COURSE SUMMARY FOR MATH 504, FALL QUARTER 2017-8: MODERN ALGEBRA

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Week 1, Sept 27, 29: Introduction to Groups

Lecture 1: Introduction to groups.

- Defined a group and discussed basic properties (e.g., uniqueness of identity and inverses)
- Discussed examples: $(\mathbb{Z}, +)$, $(\mathbb{R}^\times, \times)$, $((\mathbb{Z}/p)^\times, \times)$, symmetries of the square (i.e., the dihedral group D_8), general linear group, symmetric group
- Defined subgroups
- Discussed examples: $\mathbb{Z} \subset \mathbb{R}$, $D_8 \subset S_4$, the subgroup $\langle a_1, \ldots, a_n \rangle \subset G$ generated by elements $a_1, \ldots, a_n \in G$
- Defined the order $|G|$ of a group G and the order $|a|$ of an element $a \in G$.
- Stated Lagrange's theorem: if G is a finite group and $H \subset G$, then $|H|$ divides $|G|$.
- Showed how Lagrange's theorem implies Fermat's Little Theorem: if p is a prime and a is an integer not divisible by p, then $a^{p-1} \equiv 1 \mod p$.

Lecture 2: Group homomorphisms and normal subgroups.

- Defined group homomorphisms, injectivity, surjectivity, isomoprhisms, automorphisms, kernels, and images.
- Discussed centers, normalizers and centralizers.
- Discussed cosets and quotients. Showed if $H \subset G$ is a normal subgroup, then G/H is naturally a group such that the projection $G \to G/H$ is a surjective homomorphism.
- Discussed the four basis isomorphism theorems.
- Discussed direct and semi-direct products.

Week 2, Oct 2, 4, 6: Products, simple groups and group actions

Lecture 3: Semi-direct products.

- Discussed definition of semi-direct products.
- Proved that if G is a group, $H \trianglelefteq G$ normal subgroup and $K \subset G$ is a subgroup satisfying $HK = G$ and $H \cap K = 1$, then G is isomorphic to the semi-direct product $H \rtimes_\phi K$ where $\phi \colon K \to \mathrm{Aut}(H)$ is defined by $\phi(k)(h) = hkh^{-1}$ for $h \in H$ and $k \in K$.
- Discussed examples: $D_{2n} \cong \mathbb{Z}/n \rtimes \mathbb{Z}/2$ and $\mathbb{Q}^\times \rtimes \mathbb{Q}$.
- Discussed the universal property of the direct product.
- Defined the free product in terms of a universal property.

Lecture 4: Free groups, finitely generated groups, simple groups and the Jordan–Hölder theorem.

- Defined the free group $F(S)$ on a set S in terms of the universal property: for any map $S \to G$ to a group G, there is a unique group homomorphism $F(S) \to G$ compatible with the maps from S.

- Constructed the free group $F(S)$ directly and showed that its satisfies the universal property.
- Showed that finitely generated groups are quotients of free groups.
- Stated the classification theorem for finitely generated abelian groups (to be proved later when we discuss finitely generated modules over a PID).
- Defined simple groups and composition series.
- Stated the Jordan–Hölder theorem and proved the existence of composition series (while the uniqueness is a homework exercise).

Lecture 5: Group actions.

- Introduced the notion of a group G acting on a set Ω. Defined the stabilizer $G_x \subset G$ and orbit $O(x) \subset \Omega$ of a point $x \in \Omega$. Proved that the map $G/G_x \to O(x)$ taking a coset hG_x to hx is bijective.
- Discussed the examples of G acting on itself via left multiplication and via conjugation.
- Discussed again the symmetries of the regular n-gon and cube.
- Defined the orthogonal group $O_n(\mathbb{R})$.
- Proved that any finite subgroup $G \subset O_2(\mathbb{R})$ is isomorphic to $\mathbb{Z}/n$ or D_{2n} for some n.

Week 3, Oct 9, 11, 13: Platonic solids and the Sylow theorems

Lecture 6: Platonic solids.

