MATH509 Algebraic Structures III

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Contents

0.1 Course Outline

0.1.1 Potential Topics

- regular local rings, regular sequences
- toric varieties
- algebraic groups
- geometric invariant theory
- flatness, deformation theory
- formal schemes, ind-schemes
- (rigid) analytic spaces
- intersection theory
- positivity in algebraic geometry

0.1.2 Assignments

Weekly reflections of 3 things:

- 2 research things (theorems, facts, ...)
- 1 teaching/exposition/communication thing.

Class presentation

- 30-minute lecture.

Chapter 1

All Things Regular

Useful reference

David Eisenbud, *Commutative Algebra with a View towards Algebraic Geometry* [Eis95]

1.1 March 31: Regular Local Rings, Auslander-Buchsbaum

"Math is the art of giving the same name to different things."

— Henri Poincaré

1.1.1 Regular Local Rings

Definition 1.1. A noetherian local ring $(R, \mathfrak{m}, \kappa)$ is **regular** if $\dim_{\mathrm{Krull}} R = \dim_\kappa \mathfrak{m}/\mathfrak{m}^2$. A ring R is regular if for all $\mathfrak{p} \in \operatorname{Spec} R$, $R_\mathfrak{p}$ is a noetherian regular local ring.

M: We should create a cheat-sheet for dimension theory

Problem 1.2.

1. Find a non-noetherian ring R such that every $R_\mathfrak{p}$ is noetherian.
2. Find a noetherian ring R with $\dim R = \infty$, but every localization $R_\mathfrak{p}$ has finite Krull dimension.

Example 1.3.

1. Any field k.

2. The local rings $k[x_1, \dots, x_n]_{(x_1,\dots,x_n)}$ and $kx_1, \dots, x_n$.
3. The ring $k[x_1, \dots, x_n]$.

Example 1.4. Non-examples include $k[x, y]/(y^2)$ and $k[x, y]/xy$. The problem is we are modding out by elements in $\mathfrak{m}^2$.

Cheerful Fact

$$\dim R/I = \dim R/\sqrt{I}.$$

Main goal over the next 2 weeks:

Theorem 1.5 (**Auslander - Buchsbaum**; Serre)**.** *Let $(R, \mathfrak{m})$ be a noetherian regular local ring. Then, the following are equivalent:*

1. *The ring R is regular.*
2. *Every finitely generated R-module M has a finite free resolution*

$$0 \to R^{\oplus n_d} \to \cdots \to R^{\oplus n_1} \to R^{\oplus n_0} \to M \to 0. \tag{1.1.1}$$

3. *The R-module $\kappa = R/\mathfrak{m}$ has a finite free resolution.*

Moreover, if R is regular and there is a resolution

$$0 \to K_d \to R^{\oplus n_{d-1}} \to \cdots \to R^{\oplus n_0} \to M \to 0, \tag{1.1.2}$$

then the kernel K_d is free.

Remark. The number d in Equation (1.1.1) is called the **projective dimension** of M.

Example 1.6.

1. Any field k. Everything is automatically free.
2. The local ring $k[x]_{(x)}$. The residue field $\kappa \cong k$ has a finite free resolution

$$\begin{tikzcd}
0 \arrow[r] & k[x]_{(x)} \arrow[r, "\cdot x"] \arrow[d, "\cong"'] & k[x]_{(x)} \arrow[r] & k \arrow[r] & 0 \\
& (x) = \ker(\cdot x) \arrow[ur] & & &
\end{tikzcd}$$

so $k[x]_{(x)}$ is regular by (3) of Theorem 1.5.

3. The local ring $R = k[x,y]_{(x,y)}$. The residue field k has a finite free resolution

$$0 \longrightarrow R \xrightarrow{\begin{pmatrix} -y \\ x \end{pmatrix}} R^{\oplus 2} \xrightarrow{(x\ \ y)} R \longrightarrow k \longrightarrow 0$$

so R is regular by (3).

Example 1.7. Non-example. Let $R = k[\varepsilon]/(\varepsilon^2)$. (Some authors may simply write $k[\varepsilon]$). The obvious free resolution

$$\begin{tikzcd}
\cdots \arrow[r, "\cdot\varepsilon"] & R \arrow[r, "\cdot\varepsilon"] \arrow[d] & R \arrow[r, "\cdot\varepsilon"] \arrow[d] & R \arrow[r] & k \arrow[r] & 0 \\
(\varepsilon) \arrow[ur] & (\varepsilon) \arrow[ur] & (\varepsilon) \cong k \arrow[ur] & & &
\end{tikzcd}$$

doesn't terminate.

