Cohomology and Base Change with Applications

Lecture notes for Math 581J, *working draft*

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1 Cohomology and Base Change

Given a proper morphism $f\colon X \to Y$ of noetherian schemes and a coherent sheaf F on X, we would like to know:

(a) When is $R^i f_* F$ a vector bundle on Y?

(b) For a morphism of schemes $Y' \to Y$ inducing a cartesian diagram

$$\begin{tikzcd} X_{Y'} \arrow[r, "g'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y, \end{tikzcd}$$

when is the comparison map

$$\phi^i_{Y'} \colon g^* R^i f_* F \to R^i f'_* g'^* F \tag{1.1}$$

an isomorphism?

When $f\colon X \to Y$ is flat, Flat Base Change tells us that (1.1) is always an isomorphism. Cohomology and Base Change provides an answer when F is flat over Y.

Cohomology and Base Change is an essential tool in moduli theory. It can be applied to verify properties of families of objects and construct vector bundles on moduli spaces. For instance, for a family $\pi\colon \mathcal{C} \to S$ of smooth curves, we can verify that $\pi_* \Omega^{\otimes k}_{\mathcal{C}/S}$ is a vector bundle for $k > 0$ and that its constructions commutes with base change on S (Proposition 1.9). This in turn can be applied to show that $\mathcal{C}$ embeds canonically into $\mathbb{P}_S(\pi_* \Omega^{\otimes k}_{\mathcal{C}/S})$ allowing us to verify the algebraicity of $\mathcal{M}_g$ (**??**). Applying this result to the universal family $\pi\colon \mathcal{U}_g \to \mathcal{M}_g$ yields vector bundles $\pi_* \Omega^{\otimes k}_{\mathcal{U}_g/\mathcal{M}_g}$ on $\mathcal{M}_g$; when $k = 1$, this is a vector bundle of rank g called the *Hodge bundle*.

1.1 Algebraic input

The key algebraic input to Cohomology and Base Change is the following:

Theorem 1.1. *Let $f\colon X \to \operatorname{Spec} A$ be a proper morphism of noetherian schemes and F be a coherent sheaf on X which is flat over A. There is a complex*

$$K^\bullet\colon 0 \to K^0 \to K^1 \to \cdots \to K^n \to 0$$

of finitely generated, projective A-modules such that $H^i(X,F) = H^i(K^\bullet)$ for all i.

Moreover, for any morphism $\operatorname{Spec} B \to \operatorname{Spec} A$ of schemes, $H^i(X_B, F_B) = H^i(K^\bullet \otimes_A B)$ where $X_B := X \times_{\operatorname{Spec} A} \operatorname{Spec} B$ and F_B is the pullback of F to X_B.

Proof. See [Mum70, p.46] or [Vak17, 28.2.1]. This is established by choosing a finite affine cover $\{U_i\}$ of X and considering the corresponding alternating Céch complex $C^\bullet$ on $\{U_i\}$ with coefficients in F. Then $C^\bullet$ is a finite complex of free (but not finitely generated) A-modules and $H^i(X,F) = H^i(C^\bullet)$. The result is then obtained by inductively refining $C^\bullet$ to build a finite complex $K^\bullet$ of finitely generated, projective A-modules which is quasi-isomorphic to $C^\bullet$. $\square$

Remark 1.2 (Perfect complexes)**.** A bounded complex $K^\bullet$ of coherent sheaves on a noetherian scheme X is *perfect* if there is an affine cover $X = \bigcup_i U_i$ such that each $K^\bullet|_{U_i}$ is quasi-isomorphic to a bounded complex of vector bundles on U_i. (By a vector bundle, we mean a locally free sheaf of finite rank—this is equivalent to the corresponding module on $\Gamma(U_i, \mathcal{O}_{U_i})$) to be finitely generated, projective.) If X is affine (resp. has the resolution property, i.e. every coherent sheaf is the quotient of a vector bundle), $K^\bullet$ is perfect if and only if it is quasi-isomorphic to a bounded complex of vector bundles on X [SP, Tag 066Y] (resp. [SP, Tag 0F8F]). Moreover, the compact objects in $D_{\mathrm{QCoh}}(X)$ are precisely the perfect complexes [SP, Tag 09M8].

