COURSE SUMMARY FOR MATH 508, WINTER QUARTER 2017: ADVANCED COMMUTATIVE ALGEBRA

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WEEK 1, JAN 4, 6: DIMENSION

Lecture 1: Introduction to dimension.

- Define Krull dimension of a ring A.
- Discuss dimension 0 rings. Recall Artinian rings and various equivalences.
- Prove that a PID has dimension 1.
- Prove that if $I \subset R$ is a nilpotent ideal, then $\dim R/I = \dim R$.

Lecture 2: Conservation of dimension under integral extensions.

- Prove that if $R \to S$ is an integral ring extension, then $\dim R = \dim S$.
- Define the codimension (or height) of a prime $\mathfrak{p} \subset A$, denoted as $\operatorname{codim} \mathfrak{p}$, as the supremum of the lengths k of strictly descending chains
$$\mathfrak{p} = \mathfrak{p}_0 \supset \mathfrak{p}_1 \supset \cdots \supset \mathfrak{p}_k$$
of prime ideals. Note that $\operatorname{codim} \mathfrak{p} = \dim A_{\mathfrak{p}}$.
- Prove: if $\phi \colon R \to S$ is an integral ring homomorphism, then $\dim I = \dim \phi^{-1}(I)$.
- Discuss codimension 0 primes (i.e. minimal primes).
- Prove: if R is Noetherian and $f \in R$ is a non-unit, then any prime $\mathfrak{p} \subsetneq (f)$ has $\operatorname{codim}(\mathfrak{p}) = 0$.

WEEK 2, JAN 9, 11 (JAN 13 CANCELLED): KRULL'S HAUPTIDEALSATZ AND CONSEQUENCES

Lecture 3: Krull's Hauptidealsatz.

- State and prove Krull's Principal Ideal Theorem (a.k.a. Krull's Hauptidealsatz): if A is a Noetherian ring and $f \in A$ is not a unit, then $\operatorname{height}(f) \leq 1$; that is, for every prime ideal $\mathfrak{p}$ containing f, $\operatorname{height} \mathfrak{p} \leq 1$.
- State and prove the following generalization of Krull's Principal Ideal Theorem: if A is a Noetherian ring and $I = (x_1, \dots, x_n) \subset A$ is a proper ideal. Then $\operatorname{height} I \leq n$; that is $\operatorname{height} \mathfrak{p} \leq n$ for every prime ideal $\mathfrak{p}$ containing I (or equivalently, for every prime ideal which is minimal among prime ideals containing I).
- Prove corollary: $\dim k[x_1, \dots, x_n] = n$.

Macaulay2 Tutorial. (Evening of Jan 9)

Lecture 4: System of parameters.

- Prove the converse theorem to Krull's principal ideal theorem: if A is a Noetherian ring and $I \subset A$ is a proper ideal of height n. Then there exist $x_1, \dots, x_n \in I$ such that $\operatorname{height}(x_1, \dots, x_i) = i$ for $i = 1, \dots, n$.
- Reinterpret dimension: if $(R, \mathfrak{m})$ is a Noetherian local ring, them $\dim R$ is the smallest number n such that there exists $x_1, \dots, x_m \in \mathfrak{m}$ with $R/(x_1, \dots, x_m)$ Artinian. Such a sequence $x_1, \dots, x_m \in \mathfrak{m}$ is called a system of parameters for R.
- Prove corollary: if $(R, \mathfrak{m}) \to (S, \mathfrak{n})$ is a local homomorphism of local Noetherian rings, then $\dim S \leq \dim R + \dim S/\mathfrak{m}S$.
- Prove corollary: if R is a Noetherian ring, then $\dim R[x] = \dim R + 1$.

WEEK 3, JAN 18, 20 (JAN 16 MLK HOLIDAY): FLATNESS

Lecture 5: Basics on flatness.