1.1.2 Applications of Auslander-Buchsbaum

Corollary 1.8. *If $(R, \mathfrak{m})$ is a noetherian regular local ring and $\mathfrak{p} \in \operatorname{Spec} R$ is a prime ideal, then $(R_\mathfrak{p}, \mathfrak{p}R_\mathfrak{p})$ is regular.*

Proof

Since R is regular, by item 2 of Auslander-Buchsbaum, there is a finite free resolution

$$0 \to R^{\oplus n_d} \to \cdots \to R^{\oplus n_0} \to R/\mathfrak{p} \to 0.$$

Since localization is flat, tensoring with $R_\mathfrak{p}$ is exact. Therefore, there is a finite free resolution

$$0 \to R_\mathfrak{p}^{\oplus n_d} \to \cdots \to R_\mathfrak{p}^{\oplus n_0} \to k \to 0,$$

so $R_\mathfrak{p}$ is regular by item 3 of Theorem 1.5. ■

Corollary 1.9 (*Faithfully flat descent*)**.** *Let $R \to R'$ be a faithfully flat homomorphism of local rings. If R' is regular, then R is regular.*

Proof

Outline.

(1) If R' is noetherian, then R is noetherian.

(2) Let M be a finitely generated R-module. If $M \otimes_R R'$ is free, then M is free.

(3) Let $d = \dim R$. Choose a resolution

$$0 \to K_d \to R^{\oplus n_d} \to \cdots \to R^{\oplus n_0} \to R/\mathfrak{m} \to 0.$$

Faithful flatness implies that tensoring with R' is exact, so we get a resolution

$$0 \to (K_d)_{R'} \to (R')^{\oplus n_d} \to \cdots \to (R')^{\oplus n_0} \to (R')/\mathfrak{m} \to 0.$$

By item 3 of Theorem 1.5, $(K_d)_{R'}$ is free, so K_d is also free.

■

1.2 April 2: Regular Sequences, Cohen-Macaulay

I think some intuition leaks out in every step of an induction proof.
— Jim Propp, January 22 2000 at an AMS Special Session

Cheerful Fact

Some high-powered versions of Auslander-Buchsbaum.

1. If X is a smooth quasiprojective scheme, then every coherent sheaf has a finite resolution by vector bundles, so there is an isomorphism $K_0(X) \xrightarrow{\cong} G_0(X)$.
2. If X is smooth, then the bounded derived category $D^b(\underline{\mathrm{Coh}}_X)$ is isomorphic to the category of perfect complexes $\underline{\mathrm{Perf}_X}$ (i.e. locally free resolutions).

1.2.1 Regular Sequences

Definition 1.10. Let R be a ring and M be an R-module. An element $x \in R$ is a **nonzero-divisor** on M if the map $M \xrightarrow{\cdot x} M$ is injective.

Remark. This is nuanced when R is non-reduced!

Definition 1.11. An (*ordered*) sequence of elements $x_1, x_2, \dots, x_n$ is an M-regular sequence if

(1) The element x_i is a nonzero-divisor for $M/(x_1, \dots, x_{i-1})$.

(2) One has $M \neq (x_1, \dots, x_n)M$.

Remark. The second item is to exclude constants. By Nakayama's Lemma, (2) is also automatic for a local ring $(R, \mathfrak{m})$.

Example 1.12.

1. Any sequence in R is regular for $R^{\oplus 1}$.
2. The sequence $2, x \in \mathbb{Z}[x]$ is regular for $\mathbb{Z}[x]$.

Example 1.13. Non-examples.

1. The ring $k[x,y]/(xy, y^2)$ has no nonzero-divisors aside from field constants, so it has no regular sequences.
2. For the ring $k[x,y,z]$, the sequence $x, y(1-x), z(1-x)$ is regular, while the reordered sequence $y(1-x), z(1-x), x$ is not. Indeed, $z(1-x)$ vanishes on

the $(1-x)$ component of $y(1-x)$, so it doesn't 'cut down the correct number of dimensions.'

Cheerful Fact

(To be proven.) If $(R, \mathfrak{m})$ is local noetherian, then regularity does not depend on the ordering of the sequence. Furthermore, any two maximal regular sequences have the same length.

Definition 1.14. Let $(R, \mathfrak{m})$ be a noetherian local ring and M a finitely generated R-module. The **depth** of M is

$$\operatorname{depth}_R(M) := \max\{d \mid \text{there is a regular sequence } x_1, \ldots, x_d\}.$$

Definition 1.15. The local ring $(R, \mathfrak{m})$ is **Cohen-Macaulay (CM)** if $\dim R = \operatorname{depth}_R(R)$. A Noetherian ring S is CM if all its localizations at primes are CM.

Remark. In general, one has $\operatorname{depth} R \leqslant \dim R$.

Example 1.16.

1. The polynomial ring $k[x_1, \ldots, x_n]$ is CM.
2. The ring $\mathbb{Z}[x]$.
3. The non-reduced ring $k[x,y]/(xy, y^2)$ is not CM since it has dimension 1 but depth 0.
4. The ring $k[x,y]/(xy)$ is CM. In general, $k[x_1, \ldots, x_n]/(f)$ is CM – one first uses the Hauptidealsatz to show that the dimension is $n-1$. Then, one shows there is a nonzerodivisor.
5. Artinian local rings are CM: $\dim R = 0 \implies \operatorname{depth} R = 0$.