- Proved that any element of $\mathrm{SO}_3(\mathbb{R})$ is a rotation about some line.
- Stated the theorem that any finite subgroup $G \subset \mathrm{SO}_3(\mathbb{R})$ is isomorphic to either: (1) $\mathbb{Z}/n$ as the subgroup of rotations about a fixed line, (2) D_{2n} as the subgroup of symmetries of a regular n-gon in a plane, (3) A_4 as the subgroup of symmetries of a tetrahedron, (4) S_4 as the subgroup of symmetries of a cube or octahedron, or (5) A_5 as the subgroup of symmetries of a dodecahedron or icosahedron.
- Following the exposition in Chapter 6, Artin's *Algebra*, we proved the theorem in the case that the action of G on the set $\Omega = \{p \in S^2 \mid \exists\, g \neq 1 \in G \text{ with } gp = p\}$ has one or two orbits.

Lecture 7: More on Platonic solids and Cauchy's theorem.

- Finished the classification of finite subgroups of $\mathrm{SO}_3(\mathbb{R})$ handling the case when Ω has 3 orbits.
- Proved the following key lemma: Let H be a finite group of order p^n for a prime p and S be a finite set with an action of H. Define $S^H = \{s \in S \mid hs = s \text{ for all } h \in H\}$. Then $|S^H| \equiv |S| \mod p$.
- Using this lemma, we proved Cauchy's theorem: If G is a finite group and p is a prime dividing $|G|$, then there exists an element of order p.

Lecture 8: The Sylow theorems.

- Recalled key lemma from last lecture which was used repeatedly in the proofs of Sylow's theorems.
- Defined p-groups and p-subgroups.
- Proved that if $H \subset G$ is a p-subgroup of a finite group, then $|N_G(H) : H| \equiv |G : H| \mod p$.
- Proved the First Sylow Theorem: If G is a finite group of order $p^n m$ where p is a prime not dividing m, then for each $i = 0, \dots, n$, there exists a subgroup of G of order p^i. In fact, for each $i = 0, \dots, n-1$, any subgroup of order p^i is normal in some subgroup of order p^{i+1}.
- Defined p-Sylow subgroups.

- Proved the Second Sylow Theorem: Let H, P be p-subgroups of a finite group G such that P is a p-Sylow subgroup. Then $H \subset gPg^{-1}$ for some $g \in G$. In particular, all p-Sylow subgroups are conjugate.
- Proved the Third Sylow Theorem. Let G be a finite group of order $p^n m$ where p is a prime not dividing m. Let n_p be the number of p-Sylow subgroups of G. Then $n_p \equiv 1 \mod p$ and $n_p | m$.

Week 4, Oct 16, 18, 20: Classification of finite groups and Introduction to Rings

Lecture 9: Classification of finite groups.

- Today's lecture summarized various material on finite groups that useful to classify all finite groups of a particular order.
- Began by summarizing Sylow's theorems.
- Stated the proposition: if G is a finite group and $H \subset G$ is a subgroup such that $|G : H|$ is the smallest prime dividing $|G|$, then H is normal.
- Stated the proposition: Let H, K be groups and $\varphi \colon K \to \mathrm{Aut}(H)$ be a group homomorphism. Then
  (1) If $\phi \in \mathrm{Aut}(K)$, then $H \rtimes_\varphi K \cong H \rtimes_{\varphi \circ \phi} K$; and
  (2) If $\psi \in \mathrm{Aut}(H)$, then $H \rtimes_\varphi K \cong H \rtimes_{\psi^{-1}\varphi\psi} K$.
- Stated the proposition that
$$\mathrm{Aut}(\mathbb{Z}/p^n) = (\mathbb{Z}/p^n)^\times \begin{cases} \mathbb{Z}/((p-1)p^{n-1}) & \text{if } p \text{ is odd} \\ \mathbb{Z}/2 \times \mathbb{Z}/2^{n-1} & \text{if p=2.} \end{cases}$$
and $\mathrm{Aut}(\mathbb{Z}/p_1^{n_1} \times \cdots \times \mathbb{Z}/p_k^{n_k}) \cong (\mathbb{Z}/p_1^{n_1})^\times \times \cdots \times (\mathbb{Z}/p_k^{n_k})^\times$ if $p_1, \ldots, p_k$ are distinct primes.
- Classified groups of order pq for primes $p < q$.
- Classified groups of order p^2 for a prime p.
- Classified groups of order p^3 for a prime p.

