Hilbert and Quot Schemes

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We prove that the Grassmanian, Hilbert and Quot functors are representable by projective schemes. These results serve as the backbone of many results in moduli theory and more widely algebraic geometry. In particular, they are essential for establishing properties about the moduli stacks $\overline{\mathcal{M}}_g$ of stable curves and $\mathcal{V}^{\mathrm{ss}}_{r,d}$ of vector bundles over a curve. While the reader could safely treat these results as black boxes (and we encourage some readers to do this), it is also worthwhile to dive into the details. We follow Mumford's simplification [?] of the Grothendieck's original construction of Hilbert of Quot schemes [?]. Specifically, we exploit the theory of Castelnuovo–Mumford regularity (??) and flattening stratifications (??), which are interesting results on their own with wide-ranging applications outside moduli theory.

1 The Grassmanian, Hilbert and Quot functors

1.1 The main results

The representability theorems below are formulated for a *strongly projective* morphism $X \to S$ of noetherian schemes, i.e. there exists a closed immersion $X \hookrightarrow \mathbb{P}_S(E)$ over S where E is a vector bundle on S. This is a stronger condition than the *projectivity* of $X \to S$, which requires only the existence of a closed immersion $X \hookrightarrow \mathbb{P}(E)$ where E is a coherent sheaf [?, §II.5], [?, Tag 01W8]. On the other hand, the definition of projectivity in [?, II.4] requires that X embeds into projective space $\mathbb{P}^n_S$ over S.

Theorem 1.1. *Let S be a noetherian scheme and V be a vector bundle of rank n. For an integer $0 < k < n$, the functor*

$$\begin{aligned} \mathrm{Gr}_S(k, V) \colon \mathrm{Sch}/S &\to \mathrm{Sets} \\ (T \xrightarrow{f} S) &\mapsto \{ \text{ vector bundle quotients } V_T = f^*V \to Q \text{ of rank } k \ \} \end{aligned}$$

is represented by a scheme strongly projective over S.

If $S = \operatorname{Spec} \mathbb{Z}$ and $V = \mathcal{O}_S^n$, then $\operatorname{Gr}_S(X/S)$ is equal to the functor $\operatorname{Gr}(k,n)$ defined in **??**. In addition, when $k = 1$, the Grassmanian $\operatorname{Gr}_S(1, V)$ is identified with the projectivization $\mathbb{P}_S(V)$ of V as discussed in **??**. For arbitrary S, we sometimes denote $\operatorname{Gr}_S(k,n) := \operatorname{Gr}_S(k, \mathcal{O}_S^n)$ and we sometimes drop the subscript S when we are working over a fixed field $S = \operatorname{Spec} \Bbbk$ or $S = \operatorname{Spec} \mathbb{Z}$.

In the formulation of the following two theorems, we will use the convention that if $X \to S$ and $T \to S$ a morphisms of schemes, then $X_T := X \times_S T$. Similarly, if F is a sheaf on X, then F_T denotes the pullback of F under $X_T \to X$. If $s \in S$ is a point, then $X_s := X \times_S \operatorname{Spec} \kappa(s)$ and $F_s := F|_{X_s} = F_{\operatorname{Spec} \kappa(s)}$. If $X \to S$ is a projective morphism, $\mathcal{O}_X(1)$ is relatively ample and $s \in S$ is a point, the *Hilbert polynomial of* F_s is

$$P_{F_s}(z) = \chi(X_s, F_s(z)),$$

where $F_s(z) = F_s \otimes \mathcal{O}_{X_s}(n)$. It is a fact that this defines a polynomial $P_{F_s} \in \mathbb{Q}[z]$ (c.f. [**?**, Exer III.5.2]]); for $z \gg 0$, we have $P_{F_s}(z) = h^0(X_s, F_s(z))$.

Theorem 1.2. *Let $X \to S$ be a strongly projective morphism of noetherian schemes and $\mathcal{O}_X(1)$ be a relatively ample line bundle on X. For any polynomial $P \in \mathbb{Q}[z]$, the functor*

$$\operatorname{Hilb}^P(X/S)\colon \operatorname{Sch}/S \to \operatorname{Sets}$$

$$(T \to S) \mapsto \left\{ \begin{array}{l} \textit{subschemes } Z \subset X_T \textit{ flat and finitely presented over } T \textit{ such} \\ \textit{that } Z_t \subset X \times_S \kappa(t) \textit{ has Hilbert polynomial } P \textit{ for all } t \in T \end{array} \right\}$$

is represented by a scheme strongly projective over S.

Theorem 1.3. *Let $\pi\colon X \to S$ be a strongly projective morphism of noetherian schemes, $\mathcal{O}_X(1)$ be a relatively ample line bundle on X, and F be a coherent sheaf on X which is the quotient of $\pi^*(W)(q)$ for a vector bundle W on S and an integer q. For any polynomial $P \in \mathbb{Q}[z]$, the functor*

$$\operatorname{Quot}^P(F/X/S)\colon \operatorname{Sch}/S \to \operatorname{Sets}$$

$$(T \to S) \mapsto \left\{ \begin{array}{l} \textit{quasi-coherent and finitely presented quotients } F_T \to Q \\ \textit{on } X_T \textit{ such that } Q \textit{ is flat over } T \textit{ and } Q|_{X \times_S \kappa(t)} \\ \textit{on } X \times_S \kappa(t) \textit{ has Hilbert polynomial } P \textit{ for all } t \in T \end{array} \right\}$$

is represented by a scheme strongly projective over S.

The Grassmanian and the Hilbert scheme are special cases of the Quot scheme: $\operatorname{Gr}_S(k, V) \cong \operatorname{Quot}^P(V/S/S)$ where $P(z) = k$ is the constant polynomial and $\operatorname{Hilb}^P(X/S) = \operatorname{Quot}^P(\mathcal{O}_X/X/S)$.

Remark 1.4.

(1) In the definition of the Grassmanian and Quot functor above, two quotients $V_T \xrightarrow{q} Q$ and $V_T \xrightarrow{q'} Q'$ are identified if $\ker(q) = \ker(q')$ as subsheaves of V_T, or equivalently there exists an isomorphism $Q \xrightarrow{\alpha} Q'$ such that the composition $V_T \xrightarrow{q} Q \xrightarrow{\alpha} Q'$ is equal to $V_T \xrightarrow{q'} Q'$. In the Hilbert functor, two subschemes of X_T are identified if they are equal as subschemes (or equivalently their ideal sheaves are equal as subsheaves of $\mathcal{O}_{X_T}$).

(2) The definitions $\operatorname{Hilb}^P(X/S)$ and $\operatorname{Quot}^P(F/X/S)$ depend on the relatively ample line bundle $\mathcal{O}_X(1)$ but we have suppressed this from the notation.

(3) When T is noetherian, the conditions that Z be finitely presented and Q be of finite presentation in the definitions of $\mathrm{Hilb}^P(X/S)$ and $\mathrm{Quot}^P(F/X/S)$ are superfluous.

(4) If we do not fix P, then $\mathrm{Hilb}(X/S)$ and $\mathrm{Quot}(F/X/S)$ are representable by schemes *locally* of finite type, and there are decompositions

$$\mathrm{Hilb}(X/S) = \coprod_P \mathrm{Hilb}^P(X/S) \quad \text{and} \quad \mathrm{Quot}(F/X/S) = \coprod_P \mathrm{Quot}^P(F/X/S);$$

these functorial decompositions follows from the flatness of the quotient Q and the local constancy of the Hilbert polynomial (**??**).

(5) Suppose that S satisfies the *resolution property*, i.e. every coherent sheaf is the quotient of a vector bundle. This is satisfied if S has an ample line bundle or if S is regular. Then any projective morphism $X \to S$ is necessarily strongly projective. Moreover, if F is any coherent sheaf on X, then $\pi^*\pi_*(F(q)) \to F(q)$ is surjective for $q \gg 0$ and choosing a surjection $W \twoheadrightarrow \pi_*(F(q))$ from a vector bundle W on S, we have a surjection $\pi^*(W(-q)) \mapsto F$. **??** therefore implies that $\mathrm{Quot}^P(F/X/S)$ is strongly projective over S if $X \to S$ is projective and F is coherent.

Caution 1.5. We will abuse notation by using $\mathrm{Hilb}^P(X/S)$, $\mathrm{Quot}^P(F/X/S)$ and $\mathrm{Gr}_S(k, V)$ to denote both the functor and the scheme that represents it.

1.2 Strategy of proof

In §**??**, we show that $\mathrm{Gr}_S(k, V)$ is representable by a projective scheme by using the functorial Plücker embedding $\mathrm{Gr}_S(k, V) \to \mathbb{P}(\bigwedge^k V)$ which over an S-scheme T sends a quotient $V_T \to Q$ to the line bundle quotient $\bigwedge^k V_T \to \bigwedge^k Q$.

In §**??**, we introduce Castelnuovo–Mumford regularity and exploit Mumford's result on Boundedness of Regularity (**??**) to show that under the hypotheses of **??**, then for $d \gg 0$, the morphism of functors

$$\begin{aligned} \mathrm{Quot}^P(F/X/S) &\to \mathrm{Gr}_S(P(d), \pi_*F(d)) \\ [F_T \twoheadrightarrow Q] &\mapsto [\pi_{T,*}F_T(d) \to \pi_{T,*}Q_T(d)], \end{aligned} \tag{1.1}$$

defined over an S-scheme T, is well-defined. Note that for a field-valued point $s\colon \operatorname{Spec} \Bbbk \to S$ a quotient $[F_s \mapsto Q]$ is mapped to $[H^0(X_s, F_s(d)) \to H^0(X_s, Q(d))]$.

In fact, we show that the above functor is representable by locally closed immersions (**??**). This is established by reducing to the special case where $X = \mathbb{P}_S(V)$ and $F = \pi^*W$ where V and W are vector bundles on S; this is where Boundedness of Regularity (**??**) is applied.

Since $\mathrm{Gr}_S(P(d), \pi_*F(d))$ is representable by a projective scheme over S (**??**), this already establishes the representability and quasi-projectivity of $\mathrm{Quot}^P(F/X/S)$. Finally, we establish that $\mathrm{Quot}^P(F/X/S)$ is proper over S (**??**) by checking the valuative criterion which implies that $\mathrm{Quot}^P(F/X/S)$ is projective over S.