Proposition 1.17. *Let $(R, \mathfrak{m})$ be local noetherian.*

1. *If $x_1, \ldots, x_n$ is a regular sequence and $R/(x_1, \ldots, x_n)$ is regular, then R is regular.*
2. *If R is regular and $x_1, \ldots, x_i \in \mathfrak{m}$ is linearly independent in $\mathfrak{m}/\mathfrak{m}^2$, the $x_1, \ldots, x_i$ is a regular sequence and $R/(x_1, \ldots, x_i)$ is regular. (Nakayama: a basis of $\mathfrak{m}/\mathfrak{m}^2$ lifts to a set of generators for $\mathfrak{m}$.)*
3. *A sequence $x_1, \ldots, x_n$ is regular iff $x_1^d, \ldots, x_n^d$ is regular.*

1.2.2 Regular Sequence vs. System of Parameters

Definition 1.18. Let $(R, \mathfrak{m})$ be a noetherian local ring with Krull dimension d. A system of parameters for R is an (*unordered*) set of elements $x_1, \ldots, x_d$ satisfying either of the following conditions:

(1) The ideal $\mathfrak{m}$ is a minimal prime over $(x_1, \ldots, x_d)$.

(2) The radical $\sqrt{(x_1, \ldots, x_d)} = \mathfrak{m}$.

(3) The ideal $(x_1, \ldots, x_d)$ is $\mathfrak{m}$-*primary*, i.e. the module $R/(x_1, \ldots, x_d)$ has finite length.

Proposition 1.19. *If the noetherian local ring R is CM, then every system of parameters is a regular sequence. cf. Stack Exchange*

1.2.3 Associated Graded Modules

Let R be a ring, I an ideal, and M an R-module. There is a filtration $M \supseteq IM \supseteq I^2M \supseteq \cdots$, and we define the associated graded module to be

$$\operatorname{gr}_I M := (M/I) \oplus (IM/I^2M) \oplus \cdots = \bigoplus_{k \geqslant 0} (I^k M / I^{k+1} M).$$

Proposition 1.20. *Let R be a ring, $x_1, \ldots, x_n$ a regular sequence, and $I := (x_1, \ldots, x_n)$. Then, the map*

$$\varphi : (R/I)[y_1, \ldots, y_n] \to \operatorname{gr}_I R, y_i \mapsto \overline{x}_i \in I/I^2$$

is an isomorphism.

Proof

Proof by induction on n. ■

Remark. The above can be viewed as a flat degeneration technique, where $\operatorname{Spec} R$ degenerates into $\operatorname{Spec} \operatorname{gr}_I R$. Namely, there is a flat family $\mathscr{X} \to \mathbb{A}^1$ where the generic fiber is $\operatorname{Spec} R$ but the special fiber is $\operatorname{Spec} \operatorname{gr}_I R$.

Jarod's guess: $\mathscr{X} = \operatorname{Spec}(R \oplus I \oplus I^2 \oplus \cdots)$ Rees blow up algebra. Correct?

Corollary 1.21. *If I is generated by the regular sequence $x_1, \ldots, x_n$, then the module I^d/I^{d+1} is free of rank $\binom{n+d-1}{d}$. Geometrically, if a subscheme $Z \subseteq X$ is locally defined by regular sequences, then $\mathcal{N}_{Z/X}$ is a vector bundle.*

First notice that the associated graded ring $\operatorname{gr}_I M$ is the symmetric algebra $\operatorname{Sym}_{R/I} I/I^2$. We know the sheaf version of I/I^2 is the conormal bundle $\mathcal{N}^\vee_{Z/X} = \mathcal{I}_{Z/X}/\mathcal{I}^2_{Z/X}$. Therefore, the sheaf version of the associated graded ring w.r.t an ideal is the structure sheaf of the conormal bundle $\operatorname{Spec}\left(\operatorname{Sym}_X \mathcal{N}^\vee_{Z/X}\right)$.

Proposition 1.22. *If $(R, \mathfrak{m})$ is a regular local ring then R is a domain.*

To prove this, we will require the following lemma.

Lemma 1.23. *If $(R, \mathfrak{m})$ is a regular local ring with $\kappa = R/\mathfrak{m}$, then $\operatorname{gr}_{\mathfrak{m}} R \cong \kappa[x_1, \ldots, x_n]$.*

Proof

The notetaker is indebted to Soham Ghosh for the following argument. Let $n = \dim R = \dim_\kappa \mathfrak{m}/\mathfrak{m}^2$. The result is evident when $n = 0$. For $n > 0$, we first show that R contains a nonzero divisor x_1 that is not a unit. Assume by way of contradiction that every element of $\mathfrak{m}$ is a zero divisor. Then, $\mathfrak{m}$ is covered by the associated primes of the zero ideal [Sta25, Section 00L9]

$$\mathfrak{m} = \bigcup_{\mathfrak{p}_i \in \operatorname{Ass}(0)} \mathfrak{p}_i, \tag{1.2.1}$$

