Notes on Deformation Theory

Lecture notes for Math 581J, *working draft*

University of Washington, Fall 2021

Jarod Alper

jarod@uw.edu

February 13, 2022

Deformation theory is the study of the local geometry of a moduli space $\mathscr{M}$ near an object $E_0 \in \mathscr{M}(\Bbbk)$. We focus primarily on the following three deformation problems:

(A) Embedded deformations $Z_0 \subset X$ of a closed subscheme Z_0 in a fixed projective scheme X over $\Bbbk$. Here the moduli problem is the Hilbert functor $\mathrm{Hilb}^P(X)$ and $E_0 = [Z_0 \subset X] \in \mathrm{Hilb}^P(X)(\Bbbk)$.

(B) Deformations of a scheme E_0 over $\Bbbk$. In this section, the main example for us is when E_0 is a smooth curve in which case the moduli problem is $\mathscr{M}_g$ and $[E_0] \in \mathscr{M}_g(\Bbbk)$.

(C) Deformations of a coherent sheaf E_0 on a fixed projective scheme C over $\Bbbk$. The main example for us is when C is a smooth curve and E_0 is a vector bundle in which case the moduli problem is $\mathcal{B}\mathrm{un}_C$ and $[E_0] \in \mathcal{B}\mathrm{un}_C(\Bbbk)$.

In this chapter, we sketch the local-to-global approach of deformation theory by zooming in around $E_0 \in \mathscr{M}(\Bbbk)$ and studying successively first order neighborhoods of $\mathscr{M}$ at E_0, higher order deformations of E_0, formal neighborhoods of E_0 and eventually étale or smooth neighborhoods of E_0.

(1) A *first order deformation* of E_0 is an object $E \in \mathscr{M}(\Bbbk[\epsilon])$ over the dual numbers $\Bbbk[\epsilon] := \Bbbk[\epsilon]/(\epsilon^2)$ together with an isomorphism $\alpha\colon E_0 \to E|_{\mathrm{Spec}\,\Bbbk}$, or in other words a commutative diagram

$$\begin{tikzcd}
\mathrm{Spec}\,\Bbbk \arrow[r, hook] \arrow[dr, "{[E_0]}"'] & \mathrm{Spec}\,\Bbbk[\epsilon] \arrow[d, dashed, "{[E]}"] \\
& \mathscr{M}
\end{tikzcd}$$

allowing us to view E as a tangent vector of $\mathscr{M}$ at E_0. We classify first order deformations of Problems (A)–(C) in §1.

(2) Given a surjection $A' \twoheadrightarrow A$ of artinian local $\Bbbk$-algebras with residue field $\Bbbk$ and an object object $E \in \mathscr{M}(A)$ with an isomorphism $E_0 \to E|_{\mathrm{Spec}\,\Bbbk}$, a

deformation of E over A' is an object $E' \in \mathscr{M}(A')$ with an isomorphism $\alpha\colon E \to E'|_{\operatorname{Spec} A}$. Pictorially, this corresponds to a commutative diagram

$$\begin{tikzcd}
\operatorname{Spec} \Bbbk \arrow[r, hook] \arrow[rrd, "{[E_0]}"'] & \operatorname{Spec} A \arrow[r, hook] \arrow[rd, "{[E]}"] & \operatorname{Spec} A' \arrow[d, dashed, "{[E']}"] \\
& & \mathscr{M}.
\end{tikzcd}$$

For Problems (A)–(C), we determine when a deformation E' of E over A' exists and classify them in §2

(3) Given a noetherian complete local $\Bbbk$-algebra $(R, \mathfrak{m})$, a *formal deformation of E_0 over R* is a compatible collection of deformations $E_n \in \mathscr{M}(R/\mathfrak{m}^{n+1})$ of E_0, and a formal deformation $\{E_n\}$ is *versal* if every other deformation factors through one of the E_n (see Definition 3.5 for a precise definition). Rim–Schlessinger's Criteria (Theorem 3.11) provides criteria for the existence of a versal deformation $\{E_n\}$ of E_0, and in §3 we verify the criteria for the Problems (A)–(C).

(4) A formal deformation $\{E_n\}$ over $(R, \mathfrak{m})$ is *effective* if there exists an object $\widehat{E} \in \mathscr{M}(R)$ extending the $\{E_n\}$, or in other words there exists a commutative diagram

$$\begin{tikzcd}
\operatorname{Spec} R/\mathfrak{m} \arrow[r, hook] \arrow[rrrrd, "{[E_0]}"'] & \operatorname{Spec} R/\mathfrak{m}^2 \arrow[r, hook] \arrow[rrrd, "{[E_1]}"] & \operatorname{Spec} R/\mathfrak{m}^3 \arrow[r, hook] \arrow[rrd, "{[E_2]}"] & \cdots \arrow[r, hook] & \operatorname{Spec} R \arrow[d, dashed, "{[\widehat{E}]}"] \\
& & & & \mathscr{M}.
\end{tikzcd}$$

In §4, we show how Grothendieck's Existence Theorem (Theorem 4.4) implies that formal deformations are effective for Problems (A)–(C).

(5) In §5, we take a detour from the local-to-global approach to provide a glimpse into the role of the cotangent complex in deformation theory.

(6) Given an effective versal formal deformation $\widehat{E}$ over R, Artin Algebraization (Theorem 6.6) ensures the existence of a *finite type* $\Bbbk$-scheme U with a point $u \in U(\Bbbk)$ and an object $E \in \mathscr{M}(U)$ such that $R \cong \widehat{\mathcal{O}}_{U,u}$ and $\widehat{E}|_{\operatorname{Spec} R/\mathfrak{m}^{n+1}} \cong E|_{\operatorname{Spec} R/\mathfrak{m}^{n+1}}$ for all n.

(7) Artin's Axioms for Algebraicity (Theorems 7.1 and 7.4) provides criteria to verify the algebraicity of a moduli problem $\mathscr{M}$. Namely, it provides conditions to ensure that the morphism $[E]\colon U \to \mathscr{M}$ constructed above is a smooth morphism in an open neighborhood of E_0.

In this chapter, $\Bbbk$ will denote an algebraically closed field. In §3, §6 and §7, we work over the category of $\Bbbk$-schemes for convenience but the results hold more generally.

1 First order deformations

Denote the *dual numbers* by $\Bbbk[\epsilon] := \Bbbk[\epsilon]/(\epsilon^2)$.

1.1 First order embedded deformations

Definition 1.1. Let X be a projective scheme over a $\Bbbk$ and $Z_0 \subset X$ be a closed subscheme. A *first order deformation of* $Z_0 \subset X$ is a closed subscheme $Z \subset X_{\Bbbk[\epsilon]} := X \times_{\Bbbk} \Bbbk[\epsilon]$ flat over $\Bbbk[\epsilon]$ such that $Z_0 = Z \times_{\Bbbk[\epsilon]} \Bbbk$. Pictorially, a first order deformation is a filling of the diagram

$$\begin{tikzcd}
& X \arrow[rr, hook] \arrow[dd] & & X_{\Bbbk[\epsilon]} \arrow[dd] \\
Z_0 \arrow[ur, hook, "\mathrm{cl}"] \arrow[rr, hook, dashed] \arrow[dr] & & Z \arrow[ur, hook, dashed, "\mathrm{cl}"] \arrow[dr, dashed, "\mathrm{flat}"] & \\
& \operatorname{Spec} \Bbbk \arrow[rr, hook] & & \operatorname{Spec} \Bbbk[\epsilon].
\end{tikzcd}$$

with a scheme Z and dotted arrows making the diagram cartesian.

We say that $Z \subset X_{\Bbbk[\epsilon]}$ is *trivial* if $Z = Z_0 \times_{\Bbbk} \Bbbk[\epsilon]$.

Remark 1.2. Since Z_0 and the central fiber $Z \times_{\Bbbk[\epsilon]} \Bbbk$ of Z are embedded in X, it makes sense to require that they are equal.

Remark 1.3. The closed subscheme $Z_0 \subset X$ defines a $\Bbbk$-point $[Z_0 \subset X] \in \mathrm{Hilb}^P(X)$ of the Hilbert scheme where P is the Hilbert polynomial of Z_0 with respect to a fixed ample line bundle on X. A first order deformation corresponds to a commutative diagram

$$\begin{tikzcd}
\operatorname{Spec} \Bbbk \arrow[r, "{[Z_0 \subset X]}"] \arrow[d, hook] & \mathrm{Hilb}^P(X) \\
\operatorname{Spec} \Bbbk[\epsilon], \arrow[ur, "{[Z \subset X_{\Bbbk[\epsilon]}]}"'] &
\end{tikzcd}$$

or in other words a tangent vector $[Z \subset X_{\Bbbk[\epsilon]}] \in T_{\mathrm{Hilb}^P(X), [Z_0 \subset X]}$.

Proposition 1.4. *Let X be a projective scheme over a $\Bbbk$ and $Z_0 \subset X$ be a closed subscheme defined by a sheaf of ideals $I_0 \subset \mathcal{O}_X$. There is a bijection*

$$\{\textit{first order deformations } Z \subset X_{\Bbbk[\epsilon]}\} \cong \mathrm{H}^0(Z_0, N_{Z_0/X})$$

where $N_{Z_0/X} = \mathrm{Hom}_{\mathcal{O}_Z}(I_0/I_0^2, \mathcal{O}_Z)$ is the normal sheaf. Under this correspondence, the trivial deformation corresponds to $0 \in \mathrm{H}^0(Z_0, N_{Z_0/X})$.

Remark 1.5. In light of Remark 1.3, this proposition gives a bijection $T_{\mathrm{Hilb}^P(X), [Z_0 \subset X]} \cong \mathrm{H}^0(Z_0, N_{Z_0/X})$.

Proof. We sketch the bijection and point the reader to [Har10, Prop. 2.3] and [Ser06, Prop. 3.2.1] for details. After reducing to the affine case $X = \operatorname{Spec} B$ and $Z_0 = \operatorname{Spec} B/I_0$, we need to show that the set of first order deformations is bijective to

$$\mathrm{H}^0(Z_0, N_{Z_0/X}) \cong \mathrm{Hom}_{B/I_0}(I_0/I_0^2, B/I_0) \cong \mathrm{Hom}_B(I_0, B/I_0).$$

Given a first order deformation $Z = \operatorname{Spec} B[\epsilon]/I$, the flatness of Z over $\Bbbk[\epsilon]$ ensures that tensoring the exact sequence $0 \to I \to B[\epsilon] \to B[\epsilon]/I \to 0$ of $\Bbbk[\epsilon]$-modules with $\Bbbk = \Bbbk[\epsilon]/(\epsilon)$ yields an exact sequence $0 \to I_0 \to B \to B/I_0 \to 0$. We define $\alpha\colon I_0 \to B/I_0$ as follows: for $x_0 \in I_0$, choose a preimage $x = a + b\epsilon \in I$ and set $\alpha(x_0) := \overline{b} \in B/I_0$. Conversely, given a B-module homomorphism $\alpha\colon I_0 \to B/I_0$, we define

$$I = \{a + b\epsilon \mid a \in I_0, b \in B \text{ such that } \overline{b} = \alpha(a) \in B/I_0\} \subset B[\epsilon].$$

Then $(B[\epsilon]/I) \otimes_{\Bbbk[\epsilon]} \Bbbk = B/I_0$. To see that $B[\epsilon]/I$ is flat over $\Bbbk[\epsilon]$, we need to check that the map $B/I_0 \xrightarrow{\epsilon} B[\epsilon]/I$ is injective (see Remark 8.2): given $b \in B$ with $\epsilon b \in I$, then $b \in I_0$ by the definition of I. Thus $Z = \operatorname{Spec} B[\epsilon]/I$ defines a first order deformation of Z_0. □

1.2 Locally trivial first order deformations of schemes

Definition 1.6. Let X_0 be a scheme over a $\Bbbk$. A *first order deformation of* X_0 is a scheme X flat over $\Bbbk[\epsilon]$ together with an isomorphism $\alpha\colon X_0 \to X \times_{\Bbbk[\epsilon]} \Bbbk$, or in other words a cartesian diagram

$$\begin{tikzcd}
X_0 \arrow[r, hook, dashed] \arrow[d] \arrow[dr, phantom, "\square"] & X \arrow[d, dashed, "\text{flat}"] \\
\operatorname{Spec} \Bbbk \arrow[r, hook] & \operatorname{Spec} \Bbbk[\epsilon].
\end{tikzcd} \tag{1.1}$$

A *morphism of first order deformations* (X, α) *and* (X', α') is a morphism $\beta\colon X \to X'$ of schemes over $\Bbbk[\epsilon]$ such that $(\beta \times_{\Bbbk[\epsilon]} \Bbbk) \circ \alpha = \alpha'$, or in other words considering X and X' in cartesian diagrams (1.1), we require the restriction of β to central fiber X_0 to be the identity.

We say that X is *trivial* if X is isomorphic as first order deformations to $X \times_{\Bbbk} \Bbbk[\epsilon]$, and *locally trivial* if there exists a Zariski-cover $X = \bigcup_i U_i$ such that U_i is a trivial first order deformation of $U_i \times_{\Bbbk[\epsilon]} \Bbbk \subset X_0$.

Any morphism of deformations is necessarily an isomorphism. This is a consequence of the following algebraic fact.

Lemma 1.7. *Let A be a ring, $\mathfrak{m} \subset A$ be a nilpotent ideal (e.g. $(A, \mathfrak{m})$ is an artinian local ring) and $M \to N$ be a homomorphism of A-modules. Assume that N is flat over A. If $M/\mathfrak{m}M \to N/\mathfrak{m}N$ is an isomorphism, then so is $M \to N$.*

Proof. The right exact sequence $M \to N \to C \to 0$ becomes $M/\mathfrak{m}M \to N/\mathfrak{m}N \to C/\mathfrak{m}C \to 0$ after modding out by $\mathfrak{m}$, and we see that $C = \mathfrak{m}C$. As $\mathfrak{m}^n = 0$ for some n, we obtain that $C = \mathfrak{m}C = \mathfrak{m}^2 C = \cdots = \mathfrak{m}^n C = 0$. Considering now the exact sequence $0 \to K \to M \to N \to 0$, the flatness of N implies that we get an exact sequence $0 \to K/\mathfrak{m}K \to M/\mathfrak{m}M \to N/\mathfrak{m}N \to 0$. Thus $K = \mathfrak{m}K = \cdots = \mathfrak{m}^n K = 0$ and we see that $M \to N$ is an isomorphism. □

Proposition 1.8. *Every first order deformation of a smooth affine scheme X_0 over $\Bbbk$ is trivial. In other words, X_0 is rigid.*

Proof. Let X be a first order deformation of X_0. Since $X_0 \to \operatorname{Spec} \Bbbk$ is smooth, we may apply the Infinitesimal Lifting Criterion for Smoothness (8.4) to construct a lift $X \to X_0$ making the diagram

$$\begin{tikzcd}
X_0 \arrow[r, "\mathrm{id}"] \arrow[d, hook] & X_0 \arrow[d, "\text{smooth}"] \\
X \arrow[r] \arrow[ur, dashed] & \operatorname{Spec} \Bbbk
\end{tikzcd}$$

commute. This induces a morphism $X \to X_0 \times_{\Bbbk} \Bbbk[\epsilon]$ over $\Bbbk[\epsilon]$ which restricts to the identity on X_0, and thus is an isomorphism by Lemma 1.7.

See also [Har77, Exer. II.8.7]. □

Remark 1.9. If X_0 is not smooth or affine, then first order deformations are not necessarily trivial. For example, if $X_0 = \operatorname{Spec} \Bbbk[x, y]/(xy)$ is the nodal affine plane curve, then $X = \operatorname{Spec} \Bbbk[x, y, \epsilon]/(xy - \epsilon) \to \operatorname{Spec} \Bbbk[\epsilon]$ is a non-trivial first order deformation.

On the other hand, consider an elliptic curve $E_0 = V(y^2z - x(x - z)(x - 2z)) \subset \mathbb{P}^2$ is a elliptic curve over $\Bbbk$ with $\operatorname{char}(\Bbbk) \neq 2, 3$. It is easy to write down global deformations by deforming the coefficients of the defining equations: $\mathcal{E} = V(y^2z - (x - \lambda z)(x - z)(x - 2z)) \subset \mathbb{P}^2 \times \mathbb{A}^1$ (where $\mathbb{A}^1$ has coordinate λ) defines a flat projective morphism $\mathcal{E} \to \mathbb{A}^1$ such the central fiber $\mathcal{E}_0$ is isomorphic to E_0. Restricting $\mathcal{E}$ to the family $E := \mathcal{E} \times_{\mathbb{A}^1} \operatorname{Spec} \Bbbk[\lambda]/\lambda^2$ over the dual numbers defines a *non-trivial* first order deformation. Observe that for any affine open $U_0 \subset E_0$ and setting $U \subset E$ to be the open subscheme with the same topological space as U_0, then there is an isomorphism $U \xrightarrow{\sim} U_0 \times_{\Bbbk} \Bbbk[\epsilon]$. These local isomorphism however do not glue to a global isomorphism $E \xrightarrow{\sim} E_0 \times_{\Bbbk} \Bbbk[\epsilon]$. Since any deformation of a smooth scheme is obtained by gluing together trivial deformations, we need to understand automorphisms of trivial deformations in order to classify global deformations.

Lemma 1.10. *Let $X_0 = \operatorname{Spec} A$ be an affine scheme over $\Bbbk$ and let $X = \operatorname{Spec} A[\epsilon]$ be the trivial first order deformation. For a $\Bbbk$-algebra A, there are identifications*

$$\{\textit{automorphisms } X_0 \to X_0 \textit{ of first order defs}\} \cong \operatorname{Der}_{\Bbbk}(A, A) \cong \operatorname{Hom}_B(\Omega_{A/\Bbbk}, B).$$

Proof. The second equivalence is given by the universal property of the module of differentials. An automorphism of the trivial first order deformation corresponds to a $\Bbbk[\epsilon]$-algebra isomorphism $\phi \colon B \oplus B\epsilon \to B \oplus B\epsilon$ which is the identity modulo ϵ. Therefore, ϕ is determined by the images $\phi(b) = b + d(b)\epsilon$ of elements $b \in B$ where $d \colon B \to B$ is $\Bbbk$-linear map. Since ϕ is a ring homomorphism, for elements $b, b' \in B$, we must have that $bb' + d(bb')\epsilon = (b + d(b)\epsilon)(b' + d(b')\epsilon) = bb' + (bd(b') + b'd(b))\epsilon$ and we see that $d \colon B \to B$ is a $\Bbbk$-derivation. □

For a scheme X_0 over $\Bbbk$, let $\operatorname{Def}(X_0)$ and $\operatorname{Def}^{\mathrm{lt}}(X_0)$ denote the sets of isomorphism classes of first order and locally trivial first order deformations.

Proposition 1.11. *For a separated scheme X_0 of finite type over $\Bbbk$. There is a bijection*

$$\operatorname{Def}^{\mathrm{lt}}(X_0) \cong \mathrm{H}^1(X_0, T_{X_0}),$$

where $T_{X_0} = \mathscr{H}om_{\mathcal{O}_{X_0}}(\Omega_{X_0/\Bbbk}, \mathcal{O}_{X_0})$. The trivial deformation corresponds to $0 \in \mathrm{H}^1(X_0, T_{X_0})$.

In particular, if in addition X_0 is smooth over $\Bbbk$, then there is a bijection

$$\operatorname{Def}(X_0) \cong \mathrm{H}^1(X_0, T_{X_0}).$$

Proof. Let $X \to \operatorname{Spec} \Bbbk[\epsilon]$ be a locally trivial first order deformation of X_0. Choose an affine cover $\{U_i\}$ of X_0 and isomorphisms $\phi_i \colon U_i \times_{\Bbbk} \Bbbk[\epsilon] \xrightarrow{\sim} X|_{U_i}$, where $X|_{U_i} \subset X$ denotes the open subscheme with the same topological space as U_i. Letting $U_{ij} = U_i \cap U_j$, we have automorphisms $\phi_j^{-1}|_{U_{ij} \times_{\Bbbk} \Bbbk[\epsilon]} \circ \phi_i|_{U_{ij} \times_{\Bbbk} \Bbbk[\epsilon]}$ of the trivial deformation $U_{ij} \times_{\Bbbk} \Bbbk[\epsilon]$ which by Lemma 1.10 corresponds to elements $\psi_{ij} \in \operatorname{Hom}_{\mathcal{O}_{U_{ij}}}(\Omega_{U_{ij}/\Bbbk}, \mathcal{O}_{U_{ij}})$. Since $\phi_{ij} \circ \phi_{jk} = \phi_{ik}$ on $U_{ijk} := U_i \cap U_j \cap U_k$, we have that $\psi_{ij} + \psi_{jk} = \psi_{ik} \in T_{X_0}(U_{ijk})$. Recall that $\mathrm{H}^1(X_0, T_{X_0})$ can be computed using the Céch complex

$$\begin{tikzcd}
0 \arrow[r] & \bigoplus_i T_{X_0}(U_i) \arrow[r, "d_0"] & \bigoplus_{i,j} T_{X_0}(U_{ij}) \arrow[r, "d_1"] & \bigoplus_{i,j,k} T_{X_0}(U_{ijk}) \\
& & (s_{ij}) \arrow[r, mapsto] & (s_{ij}|_{U_{ijk}} - s_{ik}|_{U_{ijk}} + s_{jk}|_{U_{ijk}})_{ijk}
\end{tikzcd}$$

As $\{\psi_{ij}\} \in \bigoplus_{i,j} T_{X_0}(U_{ij})$ is in the kernel of d_1, it defines an element of $\mathrm{H}^1(X_0, T_{X_0}) = \ker(d_1)/\operatorname{im}(d_0)$. Conversely, given an element of $\mathrm{H}^1(X_0, T_{X_0})$ and a choice of representative $\{\psi_{ij}\} \in \ker(d_1)$, then viewing each ψ_{ij} as an automorphism ϕ_{ij} of the trivial deformation of U_{ij}, we may glue together the trivial deformations $U_i \times_{\Bbbk} \Bbbk[\epsilon]$ along $U_{ij} \times_{\Bbbk} \Bbbk[\epsilon]$ via ϕ_{ij} to construct a global first order deformation X of X_0.

For the final statement, observe that since X_0 is smooth over $\Bbbk$, any first order deformation is locally trivial by Proposition 1.8. See also [Har77, Exer. III.4.10 and Ex. III.9.13.2]. ☐

Example 1.12. If C is a smooth projective curve of genus $g \geq 2$, then we've computed that

$$T_{\mathscr{M}_g,[C]} = \mathrm{H}^1(C, T_C) \overset{\mathrm{SD}}{=} \mathrm{H}^0(C, \Omega_{C/\Bbbk}^{\otimes 2})$$

which by Riemann–Roch is a $3g-3$ dimensional vector space.

Exercise 1.13. Use the Euler exact sequence to show that $\mathrm{H}^1(\mathbb{P}^n, T_{\mathbb{P}^n}) = 0$ and conclude that every first order deformation of $\mathbb{P}^n$ is trivial, i.e. $\mathbb{P}^n$ is rigid.

1.3 First order deformations of vector bundles and coherent sheaves

Definition 1.14. Let X be a projective scheme over $\Bbbk$ and E_0 be a coherent sheaf. A *first order deformation of* E_0 is a pair (E, α) where E is a coherent sheaf on $X \times_{\Bbbk} \Bbbk[\epsilon]$ flat over $\Bbbk[\epsilon]$ and $\alpha \colon E_0 \xrightarrow{\sim} E|_X$ is an isomorphism. Pictorially, we have

$$\begin{tikzcd}
E_0 \arrow[d, dash] & E \arrow[d, dash, "\text{flat}/\Bbbk[\epsilon]"] \\
X \arrow[r, hook] & X_{\Bbbk[\epsilon]}.
\end{tikzcd}$$

A *morphism* $(E, \alpha) \to (E', \alpha')$ *of first order deformations* is a morphism $\beta \colon E \to E'$ (equivalently an isomorphism by Lemma 1.7) of coherent sheaves on X' such that $\alpha' = \beta|_X \circ \alpha$.