With this terminology in place, Theorem 1.1 has the following translation: $Rf_*F \in D^b_{\mathrm{Coh}}(\operatorname{Spec} A)$ is perfect [SP, Tag 07VK]. More generally, if $F^\bullet$ is a perfect complex on X, then $Rf_*F^\bullet$ is also perfect [SP, Tag 0A1H].

1.2 Theorems of Semicontinuity, Grauert and Cohomology and Base Change

Theorem 1.1 tells us for a proper morphism $X \to \operatorname{Spec} A$ and coherent sheaf F on X flat over A, the cohomology $H^i(X,F)$ can be computed using a perfect complex $K^\bullet$. Since Zariski-locally on the base, the complex $K^\bullet$ is a finite complex of *free* objects, this reduces cohomological questions to linear algebra.

The Semicontinuity Theorem is a direct consequence of Theorem 1.1; see [Mum70, p. 50], [Har77, Thm. 12.8] or [Vak17, 28.2.4].

Theorem 1.3 (Semicontinuity Theorem)**.** *Let $f\colon X \to Y$ be a proper morphism of noetherian schemes and F be a coherent sheaf on X which is flat over Y.*

(1) For each $i \geq 0$, the function

$$Y \to \mathbb{Z}, \quad y \mapsto H^i(X_y, F_y)$$

is upper semicontinuous.

(2) *The function*

$$Y \to \mathbb{Z}, \quad y \mapsto \sum_{i=0}^{\infty} H^i(X_y, F_y)$$

is locally constant.

When the base scheme is reduced, Grauert's Theorem provides a criterion for when the higher pushforward sheaves $R^i f_* F$ are vector bundles.

Theorem 1.4 (Grauert's Theorem)**.** *Let $f\colon X \to Y$ be a proper morphism of noetherian schemes and let F be a coherent sheaf on X which is flat over Y. Assume that Y is reduced and connected. For each integer i, the following are equivalent:*

(1) *the function $y \mapsto h^i(X_y, F_y)$ is constant; and*

(2) *$R^i f_* F$ is a vector bundle and the comparison map*

$$\phi^i_y\colon R^i f_* F \otimes \kappa(y) \to H^i(X_y, F_y)$$

is an isomorphism for all $y \in Y$.

If these conditions hold, then we have the following additional properties:

(a) *for all morphisms $g\colon Y' \to Y$ of schemes, the comparison map $\phi^i_{Y'}\colon g^* R^p f_* F \to R^i f'_* g'^* F$ of (1.1) is an isomorphism; and*

(b) *The comparison map $\phi^{i-1}_y\colon R^{i-1} f_* F \otimes \kappa(y) \to H^{i-1}(X_y, F_y)$ is an isomorphism.*

Proof. See [Mum70, p.51-2], [Har77, Cor. 12.9] and [Vak17, 28.1.5]. □

Grauert's Theorem is proved by using that $Rf_* F$ is a perfect complex and a somewhat involved linear algebra argument to show that $R^i f_* F \otimes \kappa(y)$ have constant dimension. Since Y is reduced, this implies that $R^i f_* F$ is a vector bundle. When Y is not reduced, the local criterion for flatness can be leveraged to provide the following useful criteria for (a) when the comparison maps ϕ^i_y are isomorphisms and (b) when $R^i f_* F$ is a vector bundle.