and there are finitely many associated primes. By Prime Avoidance Lemma [Sta25, Lemma 00DS], Equation (1.2.1) cannot be true unless there is only one associated prime $\{\mathfrak{p}\} = \operatorname{Ass}(0)$ for which $\mathfrak{m} = \mathfrak{p}$. This implies $\mathfrak{m}$ is a minimal prime, so $\dim R = 0$ – contradiction! Therefore, we can always find a nonzero divisor $x_1 \in \mathfrak{m}$. Take such an $x_1 \in \mathfrak{m}$ and let $\overline{x}_1 \in \mathfrak{m}/\mathfrak{m}^2$ be its image. Choose a basis $\overline{x}_1 \ldots, \overline{x}_n$ for $\mathfrak{m}/\mathfrak{m}^2$. By Nakayama's Lemma, this basis lifts to a set of generators $x_1, \ldots, x_n$ of $\mathfrak{m}$. We show that $x_1, \ldots, x_n$ is a regular sequence by induction on $n = \dim R$.
When $n = 1$, the statement is evident. Now assume the result is true for dimension $n - 1$. Let R be a regular local ring of dimension n. If R contains a nonzero divisor x_1 that is not a unit, then by item 2 of Proposition 1.17, R/x_1 is a regular local ring of dimension $n - 1$, so the inductive hypothesis applies, and we conclude that $x_1, \ldots, x_n$ is a regular sequence in R. ∎

1.3 April 7: Koszul Complex and Koszul Homology I

One way to put the dichotomy in a more philosophical or literary framework is to say that algebra is to the geometer what you might call the 'Faustian offer'. As you know, Faust in Goethe's story was offered whatever he wanted (in his case the love of a beautiful woman), by the devil, in return for selling his soul. Algebra is the offer made by the devil to the mathematician. The devil says: 'I will give you this powerful machine, it will answer any question you like. All you need to do is give me your soul: give up geometry and you will have this marvellous machine.' (Nowadays you can think of it as a computer!) Of course we like to have things both ways; we would probably cheat on the devil, pretend we are selling our soul, and not give it away. Nevertheless, the danger to our soul is there, because when you pass over into algebraic calculation, essentially you stop thinking; you stop thinking geometrically, you stop thinking about the meaning.

– *Sr.* Michael Atiyah [Ati02]

1.3.1 Regular Local Rings (Cont.)

Proposition 1.24. *A regular local ring is a integral domain.*

Proof

Let R be a regular local ring with unique maximal ideal $\mathfrak{m}$. We prove by induction on the Krull dimension $d := \dim(R)$ that R is an integral domain.

Base case. If $d = 0$, then R is a field by definition, hence a domain.

Inductive step: Suppose that every regular local ring of dimension less than d is a domain, and let $\dim(R) = d \geqslant 1$.
Since R is regular, $\mathfrak{m}/\mathfrak{m}^2$ has dimension d. In particular, by Nakayama's lemma we have $\mathfrak{m}^2 \neq \mathfrak{m}$.
Since R is noetherian, the set of minimal prime ideals is finite. Suppose, by way of contradiction, that

$$\mathfrak{m} \subseteq \mathfrak{m}^2 \cup \bigcup_{\mathfrak{q} \text{ minimal}} \mathfrak{q}.$$

Prime Avoidance Lemma [Sta25, Lemma 00DS] implies that $\mathfrak{m}$ is contained either in $\mathfrak{m}^2$ (which is impossible) or in one of the minimal primes. In the latter case, $\mathfrak{m}$ is itself a minimal prime, forcing $\dim(R) = 0$, contradicting $d \geqslant 1$. Hence, there exists an element

$$x \in \mathfrak{m} \setminus \Big(\mathfrak{m}^2 \cup \bigcup_{\mathfrak{q} \text{ minimal}} \mathfrak{q} \Big).$$

Set $S = R/(x)$ and denote by $\mathfrak{n} = \mathfrak{m}S$ the unique maximal ideal of S. By Krull's *Hauptidealsatz* [Sta25, Lemma 00KV], $\dim(S) = d - 1$. Also, the natural surjection

$$\mathfrak{m}/\mathfrak{m}^2 \to \mathfrak{n}/\mathfrak{n}^2$$

shows that $\dim_{R/\mathfrak{m}}(\mathfrak{n}/\mathfrak{n}^2) \leqslant d - 1$. Thus, S is regular and, by the induction hypothesis, an integral domain.

Since S is an integral domain, the ideal (x) in R is prime. Because x is chosen outside every minimal prime, there exists some minimal prime $\mathfrak{q}$ properly contained in (x). Now, for any $y \in \mathfrak{q}$ we have $y = ax$ for some $a \in R$. Since $x \notin \mathfrak{q}$, it must be that $a \in \mathfrak{q}$, and hence $\mathfrak{q} = x\mathfrak{q}$. By Nakayama's Lemma [Sta25, Section 07RC] it follows that $\mathfrak{q} = (0)$. Therefore, R has no nonzero minimal prime ideals and is an integral domain. ■

Remark. A scheme X of finite type over an algebraically closed field k. TFAE:

- The scheme X is smooth.
- (*locally integral*) the local ring $\mathcal{O}_{X,p}$ is regular for all $p \in X$.
- The local ring $\mathcal{O}_{X,p}$ is regular for all closed points $p \in X(k)$.

Proposition 1.25. *A regular local ring is a UFD.*

Proof

See [Sta25, Section 0FJH]. ■

Remark. A scheme X of finite type over an algebraically closed field k is *locally factorial* if every local ring is a UFD. For integral separated locally factorial schemes, the map from the Picard group Pic to the Weil class group Cl is subjective (see Figure 3.26).

