Formal Smoothness

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1 Smoothness

1.1 I-Smoothness of Algebras

Let $I \lhd B$ be an ideal of an A-algebra B. We say that B is I-smooth over A if the following condition holds, given an A-algebra C with an ideal N such that $N^2 = 0$, any A-algebra homomorphism $f: B \to C/N$ continuous for the discrete topology on C/N and the I-adic topology on B factors through C.

$$\begin{tikzcd} B \arrow[r, "f"] \arrow[dr, dotted, "g"] & C/N \\ A \arrow[u] \arrow[r] & C \arrow[u] \end{tikzcd}$$

1.1.1 Remark: Note that continuity of f is equivalent to having $f(I^n) = 0$ for some n

1.1.2 Remark: Suppose that g exists, then if $f(I^n) = 0$ we have $g(I^n) \subseteq N$ so that $g(I^{2n}) \subseteq N^2 = 0$ so that g is continuous for the discrete topology of C. This implies that for an I-smooth map we may lift over any nilpotent ideal, as if $N^k = 0$ we may look first at the map $C/N^2 \to C/N$ with kernel N/N^2 to lift to C/N^2, then continue in this fashion lifting to C/N^3 and so on until we have lifted f to $C/N^k = C$. We see that regardless of what N is, we obtain a system of maps to C/N^k for all k, so that this will define a map to $\hat{C}$ the N-adic completion of C, so in particular if C is N-adically complete this will define a map to C factoring f.

When $I = (0)$ then the notion of 0-smoothness imposes no condition on f. Clearly we see that if $I \subseteq J$, then I-smoothness implies J-smoothness. In the language of EGA I-smoothness is called formal smoothness for the I-adic topology where it was defined for arbitrary topologies on B. We will use the terminology I-smoothness, and reserve formal smoothness to mean 0-smoothness to keep in line with the language used for schemes. h

1.1.3 Theorem

Transitivity: Let $A \xrightarrow{g} B \xrightarrow{g'} B'$ be ring homomorphisms, and suppose that g' is continuous for the I-adic topology of B and I'-adic topology of B'. If B is I-smooth over A and B' is I'-smooth over B then B' is I' smooth over A.

Proof

Suppose that u is given in the following diagram.

$$\begin{tikzcd}
B' \arrow[r, "u"] \arrow[dr, dotted, "v"] & C/N \\
B \arrow[u, "g'"] \arrow[r, dotted, "w"] & C \arrow[u] \\
A \arrow[u, "g"] \arrow[ur] &
\end{tikzcd}$$

As $u \circ g' \colon B \to C/N$ is continuous, by I-smoothness of B there exists a lifting $w \colon B \to C$. Then, by I'-smoothness of B' over B we lift u to a map $v \colon B' \to C$. ■

1.1.4 Theorem

Base-change: Let A be a ring, B and A' two A-algebras, and set $B' = B \otimes_A A'$. If B is I-smooth over A, then B' is IB'-smooth over A'.

Proof

We have the diagram

$$\begin{tikzcd}
B \arrow[r, "f"] & B' \arrow[r, "u"] & C/N \\
A \arrow[u] \arrow[r] & A' \arrow[u, "g"] \arrow[r, "v"] & C \arrow[u]
\end{tikzcd}$$

With f, g the natural maps into $B' = B \otimes_A A'$. If u satisfies $u(I^k B') = 0$, there is a lifting $w \colon B \to C$ of $u \circ f$. Now define $u' \colon B' = B \otimes_A A' \to C$ by $u' = w \otimes v$, this is a lifting of u to C (use that $\otimes_A$ is a pushout). ■

1.1.5 Lemma

Let A be a ring, and S a multiplicative subset. Then $S^{-1}A$ is formally smooth over A.

Proof

Suppose we have a diagram of the form

$$\begin{tikzcd}
S^{-1}A \arrow[r, "f"] & C/N \\
A \arrow[u] \arrow[r, "g"] & C \arrow[u, "\pi"']
\end{tikzcd}$$

To define a lifting of g from $S^{-1}A \to C$ by universal property of $S^{-1}A$ we must show that S maps to units inside of C. Consider that for $s \in S$, $f(s/1) = \pi(g(s))$ is a unit, so that $g(s)$ is a unit modulo a nilpotent ideal. However anything that is a unit modulo a nilpotent ideal is a unit to begin with, so that $g(s)$ is a unit as required. ■

We now provide a few more examples of I-smoothness.