Theorem 1.5 (Cohomology and Base Change)**.** *Let $f\colon X \to Y$ be a proper and finitely presented morphism of schemes, and let F be a finitely presented sheaf on X which is flat over Y. Suppose that for a point $y \in Y$ and integer i, the comparison map $\phi^i_y\colon R^i f_* F \otimes \kappa(y) \to H^i(X_y, F_y)$ is surjective. Then the following hold*

(a) *There is an open neighborhood $V \subset Y$ of y such that for any morphism $Y' \to V$ of schemes, the comparison map $\phi^i_{Y'}$ of (1.1) is an isomorphism. In particular, ϕ^i_p is an isomorphism.*

(b) *ϕ^{i-1}_p is surjective if and only if $R^i f_* F$ is a vector bundle in an open neighborhood of p.*

Proof. See [EGA, III.7.7.5, III.7.7.10, III.7.8.4], [Har77, Thm. 12.11] and [Vak17, 28.1.6]. □

Remark 1.6. For moduli-theoretic applications, it is important to be able to apply Cohomology and Base Change in the non-noetherian setting. Using the methods of Noetherian Approximation from §**??**, it is not hard hard to see how the general statement follows from the noetherian version. Since the statement is local on Y, we can assume Y is affine and we can write $Y = \lim_i Y_i$ as a limit of affine schemes of finite type over $\mathbb{Z}$. Since $X \to Y$ is finitely presented, there exists an index 0 and a finitely presented morphism $X_0 \to Y_0$ such that $X \cong X_0 \times_{Y_0} Y$ (**??**). For each $i > 0$, we can define $X_i = X_0 \times_{Y_0} Y_i$ and we have $X \cong X_i \times_{Y_i} Y$. By **??**, $X_i \to Y_i$ is proper for $i \gg 0$. By **????**, there exist an index j and a coherent sheaf F_j on X_j that pullsback to F under $X \to X_j$. For $i \geq j$, set F_i to be the pullback of F_j under $X_i \to X_j$. By **????**, F_i is flat over Y_i for $i \gg 0$. We may know apply noetherian Cohomology and Base Change to the data of $X_i \to Y_i$ and F_i for $i \gg 0$, and we may deduce the same properties for $X \to Y$ and F under the base change $Y \to Y_i$.

Corollary 1.7. *Let $f\colon X \to Y$ be a proper morphism of noetherian schemes and let F be a coherent sheaf on X which is flat over Y. The following are equivalent:*

(1) $H^i(X_y, F_y) = 0$ for all $y \in Y$ and $i > 0$; and

(2) $R^i f_* F = 0$ for all i and $f_* F$ is a vector bundle whose construction commutes with base change on Y (i.e. for all morphisms $g\colon Y' \to Y$ of schemes, the comparison map $\phi^0_{Y'}\colon g^* f_* F \to f'_* g'^* F$ of (1.1) is an isomorphism).

Proof. The implication (2) $\implies$ (1) is clear. For the converse, since $\phi^i_y\colon R^i f_* F \otimes \kappa(y) \to H^i(X_y, F_y) = 0$ is surjective for all $y \in Y$ and $i > 0$, Cohomology and Base Change (Theorem 1.5(a)) implies that each ϕ^i_y is an isomorphism and it follows that $R^i f_* F = 0$ for $i > 0$. We now apply Cohomology and Base Change three times: Theorem 1.5(b) with $i = 1$ implies that ϕ^0_y is surjective for all $y \in Y$, Theorem 1.5(b) with $i = 0$ (as ϕ^{-1}_y is necessarily surjective) implies that $f_* F$ is a vector bundle and Theorem 1.5(a) with $i = 0$ implies that the construction of $f_* F$ commutes with base change on Y. □

In each of the above statements, the flatness of F over Y was crucial. Without flatness, one can still sometimes have base change results in special cases. The following result is sometimes quite useful:

Proposition 1.8. *Let $f\colon X \to Y$ be a projective morphism of noetherian schemes, let $\mathcal{O}_X(1)$ be a relatively ample line bundle on X, and let be a coherent sheaf on X. For any morphism of schemes $g\colon Y' \to Y$, the comparison map $\phi^0_{Y'}\colon g^* f_* F(d) \to f'_* g'^* F(d)$ of (1.1) is an isomorphism for $d \gg 0$.*

Proof. TO ADD □

1.3 Applications to moduli theory

Here is a typical application of Cohomology and Base Change to moduli theory. The following proposition is used to establish properties of smooth families of curves (**??**) and its argument applies in the same way to families of stable curves (**??**).