Corollary 1.27. *Let $(R, \mathfrak{m})$ be a regular local ring. Then, any element $x \in \mathfrak{m} - \mathfrak{m}^2$ is regular, and R/x is regular.*

Proof. Compute that $\dim R/x = \dim R - 1$ by *Hauptidealsatz*. □

X Noetherian integral.

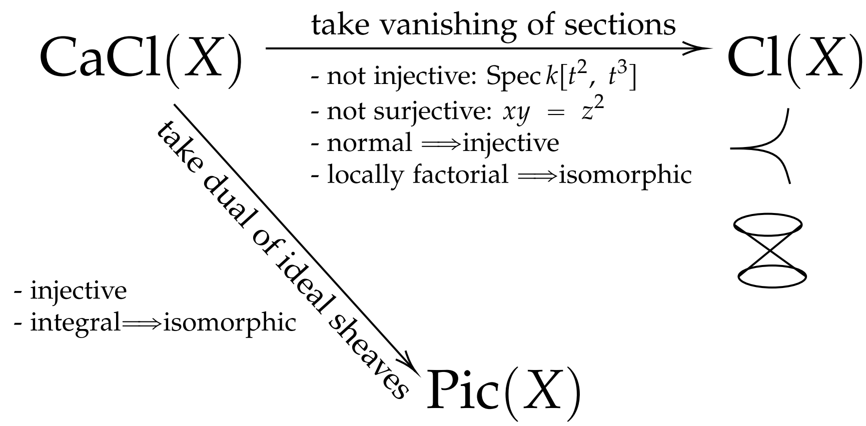

Figure 3.26: Implications on class groups

Corollary 1.28. *Let $(R, \mathfrak{m})$ be a regular local ring. If $x_1, \ldots, x_i$ are linearly independent in $\mathfrak{m}/\mathfrak{m}^2$, then $x_1, \ldots, x_i$ is regular, and $R/(x_1, \ldots, x_i)$ remains regular.*

Proof

Induction. See also Lemma 1.23. ■

Corollary 1.29. *The following are equivalent for a sequence of elements $x_1, \ldots, x_n$ in a regular local ring $(R, \mathfrak{m})$.*

- *The sequence $x_1, \ldots, x_n$ forms a basis of $\mathfrak{m}/\mathfrak{m}^2$.*
- *The sequence $x_1, \ldots, x_n$ generates $\mathfrak{m}$.*
- *The quotient $R/(x_1, \ldots, x_n)$ is zero-dimensional.*
- *The sequence $x_1, \ldots, x_n$ is a system of parameters.*

Proposition 1.30. *Let $(R, \mathfrak{m})$ be a noetherian local ring. If $x_1, \ldots, x_n$ is a regular sequence such that $R/(x_1, \ldots, x_n)$ is regular, then R is regular.*

Proof

Again, induction on the length n. If $n = 1$, then this is Corollary 1.27. Now assume the result holds for n. Then, for a sequence $x_1, \ldots, x_{n+1}$, the ring $R/(x_1)$ is regular. Let $\overline{\mathfrak{m}}$ be the image of $\mathfrak{m}$ in R/x_1. We have $\dim R/x_1 = \dim R - 1$ and $\dim \overline{\mathfrak{m}}/\overline{\mathfrak{m}}^2 \geqslant \dim \mathfrak{m}/\mathfrak{m}^2 - 1$. Then, $\dim \mathfrak{m}/\mathfrak{m}^2 \geqslant \dim R$ which implies that R is regular.

■

1.3.2 Koszul Complex & Koszul Homology

Let R be a commutative ring and M an R-module. The exterior algebra of M is

$$\begin{aligned}\bigwedge^{\bullet} M &= \bigoplus_{k\geqslant 0} \bigwedge^{k} M = R \oplus M \oplus \bigwedge^{2} M \oplus \cdots \\ &= \frac{R \oplus M \oplus M^{\otimes 2} \oplus \cdots}{\text{submodule generated by } x \otimes x}.\end{aligned} \tag{1.3.1}$$

The exterior algebra is graded commutative: $x \wedge y = (-1)^{|x|\cdot|y|} y \wedge x$.

Example 1.31. $M = R^{\oplus n} = R \cdot \{e_1, \ldots, e_n\}$. Then $\bigwedge^k M = R^{\oplus \binom{n}{k}} = R \cdot \{e_I \mid I \in \binom{[n]}{k}\}$.

Definition 1.32. [Sta25, Section 0621] Let R be a ring. Let $f : M \to R$ be an R-module map. The **Koszul complex** $K(f)_\bullet$ associated to f is the commutative differential graded algebra defined as follows:

1. the underlying graded algebra is the exterior algebra $K(f)_\bullet = \bigwedge M$,
2. the differential $d : K(f)_\bullet \to K(f)_\bullet$ is the unique derivation such that $d(x) = f(x)$ for all $x \in M = K(f)_1$.