1.1.6 Proposition
Let A be a ring, then $A[x]$ is formally smooth over A, and $A[[x]]$ is I-smooth over A where $I = (x)$.

Proof
As $A[x]$ is a free A-algebra we can prove an even stronger lifting property. Given a diagram of the form

$$\begin{array}{ccc} A[x] & \xrightarrow{u} & C/N \\ \uparrow & & \uparrow \\ A & \longrightarrow & C \end{array}$$

where C is an arbitrary A-algebra and N an arbitrary ideal we can lift u to a map $A[x] \to C$. Let $c \in C$ be a preimage of $u(x)$ via the map $C \to C/N$, then we can define the A-algebra map $A[x] \to C$ sending x to c, and this provides a map making the diagram commute.

Now assume we are given a diagram

$$\begin{array}{ccc} A[[x]] & \xrightarrow{u} & C/N \\ \uparrow & & \uparrow \\ A & \longrightarrow & C \end{array}$$

where $u(x^n) = 0$, and N is an ideal of square zero. Pick a preimage c of $u(x)$ in C. We define a map $f\colon A[[x]] \to C$ by sending $\sum a_i x^i$ to $\sum a_i c^i$, to see that this latter sum is finite we note that $c^n \in N$ so that $c^m = 0$ for $m \geq 2n$. This defines a lifting of u to C so that $A[[x]]$ is I-smooth over A. ■

1.1.7 Lemma
Let $(R, \mathfrak{m})$ and $(S, \mathfrak{n})$ be rings with finitely generated ideals. Endow R and S with the $\mathfrak{m}$ and $\mathfrak{n}$-adic topologies respectively, and let $f\colon R \to S$ be a map of topological rings (which is to say that $f(\mathfrak{m}^k) \subseteq \mathfrak{n}$ for some k, or that it is a continuous ring map). The the following are equivalent:

(a) $R \to S$ is $\mathfrak{n}$-smooth.

(b) $R \to \hat{S}$ is $\hat{\mathfrak{n}}$-smooth.

(c) $\hat{R} \to \hat{S}$ is $\hat{\mathfrak{n}}$-smooth.

where $\hat{R}, \hat{S}$ are the $\mathfrak{m}$ and $\mathfrak{n}$-adic completions of R and S respectively.

Proof
The proof is simple, albeit messy, after noting a few facts. As the ideals used are finitely generated implies that $\hat{R}$ is $\hat{\mathfrak{m}}$-adically complete (which isn't needed for the proof), that

$\mathfrak{m}\hat{R} = \hat{\mathfrak{m}}$, and $R/\mathfrak{m}^k = \hat{R}/\hat{\mathfrak{m}}^k$ in the sense that there is a commutative diagram

$$\begin{tikzcd} R \arrow[r] \arrow[d] & \hat{R} \arrow[d] \\ R/\mathfrak{m}^k \arrow[r, "\sim"] & \hat{R}/\hat{\mathfrak{m}}^k \end{tikzcd}$$

With the same statements holding for S and $\hat{S}$.

We now make an observation about formal smoothness, suppose there is a commutative diagram

$$\begin{tikzcd} S \arrow[r, "u"] & C/N \\ R \arrow[u, "f"] \arrow[r, "g"] & C \arrow[u] \end{tikzcd}$$

where $u(\mathfrak{n}^m) = 0$. Suppose that u factors through C via u', then note that as $u'(\mathfrak{n}^m)$ maps to 0 via $C \to C/N$ that it lies in N, so then $u'(\mathfrak{n}^{2m}) \subseteq N^2 = 0$ so that u' factors through $S/\mathfrak{n}^{2m}$ as does u. Similarly, if $f(\mathfrak{m}^k) \subseteq \mathfrak{n}$ then we see that $g(\mathfrak{m}^{2km}) = 0$ so that g factors through $R/\mathfrak{m}^{2km}$. We thus can form a commutative diagram

$$\begin{tikzcd} S \arrow[rr, "u"] \arrow[dr] & & C/N \\ & S/\mathfrak{n}^{2m} \arrow[ur] \arrow[ddr, dotted] & \\ & R/\mathfrak{m}^{2km} \arrow[u] \arrow[dr] & \\ R \arrow[uuu] \arrow[ur] \arrow[rr, "g"] & & C \arrow[uuu] \end{tikzcd}$$