We say that (E, α) is *trivial* if it isomorphic as first order deformations to $(p^* E_0, \mathrm{id})$ where $p \colon X_{\Bbbk[\epsilon]} \to X$.

Proposition 1.15. *Let X be a scheme over $\mathbb{k}$ and E_0 be a coherent sheaf. There is a bijection*

$$\{\textit{first order deformations } (E,\alpha) \textit{ of } E_0\}/\!\sim\ \cong \operatorname{Ext}^1_{\mathcal{O}_X}(E_0,E_0)$$

Under this correspondence, the trivial deformation corresponds to $0 \in \operatorname{Ext}^1_{\mathcal{O}_X}(E_0,E_0)$.

If in addition E_0 is a vector bundle (resp. line bundle), then the set of isomorphism classes of first order deformations of E_0 is bijective to $\mathrm{H}^1(X, \mathscr{E}nd_{\mathcal{O}_X}(E_0))$ (resp. $\mathrm{H}^1(X,\mathcal{O}_X)$).

Proof. If (E,α) is a first order deformation then since E is flat over $\mathbb{k}[\epsilon]$, we may tensor the exact sequence $0 \to \mathbb{k} \xrightarrow{\epsilon} \mathbb{k}[\epsilon] \to \mathbb{k} \to 0$ of $\mathbb{k}[\epsilon]$-modules with E to obtain an exact sequence $0 \to E_0 \xrightarrow{\epsilon} E \to E_0 \to 0$ (after identifying $E \otimes_{\mathbb{k}[\epsilon]} \mathbb{k}$ with E_0 via α). Since $\operatorname{Ext}^1_{\mathcal{O}_X}(E_0,E_0)$ parameterizes isomorphism classes of extensions [Har77, Exer. III.6.1], we have constructed an element of $\operatorname{Ext}^1_{\mathcal{O}_X}(E_0,E_0)$. Conversely, given an exact sequence $0 \to E_0 \xrightarrow{\alpha} E \to E_0 \to 0$, then E is a coherent sheaf on $X_{\mathbb{k}[\epsilon]}$ and is flat over $\mathbb{k}[\epsilon]$ by the flatness criterion over the dual numbers (see Remark 8.2). The restriction $E|_X$ is isomorphic to E_0 via α.

See also [Har10, Thm. 2.7]. ☐

Remark 1.16. The classifications of Propositions 2.2, 2.5 and 2.10 give vector space structures to the set of isomorphism classes of first order deformations. The vector space structures can also be witnessed as a consequences of Rim–Schlessinger's homogeneity condition; see Lemma 3.13.

2 Higher order deformations and obstructions

Let $\mathscr{M}$ be a moduli problem and $E \in \mathscr{M}(A)$ be an object defined over a ring A. If $A' \twoheadrightarrow A$ is a surjection of rings with square-zero kernel, in this section we address the following two questions:

(1) Does E deform to an object $E' \in \mathscr{M}(A')$?

(2) If so, can we classify all such deformations?

Pictorially, we have:

$$\begin{tikzcd} E \arrow[d, dash] & E' \arrow[d, dash, dashed] \\ \operatorname{Spec} A \arrow[r, hook] & \operatorname{Spec} A'. \end{tikzcd}$$

where $\operatorname{Spec} A \hookrightarrow \operatorname{Spec} A'$ is a closed immersion of schemes with the same topological space. Note that since $J = \ker(A' \to A)$ is square-zero, $J = J/J^2$ is naturally a module over $A = A'/J$. In other words, Question (1) is asking whether there is some “obstruction” to the existence of a deformation E' while (2) seeks to classify all higher order deformations given that there is no obstruction.

An interesting case is when A and A' are local artinian algebras with residue field $\mathbb{k}$ and the kernel $J = \ker(A' \to A)$ satisfies $\mathfrak{m}_{A'}J = 0$ (which implies that $J^2 = 0$). In this case, $J = J/\mathfrak{m}_{A'}J$ is naturally a vector space over $\mathbb{k} = A'/\mathfrak{m}_{A'}$. Setting $E_0 := E|_{\mathbb{k}} \in \mathscr{M}(\mathbb{k})$, we can view E as a deformation over E_0 over A and we are attempting to classify the higher order deformations over A'. If there are no obstructions to deforming, then the Infinitesimal Lifting Criterion for Smoothness (8.4) implies that $\mathscr{M}$ is smooth at $[E_0]$.

The previous section studied the specific case when $A = \Bbbk$ and $A' = \Bbbk[\epsilon]$ in which case deformations of an object $E_0 \in \mathscr{M}(\Bbbk)$ over A' correspond to first order deformations. In this case, the obstruction vanishes as there is always the trivial deformation (i.e. the pullback of E_0 along $\operatorname{Spec} \Bbbk[\epsilon] \to \operatorname{Spec} \Bbbk$). Other examples of $A' \to A$ to keep in mind are $\Bbbk[x]/x^{n+1} \twoheadrightarrow \Bbbk[x]/x^n$ and $\mathbb{Z}/p^{n+1} \twoheadrightarrow \mathbb{Z}/p^n$ where we inductively attempt to deform E_0 over the nilpotent thickenings $\operatorname{Spec} \Bbbk[x]/x^{n+1} \hookrightarrow \mathbb{A}^1$ and $\operatorname{Spec} \mathbb{Z}/p^{n+1} \hookrightarrow \operatorname{Spec} \mathbb{Z}$.

2.1 Higher order embedded deformations

Definition 2.1. Let $A' \twoheadrightarrow A$ be a surjection of noetherian rings with square-zero kernel. Let X' be a scheme over A' and set $X := X' \times_{A'} A$. Let $Z \subset X$ be a closed subscheme flat over A. A *deformation of $Z \subset X$ over A'* is a closed subscheme $Z' \subset X'$ flat over A' such that $Z' \times_{A'} A = Z$ as closed subschemes of X. Pictorially, a deformation is a filling of the cartesian diagram

$$\begin{tikzcd}
 & X \arrow[rr, hook] \arrow[dd] & & X' \arrow[dd] \\
Z \arrow[ur, hook, "\text{cl}"] \arrow[dr, "\text{flat}"'] \arrow[rr, dashed, hook] & & Z' \arrow[ur, dashed, hook, "\text{cl}"] \arrow[dr, dashed, "\text{flat}"] & \\
 & \operatorname{Spec} A \arrow[rr, hook] & & \operatorname{Spec} A'.
\end{tikzcd}$$

The formulation of the next proposition uses the following notion: a *torsor of a group G* is a transitive and free action of G on a set.

Proposition 2.2. *Let $A' \twoheadrightarrow A$ be a surjection of noetherian rings with square-zero kernel J. Let X' be a separated scheme over A' and $Z \subset X := X' \times_{\Bbbk} A$ be a closed subscheme flat over A. Then*

(1) If there exists a deformation $Z' \subset X'$ of $Z \subset X$ over A', then the set of such deformations is a torsor under to $\mathrm{H}^0(Z, N_{Z/X} \otimes_A J)$.

(2) If deformations of $Z \subset X$ over A' exist Zariski-locally on X, then there exists an element $\mathrm{ob}_Z \in \mathrm{H}^1(Z, N_{Z/X} \otimes_A J)$ (depending on Z and $A' \twoheadrightarrow A$) such that there exists a deformation of $Z \subset X$ over A' if and only if $\mathrm{ob}_Z = 0$.

Remark 2.3. An interesting example is when $X = X_0 \times_{\Bbbk} A$ and $X' = X_0 \times_{\Bbbk} A'$ are the base changes of a separated $\Bbbk$-scheme X_0. If the closed subscheme $Z \subset X$ has constant Hilbert polynomial P (i.e. for each $s \in \operatorname{Spec} A$, the Hilbert polynomial of $Z_s \subset X_0 \times_{\Bbbk} \kappa(s)$, with respect to a fixed ample line bundle on X_0, is independent of s), then we have an object $[Z \subset X] \in \mathrm{Hilb}^P(X_0)(A)$ of the Hilbert scheme. In this case, a deformation of $Z \subset X$ over A' is an object $[Z' \subset X'] \in \mathrm{Hilb}^P(X_0)(A')$ which restricts to $[Z \subset X]$. Note that when $A' \twoheadrightarrow A$ is a surjection of local artinian $\Bbbk$-algebras with $\mathfrak{m}_{A'} J = 0$, then there is an identification $\mathrm{H}^0(Z, N_{Z/X} \otimes_A J) = \mathrm{H}^0(Z_0, N_{Z_0/X_0} \otimes_{\Bbbk} J)$ where $Z_0 = Z \times_A \Bbbk$.

Proof. Suppose first that $X' = \operatorname{Spec} B'$, $X = \operatorname{Spec} B$ where $B = B' \otimes_{A'} A$ and $Z = \operatorname{Spec} B/I$. If there exists a deformation $Z' = \operatorname{Spec} B'/I'$, then there is an exact diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & I \otimes_A J \arrow[r] \arrow[d] & I' \arrow[r] \arrow[d] & I \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & B \otimes_A J \arrow[r] \arrow[d] & B' \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & (B/I) \otimes_A J \arrow[r] \arrow[d] & B'/I' \arrow[r] \arrow[d] & B/I \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

The exactness of the bottom row (resp. middle row) is equivalent to the flatness of B'/I' (resp. B') over A' by the Local Criterion of Flatness (Theorem 8.1) while the exactness of the left column follows from the flatness of B/I over A. Conversely, an exact diagram defines an deformation $Z' = \operatorname{Spec} B'/I'$.

We will define an action $\operatorname{Hom}_B(I, (B/I) \otimes_A J)$ on the set of deformations as follows: given $\phi \in \operatorname{Hom}_B(I, (B/I) \otimes_A J)$ and a deformation $Z' = \operatorname{Spec} B'/I'$, define $I'' \subset B'$ as the set of elements $x'' \in B'$ such that its image $\overline{x}'' \in B$ lies in I and such that a lifting $x' \in I'$ of $\overline{x}'' \in I$ satisfies $x'' - x' = \phi(\overline{x''}) \in (B/I) \otimes_A J$ (noting that this condition is independent of the choice of lifting x'). One checks that $\operatorname{Spec} B'/I''$ is another deformation.

On the other hand, given two deformations defined by ideals I' and I'', we define $\phi\colon I \to (B/I) \otimes_A J$ by $\phi(x) = \overline{x' - x''}$ where $x' \in I'$ and $x'' \in I''$ are lifts of x (which forces $x' - x'' \in B \otimes_A J$). One checks that this is a B-module homomorphism providing an inverse to the above construction. We have natural identifications

$$\operatorname{Hom}_B(I, (B/I) \otimes_A J) = \operatorname{Hom}_{B/I}(I/I^2, B/I \otimes_A J) = \mathrm{H}^0(Z, N_{Z/X} \otimes_A J).$$

The above constructions globalize to X and establishes (1).

For (2), let $\{U_i\}$ be an open cover of X such that there exists deformations $Z'_i \subset X' \cap U_i$ of $Z \cap U_i \subset X \cap U_i$ (noting that X and X' are homeomorphic). On $U_{ij} = U_i \cap U_j$, the two deformations $Z'_i|_{U_{ij}}$ and $Z'_j|_{U_{ij}}$ defines an element $\phi_{ij} \in \mathrm{H}^0(U_{ij}, N_{Z/X} \otimes_A J)$ which in turn defines a Céch 1-cocycle $(\phi_{ij}) \in \mathrm{H}^1(X, N_{Z/X} \otimes_A J)$. We leave the reader to check that the vanishing of (ϕ_{ij}) characterizes whether there is a deformation of $Z \subset X$ over A'.

See also [Har10, Thm. 6.2]. □

2.2 Higher order deformations of smooth schemes

Definition 2.4. Let $A' \twoheadrightarrow A$ be a surjection of noetherian rings with square-zero kernel and $X \to \operatorname{Spec} A$ be a flat morphism of schemes. A *deformation of* $X \to \operatorname{Spec} A$ *over* A' is a flat morphism $X' \to \operatorname{Spec} A'$ together with an isomorphism $\alpha\colon X \xrightarrow{\sim} X' \times_{A'} A$ over A, or in other words a cartesian diagram

$$\begin{tikzcd}
X \arrow[r, hook, dashed] \arrow[d, "\text{flat}"] & X' \arrow[d, dashed, "\text{flat}"] \\
\operatorname{Spec} A \arrow[r, hook] & \operatorname{Spec} A'.
\end{tikzcd} \tag{2.1}$$

A *morphism of deformations over A'* is a morphism of schemes over A' restricting to the identity on X. By Lemma 1.7, any morphism of deformations is an isomorphism.

Proposition 2.5. *Let $A' \twoheadrightarrow A$ be a surjection of noetherian rings with square-zero kernel J. If $X \to \operatorname{Spec} A$ is a smooth and separated morphism, then*

(1) The group of automorphisms of a deformation $X' \to \operatorname{Spec} A'$ of $X \to \operatorname{Spec} A$ over A' is bijective to $\mathrm{H}^0(X, T_{X/A} \otimes_A J)$.

(2) If there exists a deformation of $X \to \operatorname{Spec} A$ over A', then the set of isomorphism classes of all such deformations is a torsor under $\mathrm{H}^1(X, T_{X/A} \otimes_A J)$.

(3) There is an element $\mathrm{ob}_X \in \mathrm{H}^2(X, T_{X/A} \otimes_A J)$ with the property that there exists a deformation of $X \to \operatorname{Spec} A$ over A' if and only if $\mathrm{ob}_X = 0$.

Remark 2.6. If A and A' are local artinian rings with residue field $\Bbbk$ such that $\mathfrak{m}_{A'} J = 0$ and we set $X_0 := X \times_A \Bbbk$, then automorphisms, deformations and obstructions are classified by $\mathrm{H}^i(X_0, T_{X_0} \otimes_{\Bbbk} J)$ for $i = 0, 1, 2$.

Proof. See [Har10, Cor. 10.3]. ☐

Exercise 2.7 (Interpretation of deformations and obstruction using gerbes)**.** With the hypotheses of Proposition 2.5, consider the category $\mathcal{G}$ over Sch/X whose objects over $S \to X$ are cartesian diagrams

$$\begin{tikzcd}
S \arrow[r, hook] \arrow[d] & S' \arrow[d] \\
\operatorname{Spec} A \arrow[r, hook] & \operatorname{Spec} A'
\end{tikzcd}$$

where $S \to \operatorname{Spec} A$ is the composition $S \to X \to \operatorname{Spec} A$. A morphism $(S \to X, S \hookrightarrow S' \to \operatorname{Spec} A') \to (T \to X, T \hookrightarrow T' \to \operatorname{Spec} A')$ is the data of a morphism $\phi\colon S' \to T'$ over A' such that ϕ restricts to a morphism $S \to T$ over X.

(a) Show that $\mathcal{G}$ is a gerbe banded by the sheaf of groups $T_{X/A} \otimes_A J$ on X. (*Hint: Use Lemma 1.7 to show it is a prestack. See Definitions 8.11 and 8.14 for the definition of a banded gerbe.*)

(b) Give an alternate proof of Proposition 2.5. (*Hint:* For part (3), use Remark 8.15.)

Example 2.8 (Abelian varieties)**.** If X_0 is an abelian variety over $\mathbb{C}$ of dimension n, then it turns out that deforming X_0 as an abstract scheme is equivalent to deforming it as an abelian variety, and that obstructions to deforming X_0 as an abelian variety also live in $\mathrm{H}^2(X_0, T_{X_0})$. Using that $\Omega_{X_0} = \mathcal{O}_{X_0}^n$ is trivial and the Hodge symmetries, we see that $\mathrm{H}^2(X_0, T_{X_0}) = \mathrm{H}^2(X_0, \mathcal{O}_{X_0})^{\oplus n} = \mathrm{H}^0(X_0, \bigwedge^2 \mathcal{O}_{X_0}^n)^{\oplus n}$ is non-zero. Nevertheless, Grothendieck and Mumford showed that given any deformation problem as in (2.1), the obstruction $\mathrm{ob}_X \in \mathrm{H}^2(X, T_{X/A} \otimes_A J)$ vanishes! This shows that abelian varieties are unobstructed and their moduli is formally smooth. See [Oor71].

2.3 Higher order deformations of vector bundles

Definition 2.9. Let $A' \twoheadrightarrow A$ be a surjection of noetherian rings with square-zero kernel J. Let $X' \to \operatorname{Spec} A'$ be a separated and finite type morphism of schemes and set $X := X' \times_{A'} A$. z Given a coherent sheaf E on X flat over A, a *deformation of E over $A' \to A$* is a pair (E', α) where E' is a coherent sheaf on X' flat over A' and $\alpha \colon E \to E'|_X$ is an isomorphism. Pictorially, we have

$$\begin{tikzcd} E \arrow[d, dash, "\text{flat}/A"] & E' \arrow[d, dash, dashed, "\text{flat}/A'"] \\ X \arrow[r, hook] & X'. \end{tikzcd}$$

A *morphism* $(E, \alpha) \to (E', \alpha')$ *of deformations* is a morphism $\beta \colon E \to E'$ of coherent sheaves on $X_{A'}$ such that $\alpha' = \beta|_X \circ \alpha$. By Lemma 1.7, any morphism of deformations is an isomorphism.

Proposition 2.10. *Let $A' \twoheadrightarrow A$ be a surjection of noetherian rings with square-zero kernel J. Let $X' \to \operatorname{Spec} A'$ be a separated and finite type morphism of schemes and set $X := X' \times_{A'} A$. Let E be a vector bundle on X flat over A.*

- *(1) The group of automorphisms of a deformation E' of E over A' is bijective to $\mathrm{H}^0(X, \mathscr{E}nd_{\mathcal{O}_X}(E) \otimes_A J)$.*
- *(2) If there exists a deformation of E over A', then the set of isomorphism classes of all such deformations is a torsor under $\mathrm{H}^1(X, \mathscr{E}nd_{\mathcal{O}_X}(E) \otimes_A J)$.*
- *(3) There is an element $\mathrm{ob}_E \in \mathrm{H}^2(X, \mathscr{E}nd_{\mathcal{O}_X}(E) \otimes_A J)$ with the property that there exists a deformation of E over A' if and only if $\mathrm{ob}_E = 0$.*

Remark 2.11. If X and X' are base changes of a separated and finite type $\Bbbk$-scheme X_0, and A and A' are local artinian rings with residue field $\Bbbk$ such that $\mathfrak{m}_{A'} J = 0$, then automorphisms, deformations and obstructions are classified by $\mathrm{H}^2(X_0, \mathscr{E}nd_{\mathcal{O}_{X_0}}(E_0) \otimes_\Bbbk J)$ for $i = 0, 1, 2$ where $E_0 = E|_{X_0}$.

Proof. See [Har10, Thm. 7.1]. ☐

Exercise 2.12. Give an alternative proof of Proposition 2.10 using the technique outlined in Exercise 2.7.

3 Versal formal deformations and Rim–Schlessinger's Criteria

3.1 Functors of artin rings

For an algebraically closed field $\Bbbk$, let $\mathrm{Art}_\Bbbk$ denote the category of artinian local $\Bbbk$-algebras with residue field $\Bbbk$. The opposite category $\mathrm{Art}_\Bbbk^{\mathrm{op}}$ is equivalent to the category of local artinian $\Bbbk$-schemes (S, s) with $\kappa(s) = \Bbbk$.

Definition 3.1. We say that a covariant functor $F \colon \mathrm{Art}_\Bbbk \to \mathrm{Sets}$ is *pro-representable* if there exists a noetherian complete local $\Bbbk$-algebra R such that for all $A \in \mathrm{Art}_\Bbbk$, there is a isomorphism $F \xrightarrow{\sim} h_R$ where $h_R := \mathrm{Hom}_{\Bbbk-\mathrm{alg}}(R, -)$.

Remark 3.2. If $F\colon \mathrm{Sch}/\Bbbk \to \mathrm{Sets}$ is a contravariant functor and $x_0 \in F(\Bbbk)$, then we can consider the induced functor of artin rings

$$F_{x_0}\colon \mathrm{Art}_{\Bbbk} \to \mathrm{Sets}, \quad A \mapsto \{x \in F(A) \mid x|_{\Bbbk} = x_0 \in F(\Bbbk)\}$$

where $x|_{\Bbbk}$ denotes the image of x under $F(A) \to F(A/\mathfrak{m}_A)$. If F is representable by a scheme X and $x \in X$ is the $\Bbbk$-point corresponding to x_0, then F_{x_0} is pro-representable by $\widehat{\mathcal{O}}_{X,x}$.

Exercise 3.3. Provide an example of a non-representable contravariant functor $F\colon \mathrm{Sch}/\Bbbk \to \mathrm{Sets}$ and an object $x_0 \in F(\Bbbk)$ such that F_{x_0} is pro-representable.

Many functors of artin rings are not pro-representable. For example, if C_0 is a smooth connected projective curve with a non-trivial automorphism group, then the covariant functor $F_{C_0}\colon \mathrm{Art}_{\Bbbk} \to \mathrm{Sets}$ where $F_{C_0}(A)$ consists of isomorphism classes of smooth proper families of curves $\mathcal{C} \to \operatorname{Spec} A$ such that $\mathcal{C} \times_A A/\mathfrak{m}_A$ is isomorphic to C_0, is not pro-representable. Nevertheless many moduli functors admit *versal* deformations.

Remark 3.4. To work over a more general base (e.g. of mixed characteristic), one can consider instead the following setup: let Λ be a noetherian complete local ring with residual field $\Bbbk$ (not necessarily algebraically closed) and Art_{Λ} be the category of artinian local Λ-algebras $(A, \mathfrak{m})$ with an identification $\Bbbk \xrightarrow{\sim} A/\mathfrak{m}$. Rim–Schlessinger's Criteria (Theorem 3.11) holds after replacing $\mathrm{Art}_{\Bbbk}$ with Art_{Λ}. More generally, one can consider the setup where $A \to \Bbbk$ is a finite morphism to a field, not assumed to be the residue field.

Setting $\Lambda = \Bbbk$ recovers our setup but in many applications it is often useful to take Λ to be a ring of Witt vectors, e.g. $\Lambda = \mathbb{Z}_p$. In this way, one can consider deforming an object E_0 over $\mathbb{F}_p$ inductively along extensions $\mathbb{Z}/p^{n+1} \twoheadrightarrow \mathbb{Z}/p^n$ with the hope of applying Rim–Schlessinger's Criteria (Theorem 3.11) and Grothendieck's Existence Theorem (Theorem 4.4) to deform E_0 to an object $\widehat{E}$ over the *characteristic zero* ring $\mathbb{Z}_p$; see Section 4.1

3.2 Versal deformations

As it's important to keep track of automorphisms, we will present Rim–Schlessinger's Criteria, a generalization of Schlessinger's Criterion from functors to prestacks. Therefore we will formulate the definition of versality for prestacks $\mathcal{X}$ over $\mathrm{Art}_{\Bbbk}^{\mathrm{op}}$. We will assume that $\mathcal{X}(\Bbbk)$ is equivalent to a set consisting of a single object, i.e. there is a unique morphism between any two objects in $\mathcal{X}(\Bbbk)$.

Definition 3.5. Let $\mathcal{X}$ be a prestack over $\mathrm{Art}_{\Bbbk}^{\mathrm{op}}$ such that the groupoid $\mathcal{X}(\Bbbk)$ is equivalent to the set $\{x_0\}$.

(1) A *formal deformation* $(R, \{x_n\})$ *of* x_0 is the data of a noetherian complete local $\Bbbk$-algebra $(R, \mathfrak{m}_R)$ together with objects $x_n \in \mathcal{X}(R/\mathfrak{m}_R^{n+1})$ and morphisms $x_{n-1} \to x_n$ over $\operatorname{Spec} R/\mathfrak{m}_R^n \to \operatorname{Spec} R/\mathfrak{m}_R^{n+1}$, or in other words an element of $\varprojlim \mathcal{X}(R/\mathfrak{m}^n)$. When $\mathcal{X} = F$ is a covariant functor $\mathrm{Art}_{\Bbbk} \to \mathrm{Sets}$, a formal deformation is a compatible sequence of elements $x_n \in F(R/\mathfrak{m}_R^{n+1})$.