Explicitly, if $m_1 \wedge \ldots \wedge m_n$ is one of the generators of degree n in $K(f)_\bullet$, then

$$d\left(m_1 \wedge \ldots \wedge m_n\right) = \sum_{i=1,\ldots,n} (-1)^{i+1} f\left(m_i\right) m_1 \wedge \ldots \wedge \widehat{m_i} \wedge \ldots \wedge m_n.$$

In the case that M is a finite free module, the map f is given by a sequence of ring elements $x_1, \ldots, x_n$. In this case, the Koszul complex may be written explicitly as

$$K(x_1, \ldots, x_n)_\bullet :\to \bigwedge^k M \xrightarrow{d_k} \bigwedge^{k-1} M \to \cdots \to \bigwedge^2 M \to M \to R,$$

or

$$K(x_1, \ldots, x_n)_\bullet :\to R^{\oplus\binom{n}{k}} \xrightarrow{d_k} R^{\oplus\binom{n}{k-1}} \to \cdots \to R^{\oplus n} \xrightarrow{(x_1 \ldots x_n)} R.$$

Cheerful Fact

Michael: Let $\mathcal{O}_{\mathbb{P}^i}$ be the coherent sheaf over $\mathbb{P}^n$ that is the pushforward along the standard inclusion. Let $k[x_0, \ldots, x_n]$ be the homogenous coordinate ring of $\mathbb{P}^n$. Then, the Koszul complex $K(x_{i+1}, \ldots, x_n)_\bullet$ upgrades to a locally free resolution of $\mathcal{O}_{\mathbb{P}^i}$,

$$0 \to \mathcal{O}_{\mathbb{P}^n}(-n+i)^{\oplus\binom{n-i}{n-i}} \to \cdots \to \mathcal{O}_{\mathbb{P}^n}(-2)^{\oplus\binom{n-i}{2}} \to \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus\binom{n-i}{1}} \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^i} \to 0.$$

One uses this to show that the Grothendieck group $K_0(\mathbb{P}^n)$ is generated by $[\mathcal{O}(1)]$ as a ring.

Goal: Use $K(x_1, \ldots, x_n)_\bullet$ to test if the sequence is regular.

Definition 1.33. For an R-module N, we define the Koszul complex with coefficients in N to be

$$K(x_1, \ldots, x_n; N)_\bullet := K(x_1, \ldots, x_n)_\bullet \otimes N.$$

(In the more general case, $K(f; N)_\bullet := K(f)_\bullet \otimes N$.)

Definition 1.34. The **Koszul homology** $H_\bullet(x_1, \ldots, x_n; N)$ is the homology of the complex $K(x_1, \ldots, x_n; N)_\bullet$.

Example 1.35. $R = k[x]$. The Koszul complex $K(x)_\bullet : 0 = \bigwedge^2 R \to R \xrightarrow{\cdot x} R$ continues to the free resolution

$$0 \to R \xrightarrow{\cdot x} R \to k \to 0$$

of the ground field. We have $H_0(x) = k$ and $H_i(x) = 0$ for $i \geqslant 1$. The fact that $H_1(x) = 0$ implies that $(\cdot x)$ is injective.

Example 1.36. Here are some more examples.

$$0 \longrightarrow k[x,y] \xrightarrow{\begin{pmatrix} -y \\ x \end{pmatrix}} k[x,y]^2 \xrightarrow{\begin{pmatrix} x & y \end{pmatrix}} k[x,y] \longrightarrow k \to 0.$$

$$0 \longrightarrow k[x,y,z] \xrightarrow{\begin{pmatrix} z \\ -y \\ x \end{pmatrix}} k[x,y,z]^3 \xrightarrow{\begin{pmatrix} -y & -z & 0 \\ x & 0 & -z \\ 0 & x & y \end{pmatrix}} k[x,y,z]^3 \xrightarrow{\begin{pmatrix} x & y & z \end{pmatrix}} k[x,y,z] \longrightarrow k \to 0.$$

In all cases we have

$$H_i(x_1, \ldots, x_n) = \begin{cases} k & i = 0 \\ 0 & i > 0. \end{cases}$$

Theorem 1.37. *Let R be a ring, M an R-module, and $x_1, \ldots, x_n$ an M-regular sequence. Then,*

$$H_i(x; M) = \begin{cases} M/(x_1 \ldots, x_n) & i = 0 \\ 0 & i > 0. \end{cases}$$

In other words, the Koszul complex $K(x_1, \ldots, x_n; M)_\bullet$ is exact except at degree 0.

1.4 April 9: Koszul Homology II

Michael: what was the quote again? It was kinda long and I didn't follow

1.4.1 Koszul Complex & Koszul Homology II

Theorem 1.38. *Let R be a ring and M an R-module. Let $x_1, \ldots, x_n$ be a sequence of elements of R. Consider the following:*

(1) The sequence $x_1, \ldots, x_n$ is regular.

(2) The homology groups $H_i(x_1, \ldots, x_n; M) = 0$ for $i > 0$.

(3) The first homology group $H_1(x_1, \ldots, x_n; M) = 0$.

The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) always hold. If $(R, \mathfrak{m})$ is noetherian local and M is finitely generated, then (3) $\Rightarrow$ (1).

The idea of the proof is to show that $K(x_1, \ldots, x_n)_\bullet \cong \bigotimes_i K(x_i)_\bullet$. We need the following facts from homological algebra.