2 Representability and projectivity of the Grassmanian

The Grassmanian provides a warmup to the functorial approach of constructing projective moduli spaces in these notes and is also used in the proof of the representability of Hilb and Quot. Given its importance, we present a slow-paced expository account of the representability and projectivity of the Grassmanian. We focus first on the Grassmanian $\mathrm{Gr}(k,n)$ over $\mathbb{Z}$ parameterizing k-dimension quotients of a trivial vector bundle of rank n; see **??**. The proof of the projectivity and representability of the relative Grassmanian $\mathrm{Gr}_S(k,V)$ is shown in §**??**.

2.1 Representability by a scheme

In this subsection, we show that $\mathrm{Gr}(k,n)$ is representable by a scheme (**??**). Our strategy will be to find a Zariski-open cover of $\mathrm{Gr}(k,n)$ by representable subfunctors; see **??**. Given a subset $I \subset \{1,\dots,n\}$ of size k, let $\mathrm{Gr}_I \subset \mathrm{Gr}(k,n)$ be the subfunctor where for a scheme S, $\mathrm{Gr}(k,n)_I(S)$ is the subset of $\mathrm{Gr}(k,n)(S)$ consisting of surjections $\mathcal{O}_S^n \xrightarrow{q} Q$ such that the composition

$$\mathcal{O}_S^I \xrightarrow{e_I} \mathcal{O}_S^n \overset{q}{\twoheadrightarrow} Q$$

is an isomorphism, where e_I is the canonical inclusion.

Lemma 2.1. *For each $I \subset \{1,\dots,n\}$ of size k, the functor Gr_I is representable by affine space $\mathbb{A}_{\mathbb{Z}}^{k\times(n-k)}$*

Proof. We may assume that $I = \{1,\dots,k\}$. We define a map of functors $\phi\colon \mathbb{A}^{k\times(n-k)} \to \mathrm{Gr}_I$ where over a scheme S, a $k\times(n-k)$ matrix

$$f = \big(f_{i,j}\big)_{1\le i\le k\,,\,1\le j\le n-k}$$

of global functions on S is mapped to the quotient

$$\left(\begin{array}{cccc|ccc} 1 & & & & f_{1,1} & \cdots & f_{1,n-k} \\ & 1 & & & f_{2,1} & \cdots & f_{2,n-k} \\ & & \ddots & & \vdots & & \\ & & & 1 & f_{k,1} & \cdots & f_{k,n-k} \end{array}\right) : \mathcal{O}_S^n \to \mathcal{O}_S^k. \tag{2.1}$$

The injectivity of $\phi(S)\colon \mathbb{A}^{k\times(n-k)}(S) \to \mathrm{Gr}_I(S)$ follows from the fact that any two quotients written in the form of (**??**) which are equivalent in Gr_I are necessarily defined by the same equations. To see surjectivity, let $[\mathcal{O}_S^n \xrightarrow{q} Q] \in \mathrm{Gr}_I(S)$ where by definition $\mathcal{O}_S^I \xrightarrow{e_I} \mathcal{O}_S^n \overset{q}{\twoheadrightarrow} Q$ is an isomorphism. The tautological commutative diagram

$$\begin{array}{ccc} \mathcal{O}_S^n & \xrightarrow{\quad q \quad} & Q \\ & \searrow^{(q\circ e_I)^{-1}\circ q} & \downarrow{\scriptstyle (q\circ e_I)^{-1}} \\ & & \mathcal{O}_S^I \end{array}$$

shows that $[\mathcal{O}_S^n \overset{q}{\twoheadrightarrow} Q] = [\mathcal{O}_S^n \overset{(q\circ e_I)^{-1}\circ q}{\twoheadrightarrow} \mathcal{O}_S^I] \in \mathrm{Gr}(k,n)(S)$. Since the composition $\mathcal{O}_S^I \xrightarrow{e_I} \mathcal{O}_S^n \overset{(q\circ e_I)^{-1}}{\twoheadrightarrow} \mathcal{O}_S^I$ is the identity, the $k\times n$ matrix corresponding to $(q\circ e_I)^{-1}\circ q$ is necessarily of the same form as (**??**) for functions $f_{i,j} \in \Gamma(S, \mathcal{O}_S)$. Therefore $\phi(S)(\{f_{i,j}\}) = [\mathcal{O}_S^n \overset{q}{\twoheadrightarrow} Q] \in \mathrm{Gr}(k,n)(S)$. □

Lemma 2.2. $\{\mathrm{Gr}_I\}$ *is a Zariski-open cover of* $\mathrm{Gr}(k,n)$ *where* I *ranges over all subsets of size* k.

Proof. For a fixed subset I, we first show that $\mathrm{Gr}_I \subset \mathrm{Gr}(k,n)$ is an open subfunctor. To this end, we consider a scheme S and a morphism $S \to \mathrm{Gr}(k,n)$ corresponding to a quotient $q\colon \mathcal{O}_S^n \to Q$. Let C denote the cokernel of the composition $q \circ e_I\colon \mathcal{O}_S^I \to Q$. Notice that if $C = 0$, then q is an isomorphism. The fiber product

$$\begin{array}{ccc} F_I & \longrightarrow & S \\ \downarrow & \square & \downarrow [\mathcal{O}_S^n \overset{q}{\twoheadrightarrow} Q] \\ \mathrm{Gr}_I & \longrightarrow & \mathrm{Gr}(k,n) \end{array}$$

of functors is representable by the open subscheme $U = S \setminus \mathrm{Supp}(C)$ (the reader is encouraged to verify this claim). Note that if S is not noetherian, then $\mathrm{Supp}(C) \subset S$ is still closed as C is finitely presented as a quasi-coherent sheaf.

To check the surjectivity of $\bigsqcup_I F_I \to S$, let $s \in S$ be a point. Since $\kappa(s)^n \overset{q\otimes\kappa(s)}{\twoheadrightarrow} Q \otimes \kappa(s)$ is a surjection of vector spaces, there is a non-zero $k \times k$ minor, given by a subset I, of the $k \times n$ matrix $q \otimes \kappa(s)$. This implies that $[\kappa(s)^n \overset{q\otimes\kappa(s)}{\twoheadrightarrow} Q \otimes \kappa(s)] \in F_I(\kappa(s))$. □

???? together imply:

Proposition 2.3. *The functor* $\mathrm{Gr}(k,n)$ *is representable by a scheme.* □

Exercise 2.4. Show that $\mathrm{Gr}(k,n)$ is an integral scheme of finite type over $\mathbb{Z}$.

Exercise 2.5. Use the valuative criterion of properness to show that $\mathrm{Gr}(k,n) \to \operatorname{Spec}\mathbb{Z}$ is proper.

2.2 Projectivity of the Grassmanian

We show that the Grassmanian scheme $\mathrm{Gr}(k,n)$ is projective (**??**) by explicitly providing a projective embedding. The *Plücker embedding* is the map of functors

$$P\colon \mathrm{Gr}(k,n) \to \mathbb{P}(\textstyle\bigwedge^k \mathcal{O}^n_{\operatorname{Spec}\mathbb{Z}})$$
$$[\mathcal{O}_S^n \overset{q}{\twoheadrightarrow} Q] \mapsto [\textstyle\bigwedge^k \mathcal{O}_S^n \to \bigwedge^k Q]$$

defined above over a scheme S. As both sides are representable by schemes, the morphism P corresponds to a morphism of schemes via Yoneda's lemma.

Proposition 2.6. *The morphism* $P\colon \mathrm{Gr}(k,n) \to \mathbb{P}(\bigwedge^k \mathcal{O}^n_{\operatorname{Spec}\mathbb{Z}})$ *of schemes is a closed immersion. In particular,* $\mathrm{Gr}(k,n)$ *is a strongly projective scheme over* $\mathbb{Z}$.

Proof. A subset $I \subset \{1,\dots,n\}$ corresponds to a coordinate x_I on $\mathbb{P}(\bigwedge^k \mathcal{O}^n_{\operatorname{Spec}\mathbb{Z}})$, and we set $\mathbb{P}(\bigwedge^k \mathcal{O}^n_{\operatorname{Spec}\mathbb{Z}})_I$ to be the open locus where $x_I \neq 0$. Note that $\mathbb{P}(\bigwedge^k \mathcal{O}^n_{\operatorname{Spec}\mathbb{Z}})_I \subset \mathbb{P}(\bigwedge^k \mathcal{O}^n_{\operatorname{Spec}\mathbb{Z}})$ is the subfunctor parameterizing line bundle quotients $\bigwedge^k \mathcal{O}_S^n \to L$ such that the composition $\mathcal{O}_S \xrightarrow{e_I} \bigwedge^k \mathcal{O}_S^n \to L$ (where the first map is the inclusion of the Ith term) is an isomorphism, or in other words $\mathbb{P}(\bigwedge^k \mathcal{O}^n_{\operatorname{Spec}\mathbb{Z}})_I \cong \operatorname{Gr}(1, \binom{n}{k})_{\{I\}}$ viewing $\{I\}$ as the corresponding subset of $\{1, \ldots, \binom{n}{k}\}$ of size 1. Using these functorial descriptions, one can check that there is a cartesian diagram of functors

$$\begin{array}{ccc} \operatorname{Gr}(k,n)_I & \xrightarrow{P_I} & \mathbb{P}(\bigwedge^k \mathcal{O}^n_{\operatorname{Spec}\mathbb{Z}})_I \\ \downarrow & \square & \downarrow \\ \operatorname{Gr}(k,n) & \xrightarrow{P} & \mathbb{P}(\bigwedge^k \mathcal{O}^n_{\operatorname{Spec}\mathbb{Z}}). \end{array}$$

Since $\{\mathbb{P}(\bigwedge^k \mathcal{O}^n_{\operatorname{Spec}\mathbb{Z}})_I\}$ is a Zariski-open cover, it suffices to show that each $P_I \colon \operatorname{Gr}(k,n)_I \to \mathbb{P}(\bigwedge^k \mathcal{O}^n_{\operatorname{Spec}\mathbb{Z}})_I$ is a closed immersion.