Note that conversely, simply from $u(\mathfrak{n}^m) = 0$ that u factors through $S/\mathfrak{n}^{2m}$ as does g factor through $R/\mathfrak{m}^{2km}$ so that the diagram above will always exist with the exception of the dotted arrow. Thus the question of lifting u is exactly the same as finding the dotted arrow $S/\mathfrak{n}^{2m} \to C$ making the diagram commute. Noting that $\hat{S}/\hat{\mathfrak{n}}^{2m} = S/\mathfrak{n}^{2m}$ with a similar statement for R gives one the idea of how to prove the equivalences of the theorem as factoring the required map reduces to factoring a map through a quotient. In fact, I claim that the entire equivalence can be show by staring at certain parts of the following diagram long enough

$$\begin{tikzcd}
\hat{S} \arrow[drr] \arrow[ddr] & & \\
S \arrow[u] \arrow[rr] \arrow[dr] & & C/N \\
& S/\mathfrak{n}^{2m} \arrow[ur] \arrow[ddr, dotted] & \\
& R/\mathfrak{m}^{2km} \arrow[u] \arrow[dr] & \\
R \arrow[uuu] \arrow[ur] \arrow[rr] \arrow[d] & & C \arrow[uuu] \\
\hat{R} \arrow[uuuuu, bend left] \arrow[uur] \arrow[urr] & &
\end{tikzcd}$$

We briefly run through the arguments showing $(a) \implies (b) \implies (c) \implies (a)$.

If the map $\hat{S} \to C/N$ maps $\hat{\mathfrak{n}}^m$ to 0 then the map $S \to C/N$ maps $\mathfrak{n}^m$ to 0. By smoothness of S over R one can then form the diagram omitting $\hat{R}$ and the maps involving it, smoothness gives the existence of the dotted map. By our above discussion we see that this lifts the map $\hat{S} \to C/N$ as well. For $(b) \implies (c)$ we start with almost the same setup, the only thing that must be argued is why the map $\hat{R} \to C$ maps $\hat{\mathfrak{m}}^{2km}$ to 0. This follows as if we send $\mathfrak{m}^{2km}$ through the composition $R \to \hat{R} \to \hat{S} \to S/\mathfrak{n}^{2m} \to C$ we see that this maps $\mathfrak{m}^{2km}$ to 0. This however by commutativity is the same as sending it through $R \to C$, then as the image of $\mathfrak{m}^{2km}$ along $R \to \hat{R}$ is $\hat{\mathfrak{m}}^{2km}$ we see that $\hat{\mathfrak{m}}^{2km}$ is sent to 0 in C, letting us factor $\hat{R} \to C$ through $R/\mathfrak{m}^{2km}$. For the final direction, we start originally with the diagram not involving $\hat{S}$ and $\hat{R}$, and it is not a priori clear that we have maps $\hat{S} \to C/N$ or $\hat{R} \to C$. We can however define these as the composition $\hat{S} \to S/\mathfrak{n}^{2m} \to C/N$ and $\hat{R} \to R/\mathfrak{m}^{2km} \to C$, then by smoothness of $\hat{S}$ over $\hat{R}$ we obtain the dotted arrow showing that $R \to S$ is $\mathfrak{n}$-smooth. ∎

For fields we can characterize formal smoothness. A field extension $k \to K$ is formally smooth if and only if it is a separable extension. Here, separable must be expanded to include transcendental extensions, and the definition is that K is separable over k if for any field extension k' of k, that $k' \otimes_k K$ is reduced. In the case that K is an algebraic extension of k this notion agrees with the usual notion of separability of field extensions. If k is a perfect field, then any field extension is separable under our notion of separable so that every field is formally smooth over k. Details can be found in §26 of Matsumura's *Commutative Ring Theory*.

We now require two technical results on I-smoothness. The former is simply to be used later when discussing Cohen rings, while the latter is a very powerful form of lifting which will be required to lift a map which is not continuous for the I-adic topology only using

I-smoothness. We present a simplified version which is all that is needed for our purposes.

1.1.8 Lemma
Let $R \to S$ be a ring map, and $I \triangleleft R$ an ideal. Assume

(1) $I^2 = 0$,

(2) $R \to S$ is flat, and

(3) $R/I \to S/IS$ is formally smooth.

Then $R \to S$ is formally smooth.