Lecture 10: Introduction to Rings I.

- Corrected discussion of the classification of groups of order p^3 with an emphasis on distinguishing the case $p = 2$ from the case of odd primes p.
- Defined rings. Our definition requires that a ring has a multiplicative identity 1.
- Defined commutative rings, division rings and fields.
- Provided basic examples: $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Z}/n$
- Provided additional examples: zero ring, quaternions, matrix rings
- Discussed the group of units $R^\times$ in a ring R. Showed that the units $M_n(R)^\times$ in the matrix ring $M_n(R)$ over a ring R is precisely $\mathrm{GL}_n(R) = \{A \in M_n(R) \mid \det(A) \in R^\times$.

Lecture 11: Introduction to Rings II.

- Defined the group ring $R[G]$ of a finite group G over a ring R.
- Discussed the ring of continuous functions on a topological space X.
- Defined an integral domain and zero-divisors.
- Proved that an integral domain which is a finite set is necessarily a field.
- Defined a ring homomorphism $\phi \colon R \to S$ of rings R and S. Our definition requires that $\phi(1) = 1$.
- Defined a ring isomorphism to be a bijective ring homomorphism, and show that this implies that that there is an inverse which is also a ring isomorphism.
- Defined the kernel and image of a ring homomorphism.
- Defined left ideals, right ideals and two-sided ideals.

- Discussed examples: (1) if $\phi\colon R \to S$ is a ring homomorphism, then $\ker(\phi)$ is a 2-sided ideal, (2) if R is a ring and $a \in R$, then $Ra = \{xa \mid x \in R\}$ is a left ideal and $aR = \{ax \mid x \in R\}$ is a right ideal (and if R is commutative, we denote $(a) := Ra = aR$), (3) discussed examples of left ideals in matrix rings that are not right ideals.
- Showed that any ideal in $\mathbb{Z}$ is equal to (n) for some integer n.
- Quickly defined prime ideals and maximal ideals.

Week 5, Oct 23, 25, 27: More on Rings

Lecture 12: Quotient rings, localization and Euclidean domains.

- Discussed the quotient ring R/I of a ring R by a two-sided ideal I.
- Discussed the four isomorphism theorem for rings.
- Defined the localization $S^{-1}R$ of a commutative ring R at a multiplicative system.
- Discussed the examples: (1) if R is an integral domain, the fraction field of R is $\mathrm{Frac}(R) := S^{-1}R$ where $S = (R \setminus 0)^{-1}$, (2) if $f \in R$, the ring $R_f := S^{-1}R$ where $S = \{1, f, f^2, \ldots\}$, if $\mathfrak{p} \subset R$ is a prime ideal, then $R_{\mathfrak{p}} := S^{-1}R$ where $S = R \setminus \mathfrak{p}$.
- Defined Euclidean domains and principal ideal domains (PIDs).
- Showed that any Euclidean domain is a PID.

Lecture 13: UFDs.

- Showed that if R is a PID, then every non-zero prime is maximal.
- Defined prime and irreducible elements in an integral domain.
- Showed that in any integral domain R, prime elements are always irreducible. Moreover, we established the converse if R is a PID.
- Motivated and defined a Unique Factorization Domain (UFD).
- Showed that if R is a UFD, then every irreducible element is prime.
- Stated the main theorem: If R is a PID, then R is a UFD. This gives us the implications
$$\text{Euclidean domain} \Rightarrow \text{PID} \Rightarrow \text{UFD}.$$
- Showed that if R is a commutative ring, then the following are equivalent: (1) Every ascending chain of ideals terminates, and (2) Every ideal is finitely generated. These equivalent properties define a Noetherian ring.
- Proved that if R is Noetherian domain, then every element has a factorization into irreducible elements.

Lecture 14: Gauss's lemma.

