cong L^{\otimes m}$, then the projection formula implies that $\int_C c_1(L^{\otimes m}) = \int_{f(C)} c_1(\mathcal{O}(1)) \geq 0$. We thus have the implications

$$\text{globally generated} \implies \text{semiample} \implies \text{nef}$$

Big line bundles

A line bundle L on a normal variety X is *big* if for some $m > 0$, the rational map $\phi_m\colon X \overset{|L^{\otimes m}|}{\dashrightarrow} \mathbb{P}^N$ is birational onto its image for some $m > 0$. For a possibly non-normal variety X, we say a line bundle L is big if its pullback to the normalization is big.

Proposition F.2.8 (Kodaira's Lemma)**.** *Let X be a projective variety and L be a big line bundle. If E is an effective line bundle, then for m sufficiently divisible, $L^{\otimes m} \otimes E^{\vee}$ is effective*

Proof. See [Laz04a, Prop. 2.2.6]. □

Proposition F.2.9 (Equivalences of bigness)**.** *We have the following equivalences for a line bundle $L = \mathcal{O}_X(D)$ on an irreducible variety:*

$$\begin{aligned} L \text{ is big} &\iff \dim \operatorname{im} \phi_m = \dim X \text{ for } m \text{ sufficiently divisible} \\ &\iff \text{there exists a constant } C \text{ such that } h^0(X, L^{\otimes m}) \geq C \cdot m^{\dim X} \text{ for } m \text{ sufficiently divisible} \\ &\iff \text{for any ample divisor } A \text{ on } X, \text{ there exists a positive integer } m > 0 \text{ and an effective divisor } N \text{ on } X \text{ such that } mD = A + N \text{ (linear equivalence).} \\ &\iff \text{there exists an ample divisor } A \text{ on } X, \text{ a positive integer } m > 0 \text{ and an effective divisor } N \text{ on } X \text{ such that } mD \equiv A + N \text{ (numerical equivalence).} \end{aligned}$$

Proof. See [Laz04a, §2.2] for details; the last three equivalences follow from Kodaira's lemma. □

As a consequence of Proposition F.2.9, we see that up scaling (i.e. taking positive tensor powers), a big line bundle is the same as the sum of an ample and effective line bundle. In particular, the sum of a big and effective line bundle is also big. To summarize,

$$\text{big} \xLeftrightarrow{\text{up to scaling}} \text{ample} + \text{effective}$$

$$\text{big} + \text{effective} \implies \text{big}.$$

Finally, we mention that if H is an ample line bundle on X and E is an effective line bundle on X, then for $m \gg 0$, $mH + E$ (or $H^{\otimes m} \otimes E$ in tensor notation) is ample (in fact very ample); see [Laz04a, Ex. 1.2.10].

Proposition F.2.10. *Let $f\colon X \to Y$ be a generically quasi-finite and proper morphism of varieties such that $f_*\mathcal{O}_X = \mathcal{O}_Y$ (e.g. f is a proper birational morphism of normal varieties). For a line bundle L on Y, L is big if and only if f^*L is big.*

Proof. The projection formula

$$f_* f^* L^{\otimes m} \cong L^{\otimes m} \otimes f_*\mathcal{O}_X \cong L^{\otimes m}$$

implies that $\Gamma(Y, f^*L^{\otimes m}) = \Gamma(X, L^{\otimes m})$. The result follows from the above equivalences since X and Y have the same dimension. □

Theorem F.2.11 (Asymptotic Riemann-Roch)**.** *Let X be a proper scheme of dimension n and let L be a nef line bundle on X. Then the Euler characteristic $\chi(X, L^{\otimes m})$ is a polynomial of degree $\leq n$ in m*

$$\chi(X, L^{\otimes m}) = \frac{(c_1(L)^n)}{n!} m^n + O(m^{n-1}).$$

Remark F.2.12. See [Laz04a, Cor. 1.4.41] for a proof in the projective case and [Kol96, Thm. VI.2.15] in general.

This immediately yields the following useful characterization of bigness for nef line bundles.

Corollary F.2.13. *On a proper scheme of dimension n, a nef line bundle L is big if and only if $c_1(L)^n > 0$.*

As a further consequence, we obtain:

Proposition F.2.14. *On a proper scheme X, if B is a big and globally generated line bundle and N is a nef line bundle, then $B \otimes N$ is big.*

Remark F.2.15. In other words,

$$(\text{big } \& \text{ globally generated}) + \text{nef} \implies \text{big}.$$

Note that the pullback of an ample line bundle under a birational morphism is big and globally generated, and that the converse is true after take sufficiently high tensor power.

Proof. Using additive notation, write $D = B + N$. Since B is globally generated, B is nef and thus so is D. By Corollary F.2.13, it suffices to verify that $D^n > 0$. We compute that

$$D^n = (B+N)^n = B^n + \sum_{i=1}^{n} \binom{n}{i} B^{n-i} \cdot N^i > 0$$

where $B^n > 0$ as B is big and nef, and $B^{n-i} \cdot N^i \geq 0$ for $i \geq 1$ by Kleiman's theorem (Theorem F.2.7). □

F.2.2 Ampleness criteria

We review general techniques here to show that a line bundle L on a proper scheme X is ample. Perhaps the first strategy to keep in mind is that if L is semiample and strictly nef, then L is ample.