Proof
Tag 031L. ∎

1.1.9 Theorem
Let $(R, \mathfrak{m}) \to (S, \mathfrak{n})$ be a local ring map. Assume that $R \to S$ is $\mathfrak{n}$-smooth. Consider a solid commutative diagram

$$\begin{tikzcd} S \arrow[r, "\psi"] \arrow[dr, dotted] & A/J \\ R \arrow[u] \arrow[r] & A \arrow[u] \end{tikzcd}$$

of homomorphisms of local rings, for the complete local ring (A, I) and $(A/J, I/J)$, where J is a closed ideal of A (this is true for example when A is Noetherian). Then there exists a dotted arrow making the diagram commute which is a local map.

Proof
First note that $A = \varprojlim A/I^n$ and $A/J = \varprojlim A/(J + I^n)$ as J is a closed ideal. Let $A_{n,m} = A/(J^n + I^m)$ be R-algebras given the discrete topology, and consider the following diagram

$$\begin{tikzcd} & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] \\ \dots \arrow[r] & A/(J^3+I^3) \arrow[r] \arrow[d] & A/(J^2+I^3) \arrow[r] \arrow[d] & A/(J+I^3) \arrow[d] \\ \dots \arrow[r] & A/(J^3+I^2) \arrow[r] \arrow[d] & A/(J^2+I^2) \arrow[r] \arrow[d] & A/(J+I^2) \arrow[d] \\ \dots \arrow[r] & A/(J^3+I^1) \arrow[r] & A/(J^2+I^1) \arrow[r] & A/(J+I) \end{tikzcd}$$

Each of these squares defines a surjection $A_{n+1,m+1} \to A_{n+1,m} \times_{A_{n,m}} A_{n,m+1}$ with kernel of square zero (this is the fibered product of rings, it is a pullback in the category of rings. It is defined for rings with maps $f\colon A \to C$ and $g\colon B \to C$ as the subring of $A \times B$ of elements

(a, b) such that $f(a) = g(b)$. When A, B are Noetherian and f, g are both surjective it is also Noetherian.) We inductively define R-algebra maps $\varphi_{n,m} \colon S \to A_{n,m}$. First define $\varphi_{1,m} \colon S \to A/(J+I^m)$ as $\psi \mod J+I^m$, as ψ was a local map we see that $\varphi_{1,m}(\mathfrak{n}^m) \subseteq J+I^m$ so that each $\varphi_{1,m}$ is continuous. Define $\varphi_{n,1}$ inductively starting with $\varphi_{1,1}$ using $\mathfrak{n}$-smoothness of $R \to S$, namely as a dotted map making the following diagram commute

$$\begin{tikzcd} S \arrow[r, "\varphi_{n-1,1}"] \arrow[dr, dotted, "\varphi_{n,1}"] & A/(J^{n-1}+I) \\ R \arrow[u] \arrow[r] & A/(J^n+I) \arrow[u] \end{tikzcd}$$

We now define the rest of the $\varphi_{n,m}$ by induction on $n + m$, namely we take the surjection $(\varphi_{n+1,m}, \varphi_{n,m+1}) \colon S \to A_{n+1,m} \times_{A_{n,m}} A_{n,m+1}$ (this exists by universal property of fibered products) and lift this this map using $\mathfrak{n}$-smoothness of $R \to S$. This can be described using the following diagram

$$\begin{tikzcd} S \arrow[r, "(\varphi_{n+1,m}, \varphi_{n,m+1})"] \arrow[dr, dotted, "\varphi_{n+1,m+1}"] & A_{n+1,m} \times_{A_{n,m}} A_{n,m+1} \\ R \arrow[u] \arrow[r] & A_{n+1,m+1} \arrow[u] \end{tikzcd}$$

Because the universal property of $(\varphi_{n+1,m}, \varphi_{n,m+1})$ (or one can look at the concrete construction) this ensures that the system of maps $\varphi_{n,m}$ are compatible with the transition maps in the system $A_{n,m}$, which is to say these two triangles commute

$$\begin{tikzcd} S \arrow[r, "\varphi_{n-1,m}"] \arrow[dr] & A_{n-1,m} \\ & A_{n,m} \arrow[u] \end{tikzcd} \qquad \begin{tikzcd} S \arrow[r, "\varphi_{n,m-1}"] \arrow[dr] & A_{n,m-1} \\ & A_{n,m} \arrow[u] \end{tikzcd}$$

As a result, we can take the maps $\varphi_n = \varphi_{n,n} \mod I^n$ gives us maps $S \to A/I^n$ which are compatible so they define a map $S \to \varprojlim A/I^n = A$. When modding out by J this agrees with ψ as we know that $A/J = \varprojlim A/(J + I^n)$. As $\psi(\mathfrak{n}) \subseteq I/J$ we see that $\varphi(\mathfrak{n}) \subseteq I$ so that φ is a map of local rings. ∎