- Review of Tor. Key properties: short exact sequences induce long exact sequences of Tor groups, $\operatorname{Tor}_i(P, M) = 0$ for P projective and $i > 0$, compatibility with localization, $\operatorname{Tor}_i(M, N) = \operatorname{Tor}_i(N, M)$ and thus can be computed as a derived functor in either the first or second term.
- Examples of flat and non-flat modules
- Prove Going Down Theorem for flatness
- Prove: if $(R, \mathfrak{m}) \to (S, \mathfrak{n})$ is a flat local homomorphism of local Noetherian rings, then $\dim S = \dim R + \dim S/\mathfrak{m}S$.

Lecture 6: Homological characterization of flatness:

- Prove: Let R be a ring. An R-module M is flat if and only if $\operatorname{Tor}_1^R(R/I, M)$ for all finitely generated ideals $I \subset R$.
- Examples: flatness over the dual numbers, flatness over PIDs.
- Equational Criterion for Flatness: An R-module M is flat if and only if the following condition is satisfies: For every relation $0 = \sum_i n_i m_i$ with $m_i \in M$ and $n_i \in R$, there exist elements $m'_j \in M$ and elements $a_{ij} \in R$ such that

$$\sum_j a_{ij} m'_j = m_i \text{ for all } i \quad \text{and} \quad \sum_i a_{ij} n_i = 0 \text{ for all } j.$$

WEEK 4, JAN 23, 25, 27: ARTIN–REES LEMMA, KRULL'S INTERSECTION THEOREM, LOCAL CRITERION OF FLATNESS

Lecture 7: flatness $\iff$ projective.

- Reinterpret equational criterion for flatness using commutative diagrams.

- Prove: if M is finitely presented R-module, then M is flat $\iff$ projective. If in addition R is a local, then flat $\iff$ free $\iff$ projective.
- State and motivate simple version of Local Criterion for Flatness.

Lecture 8: Artin–Rees Lemma and Local Criterion for Flatness.

- Give motivation of the Artin–Rees Lemma.
- Prove Artin–Rees Lemma: Let R be a Noetherian ring and $I \subset R$ an ideal. Let $M' \subset M$ be an inclusion of finitely generated R-modules. If $M = M_0 \supset M_1 \supset M_2 \supset \cdots$ is an I-stable filtration, so is $M' \supset M' \cap M_1 \supset M' \cap M_2 \supset \cdots$.
- Prove Krull's Intersection Theorem: Let R be a Noetherian ring, $I \subset R$ an ideal and M a finitely generated R-module. Then there exists $x \in I$ such that

$$(1-x) \bigcap_k I^k M = 0.$$

In particular, if $(R, \mathfrak{m})$ is local, then $\bigcap \mathfrak{m}^k M = 0$ and $\bigcap \mathfrak{m}^k = 0$. Or if R is a Noetherian domain and I is any ideal, then $\bigcap I^k = 0$.
- State general version of Local Criterion for Flatness: if $(R, \mathfrak{m}) \to (S, \mathfrak{n})$ is a local homomorphism of local Noetherian rings and M is a finitely generated S-module, then

$$M \text{ is flat as an } S\text{-module} \iff \operatorname{Tor}_1^R(R/\mathfrak{m}, M) = 0.$$

Lecture 9: Fibral Flatness Theorem.

- Finish proof of Local Criterion for Flatness.
- Prove the Fibral Flatness Theorem: Consider a local homomorphisms $(R, \mathfrak{m}) \to (S, \mathfrak{n}) \to (S', \mathfrak{n}')$ of local Noetherian rings. Let M be a finitely generated S'-module which is flat over R. Then M is flat over S if and only if $M/\mathfrak{m}M$ is flat over $S/\mathfrak{m}S$.
- Discuss special case of the Fibral Flatness Theorem when $R = k[x]_{(x)}$.

WEEK 5, JAN 30, FEB 1, 3: GRADED MODULES AND COMPLETIONS

Lecture 10: Graded modules and flatness.

- Summary of flatness results.
- State Openness of Flatness and Grothendiecke's generic freeness (without proof).
- Graded modules and Hilbert functions.
- Prove: Let $R = \bigoplus_{d \geq 0} R_d$ be a graded ring which is finitely generated as an R_0-algebra by elements of degree 1. Assume R_0 is a local Noetherian domain. Let M be a finitely generated R-modules. Then M is flat$/R_0$ if and only if for $\mathfrak{p} \in \operatorname{Spec} R_0$, the Hilbert function $H_{M \otimes_{R_0} k(\mathfrak{p})} := \dim_{k(\mathfrak{p})} M_d \otimes_{R_0} k(\mathfrak{p})$ is independent of $\mathfrak{p}$.