- Finished the proof from last time that a PID is a UFD. In fact, we proved that if R is a Noetherian ring, then R is a UFD if and only if every irreducible element is prime.
- Proved Gauss's lemma.
- Discussed an important corollary of Gauss's lemma: if R is a UFD with fraction field K and $f(x) \in R[x]$ is a polynomial such that the greatest common divisor of the coefficients is 1 (e.g., f is a monic polynomial), then f is irreducible in $R[x]$ if and only if f is irreducible in $K[x]$.

Week 6, Oct 30, Nov 1, 3: Criteria for irreducibility and Introduction to modules

Lecture 15: Criteria for irreducibility.

- Proved that if R is a UFD, so is $R[x]$. Discussed examples.
- Discussed various methods to show a polynomial is irreducible.
- Proved Eisenstein's criterion.

- Discussed examples including x^4+1 and $x^{p-1}+x^{p-2}+\cdots+x+1$ for a prime p.

Lecture 16: Introduction to Modules I.

- Defined a left R-module M (also called just an R-module) over a ring R.
- Discussed the case that $R=k$ is a field. In this case modules are simply vector spaces.
- Discussed examples of R-modules: (1) R, (2) R^n, (3) left ideals $I\subset R$, (4) if $R\to S$ is a ring homomorphism, S can be viewed as an R-module.
- Discussed modules over particular rings: (1) $R=k$ is a field, (2) $R=\mathbb{Z}$, and (3) $R=k[x]$.
- Defined ring homomorphisms (and isomorphisms, injections, surjections), kernels, and images.

Lecture 17: Introduction to Modules II.

- Defined quotients M/N of an R-module M by a submodule $N\subset M$.
- Discussed the isomorphism theorems for R-modules.
- Defined a basis of an R-module and the notion of a free module.
- Given a set $\{M_i\}_{i\in I}$ of R-modules, defined direct products $\prod_{i\in I} M_i$ and direct sums $\bigoplus_{i\in I} M_i$. Discussed their universal properties.
- Proved that an R-module M is free if and only if $M\cong\bigoplus_{i\in I} R$ for some index set I.
- Defined the annihilator $\mathrm{Ann}_R(M)$ of an R-module M. Showed that if M is free, then $\mathrm{Ann}_R(M)=0$. Gave the counterexample of $M=I=(x,y)$ of a module over $k[x,y]$ which has $\mathrm{Ann}_R(M)=0$ but is not free.

Week 7, Nov 6, 8

Lecture 18:

- Defined the condition that an R-module be finitely generated. Defined the rank of a module. Given an R-module M, also discussed the submodule $\mathrm{Tor}_R(M)=\{m\in M \mid rm=0 \text{ for some } r\neq 0\}$, and the ideal $\mathrm{Ann}_R(M)=\{r\in R\,|\,rm=0 \text{ for all } m \,\mathrm{im}\, M\}$.
- Stated Theorem 1: if R is a PID and M is a finitely generated R-module, then $M\cong R^r\oplus R/(a_1)\oplus\cdots\oplus R/(a_k)$ for $r\in\mathbb{Z}$ and non-zero and non-unit elements $a_1,\dots,a_k\in R$ such that $a_1|\cdots|a_k$.
- Discussed a general sketch of the proof.
- Stated Theorem 2: If R is a PID and $N\subset R^n$ is a submodule, then (1) N is free of some rank k, and (2) there exists a basis $y_1,\dots,y_n$ of R^n and non-zero elements $a_1|\cdots|a_k$ of R such that $a_1y_1,\dots,a_ky_k$ is a basis of N.
- Showed how Theorem 2 implies Theorem 1.

Lecture 19:

- Proved Theorem 2 from last lecture.

Week 8, Nov 13, 15, 17

Lecture 20:

- Showed using the Chinese Remainder Theorem that if R is a PID and $a=p_1^{n_1}\cdots p_s^{n_s}$ is a factorization of an element $a\in R$, then $R/a\cong R/p_1^{n_1}\oplus\cdots\oplus R/p_s^{n_s}$.
- Stated a second version of the classification of finitely generated modules over a PID: if R is a PID and M is a finitely generated R-module, then $M\cong R^r\oplus R/(p_1^{n_1})\oplus\cdots\oplus R/(p_s^{n_s})$ for $r\in\mathbb{Z}$ and irreducible (but not necessarily distinct) elements $p_i\in R$.