1.2 Formal Smoothness of Schemes

We will call a morphism $f \colon X \to Y$ of schemes formally smooth if given any diagram of the following form

$$\begin{tikzcd} X \arrow[d, "f"'] & T \arrow[l] \arrow[d] \\ Y & T' \arrow[l] \end{tikzcd}$$

Where T is a closed subscheme of T' an affine scheme defined by an ideal sheaf whose square is zero, that there exists a map $T' \to X$ making the diagram commute. One notices that is X, Y are affine, equal to Spec B and Spec A, then dualizing this says that B is a formally smooth A-algebra. As was done for the smoothness of algebras one gains as a result that one may lift maps across nilpotent ideal sheaves.

1.2.1 Lemma
Let $f \colon X \to Y$ be a formally smooth map of schemes. Let $U \subseteq X$ be an open subscheme, and $V \subseteq Y$ an open subscheme such that $f(U) \subseteq V$. Then $f\lfloor_U \colon U \to V$ is formally smooth.

PROOF
Suppose there is a commutative diagram of the following form:

$$\begin{tikzcd} U \arrow[d, "f\lfloor_U"'] & T \arrow[l, "g"'] \arrow[d] \\ V & T' \arrow[l] \end{tikzcd}$$

Composing g with the inclusion into X we obtain a map g' from $T' \to X$ such that $g'\lfloor_T = g$, but as T and T' have the same topological space, we see that $g'(T') \subseteq U$ implying that g' factors through U as desired. ■

2 The Main Result

Now that we have defined formal smoothness we can state the main result, and begin a proof. The bulk of the proof is a commutative algebra result, but armed with what we already have it is easy to reduce to the commutative algebra result.

2.0.1 Theorem
Let $f \colon X \to Y$ be a formally smooth map of locally Noetherian schemes. Then f is flat.

PROOF
Let $x \in X$, and let $y = f(x)$. Take an affine neighborhood V of y, then take an affine neighborhood $U \ni x$ contained in $f^{-1}(V)$, then by Lemma 1.2.1 $f\lfloor_U \colon U \to V$ is formally smooth. So we may assume that $X = \text{Spec } B$, $Y = \text{Spec } A$ are affine schemes with A, B Noetherian. Letting $x = \mathfrak{p}$ and $y = \mathfrak{q}$ be the primes of B, A corresponding to x, y we must show that $A_\mathfrak{q} \to B_\mathfrak{p}$ is flat. We will first show that it is formally smooth.

First, note by Theorem 1.1.4 (Base-Change) that $B_\mathfrak{q}$ is formally smooth over $A_\mathfrak{q}$. Then, we can obtain $B_\mathfrak{p}$ as a localization of $B_\mathfrak{q}$, and combining Lemma 1.1.5 which says $B_\mathfrak{p}$ is formally smooth over $B_\mathfrak{q}$ with Theorem 1.1.3 (Transitivity) we see that $B_\mathfrak{p}$ is formally smooth over $A_\mathfrak{q}$. Flatness then follows from the next theorem (as formally smooth implies I-smooth for any I). ■

2.0.2 Theorem
Let $(A, \mathfrak{m}, k)$ and $(B, \mathfrak{n}, k')$ be Noetherian local rings, and $\varphi\colon A \to B$ a local homomorphism. Set $B' = B \otimes_A k = B/\mathfrak{m}B$ and $\mathfrak{n}' = \mathfrak{n}/\mathfrak{m}B$. Then the following are equivalent:

(a) B is $\mathfrak{n}$-smooth over A,

(b) B is flat over A and B' is $\mathfrak{n}'$-smooth over k.

The proof of this theorem is a hard exercise in commutative algebra which we will undertake later. The result to the author's knowledge was first proven by Grothendieck as EGA 0_{IV} (19.7.1), and the proof is very long and difficult. It was used to prove the Cohen Structure Theorem for complete Noetherian local rings. Another proof exists due to Michel Andrè in his book *Homologie des algèbres commutatives* which uses what is now called Andrè-Quillen homology which is simplicial / homotopical in nature. The third and final proof is due to Nicholae Radu, and uses the Cohen Structure Theorem which can be proven without this result to avoid a circular argument. This is the proof we have chosen to present. It is worthwhile to note that all three of these proofs are in French, and there (to the author's knowledge) doesn't appear to be a proof in English published anywhere although one direction does appear in the Stacks Project (and the other direction may well be in Stacks Project as well).