(2) A formal deformation $(R, \{x_n\})$ is *versal* if for every surjection $A \twoheadrightarrow A_0$ in $\mathrm{Art}_{\Bbbk}$ with $\mathfrak{m}_A^{n+1} = 0$, object $\eta \in \mathcal{X}(A)$ and $\Bbbk$-algebra homomorphism $\phi_0\colon R/\mathfrak{m}_R^{n+1} \to A_0$ with an isomorphism $\alpha_0\colon x_n|_{A_0} \xrightarrow{\sim} \eta|_{A_0}$ in $\mathcal{X}(A_0)$, there

exists a $\mathbb{k}$-algebra homomorphism $\phi\colon R/\mathfrak{m}_R^{n+1} \to A$ and an isomorphism $\alpha\colon x_n|_A \xrightarrow{\sim} \eta$ in $\mathcal{X}(A)$ such that ϕ_0 is the composition $R/\mathfrak{m}_R^{n+1} \xrightarrow{\phi} A \twoheadrightarrow A_0$ $\alpha|_{A_0} = \alpha_0$.

(3) A versal formal deformation $(R, \{x_n\})$ is *miniversal* (or *a pro-representable hull*) if the induced map $\mathrm{Hom}_{\mathbb{k}-\mathrm{alg}}(R, \mathbb{k}[\epsilon]) \to \mathcal{X}(\mathbb{k}[\epsilon])/{\sim}$ on isomorphism classes, defined by $(R \to R/\mathfrak{m}_R^2 \xrightarrow{\phi} \mathbb{k}[\epsilon]) \mapsto \phi(x_1)$, is bijective.

Remark 3.6. The deformation $x_n \in \mathcal{X}(R/\mathfrak{m}_R^{n+1})$ can be viewed via Yoneda's 2-Lemma as a morphism $\operatorname{Spec} R/\mathfrak{m}_R^{n+1} \to \mathcal{X}$ or more precisely as $h_{R/\mathfrak{m}_R^{n+1}} \to \mathcal{X}$. Likewise, we can view a formal deformation as a morphism $\{x_n\}\colon h_R \to \mathcal{X}$ where $h_R = \mathrm{Hom}_{\mathbb{k}-\mathrm{alg}}(R, -)$ (see Exercise 3.8). With this terminology, $\{x_n\}$ is versal if there exists a lift for every commutative diagram

$$\begin{tikzcd}
\operatorname{Spec} A_0 \arrow[r] \arrow[d, hook] & h_R \arrow[d, "\{x_n\}"] \\
\operatorname{Spec} A \arrow[r, "\eta"] \arrow[ur, dashed] & \mathcal{X}
\end{tikzcd} \tag{3.1}$$

of solid arrows where $A \twoheadrightarrow A_0$ is a surjection in $\mathrm{Art}_{\mathbb{k}}$. In this way, we see that a versal formal deformation corresponds to the Infinitesimal Lifting Criterion for Smoothness (see Smooth Equivalences 8.4(2) and Smooth Equivalences 8.4) with respect to artinian local $\mathbb{k}$-algebras. Meanwhile a miniversal deformation is a versal formal deformation inducing an isomorphism on tangent spaces $h_R(\mathbb{k}[\epsilon]) \to \mathcal{X}(\mathbb{k}[\epsilon])/{\sim}$.

Remark 3.7. The condition of versality can be checked on surjections $A \twoheadrightarrow A_0$ with $\ker(A \to A_0) \cong \mathbb{k}$. Indeed, the kernel of any surjection $A \twoheadrightarrow A_0$ in $\mathrm{Art}_{\mathbb{k}}$ is a finite dimensional $\mathbb{k}$-vector space and $A \to A_0$ can be factored into surjections where each kernel is one-dimensional.

Exercise 3.8. Let R be a noetherian complete local $\mathbb{k}$-algebra and let $h_R = \mathrm{Hom}_{\mathbb{k}-\mathrm{alg}}(R, -)$ be the covariant functor $\mathrm{Art}_{\mathbb{k}} \to \mathrm{Sets}$ which we can also view as a prestack over $\mathrm{Art}_{\mathbb{k}}^{\mathrm{op}}$. If $\mathcal{X}$ is a prestack over $\mathrm{Art}_{\mathbb{k}}^{\mathrm{op}}$, show that giving a formal deformation $(R, \{x_n\})$ is equivalent to giving a morphism $h_R \to \mathcal{X}$ of prestacks.

Remark 3.9. If F is pro-representable by R, then letting $x_n \in F(R/\mathfrak{m}_R^{n+1})$ correspond to the surjection $(R \twoheadrightarrow R/\mathfrak{m}_R^{n+1}) \in h_R(R/\mathfrak{m}_R^{n+1})$, it is easy to see that $\{x_n\}$ is a versal formal deformation. In this case, there is a unique lift in (3.1)

Remark 3.10 (Global prestacks to local deformation prestacks)**.** If $\mathcal{X}$ is a prestack over $\mathrm{Sch}/\mathbb{k}$ and $x_0 \in \mathcal{X}(\mathbb{k})$, we can consider the *local deformation prestack* $\mathcal{X}_{x_0}$ *at* x_0 as the prestack of morphisms $x_0 \to x$ over $\mathrm{Art}_{\mathbb{k}}^{\mathrm{op}}$ where a morphism $(x_0 \xrightarrow{\alpha} x) \to (x_0 \xrightarrow{\alpha'} x')$ is a morphism $\beta\colon x \to x'$ such that $\alpha' = \alpha \circ \beta$. In other words, an object of $\mathcal{X}_{x_0}$ is a pair (x, α) where $x \in \mathcal{X}(A)$ and $\alpha\colon x_0 \to x|_{\mathbb{k}}$ is an isomorphism. Note that the fiber category $\mathcal{X}_{x_0}(\mathbb{k})$ is equivalent to the set $\{x_0 \xrightarrow{\mathrm{id}} x_0\}$.

If $\mathcal{X}$ is algebraic with a smooth presentation $U \to \mathcal{X}$ from a scheme and $u \in U(\mathbb{k})$ is a point maping to x_0, then we may set $x_n \in \mathcal{X}(\mathcal{O}_{U,u}/\mathfrak{m}_u^{n+1})$ to be the composition $\operatorname{Spec} \mathcal{O}_{U,u}/\mathfrak{m}_u^{n+1} \hookrightarrow U \to \mathcal{X}$. Then $\{x_n\}$ is a versal formal deformation.

On the other hand, if $\mathcal{X}$ is not yet known to be algebraic, one can sometimes verify the existence of versal formal deformation via Rim–Schlessinger's Criteria

(Theorem 3.11) as a first step to verifying the algebraicity of $\mathcal{X}$ via Artin's Axioms for Algebraicity (Theorem 7.1).

3.3 Rim–Schlessinger's Criteria

Rim–Schlessinger's Criteria provides necessary and sufficient conditions for a prestack $\mathcal{X}$ over $\mathrm{Art}_{\mathbb{k}}^{\mathrm{op}}$ or covariant functor $F\colon \mathrm{Art}_{\mathbb{k}} \to \mathrm{Sets}$ to admit a versal formal deformation.

Theorem 3.11 (Rim–Schlessinger's Criteria)**.** *Let $\mathcal{X}$ be a prestack over $\mathrm{Art}_{\mathbb{k}}^{\mathrm{op}}$ such that the groupoid $\mathcal{X}(\mathbb{k})$ is equivalent to the set $\{x_0\}$. For morphisms $B_0 \to A_0$ and $A \to A_0$ in $\mathrm{Art}_{\mathbb{k}}$, consider the natural functor*

$$\mathcal{X}(B_0 \times_{A_0} A) \to \mathcal{X}(B_0) \times_{\mathcal{X}(A_0)} \mathcal{X}(A) \tag{3.2}$$

Then $\mathcal{X}$ admits a miniversal formal deformation if and only if

(RS$_1$) *the functor (3.2) is essentially surjective whenever $A \twoheadrightarrow A_0$ is surjection with kernel $\mathbb{k}$;*

(RS$_2$) *the map (3.2) is essentially surjective when $A_0 = \mathbb{k}$ and $A = \mathbb{k}[\epsilon]$, and given two commutative diagrams*

$$\begin{tikzcd}
x_0 \arrow[r] \arrow[d] & y_0 \arrow[d, "\alpha_1"'] \arrow[dr, "\alpha_2"] & \\
x \arrow[r] \arrow[rr, bend right] & y \arrow[r, dashed, "\beta"] & y'
\end{tikzcd}
\quad \textit{over} \quad
\begin{tikzcd}
\operatorname{Spec} \mathbb{k} \arrow[r, hook] \arrow[d] & \operatorname{Spec} B_0 \arrow[d] \\
\operatorname{Spec} \mathbb{k}[\epsilon] \arrow[r, hook] & \operatorname{Spec}(\mathbb{k}[\epsilon] \times_{\mathbb{k}} B_0)
\end{tikzcd}$$

there exists an isomorphism $\beta\colon y \to y'$ in $\mathcal{X}(\mathbb{k}[\epsilon] \times_{\mathbb{k}} B_0)$ such that $\alpha_2 = \beta \circ \alpha_1$.

(RS$_3$) $\dim_{\mathbb{k}} T_{\mathcal{X}} < \infty$ *where* $T_{\mathcal{X}} := \mathcal{X}(\mathbb{k}[\epsilon])/\!\sim$.

Moreover, $\mathcal{X}$ is prorepresentable if and only if $\mathcal{X}$ is equivalent to a functor and

(RS$_4$) *the map (3.2) is an equivalence whenever $A \twoheadrightarrow A_0$ is a surjection with kernel $\mathbb{k}$.*

Conditions (RS$_2$)–(RS$_3$) (sometimes referred to as *semi-homogeneity*) may be difficult to parse[1] but in practice it is almost always just as easy to verify the stronger condition (RS$_4$) (called *homogeneity*), and in fact the even stronger condition (RS$_4^*$) (called *strong homogeneity*) introduced in §3.4.

Remark 3.12 (Schlessinger's Criteria)**.** When $\mathcal{X}$ is a covariant functor $F\colon \mathrm{Art}_{\mathbb{k}} \to \mathrm{Sets}$ with $F(\mathbb{k}) = \{x_0\}$, then (RS$_1$)–(RS$_4$) translate into Schlessinger's conditions as introduced in [Sch68]:

(H$_1$) the map (3.2) is surjective whenever $A \twoheadrightarrow A_0$ is a surjection with kernel $\mathbb{k}$;

(H$_2$) the map (3.2) is bijective when $A_0 = \mathbb{k}$ and $A = \mathbb{k}[\epsilon]$;

(H$_3$) $\dim_{\mathbb{k}} F(\mathbb{k}[\epsilon]) < \infty$; and

(H$_4$) the map (3.2) is bijective whenever $A \twoheadrightarrow A_0$ is a surjection with kernel $\mathbb{k}$.

---

[1] The second part of (RS$_2$) is slightly stronger than requiring that two objects in $\mathcal{X}(\mathbb{k}[\epsilon] \times_{\mathbb{k}} B_0)$ are isomorphic if and only if their images are. Note that (RS$_2$) does not require the isomorphism $\beta\colon y \to y'$ to be compatible with the given morphisms $x \to y$ and $x \to y'$.

The functor F admits a miniversal formal deformation if (H_1)–(H_3) hold and is pro-representable if (H_3)–(H_4) hold.

If $\mathcal{X}$ satisfies (RS_1)–(RS_3), then the functor $F_{\mathcal{X}} \colon \mathrm{Sch}/\Bbbk \to \mathrm{Sets}$ parameterizing isomorphism classes of objects satisfies (H_1)–(H_3) but the converse does not always hold. On the other hand, the essential surjectivity of $\mathcal{X}(B_0 \times_{A_0} A) \to \mathcal{X}(B_0) \times_{\mathcal{X}(A_0)} \mathcal{X}(A)$ implies the surjectivity of $F_{\mathcal{X}}(B_0 \times_{A_0} A) \to F_{\mathcal{X}}(B_0) \times_{F_{\mathcal{X}}(A_0)} F_{\mathcal{X}}(A)$ and the fully faithfulness for $\mathcal{X}$ implies the injectivity of $F_{\mathcal{X}}$ as long as $\mathrm{Aut}_{\mathcal{X}(B_0)}(y_0) \to \mathrm{Aut}_{\mathcal{X}(A_0)}(y_0|_{A_0})$ is surjective for an object $y_0 \in \mathcal{X}(B_0)$. This latter condition holds in the case when $F_{\mathcal{X}}(A_0)$ is a set, e.g. when $A_0 = \Bbbk$. If $\mathcal{X}$ is the local deformation prestack arising from an object $x_0 \in \widetilde{\mathcal{X}}(\Bbbk)$ of an algebraic stack $\widetilde{\mathcal{X}}$ over $\mathrm{Sch}/\Bbbk$ as in Remark 3.10, then the surjectivity condition on automorphisms translates to the inertia stack $I_{\mathcal{X}} \to \mathcal{X}$ being smooth at $e(x_0)$, where $e \colon \mathcal{X} \to I_{\mathcal{X}}$ is the identity section, by the Infinitesimal Lifting Criterion for Smoothness (Smooth Equivalences 8.4).

While the existence of a miniversal formal deformation of $F_{\mathcal{X}}$ suffices for many applications, for moduli problems with automorphisms it is more natural to ask for the existence of a miniversal formal deformation of $\mathcal{X}$ and this generality is needed for some applications, e.g. Artin's Algebraization (Theorem 6.6) and Artin's Axioms for Algebraicity (Theorem 7.4).

Before proceeding to the proof, we first show that (RS_1)–(RS_2) yield natural structures on sets of deformations. In particular, they induce a vector space structure on the tangent space $T_{\mathcal{X}} = \mathcal{X}(\Bbbk[\epsilon])/\!\sim$ which allows us to make sense of condition (RS_3).

Lemma 3.13. *Let $\mathcal{X}$ be a prestack over $\mathrm{Art}_{\Bbbk}^{\mathrm{op}}$ such that the groupoid $\mathcal{X}(\Bbbk)$ is equivalent to the set $\{x_0\}$, and let $F_{\mathcal{X}} \colon \mathrm{Art}_{\Bbbk} \to \mathrm{Sets}$ be the covariant functor assigning $A \in \mathrm{Art}_{\Bbbk}$ to the set of isomorphism classes $\mathcal{X}(A)/\!\sim$. Assume that Condition (RS_2) holds for $\mathcal{X}$.*

1. *The tangent space $T_{\mathcal{X}} = F_{\mathcal{X}}(\Bbbk[\epsilon])$ has a natural structure of a $\Bbbk$-vector space. More generally, for any finite dimensional $\Bbbk$-vector space V, denoting $\Bbbk[V]$ as the $\Bbbk$-algebra $\Bbbk \oplus V$ defined by $V^2 = 0$, the set $F_{\mathcal{X}}(\Bbbk[V])$ has a natural structure of a $\Bbbk$-vector space and there is a functorial bijection $F_{\mathcal{X}}(\Bbbk[V]) = T_{\mathcal{X}} \otimes_{\Bbbk} V$.*
2. *Consider a surjection $B \twoheadrightarrow A$ in $\mathrm{Art}_{\Bbbk}$ with square-zero kernel I and an element $x \in \mathcal{X}(A)$, and let $\mathrm{Lift}_x(B)$ be the set of morphisms $x \to y$ over $\mathrm{Spec}\, A \to \mathrm{Spec}\, B$ where $x \xrightarrow{\alpha} y$ is declared equivalent to $x \xrightarrow{\alpha'} y'$ if there is an isomorphism $\beta \colon y \to y'$ such that $\alpha' = \beta \circ \alpha$. There is an action of $T_{\mathcal{X}} \otimes I$ on $\mathrm{Lift}_x(B)$ which is functorial in $\mathcal{X}$. Assuming $\mathrm{Lift}_x(B)$ is non-empty, this action is transitive if Condition (RS_1) holds for $\mathcal{X}$ and free and transitive (i.e. $\mathrm{Lift}_x(B)$ is a torsor under $T_{\mathcal{X}} \otimes I$) if Condition (RS_4) holds for $\mathcal{X}$.*

Proof. We first note if V is a finite dimensional vector space, then $\Bbbk[V] = \Bbbk[\epsilon] \times_{\Bbbk} \cdots \times_{\Bbbk} \Bbbk[\epsilon]$ and by applying (RS_2) inductively, we see that the statement of Condition (RS_2) also holds for $A_0 = \Bbbk$ and $A = \Bbbk[V]$. For $B_0 \in \mathrm{Art}_{\Bbbk}$, the first part of (RS_2) implies that $F_{\mathcal{X}}(B_0 \times_{\Bbbk} \Bbbk[V]) \xrightarrow{\sim} F_{\mathcal{X}}(B_0) \times F_{\mathcal{X}}(\Bbbk[V])$ is a bijection. In particular, $F_{\mathcal{X}}(\Bbbk[V] \times_{\Bbbk} \Bbbk[W]) \xrightarrow{\sim} F_{\mathcal{X}}(\Bbbk[V]) \times F_{\mathcal{X}}(\Bbbk[W])$ is bijective for any pair of finite dimensional vector spaces, or in other words the functor $V \mapsto F_{\mathcal{X}}(\Bbbk[V])$ commutes with finite products.

The vector space structure of $T_{\mathcal{X}} = F_{\mathcal{X}}(\Bbbk[\epsilon])$ follows from the bijectivity of

$$F_{\mathcal{X}}(\Bbbk[\epsilon] \times_{\Bbbk} \Bbbk[\epsilon']) \xrightarrow{\sim} F_{\mathcal{X}}(\Bbbk[\epsilon]) \times F_{\mathcal{X}}(\Bbbk[\epsilon']). \tag{3.3}$$

Indeed, addition of $\tau_1, \tau_2 \in F_{\mathcal{X}}(\Bbbk[\epsilon])$ is defined by using the above identification to view $(\tau_1, \tau_2) \in F_{\mathcal{X}}(\Bbbk[\epsilon] \times_{\Bbbk} \Bbbk[\epsilon'])$ and then taking its image under $F(\Bbbk[\epsilon] \times_{\Bbbk} \Bbbk[\epsilon']) \to F(\Bbbk[\epsilon])$ induced by the ring map $\Bbbk[\epsilon] \times_{\Bbbk} \Bbbk[\epsilon'] \to \Bbbk[\epsilon]$ taking $(\epsilon, 0)$ and $(0, \epsilon')$ to ϵ. Scalar multiplication of $c \in \Bbbk$ on $\tau \in F_{\mathcal{X}}(\Bbbk[\epsilon])$ is defined by taking the image of τ under $F_{\mathcal{X}}(\Bbbk[\epsilon]) \to F_{\mathcal{X}}(\Bbbk[\epsilon])$ induced by the map $\Bbbk[\epsilon] \to \Bbbk[\epsilon]$ taking ϵ to $c\epsilon$.

The same argument gives $F_{\mathcal{X}}(\Bbbk[V])$ the structure of a vector space such that the assignment $V \mapsto F_{\mathcal{X}}(\Bbbk[V])$ is a $\Bbbk$-linear functor $\mathrm{Vect}_{\Bbbk}^{\mathrm{fd}} \to \mathrm{Vect}_{\Bbbk}$ defined on finitely dimensional $\Bbbk$-vector spaces. The natural map

$$F_{\mathcal{X}}(\Bbbk[\epsilon]) \times \mathrm{Hom}_{\Bbbk}(\Bbbk[\epsilon], \Bbbk[V]) \to F_{\mathcal{X}}(\Bbbk[V]), \qquad (\tau, \phi) \mapsto \phi^* \tau$$

is $\Bbbk$-bilinear and under the equivalences $T_{\mathcal{X}} = F_{\mathcal{X}}(\Bbbk[\epsilon])$ and $V = \mathrm{Hom}_{\Bbbk}(\Bbbk[\epsilon], \Bbbk[V])$ corresponds to a linear map $T_{\mathcal{X}} \otimes V \to F_{\mathcal{X}}(\Bbbk[V])$, which is an isomorphism. This finishes the proof of (1).

For (2), we observe that the natural map

$$B \times_A B \to \Bbbk[I] \times_{\Bbbk} B, \qquad (b_1, b_2) \mapsto (\overline{b_1} + b_2 - b_1, b_1)$$

is an isomorphism. We therefore have a diagram

$$\mathcal{X}(\Bbbk[I]) \times \mathcal{X}(B) \twoheadleftarrow \mathcal{X}(\Bbbk[I] \times_{\Bbbk} B) \cong \mathcal{X}(B \times_A B) \xrightarrow{p_1^*} \mathcal{X}(B) \tag{3.4}$$

where the left functor is essentially surjective by the first part of (RS_2). Given $\tau \in T_{\mathcal{X}} \otimes I = F_{\mathcal{X}}(\Bbbk[I])$ (with a choice of representative in $\mathcal{X}(\Bbbk[I])$) and $(x \xrightarrow{\alpha} y) \in \mathrm{Lift}_x(B)$, we would like to define $\tau \cdot (x \xrightarrow{\alpha} y)$ as the image under p_1^* of a choice of preimage of (τ, y). To see that this is well-defined, consider two elements $z, z' \in \mathcal{X}(\Bbbk[I] \times_{\Bbbk} B)$ whose images in $\mathcal{X}(\Bbbk[I]) \times \mathcal{X}(B)$ are isomorphic to (τ, y). This yields a diagram

$$\begin{tikzcd}
x_0 \arrow[r] \arrow[d] & y \arrow[d, "\alpha_1"'] \arrow[dr, "\alpha_2"] & & & \operatorname{Spec} \Bbbk \arrow[r, hook] \arrow[d] & \operatorname{Spec} B \arrow[d] \\
\tau \arrow[r] \arrow[rr, bend right] & z \arrow[r, dashed, "\beta"] & z' & \text{over} & \operatorname{Spec} \Bbbk[I] \arrow[r, hook] & \operatorname{Spec}(\Bbbk[I] \times_{\Bbbk} B)
\end{tikzcd}$$

and by the second part of (RS_2), there exists a dotted arrow β such that $\alpha_2 = \beta \circ \alpha_1$. Therefore choices of pullbacks $p_1^* z$ and $p_1^* z'$ in $\mathcal{X}(B)$ defines the same element in $\mathrm{Lift}_x(B)$. If (RS_1) holds, then the statement of Condition (RS_1) also holds for any surjection $A \mapsto A_0$ (since we may factor it as a composition of surjections whose kernels are $\Bbbk$). Therefore if (RS_1) holds (resp. (RS_4) holds), then $\mathcal{X}(B \times_A B) \to \mathcal{X}(B) \times_{\mathcal{X}(A)} \mathcal{X}(B)$ is essentially surjective (resp. an equivalence) and we see that the action is transitive (resp. free and transitive). ☐

Proof Theorem 3.11. The details of the necessity of these conditions are left to the reader. We will establish the sufficiency. The tangent space $T_{\mathcal{X}} := \mathcal{X}(\Bbbk[\epsilon])/\sim$ has the structure of a vector space by Lemma 3.13(1) and is finite dimensional by (RS_2). Let $N = \dim_{\Bbbk} T_{\mathcal{X}}$ with basis $x_1, \ldots, x_N$ and define $S = \Bbbk[[x_1, \ldots, x_N]]$. We will construct inductively a decreasing sequence of ideals $J_0 \supset J_1 \supset \cdots$ and objects $\eta_n \in \mathcal{X}(S/J_n)$ together with morphisms $\eta_n \to \eta_{n+1}$ over $\operatorname{Spec} S/J_n \hookrightarrow \operatorname{Spec} S/J_{n+1}$. We set $J_0 = \mathfrak{m}_S$ and $\eta_0 = x_0 \in \mathcal{X}(\Bbbk)$. We also set $J_1 = \mathfrak{m}_S^2$ so that $S/J_1 \cong \Bbbk[T_{\mathcal{X}}]$. Using the bijection $F_{\mathcal{X}}(\Bbbk[T_{\mathcal{X}}]) \cong T_{\mathcal{X}} \otimes_{\Bbbk} T_{\mathcal{X}}$ of Lemma 3.13(1), the element $\sum_i x_i \otimes x_i$ defines an isomorphism class of an object $\eta_1 \in \mathcal{X}(S/J_1)$ such that the induced map $\operatorname{Spec} S/J_1 \to \mathcal{X}$ induces a bijection on tangent spaces. By construction, we have a morphism $\eta_0 \to \eta_1$ over $\operatorname{Spec} \Bbbk \hookrightarrow \operatorname{Spec} S/J_1$.