- Stated the uniqueness properties in the two versions of the main decompositions results for finitely generated modules over a PID. Sketched the proof of this.
- Showed how the first version of the classification theorem for finitely generated modules over a PID implies the rational canonical form of a linear transformation of a finite dimensional vector space. That is, we showed: if $T\colon V \to V$ is a linear transformation of a finite dimensional vector space V over k, then there exists polynomials $f_1|f_2|\cdots|f_k$ and a basis β of V such that the matrix representation of T with respect to β is a block diagonal matrix with blocks M_{f_i}, where each block has 1's along the subdiagonal and the negative of the coefficients of f_i in last column.

Lecture 21:

- Recalled the definition of the minimal and characteristic polynomial. With the notation of the rational canonical form (from the last lecture) of a linear transformation $T\colon V \to V$ of a finite dimensional vector space, we showed that: (1) the characteristic polynomial of T is $\prod_{i=1}^k f_k$, (2) the minimal polynomial is f_k, (3) (Cayley-Hamilton) the minimal polynomial divides the characteristic polynomial, and (4) the characteristic polynomial divides some power (in fact, the kth power suffices) of the minimal polynomial.
- Recalled the eigenvalues of a linear transformation.
- Showed how the second version of the classification theorem for finitely generated modules over a PID implies the Jordan canonical form of a linear transformation $T\colon V \to V$ of a finite dimensional vector space over k as the long as the characteristic polynomial of T splits over k.
- If R is a ring, we defined simple R-modules and composition series.
- We stated that any two composition series have the same length and the same factors after reordering. The proof follows from the same proof (from HW) for the analogous fact for composition series of finite groups.

Lecture 22:

- We discussed many examples of non-commutative rings.
- Recalled the group algebra $R[G]$ of a finite group G over a ring R.
- Recalled the matrix rings $M_n(R)$ over a ring R.
- If R is a ring and M is an R-module, we defined the endomorphism ring $\mathrm{End}_R(M)$ of M over R.
  - If $M = R$, we showed that $\mathrm{End}_R(R) = R^{\mathrm{op}}$ is the opposite ring.
  - If $I \subset R$ is a two-sided ideal, then $\mathrm{End}_R(R/I) = \mathrm{End}_{R/I}(R/I) = (R/I)^{\mathrm{op}}$.
  - If $M = R^n$, we showed that $\mathrm{End}_R(R^n) = M_n(R^{\mathrm{op}})$.
- If R is a ring, we defined the free algebra $R\langle x_1, \ldots, x_n\rangle$ over R.
- If R is a ring and M is an R-module, we showed that the following are equivalent: (1) M is simple, (2) $M = Rx$ for any non-zero $x \in M$, and (3) $M \cong R/I$ where $I \subset R$ is a maximal left ideal.
- We stated that an R-module M has a composition series if and only if M is Artinian and Noetherian. We proved the '$\Leftarrow$' direction.

Week 9, Nov 20, 22

Lecture 23:

- Recalled the proposition from last time that an R-module M has a composition series if and only if M is Artinian and Noetherian. We proved the '$\Rightarrow$' direction.
- If D is a division ring and $R = M_n(D)$, we showed that D^n is a simple R-module. Moreover, we gave a composition series of R as an R-module.
- We also discussed the simple modules over a PID.

