Math 508: Advanced Commutative Algebra, Winter Quarter 2017
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Homework 3
Due: Wednesday, March 15

Please do 6 of the following 10 problems.

Problem 3.1. Let $(R, \mathfrak{m})$ be a Noetherian local ring and let $\widehat{R}$ be the completion of R along $\mathfrak{m}$. Show that:
(a) $\dim \widehat{R} = \dim R$.
(b) R is regular if and only if $\widehat{R}$ is regular.
(c) R is Cohen-Macaulay if and only if $\widehat{R}$ is Cohen-Macaulay.

Problem 3.2. Let $(R, \mathfrak{m})$ be a Cohen–Macaulay local ring and $\mathfrak{p} \subset R$ be a prime ideal. Show that $R_{\mathfrak{p}}$ is also Cohen–Macaulay.

Problem 3.3. Let $(R, \mathfrak{m})$ be a Noetherian ring. Recall that R is called *regular* (resp. *Cohen–Macaulay*) if all localizations at prime ideals are regular (resp. Cohen–Macaulay). Show that:
(a) R is regular if and only if $R[x]$ is regular
(b) R is Cohen–Macaulay if and only if $R[x]$ is Cohen–Macaulay.

Problem 3.4. Let $(R, \mathfrak{m})$ be a Noetherian local ring and $\mathfrak{p} \subset R$ be an associated prime. Show that

$$\operatorname{depth}(\mathfrak{m}, R) \leq \dim R/\mathfrak{p}.$$

Problem 3.5 (Hilbert polynomials)**.** Let M be a finitely generated graded module over the polynomial ring $k[x_0, \ldots, x_n]$, where k is a field. Let $H_M(d) = \dim_k M_d$ be the Hilbert function. Prove that there exists a polynomial $P_M \in \mathbb{Q}[x]$ of degree $\leq n$ (called the *Hilbert polynomial of* M) such that for $d \gg 0$ sufficiently large[1], $P_M(d) = H_M(d)$.

Hint: You may want to use Hilbert's Syzygy Theorem: any finitely generated graded module M over a polynomial ring $R = k[x_1, \ldots, x_n]$ has a graded free resolution

$$0 \to F_k \to F_{k-1} \to \cdots \to F_0 \to M.$$

of length $k \leq n$. Each F_i is a finite direct sum of copies of graded free modules $R(j)$, where $R(j)_d = R_{j+d}$. (This is the graded analogue of the fact we proved in class that finitely generated modules over a Noetherian local ring have free resolutions of finite length.)

[1]Here *sufficiently large* means that there exists an integer D such that for all $d \geq D$, $P_M(d) = H_M(d)$.

Problem 3.6 (Catenary rings)**.** A Noetherian ring R is called *catenary* if for every inclusion of prime ideals $\mathfrak{p} \subset \mathfrak{q}$, every chain $\mathfrak{p} \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n \subsetneq \mathfrak{q}$ which cannot be made longer has the same length.

(a) Show that R is a catenary ring if and only if for every prime ideal $\mathfrak{p}$, the localization $R_\mathfrak{p}$ is catenary.
(b) Show that if R is a catenary ring and $J \subset R$ is any ideal, then R/J is also catenary.
(c) Show that if R is a catenary local ring which is also a domain, then for any ideal I, we have

$$\dim R = \dim R/I + \operatorname{codim} I.$$

Show that this can fail if R is not a domain.

Problem 3.7 (Universally catenary rings)**.** A Noetherian ring R is called *universally catenary* if every polynomial ring $R[x_1, \ldots, x_n]$ is catenary.

(a) Show that any Cohen–Macaulay local ring $(R, \mathfrak{m})$ is universally catenary.

Hint: By Problem 3.3, you only need to show that R is catenary. Given an inclusion of primes $\mathfrak{p} \subset \mathfrak{q}$, use the Cohen–Macaulay property of $R_\mathfrak{q}$ (Problem 3.2) to show that

$$\operatorname{codim} \mathfrak{q} = \operatorname{codim} \mathfrak{p} + \dim R_\mathfrak{q}/\mathfrak{p}R_\mathfrak{q}.$$

Use the fact that this holds for any inclusion to show that R is catenary.

(b) Show that any finitely generated algebra over a field is universally catenary.

Hint: Use the fact that $k[x_1, \ldots, x_n]$ is regular.

Problem 3.8. Compute a minimal free resolutions of the residue field of $k[x, y]_{(x,y)}/(y^2 - x^3)$.

Problem 3.9. Are the the following rings Cohen–Macaulay? If it's not Cohen–Macaulay, what is the depth of the maximal ideal? Are the rings normal?

(a) $k[x, y, z, w]_{(x,y,z,w)}/(x, y) \cap (z, w)$ (the union of two planes in 4-space at a point).
(b) The subalgebra of $k[x, y]_{(x,y)}$ generated by $x^4, x^3y, x^2y^2, xy^3, y^4$.
(c) The subalgebra of $k[x, y]_{(x,y)}$ generated by x^4, x^3y, xy^3, y^4.
(d) R/I where $R = k[\{X_{i,j}\}_{1\le i,j\le 3}]$ and I is the ideal generated by the 2×2 minors of the 3×3 matrix $X = (X_{i,j})_{1\le i,j\le 3}$ of indeterminates.

You may use Macaulay2 if you would like. In this case, you may want to load a package called "Depth.m2" available online. If you do not want to use Macualay2, that's also fine—it is of course more difficult to do some of these by hand but do as much as you can.

Problem 3.10. Are the the following rings Gorenstein?

(a) $k[x,y]/(x^2,y^2)$.
(b) $k[x,y]/(x^2,xy^2,y^3)$.
(c) $k[x,y]/(xy)$.
(d) $k[x,y,z]/(x,y)\cap(x,z)\cap(y,z)$ (union three coordinate axes in 3-space).
(e) $k[t^3,t^5]\subset k[t]$.
(f) $k[t^3,t^5,t^7]\subset k[t]$.
(*Extra credit*) How about the examples from Problem 3.9.

You may use Macaulay2 if you would like.