3 Algebra Results

3.1 The Local Criterion for Flatness

The local criterion for flatness refers to one of many variants of a theorem (much in the way Nakayama's lemma refers to several related results), we will only need one variant which we will state here.

3.1.1 Theorem
Let $R \to S$ be a local homomorphism of Noetherian local rings. Let $I \neq R$ be an ideal of R, and M a finite S-module. If $\mathrm{Tor}_1^R(M, R/I) = 0$ and M/IM is flat over R/I then M is flat over R.

As an application of the Local Criterion we can prove the following theorem which will play an important role in our proof.

3.1.2 Theorem
Let $R \to S$ be a local homomorphism of Noetherian local rings. Assume R is a regular local ring and maps a regular system of parameters of R to a regular sequence in S. Then $R \to S$ is flat.

Proof
Let $x_1, \dots, x_n$ be a regular system of parameters for R, and let $(x_1, \dots, x_n) = \mathfrak{m}$. We see that $S/\mathfrak{m}S$ is flat over $R/\mathfrak{m}$ as the latter is a field. We do induction (upwards) based on for what i we know that $S/(x_1, \dots, x_i)$ is flat over $R/(x_1, \dots, x_i)$. If we know it for i, $R' =$

$R/(x_1, \ldots, x_{i-1}) \to S' = S/(x_1, \ldots, x_{i-1})$ for Theorem 3.1.1. We see that $\mathrm{Tor}_1^R(S', R'/x_i)$ is the x_i-torsion of S' as x_i is regular on R' since $x_1, \ldots, x_n$ is a regular sequence on R, but then as it is also regular on S this torsion is 0. We can then conclude that $S/(x_1, \ldots, x_{i-1})$ is flat over $R/(x_1, \ldots, x_{i-1})$. Continuing by induction we see S is flat over R. ■

3.2 The Cohen Structure Theorem

The Cohen structure theorem provides a characterization of complete local Noetherian rings. Some of the results apply to arbitrary complete local rings, and as a result of the work done if a complete local ring has a finitely generated maximal ideal it is Noetherian. We will present relevant parts of the Cohen structure theorem that will be used in our proof.c

3.2.1 Equicharacteristic and Mixed Characteristic Rings

The first topic we must introduce is the notion of equicharacteristic and mixed characteristic rings. A local ring $(A, \mathfrak{m}, k)$ is called equicharacteristic if char $A =$ char k. If a ring is not equicharacteristic it is called mixed characteristic. We see that the only way a ring can be mixed characteristic is if it is characteristic 0 and its residue field has characteristic p, or if its characteristic is p^n for $n > 1$ and its residue field is characteristic p. There is a useful characterization of equicharacteristic rings which we present as a proposition.

3.2.1 Proposition
Let $(A, \mathfrak{m}, k)$ be a local ring. Then A is equicharacteristic if and only if it contains a field.

Proof
Suppose that A contains a field, and first that it is characteristic 0. A necessarily must contain $\mathbb{Q}$ as if it contained a positive characteristic field it too would have positive characteristic. We then see that for any $n \in \mathbb{Z}$ non-zero that n is invertible in A, so that $n \notin \mathfrak{m}$. It follows that k has characteristic 0 as if it was characteristic p we would have $p \in \mathfrak{m}$. If A instead has characteristic p^n then the field it contains must be characteristic p, but then we see that A has characteristic p as well.

Now instead suppose that A is equicharacteristic and is characteristic 0. As k is also characteristic 0 we see that $n \notin \mathfrak{m}$ for any $n \in \mathbb{Z}$ non-zero, so that n is invertible in A. Thus, A contains $\mathbb{Q}$. If A has characteristic p then it contains $\mathbb{F}_p$ as the map $\mathbb{Z} \to A$ has kernel (p). ■

3.2.2 Cohen Rings

A Cohen ring is defined to be a complete DVR Λ with uniformizer p, which is to say it has maximal ideal $p\Lambda$. We require one result on Cohen rings.

3.2.2 Proposition
Let Λ be a Cohen ring whose residue field has characteristic p. Then, for any $n \geq 1$, the map $\mathbb{Z}/p^n\mathbb{Z} \to \Lambda/p^n\Lambda$ is formally smooth. In particular, Λ is (p)-smooth over $\mathbb{Z}_p$, the p-adics.