- We proved Schur's lemma: If $\phi\colon M \to N$ is an R-module homomorphism of simple R-modules, then either $\phi = 0$ or ϕ is an isomorphism.
- We discussed the corollary: if M is a simple R-module, then $\mathrm{End}_R(M)$ is a division ring.
- If D is a division ring, $V = D^n$, and $R = \mathrm{End}_D(V) = M_n(D^{\mathrm{op}})$, then V is also an R-module. We proved that the morphism $D \to \mathrm{End}_R(V)$, which sends $d \in D$ to the R-module homomorphism $V \to V, v \mapsto dv$, is an isomorphism.
- We proved that if R is a k-algebra with k algebraically closed and M is a simple R-module of finite dimension as a k-vector space, then $\mathrm{End}_R(M) = k$.
- If G is a finite group, we defined a representation of G as a vector space V together with a group homomorphism $G \to \mathrm{GL}(V)$. We emphasized that a $\mathbb{C}[G]$-module is precisely the same as a representation of G and that under this corresponding a simple $\mathbb{C}[G]$-module corresponds to an irreducible representation of G.

Lecture 24:

- Discussed Zorn's lemma.
- Used Zorn's lemma to prove that if R is a ring and M is a semisimple R-module, then M contains a simple R-module.
- We defined an R-module M to be semisimple if and only if for every R-submodule $P \subset M$, there exists an R-submodule $Q \subset M$ such that $M \cong P \oplus Q$.
- We proved that any submodule or quotient module of a semisimple module is also semisimple.
- We proved that for an R-module M, the following are equivalent: (1) M is semisimple, (2) $M \cong \sum_{i \in I} M_i$ where each $M_i \subset M$ is simple, and (3) $M \cong \bigoplus_{i \in I} M_i$ where each $M_i \subset M$ is simple.

Week 10, Nov 27, 29, Dec 1

Lecture 25:

- We defined a semisimple module M to be of finite length if $M \cong \bigoplus_{i=1}^n M_i$ where each M_i is simple.
- We proved that if M is a semisimple R-module of finite length, then $\mathrm{End}_R(M) \cong M_{n_1}(D_1) \times \cdots \times M_{n_k}(D_k)$ for division rings $D_1, \ldots, D_k$.
- We defined a ring R to be semisimple if R is semisimple as a left R-module.
- We proved that for a ring R, the following are equivalent: (1) R is semisimple, (2) every R-module is semisimple, and (3) every short exact sequence of R-modules splits. Moreover, if these conditions hold, then R is of finite length and every simple R-module appears as an R-submodule of R.
- Discussed in detail the example of a matrix ring $M_n(D)$ over a division ring D.
- Discussed other examples.
- We proved the Artin–Wederburn theorem: if R is a semisimple ring, then R is isomorphic to a finite product of matrix rings over division rings; that is, $R \cong M_{n_1}(D_1) \times \cdots \times M_{n_k}(D_k)$ (as rings) where each D_i is a division ring.

Lecture 26: (Guest lecture by Jake Levinson)

- Proved that the ring $\mathbb{Z}[x_1, \ldots, x_n]^{S_n}$ of S_n-invariant polynomials is the polynomial ring $\mathbb{Z}]e_1, \ldots, e_n]$ genererated by the elementary symmetric polynomials $e_1, \ldots, e_n$.

Lecture 27: (Discussion section led by Charles Godfrey)

Lecture 28:

- If R is a ring, M is a right R-module, and N is a left R-module, we defined the concept of a middle linear map $M \times N \to G$ to an abelian group G.
- We defined the tensor product of M and N over R as an abelian group $M \otimes_R N$ together with a middle linear map $\phi\colon M \times N \to M \otimes_R N$ such that if $\psi\colon M \times N \to G$ is any other middle linear map, then there exists a unique group homomorphism $\chi\colon M \otimes_R N \to G$ such that $\psi = \chi \circ \phi$.
- We constructed $M \otimes_R N$ as the quotient of the free abelian group F generated by the set $M \times N$ by the subgroup generated by (1) $(x + x', y) - (x, y) - (x', y)$ for $x, x' \in M$ and $y \in N$, (2) $(x, y + y') - (x, y) - (x, y')$ for $x \in M$ and $y, y' \in N$ and (3) $(xr, y) = (x, ry)$ for $x \in M$, $y \in N$ and $r \in R$.
- Specializing to the case that R is commutative, we showed that if M and N are R-modules, then $M \otimes_R N$ is an R-module and $M \times N \to M \otimes_R N$ is bilinear. Moreover, $M \otimes_R N$ satisfies the following universal property: if $\psi\colon M \times N \to Q$ is a bilinear map to an R-module Q, then there is a unique R-module homomorphism $\chi\colon M \otimes_R N \to Q$ such that $\psi = \chi \circ \phi$.
- We established that $\mathrm{Hom}_R(M \otimes_R N, Q) \cong \mathrm{Hom}_R(M, \mathrm{Hom}_R(N, Q))$.