Proposition 1.9. *Let $\pi\colon \mathcal{C} \to S$ be a family of smooth curves of genus $g \geq 2$ (i.e. $\mathcal{C} \to S$ is a smooth, proper morphism of schemes such that every geometric fiber is a connected curve of genus g). Then*

(1) $\pi_* \mathcal{O}_{\mathcal{C}} = \mathcal{O}_S$;

(2) For $k > 1$, the pushforward $\pi_*(\Omega_{\mathcal{C}/S}^{\otimes k})$ is a vector bundle of rank $(2k-1)(g-1)$ whose construction commutes with base change on S and $R^i\pi_*(\Omega_{\mathcal{C}/S}^{\otimes k}) = 0$ for $i > 0$.

(3) The pushforward $\pi_*(\Omega_{\mathcal{C}/S})$ is a vector bundle of rank g whose construction commutes with base change on S and $R^1\pi_*(\Omega_{\mathcal{C}/S}) \cong \mathcal{O}_S$ while $R^i\pi_*(\Omega_{\mathcal{C}/S}) \cong \mathcal{O}_S = 0$ for $i \geq 2$.

Proof. To see (1), observe that the global functions $H^0(\mathcal{C}_s, \mathcal{O}_{\mathcal{C}_s}) = \kappa(s)$ are constants since $\mathcal{C}_s$ is proper and geometrically connected. It follows that $\phi_s^0 \colon \pi_*\mathcal{O}_{\mathcal{C}} \otimes \kappa(s) \to H^0(\mathcal{C}_s, \mathcal{O}_{\mathcal{C}_s})$ is surjective. Cohomology and Base Change (Theorem 1.5(a)-(b) with $i = 0$) implies that ϕ_s^0 is an isomorphism and that $\pi_*\mathcal{O}_{\mathcal{C}}$ is a line bundle. On a fiber over $s \in S$, the natural map $\mathcal{O}_S \to \pi_*\mathcal{O}_{\mathcal{C}}$ induces a surjective map $\kappa(s) \to \pi_*\mathcal{O}_{\mathcal{C}} \otimes \kappa(s)$ (as post-composing with $\pi_*\mathcal{O}_{\mathcal{C}} \otimes \kappa(s) \xrightarrow{\phi_s^0} H^0(\mathcal{C}_s, \mathcal{O}_{\mathcal{C}_s}) = \kappa(s)$ is the identity). Thus $\mathcal{O}_S \to \pi_*\mathcal{O}_{\mathcal{C}}$ is surjective morphism of line bundles, hence an isomorphism.

For (2) with $k > 1$, $H^1(\mathcal{C}_s, \Omega_{\mathcal{C}_s/\kappa(s)}^{\otimes k}) = H^0(\mathcal{C}_s, \Omega_{\mathcal{C}_s/\kappa(s)}^{\otimes(1-k)})$ for all $s \in S$ by Serre–Duality (**??**) and this vanishes as $\deg(\Omega_{\mathcal{C}_s/\kappa(s)}^{\otimes(1-k)}) < 0$. Note that we also have the vanishing of $H^i(\mathcal{C}_s, , \Omega_{\mathcal{C}_s/\kappa(s)}^{\otimes k}))$ for $i \geq 2$ since $\dim \mathcal{C}_s = 1$. Cohomology and Base Change (Theorem 1.5(a)) gives the vanishing of the higher pushforward $R^i\pi_*(\Omega_{\mathcal{C}/S}^{\otimes k}) = 0$ for $i > 0$. On the other hand, $h^0(\mathcal{C}_s, \Omega_{\mathcal{C}_s/\kappa(s)}^{\otimes k}) = \deg(\Omega_{\mathcal{C}_s/\kappa(s)}^{\otimes k}) + 1 - g = (2k-1)(g-1)$ by Easy Riemann–Roch (**??**). Corollary 1.7 implies that $\pi_*(\Omega_{\mathcal{C}/S}^{\otimes k})$ is a vector bundle of rank $(2k-1)(g-1)$.