Suppose we've constructed J_n and $\eta_{n-1} \to \eta_n$. We claim that the set of ideals

$$\begin{aligned}\Sigma = \{J \subset S \mid \mathfrak{m}_S J_n \subset J \subset J_n &\text{ and there exists } \eta_n \to \eta \\ &\text{over } \operatorname{Spec} S/J_n \hookrightarrow \operatorname{Spec} S/J\}\end{aligned} \tag{3.5}$$

has a minimal element. Indeed, it is non-empty since $J_n \in \Sigma$ and given $J, K \in \Sigma$, we must check that $J \cap K \in \Sigma$. To achieve this, choose an ideal $J' \subset S$ satisfying $J \subset J' \subset I$ with $J \cap K = J' \cap K$ and $J' + K = I$. Then $A/(J' \cap K) \cong A/J' \times_{A/I} A/K$. Letting $\eta_J \in \mathcal{X}(S/J)$ and $\eta_K \in \mathcal{X}(S/K)$ be the objects corresponding to J and K, the data of $(\eta_J|_{S/J'} \leftarrow \eta_n \to \eta_K)$ defines an object of $\mathcal{X}(S/J') \times_{\mathcal{X}(S/I)} \mathcal{X}(S/K)$. The functor $\mathcal{X}(A/(J \cap K)) \to \mathcal{X}(S/J') \times_{\mathcal{X}(S/I)} \mathcal{X}(S/K)$ is essentially surjective by (RS_1) and the existence of preimage of $(\eta_J|_{S/J'} \leftarrow \eta_n \to \eta_K)$ shows that $J \cap K \in \Sigma$.

Setting $J = \bigcap_n J_n$, then $R = S/J$ is a noetherian complete local $\Bbbk$-algebra with ideals $I_n := J_n/J$. Since $\mathfrak{m}_S J_n \subset J_{n+1}$, we have that $\mathfrak{m}_R^{n+1} \subset I_n$ and thus $\xi_n := \eta_n|_{R/\mathfrak{m}_R^{n+1}}$ defines a formal deformation of x_0 over R.

We must check that $\xi := \{\xi_n\}$ is versal. Suppose $B \twoheadrightarrow A$ is a surjection in $\mathrm{Art}_{\Bbbk}$ with kernel $\Bbbk$ and that we have a diagram

$$\begin{tikzcd} x \arrow[r] \arrow[d] & \xi & & \operatorname{Spec} A \arrow[r, "g"] \arrow[d, hook] & h_R \\ y & & \text{over} & \operatorname{Spec} B. \arrow[ur, dashed, "\widetilde{g}"'] & \end{tikzcd}$$

We need to construct a morphism $y \to \xi$ extending $x \to \xi$. We claim that it suffices to construct a morphism $\widetilde{g} \colon \operatorname{Spec} B \to h_R$ (i.e. a ring map $R \to B$) extending g. Since $h_R(\Bbbk[\epsilon]) \to T_{\mathcal{X}}$ is bijective, Lemma 3.13(2) implies that there are actions of $T_{\mathcal{X}}$ on the sets $\mathrm{Lift}_x(B)$ and $\mathrm{Lift}_g(B)$ of isomorphism classes of lifts of x and g to objects in $\mathcal{X}(B)$ and $h_R(B)$ which are compatible with the map $\mathrm{Lift}_g(B) \to \mathrm{Lift}_x(B)$ where $\widetilde{g} \mapsto \widetilde{g}^*\xi$. Thus, we can find $\tau \in T_{\mathcal{X}}$ such that $y = \tau \cdot (\widetilde{g}^*\xi) = (\tau \cdot \widetilde{g})^*\xi$. This gives an arrow $y \to \xi$ over $\tau \cdot \widetilde{g} \colon \operatorname{Spec} B \to h_R$.

To construct $\widetilde{g}$, choose n such that $R \to A$ factors as $R \to R/I_n = S/J_n \to A$. It suffices to show that $\operatorname{Spec} A \to \operatorname{Spec} S/J_n$ extends to a map $\operatorname{Spec} B \to \operatorname{Spec} S/J_{n+1}$ and for this, it suffices to show the existence of a dotted arrow making the diagram

$$\begin{tikzcd} \operatorname{Spec} A \arrow[r] \arrow[d, hook] & \operatorname{Spec} S/J_n \arrow[d, hook] \arrow[ddr, hook] & \\ \operatorname{Spec} B \arrow[r] & \operatorname{Spec} B \times_A (S/J_n) \arrow[dr, dashed] & \\ & & \operatorname{Spec} S/J_{n+1} \end{tikzcd}$$

commutative. As $S = \Bbbk[[x_1, \dots, x_n]]$, we may choose an extension $S \to B$ of $S \to S/J_n \to A$. Then $B \times_A (S/J_n) = S/K$ where K is the kernel of the induced map $S \to B \times_A (S/J_n)$. The kernel K lies in the set of ideals defined in (3.5): the inclusion $K \subset J_n$ is clear, the inclusion $\mathfrak{m}_S J_n \subset K$ is implied by the equality $\ker(B \to A) = \Bbbk$, and the existence of $\eta_n \to \eta$ over $\operatorname{Spec} S/J_n \hookrightarrow \operatorname{Spec} S/K$ follows from applying (RS_1) to the above square. Thus $J_{n+1} \subset K$ and we have a ring map $S/J_{n+1} \to S/K = B \times_A (S/J_n)$ inducing the desired dotted arrow.

Finally, we must show that if $\mathcal{X}$ is equivalent to a functor F and (RS_4) holds, then F is prorepresentable by $\xi = \{\xi_n\}$. Given a surjection $B \to A$ with kernel $\Bbbk$ and $x \in F(A)$, it suffices to show the existence of a *unique* lift in any diagram

$$\begin{tikzcd}
\operatorname{Spec} A \arrow[r, "g"] \arrow[d, hook] & h_R \arrow[d] \\
\operatorname{Spec} B \arrow[r] \arrow[ur, dashed] & F.
\end{tikzcd}$$

This holds because the map $\operatorname{Lift}_g(B) \to \operatorname{Lift}_x(B)$ is bijective by Lemma 3.13(2) as both are torsors under $T_{\mathcal{X}}$.

See also [Sch68, Thm. 2.11], [SGA7-I, Thm. VI.1.11] and [SP, Tag 06IX], where the result is established more generally for prestacks over the category Art_Λ introduced in Remark 3.4. ☐

3.4 Verifying Rim–Schlessinger's Conditions

Consider the following *strong homogeneity* condition:

(RS_4^*) $\mathcal{X}(B_0 \times_{A_0} A) \to \mathcal{X}(B_0) \times_{\mathcal{X}(A_0)} \mathcal{X}(A)$ is an equivalence for any map $B_0 \to A_0$ and surjection $A \twoheadrightarrow A_0$ of rings with square-zero kernel (where the rings are not necessarily local artinian);

If $\mathcal{X}$ is a prestack over $(\mathrm{Sch}/\Bbbk)$ satisfying (RS_4^*), then the local deformation prestack $\mathcal{X}_{x_0}$ at x_0 (see Remark 3.10) is easily checked to satisfy (RS_4). On the other hand, it turns out that any algebraic stack satisfies (RS_4^*); see [SP, Tag 07WN]. In other words, the Ferrand pushout $\operatorname{Spec}(B_0 \times_{A_0} A)$ is a pushout in the category of algebraic stacks. Condition (RS_4^*) will appear in our second version of Artin's Axioms for Algebraicity (Theorem 7.4) as it will be useful to verify openness of versality (in addition to implying (RS_2)–(RS_3) ensuring the existence of versal formal deformations).

For a moduli problem $\mathscr{M}$, it is often possible to verify (RS_4^*) (and thus (RS_4) as well as (RS_1)–(RS_2)) as a consequence of Proposition 8.9: for a ring map $B_0 \to A_0$ and surjection $A \twoheadrightarrow A_0$, the functor $\operatorname{Mod}(B_0 \times_{A_0} A) \to \operatorname{Mod}(B_0) \times_{\operatorname{Mod}(A_0)} \operatorname{Mod}(A)$ restricts to an equivalence on flat modules. When B_0, A_0 and A are artinian, there is an elementary argument for this fact since flatness translates to freeness for modules over an artinian ring (Proposition 8.3).

We say that a prestack $\mathcal{X}$ over $\mathrm{Sch}/\Bbbk$ *admits formal versal deformations* if for every $\Bbbk$-point x_0, the local deformation prestack $\mathcal{X}_{x_0}$ (Remark 3.10) admits a formal versal deformation.

Proposition 3.14. *Each of the moduli problems* $\operatorname{Hilb}^P(X)$, $\mathscr{M}_g$ *(with* $g \geq 2$*) and* $\mathcal{B}\mathrm{un}_C$ *over* $\Bbbk$ *satisfy* (RS_3) *and* (RS_4^*)*, and therefore admit formal versal deformations.*

Proof. To check (RS$_3$) for objects $[Z_0 \subset X]$, C_0 and E_0 of $\mathcal{X} = \mathrm{Hilb}^P(X)$, $\mathscr{M}_g$ and $\mathcal{B}\mathrm{un}_C$ defined over $\Bbbk$, we have identifications of the tangent spaces $T_{\mathcal{X}}$ with the *finite dimensional* $\Bbbk$-vector spaces $\mathrm{H}^0(Z_0, N_{Z_0/X})$, $H^1(C_0, T_{C_0})$ and $\mathrm{H}^1(X, \mathscr{E}nd_{\mathcal{O}_X}(E_0))$ by Propositions 1.4, 1.11 and 1.15.

For (RS$_4^*$), let $B_0 \to A_0$ be a ring map and $A \twoheadrightarrow A_0$ be a surjection with square-zero kernel. Set $B = B_0 \times_{A_0} A$. For $\mathrm{Hilb}^P(X)$, Corollary 8.10(1)–(2) implies that the diagram

$$\begin{tikzcd}
X_{A_0} \arrow[r, hook] \arrow[d] & X_A \arrow[d] \\
X_{B_0} \arrow[r, hook] & X_B
\end{tikzcd}$$

is a pushout and that the functor $\mathrm{QCoh}(X_B) \to \mathrm{QCoh}(X_{B_0}) \times_{\mathrm{QCoh}(X_{A_0})} \mathrm{QCoh}(X_A)$ restricts to an equivalence between the full subcategory of finitely presented $\mathcal{O}_{X_B}$-modules flat over B and the fiber product of the full subcategories of finitely presented $\mathcal{O}$-modules flat over B_0 and A. This implies the desired equivalence $\mathrm{Hilb}^P(X)(B) \to \mathrm{Hilb}^P(X)(B_0) \times_{\mathrm{Hilb}^P(X)(A_0)} \mathrm{Hilb}^P(X)(A)$ between closed subschemes flat over the base.

For $\mathscr{M}_g$, the essential surjectivity of $\mathscr{M}_g(B) \to \mathscr{M}_g(B_0) \times_{\mathscr{M}_g(A_0)} \mathscr{M}_g(A)$ translates into the existence of an extension

$$\begin{tikzcd}
 & \mathcal{C}_0 \arrow[rrr, hook] \arrow[dl, hook] \arrow[dd, dashed] & & & \mathcal{C} \arrow[dl, dashed, hook] \arrow[dd] \\
\mathcal{D}_0 \arrow[rrr, dashed, hook] \arrow[dd] & & & \mathcal{D} \arrow[dd, dashed] & \\
 & \mathrm{Spec}\, A_0 \arrow[rrr, hook] \arrow[dl] & & & \mathrm{Spec}\, A \arrow[dl] \\
\mathrm{Spec}\, B_0 \arrow[rrr, hook] & & & \mathrm{Spec}\, B &
\end{tikzcd}$$

of smooth families of curves. The existence of $\mathcal{D}$ as a pushout of top face follows from Theorem 8.5. The fact that $\mathcal{D}$ is smooth over B follows from Corollary 8.10(2). The properness of $\mathcal{D} \to \mathrm{Spec}\, B$ follows from the properness of $\mathcal{D}_0 \to \mathrm{Spec}\, B_0$. The fully faithfulness translates to the bijectivity of

$$\mathrm{Aut}(\mathcal{D}/B) \to \mathrm{Aut}(\mathcal{D}_0/B_0) \times_{\mathrm{Aut}(\mathcal{C}_0/A_0)} \mathrm{Aut}(\mathcal{C}/A)$$

and follows direct from the fact that $\mathcal{D}$ is a pushout of the top face. Alternatively, one can replicate the above argument for $\mathrm{Hilb}^P(X)$ using the tricanonical embedding.

For $\mathcal{B}\mathrm{un}_C$, Corollary 8.10(1) implies that the functor $\mathrm{QCoh}(X_B) \to \mathrm{QCoh}(X_{B_0}) \times_{\mathrm{QCoh}(X_{A_0})} \mathrm{QCoh}(X_A)$ restricts to an equivalence on finitely presented $\mathcal{O}$-modules flat over the base and therefore also on vector bundles. □

4 Effective formal deformations and Grothendieck's Existence Theorem

We often would like to know when a formal deformation is effective.

Definition 4.1. Let $\mathcal{X}$ be a prestack (or functor) over $(\mathrm{Sch}/\Bbbk)$. Let $x_0 \in \mathcal{X}(\Bbbk)$ and consider a formal deformation $(R, \{x_n\})$ of x_0 (or more precisely a formal deformation of the deformation stack $\mathcal{X}_{x_0}$ at x_0 as defined in Remark 3.10). We say that $\{x_n\}$ is *effective* if there exists an object $\widehat{x} \in \mathcal{X}(R)$ and compatible isomorphisms $x_n \xrightarrow{\sim} \widehat{x}|_{\mathrm{Spec}\, R/\mathfrak{m}^{n+1}}$.

Remark 4.2. A formal deformation $(R, \{x_n\})$ is effective if it is in the essential image of the natural functor $\mathcal{X}(R) \to \varprojlim \mathcal{X}(R/\mathfrak{m}^n)$ or in other words if there exists a dotted arrow making the diagram

$$\begin{tikzcd}
\operatorname{Spec} R/\mathfrak{m} \arrow[r, hook] \arrow[rrrrd, "x_0"'] & \operatorname{Spec} R/\mathfrak{m}^2 \arrow[r, hook] \arrow[rrrd, "x_1"'] & \operatorname{Spec} R/\mathfrak{m}^3 \arrow[r, hook] \arrow[rrd, "x_2"'] & \cdots \arrow[r, hook] & \operatorname{Spec} R \arrow[d, dashed, "\widehat{x}"] \\
 & & & & \mathcal{X}
\end{tikzcd}$$

commutative.

Example 4.3. If $F\colon \mathrm{Sch}/\Bbbk \to \mathrm{Sets}$ is a contravariant functor representable by a scheme X over $\Bbbk$, then any formal deformation $(R, \{x_n\})$ is effective. Indeed, x_n corresponds to a morphism $\operatorname{Spec} R/\mathfrak{m}^{n+1} \to X$ with image $x \in X(\Bbbk)$ and thus to a $\Bbbk$-algebra homomorphism $\phi_n\colon \widehat{\mathcal{O}}_{X,x} \to R/\mathfrak{m}^{n+1}$. By taking the inverse image of ϕ_n, we have a local homomorphism $\widehat{\mathcal{O}}_{X,x} \to R$ which in turn defines a morphism $\widehat{x}\colon \operatorname{Spec} R \to X$ extending $\{x_n\}$.

Grothendieck's Existence Theorem—sometimes referred to as Formal GAGA—can often be applied to show that formal deformations are effective.

Theorem 4.4 (Grothendieck's Existence Theorem)**.** *Let $X \to \operatorname{Spec} R$ be a proper morphism of schemes where $(R, \mathfrak{m})$ is a noetherian complete local ring. Set $X_n := X \times_R R/\mathfrak{m}^{n+1}$ The functor*

$$\operatorname{Coh}(X) \to \varprojlim \operatorname{Coh}(X_n), \qquad E \mapsto \{E_n\}, \tag{4.1}$$

where E_n is the pullback of E along $X_n \to X$, is an equivalence of categories.

Proof. See [EGA, III.5.1.4], [FGI⁺05, Thm. 8.4.2] and [SP, Tag 088C]. □

Remark 4.5. The essential surjectivity of (4.1) translates to an extension of the diagram

$$\begin{tikzcd}
E_0 \arrow[d, dash] & E_1 \arrow[d, dash] & E_2 \arrow[d, dash] & & E \arrow[d, dash, dashed] \\
X_0 \arrow[r, hook] \arrow[d] & X_1 \arrow[r, hook] \arrow[d] & X_2 \arrow[r, hook] \arrow[d] & \cdots \arrow[r, hook] & X \arrow[d] \\
\operatorname{Spec} R/\mathfrak{m} \arrow[r, hook] & \operatorname{Spec} R/\mathfrak{m}^2 \arrow[r, hook] & \operatorname{Spec} R/\mathfrak{m}^3 \arrow[r, hook] & \cdots \arrow[r, hook] & \operatorname{Spec} R
\end{tikzcd}$$

while the fully faithfulness of (4.1) translates to the bijectivity of the natural map $\operatorname{Hom}_{\mathcal{O}_X}(E, F) \to \varprojlim \operatorname{Hom}_{\mathcal{O}_{X_n}}(E_n, F_n)$ for coherent sheaves E and F on X.

Using the language of formal schemes and setting $\widehat{X} = X \times_{\operatorname{Spec} R} \operatorname{Spf} R$ to be the $\mathfrak{m}$-adic completion of X, then Grothendieck's Existence Theorem asserts that the functor $\operatorname{Coh}(X) \to \operatorname{Coh}(\widehat{X})$, defined by $E \mapsto \widehat{E}$, is an equivalence.

Corollary 4.6. *Let $(R, \mathfrak{m})$ be a noetherian complete local ring and $X_n \to \operatorname{Spec} R/\mathfrak{m}^{n+1}$ be a sequence of proper morphisms such that $X_n \times_{R/\mathfrak{m}^{n+1}} R/\mathfrak{m}^n \cong X_{n-1}$. If L_n is a compatible sequence of line bundles on X_n such that L_0 is ample, then there exists a projective morphism $X \to \operatorname{Spec} R$ and an ample line bundle L on X and compatible isomorphisms $X_n \cong X \times_R R/\mathfrak{m}^{n+1}$ and $L_n \xrightarrow{\sim} L|_{X_n}$.*

Remark 4.7. It follows that there an extension in the cartesian diagram

$$\begin{tikzcd}
X_0 \arrow[r, hook] \arrow[d] & X_1 \arrow[r, hook] \arrow[d] & X_2 \arrow[r, hook] \arrow[d] & \cdots \arrow[r, hook, dashed] & X \arrow[d, dashed] \\
\operatorname{Spec} R/\mathfrak{m} \arrow[r, hook] & \operatorname{Spec} R/\mathfrak{m}^2 \arrow[r, hook] & \operatorname{Spec} R/\mathfrak{m}^3 \arrow[r, hook] & \cdots \arrow[r, hook] & \operatorname{Spec} R
\end{tikzcd}$$

such that X is projective over R. We say that the formal deformation $\{X_n \to \operatorname{Spec} R/\mathfrak{m}^{n+1}\}$ of X_0 is *effective* (which is sometimes referred to as *algebraizable*.

Proof. We sketch how this follows from Grothendieck's Existence Theorem. Consider the finitely generated graded $\Bbbk$-algebra $B = \bigoplus \mathfrak{m}^n/\mathfrak{m}^{n+1}$ and the quasi-coherent graded $\mathcal{O}_{X_0}$-algebra $\mathcal{A} = B \otimes_\Bbbk \mathcal{O}_{X_0}$. By applying Serre's vanishing theorem to $\operatorname{Spec}_{X_0} \mathcal{A}$ and the ample line bundle $L_0 \otimes_{\mathcal{O}_{X_0}} \mathcal{O}_{X_0'}$, we see that $\mathrm{H}^1(X_0, \mathcal{A} \otimes L_0^{\otimes d}) = 0$ for $d \gg 0$. We have a closed immersion $X_0 \hookrightarrow \mathbb{P}^N$ defined by a basis $s_{0,0}, \dots, s_{0,N}$ of $\mathrm{H}^0(X_0, L_0^{\otimes d})$. Noting that $\mathfrak{m}^n \mathcal{O}_{X_{n+1}}/\mathfrak{m}^{n+1}\mathcal{O}_{X_{n+1}}$ is identified with $\ker(\mathcal{O}_{X_{n+1}} \to \mathcal{O}_{X_n})$, we may tensor the corresponding short exact sequence by $L_{n+1}^{\otimes d}$ to obtain a short exact sequence

$$0 \to (\mathfrak{m}^n \mathcal{O}_{X_{n+1}}/\mathfrak{m}^{n+1}\mathcal{O}_{X_{n+1}}) \otimes L_0^{\otimes d} \to L_{n+1}^{\otimes d} \to L_n^{\otimes d} \to 0,$$

where we've used that that $(\mathfrak{m}^n \mathcal{O}_{X_{n+1}}/\mathfrak{m}^{n+1}\mathcal{O}_{X_{n+1}}) \otimes L_{n+1}^{\otimes d}$ is supported on X_0 along with the identifications $L_{n+1} \otimes \mathcal{O}_{X_m} \cong L_m$ for $m \leq n$. The vanishing of $\mathrm{H}^1(X_0, \mathcal{A} \otimes L_0^{\otimes d})$ implies that we may lift the sections $s_{0,0}, \dots, s_{0,N}$ inductively to compatible sections $s_{n,0}, \dots, s_{n,N}$ of $\mathrm{H}^0(X_n, L_n^{\otimes d})$. By Nakayama's Lemma, the induced morphisms $X_n \hookrightarrow \mathbb{P}^N_{R/\mathfrak{m}^{n+1}}$ are closed immersions giving a commutative diagram

$$\begin{tikzcd}
 & \mathbb{P}^N \arrow[rr, hook] \arrow[dd] & & \mathbb{P}^n_{R/\mathfrak{m}^2_R} \arrow[r, hook] \arrow[dd] & \cdots \arrow[rr, hook] & & \mathbb{P}^N_R \arrow[dd] \\
X_0 \arrow[ur, hook, "\mathrm{cl}"] \arrow[rr, hook] \arrow[dr] & & X_1 \arrow[ur, hook, "\mathrm{cl}"] \arrow[rr, hook] \arrow[dr] & & \cdots \arrow[r, hook, dashed] & X \arrow[ur, hook, dashed, "\mathrm{cl}"] \arrow[dr, dashed] & \\
 & \operatorname{Spec} \Bbbk \arrow[rr, hook] & & \operatorname{Spec} R/\mathfrak{m}^2 \arrow[r, hook] & \cdots \arrow[rr, hook] & & \operatorname{Spec} R
\end{tikzcd}$$

Grothendieck's Existence Theorem (Theorem 4.4) gives an equivalence $\operatorname{Coh}(\mathbb{P}^N_R) \to \varprojlim \operatorname{Coh}(\mathbb{P}^N_{R/\mathfrak{m}^{n+1}})$. Essential surjectivity gives a coherent sheaf E on $\mathbb{P}^N_R$ extending $\{\mathcal{O}_{X_n}\}$ and full faithfulness gives a surjection $\mathcal{O}_{\mathbb{P}^N_R} \to E$ extending $\mathcal{O}_{\mathbb{P}^N_{R/\mathfrak{m}^{n+1}}} \to \mathcal{O}_{X_n}$. We take $X \subset \mathbb{P}^N_R$ to be the closed subscheme defined by $\ker(\mathcal{O}_{\mathbb{P}^N_R} \to E)$.