Lecture 29:

- Recalled the definition of the universal property of the tensor product $M \otimes_R N$ of modules M and N over a commutative ring R.
- Established that there are isomorphisms:
  1. $M \otimes_R N \to N \otimes_R M$ where $x \otimes y \mapsto y \otimes x$,
  2. $(M \otimes_R N) \otimes_R P \to M \otimes_R (N \otimes_R P)$ where $(x \otimes y) \otimes z \mapsto x \otimes (y \otimes z)$,
  3. $M \otimes_R R \to M$ where $x \otimes r \mapsto rx$, and
  4. $\left(\bigoplus_{i \in I} M_i\right) \otimes_R N \to \bigoplus_{i \in I}(M_i \otimes_R N)$ (here $\{M_i\}_{i \in I}$ is a set of R-modules) defined by $q_i(x) \otimes n \mapsto \tilde{q}_i(x \otimes n)$, where $q_i\colon M_i \to \bigoplus_{i \in I} M_i$ and $\tilde{q}_i\colon M_i \otimes_R N \to \bigoplus_{i \in I}(M_i \otimes_R N)$ denote the inclusions into the direct sums.
  5. Established that if M is a free R-module with basis $\{x_i\}_{i \in I}$ and N is a free R-module with basis $\{y_j\}_{j \in J}$, then $M \otimes_R N$ is a free R-module with basis $\{x_i \otimes y_j\}_{(i,j) \in I \times J}$. In particular, $R^n \otimes R^m \cong R^{nm}$.
  6. Discussed the following additional structures on tensor products. (1) If $R \to S$ is a ring homomorphism and M is an R-module, then $M \otimes_R S$ is naturally an S-module where if $s \in S$ and $m \otimes s' \in M \otimes_R S$, then $s \cdot (m \otimes s') = m \otimes (ss')$. (2) If $R \to S$ and $R \to T$ are ring homomorphisms, then $S \otimes_R T$ is naturally a ring where $(s \otimes t) \cdot (s' \otimes t') = (ss') \otimes (tt')$ for $s, s' \in S$ and $t, t' \in T$.

Lecture 30:

- We introduced the following classical question in 19th century invariant theory. Let $\mathrm{Pol}_d(\mathbb{C}^n)$ be the vector space of homogeneous degree d polynomials in $x_1, \dots, x_n$. Explicitly,
$$\mathrm{Pol}_d(\mathbb{C}^n) = \Big\{ \sum_{I = (i_1, \dots, i_n)} a_I x^I \;\mid\; a_I \in \mathbb{C} \Big\},$$
where $x^I = x_1^{i_1} \cdots x_n^{i_n}$.
- Defined the polynomial ring $\mathbb{C}[a_I]$ over the $\binom{n+d-1}{d}$ variables a_I. The ring $\mathbb{C}[a_i]$ is the ring of polynomials defined on the vector space $\mathrm{Pol}_d(\mathbb{C}^n)$. The group SL_n acts on $\mathbb{C}^n$ and therefore on $\mathrm{Pol}_d(\mathbb{C}^n)$ and therefore also on $\mathbb{C}[a_I]$.

- The question of tremendous importance in the 19th century was: What is the ring $\mathbb{C}[a_I]^{\mathrm{SL}_n}$ consisting of polynomials in the a_I's invariant under SL_n? Discussed the case of binary quadrics ($n = d = 2$) and binary quartics ($n = 2, d = 4$).
- We sketched Hilbert's original argument that the invariant ring $\mathbb{C}[a_I]^{\mathrm{SL}_n}$ is finitely generated as a $\mathbb{C}$-algebra.