For (3), observe that since $\Omega_{\mathcal{C}/S}$ is a relative dualizing sheaf, Grothendieck–Serre Duality implies that $R^1\pi_*\Omega_{\mathcal{C}/S} \cong \pi_*\mathcal{O}_{\mathcal{C}}$ and this is identified with $\mathcal{O}_S$ by (1). For $i \geq 2$, $H^i(\mathcal{C}_s, \Omega_{\mathcal{C}/S} \otimes \kappa(s)) = 0$ and Cohomology and Base Change (Theorem 1.5(a)) implies that $R^i\pi_*\Omega_{\mathcal{C}/S} = 0$. Cohomology and Base Change (Theorem 1.5(b) with $i = 2$) implies that $\phi_s^1 \colon R^1\pi_*\Omega_{\mathcal{C}/S} \otimes \kappa(s) \to H^1(\mathcal{C}_s, \Omega_{\mathcal{C}_s/\kappa(s)})$ is surjective for any $s \in S$ and thus an isomorphism (Theorem 1.5(a) with $i = 1$). Since $R^1\pi_*\Omega_{\mathcal{C}/S} \cong \pi_*\mathcal{O}_{\mathcal{C}} \cong \mathcal{O}_S$ is a line bundle, applying Theorem 1.5(b) with $i = 1$ implies that $\phi_s^0 \colon \pi_*\Omega_{\mathcal{C}/S} \otimes \kappa(s) \to H^0(\mathcal{C}_s, \Omega_{\mathcal{C}_s/\kappa(s)})$ is surjective and applying Theorem 1.5(a)-(b) with $i = 0$ implies that $\pi_*\Omega_{\mathcal{C}/S}$ is a vector bundle of rank $h^0(\mathcal{C}_s, \Omega_{\mathcal{C}_s/\kappa(s)}) = g$ whose construction commutes with base change. □

1.4 Applications to line bundles

Given a flat, proper morphism $f \colon X \to Y$, when is a line bundle L on X the pullback of a line bundle on Y? More generally, is there a largest subscheme $Z \subset Y$ where L_Z on $X_Z = X \times_Y Z$ is the pullback of a line bundle on Z? In this section, we provide three answers in increasing complexity.

As we will need to impose conditions on the fibers X_y, we first discuss relationships between various conditions.

Lemma 1.10. *Let $f \colon X \to Y$ be a flat, proper morphism of noetherian schemes. Consider the following conditions:*

(1) the geometric fibers of $f \colon X \to Y$ are connected and reduced;

(2) $h^0(X_y, \mathcal{O}_{X_y}) = 1$ for all $y \in Y$; and

(3) $\mathcal{O}_Y = f_*\mathcal{O}_X$ and this holds after arbitrary base change (i.e. $\mathcal{O}_T = f_{T,*}\mathcal{O}_{X_T}$ for a morphism $T \to Y$ of schemes).

Then (1) $\implies$ (2) $\iff$ (3).

Proof. If (1) holds, then $H^0(X_y\mathcal{O}_{X_y}) \otimes_{\kappa(y)} \overline{\kappa(y)} = H^0(X \times_Y \overline{\kappa(y)}, \mathcal{O}_{X\times_Y\overline{\kappa(y)}}) = \overline{\kappa(y)}$ by Flat Base Change and the fact that a connected, reduced and proper scheme over an algebraically closed field has only constant functions. This gives (2).

If (2) holds, then the comparison map $\phi_y^0\colon f_*\mathcal{O}_X \otimes \kappa(y) \to H^0(X_y, \mathcal{O}_{X_y}) = \kappa(y)$ is necessarily surjective as we have the global section $1 \in H^0(Y, f_*\mathcal{O}_X)$. Theorem 1.5 (with $i = 0$) implies that $f_*\mathcal{O}_X$ is a line bundle and that $\mathcal{O}_Y \to f_*\mathcal{O}_X$ is a surjection of line bundles, hence an isomorphism. Since the same argument applies to the base change $X_T \to T$, this gives (3). The converse (3) $\implies$ (2) follows by consider the base change $T = \operatorname{Spec}\kappa(y) \to Y$. $\square$

When Y is reduced, Grauert's Theorem provides a complete answer to when a line bundle is a pullback.