Proof

If $n = 1$ then this follows from our characterization of formal smoothness for fields; namely $\mathbb{Z}/p\mathbb{Z}$ is a perfect field, then as $\Lambda/p\Lambda$ is also a field it is a separable extension and thus formally smooth. For other n we argue by induction; namely if $\mathbb{Z}/p^n\mathbb{Z} \to \Lambda/p^n\Lambda$ is formally smooth we will apply 1.1.8 to the map $\mathbb{Z}/p^{n+1}\mathbb{Z} \to \Lambda/p^{n+1}\Lambda$ with the ideal $I = (p^n)$. It is clear that $I^2 = 0$, and the map is formally smooth after modding out by I by assumption, so it remains to show the map is flat. We will the use the local criterion for flatness 3.1.1 at the ideal $J = (p)$, taking $R = \mathbb{Z}/p^{n+1}\mathbb{Z}$, $S = M = \Lambda/p^{n+1}\Lambda$. Clearly M/JM is flat over R/J, the latter being $\mathbb{F}_p$, so it remains to show that $\mathrm{Tor}_1^R(M, R/J) = 0$. This being 0 is equivalent to the map $(p) \otimes_R M \to (p)M$ being injective. Any element of the former can be written as $up^k \otimes vp^m$ where u, v are units, so that we can further rewrite this as $up^k \otimes v$ where u, v are units. This then maps to uvp^k, and if this is 0 in $M = \Lambda/p^n\Lambda$ this says $k \geq n$, but then $up^k \otimes v = 0$ to begin with so the map is injective. ∎

3.2.3 The Structure Theorem

We state the necessary parts of the Cohen structure theorem. Many excellent sources on the structure theorem exists, Matsumura §28 and 29 cover it although it builds on work done in §25 and 26 on differentials, and there is the stacks project page Tag 0323.

3.2.3 Theorem

Let $(A, \mathfrak{m}, K)$ be a complete local ring. There then exists a Cohen ring Λ and a map $\Lambda \to A$ which induces an isomorphism on residue fields. If A is equicharacteristic we can take Λ to be a field isomorphic to K. If $\mathfrak{m}$ is finitely generated by generators $x_1, \ldots, x_n$ there exists a surjective map $\Lambda[[X_1, \ldots, X_n]]$ given by $X_i \mapsto x_i$ and the original map $\Lambda \to A$. As a result, $A \cong \frac{\Lambda[[X_1,\ldots,X_n]]}{I}$ and is Noetherian, if A is a regular local ring then $I = 0$.

4 The proof of Theorem 2.0.2

2.0.2 Theorem

Let $(A, \mathfrak{m}, k)$ and $(B, \mathfrak{n}, k')$ be Noetherian local rings, and $\varphi\colon A \to B$ a local homomorphism. Set $B' = B \otimes_A k = B/\mathfrak{m}B$ and $\mathfrak{n}' = \mathfrak{n}/\mathfrak{m}B$. Then the following are equivalent:

(a) B is $\mathfrak{n}$-smooth over A,

(b) B is flat over A and B' is $\mathfrak{n}'$-smooth over k.

Before we begin the proof we note that this is essentially a translation and slight expansion of the proof by Nicholae Radu entitled *Sur La Structure Des Algèbres Locales Noethériennes Formellement Lisses* which translates to *On the Structure of Formally Smooth Noetherian Algebras.*

4.1 (a) $\implies$ (b)

Proof

The first thing to note is that we may replace $A \to B$ with $\hat{A} \to \hat{B}$. Indeed, by Lemma 1.1.7 the map $\hat{A} \to \hat{B}$ is still $\hat{\mathfrak{n}}$-smooth as a local homomorphism is continuous. Noting that $A \to \hat{A}$ and $B \to \hat{B}$ are faithfully flat, we see that if we can show $\hat{A} \to \hat{B}$ is flat, then the composition $A \to \hat{A} \to \hat{B} = A \to B \to \hat{B}$ is flat, then by faithful flatness of $\hat{B}$ over B we see $A \to B$ is flat. Finally, note that $B' = \hat{B}'$ and that $\hat{\mathfrak{n}}' = \mathfrak{n}'$ so that the final condition of (b) is satisfied if we can show it for $\hat{A} \to \hat{B}$. We thus can assume that A and B are $\mathfrak{m}$ and $\mathfrak{n}$-adically complete respectively.