Lecture 11: Completions.

- Definition of the completion of a ring (and module) with respect to an ideal.
- Arithmetic and geometric examples.
- Show that if R is Noetherian and $I \subset R$ is an ideal, then $M \otimes_R \hat{R} \to \hat{M}$ is an isomorphism for all finitely generated modules M.
- Conclude that $R \to \hat{R}$ is flat.

Lecture 12: Completions continued.

- Show that if R is Noetherian, then so is the completion $\hat{R}$ of R along an ideal I.
- Conclude that R Noetherian $\implies R[[x]]$ Noetherian.
- Mention Hensel's Lemma.
- Mention Cohen's Structure Theorem.

WEEK 6, FEB 8, 10 (FEB 6 SNOW DAY): REGULAR SEQUENCES AND KOSZUL COMPLEXES

Lecture 13: regular sequences.

- Introduce regular sequences: we say $x_1, \ldots, x_n \in R$ that is a M-regular sequence if x_i is a non-zero divisor on $M/(x_1, \ldots, x_{i-1})M$ for $i = 1, \ldots, n$ and that $M \neq (x_1, \ldots, x_n)M$.
- Give examples.
- Prove: Let R be a ring and $x_1, \ldots, x_n$ be a regular sequence. Set $I = (x_1, \ldots, x_n)$. Show that the natural homomorphism
$$R/I[y_1, \ldots, y_n] \to \mathrm{Gr}_I R, \qquad y_i \mapsto x_i \in I/I^2,$$
is an isomorphism. In particular, I/I^2 is a free R/I-module of rank n.
- State more general version (which has the same proof) when M is an R-module and $x_1, \ldots, x_n$ is a M-regular sequence.

Lecture 14: Koszul complex.

- Finish proof of proposition from last class.
- Introduce alternating products.
- Give concrete definition of the Koszul complex: if R is a ring, M is an R-module and $x = (x_1, \ldots, x_n) \in R^n$, then $K(x; M)_\bullet = K(x_1, \ldots, x_n; M)_\bullet$ is the chain complex of R-modules where $K(x; M)_k = \wedge^k(R^n)$ and the differential $\wedge^k(R^n) \to \wedge^{k-1}(R^n)$ is defined by $e_{i_1} \wedge \cdots \wedge e_{i_k} \mapsto \sum_{j=1}^k (-1)^j x_{i_j} e_{i_1} \wedge \cdots \wedge \widehat{e_{i_j}} \wedge \cdots \wedge e_{i_k}$.
- Write down examples in the case $n = 1, 2, 3$.

WEEK 7, FEB 13, 15, 17: KOSZUL HOMOLOGY, DEPTH AND REGULARITY

Lecture 15: Koszul complexes and regular sequences.

- Reintrepret the Koszul complex as a tensor product of complexes.

- Define Koszul homology: if R is a ring, M is an R-module and $x = (x_1, \ldots, x_n) \in R^n$, then $H_i(x; M) := H_i(K(x; M))$.
- Prove: Let R be a ring, M be an R-module and $x = (x_1, \ldots, x_n) \in R^n$. If x is an M-regular sequence, then $H_i(x; M) = 0$ for $i \geq 1$.
- Show that the converse is true if $(R, \mathfrak{m})$ is a local Noetherian ring, M is a finitely generated R-module and $x_1, \ldots, x_n \in \mathfrak{m}$.

Lecture 16: depth.

- Finish proof of converse from previous lecture.
- State the theorem: Let R be a Noetherian ring and M be a finitely generated R-module. Let $x = (x_1, \ldots, x_n) \in R^n$ and set $I = (x_1, \ldots, x_n) \subset R$. Assume $IM \neq M$. Let d be the smallest integer such that $H_{n-d}(x; M) \neq 0$. Then any maximal M-regular sequence in I has length d.
- Define the *depth of I on M*, denoted by $\operatorname{depth}(I, M)$, as this smallest integer d.
- Give examples.
- Begin proof of theorem.