For simplicity, assume that $I = \{1, \ldots, k\}$. Under the isomorphisms $\operatorname{Gr}(k,n)_I \cong \mathbb{A}^{k\times(n-k)}_{\mathbb{Z}}$ of **??** and $\mathbb{P}(\bigwedge^k \mathcal{O}^n_{\operatorname{Spec}\mathbb{Z}})_I \cong \mathbb{A}^{\binom{n}{k}-1}_{\mathbb{Z}}$, the morphism P_I corresponds to the map

$$\mathbb{A}^{k\times(n-k)}_{\mathbb{Z}} \to \mathbb{A}^{\binom{n}{k}-1}_{\mathbb{Z}}$$

assigning a $k \times (n-k)$ matrix $A = \{x_{i,j}\}$ to the element of $\mathbb{A}^{\binom{n}{k}-1}_{\mathbb{Z}}$ whose Jth coordinate, where $J \subset \{1, \ldots, n\}$ is a subset of length k distinct from I, is the $\{1, \ldots, k\} \times J$ minor of the $k \times n$ block matrix

$$\left(\begin{array}{cccc|ccc} 1 & & & & x_{1,1} & \cdots & x_{1,n-k} \\ & 1 & & & x_{2,1} & \cdots & x_{2,n-k} \\ & & \ddots & & \vdots & & \\ & & & 1 & x_{k,1} & \cdots & x_{k,n-k} \end{array}\right).$$

The coordinate $x_{i,j}$ on $\mathbb{A}^{k\times(n-k)}_{\mathbb{Z}}$ is the pull back of the coordinate corresponding to the subset $\{1, \cdots, \widehat{i}, \cdots, k, k+j\}$ (see **??**). This shows that the corresponding ring map is surjective thereby establishing that P_I is a closed immersion. $\square$

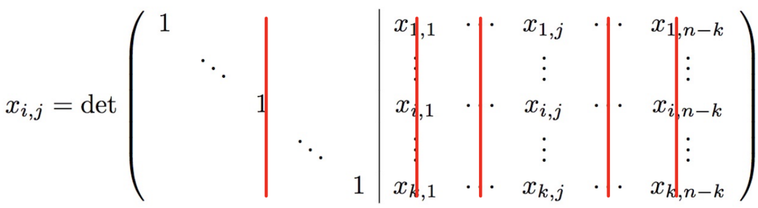

Figure 1: The minor obtained by removing the ith column and all columns $k+1, \ldots, n$ other than $k+j$ is precisely $x_{i,j}$.

Exercise 2.7. For a field $\Bbbk$, let $\operatorname{Gr}(k,n)_\Bbbk$ be the $\Bbbk$-scheme $\operatorname{Gr}(k,n) \times_{\mathbb{Z}} \Bbbk$, and $p \in \operatorname{Gr}(k,n)_\Bbbk$ be the point corresponding to a quotient $Q = \Bbbk^n/K$. Show that there is a natural bijection of the tangent space

$$T_p \operatorname{Gr}(k,n)_\Bbbk \xrightarrow{\sim} \operatorname{Hom}(K, Q).$$

with the vector space of $\Bbbk$-linear maps $K \to Q$.

Exercise 2.8. Provide an alternative proof of the projectivity of $\mathrm{Gr}(k,n)$ as follows.

(1) Show that the functor $P\colon \mathrm{Gr}(k,n) \to \mathbb{P}(\bigwedge^k \mathcal{O}^n_{\mathrm{Spec}\,\mathbb{Z}})$ is injective on points and tangent spaces.

(2) Use a criterion for being a closed immersion (c.f. [**?**, Prop. II.7.3]) to show that $P\colon \mathrm{Gr}(k,n) \to \mathbb{P}(\bigwedge^k \mathcal{O}^n_{\mathrm{Spec}\,\mathbb{Z}})$ is a closed immersion.

(Alternatively, you could show that $P\colon \mathrm{Gr}(k,n) \to \mathbb{P}(\bigwedge^k \mathcal{O}^n_{\mathrm{Spec}\,\mathbb{Z}})$ is a proper monomorphism and conclude that $\mathrm{Gr}(k,n)$ is projective over $\mathbb{Z}$.)

2.3 Relative version

We now prove the relative version of the representability and strong projectivity of the Grassmanian.

Proof of **??**. If V is a vector bundle over S of rank n, there is the relative Plücker embedding

$$P\colon \mathrm{Gr}_S(k,V) \to \mathbb{P}_S(\bigwedge^k V)$$
$$[V_T \overset{q}{\twoheadrightarrow} Q] \mapsto [\bigwedge^k V_T \to \bigwedge^k Q]$$

defined above over a S-scheme T. This is a morphism of functors over S. Since $\mathbb{P}_S(\bigwedge^k V)$ is projective over S, it suffices to show that this morphism is representable by closed immersions. This property can be checked Zariski-locally: if $U \subset S$ is an open subscheme where V is trivial, then the base change of $\mathrm{Gr}_S(k,V) \to \mathbb{P}_S(\bigwedge^k V)$ over U is the Plücker embedding $\mathrm{Gr}_U(k,\mathcal{O}^n_U) \to \mathbb{P}_S(\bigwedge^k \mathcal{O}^N_U)$ which is a closed immersion (**??**). □

3 Castelnuovo–Mumford regularity

The Cartan–Serre–Grothendieck theorem states that if F is a coherent sheaf on a projective variety $(X, \mathcal{O}_X(1))$, then for $d \gg 0$

(1) $F(d)$ is globally generated;

(2) $H^i(X, F(d)) = 0$ for $i > 0$; and

(3) the multiplication map

$$H^0(X, F(d)) \otimes H^0(X, \mathcal{O}(p)) \to H^0(X, F(d+p))$$

is surjective for all $p \geq 0$.

Castelnuovo–Mumford regularity provides a quantitative measure of the size of d necessary so that the twist $F(d)$ has the three above desired cohomological properties and in particular that the Hilbert polynomial $\chi(X, F(d))$ of F evaluated at d agrees with $h^0(X, F(d))$.

3.1 Definition and basic properties

Definition 3.1. Let F be a coherent sheaf on projective space $\mathbb{P}^n$ over a field $\Bbbk$. For an integer m, we say that F is *m-regular* if

$$H^i(\mathbb{P}^n, F(m-i)) = 0$$

for all $i \geq 1$.

The *regularity of F* is the smallest integer m such that $F(m)$ is m-regular.

While the requirement that the ith cohomology of the $(m-i)$th twist vanishes may appear mysterious at first, this definition is very convenient for induction arguments on the dimension n as indicated for instance by the following result.

Lemma 3.2. *Let F be an m-regular coherent sheaf on $\mathbb{P}^n$ over a field $\Bbbk$. If $H \subset \mathbb{P}^n$ is a hyperplane avoiding the associated points of F, then $F|_H$ is also m-regular.*

Proof. The hypotheses imply that over an affine open subscheme $U \subset \mathbb{P}^n$, the defining equation of H is a non-zero divisor for the module $\Gamma(U, F)$. Thus $F(-1) \xrightarrow{H} F$ is injective and for an integer $i > 0$ we have a short exact sequence

$$0 \to F(m-i-1) \to F(m-i) \to F|_H(m-i) \to 0$$

inducing a long exact sequence on cohomology

$$\cdots \to H^i(\mathbb{P}^n, F(m-i)) \to H^i(H, F|_H(m-i)) \to H^{i+1}(\mathbb{P}^n, F(m-i-1)) \to \cdots$$

If F is m-regular, then $H^i(\mathbb{P}^n, F(m-i)) = H^{i+1}(\mathbb{P}^n, F(m-i-1)) = 0$. It follows that $H^i(H, F|_H(m-i)) = 0$ for all $i > 0$, and thus $F|_H$ is also m-regular. $\square$

Remark 3.3. It follows from the definition of regularity that if F is m-regular, then $F(d)$ is $(m-d)$-regular. We will show in **??** that if F is m-regular, it also d-regular for all $d \geq m$.

Exercise 3.4.

(a) Show that $\mathcal{O}(d)$ is $(-d)$-regular on $\mathbb{P}^n$.

(b) Show that the structure sheaf of a hypersurface $H \subset \mathbb{P}^n$ of degree d is $(d-1)$-regular.

(c) Show that the structure sheaf of a smooth curve $C \subset \mathbb{P}^n$ of genus g is $(2g-1)$-regular.

Exercise 3.5. Let F be a coherent sheaf on $\mathbb{P}^n$ resolved by a long exact sequence of coherent sheaves. Show that if each F_i is $(m+i)$-regular, then F is m-regular.

$$\cdots \to F_2 \to F_1 \to F_0 \to F \to 0$$

Another advantage of regularity is the following lemma due to Castelnuovo.

Lemma 3.6. *Let F be an m-regular coherent sheaf on $\mathbb{P}^n$.*

(a) For $d \geq m$, F is d-regular.

(b) The multiplication map

$$H^0(\mathbb{P}^n, F(d)) \otimes H^0(\mathbb{P}^n, \mathcal{O}(k)) \to H^0(\mathbb{P}^n, F(d+k))$$

is surjective if $d \geq m$ and $k \geq 0$.

(c) For $d \geq m$, $F(d)$ is globally generated and $H^i(\mathbb{P}^n, F(d)) = 0$ for $i \geq 1$.

Proof. If $\Bbbk \to \Bbbk'$ is a field extension, then flat base change implies that $H^i(\mathbb{P}^n_{\Bbbk}, F) \otimes_{\Bbbk} \Bbbk' = H^i(\mathbb{P}^n_{\Bbbk'}, F \otimes_{\Bbbk} \Bbbk')$. As $\Bbbk \to \Bbbk'$ is faithfully flat, the assertions **??**-**??** can be checked after base change. We can thus assume that $\Bbbk$ is algebraically closed and in particular infinite.

For **??** and **??**, we will argue by induction on n with the base case of $n = 0$ being clear. If $n > 0$, since $\Bbbk$ is infinite, we may choose a hyperplane $H \subset \mathbb{P}^n$ avoiding the associated points of F. Since the restriction $F|_H$ is m-regular (**??**) on $H \cong \mathbb{P}^{n-1}$, the inductive hypothesis implies that **??** and **??** hold for $F|_H$.