Proposition 1.11 (Version 1). *Let $f\colon X \to Y$ be a flat, proper morphism of noetherian schemes such that $h^0(X_y, \mathcal{O}_{X_y}) = 1$ for all $y \in Y$. Let L be a line bundle on X. If Y is reduced, then $L = f^*M$ for a line bundle M on Y if and only if L_y is trivial for all $y \in Y$. Moreover, if these conditions hold, then $M = f_*L$ and the adjunction morphism $f^*f_*L \to L$ is an isomorphism.*

Proof. The condition on geometric fibers implies that $h^0(X_y, L_y) = 1$ and Grauert's Theorem (Theorem 1.4) implies that f_*L is a line bundle and that $f_*L \otimes \kappa(y) \xrightarrow{\sim} H^0(X_y, L_y)$ is an isomorphism. We claim that $f^*f_*L \to L$ is surjective. It suffices to show that $(f^*f_*L)|_{X_y} \to L|_{X_y}$ is surjective. Denoting $f_y\colon X_y \to \operatorname{Spec}\kappa(y)$, we have identifications $(f^*f_*L)|_{X_y} = f_y^*(f_*L \otimes \kappa(y)) = f_y^*(H^0(\widetilde{X_y}, L_y)) = \mathcal{O}_{X_y}$ and the claim follows. Since $f^*f_*L \to L$ is a surjection of line bundles, it is an isomorphism. $\square$

Exercise 1.12. Show that if Y is a connected and reduced noetherian scheme and E is a vector bundle, then $\operatorname{Pic}(\mathbb{P}_Y(E)) = \operatorname{Pic}(Y) \times \mathbb{Z}$. See also [Har77, Exer. III.12.5].

Proposition 1.13 (Version 2). *Let $f\colon X \to Y$ be a flat, proper morphism of noetherian schemes such that the geometric fibers are integral. For a line bundle L on X, the locus*

$$\{y \in Y \mid L_y \text{ is trivial}\} \subset Y$$

is closed.

Proof. The important observation here is that for a geometrically integral and proper scheme Z over field $\Bbbk$, a line bundle M is trivial if and only if $h^0(Z, M) > 0$ and $h^0(Z, M^\vee) > 0$. To see that the latter condition is sufficient, observe that we have a non-zero homomorphisms $\mathcal{O}_Z \to M$ and $\mathcal{O}_Z \to M^\vee$, the latter of which dualizes to a non-zero map $M \to \mathcal{O}_Z$. Since Z is integral, the composition $\mathcal{O}_Z \to M \to \mathcal{O}_Z$ is also non-zero and is defined by a constant in $H^0(Z, \mathcal{O}_Z) = \Bbbk$. It follows that $M \to \mathcal{O}_Z$ is a surjective map of line bundles, hence an isomorphism. By the Semicontinuity Theorem (Theorem 1.3) the condition that $h^0(X_y, L_y) > 0$ and $h^0(X_y, L_y^\vee) > 0$ are each closed, and the statement follows. See also [Mum70, p. 51]. $\square$

Remark 1.14. If the geometric fibers are only connected and reduced, the locus may fail to be closed. For example consider a family of smooth curves $f\colon X \to Y$ where Y is a curve and X is a smooth surface. For a closed point $x \in X$, consider the blow-up $\mathrm{Bl}_x X \to X$ and let E be the exceptional divisor. Then $\mathrm{Bl}_x X \to Y$ is a flat, proper morphism and the fiber over $f(x) \in Y$ is connected and reduced, but reducible. The line bundle $L = \mathcal{O}_{\mathrm{Bl}_x X}(E)$ has the property that the fiber L_y is trivial if and only if $y \neq f(x)$.