1. Shifting complexes. If $C_\bullet$ is a complex and $d \in \mathbb{Z}$ is an integer, then the complex $C[d]_\bullet$ is defined as $C[d]_i = C_{i+d}$, i.e. shifted to the left by d. Then, $H_i(C[d]_\bullet) = H_{i+d}(C_\bullet)$.

2. Tensoring complexes. Given a pair of complexes $C_\bullet$ and $D_\bullet$, the tensor product $(C \otimes D)_\bullet$ is defined as

$$\begin{aligned} (C \otimes D)_d &= \bigoplus_{d=i+j} C_i \otimes D_j \\ d_k(\alpha \otimes \beta) &= d\alpha \otimes \beta + (-1)^{|\alpha|} \alpha \otimes d\beta. \end{aligned} \tag{1.4.1}$$

Lemma 1.39.

$$K(x_1, \ldots, x_n)_\bullet \cong K(x_1, \ldots, x_{n-1}) \otimes K(x_n)_\bullet.$$

Proof

Expand definitions. write out the n = 2 case. ∎

Remark. There is a short exact sequence of complexes $0 \to R \to K(x)_\bullet \to R[-1] \to 0$ given by

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] \arrow[d] & R \arrow[r, phantom, "\cong"] \arrow[d, "\cdot x"] & R \arrow[d] \\
R \arrow[r, phantom, "\cong"] \arrow[d] & R \arrow[r] \arrow[d] & 0 \\
0 & 0 &
\end{tikzcd}$$

More generally, the sequence

$$0 \to C_\bullet \to C \otimes K(x)_\bullet \to C_\bullet[-1] \to 0$$

is short exact for any complex $C_\bullet$.

Corollary 1.40. *There is a short exact sequence*

$$0 \to K(x_1, \dots, x_{n-1}; M)_\bullet \to K(x_1, \dots, x_n; M) \to K(x_1, \dots, x_{n-1}; M)[-1]_\bullet \to 0. \tag{1.4.2}$$

Proof

Proof of Theorem 1.38. $\boxed{(1) \implies (2)}$ Induction on n. The case $n = 1$ is Example 1.35. Assume the implication holds for n. Then, the short exact sequence 1.4.2 from Corollary 1.40

$$0 \to K(x_1, \dots, x_{n-1}; M)_\bullet \to K(x_1, \dots, x_n; M) \to K(x_1, \dots, x_{n-1}; M)[-1]_\bullet \to 0$$

induces a long exact sequence

$$\cdots \to H_i(x_1, \dots, x_{n-1}; M) \to H_i(x_1, \dots, x_n; M) \to H_{i-1}(x_1, \dots, x_{n-1}; M) \to \cdots \tag{1.4.3}$$

in Koszul homology. For degrees $i > 1$, the LES forces $H_i(x_1, \dots, x_n; M)$ to be 0. For degree 1, (1.4.3) becomes

$$0 \to H_1(x_1, \dots, x_n; M) \to H_0(x_1, \dots, x_{n-1}; M) \xrightarrow{\cdot x_n} H_0(x_1, \dots, x_{n-1}; M) \to \cdots \tag{1.4.4}$$

where the connecting homomorphism $d_0 = (\cdot x_n)$ is injective since x_n is regular. Thus the kernel $H_1(x_1, \ldots, x_n; M)$ vanishes.

$\boxed{(2) \implies (3)}$ Immediate.

$\boxed{(3) \implies (1) \text{ when } (R, \mathfrak{m}) \text{ is noetherian local}}$ *Claim 1:* $H_1(x_1, \ldots, x_n) = 0$ implies that $H_1(x_1, \ldots, x_i) = 0$ for all $0 \leqslant i \leqslant n$. The proof is by descending induction: if $H_1(x_1, \ldots, x_i) = 0$, then the long exact sequence (1.4.3) becomes

$$\cdots \to H_1(x_1, \ldots, x_{i-1}) \xrightarrow{\cdot x_i} H_1(x_1, \ldots, x_{i-1}) \to 0 = H_1(x_1, \ldots, x_i) \to \cdots \tag{1.4.5}$$

so the map $(\cdot x_i)$ is surjective, but Nakayama's Lemma implies that $x_i \cdot H_1(x_1, \ldots, x_{i-1}) = 0$. Therefore, $H_1(x_1, \ldots, x_{i-1}) = 0$.

Claim 2: The sequence $x_1, \ldots, x_n$ is regular. The proof is by induction on i. The case $i = 1$ is again Example 1.35. Assuming that $x_1, \ldots, x_i$ is regular, the LES (1.4.3) at degree 0 reads

$$\cdots \to H_1(x_1, \ldots, x_{i+1}) = 0 \to \underset{\cong H_1(x_1, \ldots, x_i)}{M/(x_1, \ldots, x_i)} \xrightarrow{\cdot x_{i+1}} M/(x_1, \ldots, x_i) \to \cdots \tag{1.4.6}$$

which implies $(\cdots x_{i+1})$ is injective on $M/(x_1, \ldots, x_i)$, hence regular. ■