Lemma F.2.16. *Let X be a proper scheme. If L is a semiample line bundle and $\int_T c_1(L) = \deg L|_C > 0$ for all curves T, then L is ample.*

Proof. For some $m > 0$, $L^{\otimes m}$ defines a morphism $f\colon X \to \mathbb{P}^N$ which does not contract any curves. It follows that $f\colon X \to \mathbb{P}^N$ is a proper and quasi-finite morphism of schemes, thus finite. Therefore, $L^{\otimes m} = f^*\mathcal{O}(1)$ is ample. □

See also Lemma 4.7.1 for a similar property of algebraic spaces and Deligne–Mumford stacks.

Remark F.2.17. The semiampleness condition can be very challenging to verify in practice. However, there are powerful basepoint-free theorems in birational geometry stemming either from vanishing theorems or analytic methods that can reduce semiampleness to bigness and nefness. For instance, Kawamata's basepoint-free states that if (X, Δ) is a proper klt pair with Δ effective and D is a nef Cartier divisor such that $aD - K_X - \Delta$ is nef and big for some $a > 0$, then D is semiample [KM98, Thm. 3.3]. One can contrast this result with the Abundance Conjecture that states that if (X, Δ) is a proper log canonical pair

with Δ effective, then the nefness of $K_X + \Delta$ implies semiampleness [KM98, Conj. 3.12].

Alternatively, it is a classical result of Zariski and Wilson that if X is a normal projective variety and D is a nef and big divisor, then D is semiample if and only if its graded section ring $\bigoplus_n \Gamma(X, \mathcal{O}_X(nD))$ is finitely generated; see [Laz04a, Thm. 2.3.15]. While [BCHM10] can sometimes be apply to verify the finite generation, this result already presumes the projectivity of X; nevertheless, this can be applied for instance to show that a given birational model of X is projective.

In positive characteristic, Keel's theorem provides another technique: on a projective variety X, a nef line bundle L is semiample if and only if the restriction of L to the *exceptional locus* E is semiample, where E is defined as the union of irreducible subvarieties the $Z \subset X$ satisfying $L^{\dim Z} \cdot Z = 0$ [Kee99].

Numerical criteria for ampleness

The Nakai–Moishezon Criterion[1] for ampleness provides a convenient method to establish projectivity. We state the criteria for proper schemes but this is extended to proper algebraic spaces in Theorem 4.7.4.

Theorem F.2.18 (Nakai–Moishezon Criterion)**.** *If X is a proper scheme, a line bundle L is ample if and only if for all irreducible closed subvarieties $Z \subset X$, $c_1(L)^{\dim Z} \cdot Z > 0$*

Remark F.2.19. Using Corollary F.2.13, the Nakai–Moishezon Criterion translates to:

$$L \text{ is ample} \iff L|_Z \text{ is big for all irreducible closed subvarieties } Z \subset X.$$

Proof. Let $n = \dim X$. First, if L is very ample, then for some $m > 0$, $L^{\otimes m}$ is very ample and $m^n c_1(L)^{\dim Z} \cdot Z = c_1(L^{\otimes m})^{\dim Z} \cdot Z > 0$ as its the degree of Z under the projective embedding defined by $L^{\otimes m}$. To show the converse, we follow the proofs of [Laz04a, Thm. 1.2.23], [Kol96, Thm VI.2.18], and [Har77, Thm. V.1.10] (surface case). Since we already know that L is nef, it suffices to show that L is semiample (Lemma F.2.16).

First, by Proposition F.2.2, we may assume that X is a normal variety and we write $L = \mathcal{O}_X(D)$ for a divisor D. Since D is big on X, some positive multiple mD is effective; replacing D by mD, there exists a non-zero section $s \in H^0(X, \mathcal{O}_X(D))$. In particular, $\mathcal{O}_X(D)$ is globally generated away from the support of D. We aim to show that for $m \gg 0$, $\mathcal{O}_X(mD)$ is also globally generated on D.

By induction on $n = \dim X$, we can assume that $\mathcal{O}_X(D)|_D$ is ample; the base case for the induction is $n = 1$, where a line bundle is ample if and only if it has positive degree. Consider the exact sequence

$$0 \to \mathcal{O}_X((m-1)D) \to \mathcal{O}_X(mD) \to \mathcal{O}_D(mD) \to 0.$$

For $m \gg 0$, $\mathcal{O}_D(mD)$ is globally generated and $H^1(X, \mathcal{O}_D(mD) = 0$. It follows that $H^1(X, \mathcal{O}_X((m-1)D)) \twoheadrightarrow H^1(X, \mathcal{O}_X(mD))$ is surjective but since each vector space is finitely generated, we see that these surjections eventually become isomorphisms for $m \gg 0$. Thus, for $m \gg 0$, $H^0(X, \mathcal{O}_X(mD)) \to H^0(D, \mathcal{O}_D(mD))$ is surjective and $\mathcal{O}_X(mD)$ is globally generated on D. □

[1] This is also known as the Nakai Criterion or the Nakai–Moishezon–Kleiman Criterion. See [Laz04a, §1.2.B] for a historical account and further references.