See also [EGA, III.5.4.5], [FGI⁺05, Thm. 8.4.10] and [SP, Tag 089A]. ☐

Remark 4.8. If instead we are given only an ample line bundle L_0 on X_0 (and not the line bundles L_n), then the obstruction to deforming L_{n-1} to L_n is an element $\mathrm{ob}_{L_{n-1}} \in \mathrm{H}^2(X, \mathcal{O}_X \otimes_\Bbbk \mathfrak{m}^n)$ by Proposition 2.10. If these cohomology groups vanish (e.g. if X is of dimension 1), then there exists compatible extensions L_n and thus the formal deformation $\{X_n \to \operatorname{Spec} R/\mathfrak{m}^{n+1}\}$ are effective.

Without the existence of deformations L_n of L_0, it is not necessarily true that formal deformations are effective. For instance, there is a projective K3 surface (X_0, L_0) and a first order deformation $X_1 \to \operatorname{Spec} \Bbbk[\epsilon]$ which is not projective (so

L_0 does not deform to X_1), and a formal deformation which is not effective; see [Har10, Ex. 21.2.1]. Similarly formal deformations of abelian varieties may not be effective. Note that for both K3 surfaces and abelian varieties, Rim–Schlessinger's Criteria applies to construct versal formal deformations.

Corollary 4.9. *For each of the moduli problems* $\mathrm{Hilb}^P(X)$, $\mathscr{M}_g$ *(with* $g \geq 2$*) and* $\mathcal{B}\mathrm{un}_C$ *over* $\Bbbk$*, any formal deformation is effective. In particular, there exist effective versal formal deformations.*

Proof. For $\mathrm{Hilb}^P(X)$, we show the effectivity of a formal deformation $\{Z_n \subset X_{R/\mathfrak{m}^{n+1}}\}$ by following the argument at the end of the proof of Corollary 4.6 (with $X_n \subset \mathbb{P}^N_{R/\mathfrak{m}^{n+1}}$ replaced with $Z_n \subset X_{R/\mathfrak{m}^{n+1}}$): Grothendieck's Existence Theorem (Theorem 4.4) implies the existence of a coherent sheaf E on X_R extending $\{\mathcal{O}_{Z_n}\}$ and a surjection $\mathcal{O}_{X_R} \to E$ extending $\{\mathcal{O}_{X_n} \to \mathcal{O}_{Z_n}\}$, and we take $Z \subset X_R$ defined by $\ker(\mathcal{O}_{X_R} \to E)$.

For $\mathscr{M}_g$, the effectivity of a formal deformation $\{C_n \to \operatorname{Spec} R/\mathfrak{m}^{n+1}\}$ follows from Corollary 4.6 by taking L_n be the ample bundle $\Omega_{C_n/(R/\mathfrak{m}^{n+1})}$, or by taking L_0 to be any ample bundle on C_0 and using Proposition 2.10 and the vanishing of $\mathrm{H}^2(C_0, \mathcal{O}_{C_0})$ to inductively deform L_0 to a compatible sequence of line bundle L_n on C_n.

For $\mathcal{B}\mathrm{un}_C$, the effectivity of a formal deformation of vector bundles E_n on $C_{R/\mathfrak{m}^{n+1}}$ follows directly from Grothendieck's Existence Theorem (Theorem 4.4) noting that the coherent extension is necessarily a vector bundle.

The last statement follows from the existence of versal formal deformations of these moduli problems (Proposition 3.14). □

Exercise 4.10. If $\mathcal{X}$ is an algebraic stack locally of finite type over $\Bbbk$ and $(R, \mathfrak{m})$ is a noetherian complete local ring with residue field $\Bbbk$, show that the functor

$$\mathcal{X}(R) \to \varprojlim \mathcal{X}(R/\mathfrak{m}^{n+1})$$

is an equivalence of categories. In particular, every formal deformation is effective.

4.1 Lifting to characteristic 0

One striking application of deformation theory is to "lift" a smooth variety X_0 over a field $\Bbbk$ of $\mathrm{char}(\Bbbk) = p$ to characteristic 0. We say that X_0 is *liftable to characteristic* 0 if there exists a noetherian complete local ring $(R, \mathfrak{m})$ of characteristic 0 such that $R/\mathfrak{m} = \Bbbk$ and a smooth scheme $X \to \operatorname{Spec} R$ such that $X_0 \cong X \times_R \Bbbk$.[2] One can hope to then use characteristic 0 techniques (e.g. Hodge theory) on X and deduce properties of X_0. The strategy to lift a variety X_0 is to inductively deform X_0 to smooth schemes X_n over $R/\mathfrak{m}^{n+1}$ and then apply Grothendieck's Existence Theorem to effective the formal deformation. Note however that to achieve this, we must work over a mixed characteristic base as in Remark 3.4 rather than over a fixed field $\Bbbk$.

Smooth curves are liftable as obstructions to deforming both the curve and the ample line bundle both vanish. Serre produced an example of a non-liftable projective threefold (see [Har10, Thm. 22.4]) which Mumford extended to a non-liftable projective surface (see [FGI+05, Cor. 8.6.7]). On the other hand, Mumford

---

[2] There are some variants to this definition, e.g. when R is already given as a complete DVR with residue field $\Bbbk$.

showed that principally polarized abelian varieties are liftable [Mum69] while Deligne showed that K3 surfaces are liftable [Del81]. These examples are quite interesting as in both cases, formal deformations are not necessarily effective (see Remark 4.8) and additional techniques are needed.

5 Cotangent complex

In this chapter, we summarize properties of the cotangent complex of a morphism of schemes as introduced in [Ill71] globalizing work of André [And67] and Quillen [Qui68, Qui70] on the cotangent complex of a ring homomorphism.

5.1 Properties of the cotangent complex

Theorem 5.1. *For any morphism $f\colon X \to Y$ of schemes (resp. noetherian schemes), there exists a complex*

$$L_{X/Y}\colon \; \cdots \to L^{-1}_{X/Y} \to L^{0}_{X/Y} \to 0$$

of flat $\mathcal{O}_X$-modules with quasi-coherent (resp. coherent) cohomology, whose image in $D^-_{\mathrm{QCoh}}(\mathcal{O}_X)$ (resp. $D^-_{\mathrm{Coh}}(\mathcal{O}_X)$) is also denoted by $L_{X/Y}$. It satisfies the following properties:

(1) $\mathrm{H}^0(X, L_{X/Y}) \cong \Omega_{X/Y}$*;*

(2) f is smooth if and only if f is locally of finite presentation and $L_{X/Y}$ is a perfect complex supported in degree 0. In this case $L_{X/Y}$ is quasi-isomorphic to the complex where the vector bundle $\Omega_{X/Y}$ sits in degree 0;

(3) If f is flat and finitely presented, then f is syntomic if and only if $L_{X/Y}$ is a perfect complex supported in degrees $[-1, 0]$. Explicitly, if f factors as a local complete intersection $X \hookrightarrow \widetilde{Y}$ defined by a sheaf of ideals I and a smooth morphism $\widetilde{Y} \to Y$, then $L_{X/Y}$ is quasi-isomorphic to $0 \to I/I^2 \xrightarrow{d} \Omega_{X/Y} \to 0$ (with $\Omega_{X/Y}$ in degree 0);

(4) If

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

is a cartesian diagram with either f or g flat (or more generally f and g are tor-independent), then there is a quasi-isomorphism $g'^*L_{X/Y} \to L_{X'/Y'}$. (Note that without any flatness condition $g'^*\Omega_{X/Y} \cong \Omega_{X'/Y'}$.)

(5) If $X \xrightarrow{f} Y \to Z$ is a composition of morphisms of schemes, then there is an exact triangle in $D^-_{\mathrm{QCoh}}(\mathcal{O}_X)$

$$f^*L_{Y/Z} \to L_{X/Z} \to L_{X/Y} \to f^*L_{Y/Z}[1].$$

This induces a long exact sequence on cohomology

$$\begin{tikzcd}
& \cdots \arrow[r] & \mathrm{H}^{-2}(L_{X/Y}) \arrow[dll, out=-10, in=170] & \\
\mathrm{H}^{-1}(f^*L_{X/Z}) \arrow[r] & \mathrm{H}^{-1}(L_{X/Z}) \arrow[r] & \mathrm{H}^{-1}(L_{X/Y}) \arrow[dll, out=-10, in=170] & \\
f^*\Omega_{Y/Z} \arrow[r] & \Omega_{X/Z} \arrow[r] & \Omega_{X/Y} \arrow[r] & 0
\end{tikzcd}$$

extending the usual right exact sequence on differentials [Har77, II.8.12]. (Note that if f is smooth, then $H^{-1}(L_{X/Y}) = 0$ and $f^*\Omega_{Y/Z} \to \Omega_{X/Z}$ is injective.)

Proof. See [Ill71, II.1.2.3], [SP, Tag 08T2] for the definition of the cotangent complex of a morphism of schemes (and more generally for morphisms of ringed topoi). For (1)–(5), see [Ill71, II.1.2.4.2, II.3.1.2, II.3.2.6, II.2.2.3 and II.2.1.2] and [SP, Tags 08UV, 0D0N, 0FK3, 08QQ and 08T4] (noting that [SP, Tag 08RB] relates the *naive cotangent complex* $NL_{X/Y}$ to $L_{X/Y}$). ☐

5.2 Truncations of the cotangent complex

The definition of the cotangent complex relies on simplicial techniques and we won't attempt an exposition here. We will however give an explicit description of its truncation, which often suffice for applications.

First, if $X \to Y$ factors as a closed immersion $X \hookrightarrow P$ defined by a sheaf of ideals I and a smooth morphism $P \to Y$, then the truncation $\tau_{\geq -1}(L_{X/Y})$ of $L_{X/Y}$ in degrees $[-1, 0]$ is quasi-isomorphic to $0 \to I/I^2 \xrightarrow{d} \Omega_{X/Y} \to 0$ (with $\Omega_{X/Y}$ in degree 0). In the case that $X \to Y$ is smooth or syntomic, then $X \hookrightarrow \widetilde{Y}$ is a regular immersion, I/I^2 is a vector bundle and $L_{X/Y}$ is quasi-isomorphic to $\tau_{\geq -1}(L_{X/Y})$ (Theorem 5.1(3)).

For a morphism $X = \operatorname{Spec} A \to \operatorname{Spec} B = Y$ of affine schemes, Lichtenbaum–Schlessinger [LS67] offer an explicit description of $\tau_{\geq -2}(L_{X/Y})$. Choose a polynomial ring $P = B[x_i]$ (with possibly infinitely many generators) and a surjection $P \twoheadrightarrow A$ as B-algebras with kernel I. Choose a free P-module $F = \oplus_{\lambda \in \Lambda} P$ and a surjection $p \colon F \twoheadrightarrow I$ of P-modules with kernel $K = \ker(p)$. Let $K' \subset K$ be the submodule generated by $p(x)y - p(y)x$ for $x, y \in F$. Then the truncation $\tau_{\geq -2}(L_{X/Y})$ (or rather $\tau_{\geq -2}(L_{B/A})$) is quasi-isomorphic to the complex of A-modules

$$K/K' \to F \otimes_P A \to \Omega_{P/B} \otimes_P A \tag{5.1}$$

with the last term in degree 0; see [SP, Tag 09CG].

One defines the T^i *functors* on the category of A-modules by

$$T^i(A/B, -) := \mathrm{H}^i(\mathrm{Hom}_A(L_\bullet, -)),$$

which can be used for instance to describe deformations of schemes (see Example 5.11). See also [LS67, §2.3] and [Har10, §1.3].

5.3 Extensions of algebras and schemes

Definition 5.2. An *extension* of a ring homomorphism $R \to A$ by an A-module I is an exact sequence of R-modules

$$0 \to I \to A' \to A \to 0$$

where $A' \to A$ is an R-algebra homomorphism and $I \subset A'$ is an ideal with $I^2 = 0$. (Note that since $I^2 = 0$, $I = I/I^2$ is a module over $A = A'/I$.) The *trivial extension* is $A[I] := A \oplus I$ where multiplication is defined by $I^2 = 0$.

A morphism of extensions is a morphism of short exact sequences which is the identity on I and A. By the five lemma, any morphism of extensions is necessarily an isomorphism. We let $\underline{\mathrm{Exal}}_R(A, I)$ be the groupoid of extensions of $R \to A$ by I, and $\mathrm{Exal}_R(A, I)$ the set of isomorphism classes.

Remark 5.3. Geometrically, an extension is a commutative diagram of schemes

$$\begin{tikzcd}
\operatorname{Spec} A \arrow[r, hook] \arrow[d] & \operatorname{Spec} A' \arrow[dl] \\
\operatorname{Spec} R &
\end{tikzcd}$$

such that $I \cong \ker(A' \to A)$ and $I^2 = 0$.

The set of extensions $\mathrm{Exal}_R(A, I)$ is functorial with respect to A and I:

(a) Given a map $B \to A$ of R-algebras, there is a map $\mathrm{Exal}_R(A, I) \to \mathrm{Exal}_R(B, I)$ given by mapping a complex $0 \to I \to A' \to A \to 0$ to $0 \to I \to A' \times_A B \to B \to 0$.

(b) Given an A-module homomorphism $\alpha\colon I \to J$, there is a map $\alpha_*\colon \mathrm{Exal}_R(A, I) \to \mathrm{Exal}_R(A, J)$ given by mapping a complex $0 \to I \to A' \to A \to 0$ to $0 \to J \to (A' \oplus J)/\{(-x, \alpha(x)), x \in I\} \to A \to 0$.

(c) Given modules I and J, the natural map $(p_{1,*}, p_{2,*})\colon \mathrm{Exal}_R(A, I \oplus J) \to \mathrm{Exal}_R(A, I) \oplus \mathrm{Exal}_R(A, J)$, induced from (b) by the projections $p_1\colon I \oplus J \to I$ and $p_2\colon I \oplus J \to J$, is a bijection.

Moreover, $\mathrm{Exal}_R(A, I)$ naturally has the structure of an A-module: scalar multiplication by $x \in A$ is defined using (b) with $x\colon I \to I$ and addition is defined by $\mathrm{Exal}_R(A, I) \times \mathrm{Exal}_R(A, I) \cong \mathrm{Exal}_R(A, I \oplus I) \xrightarrow{\Sigma_*} \mathrm{Exal}_R(A, I)$ using the bijection in (c) and the map Σ_* of (b) where $\Sigma\colon I \oplus I \to I$ is addition. The maps (a)–(c) are in fact maps of A-modules. See [Ill71, §III.1.1] for details.

Proposition 5.4. *Let R be a ring.*

(1) Given a R-algebra A and an exact sequence $0 \to I' \to I \to I'' \to 0$ of A-modules, there is an exact sequence

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Der}_R(A, I') \arrow[r] & \mathrm{Der}_R(A, I) \arrow[r] & \mathrm{Der}_R(A, I'') \arrow[dll] \\
& \mathrm{Exal}_R(A, I) \arrow[r] & \mathrm{Exal}_R(A, I) \arrow[r] & \mathrm{Exal}_R(A, I'')
\end{tikzcd}$$

of A-modules.

(2) Given a homomorphism $B \to A$ of R-algebras, there is an exact sequence

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Der}_B(A, I) \arrow[r] & \mathrm{Der}_R(A, I) \arrow[r] & \mathrm{Der}_R(B, I) \arrow[dll] \\
& \mathrm{Exal}_B(A, I) \arrow[r] & \mathrm{Exal}_R(A, I) \arrow[r] & \mathrm{Exal}_R(B, I)
\end{tikzcd}$$

of A-modules.

Proof. See [EGA, 0.20.2.3] and [Ill71, III.1.2.4.3, III.1.2.5.4]. □

Remark 5.5. The top row of (5.4) can be realized using the right exact sequence $\Omega_{B/R} \otimes_B A \to \Omega_{A/R} \to \Omega_{A/B} \to 0$. Namely, apply $\mathrm{Hom}_A(-, I)$ and use the identities $\mathrm{Hom}_A(\Omega_{A/B}, I) = \mathrm{Der}_B(A, I)$, $\mathrm{Hom}_A(\Omega_{A/R}, I) = \mathrm{Der}_R(A, I)$ and $\mathrm{Hom}_A(\Omega_{B/R} \otimes_B A, I) = \mathrm{Hom}_B(\Omega_{B/R}, I) = \mathrm{Der}_R(B, I)$.

The cotangent complex can be applied to extend these sequences to long exact sequences; see Remark 5.8.

The definition of Exal extends naturally to schemes (and more generally to ringed topoi).

Definition 5.6. An *extension* of a morphism $X \to S$ of schemes by a quasi-coherent $\mathcal{O}_X$-module I is a short exact sequence

$$0 \to I \to \mathcal{O}_{X'} \to \mathcal{O}_X \to 0$$

where $X \hookrightarrow X'$ is a closed immersion of schemes defined by sheaf of ideals $I \subset \mathcal{O}_{X'}$ with $I^2 = 0$. (Note that the condition $I^2 = 0$ implies that the $I \subset \mathcal{O}_{X'}$ is naturally a $\mathcal{O}_X$-module.) The *trivial extension* is $X[I] := (X, \mathcal{O}_X \oplus I)$ where the ring structure is is defined by $I^2 = 0$.

A *morphism of extensions* is a morphism of short exact sequences which is the identity on I and $\mathcal{O}_X$. We let $\underline{\mathrm{Exal}}_S(X, I)$ be the category of extensions of $X \to S$ by I, and $\mathrm{Exal}_S(X, I)$ be the set of isomorphism classes.

The set $\mathrm{Exal}_S(X, I)$ is naturally an $\mathcal{O}_X$-module and is functorial in X and I. In fact, the groupoid $\underline{\mathrm{Exal}}_S(X, I)$ is a *Picard category*, and the prestack over Sch/S whose fiber category over $f\colon T \to S$ is $\underline{\mathrm{Exal}}_T(X_T, f^*I)$ is a *Picard stack*; see [Ill71, III.1.1.5] and [SGA4, XVIII.1.4].

5.4 The cotangent complex and deformation theory

Theorem 5.7. *If $X \to Y$ is a morphism of schemes and I is a quasi-coherent $\mathcal{O}_Y$-module, there is a natural isomorphism*

$$\mathrm{Exal}_Y(X, I) \cong \mathrm{Ext}^1_{\mathcal{O}_X}(L_{X/Y}, I).$$

Proof. See [Ill71, III.1.2.3]. □

Remark 5.8. This identification allows us to use the cotangent complex to extend the 6-term left exact sequences of Proposition 5.4 to long exact sequences. Namely, applying $\mathrm{Hom}_{\mathcal{O}_X}(L_{X/Y}, -)$ to the exact sequence $0 \to I' \to I \to I''$ extends 5.4(1) and applying $\mathrm{Hom}_{\mathcal{O}_X}(-, I)$ to the exact triangle $f^*L_{Y/Z} \to L_{X/Z} \to L_{X/Y}$ extends 5.4(2).

When $X = \mathrm{Spec}\, A \to \mathrm{Spec}\, B = Y$ is a morphism of affine schemes, using the T^i functors of §5.2, the above equivalence translates to $\mathrm{Exal}_B(A, I) = T^1(A/B, I)$. This can be established using the explicit description of the Lichtenbaum–Schlessinger truncated cotangent complex (5.1); see [LS67, 4.2.2] and [Har10, Thm. 5.1]. The T^i functors can also be used to extend the 6-term sequences of Proposition 5.4 to 9-term sequences; see [LS67, 2.3.5-6] and [Har10, Thms. 3.4-5].

Remark 5.9. More generally, there is an equivalence between the groupoid $\underline{\mathrm{Exal}}_Y(X, I)$ and the groupoid obtained from the 2-term complex $[C^{-1} \xrightarrow{d} C^0] := \tau_{\leq 0}(\mathrm{RHom}_{\mathcal{O}_X}(\tau_{\geq -1}L_{X/Y}, I)[1]))$ where objects are elements of C^0 and $\mathrm{Mor}(c, c') = \ker(d - d')$; see [Ill71, III.1.2.2].

Theorem 5.10. *Consider the following deformation problem*

$$\begin{tikzcd} X \arrow[r, hook, dashed] \arrow[d, "f"] & X' \arrow[d, dashed, "f'"] \\ Y \arrow[r, hook, "i"] & Y' \end{tikzcd}$$

where $f \colon X \to Y$ is a morphism of schemes and $i \colon Y \hookrightarrow Y'$ is a closed immersion of schemes defined by an ideal sheaf $I \subset \mathcal{O}_{Y'}$ with $I^2 = 0$. A deformation *is a morphism $f' \colon X' \to Y'$ making the above diagram cartesian and a morphism of deformations is a morphism over Y' restricting to the identity on X.*

1. *The group of automorphisms of a deformation $f' \colon X' \to Y'$ is isomorphic to $\mathrm{Ext}^0_{\mathcal{O}_X}(L_{X/Y}, f^*I)$.*
2. *If there exists a deformation, then the set of deformations is a torsor under $\mathrm{Ext}^1_{\mathcal{O}_X}(L_{X/Y}, f^*I)$.*
3. *There exists an element $\mathrm{ob}_X \in \mathrm{Ext}^2_{\mathcal{O}_X}(L_{X/Y}, f^*I)$ with the property that there exists a deformation if and only if $\mathrm{ob}_X = 0$.*

Proof. See [Ill71, III.2.1.7] and [SP, Tag 08UZ]. See also [LS67, 4.2.5] and [Har10, Thm. 10.1] for descriptions in the affine case using the truncated cotangent complex. ☐

Example 5.11. Consider a surjection $A' \twoheadrightarrow A$ of noetherian rings with square-zero kernel J. First, as a reality check, if $f \colon X \to \operatorname{Spec} A$ is smooth, we compute that

$$\mathrm{Ext}^i_{\mathcal{O}_X}(L_{X/A}, f^*J) = \mathrm{H}^i(X, T_{X/A} \otimes_A J)$$

and we recover Proposition 2.5.

The advantage of the cotangent complex is that for $\mathrm{Ext}^i_{\mathcal{O}_X}(L_{X/A}, f^*J)$ for $i = 0, 1, 2$ classifies automorphisms, deformations and obstructions for an arbitrary morphism. Moreover, the truncated cotangent complex $\tau_{\geq -2}L_{X/A}$ suffices and when $X = \operatorname{Spec} R$ is affine, we get equivalent descriptions using the T^i functors $T^i(L_{R/A}, f^*J)$.

Remark 5.12. There are analogous results for other deformation problems. For instance, for the deformation problem

$$\begin{tikzcd} X \arrow[r, hook] \arrow[dr, "f"] \arrow[ddr] & X' \arrow[dr, dashed] \arrow[ddr] & \\ & Y \arrow[r, hook] \arrow[d] & Y' \arrow[d] \\ & Z \arrow[r, hook] & Z', \end{tikzcd}$$

where the horizontal morphisms are closed immersions defined by square-zero ideal sheaves I_X, I_Y and I_Z, then automorphisms, deformations and obstructions are classified by $\mathrm{Ext}^i_{\mathcal{O}_X}(f^*L_{Y/Z}, I_X)$ for $i = -1, 0, 1$ [Ill71, III.2.2.4]. An important special case is when $Y = Y'$ and $Z = Z'$.

6 Artin Algebraization

Artin Algebraization is a procedure to "algebraize" or extend an effective versal formal deformation $\xi \in \mathscr{M}(R)$ to an object $\eta \in \mathscr{M}(U)$ over a *finite type* $\Bbbk$-scheme U. In this section, we show how Artin Algebraization follows from Artin Approximation following the ideas of Conrad and de Jong [CJ02].

