Math 508: Advanced Commutative Algebra, Winter Quarter 2017
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Homework 1
Due: Wednesday, February 1

Please do 6 of the following 10 problems.

Problem 1.1.

(a) Find a system of parameters for the local ring

$$k[x,y,z,w]_(x,y,z,w)/(xyz, xyw, xzw).$$

(b) Find a system of parameters for the local ring $k[\{X_{i,j}\}_{0\leq i,j\leq 3}]_{\mathfrak{m}}/I$ where $\mathfrak{m}$ is the maximal ideal generated by all the $X_{i,j}$ and I is the ideal generated by the 2×2 minors of

$$\begin{pmatrix} X_{0,0} & X_{0,1} & X_{0,2} & X_{0,3} \\ X_{1,0} & X_{1,1} & X_{1,2} & X_{1,3} \\ X_{2,0} & X_{2,1} & X_{2,2} & X_{2,3} \\ X_{3,0} & X_{3,1} & X_{3,2} & X_{3,3}. \end{pmatrix}$$

For (2), you may use Macaulay2 if you'd like.

Comment: We computed this ideal in the Macaulay2 tutorial in the case of 3×3 matrices. Also, you can make Macualay2 print out answers in tex with the command `tex(...)`.

Problem 1.2.

(a) Let k be a field and $R = k[\epsilon]/(\epsilon^2)$ be the dual numbers over k. Compute $\mathrm{Tor}_i^R(k,k)$ for all i. Check your answer by computing the answer for $i = 0, \ldots, 5$ using Macaulay2.

(b) Compute $\mathrm{Tor}_i^{\mathbb{Z}}(\mathbb{Z}/p, \mathbb{Z}/q)$ for every $i \geq 0$ and primes p and q. Again, check your answer in Macaulay2 by computing a few cases.

Problem 1.3.

(a) Show that $k[x,y] \to k[u,v]$ defined by $x \mapsto u, y \mapsto uv$ is *not* flat.

(b) Show that $k[x,y]/(y^2 - x^3) \to k[t]$ defined by $x \mapsto t^2, y \mapsto t^3$ is *not* flat.

Side comment: $k[t]$ is the integral closure of $k[x,y]/(y^2 - x^3)$ in its fraction field $k(t)$ where $t = y/x$.)

(c) Is the inclusion of rings $k[x,y] \hookrightarrow k[x,y,z,w]/(xz, xw, yz, yw)$ a flat homomorphism?

Problem 1.4. Fill in the missing details from the proof in class of the converse to the Principal Ideal Theorem: If R is a Noetheiran ring and $\mathfrak{p} \subset R$ is a prime ideal of codimension c, then $\mathfrak{p}$ is minimal over some ideal generated by c elements.

More specifically, in class, we had let $\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_{c-1} \subsetneq \mathfrak{p}_c = \mathfrak{p}$ be a chain of prime ideals. We constructed elements $x_1, \ldots, x_c$ inductively by letting $x_i \in \mathfrak{p}_i$ be any element not contained in any minimal prime over $(x_1, \ldots, x_{i-1})$. In class, we claimed that for each i the ideal $(x_1, \ldots, x_i)$ had codimension i or, in other words, that every prime minimal over $(x_1, \ldots, x_i)$ has codimension i. Prove this claim.

Problem 1.5. Let $\phi\colon R \to S$ be a ring homomorphism.

(a) Prove that if M is a flat R-module, then $M \otimes_R S$ is a flat S-module.
(b) Prove that $R \to S$ is flat if and only if for every prime $\mathfrak{q} \subset S$ the induced ring homomorphism $R_{\phi^{-1}(\mathfrak{q})} \to S_{\mathfrak{q}}$ is flat.

    Hint: For (2), use the fact that Tor commutes with localization, i.e., for R-modules M and N and a prime ideal $\mathfrak{p} \subset R$, then $\mathrm{Tor}_i^R(M,N)_{\mathfrak{p}} \cong \mathrm{Tor}_i^{R_{\mathfrak{p}}}(M_{\mathfrak{p}}, N_{\mathfrak{p}})$.

Problem 1.6. Let $\phi\colon R \to S$ be a ring homomorphism where R is a PID and S is Noetherian. Prove that $R \to S$ is flat if and only if for every associated prime $\mathfrak{q} \subset S$, $\phi^{-1}(\mathfrak{q}) = (0)$.

Hint: Use the fact that a prime $\mathfrak{q} \subset S$ is associated if and only if every non-zero element of $\mathfrak{q}$ is a zero-divisor.

Problem 1.7. Given an example of an injective local homomorphism $(R, \mathfrak{m}) \to (S, \mathfrak{n})$ of local rings such that

$$\dim S < \dim R + \dim S/\mathfrak{m}S.$$

Problem 1.8. (Key lemma in proof of Equational criterion for flatness) Let R be a ring. Let M and N be R-modules. Suppose that $\{n_i\}_{i\in I}$ is a generating set of N. Let $x = \sum_i m_i \otimes n_i \in M \otimes_R N$ be a finite sum where each $m_i \in M$. Show that $x = 0$ if and only if there exists elements $m'_j \in M$ for each j in some set J and $a_{i,j} \in R$ such that: (1) $\sum_j a_{i,j} m'_j = m_i$ for all i and (2) $\sum_i a_{i,j} n_i = 0$ for all j.

Problem 1.9. (Infinitesimal criterion of flatness) Let $(R, \mathfrak{m}) \to (S, \mathfrak{n})$ be a local ring homomorphism of local Noetherian rings. Suppose M is a finitely generated S-module. Prove that M is flat as an R-module if and only if $M/\mathfrak{m}^n M$ is flat as an $R/\mathfrak{m}^n$-module for every n.

Problem 1.10. Let k be a field.

(a) Let $R = k[x]_{(x)}$ and M be *any* R-module. Show that M is flat if and only if $\mathrm{Tor}_1^R(R/(x), M) = 0$.
(b) Let $R = k[x,y]_{(x,y)}$. Let $M = k(x)$ be the R-module where y acts by 0. Show that M is not flat but $\mathrm{Tor}_1^R(R/(x,y), M) = 0$.