We prove **??** by using induction also on d. The base case $d = m$ holds by hypothesis. For $d > m$, the short exact sequence $0 \to F(d-i-1) \to F(d-i) \to F|_H(d-i) \to 0$ induces a long exact sequence on cohomology

$$\cdots \to H^i(\mathbb{P}^n, F(d-i-1)) \to H^i(\mathbb{P}^n, F(d-i)) \to H^i(H, F|_H(d-i)) \to \cdots$$

For $i > 0$, the first term vanishes by the induction hypothesis on d (F is $(d-1)$-regular so $H^i(\mathbb{P}^n, F(d-1-i)) = 0$) and the third term vanishes by the inductive hypothesis on n ($F|_H$ is m-regular by **??** and thus d-regular by the inductive hypothesis on n so $H^i(H, F|_H(d-i)) = 0$). Thus, the second term vanishes and we have established **??**.

To show **??**, we use induction on k in addition to n. We denote the multiplication map by

$$\mu_{d,k} \colon H^0(\mathbb{P}^n, F(d)) \otimes H^0(\mathbb{P}^n, \mathcal{O}(k)) \to H^0(\mathbb{P}^n, F(d+k)).$$

While the base case $k = 0$ is clear, the inductive argument will require us to directly establish the case $k = 1$. To this end, we consider the commutative diagram

$$\begin{array}{ccccc}
 & & H^0(\mathbb{P}^n, F(d)) \otimes H^0(\mathbb{P}^n, \mathcal{O}(1)) & \xrightarrow{\nu_d \otimes \mathrm{res}} & H^0(H, F|_H(d)) \otimes H^0(H, \mathcal{O}_H(1)) \\
 & \nearrow^{\mathrm{id} \otimes H} & \downarrow{\scriptstyle \mu_{d,1}} & & \downarrow \\
H^0(\mathbb{P}^n, F(d)) & \xrightarrow{\alpha} & H^0(\mathbb{P}^n, F(d+1)) & \xrightarrow{\nu_{d+1}} & H^0(H, F|_H(d+1)).
\end{array} \tag{3.1}$$

As the map α is given by multiplication by $H \in H^0(\mathbb{P}^n, \mathcal{O}(1))$, α factors through the the map $\mathrm{id} \otimes H$ defined by $v \mapsto v \otimes H$. It follows that $\mathrm{im}(\alpha) \subset \mathrm{im}(\mu_{d,1})$. Since $H^1(\mathbb{P}^n, F(d)) = 0$ by **??**, the restriction map $\nu_d \colon H^0(\mathbb{P}^n, F(d)) \to H^0(H, F|_H(d))$ is surjective. Likewise, since $H^1(\mathbb{P}^n, \mathcal{O}) = 0$, $\mathrm{res} \colon H^0(\mathbb{P}^n, \mathcal{O}(1)) \to H^0(H, \mathcal{O}_H(1))$ is surjective. We conclude that the top horizontal arrow is surjective. The inductive hypothesis applied to $H = \mathbb{P}^{n-1}$ implies that the right vertical arrow is surjective. Therefore, the composition $\nu_{d+1} \circ \mu_{d,1}$ is surjective and it follows that $\mathrm{im}(\mu_{d,1})$ surjects onto $H^0(H, F|_H(d+1))$. By exactness of the bottom row, we have that

$$H^0(\mathbb{P}^n, F(d+1)) = \mathrm{im}(\mu_{d,1}) + \ker(\beta) = \mathrm{im}(\mu_{d,1}) + \mathrm{im}(\alpha) = \mathrm{im}(\mu_{d,1}),$$

which shows that $\mu_{d,1}$ is surjective.

If $k > 1$, we consider the commutative square

$$\begin{array}{ccc}
H^0(\mathbb{P}^n, F(d)) \otimes H^0(\mathbb{P}^n, \mathcal{O}(k-1)) \otimes H^0(\mathbb{P}^n, \mathcal{O}(1)) & \longrightarrow & H^0(\mathbb{P}^n, F(d)) \otimes H^0(\mathbb{P}^n, \mathcal{O}(k)) \\
\downarrow{\scriptstyle \mu_{d,k-1} \otimes \mathrm{id}} & & \downarrow{\scriptstyle \mu_{d,k}} \\
H^0(\mathbb{P}^n, F(d+k-1)) \otimes H^0(\mathbb{P}^n, \mathcal{O}(1)) & \xrightarrow{\mu_{d+k-1,1}} & H^0(\mathbb{P}^n, F(d+k)).
\end{array}$$

The left vertical map and bottom horizontal arrow are surjective by the inductive hypothesis applied to $k-1$ and $k=1$, respectively. It follows that $\mu_{d,k}$ is surjective.

To show **??**, we know that for $k \gg 0$, $F(d+k)$ is globally generated, i.e. $\gamma_{F(d+k)}\colon H^0(\mathbb{P}^n, F(d+k)) \otimes \mathcal{O}_{\mathbb{P}^n} \to F(d+k)$ is surjective. Consider the commutative square

$$\begin{array}{ccc} H^0(\mathbb{P}^n, F(d)) \otimes H^0(\mathbb{P}^n, \mathcal{O}(k)) \otimes \mathcal{O}_{\mathbb{P}^n} & \xrightarrow{\mu_{d,k}\otimes \mathrm{id}} & H^0(\mathbb{P}^n, F(d+k)) \otimes \mathcal{O}_{\mathbb{P}^n} \\ \downarrow{\scriptstyle \gamma_{F(d)}\otimes \mathrm{id}} & & \downarrow{\scriptstyle \gamma_{F(d+k)}} \\ F(d) \otimes \big(H^0(\mathbb{P}^n, \mathcal{O}(k)) \otimes \mathcal{O}_{\mathbb{P}^n}\big) & \xrightarrow{\mathrm{id}\otimes \gamma_{\mathcal{O}(k)}} & F(d) \otimes \mathcal{O}(k). \end{array}$$

Since top horizontal arrow is surjective by **??**, the composition from the top left to the bottom right is surjective. Given the nature of the bottom horizontal map, we see that $\gamma_{F(d)}$ must be surjective (indeed, if $V = \mathrm{im}(\gamma_{F(d)}) \subset F(d)$, then $\mathrm{im}(\mathrm{id} \otimes \gamma_{\mathcal{O}(k)} \circ \gamma_{F(d)} \otimes \mathrm{id}) = V \otimes \mathcal{O}(k)$). Finally, to see the vanishing of the higher cohomology of $F(d)$ observe that for each $i > 0$, the sheaf F is $(d+i)$-regular by **??** and thus $H^i(\mathbb{P}^n, F(d)) = 0$. □

One easy consequence of **??** is that if F is m-regular, then the restriction map

$$\nu_d \colon : H^0(\mathbb{P}^n, F(d)) \to H^0(H, F|_H(d))$$

is surjective for all $d \geq m$. Indeed, **??** implies that F is also d-regular and the surjectivity follows from the vanishing of $H^1(\mathbb{P}^n, F(d-1))$. The following lemma—which will be used in the proof of **??**—shows that we can still arrange for the surjectivity of ν_d under weaker hypotheses.

Lemma 3.7. *Let F be a coherent sheaf on $\mathbb{P}^n$ and H be a hyperplane avoiding the associated points of F. If $F|_H$ is m-regular and ν_d is surjective for some $d \geq m$, then ν_p is surjective for all $p \geq d$.*

Proof. By staring at the square in diagram (**??**), we see that the top arrow $\nu_d \otimes \mathrm{res}$ is surjective (as both ν_d and res are surjective) and the vertical right multiplication morphism is surjective (by applying **????** to the m-regular sheaf $F|_H$). The statement follows. □

3.2 Regularity bounds

We now turn to the following bound on the regularity of subsheaves of the trivial vector bundle established by Mumford in [**?**, p.101].

Theorem 3.8 (Boundedness of Regularity)**.** *For every pair of non-negative integers k and n and for every polynomial $P \in \mathbb{Q}[z]$, there exists an integer m_0 with the following property: for every field $\Bbbk$, any subsheaf $F \subset \mathcal{O}^k_{\mathbb{P}^n_{\Bbbk}}$ with Hilbert polynomial P is m_0-regular.*

Proof. As in the proof of **??**, we can assume that $\Bbbk$ is infinite. We will argue by induction on n. The base case of $n = 0$ holds as any sheaf F on $\mathbb{P}^0$ is m-regular for any integer m.

For $n \geq 1$ and a subsheaf $F \subset \mathcal{O}^k_{\mathbb{P}^n}$ with Hilbert polynomial P, we can choose a hyperplane $H \subset \mathbb{P}^n$ avoiding all associated points of $\mathcal{O}^k_{\mathbb{P}^n}/F$. This ensures that

$\mathrm{Tor}^1_{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{O}_H, \mathcal{O}^k_{\mathbb{P}^n}/F) = 0$ and that the short exact sequence $0 \to F \to \mathcal{O}^k_{\mathbb{P}^n} \to \mathcal{O}^k_{\mathbb{P}^n}/F \to 0$ restricts to a short exact sequence

$$0 \to F|_H \to \mathcal{O}^k_H \to \mathcal{O}^k_H/F \to 0. \tag{3.2}$$

As $H \cong \mathbb{P}^n$, this will allow us to apply the inductive hypothesis to $F|_H \subset \mathcal{O}^k_H$.

On the other hand, since $F \subset \mathcal{O}^k_{\mathbb{P}^n}$ is torsion-free, we have a short exact sequence

$$0 \to F(-1) \xrightarrow{H} F \to F|_H \to 0. \tag{3.3}$$

so that the Hilbert polynomial of $F|_H$ is $\chi(F|_H(d)) = \chi(F(d)) - \chi(F(d-1)) = P(d) - P(d-1)$.

In particular, the Hilbert polynomial of $F|_H$ only depends on P and the inductive hypothesis applied to $F|_H \subset \mathcal{O}^k_H$ gives an integer $m_0 = M_{\Bbbk,k,n-1,P'}$ such that $F|_H$ is m_0-regular.

