Math 504: Modern Algebra, Fall Quarter 2017
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Homework 9
Due: Friday, December 8

Problem 9.1.

(a) If R and S are semisimple rings, show that $R \times S$ is a semisimple ring. Conclude that any finite product $M_{n_1}(D_1) \times \cdots \times M_{n_k}(D_k)$ of matrix rings over divisions rings D_i is semisimple.
(b) If R is a PID and $a \in R$, show that R/a is a semisimple ring if and only if a is squarefree (that is, there is no prime element $p \in R$ such that p^2 divides a).

Problem 9.2. Recall that the Artin–Wedernburn theorem on semisimple rings states that any semisimple ring is a finite product of matrix rings over divisions rings. Also recall from HW Problem 8.3 that the group algebra $\mathbb{C}[G]$ of a finite group is a semisimple ring. For each of the following groups, *explicitly* give an isomorphism of $\mathbb{C}[G]$ with a product of matrix rings over divisions rings.

(a) $G = \mathbb{Z}/n$ for any positive integer n, and
(b) $G = S_3$.

Problem 9.3 (Artin–Wederburn characterization of simple rings)**.** A ring R is called *simple* if R has no non-zero, proper two-sided ideals.[1] For a ring R, show that the following are equivalent:

(i) R is a simple Artinian ring,
(ii) $R \cong M_n(D)$ for a division ring D, and
(iii) R is semisimple and all simple R-modules are isomorphic.

Hint: Show directly that (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i). You are free to use anything proven in lecture or from previous homeworks. For (i) $\Rightarrow$ (iii), first use that R is Artinian to show that there exists a simple R-submodule $M \subset R$. Second, use that R is simple to show that $\operatorname{Ann}_R(M) = 0$. Third, use that R is Artinian to show that there is an injection $R \to M^n$ for some n (for instance, use the Artinian property to choose $\phi \colon R \to M^n$ such that $\ker(\phi) \subset R$ is minimal, and then show that such a ϕ is necessarily injective). Finally, conclude that (iii) holds.

Problem 9.4.

(a) Prove that $\mathbb{Z}/n \otimes_{\mathbb{Z}} \mathbb{Z}/m \cong \mathbb{Z}/d$ where d is the greatest common divisor of n and m.
(b) Prove that $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}$.
(c) Prove that if R is a commutative ring, $I \subset R$ is an ideal and M is an R-module, then $M \otimes_R R/I = M/IM$.
(d) Prove that there is an isomorphism of rings $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C} \times \mathbb{C}$.

[1] Warning: It is not true that a ring R is simple if and only if R is simple as a left R-module. For example, a matrix ring $M_n(D)$ over a division ring D is a simple ring (from HW Problem 7.6). But $M_n(D)$ it not simple as an R-module since, for instance, the subset $M_i \subset M_n(D)$, consisting of matrices where only the ith column is nonzero, is a nonzero proper left-ideal.

(e) Let $\mathbb{F}_{p^n}$ be the finite field with p^n elements where p is prime. Prove that there is an isomorphism of rings

$$\mathbb{F}_{p^n} \otimes_{\mathbb{F}_p} \mathbb{F}_{p^n} \cong \underbrace{\mathbb{F}_{p^n} \times \cdots \times \mathbb{F}_{p^n}}_{n \text{ times}}.$$

Hint: You may use the fact that $\mathbb{F}_{p^n} \cong \mathbb{F}_p[x]/(f)$ for some irreducible monic polynomial $f(x) \in \mathbb{F}_p[x]$ of degree n.

Problem 9.5. Let $\Lambda_n = \mathbb{Z}[x_1, \ldots, x_n]^{S_n}$ be the ring of symmetric functions. The *homogeneous symmetric function* $h_i \in \Lambda_n$ is the sum of all monomials of total degree i ($i \geq 0$):

$$\begin{aligned}
h_0 &= 1, \\
h_1 &= x_1 + \cdots + x_n, \\
h_2 &= x_1^2 + \cdots + x_n^2 + x_1x_2 + \cdots + x_{n-1}x_n, \\
h_3 &= \sum_i x_i^3 + \sum_{i,j} x_i^2 x_j + \sum_{i<j<k} x_i x_j x_k,
\end{aligned}$$

and so on. In class, we proved that Λ_n is a polynomial ring on the elementary symmetric functions $e_1, \ldots, e_n$. In this problem, we'll prove the "dual fundamental theorem of symmetric polynomials", that Λ_n is *also* a polynomial ring on $h_1, \ldots, h_n$. (For deep reasons from representation theory, the e_i's are "dual" to the h_i's.)

(a) Define power series $A(t), B(t) \in \Lambda_n[[t]]$ by $A(t) = \sum_{i\geq 0} h_i t^i$ and $B(t) = \sum_{i \geq 0} e_i t^i$. Show that

$$A(t) = \prod_{i=1}^{n} \frac{1}{1 - x_i t}, \qquad B(t) = \prod_{i=1}^{n} (1 + x_i t).$$

(These factorizations take place in $\mathbb{Z}[x_1, \ldots, x_n][[t]]$.)

Hint: For $A(t)$, use geometric series.

(b) Deduce that $A(t)B(-t) = 1$. By equating t^k coefficients, deduce *Newton's identities*: $e_0 h_0 = 1$, and for $k \geq 1$,

$$h_k - h_{k-1}e_1 + h_{k-2}e_2 - \cdots + (-1)^k e_k = 0.$$

Hint: Argue by induction that each e_i is a polynomial in $h_1, \ldots, h_i$ and vice versa.

(c) Show that $\Lambda_n = \mathbb{Z}[h_1, \ldots, h_n]$.

Hint: Define an abstract ring homomorphism $\phi : \mathbb{Z}[h_1, \ldots, h_n] \to \Lambda_n$ by sending h_i to h_i. Show ϕ is surjective using part (b) and the fact that $\Lambda_n \cong \mathbb{Z}[e_1, \ldots, e_n]$. To show ϕ is injective, argue that for each d, the degree-d parts of both rings have the same rank (as free abelian groups), namely the number of partitions of d with all parts $\leq n$. Note that h_i has degree i.