Lecture 17: depth and regular local rings.

- Finish proof characterizing depth.
- State (but do not prove): Let R be a Noetherian ring, $I \subset R$ be an ideal and M be a finitely generated graded R-module. Assume $I + \operatorname{Ann}(M) \neq R$. Then $\operatorname{depth}(I, M)$ is the smallest i such that $\operatorname{Ext}^i_R(R/I, M) \neq 0$.
- Define a regular local ring $(R, \mathfrak{m})$ as a Noetherian local ring such that $\dim R = \dim_{R/\mathfrak{m}} \mathfrak{m}/\mathfrak{m}^2$.
- Give a few examples: e.g., say when $k[x_1, \ldots, x_n]_{(x_1, \ldots, x_n)}/(f)$ is regular.
- Prove that a regular local ring is a domain.

WEEK 8, FEB 22, 24 (FEB 20 PRESIDENT'S DAY): FREE RESOLUTIONS AND PROJECTIVE DIMENSION

Lecture 17: minimal free resolutions and projective dimension.

- Show that if $(R, \mathfrak{m})$ is a regular local ring and $x_1, \ldots, x_k$ are elements of $\mathfrak{m}$ which are linearly independent in $\mathfrak{m}/\mathfrak{m}^2$, then $x_1, \ldots, x_k$ is a regular sequence and $R/(x_1, \ldots, x_k)$ is a regular local ring. Such a sequence whose length is equal to $\dim R$ is called a egular system of parameters.
- Introduce the projective dimension of a module M, denoted by $\operatorname{pd} M$, as the smallest length of a projective resolution of M.
- Introduce the global dimension of a ring R, dentoed by $\operatorname{gl}\dim M$, as the supremum of $\operatorname{pd} M$ over finitely generated R-modules M.

- Define a minimal free resolution of an R-module M as a free resolution $\cdots \to F_k \xrightarrow{d_k} F_{k-1} \to \cdots \to F_0 \to M$ such that $\operatorname{im}(d_k) \subset \mathfrak{m}F_{k-1}$.
- Show: if $(R, \mathfrak{m})$ is a local Noetherian ring and M is a finitely generated R-module, then $\operatorname{pd} M$ is equal to length of any minimal free resolution and is also characterized by the smallest i such that $\operatorname{Tor}^R_{i+1}(R/\mathfrak{m}, M) = 0$. Conclude that $\operatorname{gl}\dim R = \operatorname{pd} R/\mathfrak{m}$

Lecture 18: Auslander–Buchsbaum Theorem.

- Recall notions and results from previous lecture.
- Show that if $(R, \mathfrak{m})$ is a regular local ring, then $\dim R = \operatorname{gl}\dim R/\mathfrak{m}$. Discuss examples showing that this is not true if R is not regular.
- Discuss (without proof) graded analogues: minimal graded free resolutions, graded Betti numbers and namely Hilbert's Syzygy Theorem (any finitely generated graded module over the polynomial ring $k[x_1, \ldots, x_n]$ has a finite free graded resolutions of length $\leq n$.
- Recall the notion of depth and prove that if R is a Noetherian ring and $I \subset R$ is an ideal, then $\operatorname{depth}(I, R) \leq \operatorname{codim} I$..
- Prove the Auslander-Buchsbaum Theorem: Let $(R, \mathfrak{m})$ be a Noetherian local ring and $M \neq 0$ be a finitely generated R-module with $\operatorname{pd} M < \infty$. Then

$$\operatorname{pd} M = \operatorname{depth}(\mathfrak{m}, R) - \operatorname{depth}(\mathfrak{m}, M).$$

WEEK 9, FEB 27, MAR 1 (MAR 3 CANCELLED): THE AUSLANDER–BUCHSBAUM–SERRE THEOREM AND COHEN–MACAULAY RINGS

Lecture 19: Homological characterization of regular rings.