We use this criterion to establish Kollar's Ampleness Criteria (Theorem 4.7.5), which we in turn apply to establish the projectivity of $\overline{M}_g$. The following two additional numerical criteria for ampleness will not be used in these notes but are included to offer a more complete treatment.

Theorem F.2.20 (Kleiman's Criterion)**.** *If X is a projective scheme, a divisor D is ample if and only if for all $C \in \overline{\mathrm{NE}(X)}$, $D \cdot C > 0$.*

Remark F.2.21. See [Laz04a, Thm. 1.4.23] for a proof. Note that it is not enough to check that $D \cdot C$ for only irreducible curves $C \subset X$; one must check it on curve classes in the closure $\overline{\mathrm{NE}(X)}$ of the effective cone of curves. See [Har66a, p.50-56] for a counterexample due to Mumford.

Kleiman's Criterion is not known to hold for proper schemes or algebraic spaces.

Theorem F.2.22 (Sesahdri's criterion)**.** *If X is a proper scheme, a line bundle L is ample if and only if there exists an $\epsilon > 0$ such that for every point $x \in X$ and every irreducible curve $C \subset X$, $c_1(L) \cdot C > \epsilon \operatorname{mult}_x(C)$, where $\operatorname{mult}_x(C)$ denotes the multiplicity of C at x.*

Remark F.2.23. See [Laz04a, Thm. 1.4.13] or [Kol96, Thm. 2.18] for a proof. This criterion also holds for proper algebraic spaces; see [Cor93].

F.2.3 Nef vector bundles

In Kollár's Criterion (Theorem 4.7.5), nefness of vector bundles plays an essential role:

Definition F.2.24. A vector bundle E on a scheme X is called *nef* (or *semipositive*) if for every map $f\colon C \to X$ from a proper curve, every quotient line bundle of $f^*E \twoheadrightarrow L$ has nonnegative degree.

We note that when E is a line bundle, then this is clearly equivalent to the usual notion of nefness: for all proper curves $C \subset X$, $\deg L|_C \geq 0$.

Proposition F.2.25. *Let E be a vector bundle on a proper scheme X. Then the following are equivalent:*

$$\begin{aligned} E \text{ is nef} \iff & \text{ for every map } f\colon C \to X \text{ from a proper curve, every quotient} \\ & \text{vector bundle of } f^*E \twoheadrightarrow W \text{ has nonnegative degree;} \\ \iff & \mathcal{O}_{\mathrm{Proj}_X E}(1) \text{ is nef on } \mathrm{Proj}_X E. \end{aligned}$$

Remark F.2.26. See [Har66a] or [Laz04b, Ch. 6] for details. There is a similar notion of an ample vector bundle (which we won't need in these notes) where one defines a vector bundle E to be ample if $\mathcal{O}_{\mathrm{Proj}_X E}(1)$ is ample on $\mathrm{Proj}_X E$. This notion also has some nice equivalences. If X is an irreducible projective variety and E is globally generated, then E is ample if and only if for every map $f\colon C \to X$ from a proper curve, every quotient line bundle of $f^*E \twoheadrightarrow L$ is non-trivial. There are also cohomological characterizations of ampleness for vector bundles in the same spirit as their line bundle counterparts. Moreover, nefness of E can then be characterized as requiring that for every map $f\colon C \to X$ from a proper curve and for every ample line bundle H on C, the vector bundle $H \otimes f^*E$ is ample.

Proposition F.2.27.

(1) Quotients and extensions of nef vector bundles are nef.

(2) If E is nef, then so is $\bigwedge^k E$ and $\operatorname{Sym}^k E$ for $k \geq 0$.

Proof. Part (1) follows from the definition of nefness. Part (2): to be added. □

As a consequence of Proposition F.2.4 and the first equivalence of Proposition F.2.25, we obtain that nefness is open in a proper flat family over a DVR:

Proposition F.2.28 (Openness of Nefness)**.** *Let X be a proper and flat scheme over a DVR R and E be a vector bundle on X. Let $0, \eta \in \operatorname{Spec} R$ be the closed and generic points. If $E|_{X_0}$ is nef, then so is $E|_{X_\eta}$.*

F.3 Vanishing theorems

Kollár's argument for the projectivity of $\overline{M}_g$ makes use of the following vanishing theorem in positive characteristic due to Ekedahl [Eke88]. The characteristic zero version is due to Bombieri [Bom73].

Theorem F.3.1 (Bombieri–Ekedahl vanishing)**.** *Let S be a smooth projective surface over $\Bbbk$ which is minimal and of general type. If $\operatorname{char}(\Bbbk) \neq 2$, then $H^1(S, K_S^{\otimes -n}) = 0$ for all $n \geq 1$. If $\operatorname{char}(\Bbbk) = 2$, then $h^1(S, K_S^{\otimes -n}) \leq 1$ for all $n \geq 2$.*

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