6.1 Limit preserving prestacks

Extending the definition of a limit preserving functor §**??**, we say that a prestack $\mathcal{X}$ over $\mathrm{Sch}/\Bbbk$ is *limit preserving* (or *locally of finite presentation*) if for every system B_λ of $\Bbbk$-algebras, the natural functor

$$\operatorname{colim} \mathcal{X}(B_\lambda) \to \mathcal{X}(\operatorname{colim} B_\lambda)$$

is an equivalence of categories. When $\mathcal{X}$ is an algebraic stack over $\Bbbk$, then this equivalent to the morphism $\mathcal{X} \to \operatorname{Spec} \Bbbk$ being locally of finite presentation; see **??**).

Lemma 6.1. *Each of the prestacks* $\mathrm{Hilb}^P(X)$, $\mathscr{M}_g$ *(with* $g \geq 2$*) and* $\mathcal{B}\mathrm{un}_C$ *over* $(Sch/\Bbbk)$ *are limit preserving.*

Proof. To add. ☐

6.2 Conrad–de Jong Approximation

In Artin Approximation (Theorem 8.17), the initial data is an object over a noetherian complete local $\Bbbk$-algebra $\widehat{\mathcal{O}}_{S,s}$ which is assumed to be the completion of a finitely generated $\Bbbk$-algebra at a maximal ideal. We will now see that a similar approximation result still holds if this latter hypothesis is dropped and one approximates *both* the complete local ring and the object.

Recall also that if $(A, \mathfrak{m})$ is a local ring and M is an A-module, then the *associated graded module of* M is defined as $\mathrm{Gr}_{\mathfrak{m}}(M) = \bigoplus_{n \geq 0} \mathfrak{m}^n M / \mathfrak{m}^{n+1} M$; it is a graded module over the graded ring $\mathrm{Gr}_{\mathfrak{m}}(A)$.

Theorem 6.2 (Conrad–de Jong Approximation)**.** *Let* $\mathcal{X}$ *be a limit preserving prestack over* $\mathrm{Sch}/\Bbbk$. *Let* $(R, \mathfrak{m}_R)$ *be a noetherian complete local* $\Bbbk$*-algebra and let* $\xi \in \mathcal{X}(R)$. *Then for every integer* $N \geq 0$*, there exist*

(1) *an affine scheme* $\operatorname{Spec} A$ *of finite type over* $\Bbbk$ *and a* $\Bbbk$*-point* $u \in \operatorname{Spec} A$,

(2) *an object* $\eta \in \mathcal{X}(A)$,

(3) *an isomorphism* $\phi_{N+1} \colon R/\mathfrak{m}_R^{N+1} \cong A/\mathfrak{m}_u^{N+1}$,

(4) *an isomorphism of* $\xi|_{R/\mathfrak{m}_R^{N+1}}$ *and* $\eta|_{A/\mathfrak{m}_u^{N+1}}$ *via* ϕ_N, *and*

(5) *an isomorphism* $\mathrm{Gr}_{\mathfrak{m}_R}(R) \cong \mathrm{Gr}_{\mathfrak{m}_u}(A)$ *of graded* $\Bbbk$*-algebras.*

The proof of this theorem will proceed by simultaneously approximating equations and relations defining R and the object ξ. The statements (1)–(4) will be easily obtained as a consequence of Artin Approximation. A nice insight of Conrad and de Jong is that condition (5) can be ensured by Artin Approximation, and moreover that this condition suffices to imply the isomorphism of complete local $\Bbbk$-algebras in Artin Algebraization. As such, condition (5) takes the most work to establish.

We will need some preparatory results controlling the constant appearing in the Artin–Rees lemma.

Definition 6.3 (Artin–Rees Condition)**.** Let $(A, \mathfrak{m})$ be a noetherian local ring. Let $\varphi\colon M \to N$ be a morphism of finite A-modules. Let $c \geq 0$ be an integer. We say that $(\mathsf{AR})_c$ *holds for* φ if

$$\varphi(M) \cap \mathfrak{m}^n N \subset \varphi(\mathfrak{m}^{n-c} M), \quad \forall n \geq c.$$

The Artin–Rees lemma implies that $(\mathsf{AR})_c$ holds for φ if c is sufficiently large; see [AM69, Prop. 10.9] or [Eis95, Lem. 5.1].

Lemma 6.4. *Let $(A, \mathfrak{m})$ be a noetherian local ring. Let*

$$L \xrightarrow{\alpha} M \xrightarrow{\beta} N \qquad \text{and} \qquad L \xrightarrow{\alpha'} M \xrightarrow{\beta'} N$$

be two complexes of finite A-modules. Let c be a positive integer. Assume that

(a) the first sequence is exact,

(b) the complexes are isomorphic modulo $\mathfrak{m}^{c+1}$, and

(c) $(\mathsf{AR})_c$ holds for α and β.

Then there exists an isomorphism $\mathrm{Gr}_{\mathfrak{m}}(\operatorname{coker}\beta) \to \mathrm{Gr}_{\mathfrak{m}}(\operatorname{coker}\beta')$ *of graded* $\mathrm{Gr}_{\mathfrak{m}}(A)$*-modules.*

Proof. The proof while technical is rather straightforward. One first shows that $(\mathsf{AR})_c$ holds for β' and that the second sequence is exact. Then one establishes the equality

$$\mathfrak{m}^{n+1}N + \beta(M) \cap \mathfrak{m}^n N = \mathfrak{m}^{n+1}N + \beta'(M) \cap \mathfrak{m}^n N$$

by using that $(\mathsf{AR})_c$ holds for β to show the containment "$\subset$" and then using $(\mathsf{AR})_c$ holds for β' to get the other containment. The statement then follows from the description $\mathrm{Gr}_{\mathfrak{m}}(\operatorname{coker}\beta)_n = \mathfrak{m}^n N/(\mathfrak{m}^{n+1}N + \beta(M) \cap \mathfrak{m}^n N)$ and the similar description of $\mathrm{Gr}_{\mathfrak{m}}(\operatorname{coker}\beta')_n$. For details, see [CJ02, §3] and [SP, Tag 07VF]. ☐

Proof of Conrad–de Jong Approximation (Theorem 6.2). Since $\mathcal{X}$ is limit preserving and R is the colimit of its finitely generated $\Bbbk$-subalgebras, there is an affine scheme $V = \operatorname{Spec} B$ of finite type over $\Bbbk$ and an object γ of $\mathcal{X}$ over V together with a 2-commutative diagram

$$\begin{tikzcd} \operatorname{Spec} R \arrow[r] \arrow[rr, bend left, "\xi"] & V \arrow[r, "\gamma"] & \mathcal{X}. \end{tikzcd}$$

Let $v \in V$ be the image of the maximal ideal $\mathfrak{m} \subset R$. After adding generators to the ring B if necessary, we can assume that the composition $\widehat{\mathcal{O}}_{V,v} \to R \to R/\mathfrak{m}^2$ is surjective. This implies that $\widehat{\mathcal{O}}_{V,v} \to R$ is surjective by Lemma 8.18. The goal now is to simultaneously approximate over V the equations and relations defining the closed immersion $\operatorname{Spec} R \hookrightarrow \operatorname{Spec} \widehat{\mathcal{O}}_{V,v}$ and the object ξ. In order to accomplish this goal, we choose a resolution

$$\widehat{\mathcal{O}}_{V,v}^{\oplus r} \xrightarrow{\widehat{\alpha}} \widehat{\mathcal{O}}_{V,v}^{\oplus s} \xrightarrow{\widehat{\beta}} \widehat{\mathcal{O}}_{V,v} \to R \to 0 \tag{6.1}$$

as $\widehat{\mathcal{O}}_{V,v}$-modules and consider the functor

$$\begin{aligned} F\colon (\mathrm{Sch}/V) &\to \mathsf{Sets} \\ (T \to V) &\mapsto \big\{\text{complexes } \mathcal{O}_T^{\oplus r} \xrightarrow{\alpha} \mathcal{O}_T^{\oplus s} \xrightarrow{\beta} \mathcal{O}_T\big\}. \end{aligned}$$

It is not hard to check that this functor is limit preserving. The resolution in (6.1) yields an element of $F(\widehat{\mathcal{O}}_{V,v})$. Applying Artin Approximation (Theorem 8.17) yields an étale morphism $(V' = \operatorname{Spec} B', v') \to (V,v)$ and an element

$$(B'^{\oplus r} \xrightarrow{\alpha'} B'^{\oplus s} \xrightarrow{\beta'} B') \in F(V') \tag{6.2}$$

such that α', β' are equal to $\widehat{\alpha}, \widehat{\beta}$ modulo $\mathfrak{m}^{N+1}$.

Let $U = \operatorname{Spec} A \hookrightarrow \operatorname{Spec} B' = V'$ be the closed subscheme defined by $\operatorname{im}\beta'$ and let $u = v' \in U$. Consider the composition

$$\eta\colon U \hookrightarrow V' \to V \xrightarrow{\gamma} \mathcal{X}$$

As $R = \operatorname{coker}\widehat{\beta}$ and $A = \operatorname{coker}\beta'$, we have an isomorphism $R/\mathfrak{m}^{N+1} \cong A/\mathfrak{m}_u^{N+1}$ together with an isomorphism of $\xi|_{R/\mathfrak{m}^{N+1}}$ and $\eta|_{A/\mathfrak{m}_u^{N+1}}$. This gives statements (1)–(4).

To establish (5), we need to show that there are isomorphisms $\mathfrak{m}^n/\mathfrak{m}^{n+1} \cong \mathfrak{m}_u^n/\mathfrak{m}_u^{n+1}$. For $n \leq N$, this is guaranteed by the isomorphism $R/\mathfrak{m}^{N+1} \cong A/\mathfrak{m}_u^{N+1}$. On the other hand, for $n \gg 0$, this can be seen to be a consequence of the Artin–Rees lemma. To handle the middle range of n, we need to control the constant appearing in the Artin–Rees lemma. First note that before we applied Artin Approximation, we could have increased N to ensure that $(\mathsf{AR})_N$ holds for $\widehat{\alpha}$ and $\widehat{\beta}$. We are thus free to assume this. Now statement (5) follows directly if we apply Lemma 6.4 to the exact complex $\widehat{\mathcal{O}}_{V,v}^{\oplus r} \xrightarrow{\widehat{\alpha}} \widehat{\mathcal{O}}_{V,v}^{\oplus s} \xrightarrow{\widehat{\beta}} \widehat{\mathcal{O}}_{V,v}$ of (6.1) and the complex $\widehat{\mathcal{O}}_{V,v}^{\oplus r} \xrightarrow{\widehat{\alpha}'} \widehat{\mathcal{O}}_{V,v}^{\oplus s} \xrightarrow{\widehat{\beta}'} \widehat{\mathcal{O}}_{V,v}$ obtained by restricting (6.2) to $F(\widehat{\mathcal{O}}_{V,v})$.

See also [CJ02] and [SP, Tag 07XB]. □

Exercise 6.5. Show that Conrad–de Jong Approximation implies Artin Approximation.

6.3 Artin Algebraization

Artin Algebraization has a stronger conclusion than Artin Approximation or Conrad–de Jong Approximation in that no approximation is necessary. It guarantees the existence of an object η over a pointed affine scheme $(\operatorname{Spec} A, u)$ of finite type over $\Bbbk$ which agrees with the given effective formal deformation ξ to all orders. In order to ensure this, we need to impose that ξ is *versal at* u, i.e. that the restrictions $\xi_n = \xi|_{A/\mathfrak{m}_u^{n+1}}$ define a versal formal deformation $\{\xi_n\}$ over A (Definition 3.5).

Theorem 6.6 (Artin Algebraization)**.** *Let $\mathcal{X}$ be a limit preserving prestack over* $\mathrm{Sch}/\Bbbk$*. Let $(R,\mathfrak{m})$ be a noetherian complete local $\Bbbk$-algebra and $\xi \in \mathcal{X}(R)$ be an effective versal formal deformation. There exist*

(1) an affine scheme $\operatorname{Spec} A$ *of finite type over $\Bbbk$ and a $\Bbbk$-point $u \in$* $\operatorname{Spec} A$*;*

(2) an object $\eta \in \mathcal{X}(A)$;

(3) an isomorphism $\alpha\colon R \xrightarrow{\sim} \widehat{A}_{\mathfrak{m}_u}$ of $\Bbbk$-algebras; and

(4) a compatible family of isomorphisms $\xi|_{R/\mathfrak{m}^{n+1}} \cong \eta|_{A/\mathfrak{m}_u^{n+1}}$ (under the identification $R/\mathfrak{m}^{n+1} \cong A/\mathfrak{m}_u^{n+1}$) for $n \geq 0$.

Remark 6.7. If $\mathcal{X}$ is an algebraic stack locally of finite type over $\Bbbk$, then there exists an isomorphism $\xi \cong \eta|_{\widehat{A}_{\mathfrak{m}_u}}$.

Remark 6.8. In the case that R is known to be the completion of a finitely generated $\Bbbk$-algebra, this theorem can be viewed as an easy consequence of Artin Approximation. Indeed, one applies Artin Approximation with $N = 1$ and then uses versality to obtain compatible maps $R \to A/\mathfrak{m}_u^{n+1}$ and therefore a map $R \to \widehat{A}_{\mathfrak{m}_u}$ which is an isomorphism modulo $\mathfrak{m}^2$. As R and $\widehat{A}_{\mathfrak{m}_u}$ are abstractly isomorphic, the homomorphism $R \to \widehat{A}_{\mathfrak{m}_u}$ is an isomorphism (Lemma 8.18) and the statement follows. The argument in the general case is analogous except we use Conrad–de Jong Approximation instead of Artin Approximation.

Proof of Artin Algebraization (Theorem 6.6). Applying Conrad–de Jong Approximation (Theorem 6.2) with $N = 1$, we obtain an affine scheme $\operatorname{Spec} A$ of finite type over $\Bbbk$ with a $\Bbbk$-point $u \in \operatorname{Spec} A$, an object $\eta \in \mathcal{X}(A)$, an isomorphism $\phi_2\colon \operatorname{Spec} A/\mathfrak{m}_u^2 \to \operatorname{Spec} R/\mathfrak{m}^2$, an isomorphism $\alpha_2\colon \xi|_{R/\mathfrak{m}^2} \to \eta|_{A/\mathfrak{m}_u^2}$, and an isomorphism $\operatorname{Gr}_{\mathfrak{m}}(R) \cong \operatorname{Gr}_{\mathfrak{m}_u}(A)$ of graded $\Bbbk$-algebras. We claim that ϕ_2 and α_2 can be extended inductively to a compatible family of morphisms $\phi_n\colon \operatorname{Spec} A/\mathfrak{m}_u^{n+1} \to \operatorname{Spec} R$ and isomorphisms $\alpha_n\colon \xi|_{A/\mathfrak{m}_u^{n+1}} \to \eta|_{A/\mathfrak{m}_u^{n+1}}$. Given ϕ_n and α_n, versality of ξ implies that there is a lift ϕ_{n+1} filling in the commutative diagram

$$\begin{tikzcd}
\operatorname{Spec} A/\mathfrak{m}_u^n \arrow[r, "\phi_n"] \arrow[d] & \operatorname{Spec} R \arrow[d, "\xi"] \\
\operatorname{Spec} A/\mathfrak{m}_u^{n+1} \arrow[r, "\eta|_{A/\mathfrak{m}_u^{n+1}}"'] \arrow[ur, dashed, "\phi_{n+1}"'] & \mathcal{X},
\end{tikzcd}$$

which establishes the claim. By taking the limit, we have a homomorphism $\widehat{\phi}\colon R \to \widehat{A}_{\mathfrak{m}_u}$ which is surjective since ϕ_2 is surjective (Lemma 8.18). On the other hand, for each n the $\Bbbk$-vector spaces $\mathfrak{m}^N/\mathfrak{m}^{N+1}$ and $\mathfrak{m}_u^N/\mathfrak{m}_u^{N+1}$ have the same dimension. This implies that $\widehat{\phi}$ is an isomorphism.

See also [Art69, Thm. 1.6] and [CJ02, §4], where the statement is established more generally when $\mathcal{X}$ is defined over a scheme S whose local rings are G-rings where it is required that $\operatorname{Spec} R/\mathfrak{m} \xrightarrow{\xi_0} \mathcal{X} \to S$ be of finite type. $\square$

7 Artin's Axioms for Algebraicity

A spectacular application of Artin Algebraization is to provide checkable conditions which ensure that a given stack is algebraic. This is called Artin's Axiom's for Algebraicity and we provide two versions below Theorems 7.1 and 7.4. This foundational result was proved by Artin in the very same paper [Art74] where he introduced algebraic stacks.

The first version can be proved easily using Artin Algebraization.

Theorem 7.1. *(Artin's Axioms for Algebraicity—first version) Let $\mathcal{X}$ be a stack over $\Bbbk$. Then $\mathcal{X}$ is an algebraic stack locally of finite type over $\Bbbk$ if and only if the following conditions hold:*

1. *(1) (Limit preserving) The stack $\mathcal{X}$ is limit preserving over* $\mathrm{Sch}/\Bbbk$*, i.e. for every system B_λ of $\Bbbk$-algebras, the functor*

$$\operatorname{colim} \mathcal{X}(B_\lambda) \to \mathcal{X}(\operatorname{colim} B_\lambda)$$

is an equivalence of categories.

(2) (Representability of the diagonal) The diagonal $\mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is representable.

(3) (Existence of versal formal deformations) Every $x_0 \in \mathcal{X}(\Bbbk)$ has a versal formal deformation $\{x_n\}$ over a noetherian complete local $\Bbbk$-algebra $(R, \mathfrak{m})$ with residue field $\Bbbk$.

(4) (Effectivity) For every noetherian complete local $\Bbbk$-algebra $(R, \mathfrak{m})$ with residue field $\Bbbk$, the natural functor

$$\mathcal{X}(\operatorname{Spec} R) \to \varprojlim \mathcal{X}(\operatorname{Spec} R/\mathfrak{m}^n)$$

is an equivalence of categories.

(5) (Openness of versality) For any morphism $U \to \mathcal{X}$ from a finite type $\Bbbk$-scheme which is versal at $u \in U(\Bbbk)$ (i.e. the formal deformation $\{\operatorname{Spec} \widehat{\mathcal{O}}_{U,u}/\mathfrak{m}_u^{n+1} \to \mathcal{X}\}$ is versal), there exists an open neighborhood V of u such that $U \to \mathcal{X}$ is versal at every $\Bbbk$-point of V.

Proof. We first note that for a representable and locally of finite type morphism $U \to \mathcal{X}$ from a finite type $\Bbbk$-scheme U, the Infinitesimal Lifting Criterion for Smoothness (Smooth Equivalences 8.4, Smooth Equivalences 8.4) implies that $U \to \mathcal{X}$ is smooth if and only if it is versal at all $\Bbbk$-points $u \in U$. Indeed, this is clear when $U \to \mathcal{X}$ is representable by schemes, and the general case follows as one can see that both properties are étale-local on U.

For "$\Longrightarrow$," (1) holds by **??**, (2) holds by **??** and (4) holds by Exercise 4.10. If $U \to \mathcal{X}$ is a morphism from a finite type $\Bbbk$-scheme, then it is necessarily representable and locally of finite type. By using the above equivalence between versality and smoothness, (3) holds by choosing a smooth presentation $U \to \mathcal{X}$ and a preimage $u \in U(\Bbbk)$ of x_0 and taking the formal deformation $\{\operatorname{Spec} \mathcal{O}_{U,u}/\mathfrak{m}_u^{n+1} \to \mathcal{X}\}$, and (5) holds by openness of smoothness.

For the converse, we first note that representability of the diagonal, i.e. condition (2), implies that any morphism $U \to \mathcal{X}$ from a scheme U is representable and the limit preserving property (1) implies that $U \to \mathcal{X}$ is locally of finite type. For any object $x_0 \in \mathcal{X}(\Bbbk)$, we will construct a smooth morphism $U \to \mathcal{X}$ from a scheme and a preimage $u \in U(\Bbbk)$ of x_0. Conditions (3)–(4) guarantee that there exists an effective versal formal deformation $\widehat{x}\colon \operatorname{Spec} R \to \mathcal{X}$ of x_0 where $(R, \mathfrak{m})$ is a noetherian complete local $\Bbbk$-algebra with residue field $\Bbbk$. By Artin Algebraization (Theorem 6.6), there exists a finite type $\Bbbk$-scheme U, a point $u \in U(\Bbbk)$, a morphism $p\colon U \to \mathcal{X}$, an isomorphism $R \cong \widehat{\mathcal{O}}_{U,u}$ and compatible isomorphisms $p|_{R/\mathfrak{m}^{n+1}} \xrightarrow{\sim} \widehat{x}|_{R/\mathfrak{m}^{n+1}}$. By (5), we can replace U with an open neighborhood of u so that $U \to \mathcal{X}$ is versal at every $\Bbbk$-point of U. By the equivalence in the first paragraph, we have obtained a smooth morphism $(U, u) \to (\mathcal{X}, x_0)$.

See also [Art74], [LMB, Cor. 10.11] and [SP, Tag 07Y4] where the result is established more generally. □

Remark 7.2. In practice, condition (1)–(4) are often easy to verify directly with (3) a consequence of Rim–Schlessinger's Criteria (Theorem 3.11) and (4) a consequence of Grothendieck's Existence Theorem (Theorem 4.4). Also note that (2) can sometimes be established by applying the theorem to the diagonal $\mathcal{X} \to \mathcal{X} \times \mathcal{X}$, i.e. to the Isom sheaves $\mathrm{Isom}_T(x, y)$ of objects $x, y \in \mathcal{X}(T)$ over a scheme T. In some cases, Condition (5) can be checked directly. In more general moduli problems, Condition (5) is often guaranteed as a consequence of a well-behaved deformation and obstruction theory. This will be explained in the next section.

7.1 Refinements of Artin's Axioms

We will now state refinements of Artin's Axioms for Algebraicity that are often easier to verify in practice. To formulate the statements, we will need a bit of notation. Let $\xi \in \mathcal{X}(A)$ be an object over a finitely generated $\Bbbk$-algebra A. Let M be a finite A-module and denote by $A[M]$ the ring $A \oplus M$ defined by $M^2 = 0$. Let $\mathrm{Def}_\xi(M)$ the set of isomorphism classes of diagrams

$$\begin{tikzcd}
\operatorname{Spec} A \arrow[r, "\xi"] \arrow[d, hook] & \mathcal{X} \\
\operatorname{Spec} A[M], \arrow[ur, dashed, "\eta"] &
\end{tikzcd}$$

where an isomorphism of two extensions $\eta, \eta' \colon \operatorname{Spec} A[M] \to \mathcal{X}$ is by definition an isomorphism $\eta \xrightarrow{\sim} \eta'$ in $\mathcal{X}(A[M])$ restricting to the identity on ξ. Let $\mathrm{Aut}_\xi(M)$ be the group of automorphisms of the trivial deformation $\xi' \colon \operatorname{Spec} A[M] \to \operatorname{Spec} A \to \mathcal{X}$. Note that when $\xi \in \mathcal{X}(\Bbbk)$, then $\mathrm{Def}_\xi(\Bbbk)$ is precisely the tangent space of $\mathcal{X}$ at ξ and is identified with $T\mathcal{X}_\xi = \mathcal{X}_\xi(\Bbbk[\epsilon])/\sim$ of the local deformation prestack at ξ while $\mathrm{Aut}_\xi(\Bbbk)$ is the group of infinitesimal automorphism of ξ and is identified with the kernel $\mathrm{Aut}_{\mathcal{X}(\Bbbk[\epsilon])}(\xi') \to \mathrm{Aut}_{\mathcal{X}(\Bbbk)}(\xi)$.