- Prove the Auslander–Buchsbaum–Serre Theorem: If $(R, \mathfrak{m})$ is a Noetherian local ring, the following are equivalent:
  - (i) R is regular.
  - (ii) $\operatorname{gl}\dim R < \infty$.
  - (iii) $\operatorname{pd} R/\mathfrak{m} < \infty$.

Lecture 20: Cohen-Macaulay rings.

- Prove the following corollaries of the Auslander–Buchsbaum–Serre Theorem:
  - If R is a regular local ring, then so is $R_{\mathfrak{p}}$ for every prime ideal $\mathfrak{p}$.
  - $k[x_1, \ldots, x_n]$ is a regular ring (i.e., all localizations at prime ideals are regular local rings).
- Define a Noetherian local ring $(R, \mathfrak{m})$ to be Cohen–Macaulay if $\operatorname{depth}(\mathfrak{m}, R) = \dim R$.

- Show that the following examples of Cohen–Macaulay rings: (1) regular local rings, (2) local Artinian rings, and (3) local Noetherian dimension 1 reduced rings.
- Prove: If $(R, \mathfrak{m})$ is a Cohen–Macaulay local ring and $\mathfrak{p}$ is an associated prime, then $\mathfrak{p}$ is a minimal prime and $\dim R = \dim R/\mathfrak{p}$. (In other words, Cohen–Macaulay rings can have no embedded primes (i.e. associated but not minimal primes) and is equidimensional.)
- Give some examples of rings that are not Cohen–Macaulay: $k[x, y]/(x^2, xy)$, $k[x, y, z]/(xz, yz)$,...

WEEK 10, MAR 6, 8, 10: COHEN–MACAULAY, NORMAL, COMPLETE INTERSECTIONS AND GORENSTEIN RINGS

Lecture 21: Properties of Cohen-Macaulay rings and Miracle Flatness.

- Prove: If $(R, \mathfrak{m})$ is a Cohen–Macaulay local ring, then for any ideal $I \subset R$, we have $\operatorname{depth}(I, R) = \dim R - \dim R/I = \operatorname{codim} I$. (In particular, the defining property of being Cohen–Macaulay holds for *all* ideals. Also, in the homework, we will see that Cohen–Macaulay rings satisfy a stronger dimension condition known as catenary.)
- Prove: Let $(R, \mathfrak{m})$ be a Cohen–Macaulay local ring. Then $x_1, \ldots, x_n \in \mathfrak{m}$ is a regular sequence if and only if $\dim R/(x_1, \ldots, x_n) = \dim R - n$. In other words, if $(R, \mathfrak{m})$ is a Cohen–Macaulay local ring, then any system of parameters is a regular sequence.
- Prove Miracle Flatness: Let $(R, \mathfrak{m}) \to (S, \mathfrak{n})$ be a local homomorphism of Noetherian local rings. Suppose that R is regular and S is Cohen–Macaulay. Then $R \to S$ is flat if and only if $\dim S = \dim R + \dim S/\mathfrak{m}S$.

Lecture 22: Complete Intersections and Normal rings.

- Define a Noetherian local ring $(R, \mathfrak{m})$ to be a complete intersection if the completion $\hat{R}$ is the quotient of a regular local ring by a regular sequence. Observe that any regular local ring modulo a regular sequence is a complete intersection.
- Show that any complete intersection local ring is Cohen–Macaulay. Give an example of a Cohen–Macaulay ring (e.g. $k[x, y]/(x, y)^2$) which is not a complete intersection.
- Given a Noetherian local ring $(R, \mathfrak{m})$ with residue field $k = R/\mathfrak{m}$ and minimal generators $x_1, \ldots, x_n$ of $\mathfrak{m}$, define the invariants $\epsilon_i(R) = \dim_k H_i(x_1, \ldots, x_n)$ (the Koszul homology). The number of minimal generators is called the embedding dimension of R is denoted $\operatorname{emb}\dim(R)$.
- State: R is a complete intersection if and only if $\dim R = \operatorname{emb}\dim R - \epsilon_1(R)$. Give several examples of both when this holds and doesn't.