The two versions above can be combined to the following powerful statement for a flat, proper morphism $X \to Y$. For moduli-theoretic applications, it is essential that we allow the possibly that Y is non-reduced and that the fibres X_y be reducible. The proposition will be applied in the text to show that the locus of curves C in a Hilbert scheme $\mathrm{Hilb}^P(\mathbb{P}^{5g-6}/\mathbb{Z})$ which are tri-canonically is a closed condition.

Proposition 1.15 (Version 3)**.** *Let $f\colon X \to Y$ be a flat, proper morphism of noetherian schemes such that $h^0(X_y, \mathcal{O}_{X_y}) = 1$ for all $y \in Y$ (resp. the geometric fibers are integral). For a line bundle L on X, there is a unique locally closed (resp. closed) subscheme $Z \subset Y$ such that*

(1) L_Z on $X_Z = X \times_Y Z$ is the pullback of a line bundle on Z; and

(2) if $T \to Y$ is any morphism of schemes such that L_T on X_T is the pullback of a line bundle on T, then $T \to Y$ factors through Z.

Remark 1.16. In other words, the functor

$$\mathrm{Sch}/Y \to \mathrm{Sets},$$

$$(T \to Y) \mapsto \begin{cases} \{*\} & \text{if } L_T \text{ is the pullback of a line bundle on } T \\ \emptyset & \text{otherwise} \end{cases}$$

is representable by a closed subscheme of Y.

Proof. By the Semicontinuity Theorem (Theorem 1.3), the locus $V = \{y \in Y \mid h^0(X_y, L_y) \leq 1\}$ is open. Since for points $y \notin V$, L_y is not trivial, we may replace Y with V and assume that $h^0(X_y, L_y) \leq 1$ for all $y \in Y$.

Observe that if $L = f^*M$ for a line bundle M on Y, then by using the projection formula and the fact that $\mathcal{O}_Y = f_*\mathcal{O}_X$ (Lemma 1.10), we see that $f_*L \cong f_*f^*M \cong f_*f^*\mathcal{O}_Y \otimes M \cong f_*\mathcal{O}_X \otimes M \cong M$ is a line bundle and that the adjunction map $f^*f_*L \to L$ is an isomorphism. The same holds for the base change $X_T \to T$, and we conclude that L_T is a pullback of a line bundle on T if and only if $f_{T,*}L$ is a line bundle and $f_T^*f_{T,*}L \to L$ is an isomorphism. This latter condition is Zariski-local on Y. We see therefore that the question is Zariski-local on Y and T. We will show that any point $y \in Y$ has an open neighborhood where the proposition holds.

By applying Theorem 1.1 and after replacing Y with an open affine neighborhood of y, we may assume that there is a homomorphism $d\colon A^{r_0} \xrightarrow{d} A^{r_1}$ of finitely generated and free A-modules such that for any morphism $\operatorname{Spec} B \to \operatorname{Spec} A$, $H^0(X_B, L_B) = \ker(d \otimes_A B)$. Consider the dual $d^\vee$ of d, we define M as the cokernel in the sequence

$$A^{r_1} \xrightarrow{d^\vee} A^{r_0} \to M \to 0.$$

For any ring homomorphism $A \to B$, we have a right exact sequence

$$B^{r_1} \xrightarrow{d^\vee \otimes_A B} B^{r_0} \to M \otimes_A B \to 0$$

which after applying the contravariant left-exact functor $\mathrm{Hom}_B(-, B)$ becomes

$$0 \to \mathrm{Hom}_B(M \otimes_A B, B) \to B^{r_0} \xrightarrow{d \otimes_A B} B^{r_1}.$$

We conclude that

$$H^0(X_B, L_B) = \mathrm{Hom}_B(M \otimes_A B, B) = \mathrm{Hom}_A(M, B). \tag{1.2}$$

Applying this to $A \to \kappa(y)$ for any $y \in \operatorname{Spec} A$, we have $H^0(X_y, L_y) = \mathrm{Hom}_A(M, \kappa(y)) = M \otimes_A \kappa(y)$.