Lemma 7.3. *Suppose that $\mathcal{X}$ is a prestack over $\mathrm{Sch}/\Bbbk$ satisfying the strong homogeneity condition (RS_4^*). Let $\xi \in \mathcal{X}(A)$ be an object over a finitely generated $\Bbbk$-algebra A.*

(1) For any A-module M, $\mathrm{Def}_\xi(M)$ and $\mathrm{Aut}_\xi(M)$ are naturally A-modules, and the functors

$$\mathrm{Aut}_\xi(-) \colon \mathrm{Mod}(A) \to \mathrm{Mod}(A)$$
$$\mathrm{Def}_\xi(-) \colon \mathrm{Mod}(A) \to \mathrm{Mod}(A)$$

are A-linear.

(2) Consider a surjection $B \twoheadrightarrow A$ in $\mathrm{Art}_\Bbbk$ with square-zero kernel I, and let $\mathrm{Lift}_\xi(B)$ be the set of morphisms $\xi \to \eta$ over $\operatorname{Spec} A \to \operatorname{Spec} B$ where $\xi \xrightarrow{\alpha} \eta$ is declared equivalent to $\xi \xrightarrow{\alpha'} \eta'$ if there is an isomorphism $\beta \colon \eta \to \eta'$ such that $\alpha' = \beta \circ \alpha$. There is an action of $\mathrm{Def}_\xi(I)$ on $\mathrm{Lift}_\xi(B)$ which is functorial in B and I. Assuming $\mathrm{Lift}_\xi(B)$ is non-empty, this action is free and transitive.

Proof. This can be established by arguing as in Lemma 3.13. For instance, scalar multiplication by $x \in A$ is defined by pulling back along the morphism $\operatorname{Spec} A[M] \to \operatorname{Spec} A[M]$ induced by the A-algebra homomorphism $A[M] \to A[M], a+m \mapsto a+xm$. Condition (RS_4^*) implies that $\mathcal{X}(A[M \oplus M]) \to \mathcal{X}(A[M]) \times_{\mathcal{X}(A)}$

$\mathcal{X}(A[M])$ is an equivalence. Addition $M \oplus M \to M$ induces an A-algebra homomorphism $A[M \oplus M] \to A[M]$ and thus a functor

$$\mathcal{X}(A[M]) \times_{\mathcal{X}(A)} \mathcal{X}(A[M]) \cong \mathcal{X}(A[M \oplus M]) \to \mathcal{X}(A[M])$$

which defines addition on $\mathrm{Def}_\xi(M)$ and $\mathrm{Aut}_\xi(M)$. $\square$

Theorem 7.4 (Artin's Axioms for Algebraicity—second version)**.** *A stack* $\mathcal{X}$ *over* $(\mathrm{Sch}/\Bbbk)_{\text{ét}}$ *is an algebraic stack locally of finite type over* $\Bbbk$ *if the following conditions hold:*

(AA$_1$) *(Limit preserving) The stack* $\mathcal{X}$ *is limit preserving;*

(AA$_2$) *(Representability of the diagonal) The diagonal* $\mathcal{X} \to \mathcal{X} \times \mathcal{X}$ *is representable;*

(AA$_3$) *(Finiteness of tangent spaces) For every object* $\xi\colon \operatorname{Spec}\Bbbk \to \mathcal{X}$, $\mathrm{Def}_\xi(\Bbbk)$ *is a finite dimensional* $\Bbbk$*-vector space;*

(AA$_4$) *(Strong homogeneity) For any* $\Bbbk$*-algebra homomorphism* $B_0 \to A_0$ *and surjection* $A \twoheadrightarrow A_0$ *of* $\Bbbk$*-algebras with square-zero kernel, the functor*

$$\mathcal{X}(B_0 \times_{A_0} A) \to \mathcal{X}(B_0) \times_{\mathcal{X}(A_0)} \mathcal{X}(A)$$

is an equivalence, i.e. Condition (RS$_4^*$) *holds;*

(AA$_5$) *(Effectivity) For every noetherian complete local* $\Bbbk$*-algebra* $(R, \mathfrak{m})$*, the natural functor*

$$\mathcal{X}(\operatorname{Spec} R) \to \varprojlim \mathcal{X}(\operatorname{Spec} R/\mathfrak{m}^n)$$

is an equivalence of categories;

(AA$_6$) *(Existence of an obstruction theory) For every object* $\xi \in \mathcal{X}(A)$ *over a finitely generated* $\Bbbk$*-algebra* A*, there exists the following data*

(a) there is an A*-linear functor*

$$\mathrm{Ob}_\xi(-)\colon \mathrm{Mod}(A) \to \mathrm{Mod}(A),$$

and for every surjection $B \to A$ *with square-zero kernel* I*, there is an element* $\mathrm{ob}_\xi(B) \in \mathrm{Ob}_\xi(I)$ *such that there is an extension*

$$\begin{tikzcd}
\operatorname{Spec} A \arrow[r, "\xi"] \arrow[d, hook] & \mathcal{X} \\
\operatorname{Spec} B \arrow[ur, dashed] &
\end{tikzcd}$$

if and only if $\mathrm{ob}_\xi(B) = 0$*, and*

(b) for every composition $B \to B' \to A$ *of* $\Bbbk$*-algebras such that* $B \twoheadrightarrow A$ *and* $B' \twoheadrightarrow A$ *are surjective with square-zero kernels* I *and* I'*, the image of* $\mathrm{ob}_\xi(B)$ *under* $\mathrm{Ob}_\xi(I) \to \mathrm{Ob}_\xi(I')$ *is* $\mathrm{ob}_\xi(B')$*; and*

(AA$_7$) *(Coherent deformation theory) For every object* $\xi \in \mathcal{X}(A)$ *over a* $\Bbbk$*-algebra* A*, the functors* $\mathrm{Def}_\xi(-)$ *and* $\mathrm{Ob}_\xi(-)$ *commute with products.*

Moreover (AA$_2$) *can be removed if we replace* (AA$_3$) *and* (AA$_7$) *with:*

(AA$_{3'}$) *For every object* $\xi\colon \operatorname{Spec}\Bbbk \to \mathcal{X}$, $\mathrm{Aut}_\xi(\Bbbk)$ *and* $\mathrm{Def}_\xi(\Bbbk)$ *are finite dimensional* $\Bbbk$*-vector spaces; and*

($\mathrm{AA}_{7'}$) *For every object* $\xi \in \mathcal{X}(A)$ *over a* $\Bbbk$*-algebra* A*, the functors* $\operatorname{Aut}_\xi(-)$, $\operatorname{Def}_\xi(-)$ *and* $\operatorname{Ob}_\xi(-)$ *commute with products.*

Proof. Conditions (AA_3)–(AA_4) above allow us to apply Rim–Schlessinger's Criteria (Theorem 3.11) to deduce the existence of versal formal deformations, i.e. Condition 7.1(3) holds. It remains to check openness of versality, i.e. Condition 7.1(5), in order to apply the first version (Theorem 7.1) to establish this version.

Let $\xi_0 \colon U_0 \to \mathcal{X}$ be a morphism from an affine scheme $U_0 = \operatorname{Spec} B_0$ of finite type over $\Bbbk$ which is versal at a point $u_0 \in U_0(\Bbbk)$. By (AA_1)–(AA_2), the morphism $\xi_0 \colon U_0 \to \mathcal{X}$ is representable and locally of finite type. Let $\Sigma = \{u \in U_0(\Bbbk) \mid \xi_0 \colon U_0 \to \mathcal{X}$ is not versal at $u\}$. If openness of versality does not hold, then $u_0 \in \overline{\Sigma}$ and there exists a countably infinite subset $\Sigma' = \{u_1, u_2, \dots\} \subset \Sigma$ of distinct points with $u_0 \in \overline{\Sigma'}$.

Step 1. We claim that there exists a commutative diagram

$$\begin{tikzcd}
U_0 \arrow[r, hook] \arrow[d, "\xi_0"'] & U_1 \arrow[r, hook] \arrow[dl, "\xi_1"] & U_2 \arrow[r, hook] \arrow[dll, "\xi_2"] & \cdots \\
\mathcal{X}
\end{tikzcd}$$

where each closed immersion $U_{n-1} \hookrightarrow U_n$ is defined by a short exact sequence

$$0 \to \kappa(u_n) \to \mathcal{O}_{U_n} \to \mathcal{O}_{U_{n-1}} \to 0,$$

and for each n and open neighborhood $W \subset U_n$ of u_n, the restriction $\xi_n|_W$ is not the trivial deformation of $\xi_0|_{W \cap U_0}$, i.e. there is no morphism $r \colon \xi_n|_W \to \xi_0|_{W \cap U_0}$ such that $\xi_n|_W \xrightarrow{r} \xi_0|_{W\cap U_0} \to \xi_n|_W$ is the identity. Note that for each $m \geq n$, $U_n \hookrightarrow U_m$ is a closed immersion which is square-zero (i.e. $\ker(\mathcal{O}_{U_m} \to \mathcal{O}_{U_n})$ is square-zero). We will inductively construct $U_n = \operatorname{Spec} B_n$ and $\xi_n \in \mathcal{X}(U_n)$, Since $\xi_0 \colon U_0 \to \mathcal{X}$ and $\xi_{n-1} \colon U_{n-1} \to \mathcal{X}$ are isomorphic in an open neighborhood of u_n, the morphism $\xi_{n-1} \colon U_{n-1} \to \mathcal{X}$ is also not versal at u_n. By definition of versality (using Remark 3.7) there exists a surjection $A \to A_0$ in $\mathrm{Art}_\Bbbk$ with $\ker(A \to A_0) = \Bbbk$ and a commutative diagram

$$\begin{tikzcd}
\operatorname{Spec} A_0 \arrow[r] \arrow[d, hook] & U_{n-1} \arrow[d, "\xi_{n-1}"] \\
\operatorname{Spec} A \arrow[r] \arrow[ur, dashed, "\nexists"] & \mathcal{X},
\end{tikzcd} \tag{7.1}$$

such that u_n is the image of $\operatorname{Spec} A_0 \to U_{n-1}$, which does not admit a lift $\operatorname{Spec} A \to U_{n-1}$. Using strong homogeneity (AA_4), there exists an extension of the commutative diagram

$$\begin{tikzcd}
\operatorname{Spec} A_0 \arrow[r] \arrow[d, hook] & U_{n-1} = \operatorname{Spec} B_{n-1} \arrow[d, hook] \arrow[ddr, bend left, "\xi_{n-1}"] & \\
\operatorname{Spec} A \arrow[r] \arrow[drr, bend right] & U_n = \operatorname{Spec}(A \times_{A_0} B_{n-1}) \arrow[dr, dashed, "\xi_n"] & \\
& & \mathcal{X}
\end{tikzcd}$$

yielding an object ξ_n over $U_n = \operatorname{Spec} B_n$ with $B_n := A \times_{A_0} B_{n-1}$. If ξ_n were the trivial deformation of ξ_0 in an open neighborhood of u_n, then $\operatorname{Spec} A \to \mathcal{X}$ would be the trivial deformation of $\operatorname{Spec} A_0$ contradicting the obstruction to a lift of (7.1). Finally note that $\ker(B_n \to B_{n-1}) = \Bbbk$ since $\ker(A \to A_0) = \Bbbk$. This establishes the claim.

Step 2. Letting $\widehat{B} = \varprojlim B_n$ and $\widehat{U} = \operatorname{Spec} \widehat{B}$, we claim that there exists an object $\widehat{\xi} \in \mathcal{X}(\widehat{U})$ extending each $\xi_n \in \mathcal{X}(U_n)$. Let $M_n = \ker(B_n \to B_0)$ (noting that $M_0 = 0$). Since $M_n^2 = 0$, we can view M_n as a B_0-module. The $\Bbbk$-algebra

$$\widetilde{B} := \{(b_0, b_1, \ldots) \in \prod_{n \geq 0} B_n \mid \text{the image of each } b_n \text{ under } B_n \to B_0 \text{ is } b_0\}$$

has the following properties:

- The surjective $\Bbbk$-algebra homomorphism $\widetilde{B} \to B_0$ defined by $(b_i) \mapsto b_0$ has kernel $M := \prod_{n \geq 0} M_n$;
- The map $\widetilde{B} \to B_0[M]$ defined by $(b_0, b_1, b_2, \ldots) \mapsto (b_0, b_1 - b_0, b_2 - b_1, b_3 - b_2, \ldots)$ is a surjective $\Bbbk$-algebra homomorphism with square-zero kernel;
- The composition $\widehat{B} \to \widetilde{B} \to B_0[M]$ induces a short exact sequence

$$\begin{tikzcd}
0 \arrow[r] & \ker(\widehat{B} \to B_0) \arrow[r] \arrow[d, equal] & \ker(\widetilde{B} \to B_0) \arrow[r] \arrow[d, equal] & \ker(B_0[M] \to B_0) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \varprojlim_{n \geq 0} M_n \arrow[r] & \prod_{n \geq 0} M_n \arrow[r] & \prod_{n \geq 0} M_n \arrow[r] & 0 \\
 & & (b_0, b_1, \ldots) \arrow[r, mapsto] & (b_1 - b_0, b_2 - b_1, \ldots) &
\end{tikzcd}$$

- There is an identification $\widehat{B} = \widetilde{B} \times_{B_0[M]} B_0$.

Since the lift $\xi_n \in \mathcal{X}(B_n)$ of ξ_0 exists for each n, $\operatorname{ob}_\xi(B_n) = 0 \in \operatorname{Ob}_\xi(M_n)$. By (AA$_6$)(b), the element $\operatorname{ob}_\xi(\widetilde{B})$ maps to $\operatorname{ob}_\xi(B_n)$ under $\operatorname{Ob}_\xi(M) \to \operatorname{Ob}_\xi(M_n)$. By (AA$_7$), the map $\operatorname{Ob}_\xi(M) \hookrightarrow \prod_n \operatorname{Ob}_\xi(M_n)$ is injective[3] and thus $\operatorname{ob}_\xi(\widetilde{B}) = 0 \in \operatorname{Ob}_\xi(M)$ which shows that there exists a lift $\widetilde{\xi} \in \mathcal{X}(\widetilde{B})$ of ξ_0.

The restrictions $\widetilde{\xi}|_{B_n}$ are not necessarily isomorphic to ξ_n. However, we may use the free and transitive action $\operatorname{Def}_\xi(M_n) = \operatorname{Lift}_\xi(B_0[M_n])$ on the non-empty set of liftings $\operatorname{Lift}_\xi(\widetilde{B_n})$ to find elements $t_n \in \operatorname{Def}_\xi(M_n)$ such that $\xi_n = t_n \cdot \widetilde{\xi}|_{B_n}$ (Lemma 7.3). Since $\operatorname{Def}_\xi(M) \xrightarrow{\sim} \prod_n \operatorname{Def}_\xi(M_n)$ by (AA$_7$), there exists $\widetilde{t} \in \operatorname{Def}_\xi(M)$ mapping to (t_n). After replacing $\widetilde{\xi}$ with $\widetilde{t} \cdot \widetilde{\xi}$, we can arrange that $\widetilde{\xi}|_{B_n}$ and ξ_n are isomorphic for each n.

We now show that each restriction $\widetilde{\xi}|_{B_0[M_n]} \in \operatorname{Def}_\xi(M_n)$ under the composition $\widetilde{B} \to B_0[M] \to B_0[M_n]$ is the trivial deformation. Indeed, the map $M = \ker(\widetilde{B} \to B_0) \to \ker(B_0[M_n] \to B_0) = M_n$ induces a map $\operatorname{Def}_\xi(M) \to \operatorname{Def}_\xi(M_n)$ on deformation modules which under the identification $\operatorname{Def}_\xi(M) \xrightarrow{\sim} \prod_n \operatorname{Def}_\xi(M_n)$ of (AA$_7$) sends an element $(\eta_0, \eta_1, \ldots)$ to $(\eta_{n+1}|_{B_n} - \eta_n)$. The ring map $\widetilde{B} \to B_0[M_n]$ also induces a map $\operatorname{Lift}_\xi(\widetilde{B}) \to \operatorname{Lift}_\xi(B_0[M_n])$ which is equivariant with respect to

---

[3] The hypotheses of (AA$_7$) can be weakened to only require the injectivity of $\operatorname{Ob}_\xi(M) \hookrightarrow \prod_n \operatorname{Ob}_\xi(M_n)$ although in practice one usually verifies that this map is bijective.

$\mathrm{Def}_\xi(M) \to \mathrm{Def}_\xi(M_n)$. It follows that the image of $\widetilde{\xi}$ in $\mathrm{Lift}_\xi(B_0[M_n]) = \mathrm{Def}_\xi(M_n)$ is $\xi_{n+1}|_{B_n} - \xi_n = 0$.

The existence of $\widehat{\xi} \in \mathcal{X}(\widehat{B})$ extending $(\xi_n) \in \varprojlim \mathcal{X}(B_n)$ now follows from applying the identity $\widehat{B} = \widetilde{B} \times_{B_0[M]} B_0$ and strong homogeneity (AA$_4$) to the diagram

$$\begin{tikzcd}
\operatorname{Spec} B_0[M] \arrow[r] \arrow[d, hook] & \operatorname{Spec} B_0 \arrow[d, hook] \arrow[ddr, bend left, "\xi_0"] & \\
\operatorname{Spec} \widetilde{B} \arrow[r] \arrow[drr, bend right, "\widetilde{\xi}"'] & \operatorname{Spec} \widehat{B} \arrow[dr, dashed, "\widehat{\xi}"] & \\
& & \mathcal{X}
\end{tikzcd}$$

Step 3. We now use the versality of $\xi_0 \colon U_0 \to \mathcal{X}$ at u_0 to arrive at a contradiction. Since $\mathcal{X}$ is limit preserving (AA$_1$), there exists a finitely generated $\Bbbk$-subalgebra $B' \subset \widehat{B}$ and an object $\xi' \in \mathcal{X}(B')$ together with an isomorphism $\widehat{\xi} \xrightarrow{\sim} \xi|_{\widehat{B}}$. After possibly enlarging B', we may assume that the composition $B' \hookrightarrow \widehat{B} \to B_0$ is surjective. There is thus a closed immersion $U_0 \hookrightarrow U' := \operatorname{Spec} B'$ and we can consider the commutative diagram

$$\begin{tikzcd}
U_0 \arrow[dr, "i"] \arrow[drr, hook, bend left] \arrow[ddr, bend right, "\mathrm{id}"'] & & \\
& U_0 \times_{\mathcal{X}} U' \arrow[r] \arrow[d] & U' = \operatorname{Spec} B' \arrow[d, "\xi'"] \\
& U_0 \arrow[r, "\xi_0"] & \mathcal{X}.
\end{tikzcd}$$

where the fiber product $U_0 \times_{\mathcal{X}} U'$ is an algebraic space locally of finite type over $\Bbbk$. Since $\xi_0 \colon U_0 \to \mathcal{X}$ is versal at u_0, it follows from the artinian version of the Infinitesimal Lifting Criterion for Smoothness (Smooth Equivalences 8.4) that $U_0 \times_{\mathcal{X}} U' \to U'$ is smooth at $i(u_0)$. After replacing U_0 with an open affine neighborhood of u_0, U' with the corresponding open and $\{u_1, u_2, \ldots\}$ with an infinite subsequence contained in this open, we can arrange that $U_0 \times_{\mathcal{X}} U' \to U'$ is smooth. The non-artinian version of the Infinitesimal Lifting Criterion for Smoothness implies the section of $U_0 \times_{\mathcal{X}} U' \to U'$ over U_0 extends to a global section $U' \to U_0 \times_{\mathcal{X}} U'$. This implies that ξ' is the trivial deformation of ξ_0, contradicting our choice of $\xi' \colon U' \to \mathcal{X}$.

Our exposition follows [SP, Tag 0CYF] and [Hal17, Thm. A]. See also [Art74, Thm. 5.3] and [HR19, Main Thm.]. ☐

Remark 7.5. The converse of the theorem also holds. For the necessity of the conditions, we only need to check (AA$_3$), (AA$_4$), (AA$_6$) and (AA$_7$). Condition (AA$_3$) (finiteness of the tangent spaces) holds as $\mathcal{X}$ is of finite type over $\Bbbk$. The strong homogeneity condition (AA$_4$) holds by [SP, Tag 07WN]. Condition (AA$_6$) (existence of an obstruction theory) follows from the existence of a cotangent complex $L_{\mathcal{X}/\Bbbk}$ for $\mathcal{X}$ satisfying properties analogous to Theorem 5.1; see [Ols06]. If $\xi\colon \operatorname{Spec} A \to \mathcal{X}$ is a morphism from a finitely generated $\Bbbk$-algebra A and I is an A-module, then we set $\mathrm{Ob}_\xi(I) := \mathrm{Ext}^1_A(\xi^* L_{\mathcal{X}/\Bbbk}, I)$. Property (AA$_6$)(b) holds as a consequence of [Ols06, Thm. 1.5], a generalization of [Ill71, III.2.2.4] (which was discussed in Remark 5.12) from morphisms of schemes to representable morphisms of algebraic stacks. Finally, Condition (AA$_7$) ($\mathrm{Def}_\xi(-)$ and $\mathrm{Ob}_\xi(-)$ commutes with products) follows from cohomology and base change.

7.2 Verifying Artin's Axioms

Theorem 7.6. *Each of the stacks* $\mathrm{Hilb}^P(X)$, $\mathscr{M}_g$ *(with* $g \geq 2$*) and* $\mathcal{B}\mathrm{un}_C$ *over* $(Sch/\Bbbk)_{\text{ét}}$ *are algebraic stacks locally of finite type over* $\Bbbk$.

Proof. We check condition the conditions of Theorem 7.4. Condition (AA$_1$) (limit preserving) was verified in Lemma 6.1. For (AA$_{3'}$), the finite dimensionality of the vector spaces $\mathrm{Def}_\xi(\Bbbk)$ and $\mathrm{Aut}_\xi(\Bbbk)$ for an object $\xi \in \mathcal{X}(\Bbbk)$ can be identified with:

- $\mathrm{H}^0(Z, N_{Z/X})$ and $\{0\}$ for $\xi = [Z \subset X] \in \mathrm{Hilb}^P(X)(\xi)$ (Proposition 1.4),
- $\mathrm{H}^1(C, T_C)$ and $\mathrm{H}^0(C, T_C)$ for $[C] \in \mathscr{M}_g(\Bbbk)$ (Lemma 1.10 and Proposition 1.11) and
- $\mathrm{Ext}^1_{\mathcal{O}_C}(E, E)$ and $\mathrm{Ext}^0_{\mathcal{O}_C}(E, E)$ for $[E] \in \mathcal{B}\mathrm{un}_C(\Bbbk)$ (Proposition 1.15).

Condition (AA$_4$) (the strong homogeneity condition of (RS$_4^*$)) was checked in Proposition 3.14. Condition (AA$_5$) (effectivity) was checked in Corollary 4.9 as a consequence of Grothendieck's Existence Theorem. For Condition (AA$_6$), we define obstruction theories as follows: for a finitely generated $\Bbbk$-algebra A and an A-module M, we set

- $\mathrm{Ob}_\xi(M) := \mathrm{H}^1(Z, T_{Z/X_A} \otimes_A M)$ for $\xi = [Z \subset X_A] \in \mathrm{Hilb}^P(X)(A)$,
- $\mathrm{Ob}_\xi(M) := \mathrm{H}^2(\mathcal{C}, T_{\mathcal{C}/A} \otimes_A M) = 0$ for $\xi = [\mathcal{C} \to \operatorname{Spec} A] \in \mathscr{M}_g(A)$, and
- $\mathrm{Ob}_\xi(M) := \mathrm{H}^2(C_A, \mathscr{E}nd_{\mathcal{O}_{C_A}}(E) \otimes_A M) = 0$ for $\xi = [E] \in \mathcal{B}\mathrm{un}_C(A)$.