For $m \geq m_0 - 1$, since $H^i(H, F|_H(m)) = 0$ for all $i \geq 1$, we have a long exact sequence

$$0 \to H^0(\mathbb{P}^n, F(m-1)) \to H^0(\mathbb{P}^n, F(m)) \to H^0(H, F|_H(m)) \to H^1(\mathbb{P}^n, F(m-1)) \to H^1(\mathbb{P}^n, F(m)) \to 0$$

and isomorphisms $H^i(\mathbb{P}^n, F(m-1)) \to H^i(\mathbb{P}^n, F(m))$ for $i \geq 2$. For $i \geq 2$, we have that $H^i(\mathbb{P}^n, F(d))$ vanishes for d sufficiently large and thus $H^i(\mathbb{P}^n, F(m-1)) \cong H^i(\mathbb{P}^n, F(m)) \cong \cdots$ are all zero. We conclude that $H^i(\mathbb{P}^n, F(m)) = 0$ for all $i \geq 2$ and $m \geq m_0 - 2$.

To handle H^1, we use the inequalities $h^1(\mathbb{P}^n, F(m_0-1)) \geq h^1(\mathbb{P}^n, F(m_0)) \geq h^1(\mathbb{P}^n, F(m_0+1)) \geq \cdots$ which eventually stabilize to 0. We claim that in fact that we have strict inequalities $h^1(\mathbb{P}^n, F(m_0-1)) > h^1(\mathbb{P}^n, F(m_0)) > h^1(\mathbb{P}^n, F(m_0+1)) > \cdots$ until they become 0. To see this, we observe that there is an equality $h^1(\mathbb{P}^n, F(m_0+j-1)) = h^1(\mathbb{P}^n, F(m_0+j))$ for $j \geq 0$ if and only if $\nu_{m_0+j} \colon H^0(\mathbb{P}^n, F(m_0+j)) \to H^0(H, F|_H(m_0+j))$ is surjective. Once $h^1(\mathbb{P}^n, F(m_0+j-1)) = h^1(\mathbb{P}^n, F(m_0+j))$ for some $j \geq 0$, then ν_{m_0+j} is surjective. Since $F|_H$ is m_0-regular, we may apply **??** to conclude that $\nu_{m_0+j'}$ is surjective for all $j' \geq j$ which further implies that all $h^1(\mathbb{P}^n, F(m_0+j'))$ are equal for all $j' \geq j$ and thus necessarily 0. This establishes the claim and setting $m_1 = m_0 + h^1(\mathbb{P}^n, F(m_0-1))$, we see that $h^1(\mathbb{P}^n, F(m_1-1)) = 0$ and that F is m_1-regular.

To see that m_1 is independent of F, we use that since $F \subset \mathcal{O}^k_{\mathbb{P}^n}$, we have $h^0(\mathbb{P}^n, F(d)) \leq kh^0(\mathbb{P}^n, \mathcal{O}(d)) = k\binom{n+d}{n}$. Using the vanishing of $h^i(\mathbb{P}^n, F(m_0))$ for $i \geq 2$, we have

$$h^1(\mathbb{P}^n, F(m_0-1)) = h^0(\mathbb{P}^n, F(m_0-1)) - \chi(F(m_0-1)) \leq k\binom{n+m_0-1}{n} + P(m_0-1)$$

and $m_1 \leq m_0 + k\binom{n+m_0-1}{n} + P(m_0-1)$. □

Remark 3.9. The above proof establishes in fact a stronger statement. In order to formulate the result, we recall that any numerical polynomial $P \in \mathbb{Q}[z]$ (i.e. $P(d) \in \mathbb{Z}$ for integers $d \gg 0$) of degree n can be uniquely written as

$$P(d) = \sum_{i=0}^{n} a_i \binom{d}{i}$$

for $a_i \in \mathbb{Z}$; this follows from a straightforward inductive argument (c.f. [**?**, Prop. I.7.3]). For non-negative integers k and n, there exists a polynomial $\Lambda_{k,n} \in \mathbb{Z}[x_0, \dots, x_n]$ with the following property: for every field $\Bbbk$, any subsheaf $F \subset \mathcal{O}^k_{\mathbb{P}^n_{\Bbbk}}$ with Hilbert polynomial $P(d) = \sum_{i=0} a_i \binom{d}{i}$ is m_0-regular for $m_0 = \Lambda_{k,n}(a_0, \dots, a_n)$.

Remark 3.10 (Optimal bounds)**.** Although Mumford's result on Boundedness of Regularity (**??**) provides an explicit bound and is sufficient for many applications including the construction of the Quot scheme as well as for other applications, there is a more optimal bound established by Gotzmann: for a projective scheme $X \subset \mathbb{P}^N$ over a field $\Bbbk$ with Hilbert polynomial P, there are unique integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \geq 1$ such that P can be expressed as

$$P(d) = \binom{d + \lambda_1 - 1}{\lambda_1 - 1} + \binom{d + \lambda_2 - 2}{\lambda_2 - 1} + \cdots + \binom{d + \lambda_r - r}{\lambda_r - 1},$$

and the ideal sheaf $\mathcal{I}_X$ of X is r-regular. See [**?**], [**?**], [**?**, §3] and [**?**, §4.3].

Exercise 3.11. Let $C \subset \mathbb{P}^n$ be a curve of degree d and genus g. Show that Gotzmann's bound implies that the ideal sheaf I_C of C is $(\binom{d}{2} + 1 - g)$-regular. Can you compare this to the bound given by the proof of **??**, i.e. can you compute $\Lambda_{1,n}(1 - g, d)$ for an explicit polynomial satisfying **??**?

Remark 3.12. It was shown in [**?**] that the ideal sheaf I_C of an integral, non-degenerate curve $C \subset \mathbb{P}^N$ of degree d is $(d - N + 2)$-regular. It is conjectured more generally that the ideal sheaf of any smooth, non-degenerate projective variety $X \subset \mathbb{P}^N$ of dimension n and degree d is $(d - (N - n)) + 1)$-regular; see [**?**] and [**?**].

Corollary 3.13. *Let $\pi \colon X \to S$ be a strongly projective morphism of noetherian schemes and $\mathcal{O}_X(1)$ be a relatively ample line bundle on X. Let F be quotient sheaf of $\pi^*(W)(q)$ for some vector bundle W on S and integer q. Let $P \in \mathbb{Q}[z]$ be a polynomial. There exists an integer m_0 satisfying the following property for any $d \geq m_0$: for any morphism $f \colon T \to S$ inducing a cartesian square*

$$\begin{array}{ccc} X_T & \xrightarrow{f_T} & X \\ \downarrow{\scriptstyle \pi_T} & & \downarrow{\scriptstyle \pi} \\ T & \xrightarrow{f} & S \end{array}$$

and any finitely presented quotient $Q = F_T/K$ flat over S such that every fiber Q_t on X_t has Hilbert polynomial P, then

(a) *$\pi_{T,*}Q(d)$ is a vector bundles of rank $P(d)$*

(b) *the comparison maps $f^*\pi_*Q(d) \to \pi_{T,*}f_T^*Q(d)$, $f^*\pi_*F(d) \to \pi_{T,*}f_T^*F(d)$ and $f^*\pi_*K(d) \to \pi_{T,*}f_T^*K(d)$ are isomorphisms;*

(c) *$R^1\pi_{T,*}K(d) = 0$ for $i > 0$; and*

(d) *the adjunction maps $\pi_T^*\pi_{T,*}Q(d) \to Q(d)$, $\pi_T^*\pi_{T,*}F_T(d) \to F_T(d)$ and $\pi_T^*\pi_{T,*}K(d) \to K(d)$ are surjective.*

Proof. For **??**, since $\pi \colon X \to S$ is strongly projective, there is a closed immersion $i \colon X \hookrightarrow \mathbb{P}_S(V)$ where V is a vector bundle on S. Since the statement is local on S (and S is quasi-compact), we may assume that S is affine and that V is the trivial vector bundle of rank $n+1$. We are given a surjection $\pi^*(W)(q) \twoheadrightarrow F$, and if Q is a quotient of i_*F with Hilbert polynomial P, then $Q(-q)$ is a quotient of π^*W with Hilbert polynomial P' where $P'(z) = P(z+d)$. We can therefore replace (F, X, P) with $(\pi^*(W), \mathbb{P}_S(V), P')$. In particular, for every field-valued point $s\colon \operatorname{Spec} \Bbbk \to S$, $\mathbb{P}_S(V)_s \cong \mathbb{P}^n_{\Bbbk}$ and $F_s \cong \mathcal{O}^k_{\mathbb{P}^n_{\Bbbk}}$ where $\operatorname{rk}(V) = n+1$ and $\operatorname{rk}(W) = k$.

By Boundedness of Regularity (**??**), there exists an integer m_0 depending on n, r and P such that for any every field-valued point $s\colon \operatorname{Spec} \Bbbk \to S$, the kernel K_s is m_0-regular. As K_s is also (m_0+2)-regular (**??**) and $F_s \cong \mathcal{O}^k_{\mathbb{P}^n_{\Bbbk}}$ is (m_0+1)-regular (in fact, it is 0-regular), it follows that Q_s is m_0-regular (**??**). By **??**, for $d \geq m_0+2$, $K_s(d)$, $F_s(d)$ and $Q_s(d)$ are each globally generated with vanishing higher cohomology. Since K, F and Q are flat over S, statements **??**-**??** follow from applying Cohomology and Base Change in the form of **??**. For **??**, to verify the surjectivity of the adjunction map $\pi_T^*\pi_{T,*}K(d) \to K(d)$ (and likewise for F_T and Q), it suffices to check that the restriction

$$(\pi_T^*\pi_{T,*}K(d))|_{X_t} \to K_t(d) \tag{3.4}$$

is surjective one each fiber X_t over $t \in T$. Using **??**, we have identifications

$$(\pi_T^*\pi_{T,*}K(d))|_{X_t} \cong \pi_t^*(\pi_{T,*}K(d) \otimes \kappa(t)) \cong \pi_t^*\pi_{t,*}K_t(d),$$

where $\pi_t\colon X_t \to \operatorname{Spec} \kappa(t)$ and thus (**??**) corresponds to the adjunction map $\pi_t^*\pi_{t,*}K_t(d) \to K_t(d)$, which we know is surjective as $K_t(d)$ is globally generated. □

4 Representability and projectivity of Hilb and Quot

In this section, we prove the representability and projectivity of Quot (**??**) and as a consequence we obtain the same for the Hilbert scheme (**??**).