If $h^0(X_y, L_y) = 0$, then L_y is not trivial and there is an open neighborhood U of y such that $\widetilde{M}|_U = 0$. The proposition holds over U since there are no morphisms $T \to U$ from a non-empty scheme such that L_T is a pullback. On the other hand, if $h^0(X_y, L_y) = 1$, then $M \otimes_A \kappa(y) = \kappa(y)$ and by Nakayama's lemma, after replacing Y with an open affine neighborhood of y, there is a surjection $A \to M$. Write $M = A/I$ for an ideal I and define the closed subscheme $Z = V(I) \subset Y$. Observe that $H^0(Z, L_Z) = \mathrm{Hom}_A(A/I, A/I) = A/I$ so that $f_{Z,*}L_Z$ is the trivial line bundle. For an A/I-algebra B, we have that $H^0(X_B, L_B) = \mathrm{Hom}_A(A/I, B) = B$. It follows that the comparison map $H^0(X_Z, L_Z) \otimes_{A/I} B \to H^0(X_B, L_B)$ is an isomorphism, or in other words the construction of $f_{Z,*}L_Z$ commutes with base change.

We claim that $T \to Y$ factors through Z if and only if $f_{T,*}L_T$ is a line bundle. This question is Zariski-local on T so we may assume $T = \operatorname{Spec} B$ is affine. If $f_{T,*}L_T$ is a line bundle, we may assume $f_{T,*}L_T = \mathcal{O}_T$ is trivial since the question is local on T. Then $B = \mathrm{Hom}_A(A/I, B)$ implies that $I \subset \ker(A \to B)$ or in other words that $A \to B$ factors as $A \to A/I \to B$.

Finally, considering the adjunction morphism $\lambda\colon f_Z^* f_{Z,*}L_Z \to L_Z$ on X_Z, we claim that for $y \in Z$, L_y is trivial if and only if $\lambda|_{X_y}$ is surjective. If $\lambda|_{X_y}$ is surjective, then using that $f_{Z,*}L_Z = \mathcal{O}_Z$, we have a surjection $\mathcal{O}_{X_y} \to L_y$ of line bundles, hence an isomorphism. For converse, observe that since $f_{Z,*}L_Z$ commutes with base change, the comparison map $f_{Z,*}L_Z \otimes \kappa(y) = H^0(X_y, L_y)$ is an isomorphism. Denoting $f_y\colon X_y \to \operatorname{Spec} \kappa(y)$, we have identifications $(f_Z^* f_{Z,*}L_Z)|_{X_y} = f_y^*(f_{Z,*}L_Z \otimes \kappa(y)) = f_y^* f_{y,*}L_y$ under which $\lambda|_{X_y}$ corresponds to the adjunction map $f_y^* f_{y,*}L_y \to L_y$ which is an isomorphism. Replacing Z with $Z \setminus \operatorname{Supp}(\operatorname{coker}(\lambda))$ establishes the proposition in the case that $h^0(X_y, \mathcal{O}_{X_y}) = 1$ for all $y \in Y$. If the fibers are geometrically integral, then Proposition 1.13 implies that Z is closed.

See also [Mum70, p. 90], [Vie95, Lem. 1.19] and [SP, Tags 0BEZ and 0BF0]. $\square$

Remark 1.17. Note that to prove the strongest version, we needed the strongest version of our various cohomology and base change results, namely Theorem 1.1.

Remark 1.18. For a flat, proper morphism $X \to S$, define the *Picard functor* as

$$\mathrm{Pic}_{X/S}\colon \mathrm{Sch}/S \to \mathrm{Sets}, \quad T \mapsto \mathrm{Pic}(X_T)/f_T^* \mathrm{Pic}(T).$$

If $f\colon X \to S$ has geometrically integral fibers, then the existence of a closed subscheme $Z \subset Y$ characterized by Proposition 1.15 is equivalent to the diagonal morphism $\mathrm{Pic}_{X/S} \to \mathrm{Pic}_{X/S} \times_S \mathrm{Pic}_{X/S}$ of presheaves over Sch/S being representable by closed immersions, i.e. $\mathrm{Pic}_{X/S}$ is separated over S.

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