Math 508: Advanced Commutative Algebra, Winter Quarter 2017
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Homework 2
Due: Wednesday, February 22

Please do 6 of the following 10 problems.

Problem 2.1 (Faithfully flat ring homomorphisms)**.**
Let $\phi \colon R \to S$ be a flat ring homomorphism. Show that the following statements are equivalent:

(i) an R-module M is zero if and only if $M \otimes_R S$ is zero.
(ii) a complex $0 \to M' \to M \to M'' \to 0$ of R-modules is exact if and only if $0 \to M' \otimes_R S \to M \otimes_R S \to M'' \otimes_R S \to 0$ is exact.
(iii) for every R-module M, the induced map $M \to M \otimes_R S, m \mapsto m \otimes 1$ is injective.
(iv) $\operatorname{Spec} S \to \operatorname{Spec} R$ is surjective.
(v) any maximal ideal $\mathfrak{m} \subset R$ is in the image of the map $\operatorname{Spec} S \to \operatorname{Spec} R$.
(vi) For every ideal $I \subset R$, there is an equality $I = \phi^{-1}(IR)$.

If the above equivalent conditions are satisfied, we say that $R \to S$ is *faithfully flat*.

Problem 2.2 (Properties of faithful flatness)**.**

(a) Let $R \to S$ be a flat local homomorphism of local rings. Show that $R \to S$ is faithfully flat.
(b) Let $(R, \mathfrak{m})$ be a Noetherian local ring and $\widehat{R} = \varprojlim R/\mathfrak{m}^n$. Show that $R \to \widehat{R}$ is faithfully flat.
(c) Let $R \to S$ be a faithfully flat ring homomorphism. Show that an R-module M is flat if and only if the S-module $M \otimes_R S$ is flat.
(d) Let $R \to S$ be a faithfully flat ring homomorphism. Show that if S is Noetherian, so is R.

Problem 2.3 (Hensel's Lemma)**.** Prove the following version of Hensel's Lemma: Let $(R, \mathfrak{m})$ be a complete local ring and $f \in R[x]$. Denote by $f' \in R[x]$ the derivative of f. Suppose that $a_0 \in R/\mathfrak{m}$ is a solution to f modulo $\mathfrak{m}$ such that $f'(a_0) \notin \mathfrak{m}$. Then there exists a solution $a \in R$ to f such that $a \cong a_0 (\text{mod } \mathfrak{m})$.

Hint: Let $f_n \in R/\mathfrak{m}^n[x]$ be the image of f. Inductively define $a_{n+1} \in R/\mathfrak{m}^{n+1}$ as follows: first pick an arbitrary lift $b \in R/\mathfrak{m}^{n+1}$ of a_n and then set

$$a_{n+1} := b - \frac{f_{n+1}(b)}{f'_{n+1}(b)} \in R/\mathfrak{m}^{n+1}.$$

Show that this makes sense and that $a = (a_0, a_1, , \ldots) \in R$ is a solution.

Problem 2.4 (Cohen's Structure Theorem)**.**

(a) Let $R \to S$ be a ring homomorphism. Suppose that S is complete with respect to an ideal J. Show that if $a_1, \ldots, a_n \in J$, there is an induced ring homomorphism

$$R[[x_1, \ldots, x_n]] \to S$$

such that the image of x_i is a_i.

(b) Let $(R, \mathfrak{m}) \to (S, \mathfrak{n})$ be a local ring homomorphism of complete local Noetherian rings. Suppose that $R/\mathfrak{m} \to S/\mathfrak{n}$ is an isomorphism and that $\mathfrak{m}/\mathfrak{m}^2 \to \mathfrak{n}/\mathfrak{n}^2$ is surjective. Show that $R \to S$ is surjective.

Hint: Show that the graded ring homomorphism $\mathrm{gr}_\mathfrak{m} R \to \mathrm{gr}_\mathfrak{n} S$ *is surjective and appeal to a lemma from a lecture to conclude that* $R \to S$ *is surjective as well.*

(c) Use Parts (1) and (2) to show the following special case of Cohen's Structure Theorem: Let $(R, \mathfrak{m})$ be a complete local Noetherian ring with residue field $k = R/\mathfrak{m}$. Suppose that there is an inclusion $k \subset R$ such that $k \to R \to R/\mathfrak{m}$ is an isomorphism. Show that $R \cong k[[x_1, \ldots, x_n]]/I$ for some ideal I.

Problem 2.5 (Nodal cubic)**.** Let k be a field and $R = k[x, y]/(y^2 - x^2 - x^3)$. Let $\widehat{R}$ be the completion of R along (x, y). Show that $\widehat{R} \cong k[u, v]/(uv)$.