As before, $\pi\colon X \to S$ is a strongly projective morphism of noetherian schemes, $\mathcal{O}_X(1)$ is a relatively ample line bundle on X, F is a quotient sheaf of $\pi^*(W)(q)$ for some vector bundle W on S and integer q, and $P \in \mathbb{Q}[z]$ is a polynomial. Our strategy is to use the morphism of functors

$$\begin{aligned} \operatorname{Quot}^P(F/X/S) &\to \operatorname{Gr}_S(P(d), \pi_*F(d)) \\ [F_T \twoheadrightarrow Q] &\mapsto [\pi_{T,*}F_T(d) \to \pi_{T,*}Q_T(d)], \end{aligned} \tag{4.1}$$

defined above over an S-scheme T. For $d \gg 0$, **??** implies that the above morphism is well-defined: indeed part **??** shows that $\pi_{T,*}Q(d)$ is a vector bundle of rank $P(d)$, part **??** shows the pullback of the coherent sheaf $\pi_*F(d)$ under $T \to S$ is identified with $\pi_{T,*}F_T(d)$, and part **??** shows that $R^1\pi_{T,*}K(d) = 0$ which implies the surjectivity of $\pi_{T,*}F_T(d) \to \pi_{T,*}Q(d)$.

4.1 $\operatorname{Quot}^P(F/X/S) \to \operatorname{Gr}_S(P(d), \pi_*F(d))$ is a locally closed immersion

Proposition 4.1. *Let $\pi\colon X \to S$ be a strongly projective morphism of noetherian schemes, $\mathcal{O}_X(1)$ be a relatively ample line bundle on X, and F be a coherent* *sheaf on X which is the quotient of $\pi^*(W)(q)$ for a vector bundle W on S and an integer q. For $d \gg 0$, the morphism $\operatorname{Quot}^P(F/X/S) \to \operatorname{Gr}_S(P(d), \pi_*F(d))$ is representable by locally closed immersions, i.e. for any morphism $T \to \operatorname{Gr}_S(P(d), \pi_*F(d))$ from a scheme, the fiber product $T \times_{\operatorname{Gr}_S(P(d),\pi_*F(d))} \operatorname{Quot}^P(F/X/S)$ is representable by a locally closed subscheme of T.*

Proof. We first reduce to the special case that $X = \mathbb{P}_S(V)$ and $F = \pi^*W$ for trivial vector bundles V and W. Let $i\colon X \hookrightarrow \mathbb{P}_S(V)$ be a closed immersion where V is a vector bundle on S. The morphism of functors $\operatorname{Quot}^P(F/X/S) \to \operatorname{Gr}_S(P(d), \pi_*F(d))$ is defined over S and its base change to any open subscheme $U \subset S$ is identified with the morphism $\operatorname{Quot}^P(F_U/X_U/U) \to \operatorname{Gr}_S(P(d), \pi_{U,*}F_U(d))$. Since the property of being a locally closed immersion is Zariski-local on the target, the statement is Zariski-local on S. We may therefore assume that S is affine and that V is the trivial vector bundle of rank $n+1$.

First, observe that since there is an isomorphism of functors $\operatorname{Quot}^P(F/X/S) \to \operatorname{Quot}^P(i_*F/\mathbb{P}_S(V)/S)$, we may replace (F, X) with $(i_*F, \mathbb{P}_S(V))$. Next using the surjection $\pi^*(W)(q) \twoheadrightarrow F$, we obtain a morphism of functors

$$\begin{aligned} \operatorname{Quot}^P(F/\mathbb{P}_S(V)/S) &\to \operatorname{Quot}^{P'}(\pi^*W/\mathbb{P}_S(V)/S) \\ [F_T \to Q] &\mapsto [(\pi^*W)_T \to F(-q)_T \to Q(-q)], \end{aligned}$$

defined over an S-scheme T, where $P'(z) = P(z-q)$. We claim that this morphism is representable by closed immersions. This claims boils down to the statement that for an S-scheme T and quotient $\pi^*W(q)_T \twoheadrightarrow Q$, there is a closed subscheme $Z \subset T$ such that a morphism $U \to T$ factors through Z if and only if the restriction $\pi^*W(q)_U \twoheadrightarrow G_U$ factors through F_U. Defining $K = \ker(\pi^*W(q)_T \to F_T)$ and considering the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \longrightarrow & \pi^*W(q)_T & \longrightarrow & F_T & \longrightarrow & 0 \\ & & & \searrow & \downarrow\!\!\downarrow & & & & \\ & & & & G, & & & & \end{array}$$

we see that the claim is satisfied by taking $Z \subset T$ to be vanishing scheme of the morphism $K \to G$ (see **????**).

Finally, using that $\pi_*(\pi^*W(d)) = W \otimes \operatorname{Sym}^d V$, we have a commutative diagram

$$\begin{array}{ccc} \operatorname{Quot}^P(F/\mathbb{P}_S(V)/S) & \hookrightarrow & \operatorname{Quot}^{P'}(\pi^*W/\mathbb{P}_S(V)/S) \\ \downarrow & & \downarrow \\ \operatorname{Gr}_S(P(d), \pi_*F(d)) & \longrightarrow & \operatorname{Gr}_S(P'(d), W \otimes \operatorname{Sym}^d V). \end{array}$$

By the above claim, the top horizontal map is a closed immersion. As $\operatorname{Gr}_S(P(d), \pi_*F(d))$ and $\operatorname{Gr}_S(P'(d), W \otimes \operatorname{Sym}^d V)$ are projective (**??**), the bottom horizontal map is projective and in particular separated. If the proposition holds for $\operatorname{Quot}^{P'}(\pi^*W/\mathbb{P}_S(V)/S)$ and the right vertical map is a locally closed immersion, then the left vertical map is also a closed immersion by the cancellation property.

We now handle the special case. We first claim that $\operatorname{Quot}^P(F/X/S) \to \operatorname{Gr}_S(P(d), W \otimes \operatorname{Sym}^d V)$ is a monomorphism, i.e. $\operatorname{Quot}^P(F/X/S)(T) \to \operatorname{Gr}_S(P(d), \pi_*F(d))(T)$

is injective for each scheme T. To see this, observe that if $F_T = Q/K$ is a quotient with Hilbert polynomial P, then **??** implies that there is a map of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \pi_T^*\pi_{T,*}K(d) & \longrightarrow & \pi_T^*\pi_{T,*}F_T(d) & \longrightarrow & \pi_T^*\pi_{T,*}Q(d) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & K(d) & \longrightarrow & F_T(d) & \longrightarrow & Q(d) & \longrightarrow & 0 \end{array}$$

where the vertical maps are surjections. Thus $F_T(d) \to Q(d)$ can be recovered from $\pi_{T,*}F_T(d) \to \pi_{T,*}Q(d)$ by taking the cokernel of the composition $\pi_T^*\pi_{T,*}K(d) \to \pi_T^*\pi_{T,*}F_T(d) \to F_T(d)$.

Let $T \to \mathrm{Gr}_S(P(d), W \otimes \mathrm{Sym}^d V)$ be a morphism determined by a vector bundle quotient $\gamma\colon \pi_{T,*}F_T(d) = W_T \otimes \mathrm{Sym}^d V_T \to G$ of rank $P(d)$. Define Q as the quotient sheaf of F_T with the property that $F_T(d) \twoheadrightarrow Q(d)$ is identified with the cokernel of $\ker(\pi_T^*\gamma) \to \pi_T^*\pi_{T,*}F_T(d) \to F_T(d)$. The fiber product

$$\begin{array}{ccc} Z & \longrightarrow & T \\ \downarrow & & \downarrow \\ \mathrm{Quot}^P(F/X/S) & \longrightarrow & \mathrm{Gr}_S(P(d), W \otimes \mathrm{Sym}^d V) \end{array}$$

is identified with the subfunctor of T (or more precisely the subfunctor of $\mathrm{Mor}_S(-, T)$) consisting of morphisms $T' \to T$ such that $Q_{T'}$ is flat over T' with Hilbert polynomial P (in other words, a map $T' \to T$ factors through Z if and only if $Q_{T'}$ is flat over T' with Hilbert polynomial P). By Existence of Flattenning Stratifications (**??**), Z is representable by a locally closed subscheme of T. □

4.2 Valuative Criterion for Quot

In order to establish that Quot is projective, it will be sufficient to know that it is proper.

Proposition 4.2. *For any projective morphism $X \to S$ of noetherian schemes, relatively ample line bundle $\mathcal{O}_X(1)$, coherent sheaf F on X and polynomial $P \in \mathbb{Q}[x]$, the functor $\mathrm{Quot}^P(F/X/S)$ satisfies the valuative criterion for properness, i.e. for any DVR R over S with fraction field K, any flat coherent quotient $F_K \to Q^\times$ on X_K with Hilbert polynomial P extends uniquely to a flat coherent quotient $F_R \to Q$ on X_R with Hilbert polynomial P.*

Remark 4.3. In other words, the proposition implies that for any commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}\, K & \longrightarrow & \mathrm{Quot}^P(F/X/S) \\ \downarrow & \nearrow & \downarrow \\ \mathrm{Spec}\, R & \longrightarrow & S, \end{array}$$

of solid arrows, there is a unique dotted arrow filling in the diagram. See §**??** for a further discussion of the valuative criterion for functors and stacks.

Proof. If we write $j\colon X_K \hookrightarrow X_R$ as the open immersion, we define Q as the image of the composition $F_R \to j_*F_K \to j_*Q^\times$ (where the first map is given by adjunction $F_R \to j_*j^*F_R = j_*F_K$). Since Q is a subsheaf of j_*Q^*, it is torsion free over R and thus flat (as R is a DVR). (Locally, if $S = \operatorname{Spec} B$ is affine and $U = \operatorname{Spec} A \subset X$ is an affine open, then we can write $F|_U = \widetilde{M}$ for a finitely generated A-module M and we have a quotient $M \otimes_B K \to N^\times$ of $A \otimes_B K$-modules where $Q^\times|_{U_K} = \widetilde{N^\times}$. Then $Q = \widetilde{N}$ where N is the $A \otimes_B R$-module defined by $N := \operatorname{im}(M \otimes_B R \to M \otimes_B K \to N^\times)$. Since the R-module N is a subsheaf of the K-module $N^\times$, we see that N is torsion free and thus flat.) Finally, since Q if flat over R and $\operatorname{Spec} R$ is connected, its Hilbert polynomial is constant. □

Remark 4.4. For $\operatorname{Hilb}^P(X/S)$, the argument translates into the following: the unique extension of a closed subscheme $Z^\times \subset X_K$ is the scheme-theoretic image $Z = \operatorname{im}(Z^\times \to X_K \hookrightarrow X_R)$. The scheme Z is flat over R as all associated points live over the generic point of $\operatorname{Spec} R$.

