Math 504: Modern Algebra, Fall Quarter 2017
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Homework 8
Due: Wednesday, November 29

Problem 8.1. Use Zorn's lemma to show the following: if R is a ring and $I \subset R$ is a proper left ideal, then there exists a maximal left ideal $\mathfrak{m} \subset R$ containing I. (In particular, every non-zero ring has a maximal left ideal.)

Problem 8.2. Let k be a field and D a division algebra over k (i.e. D is a division ring and there is a ring homomorphism $k \to D$). Suppose that D is finite dimensional over k as a k-vector space.

(a) Show that for every element $a \in D$, there exists a monic polynomial $f(x) \in k[x]$ with $f(a) = 0$.
(b) Conclude that if k is algebraically closed, then $D = k$.

Problem 8.3 (Maschke's Theorem)**.** Let k be a field and G be a finite group such that $|G|$ is invertible in k (that is, the characteristic of k does not divide the order of G). Show that any $k[G]$-module is semisimple. (Note that this establishes that $k[G]$ is a semisimple ring.)

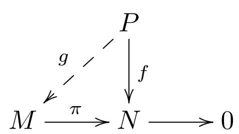
Hint: If M is a $k[G]$-module and $N \subset M$ is a $k[G]$-submodule, consider the quotient $\pi \colon M \to M/N$. First choose a section $s \colon M/N \to M$ as k-vector spaces (that is, s is a linear transformation such that $\pi \circ s = \mathrm{id}$). This section will not in general be a $k[G]$-module homomorphism but we can define a new section (which you need to check is a $k[G]$-module homomorphism) as follows:

$$s' \colon M/N \to M \qquad x \mapsto \frac{1}{|G|} \sum_{g \in G} e_g s(e_{g^{-1}} x).$$

Problem 8.4. Let R be a ring and M be an R-module. Provide a counterexample to the following assertion: if $N \subset M$ is an R-submodule such that both N and M/N are semisimple, then M is semisimple.

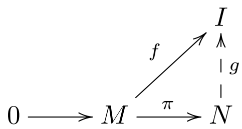
Problem 8.5 (Sorry—this is a long exercise!)**.** Let R be a ring.

(a) An R-module P is called *projective* if given any surjective homomorphism $\pi \colon M \to N$ of R-modules and an R-module homomorphism $f \colon P \to N$, there exists an R-module homomorphism $g \colon P \to M$ such that $f = \pi \circ g$. In other words, there is a dotted arrow g making the diagram



commute. Show that the following are equivalent for an R-module P:
  (i) P is projective;
  (ii) Every surjection $p \colon M \to P$ splits; that is, there is an R-module homomorphism $s \colon P \to M$ with $p \circ s = \mathrm{id}$; and
  (iii) There exists an R-module Q such that $P \oplus Q$ is a free R-module.

(b) Similarly, an R-module I is called *injective* if given any injective homomorphism $\pi\colon M \to N$ of R-modules and an R-module homomorphism $f\colon M \to I$, there exists an R-module homomorphism $g\colon N \to I$ such that $f = g \circ \pi$. In other words, there is a dotted arrow g making the diagram



commute. Show that the following are equivalent for an R-module I:

(i) I is injective; and

(ii) Every injection $s\colon I \to M$ splits; that is, there is an R-module homomorphism $p\colon M \to I$ with $p \circ s = \mathrm{id}$

Hint: To show that (ii) $\Rightarrow$ (i), given R-module homomorphisms $\pi\colon M \to N$ and $f\colon M \to I$, define

$$N \times_M I = (N \times I)/Q,$$

where $Q \subset N \times I$ is the R-submodule generated by elements of the form $(\pi(m), -f(m))$ for $m \in M$. Show that there is an injection $I \to N \times_M I$ and use this to get the desired homomorphism $N \to I$. (The construction $N \times_M I$ is an example of a *push-out* and it satisfies a universal property.)

Problem 8.6.

(a) Show directly that any module over a division ring is both projective and injective.

(b) If R is a PID, show that a finitely generated R-module is projective if and only if it is free.

(c) Show that $\mathbb{Q}$ is an injective $\mathbb{Z}$-module but is not a projective $\mathbb{Z}$-module.

Problem 8.7. Let R be a ring. Show that the following are equivalent:

(i) R is semisimple;

(ii) Every R-module is projective; and

(iii) Every R-module is injective.