Property (AA$_6$)(a) holds for these obstruction theories as a consequence of Propositions 2.2, 2.5 and 2.10; these results also show that $\mathrm{Aut}_\xi(M)$ and $\mathrm{Def}_\xi(M)$ are identified with the analogous cohomology groups. Condition (AA$_{7'}$) ($\mathrm{Aut}_\xi(-)$, $\mathrm{Def}_\xi(-)$ and $\mathrm{Ob}_\xi(-)$ commutes with products) follows from cohomology and base change (Corollary 8.19). □

8 Appendix

8.1 Properties of flatness and smoothness

Theorem 8.1 (Local Criterion for Flatness)**.** *Let* $A' \twoheadrightarrow A$ *be a surjective homomorphism of noetherian rings with kernel* I *such that* $I^2 = 0$*. An* A'*-module* M' *is flat over* A' *if and only if*

(1) $M := M \otimes_{A'} A$ *is flat over* A*; and*

(2) the map $M \otimes_A I \to M'$ *is injective.*

Remark 8.2. Applying this with $A' = \Bbbk[\epsilon]/(\epsilon^2)$ being the dual numbers and $A = \Bbbk$, we recover the fact that an A'-module M' is flat if and only if $M \otimes_{\Bbbk[\epsilon]/(\epsilon^2)} \Bbbk \xrightarrow{\epsilon} M$ is injective. This also follows from the fact that a module N over a ring B is flat if and only if for every ideal $I \subset B$, the map $I \otimes_B M \to M$ is injective [SP, Tag 00HD], and using that the only ideal in $\Bbbk[\epsilon]/(\epsilon^2)$ is (ϵ).

Proposition 8.3 (Flatness Criterion over Artinian Rings)**.** *A module over an artinian ring is flat if and only if it is free.*

Smooth Equivalences 8.4. Let $f\colon X \to Y$ be morphism of (resp. noetherian) schemes locally of finite presentation. The following are equivalent:

(1) f is smooth;

(2) f satisfies the *Infinitesimal Lifting Criterion for Smoothness* (sometimes referred to as *formal smoothness*): for any surjection $A \to A_0$ of rings with nilpotent kernel (resp. surjection $A \to A_0$ of local artinian rings whose kernel is isomorphic to the residue field $A/\mathfrak{m}_A$) and any commutative diagram

$$\begin{tikzcd}
\operatorname{Spec} A_0 \arrow[r] \arrow[d, hook] & X \arrow[d, "f"] \\
\operatorname{Spec} A \arrow[r] \arrow[ur, dashed] & Y
\end{tikzcd}$$

of solid arrows, there exists a dotted arrow filling in the diagram;

(3) f satisfies the *Jacobi Criterion for Smoothness*: for every point $x \in X$, there exist affine open neighborhoods $\operatorname{Spec} B$ of $f(x)$ and $\operatorname{Spec} A \subset f^{-1}(\operatorname{Spec} B)$ of x and an A-algebra isomorphism

$$B \cong \big(A[x_1, \dots, x_n]/(f_1, \dots, f_r)\big)_g$$

for some $f_1, \dots, f_r, g \in A[x_1, \dots, x_n]$ with $r \leq n$ such that the determinant $\det(\frac{\delta f_j}{\delta x_i})_{1 \leq i,j \leq r} \in B$ of the Jacobi matrix, defined by the partial derivatives with respect to the *first* r x_i's, is a unit.

If in addition X and Y are locally of finite type over an algebraically closed field $\Bbbk$, then the above are equivalent to:

(4) for all $x \in X(\Bbbk)$, there is an isomorphism $\widehat{\mathcal{O}}_{X,x} \cong \widehat{\mathcal{O}}_{Y,y}[[x_1, \dots, x_r]]$ of $\widehat{\mathcal{O}}_{Y,y}$-algebras.

8.2 Pushouts

Pushouts are the dual notion of fiber product. Unlike fiber products, pushouts may not exist. However, Ferrand showed that they often exist when one of the maps is a closed immersion and the other is an affine morphism.

Theorem 8.5 (Ferrand's Theorem on the Existence of Pushouts)**.** *Consider a diagram*

$$\begin{tikzcd}
X_0 \arrow[r, hook, "i"] \arrow[d, "f_0"'] & X \arrow[d, dashed, "f"] \\
Y_0 \arrow[r, hook, dashed, "j"] & Y
\end{tikzcd} \tag{8.1}$$

of schemes where $i\colon X_0 \hookrightarrow X$ is a closed immersion and $f_0\colon X_0 \to Y_0$ is affine. If

($\star$) *for any point $y_0 \in Y_0$, the subspace $f_0^{-1}(\operatorname{Spec} \mathcal{O}_{Y_0,y_0}) \subset X_0$ has a basis of open affine neighborhoods of X,*

then there exists a closed immersion $j\colon Y_0 \hookrightarrow Y$ and an affine morphism $f\colon X \to Y$ of schemes such that (8.1) is cocartesian (i.e. a pushout). Moreover, we have the following properties:

(a) *the square (8.1) is cartesian, $X \to Y$ restricts to an isomorphism $X \setminus X_0 \to Y \setminus Y_0$ and the induced map $X \coprod Y_0 \to Y$ is universally submersive;*

(b) *the induced map*
$$\mathcal{O}_Y \to j_*\mathcal{O}_{Y_0} \times_{(j\circ f_0)_*\mathcal{O}_{X_0}} f_*\mathcal{O}_X$$
is an isomorphism of sheaves; and

(c) *if f_0 is finite (resp. integral), then so is f. In this case, Condition ($\star$) can be replaced with the condition that every finite set of points in X_0 and Y is contained in an open affine (resp. for every $y_0 \in Y_0$, $f_0^{-1}(y_0)$ is contained in an open affine). Finally if X_0, X and Y_0 are of finite type over a noetherian scheme, then so is Y.*

Proof. See [Fer03, Thm. 5.4 and 7.1] and [SP, Tag 0ECH]. ☐

Example 8.6 (Affine case)**.** In the affine case where $X = \operatorname{Spec} A$, $X_0 = \operatorname{Spec} A_0$, $Y_0 = \operatorname{Spec} B_0$, then $\operatorname{Spec}(A \times_{A_0} B_0)$ is the pushout $X \coprod_{X_0} Y_0$.

Example 8.7 (Gluing and pinching)**.** If $X_0 \hookrightarrow X$ and $X_0 \hookrightarrow Y_0$ are closed immersions, the pushout $X \coprod_{X_0} Y_0$ can be viewed as the gluing of X and Y_0 along X_0. For example, the nodal curve $\operatorname{Spec} k[x,y]/xy$ is the union of $\mathbb{A}^1$ and $\mathbb{A}^1$ along their origins. If $X_0 = Z \sqcup Z$ is the union of two isomorphic disjoint subschemes of X and $X_0 \to Z$ is the projection, then the pushout $X \coprod_{Z \sqcup Z} Z$ can viewed as the pinching of the two copies of Z in X. For example, the nodal cubic curve is the pinching of 0 and ∞ in $\mathbb{P}^1$.

Example 8.8 (Non-noetherianness)**.** When $f_0\colon X_0 \to Y_0$ is affine but not finite, then the pushout $X \coprod_{X_0} Y_0$ is often not noetherian. For example, if $X_0 = V(x) \subset X = \mathbb{A}^2_{\Bbbk}$ and $f_0\colon X_0 \to \operatorname{Spec} \Bbbk$, the pushout is the non-noetherian affine scheme defined by
$$k[x,y] \times_{k[x]} k = k[x, xy, xy^2, xy^3, \dots] \subset k[x,y].$$
On the other hand, we wouldn't expect a finite type pushout as one cannot contract the y-axis in $\mathbb{A}^2_{\Bbbk}$.

Given a fiber product diagram of rings
$$\begin{tikzcd} B \arrow[r] \arrow[d, two heads] & A \arrow[d, two heads] \\ B_0 \arrow[r] & A_0 \end{tikzcd}$$
with $A \twoheadrightarrow A_0$ surjective and $B := B_0 \times_{A_0} A$, the fiber product $\operatorname{Mod}(B_0) \times_{\operatorname{Mod}(A_0)} \operatorname{Mod}(A)$ is the category of triples (N_0, M, α) consisting of a B_0-module N_0, an A-module M and isomorphism $\alpha\colon N_0 \otimes_{B_0} A_0 \xrightarrow{\sim} M \otimes_A A_0$. Equivalently, an object

is a diagram

$$\begin{tikzcd}
 & & M \arrow[dl] \arrow[dd, dash] \\
N_0 \arrow[r] \arrow[dd, dash] & M_0 \arrow[dd, dash] & \\
 & & A \arrow[dl] \\
B_0 \arrow[r] & A_0 &
\end{tikzcd} \tag{8.2}$$

where N_0, M_0 and M are modules over B_0, A_0 and A, and the maps $N_0 \to M_0$ and $M \to M_0$ are morphisms of B_0 and A-modules inducing isomorphisms $N_0 \otimes_{B_0} A_0 \to M_0$ and $M \otimes_A A_0 \to M_0$.

We define functors

$$\mathrm{Mod}(B) \underset{L}{\overset{R}{\leftrightarrows}} \mathrm{Mod}(B_0) \times_{\mathrm{Mod}(A_0)} \mathrm{Mod}(A) \tag{8.3}$$

where for a B-module N, $L(N) := (N \otimes_B B_0, N \otimes_B A, \alpha)$ with α being the canonical isomorphism $(N \otimes_B B_0) \otimes_{B_0} A_0 \xrightarrow{\sim} (N \otimes_B A) \otimes_A A_0$. For an object (N_0, M, α) corresponding to a diagram (8.2), we define $R(N_0, M, \alpha) := N_0 \times_{M_0} M$, which we can view as:

$$\begin{tikzcd}
 & N_0 \times_{M_0} M \arrow[rr] \arrow[dl] \arrow[dd, dash] & & M \arrow[dl] \arrow[dd, dash] \\
N_0 \arrow[rr] \arrow[dd, dash] & & M_0 \arrow[dd, dash] & \\
 & B_0 \times_{A_0} A \arrow[rr] \arrow[dl] & & A \arrow[dl] \\
B_0 \arrow[rr] & & A_0. &
\end{tikzcd}$$

Proposition 8.9. *The functors L and R restrict to an equivalence on the full subcategories of flat (resp. finite) modules.*

Corollary 8.10. *Consider a commutative cube of schemes*

$$\begin{tikzcd}
 & X'_0 \arrow[rr, hook] \arrow[dl] \arrow[dd] & & X' \arrow[dl] \arrow[dd] \\
X_0 \arrow[rr, hook] \arrow[dd] & & X \arrow[dd] & \\
 & Y'_0 \arrow[rr, hook] \arrow[dl] & & Y' \arrow[dl] \\
Y_0 \arrow[rr, hook] & & Y &
\end{tikzcd}$$

of schemes where $X_0 \hookrightarrow X$ is a closed immersion and $X_0 \to Y_0$ is affine.

(1) *Assume that $Y' \to Y$ is a flat morphism of schemes and X'_0, Y'_0 and X' are the base changes under $Y' \to Y$ (i.e. the bottom, left, top and right faces are cartesian).*

   (a) *If the front face is a pushout, then so is the back face and the natural functor*

$$\mathrm{QCoh}(Y') \to \mathrm{QCoh}(Y'_0) \times_{\mathrm{QCoh}(X'_0)} \mathrm{QCoh}(X'),$$

   restricts to an equivalence on the full subcategories of $\mathrm{QCoh}(Y')$, $\mathrm{QCoh}(Y'_0)$ *and* $\mathrm{QCoh}(X')$ *finitely presented $\mathcal{O}$-modules flat over Y', Y'_0 and X'.*

   (b) *If in addition $Y' \to Y$ is faithfully flat and locally of finite presentation, then back face being a pushout implies that the front face is as well.*

(2) *If the top and left faces are cartesian, and the front and back faces are pushouts, then all faces are cartesian. Moreover, if $Y_0' \to Y_0$ and $X' \to X$ are étale (resp. smooth, flat), then so is $Y' \to Y$.*

8.3 Gerbes

Definition 8.11 (Gerbes)**.** A stack $\mathcal{X}$ over a site $\mathcal{S}$ is called a *gerbe* if

(1) for every object $T \in \mathcal{S}$, there exists a covering $\{T_i \to T\}$ in $\mathcal{S}$ such that each fiber category $\mathcal{X}(T_i)$ is non-empty; and

(2) for objects $x, y \in \mathcal{X}$ over $T \in \mathcal{S}$, there exists a covering $\{T_i \to T\}$ and isomorphisms $x|_{T_i} \xrightarrow{\sim} y|_{T_i}$ for each i.

A *morphism of gerbes* is defined as a morphism of stacks, and is necessarily an isomorphism.

We say that a gerbe $\mathcal{X}$ is *abelian* if $\underline{\mathrm{Aut}}_T(x)$ is a sheaf of abelian groups for every object $x \in \mathcal{X}$ over $T \in \mathcal{S}$.

We say that a gerbe $\mathcal{X}$ is *trivial* if there is a section $\mathcal{S} \to \mathcal{X}$ of $\mathcal{X} \to \mathcal{S}$; if $\mathcal{S}$ has a final object X, then this is equivalent to the existence of an element of $\mathcal{X}(X)$.

Definition 8.12 (Smooth gerbes)**.** We say that a morphism $\mathcal{X} \to \mathcal{Y}$ of algebraic stacks is a *smooth gerbe* if there exists a smooth presentation $V \to \mathcal{Y}$ such that $\mathcal{X} \times_{\mathcal{Y}} V \cong \mathbf{B}G$ for a smooth group algebraic space $G \to V$.

We say that an algebraic stack $\mathcal{X}$ is a *smooth gerbe* if there exists an algebraic space X and a morphism $\mathcal{X} \to X$ which is a smooth gerbe.

A smooth gerbe $\mathcal{X} \to \mathcal{Y}$ is *trivial* if there exists a section.

Remark 8.13. Sometimes the group algebraic space $G \to V$ will be pulled back from a base scheme $H \to S$ and in such cases, we can think of a smooth gerbe as an algebraic $\mathcal{X}$ over $\mathcal{Y}$ which is smooth locally on $\mathcal{Y}$ isomorphic to $\mathbf{B}H \times \mathcal{Y}$.

Definition 8.14 (Banded G-gerbes)**.** Let $G \to S$ be a smooth *commutative* group algebraic space. We say that a morphism $\mathcal{X} \to \mathcal{Y}$ of algebraic stacks over S is a *gerbe banded by G* or *G-banded gerbe* if $\mathcal{X} \to \mathcal{Y}$ is a smooth gerbe together with the data of isomorphisms $\psi_x \colon G|_T \to \underline{\mathrm{Aut}}_T(x)$ of sheaves for each object $x \in \mathcal{X}$ over an S-scheme T such that for each isomorphism $\alpha \colon x \xrightarrow{\sim} y$ over T, the diagram

$$\begin{tikzcd} & G|_T \arrow[dl, "\psi_x"'] \arrow[dr, "\psi_y"] & \\ \underline{\mathrm{Aut}}_T(x) \arrow[rr, "\mathrm{Inn}_\alpha"] & & \underline{\mathrm{Aut}}_T(y). \end{tikzcd} \tag{8.4}$$

commutes, where $\mathrm{Inn}_\alpha(\tau) = \alpha\tau\alpha^{-1}$.

The compatibility condition (8.4) ensures that if $z \colon \mathcal{Z} \to \mathcal{X}$ any morphism of algebraic stacks, then there is functorial isomorphism $\phi_z \colon G|_{\mathcal{Z}} \xrightarrow{\sim} \mathrm{Aut}_{\mathcal{Z}}(z)$ of sheaves of groups on $(\mathrm{Sch}/\mathcal{Z})_{\text{ét}}$. In particular, taking the identity morphism $\mathrm{id} \colon \mathcal{X} \to \mathcal{X}$, there is isomorphism $G|_{\mathcal{X}} \xrightarrow{\sim} I_{\mathcal{X}}$.

Remark 8.15. If G is an sheaf of *abelian* groups on a site $\mathcal{S}$, then the cohomology groups $H^1(\mathcal{S}, G)$ and $H^2(\mathcal{S}, G)$ classify isomorphism classes of G-torsors and banded

G-gerbes. When G is non-abelian, then one can define *non-abelian cohomology pointed sets* $H^1(\mathcal{S}, G)$ and $H^2(\mathcal{S}, G)$ as isomorphism classes of G-torsors and banded G-gerbes and these sets have equivalent descriptions in terms of cocycles. Moreover, given a short exact sequence of sheaves of groups $1 \to G' \to G \to G'' \to 1$ with G' abelian and central in G, then there is an induced long exact sequence on the ith cohomology groups for $i \leq 2$. See [Gir71], [Mil80, pp. 144-145] and [Ols16, §12].

8.4 Artin Approximation

Definition 8.16. A noetherian local ring A is called a *G-ring* if the homomorphism $A \to \widehat{A}$ is geometrically regular.

Theorem 8.17 (Artin Approximation)**.** *Let S be a scheme and $s \in S$ a point such that $\mathcal{O}_{S,s}$ is a G-ring (Definition 8.16), e.g. a scheme of finite type over a field or $\mathbb{Z}$. Let*

$$F\colon \mathsf{Sch}/S \to \mathsf{Sets}$$

be a limit preserving contravariant functor and $\widehat{\xi} \in F(\operatorname{Spec} \widehat{\mathcal{O}}_{S,s})$. For any integer $N \geq 0$, there exist an étale morphism

$$(S', s') \to (S, s) \qquad \textit{and} \qquad \xi' \in F(S')$$

with $\kappa(s) = \kappa(s')$ such that the restrictions of $\widehat{\xi}$ and ξ' to $\operatorname{Spec}(\mathcal{O}_{S,s}/\mathfrak{m}_s^{N+1})$ are equal.

Lemma 8.18. *Let $(A, \mathfrak{m}_A) \to (B, \mathfrak{m}_B)$ be a local homomorphism of noetherian complete local rings. If $A/\mathfrak{m}_A^2 \to B/\mathfrak{m}_B^2$ is surjective, so is $A \to B$. If in addition $A = B$, then $A \to B$ is an isomorphism.*

8.5 Cohomology and Base Change

We have used the following consequence of Cohomology and Base Change:

Corollary 8.19. *Let $X \to \operatorname{Spec} A$ be a proper morphism of noetherian schemes and F be a coherent sheaf on X which is flat over A. Then the functor*

$$\mathrm{H}^i(X, F \otimes_A -)\colon \operatorname{Mod}(A) \to \operatorname{Mod}(A)$$

commutes with products.

References

[AM69] M. F. Atiyah and I. G. Macdonald, *Introduction to commutative algebra*, Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1969.

[And67] Michel André, *Méthode simpliciale en algèbre homologique et algèbre commutative*, Lecture Notes in Mathematics, Vol. 32, Springer-Verlag, Berlin-New York, 1967.

[Art69] M. Artin, *Algebraization of formal moduli. I*, Global Analysis (Papers in Honor of K. Kodaira), Univ. Tokyo Press, Tokyo, 1969, pp. 21–71.

[Art74] ———, *Versal deformations and algebraic stacks*, Invent. Math. **27** (1974), 165–189.

[CJ02] B. Conrad and A. J. de Jong, *Approximation of versal deformations*, J. Algebra **255** (2002), no. 2, 489–515.

[Del81] P. Deligne, *Relèvement des surfaces $K3$ en caractéristique nulle*, Algebraic surfaces (Orsay, 1976–78), Lecture Notes in Math., vol. 868, Springer, Berlin-New York, 1981, Prepared for publication by Luc Illusie, pp. 58–79.

[EGA] A. Grothendieck, *Éléments de géométrie algébrique*, I.H.E.S. Publ. Math. **4, 8, 11, 17, 20, 24, 28, 32** (1960, 1961, 1961, 1963, 1964, 1965, 1966, 1967).

[Eis95] D. Eisenbud, *Commutative algebra*, Graduate Texts in Mathematics, vol. 150, Springer-Verlag, New York, 1995, With a view toward algebraic geometry.

[Fer03] D. Ferrand, *Conducteur, descente et pincement*, Bull. Soc. Math. France **131** (2003), no. 4, 553–585.

[FGI$^+$05] Barbara Fantechi, Lothar Göttsche, Luc Illusie, Steven L. Kleiman, Nitin Nitsure, and Angelo Vistoli, *Fundamental algebraic geometry*, Mathematical Surveys and Monographs, vol. 123, American Mathematical Society, Providence, RI, 2005, Grothendieck's FGA explained.

[Gir71] Jean Giraud, *Cohomologie non abélienne*, Springer-Verlag, Berlin-New York, 1971, Die Grundlehren der mathematischen Wissenschaften, Band 179.

[Hal17] Jack Hall, *Openness of versality via coherent functors*, J. Reine Angew. Math. **722** (2017), 137–182.

[Har77] Robin Hartshorne, *Algebraic geometry*, Springer-Verlag, New York-Heidelberg, 1977, Graduate Texts in Mathematics, No. 52.

[Har10] ———, *Deformation theory*, Graduate Texts in Mathematics, vol. 257, Springer, New York, 2010.

[HR19] Jack Hall and David Rydh, *Artin's criteria for algebraicity revisited*, Algebra Number Theory **13** (2019), no. 4, 749–796.

[Ill71] L. Illusie, *Complexe cotangent et déformations. I*, Lecture Notes in Mathematics, Vol. 239, Springer-Verlag, Berlin, 1971.

[LMB] G. Laumon and L. Moret-Bailly, *Champs algébriques*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge., vol. 39, Springer-Verlag, Berlin, 2000.

[LS67] S. Lichtenbaum and M. Schlessinger, *The cotangent complex of a morphism*, Trans. Amer. Math. Soc. **128** (1967), 41–70.

[Mil80] James S. Milne, *Étale cohomology*, Princeton Mathematical Series, vol. 33, Princeton University Press, Princeton, N.J., 1980.

[Mum69] David Mumford, *Bi-extensions of formal groups*, Algebraic Geometry (Internat. Colloq., Tata Inst. Fund. Res., Bombay, 1968), Oxford Univ. Press, London, 1969, pp. 307–322.

[Ols06] M. Olsson, *Deformation theory of representable morphisms of algebraic stacks*, Math. Z. **253** (2006), no. 1, 25–62.

[Ols16] Martin Olsson, *Algebraic spaces and stacks*, American Mathematical Society Colloquium Publications, vol. 62, American Mathematical Society, Providence, RI, 2016.

[Oor71] Frans Oort, *Finite group schemes, local moduli for abelian varieties, and lifting problems*, Compositio Math. **23** (1971), 265–296.

[Qui68] Daniel Quillen, *Homology of commutative rings*, Unpublished, 1968, pp. 1–81.

[Qui70] ———, *On the (co-) homology of commutative rings*, Applications of Categorical Algebra (Proc. Sympos. Pure Math., Vol. XVII, New York, 1968), Amer. Math. Soc., Providence, R.I., 1970, pp. 65–87.

[Sch68] Michael Schlessinger, *Functors of Artin rings*, Trans. Amer. Math. Soc. **130** (1968), 208–222.

[Ser06] E. Sernesi, *Deformations of algebraic schemes*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 334, Springer-Verlag, Berlin, 2006.

[SGA4] *Théorie des topos et cohomologie étale des schémas.*, Lecture Notes in Mathematics, Vol. 269, 270, 305, Springer-Verlag, Berlin, 1972, Séminaire de Géométrie Algébrique du Bois-Marie 1963–1964 (SGA 4), Dirigé par M. Artin, A. Grothendieck, et J. L. Verdier. Avec la collaboration de N. Bourbaki, P. Deligne et B. Saint-Donat.

[SGA7-I] Alexander Grothendieck, Michel Raynaud, and Dock Sang Rim, *Groupes de monodromie en géométrie algébrique. I*, Lecture Notes in Mathematics, Vol. 288, Springer-Verlag, 1972, Séminaire de Géométrie Algébrique du Bois-Marie 1967–1969 (SGA 7 I).

[SP] The Stacks Project Authors, *Stacks Project*, `http://stacks.math.columbia.edu`, 2020.