4.3 Projectivity

We can finally wrap up the proof of the main theorem of this section.

Proof of **??**. For $d \gg 0$, the morphism

$$\operatorname{Quot}^P(F/X/S) \to \operatorname{Gr}_S(P(d), \pi_*F(d))$$

is a locally closed immersion of schemes defines over S (**??**). Since $\operatorname{Quot}^P(F/X/S)$ is proper over S (**??**), this map is a closed immersion. Since $\operatorname{Gr}_S(P(d), \pi_*F(d))$ is strongly projective over S (**??**), so is $\operatorname{Quot}^P(F/X/S)$. □

Exercise 4.5. (1) Show that if S is a noetherian scheme and V is a *coherent* sheaf on S, then functor $\operatorname{Gr}_S(k, V)$ defined analogously to **??** is represented by a scheme projective (but not necessarily strongly projective) over S.

(2) Show that if $X \to S$ is a projective morphism of noetherian scheme and F is a coherent sheaf on X *flat* over S, then $\operatorname{Quot}^P(F/X/S) \to \operatorname{Gr}_S(P(d), \pi_*F(d))$ is well-defined for $d \gg 0$ and $\operatorname{Quot}^P(F/X/S)$ is projective over S.

4.4 Generalizations

If $\pi\colon X \to S$ is a strongly quasi-projective morphism of noetherian schemes (i.e. there is a locally closed immersion $X \hookrightarrow \mathbb{P}_S(V)$ where V is a vector bundle on S), $\mathcal{O}_X(1)$ is a relatively ample line bundle, F is a coherent sheaf on X which is a quotient of $\pi^*(W)(q)$ for a vector bundle W on S and integer q, and $P \in \mathbb{Q}[z]$ is a polynomial, we can modify the functors of Hilb and Quot as follows:

$$\operatorname{Hilb}^P(X/S)\colon \operatorname{Sch}/S \to \operatorname{Sets}$$

$$(T \to S) \mapsto \left\{ \begin{array}{l} \text{subschemes } Z \subset X_T \text{ flat, proper and finitely presented over } T \text{ such} \\ \text{that } Z_t \subset X \times_S \kappa(t) \text{ has Hilbert polynomial } P \text{ for all } t \in T \end{array} \right\}$$

$$\operatorname{Quot}^P(F/X/S)\colon \operatorname{Sch}/S \to \operatorname{Sets}$$

$$(T \to S) \mapsto \left\{ \begin{array}{l} \text{quasi-coherent quotients } F_T \to Q \text{ on } X_T \text{ of finite presentation} \\ \text{with proper support over } T \text{ such that } Q|_{X \times_S \kappa(t)} \text{ on } X \times_S \kappa(t) \\ \text{has Hilbert polynomial } P \text{ for all } t \in T \end{array} \right\}$$

Then $\mathrm{Hilb}^P(X/S)$ and $\mathrm{Quot}^P(F/X/S)$ are represented by strongly quasi-projective schemes over S; see [?, §4], [?] or [?, §5.6]

If $X \to S$ is merely a separated morphism of noetherian schemes, then one can define functors $\mathrm{Hilb}(X/S)$ and $\mathrm{Quot}(F/X/S)$ as above dropping the condition on the Hilbert polynomial P. These functors are representable by algebraic spaces separated and locally of finite type over S; see [?, Thm. 6.1][1] and [?, Tag 09TQ]. Examples of Hironaka produce smooth proper (but not projective) 3-folds X over $\mathbb{C}$ such that $\mathrm{Hilb}^P(X/S)$ is not a scheme.

There are further variants and generalizations:

- Vistoli's Hilbert stack parameterizing finite and unramified morphisms to a separated scheme X (or stack) [?].
- Alexeev and Knutson's moduli of branch varieties parameterizing finite morphisms from a geometrically reduced proper scheme to a separated scheme X [?].
- If $X \to S$ is not separated, then Hall and Rydh show that there is an algebraic stack locally of finite type over S parameterizing quasi-finite morphism $Z \to X$ from a proper scheme [?].

Exercise 4.6 (Schemes of morphisms)**.** For projective morphisms $X \to S$ and $Y \to S$ of noetherian schemes, consider the functor

$$\begin{aligned} \underline{\mathrm{Mor}}_S(X,Y)\colon \mathrm{Sch}/S &\to \mathrm{Sets} \\ (T \to S) &\mapsto \mathrm{Mor}_T(X_T, Y_T) \end{aligned}$$

assigning an S-scheme T to the set of T-morphisms $X_T \to Y_T$. By using a suitable Hilbert scheme $\mathrm{Hilb}^P(X \times_S Y/X)$ parameterizing graphs $X \subset X \times_S Y$ of morphisms $X \to Y$, show that $\underline{\mathrm{Mor}}_S(X,Y)$ is representable by a projective scheme over S. Can we weaken the hypothesis on X and Y?

5 An invitation to the geometry of Hilbert schemes

In this section, we work over an algebraically closed field $\Bbbk$.

The Hilbert polynomial $P(z) = \sum_{i=0}^d a_i z^i$ of a projective scheme $X \subset \mathbb{P}^n$ encodes invariants of X. For instance, $\dim X$ is the degree d of P and $\deg X$ is normalized leading coefficient $d!a_d$. Applying Riemann–Roch in the case of a smooth curve $C \subset \mathbb{P}^n$ gives $P(z) = \deg(C)z + (1-g)$ and for a surface $S \subset \mathbb{P}^n$ gives $P(z) = \frac{1}{2}(nH \cdot (nH - K)) + (1 - p_a)$ where H is a hyperplane divisor, K is the canonical divisor and $p_a = 1 - \chi(\mathcal{O}_S)$ is the arithmetic genus. In arbitrary dimension, Hirzebruch–Riemann–Roch implies that $P(z) = \int_X \mathrm{ch}(\mathcal{O}_X(z))\mathrm{td}(X)$, where $\mathrm{ch}(\mathcal{O}_X(z))$ is the Chern character and $\mathrm{td}(X)$ the Todd class.

5.1 Hilbert scheme of hypersurfaces and linear subspaces

A hypersurface $H \subset \mathbb{P}^n$ of degree d has Hilbert polynomial

$$P(z) = \chi(\mathcal{O}_{\mathbb{P}^n}(z)) - \chi(\mathcal{O}_{\mathbb{P}^n}(z-d)) = \binom{n+z}{n} - \binom{n+z-d}{n}$$

[1] As pointed out in [?, Appendix], the representability is not true without the separated hypothesis on $X \to S$.

(coming from the exact sequence $0 \to \mathcal{O}_{\mathbb{P}^n}(-d) \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_H \to 0$). We claim that $\mathrm{Hilb}^P(\mathbb{P}^n) \cong \mathbb{P}(\Gamma(\mathbb{P}^n, \mathcal{O}(d)))$. We encourage the reader to show this and in particular establish that any subscheme $Z \subset \mathbb{P}^n$ with Hilbert polynomial P is a hypersurface.

Similarly, a linear subspace $L \subset \mathbb{P}^n$ of dimension k has Hilbert polynomial $P(z) = \binom{z+k}{k}$ and $\mathrm{Hilb}^P(\mathbb{P}^n) = \mathrm{Gr}(k+1, n+1)$.

5.2 Hilbert scheme of points on a curve

If C is a smooth projective curve, then the Hilbert scheme of n points $\mathrm{Hilb}^n(C)$ (viewing n as the constant polynomial) is a smooth irreducible projective variety isomorphic to the symmetric product

$$\mathrm{Sym}^n C := \underbrace{C \times \cdots \times C}_{n} / S_n,$$

where S_n acts by permuting the factors. The quotient exists as a projective variety since $C \times \cdots \times C$ is projective; see **??**.

5.3 Hilbert scheme of points on a surface

If S is a smooth irreducible projective surface, then the Hilbert scheme of n points $\mathrm{Hilb}^n(S)$ is a smooth irreducible projective variety [**?**]. See also [**?**, §4] and [**?**, §5]. There is a birational morphism

$$\mathrm{Hilb}^n(S) \to \mathrm{Sym}^n(S) := \underbrace{S \times \cdots \times S}_{n} / S_n,$$

of projective varieties. The symmetric product $\mathrm{Sym}^n(S)$ is not smooth for $n > 1$ and this provides a resolution of singularities. For an unordered collection of (possibly non-distinct) points $(p_1, \ldots, p_n) \in \mathrm{Sym}^n(S)$, the fiber consists of all possible scheme structures on $\{p_1, \ldots, p_n\}$ of length n. For example, when $n = 2$ and $p_1 = p_2$, then there is a $\mathbb{P}^1$ of scheme structures given by $k[x, y]/(x^2, xy, y^2, ay - bx)$ (with coordinates such that $p_1 = p_2 = 0$) parameterized by their "tangent direction" $[a : b] \in \mathbb{P}^1$. In this case, $\mathrm{Hilb}^2(S) \to \mathrm{Sym}^2(S)$ is the blow-up of the diagonal $S \hookrightarrow \mathrm{Sym}^2(S)$ given by $p \mapsto (p, p)$. In fact, for $n > 2$, the map $\mathrm{Hilb}^n(S) \to \mathrm{Sym}^n(S)$ is a blow-up along some ideal sheaf [**?**] but the description of the ideal sheaf is more complicated. Even more generally, for a scheme X of arbitary dimension, an irreducible component $\mathrm{Hilb}^n(X)$, called the "good component," can be identified with the blow-up of $\mathrm{Sym}^n(S)$ along some ideal sheaf [**?**].

5.4 Twisted cubics

The Hilbert scheme $\mathrm{Hilb}^{3z+1}(\mathbb{P}^3)$ consists of the union of two smooth rational irreducible components H and H' of dimensions 12 and 15 intersecting transversely along a smooth rational subvariety of dimension 11 [**?**].