Corollary 1.41. *Let $(R, \mathfrak{m})$ be noetherian local and M be a finitely generated R-module. Then, the fact that $x_1, \ldots, x_n$ is regular does not depend on the ordering of $x_1, \ldots, x_n$.*

Proof

Since tensoring is commutative, we have

$$K(x_1, \ldots, x_n)_\bullet \underset{\text{Lemma 1.39}}{\cong} \bigotimes K(x_i)_\bullet \cong K(x_{w_1}, \ldots, x_{w_n})_\bullet$$

for any permutation $w \in \mathcal{S}_n$. Then, implication $(3) \implies (1)$ of Theorem 1.38 implies that both sequences of regular. ■

Corollary 1.42. *For the sequence $x_1, \ldots, x_n \in k[x_1, \ldots, x_n]$, the Koszul complex $K(x_1, \ldots, x_n)_\bullet$ is a free resolution of the ground field k.*

1.4.2 Regularity & Dimension Theory

Definition 1.43. Let R be a noetherian local ring and M a finitely generated R -module. The **projective dimension** of M, denoted $\operatorname{pd}(M)$, is the minimum length of finite free resolutions of M. The **global dimension** R, denoted $\operatorname{gldim}(R)$, is the supremum of $\operatorname{pd}(M)$ for all finitely generated modules M.

Definition 1.44. Let $(R, \mathfrak{m})$ be a noetherian local ring. A **minimal free resolution** of an R-module M is a finite free resolution $0 \to F_l \xrightarrow{d_l} \cdots \to F_0 \to M \to 0$ such that $\operatorname{im} d_i \subseteq \mathfrak{m}F_i$.

Cheerful Fact

From a minimal resolution, one gets a list of ranks$(\operatorname{rk} F_0, \ldots, \operatorname{rk} F_l)$ called the **Betti numbers** of M. For a graded ring R, one instead obtains a two-dimensional array called the **Betti table**. The `Macaulay2` language is a computer algebra system that offers many amazing functionalities including the computation of Betti tables [GS, `betti`]. Below is a small example from the documentation page (linked above):

```
i6 : R = QQ[a,b,c, Degrees => {-1,-2,-3}];

i7 : heft R
o7 = {-1}
o7 : List

i8 : betti koszul vars R
            0 1 2 3
o8 = total: 1 3 3 1
         0: 1 1 . .
         1: . 1 1 .
         2: . 1 1 .
         3: . . 1 1
o8 : BettiTally
```

Proposition 1.45. *Let $(R, \mathfrak{m}, \kappa)$ be a noetherian local ring.*

1. *For a finitely generated module M, the projective dimension $\operatorname{pd}(M)$ is equal to the length of any minimal free resolution which is equal to the minimal i such that $\operatorname{Tor}_{i+1}(\kappa, M) = 0$.*
2. *If $F_\bullet \to M$ is a minimal free resolution, then $\operatorname{rk} F_i = \dim \operatorname{Tor}_{i+1}(\kappa, M)$.*

Proof

$\boxed{2}$ Any minimal free resolution

$$0 \to R^{\oplus d_l} \to \cdots \to R^{\oplus d_0} \to M \to 0$$

becomes

$$0 \to \kappa^{\oplus d_l} \xrightarrow{0} \cdots \xrightarrow{0} \kappa^{\oplus d_0} \xrightarrow{0} M_\kappa \to 0$$

so the ranks d_i are equal to the dimensions of $\operatorname{Tor}_{i+1}(\kappa, M) \cong \kappa^{\oplus d_i}$. ■

Remark. Tor is a *bifunctor*, and $\operatorname{Tor}(A, B)$ can be computed by either $E_\bullet \to A$ or $F_\bullet \to B$.

Proposition 1.46.

$$\operatorname{gldim}(R) = \operatorname{pd}(\kappa).$$

Proof

$\boxed{\geqslant}$ This direction follows from definition.

$\boxed{\leqslant}$ Let $n = \operatorname{pd}(\kappa)$. Then, there is a free resolution $0 \to F_n \to \cdots \to F_0 \to \kappa \to 0$, which implies $\operatorname{Tor}_i(\kappa, M) = 0$ for all $i > n$. See **??** . By Proposition 1.45, it follows that $\operatorname{pd}(M) \leqslant n$. ■

Theorem 1.47. *Let $(R, \mathfrak{m}, \kappa)$ be a noetherian local ring. The following are equivalent.*

(1) *The ring R is regular.*

(2) *The global dimension $\operatorname{gldim}(R)$ is finite.*

(3) *The projective dimension $\operatorname{pd}(\kappa)$ is finite.*

Proof

We prove the two important implications, noting that the other implications follows from definition.

$\boxed{\text{Regular} \implies \mathrm{pd}(\kappa) < \infty}$ Let $x_1, \ldots, x_n$ be a minimal set of generators for $\mathfrak{m}$. Then, by Corollary 1.29, $x_1, \ldots, x_n$ is a regular sequence, so the Koszul resolution is a finite free resolution of κ, the length of which upper bounds $\mathrm{pd}(\kappa)$.

$\boxed{\mathrm{pd}(\kappa) < \infty \implies R \text{ regular}}$ Jarod: Good topic for a student lecture! ■

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