By Theorem 1.1.4 (Base-Change) we see that B' is $\mathfrak{n}'$-smooth over k. Suppose that p is the characteristic of k. Let W be a Cohen ring such that $W[[x_1, \ldots, x_n]] \to A$ is surjective, and W' a Cohen ring such that $W'[[y_1, \ldots, y_m]] \to B$ is surjective. Let $W_0 = \mathbb{Z}_p$ the p-adic integers if W is not a field, and W_0 the prime field of W otherwise. Note that W and W' are W_0 algebras, indeed in the case that W_0 is a field, then A is equicharacteristic so that $p = 0$ in A, but then $\varphi(p) = p = 0$ in B as well. Then B is equicharacteristic which means that W' is also a field of characteristic p so that it contains W_0 as well. The map $W \to A \to B$ induces a map $W \to W'[[y_1, \ldots, y_m]]$ making the following diagram commute,

$$\begin{tikzcd} W \arrow[r, "\psi"] \arrow[d, "\pi"] & W'[[y_1, \ldots, y_m]] \arrow[d, "\pi'"] \\ A \arrow[r, "\varphi"] & B \end{tikzcd}$$

To see this, we will first handle the case where W_0 is a field. In this case, as W_0 is a perfect field, W is formally smooth over it, then by examining the following diagram

$$\begin{tikzcd} W \arrow[r, "\varphi\pi"] & B \\ W_0 \arrow[u] \arrow[r] & W'[[y_1, \ldots, y_m]] \arrow[u, "\pi'"] \end{tikzcd}$$

We note that $W'[[y_1, \ldots, y_m]]$ is a local ring with maximal ideal $I = (p, y_1, \ldots, y_m)$, and as W' was pW-adically complete this is I-adically complete, so that it is $\ker \pi'$-adically complete. Thus by formal smoothness of W over W_0, $\varphi\pi$ lifts to a map to $W'[[y_1, \ldots, y_m]]$.

In the case that $W_0 = \mathbb{Z}_p$, we consider the same diagram, but as W is not formally smooth over $\mathbb{Z}_p$ we run into an issue. We note however that W is (p)-smooth over $\mathbb{Z}_p$, and the map $W \to B$ is a local map. As $W'[[y_1, \ldots, y_m]]$ is a complete local ring we can apply Theorem 1.1.9 to lift the map $W \to B$ to a map to $W'[[y_1, \ldots, y_m]]$.

We can then define a new map $\Psi \colon W[[x_1, \ldots, x_n]] \to W'[[x_1, \ldots, x_n, y_1, \ldots, y_m]]$ by sending x_i to x_i and then defining it on W by sending a to $\psi(a)$. Note that a regular system of parameters of $W[[x_1, \ldots, x_n]]$ is given by $p, x_1, \ldots, x_n$ and one of $W'[[x_1, \ldots, x_n, y_1, \ldots, y_m]]$ by $p, x_1, \ldots, x_n, y_1, \ldots, y_m$ so that Ψ sends a regular system of parameters to a subset of a regular sequence so that it is flat. Define a map $W'[[x_1, \ldots, x_n, y_1, \ldots, y_m]] \to B$ by sending

x_i to $\varphi((W[[x_1,\dots,x_n]] \to A)(x_i))$, then defining the rest via π', then this is a surjective map as π' was surjective. We then obtain a commutative diagram with surjective vertical maps

$$\begin{tikzcd} A' = W[[x_1,\dots,x_n]] \arrow[r, "\Psi"] \arrow[d] & W'[[x_1,\dots,x_n,y_1,\dots,y_m]] = B' \arrow[d] \\ A \arrow[r, "\varphi"] & B \end{tikzcd}$$

Now $C = B' \otimes_{A'} A$ is a complete local ring as it is a quotient of B' (as A is a quotient of A') which is complete. There is a map $h\colon C \to B$ which is surjective as $B' \to B$ is surjective, then note that we can apply Theorem 1.1.9 to the identity map $B \to B$ to obtain a dotted arrow making the following diagram commute

$$\begin{tikzcd} B \arrow[r, "\mathrm{id}"] \arrow[dr, dotted] & B \\ A \arrow[u] \arrow[r] & C \arrow[u] \end{tikzcd}$$

which is to say that $C \to B$ splits, so B embeds as a summand of C as an A-module. As B' is flat over A', we see that $C = B' \otimes_{A'} A$ is flat over A, then as B is a direct summand of C we see that B is flat over A as desired. ■

4.2 (b) $\implies$ (a)