The locus H is the closure of the locus H_0 consisting of *twisted cubics*, i.e. rational smooth curves in $\mathbb{P}^3$ of degree 3. Each twisted cubic can be represented by a map $\mathbb{P}^1 \to \mathbb{P}^3$ given by the line bundle $\mathcal{O}_{\mathbb{P}^1}(3)$ and a choice of basis of $\Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(3))$, and this representation is unique up to automorphisms of $\mathbb{P}^1$. All such curves are projectively equivalent, i.e. differ by an automorphism of $\mathbb{P}^3$, so we see that H_0 is identified with the homogeneous space $\operatorname{Aut}(\mathbb{P}^3)/\operatorname{Aut}(\mathbb{P}^1) = \mathrm{SL}_4/\mathrm{SL}_2$, which is smooth and irreducible of dimension 12. The locus H_0 is not proper as it includes families such as $\mathbb{P}^1 \hookrightarrow \mathbb{P}^3$ given by $[x, y] \mapsto [x^3, x^2y, xy^2, ty^3]$ parameterized by $t \in \mathbb{A}^1$ whose limit is a singular curve C_0 supported on a nodal cubic in $V(w) = \mathbb{P}^2$ (where w is the 4th coordinate) but with an embedded point at the node; see [**?**, Ex. 9.8.4].

The locus H' is the closure of the locus H'_0 consisting of subschemes $C \sqcup \{p\}$ where C is a smooth cubic curve contained in a hyperplane H and $p \in \mathbb{P}^3 \setminus C$. To count the dimension, observe that the choice of hyperplane $H \in \mathbb{P}(H^0(\mathbb{P}^3, \mathcal{O}(1)))$ is given by 3 parameters, the choice of plane cubic $C \in \mathbb{P}(H^0(H, \mathcal{O}_H(3)))$ is given by 9 parameters and the point $p \in \mathbb{P}^3 \setminus C$ is given by 3 parameters. The locus H'_0 is smooth and irreducible of dimension 15. Again, the locus H'_0 is not proper and its closure contains the limits of for instance degenerating the point p to lie on the curve whose limit can be curves like C_0.

The intersection $H \cap H'$ consists of plane, singular cubic curves with an embedded point at the singular point. This locus contains curves such as C_0 above but it also contains even more degenerate curves such as a triple line with an embedded point. Any curve $C \in H \cap H'$ is in fact projectively equivalent to the curve defined by $V(xz, yz, z^2, q(x, y, w))$ where $q(x, y, w)$ is a homogeneous cubic polynomial with a singular point at $(0, 0, 1)$. This depends on 11 parameters.

5.5 Non-emptyness

The Hilbert scheme $\operatorname{Hilb}^P(\mathbb{P}^n)$ is non-empty if and only if the Hilbert polynomial P can be written as as

$$P(z) = \binom{z + \lambda_1 - 1}{\lambda_1 - 1} + \binom{z + \lambda_2 - 2}{\lambda_2 - 1} + \cdots + \binom{z + \lambda_r - r}{\lambda_r - 1}, \tag{5.1}$$

integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \geq 1$. This is a result of Hartshorne [**?**, Cor. 5.7] The necessity of this condition was already mentioned in **??** in the context of Gotzmann's bounds on regularity.

5.6 Connectedness

Hartshorne's Connectedness Theorem asserts that the Hilbert scheme $\operatorname{Hilb}^P(\mathbb{P}^n)$ is connected for every Hilbert polynomial P [**?**]. More generally, if any connected noetherian scheme S, $\operatorname{Hilb}^P(\mathbb{P}^n_S/S)$ is connected.

The strategy of argument is to show that any closed subscheme $Z \subset \mathbb{P}^n$ degenerates to a subscheme $V(I)$ defined by a monomial idea. This reduces the question to the combinatorial question of connecting any two monomial ideals by a family over $\mathbb{A}^1$. It turns out that there is a purely deformation and combinatorial question or as Hartshorne writes: "It also appears that the Hilbert scheme is never actually needed in the proof."

See also [**?**, §3].

5.7 Murphy's Law

> Murphy's Law for Hilbert Schemes: There is no geometric possibility so horrible that it cannot be found generically on some component of the Hilbert scheme. [**?**, p.18]

The first pathology was exhibited by Mumford: there is an irreducible component of $\mathrm{Hilb}^{14z-23}(\mathbb{P}^3)$ which is generically non-reduced [**?**]. Ellia-Hirschowitz-Mezzetti show that the number of irreducible components in $\mathrm{Hilb}^{az+b}(\mathbb{P}^3)$ is not bounded by a polynomial in a, b [**?**].

Murphy's Law was made precise by Vakil [**?**]: for every scheme X finite type over $\mathbb{Z}$ and point $x \in X$, there exists a point $q = [Z \subset \mathbb{P}^n] \in \mathrm{Hilb}^P(\mathbb{P}^n)$ of some Hilbert scheme and an isomorphism

$$\widehat{\mathcal{O}}_{X,p}[[x_1, \ldots, x_s]] \cong \widehat{\mathcal{O}}_{\mathrm{Hilb}^P(\mathbb{P}^n),q}[[y_1, \ldots, y_t]]$$

for integers s, t. In other words, if we introduce the equivalence relation on pointed schemes (Z, z) generated by $(Z, z) \sim (Z', z')$ if there exists a smooth pointed morphism $(Z', z') \to (Z, z)$, then (X, p) is equivalent to to $(\mathrm{Hilb}^P(\mathbb{P}^n), q)$.

In fact, it can be arranged that the Hilbert scheme parameterizes smooth curves in $\mathbb{P}^n$ for some n, or that it parameterizes smooth surfaces in $\mathbb{P}^5$ (resp. surfaces in $\mathbb{P}^4$). It turns out that various other moduli spaces also satisfy Murphy's Law: Kontsevich's moduli space of maps, moduli of canonically polarized smooth surfaces, moduli of curves with linear systems and the moduli space of stable sheaves.

5.8 Smoothness

A theorem of Skjelnes–Smith [**?**] states that the Hilbert scheme $\mathrm{Hilb}^P(\mathbb{P}^n)$ is smooth if and only if $P(z)$ can be written as (**??**) for a partition $\lambda = (\lambda_1, \ldots, \lambda_r)$ of integers satisfying $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \geq 1$ such that one of the seven condition holds:

(1) $n = 2$;

(2) $\lambda_r \geq 2$;

(3) $\lambda = (1)$ or $\lambda = (n^{r-2}, \lambda_{r-1}, 1) = (\underbrace{n, \ldots, n}_{r-2}, \lambda_{r-1}, 1)$ where $r \geq 2$ and $n \geq \lambda_{r-1} \geq 1$;

(4) $\lambda = (n^{r-s-3}, \lambda_{r-s-2}^{s+2}, 1)$ where $r - s \geq s \geq 0$ and $n - 1 \geq \lambda_{r-s-2} \geq 3$;

(5) $\lambda = (n^{r-s-5}, 2^{s+4}, 1)$ where $r - 5 \geq s \geq 0$;

(6) $\lambda = (n^{r-3}, 1^s)$ where $r \geq 3$;

(7) $\lambda = (n + 1)$ or $r = 0$.

Notes

Grothendieck established the representability and projectivity of $\mathrm{Quot}^P(F/\mathbb{P}^n_A/\operatorname{Spec} A)$ where F is coherent sheaf on $\mathbb{P}^n_A$ and A is a noetherian ring [**?**, Thm. 3.2]. Our exposition follows Grothendieck's strategy deviating in only our use of Mumford–Castelnuovo regularity to establish boundedness. Grothendieck's original approach established the boundedness of $\mathrm{Quot}^P(F/\mathbb{P}^n_A/\operatorname{Spec} A)$ by reducing it to the case when $F = \mathcal{O}_X$ and relying on Chow's result on the boundedness of reduced, pure-dimensional subscheme $Y \subset X$ of fixed degree. We have following Mumford's argument for Boundedness of Regularity (**??**) in [**?**] which Mumford applies to construct the Hilbert scheme of curves on a surface (but applies equally to $\mathrm{Quot}^P(F/\mathbb{P}^n_A/\operatorname{Spec} A)$). Our formulation of **??** using the strong projectivity of

$X \to S$ follows [?, Thm. 2.6]. This chapter follows closely the excellent expositions of [?, §14-15], [?, §6], [?, §1], [?, §1.8], and [?, §2].

6 Appendix

Proposition 6.1 (Flatness via the Hilbert Polynomial)**.** *Let S be a connected and reduced scheme and $X \subset \mathbb{P}^n_S$ a closed subscheme. A quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ is flat over S if and only if the function*

$$S \to \mathbb{Q}[z], \qquad s \mapsto P_{\mathcal{F}|_{X_s}}$$

assigning a point $s \in S$ to the Hilbert polynomial of the restriction $\mathcal{F}|_{X_s}$ to the fiber $X_s \subset \mathbb{P}^n_{\kappa(s)}$ is constant.

Theorem 6.2 (Existence of Flattenning Stratifications)**.** *Let $X \to S$ be a projective morphism of noetherian schemes, $\mathcal{O}_X(1)$ be a relatively ample line bundle and $\mathcal{F}$ be a coherent sheaf on X. For each polynomial $P \in \mathbb{Q}[z]$, there exists a locally closed subscheme $S_P \hookrightarrow S$ such that a morphism $T \to S$ factors through S_P if and only if the pullback $\mathcal{F}_T$ of $\mathcal{F}$ to X_T is flat over T with Hilbert polynomial P.*

Moreover, there exists a finite indexing set I of polynomials such that $S = \coprod_{P \in I} S_P$ set-theoretically. The closure of S_P in S is contained set-theoretically in the union $\bigcup_{P \leq Q} S_Q$, where $P \leq Q$ if and only if $P(z) \leq Q(z)$ for $z \gg 0$.

Exercise 6.3. Let $E \to F$ be a morphism of coherent sheaves on a noetherian scheme X. Show that the subfunctor of X (or more precisely of $h_X = \mathrm{Mor}(-, X)$) defined by

$$\mathrm{Sch} \to \mathrm{Sets}, \quad T \mapsto \{\text{morphisms } T \to X \text{ such that } E_T \to F_T \text{ is zero}\}$$

is representable by a closed subscheme of X.