Problem 2.6 (Hilbert polynomials)**.** In class, we will show that if M is a finitely generated graded module over the polynomial ring $k[x_0, \ldots, x_n]$, then for sufficiently large $d \gg 0$, the Hilbert function $H_M(d) = \dim_k M_d$ agrees with a polynomial. This polynomial is called the *Hilbert polynomial of* M and is denoted by P_M.

Compute the following Hilbert polynomials:

(a) $k[x, y, z]/(xz - y^2)$
(b) $k[x, y, z]/(xz)$
(c) $k[x, y, z, w]/(yw - z^2, xw - yz, xz - y^2)$
(d) $k[x, y, z, w]/(yw - z^2, xw - yz, xz)$
(e) $k[x_0, \ldots, x_3]/(x_0, x_1) \cap (x_2, x_3)$
(f) $k[x_0, \ldots, x_4]/(x_0, x_1) \cap (x_2, x_3)$
(g) $k[x_0, \ldots, x_5]/(x_0, x_1, x_2) \cap (x_3, x_4, x_5)$

You may use Macaulay2 if you'd like. Macaulay2 has a function `hilbertPolynomial` which computes the Hilbert polynomial. The output is often expressed in terms of the Hilbert polynomial P_n of $k[x_0, \ldots, x_n]$ but you expand out the expression and write out the coefficients of the polynomial.

Side comment: Part (f) corresponds to the union two projective planes in $\mathbb{P}^4$ *along a point, and Part (g) corresponds to the disjoint union of two projective planes in* $\mathbb{P}^5$*.*

Problem 2.7 (Equivalences between graded modules)**.** Let $R = \oplus_{d \geq 0} R_d$ be a graded ring such that R is finitely generated as an algebra over R_0 by elements of degree 1 and that R_1 is finitely generated as an R_0-module.

If M is a finitely generated graded R-module and N is a positive integer, we define $M^{\geq K} := \bigoplus_{d \geq 0} (M^{\geq K})_d$ by

$$(M^{\geq K})_d := \begin{cases} 0 & \text{if } d \geq K \\ M_d & \text{if } d > K \end{cases}$$

(a) Show that $M^{\geq K}$ is a finitely generated graded R-submodule of M. If $R_0 = k$ is a field, show that M and $M^{\geq K}$ have the same Hilbert polynomials but in general different Hilbert functions.

(b) Let $M \to N$ be a homomorphism of finitely generated graded R-modules. Show that the following two statements are equivalent:
  (i) the induced map $M^{\geq N} \to N^{\geq N}$ is an isomorphism for some N, and
  (ii) the induced map

$$(M_f)_0 \to (N_f)_0$$

  is an isomorphism for each $f \in R_1$. Here $(M_f)_0$ is the degree 0 component of the localization M_f (which is naturally a graded R-module).

Problem 2.8 (Hilbert polynomials and flatness)**.** Let $R = \oplus_{d \geq 0} R_d$ be a graded ring such that R is finitely generated as an algebra over R_0 by elements of degree 1 and that R_1 is finitely generated as an R_0-module.

(a) Suppose R_0 be a local Noetherian domain. Let M be a finitely generated graded R-module. Show that the following two statements are equivalent:
  (a) for each prime ideal $\mathfrak{p} \subset R_0$, the Hilbert polynomial $P_{M \otimes_{R_0} k(\mathfrak{p})}$ is independent of $\mathfrak{p}$, and
  (b) for each $f \in R_1$, the R_0-module $(M_f)_0$ is flat.

  Side comment: You should compare this result to the theorem from lecture where we showed that M is flat if and only if the Hilbert function $H_{M \otimes_{R_0} k(\mathfrak{p})}$ is independent of $\mathfrak{p}$.

(b) Is the ring homomorphism $k[t]_{(t)} \to k[x, y, z, t]_{(t)}/(xz - ty^2)$ flat?

(c) Is the ring homomorphism $k[t]_{(t)} \to k[x, y, z, w, t]_{(t)}/(yw - z^2, xw - yz, xz - ty^2)$ flat?

Problem 2.9 (Regular sequences)**.**

(a) Let $R = k[x, y, z]$. Show that $x, (x-1)y, (x-1)z$ is a regular sequence but that $(x-1)y, (x-1)z, x$ is not.

(b) Let R be a Noetherian local ring and $x_1, \ldots, x_n$ be a regular sequence. Show that $\dim R/(x_1, \ldots, x_n) = \dim R - n$.

Problem 2.10 (Free resolutions and Betti tables)**.** Using the Macaulay2 commands `res` and `betti`, compute the minimal graded free resolutions and graded Betti tables for each of the rings in Problem 2.6(a)-(g).