- Recall that a domain R is called normal if it is integrally closed in its fraction field.
- For a Noetherian ring R, introduce Serre's properties:
  (R_i) for all $\mathfrak{p} \in \operatorname{Spec} R$ with $\operatorname{codim}(\mathfrak{p}) \leq i$, $R_\mathfrak{p}$ is regular.
  (S_i) for all $\mathfrak{p} \in \operatorname{Spec} R$, $\operatorname{depth} R_\mathfrak{p} \geq \min(\operatorname{codim}(\mathfrak{p}), i)$.
  Note that: R is regular if and only if (R_i) holds for all i and R is Cohen–Macaulay if and only if (S_i) holds for all i.
- Reinterpret the Conditions (S_0), (S_1) and (S_2). State that R is reduced if and only if (R_0) and (S_1) hold.
- State Serre's Normality Criterion: Let R be a Noetherian ring. Then R is normal if and only if (R_1) and (S_2) hold. (Prove the $\Leftarrow$ implication.)
- Mention Algebraic Hartog's: If R is a normal Noetherian domain, then $R = \bigcap_{\operatorname{codim}(\mathfrak{p})=1} R_\mathfrak{p}$.

Lecture 23: Gorenstein rings.

- Recall the notion of an injective module and an injective resolution. If R is a ring, define the injective dimension of an R-module, denoted by $\operatorname{inj}\dim_R M$, as the smallest length of an injective resolution of M.
- State (and explain some of the implications in) the following lemma: If R is a ring and M is an R-module, then the following are equivalent:
  (i) $\operatorname{inj}\dim_R M \leq n$.
  (ii) $\operatorname{Ext}^{n+1}_R(N, M) = 0$ for all R-modules N.
  (iii) $\operatorname{Ext}^{n+1}_R(R/I, M) = 0$ for all ideals $I \subset R$.
  If, in addition, $(R, \mathfrak{m})$ is local and M is finitely generated, then the above is also equivalent to:
  (iv) $\operatorname{Ext}^{n+1}_R(R/\mathfrak{p}, M) = 0$ for all $\mathfrak{p} \in \operatorname{Spec} R$.
  If, in addition, $(R, \mathfrak{m})$ is a Noetherian local ring and M is finitely generated, then the above is also equivalent to:
  (iv) $\operatorname{Ext}^{n+1}_R(R/\mathfrak{m}, M) = 0$.
- Conclude: $(R, \mathfrak{m})$ is a Noetherian local and M is a finitely generated R-module, then $\operatorname{inj}\dim_R M$ is the largest i such that $\operatorname{Ext}^i_R(R/\mathfrak{m}, M) \neq 0$.
- State: If $(R, \mathfrak{m})$ is a Noetherian local and M is a finitely generated R-module with $\operatorname{inj}\dim_R M < \infty$, then $\dim M \leq \operatorname{inj}\dim_R M = \operatorname{depth}(\mathfrak{m}, R)$.
- Compare the above characterizations and properties of injective dimension with what we've seen for projective dimension.
- Define a Noetherian local ring $(R, \mathfrak{m})$ to be Gorenstein if $\operatorname{inj}\dim_R R < \infty$.
- Give the following equivalences (explaining some of the implications): Let $(R, \mathfrak{m})$ is a Noetherian local ring of dimension n with residue field $k = R/\mathfrak{m}$, then the following are equivalent

(i) R is Gorenstein.
(ii) $\operatorname{inj}\dim_R R = n$.
(iii) $\operatorname{Ext}^i_R(k, M) = 0$ for some $i > n$.
(iv)

$$\operatorname{Ext}^i_R(k, M) = \begin{cases} 0 & i \neq n \\ k & i = n. \end{cases}$$

(v) R is Cohen–Macaulay and $\operatorname{Ext}^n_R(k, R) = k$.
(vi) There exists a regular sequence $x_1, \ldots, x_n \in \mathfrak{m}$ such that $R/(x_1, \ldots, x_n)$ is Gorenstein and dimension 0.

- Prove: Let $(R, \mathfrak{m})$ be a Noetherian local ring. Then

regular $\implies$ complete intersection $\implies$ Gorenstein $\implies$ Cohen–Macaulay

Give examples showing that each implication is strict.